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Poissonian actions of Polish groups [☆]

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ABSTRACT

We define and study Poissonian actions of Polish groups as a framework to Poisson suspensions, characterize them spectrally, and provide a complete characterization of their ergodicity. We further construct *spatial* Poissonian actions, answering partially a question of Glasner, Tsirelson & Weiss about Lévy groups. We also construct for every diffeomorphism group a weakly mixing free spatial probability-preserving action. This constitutes a new class of Polish groups admitting non-essentially countable orbit equivalence relations, obtaining progress on a problem of Kechris.

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Contents

1. Introduction

A well-known construction in probability theory is the Poisson point process, in which every standard (typically nonatomic and infinite) measure space $(X, \mathcal{B}, \mu)$ is associated a standard probability space $(X^*, \mathcal{B}^*, \mu^*)$, whose points are configurations of points from X and their distribution is governed by Poisson distribution with intensity μ. In ergodic theory, one naturally associates with every measure-preserving transformation T of $(X, \mathcal{B}, \mu)$ a probability-preserving transformation T^* of $(X^*, \mathcal{B}^*, \mu^*)$, namely the **Poisson suspension**. In the first part of this work we aim to put the constructions of Poisson point process and the Poisson suspension in a general, more axiomatic framework, thus defining the notion of measure-preserving **Poissonian action** of Polish groups.

For a parameter $0 \leqslant \alpha \leqslant \infty$, denote by $\mathrm{Poiss}\,(\alpha)$ the Poisson distribution with mean α, with the convention that for $\alpha \in \{0, \infty\}$ it is the distribution of the constant α.

Definition 1.1 *(Poisson point process).* Let $(X, \mathcal{B}, \mu)$ be a standard nonatomic[1] measure space, and $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space. A collection

$$\mathcal{P} = \{P_A : A \in \mathcal{B}\}$$

of random variables that are defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **Poisson point process** with the **base space** $(X, \mathcal{B}, \mu)$, if the following properties hold:

(1) P_A has distribution $\mathrm{Poiss}\,(\mu\,(A))$ for every $A \in \mathcal{B}$.

(2) $P_{A \cup B} = P_A + P_B$ $\mathbb{P}$-a.e. whenever $A, B \in \mathcal{B}$ are disjoint.

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[1] See Remark 5.9.

Such a Poisson point process $\mathcal{P}$ will be called **generative** if, in addition,

(3) The members of $\mathcal{P}$ generate $\mathcal{F}$ modulo $\mathbb{P}$.

The measure μ is referred to as the **intensity** of $\mathcal{P}$.

Remark 1.2. By the famous Rényi Theorem [37, Theorem 1], which is valid in our general setting, a Poisson point process $\mathcal{P}$ as in Definition 1.1 automatically satisfies that $P_{A_1},\dots,P_{A_n}$ are independent whenever $A_1,\dots,A_n\in\mathcal{B}$ are disjoint.

The classical (generative) Poisson point process with an arbitrary base space $(X,\mathcal{B},\mu)$, is usually constructed to be the standard probability space $(X^*,\mathcal{B}^*,\mu^*)$, where X^* consists of Borel simple counting measures on X, and it is the collection

$$\mathcal{N}=\{N_A:A\in\mathcal{B}\}\ \text{ given by }\ N_A(\omega)=\omega(A).$$

As we shall see in Proposition 3.1, this construction of Poisson point process amounts to a choice of topology which is not always canonical, and $\mathcal{N}$ in its A-variable becomes a *random measure* on X, a property that is not assumed a priori for general Poisson point process as in Definition 1.1. Nevertheless, as we shall see in Proposition 3.3, all Poisson point processes are essentially unique, and in particular all form random measures in a precise sense. Despite this universality of the Poisson point process, the ability to deviate oneself from the aforementioned concrete construction will be of great importance to us as we shall see in Theorems 3 and 4.

Given a Poisson point process $\mathcal{P}$ as in Definition 1.1, let $\mathcal{B}_\mu$ be the Borel sets in $\mathcal{B}$ with finite measure, and look at the real Hilbert space

$$H(\mathcal{P}):=\overline{\operatorname{span}}\{P_A:A\in\mathcal{B}_\mu\}\subset L^2_{\mathbb{R}}(\Omega,\mathcal{F},\mathbb{P}).$$

Definition 1.3 *(Poissonian action).* Let $\mathcal{P}$ be a generative Poisson point process as in Definition 1.1. A probability-preserving action $\mathbf{S}:G\curvearrowright(\Omega,\mathcal{F},\mathbb{P})$ of a Polish group G is said to be a **Poissonian action** with respect to $\mathcal{P}$, if its Koopman representation preserves $H(\mathcal{P})$ within $L^2_{\mathbb{R}}(\Omega,\mathcal{F},\mathbb{P})$.

In the next theorem we provide a characterization of Poissonian actions. We will start by introducing the natural source for Poissonian actions, namely the **Poisson suspension** construction, omitting essential technical details that will be treated with great care in Proposition 3.1. Observe that if T is a measure-preserving transformation of $(X,\mathcal{B},\mu)$, one may define a probability-preserving transformation T^* of $(X^*,\mathcal{B}^*,\mu^*)$ by the property that for every ω in an appropriate μ^*-conull set,

$$T^*(\omega)=\omega\circ T^{-1}.$$

Evidently, we have the property

$$N_A\left(T^*\left(\omega\right)\right)=N_{T^{-1}(A)}\left(\omega\right)\ \text{ for } A\in\mathcal{B} \text{ and for } \mu^*\text{-a.e. } \omega\in X^*.$$

This readily implies that T^* is a Poissonian transformation with respect to the Poisson point process $\mathcal{N}$. As we shall see later on, this source of Poissonian actions is not limited to a single transformation but for every measure-preserving action $\mathbf{T}:G\curvearrowright(X,\mathcal{B},\mu)$ of an arbitrary Polish group G we obtain a Poissonian action $\mathbf{T}^*:G\curvearrowright(X^*,\mathcal{B}^*,\mu^*)$ with respect to the Poisson point process $\mathcal{N}$. In the Poisson suspension construction, the action $\mathbf{T}$ is referred to as a **base action** of $\mathbf{T}^*$, and for a general Poissonian action this will be put as follows.

Definition 1.4 *(Base of a Poissonian action).* Let $\mathbf{S}:G\curvearrowright(\Omega,\mathcal{F},\mathbb{P})$ be a Poissonian action of a Polish group G with respect to a generative Poisson point process $\mathcal{P}$ as in Definition 1.3. An action $\mathbf{T}:G\curvearrowright(X,\mathcal{B},\mu)$ is called a **base action** for $\mathbf{S}$ if

$$P_A\circ S_g=P_{T_g^{-1}(A)}\quad\mathbb{P}\text{-a.e. for every } g\in G \text{ and } A\in\mathcal{B}.$$

The following theorem is our main result about Poissonian actions. We put it in a principled form so as to make things clear and the precise formulations can be found in Theorems 4.4 and 4.5 and Corollary 4.6.

Theorem 1. *Suppose $\mathcal{P}$ is a generative Poisson point process as in Definition 1.1, and that a Polish group – the acting group – is given. Then:*

(1) Every measure-preserving action on the base space of $\mathcal{P}$ is a base action of an essentially unique Poissonian action with respect to $\mathcal{P}$.
(2) Every Poissonian action with respect to $\mathcal{P}$ admits an essentially unique base action on the base space of $\mathcal{P}$.

In the next we completely characterize the ergodicity of Poissonian actions in terms of their base actions, to the generality of measure-preserving actions of Polish groups. For $G=\mathbb{Z}$ it was proved by Marchat [30] and other proofs were given later by Grabinsky [17, Theorem 1] and Roy [39, §4.5] (see also Remark 4.12 below).

Theorem 2. *Suppose* $\mathbf{S}$ *is a Poissonian action with a base action* $\mathbf{T}$*. The following are equivalent.*

(1) $\mathbf{S}$ *is ergodic.*
(2) $\mathbf{S}$ *is weakly mixing.*
(3) $\mathbf{T}$ *admits no invariant set of a positive finite measure.*

The continuation of our study of Poissonian actions is in the more restrictive framework of **spatial actions** of Polish groups. As opposed to the general notion of measure-preserving actions, in which every group element corresponds to a transformation that is defined almost everywhere, in spatial actions one requires an actual Borel action which happens to admit an invariant (or quasi-invariant) measure. In many common cases, such as locally compact Polish groups, the *Mackey property* ensures that there is no essential difference between the two notions. However, this completely fails for general Polish groups, as was demonstrated by Becker and by Glasner, Tsirelson & Weiss in *Lévy groups*. We discuss this in Section 5.1.

Observe that the usual construction of the Poisson suspension $\mathcal{N}$ fails in the spatial category. Indeed, the standard Borel space $(X^{*},\mathcal{B}^{*})$ is classically defined as the space of simple counting Radon measures, namely Radon measures that taking nonnegative integer values and are finite on bounded sets, with respect to an appropriate choice of a metric. Then for a general Borel transformation T of $(X,\mathcal{B})$, since it may not preserve the class of bounded sets, there is no apparent way to identify T with a Borel transformation of $(X^{*},\mathcal{B}^{*})$. In order to deal with it, we extend the classical construction of the **Poisson random set** (see e.g. [32, §1.9]) from locally compact topologies to general Polish topologies. This provides a construction of Poisson point processes in Polish topologies that may not be locally compact, and manifests the advantage of treating Poisson point processes abstractly.

Recall that if $(X,\mathcal{B})$ is a standard Borel space, then with every Polish topology τ on X that generates $\mathcal{B}$ is associated the **Effros Borel space**,

$$\mathbf{F}_{\tau}\left(X\right):=\left\{F\subseteq X:X\backslash F\in\tau\right\},$$

whose points are the τ-closed sets. It is well-known that $\mathbf{F}_{\tau}\left(X\right)$ has a structure of a standard Borel space, that will be referred to as the *Effros σ-algebra* and denoted $\mathcal{E}_{\tau}\left(X\right)$. See the presentation in Section 5.2. Thus, a **random closed set** is a probability measure on $\left(\mathbf{F}_{\tau}\left(X\right),\mathcal{E}_{\tau}\left(X\right)\right)$.

Theorem 3 *(Poisson random set). Let $(X,\mathcal{B})$ be a standard Borel space. For every Polish topology τ that generates $\mathcal{B}$, there are random variables*

$$\left\{\Xi_{A}:A\in\mathcal{B}\right\}\ \text{ of the form }\Xi_{A}:\mathbf{F}_{\tau}\left(X\right)\to\mathbb{Z}_{\geqslant0}\cup\left\{\infty\right\},$$

with the following property:

> *For every nonatomic Borel measure μ on $(X,\mathcal{B})$ which is τ-locally finite, there exists a unique random closed set μ_{τ} on $\left(\mathbf{F}_{\tau}\left(X\right),\mathcal{E}_{\tau}\left(X\right)\right)$, with respect to which $\left\{\Xi_{A}:A\in\mathcal{B}\right\}$ forms a Poisson point process with base space $(X,\mathcal{B},\mu)$.*

Our first main result about spatial Poissonian actions allows one to construct spatial probability-preserving actions out of spatial infinite measure-preserving actions. This will be established for actions with an appropriate Polish topology in the following sense.

Definition 1.5. A **locally finite Polish action** is a measure-preserving action $\mathbf{T}: G \curvearrowright (X, \mathcal{B}, \mu)$ with a Polish topology for which, simultaneously, $\mathbf{T}$ is continuous and μ is locally finite.

Theorem 4. *Every locally finite Polish action of a Polish group is a base action of a* spatial *Poissonian action.*

The construction of spatial Poissonian actions in Theorem 4 provides a tool to construct probability-preserving spatial actions for Polish groups without appealing to the Mackey property. Our following results demonstrate the strength of this construction.

In their work on probability-preserving spatial actions of Lévy groups, Glasner, Tsirelson & Weiss showed that all such actions are necessarily trivial in that the measure must be supported on the set of fixed points [14, Theorem 1.1], and they left it as an open question whether Lévy groups may admit nontrivial nonsingular spatial actions [14, Question 1.2], [35, Open Problem 7.1.19]. Using Theorem 4 we obtain the following partial answer:

Theorem 5. *A Polish group admits a nontrivial probability-preserving spatial action if it admits any of the following nontrivial actions:*

(1) A locally finite Polish measure-preserving action.
(2) A locally finite Polish nonsingular action with a spatially continuous *Radon–Nikodym cocycle (see Definition 6.2 for the precise meaning).*

In particular, Lévy groups admit no such nontrivial actions.

An immediate strengthening of Theorem 5 can be obtained using a recent result of Kechris, Malicki, Panagiotopoulos & Zielinski [24, Theorem 2.3]. Recall that an action is **faithful** if each group element, except for the identity, acts nontrivially on a positive measure set.

Corollary 1.6. *Every Polish group G admitting one of the actions (1) or (2) as in Theorem 5 which is also faithful, admits a free probability-preserving spatial action.*

We now move to use the construction of spatial Poissonian action in Theorem 4 to construct nontrivial spatial actions in the class of diffeomorphism groups. Let M be a Hausdorff connected compact finite dimensional smooth manifold. We call by a **diffeomorphism group** of M, for some $1 \leqslant r \leqslant \infty$, the group

$$\mathrm{Diff}^r(M)$$

of all C^r-diffeomorphisms of M to itself, considered as a (non-locally compact) Polish group with the compact-open C^r-topology.

The aforementioned Mackey property, established by Mackey for locally compact groups, was generalized by Kwiatkowska & Solecki to groups of isometries of locally compact metric spaces and, to the best of our knowledge, currently this is the largest class of Polish groups for which the Mackey property is known to hold. The following theorem shows that diffeomorphism groups are not in this class whenever $r \neq d+1$. It is the consequence of two highly nontrivial results: one is a theorem due to Herman, Thurston and Mather, following a theorem by Epstein, about the simplicity of the identity component in diffeomorphism groups, and another by Kwiatkowska & Solecki, following Gao & Kechris, about the topological structure of isometry groups of locally compact metric spaces.

Theorem 6. *For every compact smooth d-dimensional manifold M and every $1 \leqslant r \leqslant \infty$ with $r \neq d+1$, the diffeomorphism group $\mathrm{Diff}^r(M)$ is* not *a group of isometries of a locally compact metric space.*

Although the Mackey property for diffeomorphism groups is unknown, we use the construction of spatial Poissonian actions to define spatial actions of such groups. In this context, it is worth noting that, by a result of Megrelishvili [31, Theorem 3.1] (see also [14, Remark 1.7]), the homeomorphism group of the unit interval admits no nontrivial nonsingular actions, not even nontrivial Boolean actions. The situation for diffeomorphism groups, however, turns out to be quite different.

Theorem 7. *Every diffeomorphism group admits a weakly mixing free probability-preserving spatial action. Hence, diffeomorphism groups are never Lévy.*

Theorem 7 has a consequence in the theory of equivalence relations on standard Borel spaces. An open problem of Kechris asks whether every non-locally compact Polish group admits a Borel action on a standard Borel space whose orbit equivalence relation is not *essentially countable*. See the details in Section 7.1. Recently, this question was answered affirmatively for groups of isometries of a locally compact metric space by Kechris, Malicki, Panagiotopoulos & Zielinski. While diffeomorphism groups, at least when $r \neq d+1$, do not belong to this class by Theorem 6, we have the following corollary, which is the result of Theorem 7 together with a theorem by Feldman & Ramsay:

Corollary 1.7. *Every diffeomorphism group admits a non-essentially countable orbit equivalence relation.*

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Post-publication Note: Subsequent to the first publication of this work, Fabien Hoareau and François Le Maître independently published related work [19]. Their *Theorem E* is essentially the same as our Theorem 5 (the relations between the proofs are discussed there). Their *Question 2* is the same as our Question 8.1.

