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Relations between Schramm spaces and generalized Wiener classes

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ABSTRACT

We give necessary and sufficient conditions for the embeddings $\Lambda\mathrm{BV}^{(p)} \subseteq \Gamma\mathrm{BV}^{(q_n\uparrow q)}$ and $\Phi\mathrm{BV} \subseteq \mathrm{BV}^{(q_n\uparrow q)}$. As a consequence, a number of results in the literature, including a fundamental theorem of Perlman and Waterman, are simultaneously extended.

1. Introduction and main results

Let $\Lambda = \{\lambda_j\}_{j=1}^{\infty}$ be a nondecreasing sequence of positive numbers such that $\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty$. Following [1], we call Λ a Waterman sequence. Let $\Phi = \{\phi_j\}_{j=1}^{\infty}$ be a sequence of increasing convex functions on $[0, \infty)$ with $\phi_j(0) = 0$. We say that Φ is a Schramm sequence if $0 < \phi_{j+1}(x) \leq \phi_j(x)$ for all j and $\sum_{j=1}^{\infty} \phi_j(x) = \infty$ for all $x > 0$. This terminology is used throughout.

We begin by recalling two generalizations of the concept of bounded variation which are central to our work.

Definition 1.1. A real-valued function f on $[a, b]$ is said to be of Φ-bounded variation if

$$V_\Phi(f) = V_\Phi(f; [a, b]) = \sup \sum_{j=1}^{n} \phi_j(|f(I_j)|) < \infty,$$

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where the supremum is taken over all finite collections $\{I_j\}_{j=1}^n$ of nonoverlapping subintervals of $[a,b]$ and $f(I_j) = f(\sup I_j) - f(\inf I_j)$. We denote by ΦBV the linear space of all functions f such that cf is of Φ-bounded variation for some $c > 0$.

If for every $f \in \Phi\mathrm{BV}$, we define

$$\|f\| := |f(a)| + \inf\{c > 0 : V_\Phi(f/c) \le 1\},$$

then it is easily seen that $\|\cdot\|$ is a norm, and ΦBV endowed with this norm turns into a Banach space. The space ΦBV is introduced in Schramm's paper [15]. For more information about ΦBV, the reader is referred to [1].

If ϕ is a strictly increasing convex function on $[0,\infty)$ with $\phi(0) = 0$, and if $\Lambda = \{\lambda_j\}_{j=1}^\infty$ is a Waterman sequence, by taking $\phi_j(x) = \phi(x)/\lambda_j$ for all j, we get the class $\phi\Lambda$BV of functions of $\phi\Lambda$-bounded variation. This class was introduced by Schramm and Waterman in [16] (see also [17] and [11]). More specifically, if $\phi(x) = x^p$ ($p \ge 1$), we get the Waterman–Shiba class $\Lambda\mathrm{BV}^{(p)}$, which was introduced by Shiba in [18]. When $p = 1$, we obtain the well-known Waterman class ΛBV.

In the case $\lambda_j = 1$ for all j, we obtain the class ϕBV of functions of ϕ-bounded variation introduced by Young [26]. More specifically, when $\phi(x) = x^p$ ($p \ge 1$), we obtain the Wiener class BV_p (see [24]), and taking $p = 1$, we have the well-known Jordan class BV.

Remark 1.2. One can easily observe that functions of Φ-bounded variation are bounded and can only have simple discontinuities (countably many of them, indeed). The class ΦBV has many applications in Fourier analysis as well as in treating topics such as convergence, summability, etc. (see [24,26,21–23,12,15]).

