# Freeway traffic management via kinetic compartmental models

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*July 4, 2025*

# Outline

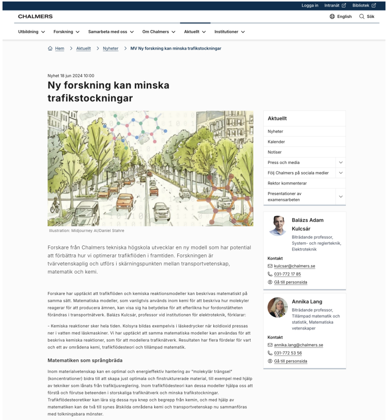
# OUTLINE

- Aims and motivation
- Road traffic models
- Traffic Reaction Model
- Traffic density estimation
- Summary and future directions

# Aims and motivation

# AIMS AND MOTIVATION





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Nyhet 18 jun 2024 10:00

**Ny forskning kan minska trafikstockningar**

Illustration: Midjourney AI/Daniel Stahre

Forskare från Chalmers tekniska högskola utvecklar en ny modell som har potential att förbättra hur vi optimerar trafikflöden i framtiden. Forskningen är tvärvetenskaplig och utförs i skärningspunkten mellan transportvetenskap, matematik och kemi.

Forskare har upptäckt att trafikflöden och kemiska reaktionsmodeller kan beskrivas matematiskt på samma sätt. Matematiska modeller, som vanligtvis används inom kemi för att beskriva hur molekyler reagerar för att producera ämnen, kan visa sig ha betydelse för att efterlikna hur fordonstätheten förändras i transportnätverk. Balázs Kulcsár, professor vid institutionen för elektroteknik, förklarar:

- Kemiska reaktioner sker hela tiden. Kolsyra bildas exempelvis i läskedrycker när koldioxid pressas ner i vatten med läskmaskiner. Vi har upptäckt att samma matematiska modeller kan användas för att beskriva kemiska reaktioner, som för att modellera trafiknätverk. Resultaten har flera fördelar för vart och ett av områdena kemi, trafikflödesteori och tillämpad matematik.

**Matematiken som språngbräda**

Inom materialvetenskap kan en optimal och energieffektiv hantering av "molekylär trängsel" (koncentrationer) bidra till att skapa just optimala och finstrukturerade material, till exempel med hjälp av tekniker som lånats från trafikljusreglering. Inom trafikflödesteori kan dessa modeller hjälpa oss att förstå och förutse beteenden i storskaliga trafiknätverk och minska trafikstockningar. Trafikflödesteoretiker kan lära sig dessa nya knep och begrepp från kemin, och med hjälp av matematiken kan de två till synes åtskilda områdena kemi och transportvetenskap nu sammanföras med tolkningsbara mönster.

Aktuellt
- Nyheter
- Kalender
- Notiser
- Press och media
- Följ Chalmers på sociala medier
- Rektor kommenterar
- Presentationer av examensarbeten

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- Unusual way of interpreting traffic
- Chalmers centered development from 2020
- Multidisciplinary: math, chemistry, traffic flow theory
- Molecules as cars? Cars as molecules?
- Joint cluster work with Mike Pereira (Mines Paris - PSL, France), Jean Auriol (Supelec, France), Gyorgy Liptak, Gabor Szederkenyi, Mihaly Kovacs (Hungarian Academy of Sciences, Pazmany P Catholic University), Annika Lang (Mathematics, Chalmers), Pinar Boyraz Baykas (Mechanical Engineering, Chalmers), Sondre Wiersdalen (Electrical Engineering, Chalmers).