2. General preliminaries

Let $(X, \mathcal{B})$ be a standard Borel space. Thus, X is equipped with a σ-algebra $\mathcal{B}$ which is the Borel σ-algebra of some unspecified Polish topology τ on X. A **transformation** of $(X, \mathcal{B})$ is an invertible Borel map $T : X \to X$. A **Borel** (τ-**Polish**) **action** of a Polish group G on X is a jointly Borel (jointly τ-continuous, resp.) map $\mathbf{T} : G \times X \to X$, $\mathbf{T} : (g, x) \mapsto T_g(x)$, such that $T_e = \mathrm{Id}_X$, where $e \in G$ is the identity, and $T_g \circ T_h = T_{gh}$ for every $g, h \in G$. For the following see [20, Exercise (9.16)] or [44, Theorem 4.8.6].

Fact 2.1. Let G be a Polish group and $\mathbf{T}$ an action of G on a Polish space X by homeomorphisms. If the map $g \mapsto T_g(x)$ is Borel for each $x \in X$, then $\mathbf{T}$ is a Polish action.

By a **standard measure space** we refer to $(X, \mathcal{B}, \mu)$, where $(X, \mathcal{B})$ is a standard Borel space and μ is a measure that belongs to one of the following of classes:

$\mathcal{M}_1(X, \mathcal{B})$**:** nonatomic Borel probability measures on X.
$\mathcal{M}_\sigma(X, \mathcal{B})$**:** nonatomic infinite σ-finite Borel measures on X.
$\mathcal{M}_\sigma^\tau(X, \mathcal{B})$**:** those measures in $\mathcal{M}_\sigma(X, \mathcal{B})$ that are τ-locally finite.

Being τ-locally finite, by definition, means that every point in X has a τ-neighborhood of finite measure. By the Lindelöf property of Polish spaces, τ-local finiteness is equivalent to the existence of a countable base, or a countable open cover, consisting of sets with finite measure. The general theory of point processes is typically developed for Radon measures, which are locally finite measures with respect to a locally compact topology (see, e.g., [11, Theorem 7.8]). Here we deal with general Polish topologies.

If $(X, \mathcal{B}, \mu)$ is a standard measure space, we denote by $\mathcal{B}_\mu$ the ideal of sets $A \in \mathcal{B}$ for which $\mu(A) < \infty$. We use the common terminology of μ**-a.e.** or μ**-conull**, applied to a specified property of the elements of X, to indicate that the property holds true for all the members of some set in $\mathcal{B}$ whose complement is μ-null, namely is assigned zero by μ. By a **transformation** of $(X, \mathcal{B}, \mu)$ we refer to a bi-measurable bijective map between two μ-conull sets of X. A transformation T of $(X, \mathcal{B})$ is said to be **measure-preserving** if $\mu \circ T^{-1} = \mu$, and **nonsingular** if $\mu \circ T^{-1}$ and μ are in the same measure class, namely they are mutually absolutely continuous. We denote by

$$\mathrm{Aut}\,(X, \mathcal{B}, \mu) \text{ and } \mathrm{Aut}\,(X, \mathcal{B}, [\mu])$$

the groups of equivalence classes, up to equality on a μ-conull set, of measure-preserving and nonsingular transformations, respectively. The latter group clearly depends only on the measure class $[\mu]$ of μ rather than μ itself, and it becomes a Polish group with the weak topology, in which $S_n \xrightarrow[n\to\infty]{} \mathrm{Id}$ if $\mu\,(S_n A \triangle A) \xrightarrow[n\to\infty]{}$ for every $A \in \mathcal{B}$ with $\mu\,(A) < \infty$ and $\frac{d\mu \circ S_n}{d\mu} \xrightarrow[n\to\infty]{} 1$ in measure. The former group then becomes a closed, hence Polish, subgroup of the latter (see e.g. [1, end of §1.0], [20, Exercise (17.46)]).

3. Foundations of Poisson point processes

Recall Definition 1.1 for the general notion of Poisson point process. In the following we introduce the usual concrete construction of the Poisson point process that should be regarded as a folklore. The details of this construction will be important to us for the general context and later uses.

Let $(X, \mathcal{B})$ be a standard Borel space. By the Isomorphism Theorem of standard Borel spaces (see [20, §15.B]) there can be found a complete metric on X that induces a locally compact Polish topology τ on X, and thus we may relate to bounded Borel sets in X with respect to a fixed choice of such a metric. We may further assume that τ admits a countable base that consists of bounded sets and, in fact, we may assume that this topology has all properties of the usual topology on $\mathbb{R}$. Denote by X_τ^* the space of simple counting Radon measures on X. That is, a Borel measure ω on X is in X_τ^* if it satisfies the following properties:

(1) (Radon) $\omega\,(A) < \infty$ for every bounded set $A \in \mathcal{B}$.
(2) (counting) $\omega\,(A) \in \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ for every $A \in \mathcal{B}$.
(3) (simple) $\omega\,(\{x\}) \in \{0, 1\}$ for every $x \in X$.

The space X_τ^* becomes a standard Borel space with the σ-algebra $\mathcal{B}_\tau^*$ generated by the canonical mappings

$$\mathcal{N} = \{N_A : A \in \mathcal{B}\}, \quad N_A : X_\tau^* \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}, \quad N_A : \omega \mapsto \omega\,(A). \tag{3.0.1}$$

For details on the standard Borel structure of X_τ^* see [6, §9.1].

Suppose now that $(X, \mathcal{B}, \mu)$ is a standard infinite measure space, thus $\mu \in \mathcal{M}_\sigma\,(X, \mathcal{B})$. By the Isomorphism Theorem for standard measure spaces (see e.g. [20, §17.F]) there can be found a complete metric that induces a Polish topology τ that satisfies all of the above and, at the same time, turns μ into a Radon measure, i.e. $\mu \in \mathcal{M}_\sigma^\tau\,(X, \mathcal{B})$. In fact, up to a Borel isomorphism, we may assume that $(X,\mathcal{B},\mu)$ is $\mathbb{R}$ with its usual Borel structure and the Lebesgue measure. By the classical construction of the Poisson point process, there exists a unique probability measure $\mu_{\tau}^{*}\in\mathcal{M}_{1}\left(X_{\tau}^{*},\mathcal{B}_{\tau}^{*}\right)$ with respect to which the random variables $\mathcal{N}$ as in (3.0.1) form a generative Poisson point process with base space $(X,\mathcal{B},\mu)$. For details about this classical construction we refer to [5, §2.4], [26, §3], [42, Proposition 19.4]. In our context we put this as follows:

Proposition 3.1. *For every standard measure space $(X,\mathcal{B},\mu)$ there exists a Polish topology τ and a standard probability space $\left(X_{\tau}^{*},\mathcal{B}_{\tau}^{*},\mu_{\tau}^{*}\right)$ defined uniquely by the property that the collection of random variables $\mathcal{N}$ as in (3.0.1) forms a generative Poisson point process with base space $(X,\mathcal{B},\mu)$. Moreover, there is a continuous embedding of Polish groups*

$$\operatorname{Aut}(X,\mathcal{B},\mu)\hookrightarrow\operatorname{Aut}\left(X_{\tau}^{*},\mathcal{B}_{\tau}^{*},\mu_{\tau}^{*}\right),\quad T\mapsto T^{*},$$

such that for every $T\in\operatorname{Aut}(X,\mathcal{B},\mu)$ there is a μ_{τ}^{*}-conull set on which

$$N_{A}\circ T^{*}=N_{T^{-1}(A)}\text{ for every }A\in\mathcal{B}. \tag{3.1.1}$$

Proof. Thanks to the Isomorphism Theorem for standard measure spaces, we may assume that $(X,\mathcal{B},\mu)$ is nothing but the real line with its Lebesgue measure, for which the aforementioned classical construction of the Poisson point process with respect to the usual topology is well known. Let us show the second part. Pick a countable base $\mathcal{O}$ for τ consisting of μ-finite measure sets. An arbitrary element $[T]_{\mu}\in\operatorname{Aut}(X,\mathcal{B},\mu)$ will be mapped to $[T^{*}]_{\mu_{\tau}^{*}}\in\operatorname{Aut}\left(X_{\tau}^{*},\mathcal{B}_{\tau}^{*},\mu_{\tau}^{*}\right)$ as follows. Pick a representative $T\in[T]_{\mu}$ and consider the Borel set

$$X_{\tau}^{*}(T):=\bigcap\nolimits_{O\in\mathcal{O}}\bigcap\nolimits_{n\in\mathbb{Z}}\left\{N_{T^{n}(O)}<\infty\right\}.$$

By the construction of μ_{τ}^{*}, as T preserves μ we see that $\mu_{\tau}^{*}\left(X_{\tau}^{*}\triangle X_{\tau}^{*}(T)\right)=0$. Let T^{*} be the automorphism of $\left(X_{\tau}^{*}(T),\mathcal{B}_{\tau}^{*}(T),\mu_{\tau}^{*}\mid_{X_{\tau}^{*}(T)}\right)$ given by

$$T^{*}(\omega)=\omega\circ T^{-1},\quad\omega\in X_{T}^{*}.$$

As $\mu_{\tau}^{*}\left(X_{\tau}^{*}\triangle X_{\tau}^{*}(T)\right)=0$, the element $[T^{*}]_{\mu_{\tau}^{*}}\in\operatorname{Aut}\left(X_{\tau}^{*},\mathcal{B}_{\tau}^{*},\mu_{\tau}^{*}\right)$ is well-defined. This defines the desired mapping, that from now on we abbreviate without the equivalence class notations, i.e. $\operatorname{Aut}(X,\mathcal{B},\mu)\to\operatorname{Aut}\left(X_{\tau}^{*},\mathcal{B}_{\tau}^{*},\mu_{\tau}^{*}\right)$, $T\mapsto T^{*}$. It is clearly a homomorphism. In order to see that it is injective, note that if $T\neq\operatorname{Id}_{X}\in\operatorname{Aut}(X,\mathcal{B},\mu)$ there is a Borel set of the form $A=T^{-1}(B)\backslash B$ with $\mu(A)>0$. Since $\mu\left(A\cap T^{-1}(A)\right)=0$ we have $\mu_{\tau}^{*}\left(N_{A}\circ T^{*}>0,N_{A}=0\right)>0$, hence $T^{*}\neq\operatorname{Id}_{X_{\tau}^{*}}\in\operatorname{Aut}\left(X_{\tau}^{*},\mathcal{B}_{\tau}^{*},\mu_{\tau}^{*}\right)$. The equivariance property (3.1.1) is verified by noting that for every $A\in\mathcal{B}_{\mu}$, for ω in an appropriate μ^{*}-conull set,

$$N_{A}\circ T_{g}^{*}(\omega)=N_{A}\left(\omega\circ T_{g}^{-1}\right)=\omega\left(T_{g}^{-1}(A)\right)=N_{T_{g}^{-1}(A)}(\omega).$$

The continuity of this embedding can be verified using elementary considerations, but it is also an immediate consequence of the automatic continuity property of $\operatorname{Aut}(X, \mathcal{B}, \mu)$ by Le Maître [27, Theorem 1.2]. □

Definition 3.2 *(Classical Poisson point process).* The construction in Proposition 3.1, while depending on the highly non-canonical choice of τ, serves as a concrete Poisson point process with an arbitrary base space $(X, \mathcal{B}, \mu)$. Ignoring τ, we will refer to it as the **classical Poisson point process** and denote it by

$$(X^*, \mathcal{B}^*, \mu^*) \text{ and } \mathcal{N} = \{N_A : A \in \mathcal{B}\}.$$

While the choice of τ affects directly $X^* = X^*_\tau$ as a subspace of the τ-Radon measures on X, the following proposition shows that the Poisson point process is a universal object to which the choice of τ is irrelevant up to a Borel isomorphism. In fact, we will show that the Poisson point process is universal in the widest sense of Definition 1.1 up to a Borel isomorphism.

Proposition 3.3. *All generative Poisson point processes on the same base space are isomorphic. More explicitly, let $(X, \mathcal{B}, \mu)$ be a standard measure space and $\mathcal{P} = \{P_A : A \in \mathcal{B}\}$ be a generative Poisson point process that is defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with base space $(X, \mathcal{B}, \mu)$. There is an isomorphism of measure spaces*

$$\varphi : (\Omega, \mathcal{F}, \mathbb{P}) \to (X^*, \mathcal{B}^*, \mu^*),$$

such that on a $\mathbb{P}$-conull set,

$$N_A \circ \varphi = P_A \textit{ for all } A \in \mathcal{B}.$$

Thus, $\mathcal{P}$ is a random measure on X in that for every ω in an appropriate $\mathbb{P}$-conull set, the map $A \mapsto P_A(\omega)$ defines a measure on $(X, \mathcal{B})$.

Proof. For $\omega \in \Omega$ define $\varphi(\omega) : \mathcal{B} \to \mathbb{R}_{\geqslant 0}$ by

$$\varphi(\omega)(A) = P_A(\omega).$$

First we prove that for ω in a $\mathbb{P}$-conull set it holds that $\varphi(\omega)$ extends to a genuine simple counting measure on $\mathcal{B}$. To this end we verify the conditions appear in [6, p. 17]. The finite additivity of $\varphi(\omega)$ for every $\omega \in \Omega$ is immediate from the definition of $\mathcal{P}$ as a Poisson point process. As for the continuity, note that the finite additivity implies that $P_A \leqslant P_B$ whenever $A \subseteq B$, hence if $\mathcal{B}_\mu \ni A_n \searrow \emptyset$ as $n \nearrow \infty$ then the pointwise limit of the monotone descending sequence P_{A_n} as $n \to \infty$ is a nonnegative random variable and, using the dominated convergence theorem, it has zero mean hence $P_{A_n} \searrow 0$ as $n \nearrow \infty$, establishing the continuity. It follows from [6, Lemma 9.1.XIV] that $\varphi(\omega)$ is a measure on $\mathcal{B}$ for every ω on a $\mathbb{P}$-conull set. Since μ is nonatomic it follows that $\varphi(\omega)$ is further a simple counting measure (see the proof of [37, Theorem 1]).

Thus we obtain a map $\varphi : \Omega \to X^*$ defined in a $\mathbb{P}$-conull set. In order to see that $\mathbb{P} \circ \varphi^{-1} = \mu^*$, note that for every ω in a $\mathbb{P}$-conull set,

$$N_A(\varphi(\omega)) = \varphi(\omega)(A) = P_A(\omega), \quad A \in \mathcal{B},$$

from which it readily follows that $\mathcal{N}$ forms a Poisson point process with respect to $\mathbb{P} \circ \varphi^{-1}$ as well. By the uniqueness of the classical Poisson point process, $\mathbb{P} \circ \varphi^{-1} = \mu^*$. □

4. Foundations of Poissonian actions

4.1. Preliminaries: actions and representations of Polish groups

The very definition of a 'measure-preserving action' of a Polish group has more than one possible meaning, essentially leading to two main formulations, which form a substantial part of our study. The first, more general approach can be described in two ways: as *near actions*, following Zimmer, or as *Boolean actions*, a classical object that admits a convenient formulation due to Glasner, Tsirelson, and Weiss (see [14, Introduction] and references therein). The more restrictive notion of *spatial actions* will be introduced in the second part of this work, beginning in Section 5.