Definition 1.3. Let $\{q_n\}_{n=1}^\infty$ and $\{\delta_n\}_{n=1}^\infty$ be sequences of positive real numbers such that $1 \le q_n \uparrow q \le \infty$ and $2 \le \delta_n \uparrow \infty$. A real-valued function f on $[a,b]$ is said to be of q_n-Λ-bounded variation if

$$V_\Lambda(f) = V_\Lambda(f; q_n \uparrow q; \delta) := \sup_{n\ge1} \sup_{\{I_j\}} \Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{q_n}}{\lambda_j} \Big)^{\frac{1}{q_n}} < \infty,$$

where the $\{I_j\}_{j=1}^s$ are collections of nonoverlapping subintervals of $[a,b]$ such that $\inf_j |I_j| \ge \frac{b-a}{\delta_n}$. The class of functions of q_n-Λ-bounded variation is denoted by $\Lambda\mathrm{BV}^{(q_n\uparrow q)}$ $(= \Lambda\mathrm{BV}_\delta^{(q_n\uparrow q)})$. In the sequel, we suppose that $[a,b] = [0,1]$.

The class $\Lambda\mathrm{BV}^{(q_n\uparrow q)}$ was introduced by Vyas in [19]. When $\lambda_j = 1$ for all j and $\delta_n = 2^n$ for all n, we get the class $\mathrm{BV}^{(q_n\uparrow q)}$—introduced by Kita and Yoneda (see [9])—which in turn recedes to the Wiener class BV_q, when $q_n = q$ for all n.

A natural and important problem is to determine relations between the above-mentioned classes; see [21,12,4,9,6,13,8,5] for some results in this direction. In particular, Perlman and Waterman found the fundamental characterization of embeddings between ΛBV classes in [12]. Ge and Wang characterized the embeddings $\Lambda\mathrm{BV} \subseteq \phi\mathrm{BV}$ and $\phi\mathrm{BV} \subseteq \Lambda\mathrm{BV}$ (see [5]). It was shown by Kita and Yoneda in [9] that the embedding $\mathrm{BV}_p \subseteq \mathrm{BV}^{(p_n\uparrow\infty)}$ is both automatic and strict for all $1 \le p < \infty$. Furthermore, Goginava characterized the embedding $\Lambda\mathrm{BV} \subseteq \mathrm{BV}^{(q_n\uparrow\infty)}$ in [6], and a characterization of the embedding $\Lambda\mathrm{BV}^{(p)} \subseteq \mathrm{BV}^{(q_n\uparrow q)}$ $(1 \le q \le \infty)$ was given by Hormozi, Prus-Wiśniowski and Rosengren in [8]. In this paper, we investigate the embeddings $\Lambda\mathrm{BV}^{(p)} \subseteq \Gamma\mathrm{BV}^{(q_n\uparrow q)}$ and $\Phi\mathrm{BV} \subseteq \mathrm{BV}^{(q_n\uparrow q)}$ $(1 \le q \le \infty)$. The problem as to when the reverse embeddings hold is also considered, which turns out to have a simple answer (see Remark 1.10(ii) below).

Throughout this paper, the letters Λ and Γ are reserved for a typical Waterman sequence. We associate to Λ a function which we still denote by Λ and define it as $\Lambda(r) := \sum_{j=1}^{[r]} \frac{1}{\lambda_j}$ for $r \ge 1$. The function $\Lambda(r)$ is clearly nondecreasing and $\Lambda(r) \to \infty$ as $r \to \infty$. Our first main result reads as follows.

Theorem 1.4. *Let $1 \leq p \leq q_n \uparrow q \leq \infty$. Then, a necessary and sufficient condition for the embedding $\Lambda\mathrm{BV}^{(p)} \subset \Gamma\mathrm{BV}^{(q_n \uparrow q)}$ is*

$$\limsup_{n\to\infty} \max_{1\leq k\leq \delta_n} \Gamma(k)^{\frac{1}{q_n}} \Lambda(k)^{-\frac{1}{p}} < \infty. \tag{1.1}$$

Moreover, if the hypothesis is replaced by the condition that $\{\Gamma(n)/\Lambda(n)\}_{n=1}^{\infty}$ be nondecreasing, then the conclusion of the theorem still holds true.