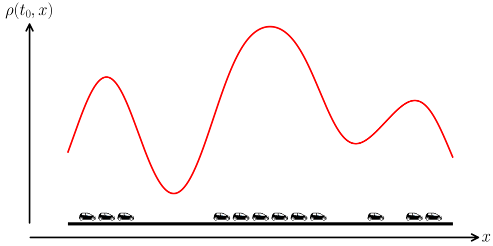
# Road traffic models

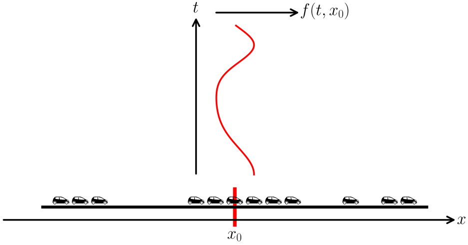
# ROAD TRAFFIC MODELS

- Two quantities of interest defined for any time $t \geq 0$ and road location $x \in \mathbb{R}$

– Density of vehicles $\rho(t,x)$

– Flux of vehicles $f(t,x)$




- Q2. Conservation law on the number of vehicles in a road section $[x_1, x_2]$:

$$\forall t_0 \geq 0, \qquad \frac{d}{dt}\left(\int_{x_1}^{x_2} \rho(t,x)\,\mathrm{d}x\right)\Bigg|_{t=t_0} = f(t_0,x_1) - f(t_0,x_2)$$

# (FIRST-ORDER) MACROSCOPIC TRAFFIC MODELS

- PDE satisfied by the density $\rho$ and the flux $f$:

$$\frac{\partial \rho}{\partial t}(t,x) + \frac{\partial f}{\partial x}(t,x) = 0$$

# (FIRST-ORDER) MACROSCOPIC TRAFFIC MODELS

- PDE satisfied by the density $\rho$ and the flux $f$:

$$\frac{\partial \rho}{\partial t}(t,x) + \frac{\partial f}{\partial x}(t,x) = 0$$

- Flux and density are not independent variables...

Some flux/density conditions:

– They are both bounded and non-negative

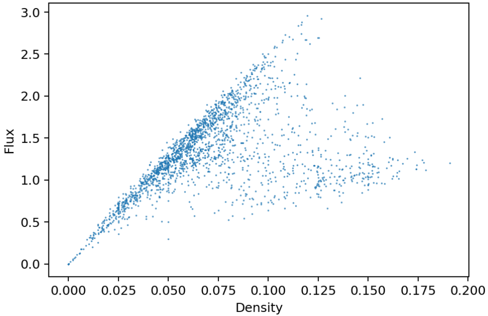
$$\exists\ \rho_m, f_m > 0, \quad \rho \in [0, \rho_m], \quad f \in [0, f_m]$$

– No vehicles $\Rightarrow$ No flux

$$\rho = 0 \Rightarrow f = 0$$

– Road at capacity $\Rightarrow$ No flux

$$\rho = \rho_m \Rightarrow f = 0$$




Plot Flux Vs Density from measurements done in a German highway (HighD dataset)

# (FIRST-ORDER) MACROSCOPIC TRAFFIC MODELS

- PDE satisfied by the density $\rho$ and the flux $f$:

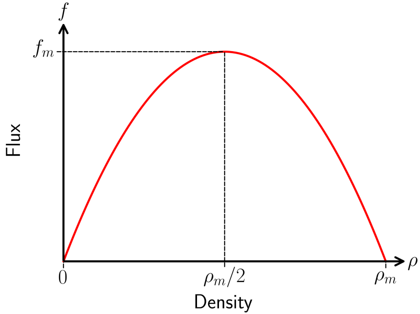
$$\frac{\partial \rho}{\partial t}(t,x) + \frac{\partial f}{\partial x}(t,x) = 0$$

**Lighthill–Whitham–Richards (LWR) model**
Simplest model satisfying the flux/density conditions

$$\exists v_m > 0, \quad \boxed{f \equiv f(\rho) = \rho(t,x) \cdot v_m \left(1 - \frac{\rho(t,x)}{\rho_m}\right)}$$

In particular,

$$f_m = f(\rho_m/2) = \frac{v_m \rho_m}{4}$$



# (FIRST-ORDER) MACROSCOPIC TRAFFIC MODELS

- PDE satisfied by the density $\rho$ and the flux $f$:

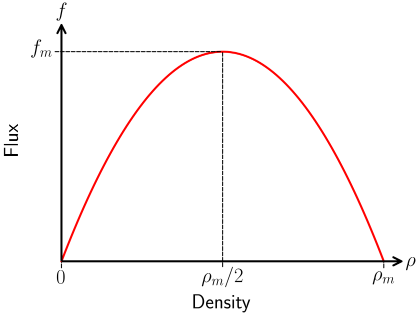
$$\frac{\partial \rho}{\partial t}(t,x) + \frac{\partial f}{\partial x}(t,x) = 0$$

- LWR model $\rightarrow$ Nonlinear (hyperbolic) PDE satisfied by the density $\rho$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(f(\rho)\right) = 0} \quad \left(f(\rho) = v_m \rho \left(1 - \frac{\rho}{\rho_m}\right)\right)$$

Parameters:

- Maximal density $\rho_m$
- Critical speed $v_m$

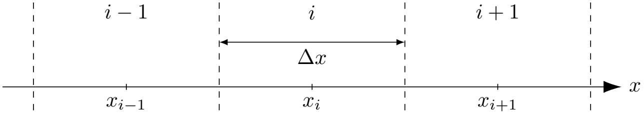


# Traffic Reaction Model

# FINITE-VOLUME DISCRETIZATION

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(f(\rho)\right) = 0}$$

- Discretized space




- Finite Volume method $\rightarrow$ Approximation of cell averages $\rho_i$

$$\rho_i(t) = \frac{1}{\Delta x} \int_{x_i - \Delta x/2}^{x_i + \Delta x} \rho(t, x)\, \mathrm{d}x$$

- Remark: $\rho$ = density gives

$$\boxed{\rho_i(t) = \frac{\text{Number of vehicles in the } i\text{-th cell at } t}{\Delta x}}$$

# ■ FINITE-VOLUME DISCRETIZATION

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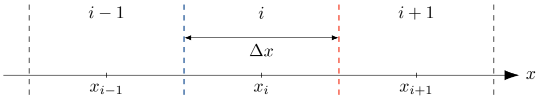


- Integration of PDE on each cell gives

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}t}(t) = \frac{1}{\Delta x}\left[ f(\rho)\big|_{t,x=x_i-\Delta x/2} - f(\rho)\big|_{t,x=x_i+\Delta x/2} \right]$$

- Finite volume scheme: replace true fluxes by approximations

$$\frac{\mathrm{d}\rho_i}{\mathrm{d}t}(t) = \frac{1}{\Delta x}\left[F(\rho_{i-1},\rho_i) - F(\rho_i,\rho_{i+1})\right]$$

- Time discretization (Euler method) $\rightarrow$ recurrence relation

$$\boxed{\rho_i^{n+1} \equiv \rho_i(t_n + \Delta t) = \rho_i^n + \frac{\Delta t}{\Delta x}\left[F(\rho_{i-1}^n,\rho_i^n) - F(\rho_i^n,\rho_{i+1}^n)\right]}$$

Recall the PDE of interest

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(f(\rho)\right) = 0$$

Let $\mathcal{H}$ denote the recurrence relation of finite volume scheme, i.e.

$$\rho_i^{n+1} = \mathcal{H}(\rho^n\ ; i) = \rho_i^n + \frac{\Delta t}{\Delta x}\left[F(\rho_{i-1}^n, \rho_i^n) - F(\rho_i^n, \rho_{i+1}^n)\right], \quad n \in \mathbb{N}, \quad i \in \mathbb{Z}$$

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**Consistency** The scheme is consistent (with the flux function $f$) if

$$F(u, u) = f(u), \quad \forall u$$

## FINITE-VOLUME DISCRETIZATION: PROPERTIES

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Recall the PDE of interest

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(f(\rho)\right) = 0$$

Let $\mathcal{H}$ denote the recurrence relation of finite volume scheme, i.e.

$$\rho_i^{n+1} = \mathcal{H}(\rho^n\,; i) = \rho_i^n + \frac{\Delta t}{\Delta x}\left[F(\rho_{i-1}^n, \rho_i^n) - F(\rho_i^n, \rho_{i+1}^n)\right], \quad n \in \mathbb{N}, \quad i \in \mathbb{Z}$$