Definition 4.1 *(Zimmer).* A **near action** of a Polish group G on a standard measure space $(X, \mathcal{B}, \mu)$ is a jointly Borel map $\mathbf{T} : G \times X \to X$, $\mathbf{T} : (g, x) \mapsto T_g(x)$, such that:

(1) $T_e = \mathrm{Id}_X$ on a μ-conull set, where $e \in G$ is the identity element.
(2) $T_g \circ T_h = T_{gh}$ on a μ-conull set for every $g, h \in G$.[2]
(3) $\mu \circ T_g^{-1} = \mu$ for every $g \in G$.

There is a natural way to view $\mathrm{Aut}(X, \mathcal{B}, \mu)$ as the group of Boolean isometries of the measure algebra associated with $(X, \mathcal{B}, \mu)$ (i.e. the Boolean algebra of Borel sets in X modulo μ-null sets, with its natural complete metric). With this point of view, Glasner, Tsirelson & Weiss put the notion of Boolean action as follows.

Definition 4.2 *(Glasner, Tsirelson & Weiss).* A **Boolean action** of a Polish group G on a standard measure space $(X, \mathcal{B}, \mu)$ is a continuous (equivalently, measurable)[3] homomorphism $\mathbf{T} : G \to \mathrm{Aut}(X, \mathcal{B}, \mu)$.

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[2] It is crucial here that the μ-conull set may depend on g, h.

[3] The equivalence of measurability and continuity for homomorphisms between Polish group is by Pettis' automatic continuity (see [20, §9.C]).

The reader may recall Proposition 3.1 that suggests why the formulation of Boolean action is more convenient for our purposes. It was observed by Glasner, Tsirelson & Weiss [14, Introduction] that both definitions are essentially the same. Thus, we relate to near actions and Boolean actions simply as **actions**, and denote this unified object by

$$\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu) .$$

We may sometimes refer to an action as a *probability-preserving action* or an *infinite measure-preserving action*, depending on whether the underlying measure is a probability measure or an infinite one.

A pair of actions $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$ and $\mathbf{T}' : G \curvearrowright (X', \mathcal{B}', \mu')$ are considered to be isomorphic, if there exists a bi-measurable bijection $\varphi : X \to X'$, possibly defined only on corresponding conull sets, such that $\mu \circ \varphi^{-1} = \mu'$ and $T'_g \circ \varphi = \varphi \circ T_g$ on a μ-conull set for each $g \in G$.

Recall that a unitary representation $\mathbf{U}$ of a Polish group G in a separable Hilbert space $\mathcal{H}$ is a group homomorphism $\mathbf{U} : G \to \mathrm{U}(\mathcal{H})$, $\mathbf{U} : g \mapsto U_g$, where $\mathrm{U}(\mathcal{H})$ denotes the unitary group of $\mathcal{H}$, such that the mapping $(g, f) \mapsto U_g f$ is jointly continuous.[4] When $\mathcal{H}$ is a subspace of some $L^2_{\mathbb{R}}$-space, we say that $\mathbf{U}$ is **positivity preserving** if $f \geqslant 0$ implies $U_g f \geqslant 0$ for every $g \in G$ and $f \in \mathcal{H}$ (see [42, p. 207]) and, when $1 \in \mathcal{H}$, we say that $\mathbf{U}$ is **unital** if $U_g 1 = 1$ for every $g \in G$. The (real) **Koopman representation** of an action $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$ is the unitary representation

$$\mathbf{U} : G \to \mathrm{U}\left(L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)\right), \quad U_g : f \mapsto f \circ T_g^{-1}, \quad g \in G.$$

4.2. Poissonian actions

Let us now recall the fundamental object of Poissonian action, which we are about to discuss in detail. For a generative Poisson point process $\mathcal{P}$ as in Definition 1.1 we have the real Hilbert space

$$H(\mathcal{P}) := \overline{\mathrm{span}}\{P_A : A \in \mathcal{B}_\mu\} \subset L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P}) .$$

An action $\mathbf{S} : G \curvearrowright (\Omega, \mathcal{F}, \mathbb{P})$ of a Polish group G is said to be a **Poissonian action** with respect to $\mathcal{P}$, if its Koopman representation preserves $H(\mathcal{P})$ within $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$.

Let us now introduce the Poisson suspension construction in a precise way as a natural source for Poissonian actions. Suppose $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$ is an action of a Polish group G on a standard measure space $(X, \mathcal{B}, \mu)$. Consider the classical Poisson point process $\mathcal{N} = \{N_A : A \in \mathcal{B}\}$ defined on $(X^*, \mathcal{B}^*, \mu^*)$. Using the continuous embedding introduced in Proposition 3.1, we obtain an action $\mathbf{T}^* : G \curvearrowright (X^*, \mathcal{B}^*, \mu^*)$ by composing

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[4] Since U_g is a unitary operator and therefore continuous for each $g \in G$, by Fact 2.1 if $g \mapsto U_g f$ is Borel for each $f \in \mathcal{H}$, then $(g, f) \mapsto U_g f$ is jointly continuous.

$$\mathbf{T}^{*}: G \to \operatorname{Aut}(X, \mathcal{B}, \mu) \hookrightarrow \operatorname{Aut}(X^{*}, \mathcal{B}^{*}, \mu^{*}), \quad g \mapsto T_g \mapsto T_g^{*}.$$

The equivariance property (3.1.1) readily implies that the Koopman representation of $\mathbf{T}^{*}$ preserves $H(\mathcal{P})$, thus $\mathbf{T}^{*}$ is a Poissonian action with respect to $\mathcal{N}$, and its base action is nothing but $\mathbf{T}$.

Remark 4.3. In the setting of Definition 1.4, if $\mathcal{P}$ is not generative we may pass to the sub-σ-algebra on Ω generated by $\mathcal{P}$. Then if the equivariance relations of $\mathbf{S}$ and $\mathbf{T}$ hold, this sub-σ-algebra is $\mathbf{S}$-invariant and we obtain a factor of $\mathbf{S}$ which is a Poissonian action.

We now formulate and prove each of the statements in Theorem 1.

Theorem 4.4. *Let $\mathbf{T}: G \curvearrowright (X, \mathcal{B}, \mu)$ be an action and $\mathcal{P} = \{P_A : A \in \mathcal{B}\}$ be a generative Poisson point process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with base space $(X, \mathcal{B}, \mu)$. There exists a Poissonian action $\mathbf{S}: G \curvearrowright (\Omega, \mathcal{F}, \mathbb{P})$ with respect to $\mathcal{P}$ whose base action is $\mathbf{T}$. Moreover, $\mathbf{S}$ is essentially unique in that Poissonian actions associated with isomorphic base actions are isomorphic.*

Proof. By Proposition 3.3 we may assume without loss of generality that $(\Omega, \mathcal{F}, \mathbb{P}) = (X^{*}, \mathcal{B}^{*}, \mu^{*})$ and that $\mathcal{P} = \mathcal{N}$. Then the aforementioned construction of the Poisson suspension, using the embedding described in Proposition 3.1, is a Poissonian action with respect to $\mathcal{N}$ whose base action is $\mathbf{T}$.

In order to show the uniqueness, we start by showing that all Poissonian actions with base action $\mathbf{T}$ are isomorphic to the Poisson suspension $\mathbf{T}^{*}: G \curvearrowright (X^{*}, \mathcal{B}^{*}, \mu^{*})$. Let $\mathbf{S}: G \curvearrowright (\Omega, \mathcal{F}, \mathbb{P})$ be such a Poissonian action with respect to a generative Poisson point process $\mathcal{P} = \{P_A : A \in \mathcal{B}\}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. By Proposition 3.3 there is an isomorphism of probability spaces $\varphi: \Omega \to X^{*}$ such that $\mathbb{P} \circ \varphi^{-1} = \mu^{*}$, that interchanges $\mathcal{P}$ and $\mathcal{N}$ in that $N_A \circ \varphi = P_A$ for every $A \in \mathcal{B}$. In order to verify that indeed $\varphi \circ S_g = T_g^{*} \circ \varphi$ on a $\mathbb{P}$-conull set for every $g \in G$, note that

$$N_A \circ \varphi \circ S_g = P_A \circ S_g = P_{T_g^{-1}(A)} = N_A \circ T_g^{*} \circ \varphi \quad \text{for every } A \in \mathcal{B},$$

and since $\mathcal{N}$ is generative the desired property follows. Thus, φ is an isomorphism of actions.

For the general case, suppose $\mathbf{T}': G \curvearrowright (X', \mathcal{B}', \mu')$ is an infinite measure-preserving action that is isomorphic to $\mathbf{T}: G \curvearrowright (X, \mathcal{B}, \mu)$ via $\psi: X' \to X$. From the classical Poisson point process $\mathcal{N} = \{N_A : A \in \mathcal{B}\}$ on the base space $(X, \mathcal{B}, \mu)$ we obtain the Poisson point process $\mathcal{P} := \left\{N_{\psi(A')} : A' \in \mathcal{B}'\right\}$ on the base space $(X', \mathcal{B}', \mu')$. It is then evident that the Poisson suspension $\mathbf{T}'^{*}: G \curvearrowright (X'^{*}, \mathcal{B}'^{*}, \mu'^{*})$ is a Poissonian action with respect to $\mathcal{P}$ with base action $\mathbf{T}: G \curvearrowright (X, \mathcal{B}, \mu)$. Hence, by the previous part of the proof it is isomorphic to the Poisson suspension $\mathbf{T}^{*}: G \curvearrowright (X^{*}, \mathcal{B}^{*}, \mu^{*})$. $\square$

The following theorem generalizes the second statement of Theorem 1, which can be seen as a statement on Koopman representations, to a larger family of unitary representations of the real Hilbert space

$$H(\mathcal{P}) := \overline{\operatorname{span}}\left\{P_A : A \in \mathcal{B}_\mu\right\} \subset L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P}),$$

defined for a Poisson point process $\mathcal{P}$ as in Definition 1.1.

Theorem 4.5. *Let $\mathcal{P}$ be a generative Poisson point process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with base space $(X, \mathcal{B}, \mu)$. For every unital positivity preserving unitary representation $\mathbf{U} : G \to \mathrm{U}(H(\mathcal{P}))$ there exists a unique action $\mathbf{S} : G \curvearrowright (\Omega, \mathcal{F}, \mathbb{P})$ whose Koopman representation coincides with $\mathbf{U}$ on $H(\mathcal{P})$. This $\mathbf{S}$ is a Poissonian action with respect to $\mathcal{P}$, admitting a unique base action $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$.*

Noting that Koopman representations are unital and positivity preserving, we obtain from Theorem 4.5 for Koopman representations:

Corollary 4.6. *Every Poissonian action arises from a unique base action.*

For the proof of Theorem 4.5 we introduce two lemmas. First, we will use the following substitution for the notion of unital operators when dealing with infinite measure spaces. A unitary operator U of $L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$ for some standard measure space $(X, \mathcal{B}, \mu)$ will be called **quasi-unital** if

$$\int_X U f \, d\mu = \int_X f \, d\mu \text{ for every } f \in L^1_{\mathbb{R}}(X, \mathcal{B}, \mu) \cap L^2_{\mathbb{R}}(X, \mathcal{B}, \mu).$$

In particular, U preserves $L^1_{\mathbb{R}}(X, \mathcal{B}, \mu) \cap L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$ within $L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$. The quasi-unital positivity preserving unitary operators of $L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$ form a subgroup of the unitary group of $L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$.

In the following we formulate in our terminology a well-known fact which is a version of the Banach–Lamperti Theorem for L^2. As it was observed in [45, footnote 3], while the general Banach–Lamperti Theorem is formulated for unitary operators of L^p-spaces for $p \neq 2$, for positivity preserving unitary operators the proof of Lamperti applies for L^2-spaces as well. A byproduct of the following lemma is that when μ is a probability measure, for a positivity preserving unitary operator of $L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$ being unital and quasi-unital is the same.

Lemma 4.7. *For every standard measure space $(X, \mathcal{B}, \mu)$, the Koopman embedding*

$$T \mapsto U_T, \quad U_T f = f \circ T^{-1},$$

forms a bijection between Aut $(X, \mathcal{B}, \mu)$ *and the quasi-unital positivity preserving unitary operators of* $L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$. *Consequently, the latter is a closed subgroup of the unitary group of* $L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$ *in the strong operator topology.*

Proof. By the Banach–Lamperti Theorem in L^2 (see the aforementioned [45, footnote 3]), there is a bijective correspondence between the group of nonsingular transformations of $(X, \mathcal{B}, \mu)$ and the group of positivity preserving unitary operators of $L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$, given by

$$T \mapsto U_T, \quad U_T f = \sqrt{\frac{d\mu \circ T^{-1}}{d\mu}} \cdot f \circ T^{-1}.$$

This is a homomorphism of groups and, since T is necessarily measure-preserving when U_T is quasi-unital, the measure-preserving transformations correspond to the quasi-unital positivity preserving unitary operators. Thus, the restriction of this homomorphism to Aut $(X, \mathcal{B}, \mu)$, which is the usual Koopman embedding, forms a bijective homomorphism from Aut $(X, \mathcal{B}, \mu)$ onto the closed subgroup of quasi-unital positivity preserving unitary operators of $L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$. Finally, the Polish topology of Aut $(X, \mathcal{B}, \mu)$ is, by definition, induced from this correspondence, so this is a homeomorphism. □

For the next step towards proving Theorem 4.5 we will take a further look into unitary operators of $H(\mathcal{P})$. The following objects and their basic properties are presented in more details in Appendix A. Fix a Poisson point process $\mathcal{P}$ as in Definition 1.1. The **first chaos** of $\mathcal{P}$ is the space

$$H_1(\mathcal{P}) := \overline{\operatorname{span}}\{P_A - \mu(A) : A \in \mathcal{B}_\mu\} \subset L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P}).$$

As part of the Fock space structure associated with $\mathcal{P}$, there is an isometric isomorphism of Hilbert spaces

$$I_\mu : L^2_{\mathbb{R}}(X, \mathcal{B}, \mu) \to H_1(\mathcal{P}), \quad I_\mu : f \mapsto I_\mu(f),$$

given by a stochastic integral against $\mathcal{P}$. Recalling the space

$$H(\mathcal{P}) = \overline{\operatorname{span}}\{P_A : A \in \mathcal{B}_\mu\} \subset L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P}),$$

the first chaos is its orthogonal summand,

$$H(\mathcal{P}) = H_1(\mathcal{P}) \oplus \mathbb{R},$$

where $\mathbb{R}$ is understood as the subspace of constant functions in $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$. For every unital positivity preserving unitary operator U of $H(\mathcal{P})$ we have

$$\langle U(I_\mu(f)),1\rangle = \left\langle I_\mu(f), U^{-1}(1)\right\rangle = \langle I_\mu(f),1\rangle, \quad f \in L^2_{\mathbb{R}}(X,\mathcal{B},\mu).$$

Thus, U preserves $H_1(\mathcal{P})$ as a direct summand of $H(\mathcal{P})$. Since I_μ is an isometric isomorphism of Hilbert spaces, the conjugation by I_μ forms a map between the unitary groups,

$$\mathrm{U}(H(\mathcal{P})) \to \mathrm{U}\left(L^2_{\mathbb{R}}(X,\mathcal{B},\mu)\right), \quad U \mapsto U^\mu := I_\mu^{-1} \circ U \circ I_\mu. \tag{4.7.1}$$

This map will be important to the proof of Theorem 4.5. The following lemma deals with its fundamental properties, with its proof following a similar approach to the proof of [40, Proposition 4.4].