An important consequence of Theorem 1.4 is the following corollary, which is indeed a nontrivial extension of [12, Theorem 3].

Corollary 1.5. *Let $1 \leq p \leq q < \infty$. Then, a necessary and sufficient condition for the embedding $\Lambda\mathrm{BV}^{(p)} \subseteq \Gamma\mathrm{BV}^{(q)}$ is*

$$\sup_{1\leq n<\infty} \frac{\Gamma(n)^{\frac{1}{q}}}{\Lambda(n)^{\frac{1}{p}}} < \infty.$$

Corollary 1.6. ([8, Theorem 1]) *Let $1 \leq p < \infty$. Then, a necessary and sufficient condition for the embedding $\Lambda\mathrm{BV}^{(p)} \subseteq \mathrm{BV}^{(q_n \uparrow q)}$ is*

$$\limsup_{n\to\infty} \max_{1\leq k\leq \delta_n} \frac{k^{\frac{1}{q_n}}}{\left(\sum_{i=1}^{k} \frac{1}{\lambda_i}\right)^{\frac{1}{p}}} < \infty.$$

Next corollary extends [9, Lemma 2.1].

Corollary 1.7. *Let $1 < q \leq \infty$. Then, we have*

$$\bigcup_{1\leq p<q} \Lambda\mathrm{BV}^{(p)} \subseteq \Lambda\mathrm{BV}^{(q_n \uparrow q)}.$$

If $\Phi = \{\phi_j\}_{j=1}^{\infty}$ is a Schramm sequence, we define $\Phi_k(x) := \sum_{j=1}^{k} \phi_j(x)$ for $x \geq 0$. Then $\Phi_k(x)$ is clearly an increasing convex function on $[0, \infty)$ such that $\Phi_k(0) = 0$ and $\Phi_k(x) > 0$ for $x > 0$. Without loss of generality we assume that $\Phi_k(x)$ is strictly increasing on $[0, \infty)$. Let $\Phi_k^{-1}(x)$ be the inverse function of $\Phi_k(x)$. Our next main result can be formulated as follows.

Theorem 1.8. *A necessary and sufficient condition for the embedding $\Phi\mathrm{BV} \subset \mathrm{BV}^{(q_n \uparrow q)}$ is*

$$\limsup_{n\to\infty} \max_{1\leq k\leq \delta_n} k^{\frac{1}{q_n}} \Phi_k^{-1}(1) < \infty. \tag{1.2}$$

Corollary 1.9. *A necessary and sufficient condition for the embedding $\phi\Lambda\mathrm{BV} \subset \mathrm{BV}^{(q_n \uparrow q)}$ is*

$$\limsup_{n\to\infty} \max_{1\leq k\leq \delta_n} k^{\frac{1}{q_n}} \phi^{-1}(\Lambda(k)^{-1}) < \infty.$$

Remark 1.10. (i) When $\phi(x) = x^p$, $1 \leq p < \infty$, Corollary 1.9 yields Corollary 1.6 as a special case.

(ii) By [9, Theorem 3.3], the class $\mathrm{BV}^{(q_n \uparrow \infty)}$ always contains a function with nonsimple discontinuities. Since clearly $\mathrm{BV}^{(q_n \uparrow \infty)} \subseteq \Lambda\mathrm{BV}^{(q_n \uparrow \infty)}$, this is also the case for the class $\Lambda\mathrm{BV}^{(q_n \uparrow \infty)}$. On the other hand, as pointed out in Remark 1.2, the functions in the classes $\Phi\mathrm{BV}$ and $\Lambda\mathrm{BV}^{(p)}$ can only have simple discontinuities. Hence, the corresponding reverse embeddings can never happen.

2. An auxiliary inequality

In this section we establish an inequality (see (2.1) below) which plays a crucial role in the sufficiency part of the proof of Theorem 1.4. Also some applications of it are presented in Corollary 2.2 and Remark 2.3. The following proposition is indeed a generalization of [10, Lemma].