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**Monotonicity** The scheme is monotone if for any $n \geq 0$,

$$[\forall i, \quad \rho_i^n \leq r_i^n] \Rightarrow \left[\forall i, \quad \rho_i^{n+1} = \mathcal{H}(\rho^n\,; i) \leq \mathcal{H}(r^n\,; i) = r_i^{n+1}\right]$$

**($L^\infty$-)Stability** The scheme is stable if

$$\left[\exists m, M \in \mathbb{R}, \quad \forall i \in \mathbb{Z},\ \rho_i^0 \in [m, M]\right] \Rightarrow [\forall n \in \mathbb{N},\ \forall i \in \mathbb{Z},\ \rho_i^n \in [m, M]]$$

# FINITE-VOLUME DISCRETIZATION: PROPERTIES

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Recall the PDE of interest

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(f(\rho)\right) = 0$$

Let $\mathcal{H}$ denote the recurrence relation of finite volume scheme, i.e.

$$\rho_i^{n+1} = \mathcal{H}(\rho^n\ ; i) = \rho_i^n + \frac{\Delta t}{\Delta x}\left[F(\rho_{i-1}^n, \rho_i^n) - F(\rho_i^n, \rho_{i+1}^n)\right], \quad n \in \mathbb{N}, \quad i \in \mathbb{Z}$$

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**Convergence** For any time horizon $T > 0$, the discrete solution $\{\rho_i^n\}_{i,n}$ converges (in $L^1$) to "entropy solution" $\rho$ of the PDE if

$$\sum_{\substack{n\in\mathbb{N} \\ t_{n+1}\leq T}} \int_{t_n}^{t_{n+1}} \left( \sum_{j\in\mathbb{Z}} \int_{x_i-\Delta x/2}^{x_i+\Delta x/2} |\rho(t,x) - \rho_i^n| dx \right) dt \longrightarrow 0 \text{ as } \Delta t, \Delta x \to 0$$

(with $\Delta t/\Delta x =$Constant)

# TRAFFIC REACTION MODEL

- **Assumption**: The flux function $f$ can be written as

$$\boxed{f(\rho) = g(\rho, \rho_m - \rho)}$$

for some $g : [0, \rho_m] \times [0, \rho_m] \mapsto \mathbb{R}_+$ Lipschitz, non-decreasing w.r.t. both arguments, and satisfying $g(\rho, 0) = g(0, \rho_m - \rho) = 0$

$\rightarrow$ Remark: It is enough to have $\boxed{f(\rho) = g_1(\rho) g_2(\rho_m - \rho)}$ where $g_1, g_2$ Lipschitz, non-decreasing, and satisfying $g_1(0) = g_2(0) = 0$.

- For the LWR model, i.e. taking $f(\rho) = v_m \rho \left(1 - \frac{\rho}{\rho_m}\right)$, some choices are

$$f(\rho) = g(\rho, \rho_m - \rho) = v_m \rho \left(\frac{\rho_m - \rho}{\rho_m}\right) = D(\rho) Q(\rho_m - \rho)/f_m = \min(D(\rho), Q(\rho_m - \rho))^1$$

with $D(\nu) = f(\min\{\nu, \frac{\rho_m}{2}\})$ and $Q(\nu) = f(\max\{\rho_m - \nu, \frac{\rho_m}{2}\})$

[1] link to Cell Transmission Model

# TRAFFIC REACTION MODEL

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Using more general flux functions than the LWR flux $f(\rho) = v_m \rho \left(1 - \frac{\rho}{\rho_m}\right) = g_1(\rho) g_2(\rho - \rho_m)$:

$$f(\rho) = g(\rho, \rho_m - \rho)$$

where $g$ is non-decreasing with respect to both its argument.