Lemma 4.8. *The map $U \mapsto U^\mu$ as in (4.7.1) forms a bijection between unital positivity preserving unitary operators of $H(\mathcal{P})$ and quasi-unital positivity preserving unitary operator of $L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$.*

The proof makes essential use of the properties of the stochastic integral I_μ as well as the fundamentals of IDp random variables. We present this in Appendix A.

Proof. Since I_μ is an isometric isomorphism of Hilbert spaces, U^μ is a unitary operator. Let us fix an arbitrary $f \in L^1_{\mathbb{R}}(X,\mathcal{B},\mu) \cap L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$. As described in Proposition A.1, we have the IDp random variable

$$W_f := I_\mu(U^\mu f) + \int_X f d\mu = U(I_\mu(f)) + \int_X f d\mu = U\Big(\int_X f d\mathcal{P}\Big),$$

whose Lévy measure is given by

$$\ell_{U^\mu f} = \mu\mid_{\{U^\mu f \neq 0\}} \circ (U^\mu f)^{-1}.$$

Assume further that $f \geqslant 0$ so that also $\int_X f d\mathcal{P} \geqslant 0$ and, since U is positivity preserving, also $W_f \geqslant 0$. From Proposition A.1(3) it follows that

$$\mu(U^\mu f < 0) = \ell_{U^\mu f}(\mathbb{R}_{<0}) = 0 \text{ and } \int_X U^\mu f d\mu = \int_{\mathbb{R}_{\geqslant 0}} t d\ell_{U^\mu f}(t) < \infty.$$

The first property shows that U^μ preserves positivity for every f in a dense subspace, hence it is positivity preserving. The second property shows that $U^\mu f \in L^1_{\mathbb{R}}(X,\mathcal{B},\mu) \cap L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$, so that by Proposition A.1(1),

$$I_\mu(U^\mu f) = \int_X U^\mu f d\mathcal{P} - \int_X U^\mu f d\mu.$$

Plugging this into the definition of W_f and using that $W_f \geqslant 0$, we obtain

$$\int\limits_X U^\mu f d\mu - \int\limits_X f d\mu = \int\limits_X U^\mu f d\mathcal{P} - W_f \leqslant \int\limits_X U^\mu f d\mathcal{P}.$$

With f being fixed, the left hand side is a constant, while the right hand side is a nonnegative IDp random variable that is obtained as a stochastic integral of an integrable function. It follows from [42, Corollary 24.8] that the infimum of the right hand side (as a random variable) is zero, and we conclude that

$$\int\limits_X U^\mu f d\mu \leqslant \int\limits_X f d\mu.$$

Since the map (4.7.1) respects inverses, the same proof shows that the same inequality holds when U^μ is replaced by $(U^\mu)^{-1}$. It follows that

$$\int\limits_X U^\mu f d\mu = \int\limits_X f d\mu,$$

hence U^μ is quasi-unital. □

Proof of Theorem 4.5. Suppose $\mathbf{U} : G \to \mathrm{U}(H(\mathcal{P}))$ is a unital positivity preserving unitary representation of a Polish group G as in the theorem. We construct a continuous homomorphism along the following arrows

$$\begin{tikzcd} G \arrow[r, "\mathbf{U}"] & \mathrm{U}(H(\mathcal{P})) \arrow[r, "(4.7.1)"] & \mathrm{U}\left(L^2_{\mathbb{R}}(X,\mathcal{B},\mu)\right) \arrow[r, "\text{Banach–Lamperti}"] & \mathrm{Aut}(X,\mathcal{B},\mu) \end{tikzcd}$$

as follows. The first arrow is the given representation $g \mapsto U_g$. The second arrow is the map $U_g \mapsto U_g^\mu$ as in (4.7.1), whose image lies in the closed subgroup of quasi-unital positivity preserving unitary operators by Lemma 4.8. Then on the image of the second arrow, the third arrow $U_g^\mu \mapsto T_g$ is given by Lemma 4.7. Since all the arrows are continuous, the map $g \mapsto T_g$ constitutes an action $\mathbf{T} : G \curvearrowright (X,\mathcal{B},\mu)$. We now use Theorem 4.4 to pick a Poissonian action $\mathbf{S} : G \curvearrowright (\Omega,\mathcal{F},\mathbb{P})$ whose base action is $\mathbf{T}$. The equivariance property that relates $\mathbf{T}$ and $\mathbf{S}$ as in Definition 1.4 implies that

$$U_g(I_\mu(f)) = I_\mu\left(U_g^\mu f\right) = I_\mu\left(f \circ T_g^{-1}\right) = I_\mu(f) \circ S_g, \quad g \in G,$$

for every $f \in L^1_{\mathbb{R}}(X,\mathcal{B},\mu) \cap L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$. Thus, $\mathbf{U}$ coincides with the Koopman representation of $\mathbf{S}$ on $H(\mathcal{P})$, which is a Poissonian action with base action $\mathbf{T}$ as required in the theorem.

We are now left with proving the two uniqueness properties stated in the theorem. To this end, it will be convenient to use Proposition 3.3 and assume, without loss of generality, that the generative Poisson point process $\mathcal{P}$ is the classical Poisson point process $\mathcal{N}$, defined on $(X^*, \mathcal{B}^*, \mu^*)$.

For the uniqueness of $\mathbf{S}$, assume that $\mathbf{S}' : G \curvearrowright (X^*, \mathcal{B}^*, \mu^*)$ is an action whose Koopman representation coincides with $\mathbf{U}$ on $H(\mathcal{N})$. Then:

$$N_A \circ S'_g = U_{g^{-1}} N_A = N_A \circ S_g \quad \mu^*\text{-a.e. for every } g \in G \text{ and } A \in \mathcal{B}.$$

Explicitly, this means that for every $g \in G$ and every $A \in \mathcal{B}$ it holds that

$$S_g(\omega)(A) = S'_g(\omega)(A) \text{ for } \mu^*\text{-a.e. } \omega \in X^*.$$

Since Radon measures are completely determined by their values on countably many elements of $\mathcal{B}$, it follows that $S_g(\omega)$ and $S'_g(\omega)$ coincide as Radon measures for μ^*-a.e. $\omega \in X^*$, meaning that $S_g = S'_g$ in $\operatorname{Aut}(X^*, \mathcal{B}^*, \mu^*)$. This shows that $\mathbf{S} = \mathbf{S}'$.

For the uniqueness of $\mathbf{T}$, assume that $\mathbf{T}' : G \curvearrowright (X, \mathcal{B}, \mu)$ is a base action for $\mathbf{S}$ with respect to $\mathcal{N}$. Then:

$$N_{T_g^{-1}(A)} = N_A \circ S_g = N_{T_g'^{-1}(A)} \quad \mu^*\text{-a.e. for every } g \in G \text{ and } A \in \mathcal{B}.$$

This means that $T_g^* = T_g'^*$ in $\operatorname{Aut}(X^*, \mathcal{B}^*, \mu^*)$, and by Proposition 3.1 we deduce that $T_g = T'_g$ in $\operatorname{Aut}(X, \mathcal{B}, \mu)$. This shows that $\mathbf{T} = \mathbf{T}'$. $\square$

4.3. Ergodicity of Poissonian actions

Here we prove Theorem 2. Let us first recall some basic definitions. For an action $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$ we denote the invariant σ-algebra

$$\mathcal{I}_{\mathbf{T}} := \left\{A \in \mathcal{B} : \mu\left(A \triangle T_g^{-1}(A)\right) = 0 \text{ for every } g \in G\right\}.$$

We say that $\mathbf{T}$ is:

- **Null:** for every $A \in \mathcal{I}_{\mathbf{T}}$ either $\mu(A) = 0$ or $\mu(A) = \infty$;
- **Ergodic:** for every $A \in \mathcal{I}_{\mathbf{T}}$ either $\mu(A) = 0$ or $\mu(X \backslash A) = 0$;
- **Doubly Ergodic:** the diagonal action $\mathbf{T} \otimes \mathbf{T} : G \curvearrowright (X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \otimes \mu)$ is ergodic;
- **Weakly mixing:** for every ergodic action $\mathbf{S} : G \curvearrowright (Y, \mathcal{C}, \nu)$ of G, the diagonal action $\mathbf{T} \otimes \mathbf{S} : G \curvearrowright (X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ is ergodic.

Remark 4.9. A few general remarks about those properties:

1. Being null is equivalent to the non-existence of a $\mathbf{T}$-invariant probability measure absolutely continuous with respect to μ.
2. Double ergodicity and weak mixing are equivalent in probability-preserving actions of general groups. For locally compact groups this is a classical fact, and it was

pointed out to us by Benjamin Weiss in a personal communication that this is true in full generality. Indeed, one implication is obvious, and the other was proved by Glasner & Weiss in [16, Theorem 2.1], that while is formulated for locally compact groups the proof holds in full generality. With the terminology of [16, Definition 1.1], this can be seen by looking at the proofs of the implications $DE \implies EIC \implies EUC \implies WM$.

(3) For a probability-preserving action $\mathbf{T}$ of G, let $\mathbf{T}^{\otimes\mathbb{N}}$ denote the infinite diagonal action of G on the infinite product of the underlying probability space. Note that a probability-preserving action $\mathbf{T}$ is weakly mixing precisely when $\mathbf{T}^{\otimes\mathbb{N}}$ is ergodic (this follows from Lévy's 0-1 Law). Furthermore, since $\mathbf{T}^{\otimes\mathbb{N}}$ and $\mathbf{T}^{\otimes\mathbb{N}}\otimes\mathbf{T}^{\otimes\mathbb{N}}$ are isomorphic, in this case $\mathbf{T}^{\otimes\mathbb{N}}$ is further weakly mixing. This will be useful in the proof of Theorem 7.

In proving Theorem 2 we will need the following key result established in [34, Theorem 1]. We will refer to the chaos structure of $L^2_{\mathbb{R}}(X^*,\mathcal{B}^*,\mu^*)$, as the orthogonal sum

$$L^2_{\mathbb{R}}(X^*,\mathcal{B}^*,\mu^*)=\bigoplus\nolimits_{n=0}^{\infty}H_n(\mathcal{N}),$$

which is its structure as the Fock space of $L^2_{\mathbb{R}}(X,\mathcal{B},\mu)$. For details see Appendix A.0.1.

Theorem 4.10 *(Parreau & Roy). Let $(X,\mathcal{B},\mu)$ be a standard nonatomic infinite measure space. Every conditional expectation of $L^2_{\mathbb{R}}(X^*,\mathcal{B}^*,\mu^*)$ that preserves its chaos structure and vanishes on the first chaos $H_1(\mathcal{N})$, necessarily vanishes on all higher chaoses, i.e. it is the projection to the constants.*

The following technical lemma was proved for a single transformation in [41, §3.2], and the following extension to general groups is straightforward.

Lemma 4.11. *Let $\mathbf{T}:G\curvearrowright(X,\mathcal{B},\mu)$ be an action of a Polish group G. Then the conditional expectation $\pi_{\mathbf{T}^*}$ of $L^2_{\mathbb{R}}(X^*,\mathcal{B}^*,\mu^*)$ with respect to $\mathcal{I}_{\mathbf{T}^*}$ preserves the chaos structure of $L^2_{\mathbb{R}}(X^*,\mathcal{B}^*,\mu^*)$.*

Proof. We start with a single transformation $T\in\operatorname{Aut}(X,\mathcal{B},\mu)$ and consider $T^*\in\operatorname{Aut}(X^*,\mathcal{B}^*,\mu^*)$. Let U_{T^*} be the Koopman operator of T^* and set $\Psi_{T^*}=U_{T^*}-\mathrm{Id}$, noting that $\mathrm{Im}\pi_{T^*}=\ker\Psi_{T^*}$. Since U_{T^*} preserves the chaos structure of $L^2_{\mathbb{R}}(X^*,\mathcal{B}^*,\mu^*)$ so does Ψ_{T^*}. Then for every $f=\sum_{n=0}^{\infty}f_n\in L^2_{\mathbb{R}}(X^*,\mathcal{B}^*,\mu^*)$, where $f_n\in H_n(\mathcal{N})$ for every n, we have

$$\Psi_{T^*}f=\sum\nolimits_{n=0}^{\infty}\Psi_{T^*}f_n \text{ and } \Psi_{T^*}f_n\in H_n(\mathcal{N}) \text{ for every } n.$$

Hence $f\in\ker\Psi_{T^*}$ if and only if $\Psi_{T^*}f_n=0$ for every n, meaning that

$$\ker\Psi_{T^*}=\bigoplus\nolimits_{n=0}^{\infty}\left(\ker\Psi_{T^*}\cap H_n(\mathcal{N})\right).$$

However, since $\mathrm{Im}\pi_{T^*} = \ker \Psi_{T^*}$ on each $H_n(\mathcal{N})$ we obtain

$$\mathrm{Im}\pi_{T^*} = \bigoplus\nolimits_{n=0}^{\infty} \left(\mathrm{Im}\pi_{T^*} \cap H_n(\mathcal{N})\right). \tag{4.11.1}$$

Since $\ker \pi_{T^*}$ is the orthogonal complement of $\mathrm{Im}\pi_{T^*}$ on each $H_n(\mathcal{N})$, also

$$\ker \pi_{T^*} = \bigoplus\nolimits_{n=0}^{\infty} \left(\ker \pi_{T^*} \cap H_n(\mathcal{N})\right). \tag{4.11.2}$$

Then (4.11.1) and (4.11.2) together imply that π_{T^*} preserves the chaos structure of $L^2_{\mathbb{R}}(X^*, \mathcal{B}^*, \mu^*)$, proving the lemma for a single transformation.

Now for an action $\mathbf{T}$, if we abbreviate $\Psi^*_g := \Psi_{T^*_g} = U_{T^*_g} - \mathrm{Id}$ for $g \in G$, then by the same reasoning

$$\bigcap\nolimits_{g\in G} \ker \Psi^*_g = \bigoplus\nolimits_{n=0}^{\infty} \bigcap\nolimits_{g\in G} \left(\ker \Psi^*_g \cap H_n(\mathcal{N})\right).$$

Since $\mathrm{Im}\pi_{\mathbf{T}^*} = \bigcap_{g\in G} \ker \Psi^*_g$ on each $H_n(\mathcal{N})$, it similarly follows that

$$\mathrm{Im}\pi_{\mathbf{T}^*} = \bigoplus\nolimits_{n=0}^{\infty} \left(\mathrm{Im}\pi_{\mathbf{T}^*} \cap H_n(\mathcal{N})\right)$$

and then that

$$\ker \pi_{\mathbf{T}^*} = \bigoplus\nolimits_{n=0}^{\infty} \left(\ker \pi_{\mathbf{T}^*} \cap H_n(\mathcal{N})\right).$$

This completes the proof of the lemma also for actions. $\square$

Proof of Theorem 2. By Proposition 3.3 and Theorem 1 it is sufficient to prove the theorem for Poisson suspensions $\mathbf{T}^* : G \curvearrowright (X^*, \mathcal{B}^*, \mu^*)$. If the action $\mathbf{T}$ is null there are no nonzero $\mathbf{T}$-invariant function in $L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$. By the Fock space structure, there are no nontrivial $\mathbf{T}^*$-invariant functions in the first chaos $H_1(\mathcal{N})$. Thus, the conditional expectation $\pi_{\mathbf{T}^*}$ as in Lemma 4.11 vanishes on $H_1(\mathcal{N})$, and by Theorem 4.10 we obtain that $\mathrm{Im}\pi_{\mathbf{T}^*}$ consists of constant functions, meaning that the Poisson suspension is ergodic. Conversely, if the action is not null so that $\mathcal{I}_{\mathbf{T}}$ contains a set A with $0 < \mu(A) < \infty$, then N_A is a non-constant, $\mathbf{T}^*$-invariant function in $L^2_{\mathbb{R}}(X^*, \mathcal{B}^*, \mu^*)$, so the Poisson suspension is not ergodic.