Proposition 2.1. *Let $1 \leq q < \infty$ and $n \in \mathbb{N}$. Then*

$$\Big(\sum_{j=1}^{n} x_j^q z_j\Big)^{\frac{1}{q}} \leq \sum_{j=1}^{n} x_j y_j \max_{1\leq k\leq n} \Big(\sum_{j=1}^{k} z_j\Big)^{\frac{1}{q}} \Big(\sum_{j=1}^{k} y_j\Big)^{-1}, \tag{2.1}$$

where $\{x_j\}$, $\{y_j\}$ and $\{z_j\}$ are positive nonincreasing sequences.

Proof. Without loss of generality we may assume that $\sum_{j=1}^{n} x_j y_j = 1$. With this in mind, it is enough to prove that the maximum value of $\sum_{j=1}^{n} x_j^q z_j$ under above assumptions is

$$\max_{1\leq k\leq n} \Big(\sum_{j=1}^{k} z_j\Big)\Big(\sum_{j=1}^{k} y_j\Big)^{-q}.$$

We claim that the solution to this problem satisfies condition

$$x_1 = x_2 = \cdots = x_k > x_{k+1} = x_{k+2} = \cdots = x_n = 0 \tag{2.2}$$

for some $1 \leq k \leq n$. To prove our claim, we suppose to the contrary that there exists a solution which does not satisfy condition (2.2). Then for some $1 \leq k \leq n$, we have $x_{k+1} > 0$ and

$$x_1 = x_2 = \cdots = x_k > x_{k+1} \geq x_{k+2} \geq \cdots \geq x_n \geq 0.$$

Put

$$A := \sum_{j=1}^{k} x_j y_j, \quad B := \sum_{j=k+1}^{n} x_j y_j, \quad C := \frac{x_{k+1}}{x_k},$$

and define

$$A\eta(t) + Bt = 1.$$

Then the n-tuple

$$(\eta(t)x_1, \eta(t)x_2, \cdots, \eta(t)x_k, tx_{k+1}, \cdots, tx_n)$$

satisfies conditions of the problem, whenever $0 \leq t < 1/AC + B$. Now define

$$f(t) := \eta(t)^q \sum_{j=1}^{k} x_j^q z_j + t^q \sum_{j=k+1}^{n} x_j^q z_j$$

and consider two possibilities:

1) If $q > 1$ then

$$f''(t) = q(q-1)\Big(\eta(t)^{q-2}(\eta'(t))^2 \sum_{j=1}^{k} x_j^q z_j + t^{q-2} \sum_{j=k+1}^{n} x_j^q z_j\Big)$$

and hence $f''(1) > 0$ which in turn implies that f has a local minimum at $t = 1$. This is a contradiction.

2) If $q = 1$ then $f(t)$ is linear. Consequently,

$$A \sum_{j=k+1}^{n} x_j^q z_j - B \sum_{j=1}^{k} x_j^q z_j = 0$$

which implies that the problem has a solution satisfying condition (2.2). This completes the proof. □

Let f be a bounded function on $[0, 1]$. The modulus of variation of f is the sequence ν_f and is defined by

$$\nu_f(n) := \sup \sum_{j=1}^{n} |f(I_j)|,$$

where the supremum is taken over all finite collections $\{I_j\}_{j=1}^n$ of nonoverlapping subintervals of $[0, 1]$. The modulus of variation of f is nondecreasing and concave. A sequence ν with such properties is called a modulus of variation. The symbol $V[v]$ denotes the class of all functions f for which there exists a constant $C > 0$ (depending on f) such that $\nu_f(n)/\nu(n) \leq C$ for all n (see [3]). The following corollary is an immediate consequence of inequality (2.1).