**Traffic Reaction Model (TRM)**

$$\boxed{F(\rho_l, \rho_r) = g(\rho_l, \rho_m - \rho_r)}$$

$$F_i^n(\rho_i^n, \rho_{i+1}^n) = g(\rho_i^n, \rho_m - \rho_{i+1}^n) = g_1(\rho_i^n) g_2(\rho_m - \rho_{i+1}^n)$$

- → Family of numerical schemes
- → $F_i^n(\rho_i^n, \rho_{i+1}^n)$ can be selected smooth (no "jumpy" nonlinear components in the ODEs)
- → Consistency, Stability, Monotonicity, Convergence imposing a Courant–Friedrichs–Lewy condition.
- → Traffic Flow Theory + Chemical Reaction Model

# TRAFFIC REACTION MODEL: INTERPRETATION

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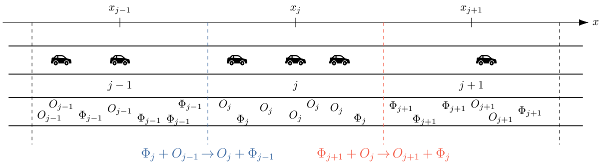


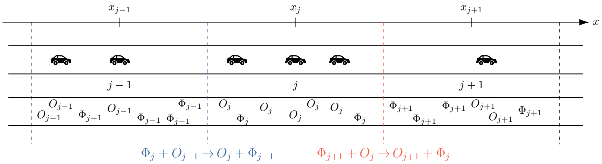

2 "chemical species" present in each cell/compartment: Occupied slots $O_j$ and Free slots $\Phi_j$

$\rightarrow$ Species concentrations in the $j$-th cell

$$\text{Occupied slots :} \quad \frac{\text{Amount of Occupied slots } O_j}{\text{Volume of the compartment}} = \text{Density of vehicles} = \rho_j(t)$$

$$\text{Free slots :} \quad \frac{\text{Amount of Free slots } \Phi_j}{\text{Volume of the compartment}} = \text{Density of free space} \equiv \rho_m - \rho_j(t)$$

# ■ TRAFFIC REACTION MODEL: INTERPRETATION




2 "chemical species" present in each cell/compartment: Occupied slots $O_j$ and Free slots $\Phi_j$

- Reaction model for vehicle transfer between cells

$$\Phi_j + O_{j-1} \xrightarrow{k_{j-1\to j}} O_j + \Phi_{j-1} \qquad \Phi_{j+1} + O_j \xrightarrow{k_{j\to j+1}} O_{j+1} + \Phi_j$$

where $k_{j\to j+1} = g(\rho_i, \rho_m - \rho_{i+1})/\Delta x$ = reaction rate depending on the "concentrations" $\rho_i$ (of occupied space in $i$) and $\rho_m - \rho_{i+1}$ (of free space in $i+1$)

- (Discretized) Reaction kinetics then give

$$\rho_j^{n+1} = \rho_j^n + \frac{\Delta t}{\Delta x}\left[g(\rho_{j-1}^n, \rho_m \rho_j^n) - g(\rho_j^n, \rho_m - \rho_{j+1}^n)\right]$$

$\to$ Finite-volume scheme!

# Traffic Density Estimation

## ■ PROBLEM STATEMENT

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**Estimation Problem:** Consider a stretch of highway with the length $L > 0$ km represented by the interval $[a, b]$. Let $\rho(x, t)$ denote the density at time $t$ and point $x$ on the highway. Given the density of $\rho(t, x)$ for $t \geq 0$ and $x$ in the vicinity of the points $a$ and $b$, we seek an approximation $\mu(t, x)$ such that

$$\mu(x, t) \approx \rho(x, t) \text{ for } t \geq 0 \text{ and } x \in [a, b]. \tag{1}$$

**Initial Assumption:** The density on the highway is governed by the LWR model and the flux function $f$ can be written as

$$f(\rho) = \rho v(\rho), \quad \rho \in \mathbb{R} \tag{2}$$

where the function $v$ is Lipschitz, nonincreasing and such that

$$v(\rho_{\max}) = 0 \text{ and } v(\rho) > 0, \quad \rho < \rho_{\max}. \tag{3}$$