Let us now show that for the Poisson suspension ergodicity implies weak mixing, and in doing so we will use the previous part of the proof twice. The Poisson suspension being ergodic implies that $\mathbf{T} : G \curvearrowright (X, \mathcal{B}, \mu)$ is null, and it is clear that in this case also the action

$$\mathbf{T} \otimes \mathrm{Id} : G \curvearrowright \left(X \times \{0,1\}, \mu \otimes \tfrac{1}{2}(\delta_0 + \delta_1)\right)$$

is null, so that the Poisson suspension associated with this latter action is ergodic as well. However, this Poisson suspension, $(\mathbf{T} \otimes \mathrm{Id})^*$, when taken with respect to the product of the topology τ for which X^* was defined with the discrete topology of $\{0,1\}$, is isomorphic to the diagonal action

$$\mathbf{T}^* \otimes \mathbf{T}^* : G \curvearrowright (X^* \times X^*, \mathcal{B}^* \otimes \mathcal{B}^*, \mu^* \otimes \mu^*) ,$$

hence $\mathbf{T}^*$ is weakly mixing. □

Remark 4.12. It was suggested to us by Alexandre Danilenko that, since the ergodicity of a probability-preserving action of a Polish group is determined by the ergodicity of the action restricted to a countable dense subgroup, it is sufficient to prove Theorem 2 for countable (non-topological) groups. Now for countable groups, Theorem 2 can be deduced from known results mentioned in [8, Facts 2.2 & 2.4].

5. Spatial Poissonian actions

Our main object of study in this part is a more restrictive notion of measure-preserving actions, namely spatial actions.

5.1. Preliminaries: spatial actions and the point-realization problem

We recall that in our terminology, an *action* refers to a near/Boolean action as in Definition 4.2. In contrast, we have the following more restrictive notion of a measure-preserving action.

Definition 5.1. A (measure-preserving) **spatial action** of a Polish group G on a standard measure space $(X, \mathcal{B}, \mu)$ is a Borel action $\mathbf{T} : G \curvearrowright (X, \mathcal{B})$, $\mathbf{T} : (g, x) \mapsto T_g(x)$, such that $\mu \circ T_g^{-1} = \mu$ for every $g \in G$. We denote spatial actions by

$$\mathbf{T} : G \overset{\text{sp}}{\curvearrowright} (X, \mathcal{B}, \mu) .$$

Every spatial action induces an action in an obvious way, and in this case, it can be regarded as a *point-realization* of the resulting action. However, in general, not every action admits a point-realization. This leads to the point-realization problem in ergodic theory, which concerns whether a given action admits such a realization. Remarkably, for certain important classes of Polish groups, this problem has a striking solution.

A Polish group G is said to possess the **Mackey property** (following [21]) if every probability-preserving action of G admits a point-realization. The following classes of groups are known to possess the Mackey property:

- **Locally compact Polish groups:** This is the celebrated Mackey–Ramsay Point-Realization Theorem [36, Theorem 3.3].
- **Non-Archimedean groups:** Polish groups with a base of clopen subgroups at the identity. This is Glasner & Weiss' [15, Theorem 4.3].

- **Groups of isometries of a locally compact metric space:** closed subgroups of the group of isometries of a locally compact metric space, with the Polish topology of pointwise convergence. This is Kwiatkowska & Solecki's [25, Theorem 1.1].

The class of groups of isometries of a locally compact metric space includes both locally compact Polish groups and non-Archimedean groups, and to the best of our knowledge this is the largest class of Polish groups on which the Mackey property is known to hold (beyond Polish groups see [7]). It is worth mentioning that in locally compact Polish groups, the point-realization is also unique up to an isomorphism of spatial actions [36, Theorem 3.5], but for general Polish groups this is no longer true (see [19, §4.3]).

The fact that there are Polish groups without the Mackey property was demonstrated by Becker for the Abelian group $L^0\left(\left[0,1\right],S^1\right)$ of Borel functions $[0,1]\to S^1$, identified up to equality on a Lebesgue-conull set, with the topology of convergence in measure (see [14, Appendix A], [35, §7.1] and the references therein).

This was vastly generalized by Glasner, Tsirelson & Weiss [14, Theorem 1.1] in showing that the Mackey property fails completely for the class of *Lévy groups*, a class of groups that was studied by many following Gromov & Milman (see [35, §4] and the references therein). This class includes, among others, the group $\mathrm{Aut}\left(X,\mathcal{B},\mu\right)$ itself with the weak topology; the unitary group $\mathrm{U}\left(\mathcal{H}\right)$ of a separable Hilbert space $\mathcal{H}$ with the strong operator topology; and, the aforementioned $L^0\left(\left[0,1\right],S^1\right)$. Thus, it was shown by Glasner, Tsirelson & Weiss that Lévy groups admit no probability-preserving spatial actions except for trivial ones, and a fortiori do not possess the Mackey property. There are also non-Lévy Polish groups that do not possess the Mackey property [15, §6], [33].

Remark 5.2. A spatial action is considered to be trivial if the set of fixed points of the action has full measure. Note that the set of fixed points of a Borel action of a Polish group is a Borel set. This is clearly true for Polish actions, and for Borel action this follows from the theorem of Becker & Kechris [4, §5.2] by which every Borel action is a Polish action with respect to some Polish topology that induces the given Borel σ-algebra.

Every Poissonian action of a group that possesses the Mackey property, admits a point-realization that serves as a spatial Poissonian action. By a *spatial Poissonian action* we refer to a Poissonian action (as in Definition 1.3) which is also a spatial action. Our goal here is to show that the point-realization problem in Poissonian actions can be solved without appealing to the Mackey property.

5.2. Poisson random set

For a standard Borel space $(X,\mathcal{B})$, with every Polish topology τ that generates $\mathcal{B}$ is associated the **Effros Borel space**

$$\mathbf{F}_{\tau}\left(X\right)=\left\{F\subseteq X:X\backslash F\in\tau\right\}.$$

It is a standard Borel space in the **Effros σ-algebra** $\mathcal{E}_\tau(X)$, generated by

$$B_O := \{F \in \mathbf{F}_\tau(X) : F \cap O \neq \emptyset\}, \quad O \in \tau.$$

See [20, Section (12.C)]. A **random closed set** is nothing but a probability measure on $(\mathbf{F}_\tau(X), \mathcal{E}_\tau(X))$, namely an element of $\mathcal{M}_1(\mathbf{F}_\tau(X), \mathcal{E}_\tau(X))$. A common way to construct random closed sets for locally compact topologies is the Choquet Capacity Theorem (see [32, §1.1.3]), but here we shall construct the Poisson random closed set also for Polish topologies that may not be locally compact.

Let us denote by

$$(\mathbf{F}^*_\tau(X), \mathcal{E}^*_\tau(X)) \subset (\mathbf{F}_\tau(X), \mathcal{E}_\tau(X))$$

the subspace of infinite sets in $\mathbf{F}_\tau(X)$ with its induced Effros σ-algebra. Note that $\mathbf{F}^*_\tau(X)$ is Effros-measurable: fixing a countable base $\mathcal{O}$ for τ, for every $n \in \mathbb{Z}_{\geqslant 0}$ the property $\#F \geqslant n$ of $F \in \mathbf{F}_\tau(X)$ is witnessed by the existence of pairwise disjoint $O_1, \ldots, O_n \in \mathcal{O}$ such that $F \in B_{O_1} \cap \cdots \cap B_{O_n}$.

Theorem 5.3 *(Poisson random set). Let $(X, \mathcal{B})$ be a standard Borel space. For every Polish topology τ that generates $\mathcal{B}$, there are random variables*

$$\{\Xi_A : A \in \mathcal{B}\} \text{ of the form } \Xi_A : \mathbf{F}^*_\tau(X) \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\},$$

and a one-to-one correspondence

$$\mathcal{M}^\tau_\sigma(X, \mathcal{B}) \to \mathcal{M}_1(\mathbf{F}^*_\tau(X), \mathcal{E}^*_\tau(X)), \quad \mu \mapsto \mu^*_\tau,$$

such that $\{\Xi_A : A \in \mathcal{B}\}$ forms a Poisson point process with base space $(X, \mathcal{B}, \mu)$. Furthermore, for every μ the following properties hold.

(1) μ^*_τ is supported on the class of τ-discrete sets.[5]
(2) On the support of μ^*_τ it holds that $\Xi_A(\cdot) = \#(A \cap \cdot)$ for every $A \in \mathcal{B}$.

In order to spot the inherent difficulty in defining the Poisson point process as a random closed set in $\mathbf{F}_\tau(X)$, it should be noted that as long as τ is not locally compact, then for a general $A \in \mathcal{B}$, the function

$$\mathbf{F}_\tau(X) \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}, \quad F \mapsto \#(A \cap F),$$

may not be Effros-measurable unless $A \in \tau$. In fact, by a theorem of Christensen this function may fail to be Effros-measurable even if $A \in \mathbf{F}_\tau(X)$ [20, Theorem (27.6)]. In

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[5] While the class of τ-discrete sets is not Effros-measurable, by a theorem of Hurewicz it is co-analytic (see [20, Theorem (27.5), Exercise (27.8)]) hence universally measurable.

order to resolve this issue we use the following simple modification on the Kuratowski–Ryll-Nardzewski's Selection Theorem.

Proposition 5.4. *Let $(X, \mathcal{B})$ be a standard Borel space. For every Polish topology τ that generates $\mathcal{B}$ there is a countable collection of mappings*

$$\{\xi_n : n \in \mathbb{N}\} \text{ of the form } \xi_n : \mathbf{F}_\tau^* (X) \to X,$$

such that the following properties hold.

1. *(Measurability) Each ξ_n is Effros-measurable.*
2. *(Injectivity) For each $F \in \mathbf{F}_\tau^* (X)$, the mapping $\mathbb{N} \to X$ given by $n \mapsto \xi_n (F)$ is injective.*
3. *(Selectivity) For each $F \in \mathbf{F}_\tau^* (X)$, the set $\{\xi_n (F) : n \in \mathbb{N}\}$ is a dense subset of F.*

We will refer to such $\{\xi_n : n \in \mathbb{N}\}$ as a ***measurable injective selection****.*

Proof. By the Kuratowski–Ryll-Nardzewski's Selection Theorem (see [20, Theorem (12.13)]) there exists a measurable selection: a collection of mappings $\theta_n : \mathbf{F}_\tau (X) \setminus \{\emptyset\} \to X$, $n \in \mathbb{N}$, satisfying the first and the third properties. For each $n \in \mathbb{N}$, restrict the mapping θ_n to $\mathbf{F}_\tau^* (X)$ and modify it into the mapping $\xi_n : \mathbf{F}_\tau^* (X) \to X$ by letting

$$\xi_n (F) = \theta_{\pi_n(F)} (F), \quad F \in \mathbf{F}_\tau^* (X),$$

where $\pi_n : \mathbf{F}_\tau^* (X) \to \mathbb{N}$ is given by

$$\pi_n (F) = \inf \{k \in \mathbb{N} : \# \{\theta_1 (F), \ldots, \theta_k (F)\} = n\}.$$

Clearly, $\{\xi_n : n \in \mathbb{N}\}$ forms an injective selection, and an elementary proof by induction on π_n and ξ_n shows that each ξ_n is Effros-measurable. □

Proposition 5.5. *Let $(X, \mathcal{B})$ be a standard Borel space. For every Polish topology τ that generates $\mathcal{B}$ there are mappings*

$$\{\Xi_A : A \in \mathcal{B}\} \text{ of the form } \Xi_A : \mathbf{F}_\tau^* (X) \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\},$$

with the following properties:

1. *$\Xi_{A \cup B} = \Xi_A + \Xi_B$ for every disjoint $A, B \in \mathcal{B}$.*
2. *Each Ξ_A is Effros-measurable, and $\{\Xi_A : A \in \mathcal{B}\}$ generate $\mathcal{E}_\tau^* (X)$.*
3. *The identity $\Xi_A (F) = \# (A \cap F)$ holds in the following cases:*
    - *(i) $A \in \tau$ (and every F).*
    - *(ii) There is $O \in \tau$ such that $A \subseteq O \in \tau$ and $\# (F \cap O) < \infty$.*
    - *(iii) $A \in \mathcal{B}$ and F is τ-discrete.*

Proof. Pick a measurable injective selection $\{\xi_n : n \in \mathbb{N}\}$ of $\mathbf{F}_\tau^*(X)$ as in Proposition 5.4, and for $A \in \mathcal{B}$ put

$$\Xi_A(F) = \#\{n \in \mathbb{N} : \xi_n(F) \in A\} = \sum\nolimits_{n \in \mathbb{N}} 1_{\{\xi_n \in A\}}(F).$$

Property (1) follows from the injectivity of the measurable injective selection. As for Property (2), the Effros-measurability of each Ξ_A follows from the measurability of the measurable injective selection. We now make the observation that if $A \in \tau$ and $C \subseteq A$ is any countable set, then $\#\left(A \cap \overline{C}\right) = \#(A \cap C)$, where $\overline{C}$ denotes the closure of C, in the sense that they are either infinite together and otherwise both are finite and of the same cardinality. The same is true when $A \subseteq O \in \tau$ and $C \subseteq O$ is a finite set, for some $O \in \tau$. Thus, using the properties of measurable injective selection, this establishes Property (3)(i) and (ii), and in a straightforward way also Property (3)(iii). Finally, note that by Property (3)(i) we have

$$B_O \cap \mathbf{F}_\tau^*(X) = \{\Xi_O > 0\}, \quad O \in \tau,$$

and this completes the proof of the second part of Property (2). □

We can now prove Theorem 5.3. To this end we exploit the existence of the classical Poisson point process for locally compact Polish topologies as follows. Let $(X, \mathcal{B}, \mu)$ be a standard infinite nonatomic measure space, and consider the classical construction of the Poisson point process: Following Proposition 3.1, pick a locally compact Polish topology ϑ for which μ is Radon, and let $(X_\vartheta^*, \mathcal{B}_\vartheta^*, \mu_\vartheta^*)$ with the random variables $N_A : \omega \mapsto \omega(A)$ for $A \in \mathcal{B}$. Consider the map

$$\Phi : X_\vartheta^* \to \mathbf{F}_\vartheta^*(X), \quad \Phi(\omega) = \mathrm{Supp}(\omega).$$

Since the Poisson point process consists of simple counting Radon measures, Φ is well-defined. It is injective and, since $\Phi^{-1}(B_O) = \{N_O > 0\}$ for every $O \in \tau$, it is measurable. Define now

$$\Xi_A^\vartheta : \Phi(X_\vartheta^*) \subseteq \mathbf{F}_\vartheta^*(X) \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}, \quad \Xi_A^\vartheta = N_A \circ \Phi^{-1}, \quad A \in \mathcal{B}.$$

Evidently, the random variables $\left\{\Xi_A^\vartheta : A \in \mathcal{B}\right\}$ satisfy the condition of Proposition 5.5. Thus, the classical Poisson point process

$$(X_\vartheta^*, \mathcal{B}_\vartheta^*, \mu_\vartheta^*) \text{ with } \{N_A : A \in \mathcal{B}\}$$

can be naturally identified via Φ with the Poisson random set

$$\left(\mathbf{F}_\vartheta^*(X), \mathcal{E}_\vartheta^*(X), \mu_\vartheta^*\right) \text{ with } \left\{\Xi_A^\vartheta : A \in \mathcal{B}\right\}. \tag{5.5.1}$$

Proof of Theorem 5.3. Let $(X,\mathcal{B},\mu)$ be a standard infinite measure space, and τ a Polish topology τ on X for which $\mu\in\mathcal{M}_{\sigma}^{\tau}(X,\mathcal{B})$. The proof will be divided into four parts. In the first we construct a generating algebra of sets $\mathbf{A}$ on $\mathbf{F}_{\tau}^{*}(X)$. In the second we define μ_{τ}^{*} by defining it as a pre-measure on $\mathbf{A}$ and extending using the Hahn–Kolmogorov Extension Theorem. In the third and fourth we show the desired properties of μ_{τ}^{*}.