Corollary 2.2. ([2, Theorem 1]) *The following inclusion holds.*

$$\Lambda\mathrm{BV} \subseteq V[n\Lambda(n)^{-1}].$$

Proof. Let $\{I_j\}_{j=1}^n$ be a collection of nonoverlapping subintervals of $[0, 1]$. If $f \in \Lambda\mathrm{BV}$, $q = 1$, $x_j = |f(I_j)|$, $y_j = 1/\lambda_j$ and $z_j = 1$, from (2.1) we obtain

$$\sum_{j=1}^{n} |f(I_j)| \leq \sum_{j=1}^{n} \frac{|f(I_j)|}{\lambda_j} \max_{1\leq k\leq n} k\Lambda(k)^{-1} \leq V_\Lambda(f) n\Lambda(n)^{-1},$$

which means that $f \in V[n\Lambda(n)^{-1}]$. □

Remark 2.3. Let $\Lambda = \{\lambda_j\}$ and $\Gamma = \{\gamma_j\}$ be Waterman sequences. As stated on page 181 of [14], Perlman and Waterman have shown, in the course of the proof of [12, Theorem 3], that if there is a constant C such that

$$\sum_{j=1}^{n} \frac{1}{\gamma_j} \leq C \sum_{j=1}^{n} \frac{1}{\lambda_j} \quad \text{for all } n,$$

then, given any nonincreasing sequence $\{a_j\}$ of nonnegative numbers,

$$\sum_{j=1}^{n} \frac{a_j}{\gamma_j} \leq C \sum_{j=1}^{n} \frac{a_j}{\lambda_j}.$$

It is worth mentioning that one can easily see that this is a simple consequence of inequality (2.1) above.

3. Proofs of main results

Proof of Theorem 1.4. Necessity. We proceed by contraposition. If (1.1) does not hold, using the fact that $\Gamma(r) \to \infty$ as $r \to \infty$, we may, without loss of generality, assume that $\gamma_1 = 1$ and for each n

$$\Gamma(\delta_n) \geq 2^{n+2}, \tag{3.1}$$

and

$$\Gamma(r_n)^{\frac{1}{q_n}} \Lambda(r_n)^{-\frac{1}{p}} > \; 2^{4n} \tag{3.2}$$

for some integer r_n, $1 \leq r_n \leq \delta_n$.

We are going to construct a function f in $\Lambda \mathrm{BV}^{(p)}$ that does not belong to $\Gamma \mathrm{BV}^{(q_n \uparrow q)}$. To this end, let s_n be the greatest integer such that $2s_n - 1 \leq 2^{-n}\Gamma(\delta_n)$ and put $t_n = \min\{r_n, s_n\}$. We define a sequence of functions $\{f_n\}_{n=1}^{\infty}$ on $[0,1]$ as follows:

$$f_n(x) := \begin{cases} 2^{-n}\Lambda(r_n)^{-\frac{1}{p}} \;, & x \in [2^{-n} + \frac{2j-2}{\delta_n}, 2^{-n} + \frac{2j-1}{\delta_n});\ 1 \leq j \leq t_n, \\ 0 & \text{otherwise.} \end{cases}$$

The functions f_n, defined in this fashion, have disjoint supports and therefore $f(x) := \sum_{n=1}^{\infty} f_n(x)$ is a well-defined function on $[0,1]$. In addition, we have

$$V_\Lambda(f) \leq \sum_{n=1}^{\infty} V_\Lambda(f_n) = \sum_{n=1}^{\infty} \Big(\sum_{j=1}^{2t_n} \frac{(2^{-n}\Lambda(r_n)^{-\frac{1}{p}})^p}{\lambda_j} \Big)^{\frac{1}{p}}$$

$$\leq \sum_{n=1}^{\infty} 2^{-n+1} \Big(\sum_{j=1}^{r_n} \frac{\Lambda(r_n)^{-1}}{\lambda_j} \Big)^{\frac{1}{p}} = \sum_{n=1}^{\infty} 2^{-n+1} \Big(\frac{\Lambda(r_n)}{\Lambda(r_n)} \Big)^{\frac{1}{p}} < \infty,$$

since the sequence $\{\Lambda(r_n)^{-1}\}_{n=1}^{\infty}$ is nonincreasing and $t_n \leq r_n$. This means that $f \in \Lambda \mathrm{BV}^{(p)}$.