## ■ PROPOSED SOLUTION PART 1 (SEMI-DISCRETIZATION)

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We discretize the LWR model (in space) over $n+2$ consecutive cells $\{C_i\}_{i=0}^{N+1}$ covering $[a,b]$:

$$\dot{\rho}_i = \frac{1}{\Delta x}\big(\rho_{i-1}v(\rho_i) - \rho_i v(\rho_{i+1})\big), \quad i = 1,\dots,N \tag{4}$$

$$d_i = \frac{1}{\Delta x}\int_{C_i} \rho(x,0)\mathrm{d}x, \quad i = 1,\dots,N \tag{5}$$

The cells $C_0$, $C_{N+1}$ are centered about the points $a, b$ and we *define*

$$\rho_0(t) := \int_{C_0} \rho(x,t)\mathrm{d}x, \quad \rho_{N+1}(t) := \int_{C_{N+1}} \rho(x,t)\mathrm{d}x, \quad t \geq 0 \tag{6}$$

which by assumption are measured. If the spatial discretization length $\Delta x$ is small (corresponding with large $N$), then

$$\rho_i(t) \approx \frac{1}{\Delta x}\int_{C_i} \rho(x,t)\mathrm{d}x, \quad t \geq 0, \quad i = 1,\dots,N. \tag{7}$$

# PROPOSED SOLUTION PART 2 (THE ESTIMATE)

Now we define $(\mu_1, \ldots, \mu_n)$ to be **any** solution to (4) starting in $[0, \rho_{\max}]^N$ and set

$$\mu(x,t) \coloneqq \mu_i(t), \quad t \geq 0, \quad x \in C_i. \tag{8}$$

If there exists $\bar{\rho}$ such that

$$\rho_{N+1}(t) \leq \bar{\rho} < \rho_{\max}, \quad t \geq 0, \tag{9}$$

then there exist $\lambda > 0$, $\gamma \geq 1$ such that

$$\sum_{i=1}^{N} |\rho_i(t) - \mu_i(t)| \leq \gamma e^{-\lambda t} \sum_{i=1}^{N} |\rho_i(0) - \mu_i(0)|, \quad t \geq 0. \tag{10}$$

This solves the proposed estimation problem, granted the *additional* assumption (9).

# PROOF: MAIN STEPS

$$\sum_{i=1}^{N} |\rho_i(t) - \mu_i(t)| \leq \gamma e^{-\lambda t} \sum_{i=1}^{N} |\rho_i(0) - \mu_i(0)|, \quad t \geq 0. \tag{11}$$

**Step 0:** The Traffic Reaction Model gives rise to a cooperative system of ODEs associated with the statespace $[0, \rho_{\max}]^N$.
**Step 1:** The exponential convergence (11) is shown directly with a linear Lyapunov function whenever $0 \leq \rho_i(0) \leq \mu_i(0) \leq \rho_{\max}$ for $i = 1, \ldots, n$.
**Step 2:** The exponential convergence (11) can be shown to hold for all initial conditions in $[0, \rho_{\max}]^N$ owing to step 1 and the fact that the system is cooperative[2].

[2]See Theorem 2 in Wiersdalen, S. (2025). *Incremental Stability of Traffic Reaction Models* [Licentiate Thesis, Chalmers University of Technology]

# NUMERICAL EXPERIMENT

- Greenshields flux function

$$f(\rho) = v_{\max}\rho(1 - \frac{\rho}{\rho_{\max}}), \quad v_{\max} = 110\frac{\text{km}}{\text{h}}, \quad \rho_{\max} = 200\frac{\text{veh}}{\text{km}} \tag{12}$$

- We take the initial condition

$$u(x) = \begin{cases} 0, & x \leq 0 \\ 150, & 0 < x \leq 1 \\ 50, & 1 < x \leq 2 \\ 200, & x > 2 \end{cases} \tag{13}$$

and simulate for 30 seconds and provide an estimate $\mu(x,t)$ of $\rho(x,t)$ for $x \in [a,b] = [0.25, 0.75]$ and $t \geq 0$ ($\Delta x = 5$ meters).