Part 1. Since μ is τ-locally finite, we can fix a countable base for τ,

$$\mathcal{O}=\{O_1,O_2,\dots\},\ \text{such that } \mu(O_n)<\infty \text{ for every } n.$$

Since closed subsets of Polish spaces are Polish in the subspace topology, by removing the largest τ-open set which is μ-null, we may assume that $\mu(O_n)>0$ for every n. For every n, let ρ_n be the finest partition of $O_1\cup\dots\cup O_n$ generated by $\{O_1,\dots,O_n\}$. Every atom of ρ_n, being a subset of $O_1\cup\dots\cup O_n$, belongs to $\mathcal{B}_{\mu}$. Since $\mathcal{O}$ is a base for τ, the ascending sequence of partitions $(\rho_n)_{n=1}^{\infty}$ separates points in X, and hence the ascending sequence of σ-algebras $(\sigma(\rho_n))_{n=1}^{\infty}$ generates $\mathcal{B}$ (see [20, Exercise (15.4)]).

Fix a collection of random variables $\{\Xi_A : A\in\mathcal{B}\}$ as in Proposition 5.5. For every nonempty $A\in\mathcal{B}$, define a partition Π_A of $\mathbf{F}_{\tau}^{*}(X)$ to consist of two atoms: $\{\Xi_A=0\}$ and $\{\Xi_A>0\}$. More generally, for a partition ρ of a nonempty $A\in\mathcal{B}$, let Π_{ρ} be the finest partition of $\mathbf{F}_{\tau}^{*}(X)$ generated by the partitions Π_B, $B\in\rho$, of A. For every n, recalling the partition ρ_n, put

$$\Pi_n:=\Pi_{\rho_n}.$$

Thus, ρ_n is a partition of $O_1\cup\dots\cup O_n\subset X$ and Π_n is a partition of $\mathbf{F}_{\tau}^{*}(X)$. We note that $\#\Pi_n=2^{\#\rho_n}$ for every n. More specifically, for every subset $S\subset\rho_n$ there corresponds the unique nonempty atom

$$\bigcap\nolimits_{B\in S}\{\Xi_B>0\}\cap\bigcap\nolimits_{B\in\rho_n\setminus S}\{\Xi_B=0\}\in\Pi_n.$$

To see why every such an atom is nonempty, for every $B\in S$ pick an arbitrary point $x_B\in B$ and put $F:=\{x_B:B\in S\}\cup(X\backslash(O_1\cup\dots\cup O_n))$. One can then see that $F\in\mathbf{F}_{\tau}^{*}(X)$ and it belongs to the atom defined by S.

Since $\mathcal{E}_{\tau}^{*}(X)$ is generated by $\{\Xi_O : O\in\tau\}$, it follows that the ascending sequence of partitions $(\Pi_n)_{n=1}^{\infty}$ separates points in $\mathbf{F}_{\tau}^{*}(X)$, and hence the ascending sequence of σ-algebras $(\sigma(\Pi_n))_{n=1}^{\infty}$ generates $\mathcal{E}_{\tau}^{*}(X)$ (as in the aforementioned [20, Exercise (15.4)]). We obtain that

$$\mathbf{A}:=\sigma(\Pi_1)\cup\sigma(\Pi_2)\cup\dots\subset\mathcal{E}_{\tau}^{*}(X),$$

is an algebra of sets that generates $\mathcal{E}_{\tau}^{*}(X)$.

Part 2. We now aim to define a pre-measure μ_0^{*} on $\mathbf{A}$, and to this end we use that $\{\Xi_A^{\vartheta} : A\in\mathcal{B}\}$, which are defined with respect to some locally compact Polish topology ϑ

for which μ becomes a Radon measure, already forms a Poisson point process with respect to μ_{ϑ}^{*} as in (5.5.1). Recall that in constructing the algebra $\mathbf{A}$ we used $\{\Xi_A : A \in \mathcal{B}\}$ as an input. Repeating this construction with $\{\Xi_A^{\vartheta} : A \in \mathcal{B}\}$ instead, we have for each n the partition Π_n^{ϑ} of $\mathbf{F}_{\vartheta}^{*}(X)$ generated by $\{\Xi_A^{\vartheta} = 0\}$ and $\{\Xi_A^{\vartheta} > 0\}$ for $A \in \rho_n$, and then we obtain the algebra

$$\mathbf{A}^{\vartheta} := \sigma\left(\Pi_1^{\vartheta}\right) \cup \sigma\left(\Pi_2^{\vartheta}\right) \cup \cdots \subset \mathcal{E}_{\vartheta}^{*}(X).$$

One can naturally identify $\{\Xi_A = 0\}$ with $\{\Xi_A^{\vartheta} = 0\}$ and $\{\Xi_A > 0\}$ with $\{\Xi_A^{\vartheta} > 0\}$ for each $A \in \rho_1 \cup \rho_2 \cup \cdots$, so we obtain a natural correspondence $E \mapsto E^{\vartheta}$ between the algebras $\mathbf{A}$ and $\mathbf{A}^{\vartheta}$. Note that in this correspondence $\sigma(\Pi_n)$ corresponds to $\sigma\left(\Pi_n^{\vartheta}\right)$ for each n, and that this correspondence respects complements and disjoint unions.

Having the algebras $\mathbf{A}$ and $\mathbf{A}^{\vartheta}$ in hands, we exploit the existence of μ_{ϑ}^{*} as a measure on $\mathcal{E}_{\vartheta}^{*}(X)$, and define a set-function μ_0^{*} on $\mathbf{A}$ by

$$\mu_0^{*}(E) := \mu_{\vartheta}^{*}\left(E^{\vartheta}\right), \quad E \in \mathbf{A}.$$

Then μ_0^{*} is a pre-measure on $\mathbf{A}$ simply because μ_{ϑ}^{*} is already a measure on $\mathcal{E}_{\vartheta}^{*}(X)$ which contains $\mathbf{A}^{\vartheta}$.

Finally, since $\mathbf{A}$ generates $\mathcal{E}_{\tau}^{*}(X)$, by the Hahn–Kolmogorov Extension Theorem μ_0^{*} extends to a genuine probability measure on $\mathbf{F}_{\tau}^{*}(X)$, and we denote this probability measure by μ_{τ}^{*}.

Part 3. Let us now show that

$$\mu_{\tau}^{*}(\Xi_A = 0) = e^{-\mu(A)} \text{ for every } A \in \mathcal{B}. \tag{5.5.2}$$

By the construction of μ_{τ}^{*}, (5.5.2) holds for all $A \in \mathbf{A}$. Given any $A' \in \mathcal{B}$, assume first that A' is contained in $O_1 \cup \cdots \cup O_{n'}$ for some n'. We can then replace the partitions $(\rho_n)_{n=1}^{\infty}$ in the above construction with the finer partitions $(\rho_n')_{n=1}^{\infty}$, where $\rho_n' = \rho_n$ for $n = 1, \ldots, n'-1$ and ρ_n' is the finest partition of $O_1 \cup \cdots \cup O_n$ generated by $A', O_1, \ldots, O_n$ for $n \geqslant n'$. Repeating the above construction, we obtain an algebra $\mathbf{A}'$ containing $\mathbf{A}$, and a probability measure μ_{τ}' on $\mathbf{F}_{\tau}^{*}(X)$ with respect to which (5.5.2) holds for every $A \in \mathbf{A}'$, and in particular for A'. However, by the uniqueness in the Hahn–Kolmogorov Theorem, since μ_{τ}' and μ_{τ}^{*} coincide on $\mathbf{A}$, necessarily $\mu_{\tau}' = \mu_{\tau}^{*}$. Thus, (5.5.2) holds every A' contained in some $O_1 \cup \cdots \cup O_n$. For a general $A' \in \mathcal{B}$, since $\Xi_{A' \cap (O_1 \cup \cdots \cup O_n)} \nearrow \Xi_{A'}$ pointwise and $\mu(A' \cap (O_1 \cup \cdots \cup O_n)) \nearrow \mu(A')$ as $n \nearrow \infty$, by the monotone convergence theorem $\mu_{\tau}^{*}(\Xi_{A'} = 0) = e^{-\mu(A')}$, concluding that (5.5.2) holds for all $A' \in \mathcal{B}$.

Part 4. We now show that $\{\Xi_A : A \in \mathcal{B}\}$ forms a Poisson point process on the base space $(X, \mathcal{B}, \mu)$ with respect to μ_{τ}^{*}. Let $A' \in \mathcal{B}$. Since μ is nonatomic, for every n we can pick a partition $\{A_i^n : 1 \leqslant i \leqslant n\}$ of A' such that $\mu(A_i^n) = \mu(A')/n$ for every $1 \leqslant i \leqslant n$. We do that in such a way that $\{A_i^n : 1 \leqslant i \leqslant n, n \in \mathbb{N}\}$ separates points in A', and hence

$$\Xi_{A'} = \lim_{n\to\infty} \sum\nolimits_{i=1}^{n} 1_{\{\Xi_{A_i^n}>0\}}. \tag{5.5.3}$$

For every n, by (5.5.2) one easily sees that the Bernoulli random variables $1_{\{\Xi_{A_i^n}>0\}}$, $1 \leqslant i \leqslant n$, are independent with probability $1 - e^{\mu(A)/n}$ of success with respect to μ_τ^*. Since $\lim_{n\to\infty} n \cdot \left(1 - e^{-\mu(A)/n}\right) = \mu(A)$, by the Poisson limit theorem and (5.5.3) we deduce that Ξ_A has distribution $\mathrm{Poiss}\,(\mu(A))$.

Finally, to see the two further properties in the theorem, let

$$\mathbf{E} := \bigcap\nolimits_{O\in\mathcal{O}} \{\Xi_O < \infty\} \subset \mathbf{F}_\tau^*(X).$$

Then $\mathbf{E}$ is μ_τ^*-conull since $\mu_\tau^*(\Xi_O < \infty) = 1$ for every $O \in \mathcal{O}$. Also, since by Proposition 5.5(3) we have $\Xi_O(\cdot) = \#(O \cap \cdot)$ for every $O \in \mathcal{O}$ while $\mathcal{O}$ is a base for τ, it follows that $\mathbf{E}$ consists of τ-discrete sets. □

5.3. Spatial Poissonian actions

In this section we prove Theorem 4. Let $(X, \mathcal{B})$ be a standard Borel space, and τ be a Polish topology on X that generates $\mathcal{B}$. For every τ-homeomorphism T of X we define

$$T^* : \mathbf{F}_\tau^*(X) \to \mathbf{F}_\tau^*(X), \quad T^*(F) := T(F) = \{T(x) \in X : x \in F\}.$$

Lemma 5.6. *Let $\mu \in \mathcal{M}_\sigma(X, \mathcal{B})$. If T is a τ-homeomorphism that preserves μ, then T^* is a transformation that preserves μ_τ^* (as in Theorem 5.3).*

Proof. For every $O \in \tau$ we have

$$\begin{aligned} \mu_\tau^*\left(T^{*-1}(B_O)\right) &= \mu^*\left(B_{T^{-1}(O)}\right) = \mu^*\left(\Xi_{T^{-1}(O)} > 0\right) = 1 - e^{-\mu\left(T^{-1}(O)\right)} \\ &= 1 - e^{-\mu(O)} = \mu^*(\Xi_O > 0) = \mu_\tau^*(B_O), \end{aligned}$$

thus $\mu_\tau^* \circ T^{*-1}$ and μ_τ^* coincide on B_O, $O \in \tau$. Since the values on B_O, $O \in \tau$, determine random closed sets uniquely (see [32, Theorem 1.3.20]) it follows that T^* preserves μ_τ^*. □

Note that $(T \circ S)^* = S^* \circ T^*$ whenever T, S are τ-homeomorphism, and in particular $(T^*)^{-1} = \left(T^{-1}\right)^*$. Obviously, $\mathbf{F}_\tau^*(X)$ is T^*-invariant for whatever τ-homeomorphism T of X. Then from a τ-Polish action $\mathbf{T} : G \curvearrowright (X, \mathcal{B})$ we obtain the action

$$\mathbf{T}^* : G \curvearrowright \left(\mathbf{F}_\tau^*(X), \mathcal{E}_\tau^*(X)\right), \quad \mathbf{T}^* : (g, F) \mapsto T_g^*(F). \tag{5.6.1}$$

Lemma 5.7. *If $\mathbf{T} : G \curvearrowright (X, \mathcal{B})$ is a τ-Polish action then $\mathbf{T}^* : G \curvearrowright (\mathbf{F}_\tau(X), \mathcal{E}_\tau(X))$, and in particular $\mathbf{T}^* : G \curvearrowright \left(\mathbf{F}_\tau^*(X), \mathcal{E}_\tau^*(X)\right)$ as in (5.6.1), is a Borel action.*

Remark 5.8. An alternative proof of this lemma can be found within the proof of [19, Theorem 6.10]. It is worth mentioning that when τ is locally compact (and only then), the *Fell topology* on $\mathbf{F}_{\tau}(X)$ is Polish and generates $\mathcal{E}_{\tau}(X)$ (see [20, Exercise (12.7)]). It can be verified using Fact 2.1 that in this case $\mathbf{T}^{*}$ is further a Polish action in the Fell topology.