On the other hand, $f \notin \Gamma \mathrm{BV}^{(q_n \uparrow q)}$. To see this, note that the definition of s_n implies $2(s_n + 1) - 1 > 2^{-n}\Gamma(\delta_n)$. Combining this with (3.1), we obtain $\Gamma(2s_n - 1) \geq 2^{-n-1}\Gamma(\delta_n)$. Consequently, if $t_n = s_n$, then the preceding inequality means that

$$\Gamma(2t_n - 1) \geq 2^{-n-1}\Gamma(\delta_n) \geq 2^{-n-1}\Gamma(r_n),$$

since $r_n \leq \delta_n$. Also, if $t_n = r_n$, clearly $2t_n - 1 \geq r_n$ and hence $\Gamma(2t_n - 1) \geq \Gamma(r_n)$, since $\Gamma(r)$ is increasing. Thus, we have shown

$$\Gamma(2t_n - 1) \geq 2^{-n-1}\Gamma(r_n), \qquad \text{for all } n. \tag{3.3}$$

Finally, the intervals

$$I_j := \Big[2^{-n} + \tfrac{j-1}{\delta_n}, 2^{-n} + \tfrac{j}{\delta_n}\Big], \qquad j = 1, \ldots, 2t_n - 1,$$

have length $\frac{1}{\delta_n}$ for each n, and thus

$$V_\Gamma(f) \geq \Big(\sum_{j=1}^{2t_n-1} \frac{|f(I_j)|^{q_n}}{\gamma_j} \Big)^{\frac{1}{q_n}} = \Big(\Gamma(2t_n - 1)(2^{-n}\Lambda(r_n)^{-\frac{1}{p}})^{q_n} \Big)^{\frac{1}{q_n}}$$

$$\geq 2^{-n} \Big(2^{-n-1}\Gamma(r_n)(\Lambda(r_n)^{-\frac{1}{p}})^{q_n} \Big)^{\frac{1}{q_n}} \geq 2^n,$$

where the last two inequalities are due to (3.3) and (3.2), respectively. As a result, $V_\Gamma(f)$ is not finite.

Sufficiency. Assume (1.1) and let $f \in \Lambda\mathrm{BV}^{(p)}$. Let $\{I_j\}_{j=1}^s$ be a nonoverlapping collection of subintervals of $[0,1]$ with $\inf |I_j| \geq 1/\delta_n$, and let $q = q_n/p \geq 1$, $x_j = |f(I_j)|^p$, $y_j = 1/\lambda_j$, $z_j = 1/\gamma_j$. By [7, Theorem 368], we may also assume that the x_j's are arranged in descending order. Now, we can apply (2.1) to get

$$\Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{q_n}}{\gamma_j} \Big)^{\frac{1}{q_n}} \leq \Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{p}}{\lambda_j} \Big)^{\frac{1}{p}} \max_{1\leq k\leq s} \Gamma(k)^{\frac{1}{q_n}}\Lambda(k)^{-\frac{1}{p}}$$

$$\leq \Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{p}}{\lambda_j} \Big)^{\frac{1}{p}} \max_{1\leq k\leq \delta_n} \Gamma(k)^{\frac{1}{q_n}}\Lambda(k)^{-\frac{1}{p}},$$

where the second inequality is a consequence of $s \leq \delta_n$. Taking suprema over all collections $\{I_j\}_{j=1}^s$ as above, and over all n yields

$$V_\Gamma(f) \leq V_\Lambda(f) \sup_n \max_{1\leq k\leq \delta_n} \Gamma(k)^{\frac{1}{q_n}}\Lambda(k)^{-\frac{1}{p}} < \infty.$$

Hence $f \in \Gamma\mathrm{BV}^{(q_n\uparrow q)}$ and the first part of the theorem is proved.