# ■ DENSITY PROFILES

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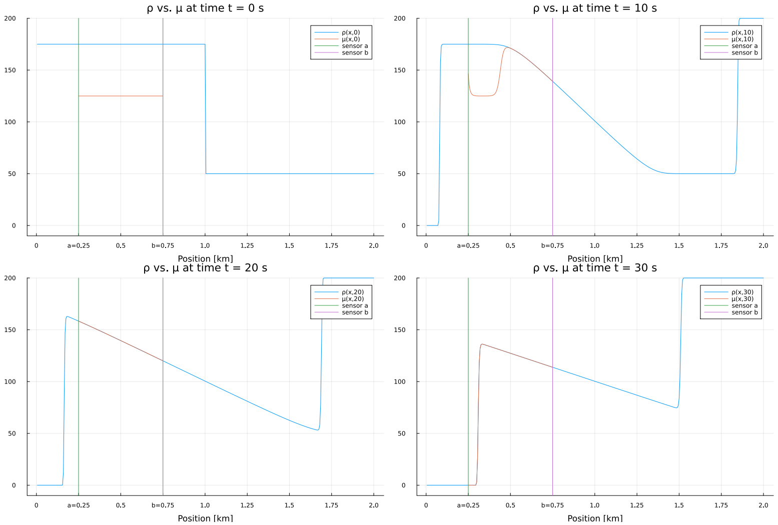
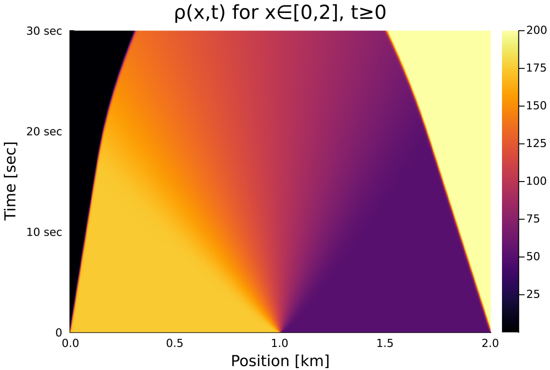
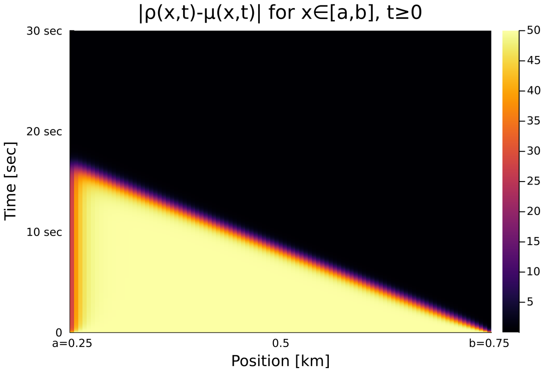


# ■ HEAT MAP



ρ(x,t) for x∈[0,2], t≥0
30 sec
20 sec
10 sec
0
Time [sec]
0.0
0.5
1.0
1.5
2.0
Position [km]
200
175
150
125
100
75
50
25

|ρ(x,t)-μ(x,t)| for x∈[a,b], t≥0
30 sec
20 sec
10 sec
0
Time [sec]
a=0.25
0.5
b=0.75
Position [km]
50
45
40
35
30
25
20
15
10
5

# Summary and future directions

# SUMMARY

- family of finite volume discretization: nonnegatity, capacity inherited by the ODEs.
- $f(\rho)$ can be over-parametrized by the dual variable $\rho_m - \rho$ via $g(\rho, \rho_m - \rho)$ to *gate* the transfer of the conserved quantity.
- mass kinetic discretization, equivalence to chemical reaction networks
- open loop traffic state estimation

# FUTURE WORK

- PDEs on a network
- Controlled TRM: change of the reaction rate
- Stochastic TRM

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