Proof. Evidently $\mathbf{T}^{*}$ is an action. To see that it is Borel, fix a measurable selection $\{\theta_n : n \in \mathbb{N}\}$ for $\mathbf{F}_{\tau}(X) \setminus \{\emptyset\}$ by the aforementioned Kuratowski–Ryll-Nardzewski's Selection Theorem and, disregarding the empty set, decompose $\mathbf{T}^{*}$ into the maps

$$G \times \left(\mathbf{F}_{\tau}(X) \setminus \{\emptyset\}\right) \to G \times X^{\mathbb{N}} \to X^{\mathbb{N}} \to \mathbf{F}_{\tau}(X) \setminus \{\emptyset\},$$

the first is given by $(g, F) \mapsto \left(g, \left(\theta_n(F)\right)_{n\in\mathbb{N}}\right)$; the second is given by $\left(g, (x_n)_{n\in\mathbb{N}}\right) \mapsto \left(T_g(x_n)\right)_{n\in\mathbb{N}}$; and the third is given by $(x_n)_{n\in\mathbb{N}} \mapsto$ the τ-closure of $\{x_n : n \in \mathbb{N}\}$. It is elementary to verify that each of these maps is Borel, hence $\mathbf{T}^{*}$ is Borel. □

Proof of Theorem 4. Given a locally finite Polish action $\mathbf{T} : G \overset{\text{sp}}{\curvearrowright} (X, \mathcal{B}, \mu)$ with respect to a Polish topology τ, using the construction of Theorem 5.3, the construction of the action as in (5.6.1), together with Lemmas 5.6 and 5.7, we obtain the spatial action

$$\mathbf{T}^{*} : G \overset{\text{sp}}{\curvearrowright} \left(\mathbf{F}_{\tau}^{*}(X), \mathcal{E}_{\tau}^{*}(X), \mu_{\tau}^{*}\right).$$

We complete the proof by showing that this is a Poissonian action, whose base action is $\mathbf{T} : G \overset{\text{sp}}{\curvearrowright} (X, \mathcal{B}, \mu)$, with respect to the Poisson point process $\{\Xi_A : A \in \mathcal{B}\}$ defined on $\left(\mathbf{F}_{\tau}^{*}(X), \mathcal{E}_{\tau}^{*}(X), \mu_{\tau}^{*}\right)$ as in Theorem 5.3.

Recall that by Proposition 5.5, for every $A \in \mathcal{B}$, we have that $\Xi_A(F) = \#(A \cap F)$ whenever $F \in \mathbf{F}_{\tau}^{*}(X)$ is τ-discrete. Since μ_{τ}^{*} is supported on the class of τ-discrete sets, it follows that for F in a μ_{τ}^{*}-conull set,

$$\Xi_A\left(T^{*}(F)\right) = \#\left(A \cap T^{*}(F)\right) = \#\left(T^{-1}(A) \cap F\right) = \Xi_{T^{-1}(A)}(F).$$

This shows that $\mathbf{T}$ is a base action for $\mathbf{T}^{*}$, completing the proof. □

Remark 5.9. In defining Poisson point processes in Definition 1.1, we consider only base spaces with a nonatomic intensity μ. For a purely atomic intensity μ, with atoms $a_1, a_2, \ldots$, we define a Poisson point process as a collection of independent random variables $P_{a_1}, P_{a_2}, \ldots$, where each P_{a_i} is Poisson distributed with mean $\mu(a_i)$. For a general intensity μ, the Poisson point process is an independent splitting of the Poisson point processes on the nonatomic and atomic parts of μ. When a group G acts on the base space in a measure-preserving way, the decomposition of the space into nonatomic and atomic parts is G-invariant. Furthermore, if G acts spatially, the decomposition is pointwise G-invariant, and G's action on the atomic part is spatial.

6. Constructing spatial actions from nonsingular spatial actions

Here we aim to establish Theorem 5. Recall the Polish group

$$\mathrm{Aut}\left(X,\mathcal{B},[\mu]\right)$$

of the (equivalence classes of) nonsingular transformations of a standard measure space $(X,\mathcal{B},\mu)$. It is worth mentioning that similarly to $\mathrm{Aut}\left(X,\mathcal{B},\mu\right)$, also $\mathrm{Aut}\left(X,\mathcal{B},[\mu]\right)$ is a Lévy group [13, Theorem 6.1], [35, §4.5]. Let us start with the important construction of the Maharam Extension following [29], which allows one to realize the nonsingular transformations of one space as measure-preserving transformations of another space.

Proposition 6.1 *(Maharam Extension). For every standard (probability or infinite) measure space $(X,\mathcal{B},\mu)$ there exists a standard infinite measure space $\left(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu}\right)$, with a continuous embedding of Polish groups*

$$\mathrm{Aut}\left(X,\mathcal{B},[\mu]\right)\hookrightarrow\mathrm{Aut}\left(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu}\right),\quad T\mapsto\widetilde{T}.$$

Proof. For a standard measure space $(X,\mathcal{B},\mu)$ define $\left(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu}\right)$ by

$$\widetilde{X}:=X\times\mathbb{R}_{>0},\quad \widetilde{\mathcal{B}}:=\mathcal{B}\otimes\mathcal{B}\left(\mathbb{R}_{>0}\right),\quad d\widetilde{\mu}\left(x,t\right)=d\mu\left(x\right)dt,$$

where dt refers to the usual Lebesgue measure on $\mathbb{R}_{>0}$. Obviously, this is a standard infinite measure space. In order to define the desired embedding, let us denote the Radon–Nikodym cocycle by

$$\nabla:\mathrm{Aut}\left(X,\mathcal{B},[\mu]\right)\times X\to(0,\infty),\quad \nabla_T\left(\cdot\right)=\frac{d\mu\circ T^{-1}}{d\mu}\left(\cdot\right)\in L^0\left(X,\mathcal{B},\mu\right).$$

Note that this is only an *almost cocycle* in the sense that for every $T,S\in\mathrm{Aut}\left(X,\mathcal{B},[\mu]\right)$ it holds that

$$\nabla_{T\circ S}=\nabla_T\circ S\cdot\nabla_S\text{ in }L^0\left(X,\mathcal{B},\mu\right).$$

We now define the embedding $\mathrm{Aut}\left(X,\mathcal{B},[\mu]\right)\hookrightarrow\mathrm{Aut}\left(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu}\right)$ by

$$T\mapsto\widetilde{T},\quad \widetilde{T}\left(x,t\right)=\left(T\left(x\right),t/\nabla_T\left(x\right)\right).$$

It is a classical observation of Maharam [29] that $\widetilde{T}$ is measure-preserving. One can verify that this group embedding is continuous (for a description of the topology of $\mathrm{Aut}\left(X,\mathcal{B},[\mu]\right)$ see [1, pp. 13-14], [20, Exercise (17.46)]). □

Suppose G is a Polish group. A **nonsingular** (Boolean) **action** of G on a standard measure space $(X,\mathcal{B},\mu)$ is a continuous (equivalently, measurable) homomorphism $\mathbf{T}: G \to \mathrm{Aut}\,(X,\mathcal{B},[\mu])$. We denote such an action by

$$\mathbf{T}: G \curvearrowright (X,\mathcal{B},[\mu])\,.$$

A **nonsingular spatial action** of G on a standard measure space $(X,\mathcal{B},\mu)$ is a Borel action $\mathbf{T}: G \curvearrowright (X,\mathcal{B})$, $\mathbf{T}: (g,x) \mapsto T_g\,(x)$, such that for every $g \in G$, T_g is a nonsingular transformation of $(X,\mathcal{B},\mu)$. We denote such an action by

$$\mathbf{T}: G \overset{\mathrm{sp}}{\curvearrowright} (X,\mathcal{B},[\mu])\,.$$

Using the Maharam Extension construction, from every nonsingular action we obtain a (measure-preserving) action, however this does not work in the spatial category since the Radon–Nikodym cocycle is merely an almost cocycle. When G is locally compact, by the Mackey Cocycle Theorem the Radon–Nikodym cocycle admits a pointwise version (*strict version* in the terminology of [3]) and then the Maharam Extension does admit a point-realization. However, as it was shown by Becker [3, Section E], in general a pointwise version of the Radon–Nikodym cocycle may not exist.

In formulating Theorem 5, we consider the following property of (pointwise defined) cocycles:

Definition 6.2 *(Spatially continuous cocycle).* Let G be a Polish group and X be a Polish G-space. A cocycle $c: G\times X \to \mathbb{R}_{>0}$, $c: (g,x) \mapsto c_g\,(x)$, is said to be **spatially continuous** if the map $x \mapsto c_g\,(x)$ is continuous for each $g \in G$, and the map $g \mapsto c_g\,(x)$ is Borel for each $x \in X$.

The following general fact is a direct consequence of Fact 2.1.

Fact 6.1. Let G be a Polish group, and $\mathbf{T}$ a Polish action of G on a Polish space X with a spatially continuous cocycle c. Then the action of G on $X \times \mathbb{R}_{>0}$ given by $g: (x,t) \mapsto (T_g\,(x)\,, t/c_g\,(x))$ is a Polish action.

Proof of Theorem 5. If G admits a locally finite Polish action, then by Theorem 4 the spatial Poissonian action of this locally finite Polish action, with an appropriate Polish topology, is a probability-preserving spatial action. If G admits a locally finite Polish nonsingular action $\mathbf{T}: G \overset{\mathrm{sp}}{\curvearrowright} (X,\mathcal{B},[\mu])$, $\mathbf{T}: (g,x) \mapsto T_g\,(x)$, such that the Radon–Nikodym cocycle ∇ is pointwise defined and spatially continuous, by Fact 6.1 the Maharam Extension

$$\widetilde{\mathbf{T}}: G \overset{\mathrm{sp}}{\curvearrowright} \left(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu}\right),\quad \widetilde{\mathbf{T}}: (g,(x,t)) \mapsto (T_g\,(x)\,, t/\nabla_g\,(x))\,,$$

is a Polish action with respect to $\widetilde{\tau} := \tau \otimes \tau_{\mathbb{R}}$, where τ is the given Polish topology on X and $\tau_{\mathbb{R}}$ is the usual topology of $\mathbb{R}$. Moreover, since μ is τ-locally finite then $\widetilde{\mu}$ is $\widetilde{\tau}$-locally finite. Thus, the Maharam Extension $\mathbf{T}$ becomes a locally finite Polish action, and by Theorem 4 the spatial Poissonian action

$$\widetilde{\mathbf{T}}^* : G \overset{\text{sp}}{\curvearrowright} \left(\mathbf{F}^*_{\widetilde{\tau}}(\widetilde{X}), \mathcal{E}^*_{\widetilde{\tau}}(\widetilde{X}), \widetilde{\mu}^*_{\widetilde{\tau}}\right)$$

is a probability-preserving spatial action of G. □

Proof of Corollary 1.6. By [24, Theorem 2.3], if $\mathbf{S} : G \overset{\text{sp}}{\curvearrowright} (Y, \nu)$ is a faithful probability-preserving spatial action, the infinite diagonal action $\mathbf{S}^{\otimes\mathbb{N}} : G \overset{\text{sp}}{\curvearrowright} \left(Y^{\mathbb{N}}, \nu^{\otimes\mathbb{N}}\right)$ is essentially free in the sense that there is a $\nu^{\otimes\mathbb{N}}$-conull invariant subset of $Y^{\mathbb{N}}$ (where the invariance is pointwise) on which the action $\mathbf{S}^{\otimes\mathbb{N}}$ is free. Then the restriction of $\mathbf{S}^{\otimes\mathbb{N}}$ to this ν-conull set is a free probability-preserving spatial action. □

7. Diffeomorphism groups: classification and spatial actions

Let M be a compact smooth manifold. We will always assume that M is d-dimensional Hausdorff connected and without boundary. Let a parameter $1 \leqslant r \leqslant \infty$ be fixed, and denote by

$$\mathrm{Diff}^r(M)$$

the group of all C^r-diffeomorphisms from M onto itself. It becomes a Polish group with the compact-open C^r-topology. Denote the connected component of the identity, as a normal subgroup, by

$$\mathrm{Diff}^r_o(M) \lhd \mathrm{Diff}^r(M).$$

Recall that in any locally connected group, the connected component of the identity is clopen. As Diffeomorphism groups are locally connected (see [2, Proposition 1.2.1]), $\mathrm{Diff}^r_o(M)$ is a clopen subgroup, hence a non-locally compact Polish group in the subspace topology.

Observe that the local-connectedness of $\mathrm{Diff}^r(M)$ implies that it is not a non-Archimedean group. Indeed, non-Archimedean groups, admitting a base of clopen subgroups, are totally disconnected. Here we aim to show in the case of $r \neq d+1$ the stronger statement of Theorem 6. To this end we exploit the following celebrated result that was established by Herman in the case when M is a torus and $r = \infty$; by Thurston for every M and $r = \infty$; and by Mather for every M and $1 \leqslant r < \infty$, $r \neq d+1$. See [2, §2] and the references therein.

Theorem 7.1 *(Herman, Thurston, Mather). For all manifold M as above, $\mathrm{Diff}^r_o(M)$ is a simple group whenever $r \neq d+1$.*

The second tool we need is the following characterization of groups of isometries of a locally compact metric space [25, Theorem 1.2].

Theorem 7.2 *(Kwiatkowska & Solecki). A Polish group G is a group of isometries of a locally compact metric space if and only if it possesses the following property:*

> *Every identity neighborhood contains a closed subgroup H, such that G/H is a locally compact space and the normalizer*
>
> $$N_G(H) := \left\{g \in G : gHg^{-1} = H\right\} \text{ is clopen.}$$

We can now prove Theorem 6.

Proof of Theorem 6. Let $1 \leqslant r \leqslant \infty$, $r \neq d+1$, be fixed. Since $\mathrm{Diff}_o^r(M)$ is a clopen subgroup of $\mathrm{Diff}^r(M)$, it is sufficient to show that $\mathrm{Diff}_o^r(M)$ is not a group of isometries of some locally compact metric space. Suppose otherwise towards a contradiction. Then by Theorem 7.2 there exists a closed subgroup $H \leqslant \mathrm{Diff}_o^r(M)$ for which $\mathrm{Diff}_o^r(M)/H$ is a locally compact space and the normalizer $N_{\mathrm{Diff}_o^r(M)}(H)$ is clopen. Since every identity neighborhood of $\mathrm{Diff}_o^r(M)$ contains such H, we may assume $H \neq \mathrm{Diff}_o^r(M)$. Since $\mathrm{Diff}_o^r(M)/H$ is a locally compact space while $\mathrm{Diff}_o^r(M)$ is not, it follows that H cannot be the trivial group. Since $N_{\mathrm{Diff}_o^r(M)}(H)$ is clopen while $\mathrm{Diff}_o^r(M)$ is connected, it follows that $N_{\mathrm{Diff}_o^r(M)}(H) = \mathrm{Diff}_o^r(M)$, namely H is a normal subgroup of $\mathrm{Diff}_o^r(M)$. This contradicts Theorem 7.1. $\square$

We move now to construct spatial actions of diffeomorphism groups.

Proof of Theorem 7. Let M be a compact smooth manifold and pick a volume form Vol on M. Look at the tautological action of the group $\mathrm{Diff}^r(M)$ on M which is obviously nonsingular with respect to Vol. Thus, we have a nonsingular locally finite Polish action

$$\mathbf{D} : \mathrm{Diff}^r(M) \overset{\mathrm{sp}}{\curvearrowright} (M, \mathrm{Vol}), \quad f.x = f(x).$$

The Radon–Nikodym cocycle of this action is the corresponding Jacobian, which is jointly continuous as a map $\mathrm{Diff}^r(M) \times M \to (0, \infty)$, hence by Theorem 5 we conclude that $\mathrm{Diff}^r(M)$ admits a nontrivial probability-preserving spatial action. Using Corollary 1.6 we immediately obtain that there exists also such a free action.