To prove the second part, let us assume that $\{\Gamma(n)/\Lambda(n)\}_{n=1}^\infty$ is nondecreasing. Observe that the proof of necessity is identical to that given in the first part. For sufficiency, note that the only case which needs to be justified is when $q_n < p$ for some n. If this is the case, we first apply (2.1) with $q = 1$ to obtain

$$\sum_{j=1}^{s} \frac{|f(I_j)|^{p}}{\gamma_j} \leq \sum_{j=1}^{s} \frac{|f(I_j)|^{p}}{\lambda_j} \max_{1\leq k\leq s} \Gamma(k)\Lambda(k)^{-1}. \tag{3.4}$$

Then an application of Hölder's inequality yields

$$\sum_{j=1}^{s} \frac{|f(I_j)|^{q_n}}{\gamma_j} = \sum_{j=1}^{s} \Big(\frac{|f(I_j)|^{p}}{\gamma_j} \Big)^{\frac{q_n}{p}} \gamma_j^{\frac{q_n}{p}-1}$$

$$\leq \Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{p}}{\gamma_j} \Big)^{\frac{q_n}{p}} \Gamma(s)^{1-\frac{q_n}{p}}$$

$$\leq \Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{p}}{\lambda_j} \Big)^{\frac{q_n}{p}} \Gamma(s)^{1-\frac{q_n}{p}} \max_{1\leq k\leq s} \Gamma(k)^{\frac{q_n}{p}}\Lambda(k)^{-\frac{q_n}{p}}$$

$$\leq \Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{p}}{\lambda_j} \Big)^{\frac{q_n}{p}} \max_{1\leq k\leq \delta_n} \Gamma(k)\Lambda(k)^{-\frac{q_n}{p}},$$

where the last two inequalities are due, respectively, to (3.4) and the fact that $\{\Gamma(n)/\Lambda(n)\}_{n=1}^\infty$ is nondecreasing. □

Proof of Theorem 1.8. Necessity. Suppose (1.2) does not hold. Then, without loss of generality, we may assume that for each n

$$\delta_n \geq 2^{n+2},$$

and

$$r_n^{\frac{1}{q_n}} \Phi_{r_n}^{-1}(1) > \ 2^{4n} \tag{3.5}$$

for some integer r_n, $1 \leq r_n \leq \delta_n$.

We will now construct a function $f \in \Phi\mathrm{BV}$ such that $f \notin \mathrm{BV}^{(q_n \uparrow q)}$. To do so, let s_n be the greatest integer such that $2s_n - 1 \leq 2^{-n}\delta_n$, let $t_n = \min\{r_n, s_n\}$ and consider the sequence $\{f_n\}_{n=1}^{\infty}$ of functions on $[0,1]$ defined in the following way:

$$f_n(x) := \begin{cases} 2^{-n}\Phi_{r_n}^{-1}(1)\ , & x \in [2^{-n} + \frac{2j-2}{\delta_n}, 2^{-n} + \frac{2j-1}{\delta_n});\ 1 \leq j \leq t_n, \\ 0 & \text{otherwise.} \end{cases}$$

Since the f_n's have disjoint supports, $f(x) := \sum_{n=1}^{\infty} f_n(x)$ is a well-defined function on $[0,1]$. Thus, using convexity of the Φ_{r_n}'s we have

$$\begin{aligned} V_\Phi(f) \leq \ \sum_{n=1}^{\infty} V_\Phi(f_n) \ &= \sum_{n=1}^{\infty}\sum_{j=1}^{2t_n} \phi_j(2^{-n}\Phi_{r_n}^{-1}(1)) = \sum_{n=1}^{\infty} \Phi_{2t_n}(2^{-n}\Phi_{r_n}^{-1}(1)) \\ &\leq \sum_{n=1}^{\infty} \Phi_{2r_n}(2^{-n}\Phi_{r_n}^{-1}(1)) \leq \sum_{n=1}^{\infty} 2\Phi_{r_n}(2^{-n}\Phi_{r_n}^{-1}(1)) < \infty, \end{aligned}$$

that is, $f \in \Phi\mathrm{BV}$.

In conclusion, let us show that $f \notin \mathrm{BV}^{(q_n \uparrow q)}$. To this end, proceeding in the same way as in the proof of Theorem 1.4, we obtain

$$2t_n - 1 \geq 2^{-n-1} r_n, \qquad \text{for all } n. \tag{3.6}$$

Since for every n, all intervals

$$I_j := \left[2^{-n} + \tfrac{j-1}{\delta_n}, 2^{-n} + \tfrac{j}{\delta_n}\right], \qquad j = 1, ..., 2t_n - 1,$$

have length $\frac{1}{\delta_n}$, we get

$$\begin{aligned} V(f; q_n \uparrow q, \delta) \geq \Big(\sum_{j=1}^{2t_n-1} |f(I_j)|^{q_n}\Big)^{\frac{1}{q_n}} &= \Big((2t_n - 1)(2^{-n}\Phi_{r_n}^{-1}(1))^{q_n}\Big)^{\frac{1}{q_n}} \\ &\geq 2^{-n}\Big(2^{-n-1} r_n (\Phi_{r_n}^{-1}(1))^{q_n}\Big)^{\frac{1}{q_n}} \geq 2^n, \end{aligned}$$

where the last two inequalities are results of (3.6) and (3.5), respectively. Therefore, $f \notin \mathrm{BV}^{(q_n \uparrow q)}$.

Sufficiency. Let $f \in \Phi\mathrm{BV}$. To show that $f \in \mathrm{BV}^{(q_n \uparrow q)}$, it suffices to prove the inequality

$$V(f; q_n \uparrow q; \delta) \leq C \sup_n \max_{1 \leq k \leq \delta_n} k^{\frac{1}{q_n}} \Phi_k^{-1}(1), \tag{3.7}$$

where C is a positive constant depending solely on f.

In the course of the proof of Theorem 2.1 in [25], the author proceeds to estimate $(\sum_{j=1}^{n} x_j^q)^{\frac{1}{q}}$ under the restriction

$$\sum_{j=1}^{n} \phi_j(x_{\tau(j)}) \le V_\Phi(f),$$

where the x_j's are arranged in descending order and τ is any permutation of n letters. Using Wang's approach in [20], he finds the following:

$$\Big(\sum_{j=1}^{n} x_j^q\Big)^{\frac{1}{q}} \le 16 \max_{1\le k\le n} k^{\frac{1}{q}} \Phi_k^{-1}(V_\Phi(f)). \tag{3.8}$$

To prove (3.7), consider a nonoverlapping collection $\{I_j\}_{j=1}^{s}$ of subintervals of $[0,1]$ with $\inf|I_j| \ge 1/\delta_n$. If we put $q = q_n$, $x_j = |f(I_j)|$, and if the x_j's are rearranged in descending order, then we may apply (3.8) to obtain

$$\begin{aligned}\Big(\sum_{j=1}^{s} |f(I_j)|^{q_n}\Big)^{\frac{1}{q_n}} &\le 16 \max_{1\le k\le s} k^{\frac{1}{q_n}} \Phi_k^{-1}(V_\Phi(f)) \\ &\le 16 \max_{1\le k\le \delta_n} k^{\frac{1}{q_n}} \Phi_k^{-1}(V_\Phi(f)).\end{aligned}$$

Taking suprema and using concavity of the Φ_k^{-1}'s yields (3.7) with $C = 16(1 + V_\Phi(f))$. □

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