In order to obtain also weak mixing, we look back at the steps in the construction as in the proof of Theorem 5: starting with the locally finite Polish nonsingular action $\mathbf{D}$, we constructed the Maharam Extension $\widetilde{\mathbf{D}}$, from which we constructed the spatial Poissonian action $\widetilde{\mathbf{D}}^*$ (with respect to the obvious Polish topology on $\widetilde{M}$). Now using [24, Theorem 2.3], the infinite diagonal product action $\left(\widetilde{\mathbf{D}}^*\right)^{\otimes \mathbb{N}}$ is essentially free in the sense that there is a conull invariant subset of (where the invariance is pointwise) on which the action is free. Thus, the restriction of $\left(\widetilde{\mathbf{D}}^*\right)^{\otimes \mathbb{N}}$ to this conull set forms a free probability-preserving spatial action. We then argue that $\left(\widetilde{\mathbf{D}}^*\right)^{\otimes \mathbb{N}}$ is weakly mixing. To this end we exploit the fact that the Maharam Extension $\widetilde{\mathbf{D}}$ is ergodic, which will be proved in

Appendix B, and then, as every infinite measure-preserving action, the ergodicity of $\widetilde{\mathbf{D}}$ implies that it is null. From Theorem 2 it follows that the Poisson suspension $\widetilde{\mathbf{D}}^*$ is weakly mixing. Recalling Remark 4.9(3), this implies that $\left(\widetilde{\mathbf{D}}^*\right)^{\otimes\mathbb{N}}$ is weakly mixing. This completes the proof. □

7.1. Non-essentially countable orbit equivalence relations

We introduce the necessary background to Corollary 1.7. A general reference to the subject, with many references therein, is [23]. Here we follow the terminology of [24].

An equivalence relation E on a standard Borel space $(X,\mathcal{B})$ is a set $E\subseteq X\times X$ such that the condition $x\sim x' \iff (x,x')\in E$ defines an equivalence relation on X. Such an equivalence relation is said to be **Borel** if it is a Borel subset of $X\times X$, and **countable** if each of its equivalence classes is countable. The Borel complexity of one equivalence relation relative to another can be tested by the possibility of producing a Borel reduction of the former into the latter in the following sense. A **Borel reduction** of an equivalence relation E on a standard Borel space $(X,\mathcal{B})$ into an equivalence relation F on a standard Borel space $(Y,\mathcal{C})$, is a Borel function $f:X\to Y$ such that

$$(x,x')\in E \iff (f(x),f(x'))\in F \text{ for all } x,x'\in X.$$

An equivalence relation is said to be **essentially countable** if it admits a Borel reduction into a countable Borel equivalence relation.

An important class of equivalence relations are **orbit equivalence relations**. If $\mathbf{T}:G\curvearrowright(X,\mathcal{B})$ is a Borel action of a Polish group G on a standard Borel space $(X,\mathcal{B})$, the associated orbit equivalence relation is

$$E_{\mathbf{T}}^{X}=\{(x,T_g(x)):x\in X,g\in G\}\subseteq X\times X.$$

It was shown by Kechris (see [22, Theorem 4.10]) that if G is locally compact then $E_{\mathbf{T}}^{X}$ is always essentially countable, and he left as open question whether this property characterizes locally compact groups among the Polish groups (see [24, Problem 1.2], [23, Problem 4.16] and the references therein).

Proof of Corollary 1.7. By Theorem 7, every diffeomorphism group admits a free spatial action on a standard probability space. If the orbit equivalence relation of this action would be essentially countable, then by a theorem of Feldman & Ramsay [10, Theorem A] (cf. [24, Theorem 2.1]) it would follow that the acting group is locally compact, which is false, hence this orbit equivalence relation is non-essentially countable. □

8. Open problems

We revisit Glasner, Tsirelson & Weiss' [14, Question 1.2] [35, Open Problem 7.1.19] of whether Lévy groups admit nonsingular spatial action. In their proof of the non-existence of probability-preserving actions of Lévy groups, it was sufficient to show the non-existence of such *Polish* actions. This reduction was possible thanks to a theorem of Becker & Kechris [4, §5.2], by which every Borel action of a Polish group admits a Polish topology with respect to which it becomes a Polish action. In locally finite Polish actions, by definition, there exists such a topology that, simultaneously, makes the action Polish and the measure locally finite. It is then natural to ask about the following refinements of Becker & Kechris' theorem:

Question 8.1. Let $\mathbf{T} : G \curvearrowright (X, \mathcal{B})$ be a Borel action of a Polish group G, preserving an infinite measure μ. When does there exist a Polish topology on X with respect to which, simultaneously, $\mathbf{T}$ is Polish and μ is locally finite?

Diffeomorphism groups were shown, in Theorems 6 and 7, to belong neither to the class of groups of isometries of a locally compact metric space nor the class of Lévy groups. Thus it is natural to ask:

Question 8.2. Do diffeomorphism groups possess the Mackey property?

The following related question was posed by Moore & Solecki [33, §4]. For a compact smooth manifold M, let $C^{\infty}\left(M, S^1\right)$ be the group of C^{∞}-functions from M to the unit circle S^1 with pointwise multiplication and the compact-open C^{∞}-topology. It was shown by Moore & Solecki [33, Theorem 1.1] that the Mackey property fails for continuous functions from M to S^1, and they ask about the Mackey property for $C^{\infty}\left(M, S^1\right)$, describing it as 'tempting to conjecture'.

The Mackey property of a Polish group refers to all of its actions at once. However, it is possible for a Polish group without the Mackey property to admit spatial actions. Indeed, the homeomorphism group of the circle S^1 does not possess the Mackey property by the aforementioned result of Moore & Solecki, and yet it acts spatially and ergodically on S^1 with its Lebesgue measure by $f.z = f(1)\, z$. This suggests to deviate from the general Mackey property and raise the following problem that seems to be widely open:

Problem 8.1. Consider the following types of actions a Polish group may possess:

- **PMP:** Spatial ergodic probability-preserving actions.
- **IMP:** Spatial ergodic infinite measure-preserving actions.
- **NS:** Spatial ergodic nonsingular actions without an absolutely continuous invariant measure.

Beyond the locally compact case, little is known about whether a given Polish group admits PMP, IMP or NS actions. A particular important case is Glasner, Tsirelson & Weiss' question: Lévy groups do not admit PMP actions, but do they admit IMP actions? What about NS actions?

There are also classification aspects of this problem: does the admission of one type of an action imply the admission of another type? On the contrary, is there a Polish group that admits an action of a certain type but not an action of some other type?

Remark 8.1. We comment on known results in the above problem.

(1) Locally compact Polish groups admit IMP actions (there is a Haar measure) and PMP (the Poissonian action based on the group with the Haar measure, which is spatial by the Mackey property). It is in fact true for all groups of isometries of locally compact metric spaces, since they also admit an invariant Radon measure (see [12, §44, Theorem 441H]). With regard to NS actions in locally compact Polish groups, see [8].

(2) Diffeomorphism groups of compact smooth manifolds admit NS actions (tautologically), IMP actions (the Maharam Extension), and PMP actions (the spatial Poissonian action as in Theorem 7).

(3) The group $\mathrm{Homeo}_{+}([0,1])$ of orientation preserving homeomorphisms admits no nontrivial Boolean action whatsoever by a theorem of Megrelishvili [31, Theorem 3.1] (see also [14, Remark 1.7]).

(4) The Maharam Extension construction demonstrates that in certain cases, having a NS action implies the admission of a IMP action. The spatial Poissonian action construction demonstrates that in certain cases, having a IMP action implies the admission of a PMP action. By Theorem 5, these implications are valid in actions with appropriate continuity properties.

Appendix A. The first chaos of Poisson point processes

Fix a generative Poisson point process $\mathcal{P}=\{P_A : A\in\mathcal{B}\}$ defined on $(\Omega,\mathcal{F},\mathbb{P})$ with base space $(X,\mathcal{B},\mu)$ as in Definition 1.1. In the following we describe some fundamental properties of the **first chaos** of $\mathcal{P}$, namely the real Hilbert space

$$H_1(\mathcal{P}) := \overline{\operatorname{span}}\left\{P_A-\mu(A) : A\in\mathcal{B}_\mu\right\}\subset L^2_{\mathbb{R}}(\Omega,\mathcal{F},\mathbb{P}).$$

A.0.1. Fock space and chaos decomposition

The (real, symmetric) **Fock space** associated with a Hilbert space $\mathcal{H}$ is, by definition, the Hilbert space

$$\mathrm{F}(\mathcal{H})=\bigoplus\nolimits_{n=0}^{\infty}\mathcal{H}^{\odot n},$$

where $\mathcal{H}^{\odot 0}=\mathbb{R}$ and $\mathcal{H}^{\odot n}$ for $n\geqslant 1$ is the symmetric n^{th} tensor product of $\mathcal{H}$, i.e. its elements are the vectors in the usual tensor product $\mathcal{H}^{\otimes n}$ that are invariant to permutations of their coordinates, which are generated by elements of the form

$$u_1 \odot \cdots \odot u_n := \frac{1}{n!} \sum\nolimits_{\sigma \in \mathrm{Sym}(n)} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)},$$

and with the inner product given by

$$\langle u_1 \odot \cdots \odot u_n, v_1 \odot \cdots \odot v_n \rangle_{\mathcal{H}^{\odot n}} := \frac{1}{n!} \sum\nolimits_{\sigma \in \mathrm{Sym}(n)} \prod\nolimits_{j=1}^{n} \left\langle u_j, v_{\sigma(j)} \right\rangle_{\mathcal{H}}.$$

Here the operation $\bigoplus$ denotes the operation of taking the Hilbert space obtained as the completion of the direct sum.

For the classical Poisson point process $\mathcal{N}$ that is defined on $(X^*, \mathcal{B}^*, \mu^*)$ with the base space $(X, \mathcal{B}, \mu)$, the Fock space decomposition refers to the isometric isomorphism between $L^2_{\mathbb{R}}(X^*, \mathcal{B}^*, \mu^*)$ and $\mathrm{F}\left(L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)\right)$. See [28] (see also [26, §18]). For the given $\mathcal{P}$ as in Definition 1.1, using Proposition 3.3 we obtain an isometric isomorphism between $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathrm{F}\left(L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)\right)$ that we will present briefly.

The Hilbert space $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$ has the **chaos decomposition** into

$$L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus\nolimits_{n=0}^{\infty} H_n(\mathcal{P}),$$

each $H_n(\mathcal{P})$ is called the n^{th} *chaos* with respect to $\mathcal{P}$. In the following description of this chaos structure, we use stochastic integration against $\mathcal{P}$ as a random measure, which is justified in Proposition 3.3.

- $H_0(\mathcal{P}) := \mathbb{R}$ viewed as the subspace of constant functions in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.
- $H_1(\mathcal{P}) := \overline{\mathrm{span}}\{P_A - \mu(A) : A \in \mathcal{B}_\mu\} \subset L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$.
- $\cdots$
- $H_n(\mathcal{P}) = \overline{\mathrm{span}}\left\{\int_{X^{\odot n}} 1_A^{\odot n} d(\mathcal{P} - \mu)^{\odot n} : A \in \mathcal{B}_\mu\right\} \subset L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$, where $X^{\odot n}$ is the set of sequences in X^n consisting of n distinct elements.

With this chaos decomposition of $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$, the isometric isomorphism

$$I_\mu : \mathrm{F}\left(L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)\right) \to L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P}),$$

is given by stochastic integration against $\mathcal{P}$ as follows.

- $I_\mu : \mathbb{R} \to H_0(\mathcal{P})$ is defined in the obvious way.
- $I_\mu : L^2_{\mathbb{R}}(X, \mathcal{B}, \mu) \to H_1(\mathcal{P})$ is given by the stochastic integral

$$I_\mu : 1_A \mapsto \int\limits_X 1_A d(\mathcal{P} - \mu) = P_A - \mu(A), \quad A \in \mathcal{B}_\mu.$$

- $\cdots$

- $I_\mu : L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)^{\odot n} \to H_n(\mathcal{P})$ is given by the stochastic integral

$$I_\mu : 1_A^{\odot n} \mapsto \int_{X^{\odot n}} 1_A^{\odot n} d(\mathcal{P} - \mu)^{\odot n}, \quad A \in \mathcal{B}_\mu.$$

In general, every unitary operator U of a Hilbert space $\mathcal{H}$ induces a unitary operator $\mathrm{F}(U)$ of the Fock space $\mathrm{F}(\mathcal{H})$, that is defined by letting $\mathrm{F}(U)$ acts on $\mathcal{H}^{\odot n}$ as the n^{th} tensor product $U^{\otimes n}$. Thus, for every $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$, by looking at the Koopman operator U_T we obtain the operator $\mathrm{F}(U_T)$ of $\mathrm{F}\left(L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)\right) \cong L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$. This operator can also be described without reference to the Fock space structure: if $S \in \mathrm{Aut}(\Omega, \mathcal{F}, \mathbb{P})$ is a Poissonian transformation with base transformation T, thus $S = T^*$ as in Proposition 3.1, we have the Koopman operator U_S of $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$. The equivariance property that relates T and S as in Definition 1.4 implies that

$$\mathrm{F}(U_T) = U_S.$$

A.0.2. Infinitely divisible distributions and the first chaos

We start by recalling some of the basics of infinitely divisible distributions. A general reference to the subject is [42].

A distribution is **infinitely divisible** if for every positive integer n it can be presented as the n^{th}-power convolution of some other distribution. By the fundamental Lévy–Khintchine Representation (see [42, Chapter 2, §8]), every infinitely divisible distribution is completely determined by a triplet

$$(\sigma, \ell, b), \text{ where } \sigma \geqslant 0,\ b \in \mathbb{R} \text{ and } \ell \text{ is a } \textbf{Lévy measure}.$$

By definition, a Lévy measure is a σ-finite Borel (possibly with atoms) measure ℓ on $\mathbb{R}$ with

$$\ell(\{0\}) = 0 \text{ and } \int_{\mathbb{R}} t^2 \wedge 1 d\ell(t) < \infty.$$

According to the Lévy–Khintchine Representation, a random variable W has infinitely divisible distribution associated with a triplet (σ, ℓ, b) if its characteristic function takes the form

$$\mathbb{E}(\exp(itW)) = \exp\big(- t^2\sigma^2/2 + \int_{\mathbb{R}} \left(e^{itx} - 1 - itx \cdot 1_{[-1,+1]}(x)\right) d\ell(x) + itb\big).$$

Thus, every infinitely divisible distribution is the convolution of a centered Gaussian distribution with variance σ^2, and another infinitely divisible distribution that is determined by a Lévy measure ℓ and a constant b via the Lévy–Khintchine Representation.

An infinitely divisible distribution for which the Gaussian part vanishes, namely $\sigma = 0$, is referred to as IDp distribution. A random variable is IDp if its distribution is IDp.

Given a generative Poisson point process $\mathcal{P}$, the restriction of the stochastic integral in the aforementioned chaos decomposition to the first chaos $H_1(\mathcal{P})$, forms an isometric isomorphism of the Hilbert spaces

$$I_\mu : L^2_{\mathbb{R}}(X, \mathcal{B}, \mu) \xrightarrow{\sim} H_1(\mathcal{P}).$$

We introduce some useful properties of this stochastic integral.

Proposition A.1. *In the above setting the following hold.*

(1) For every $f \in L^1_{\mathbb{R}}(X, \mathcal{B}, \mu) \cap L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$,

$$I_\mu(f) = \int_X f d\mathcal{P} - \int_X f d\mu.$$

(2) For every $f \in L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$, $I_\mu(f)$ is an IDp random variable whose Lévy measure is given by

$$\ell_f := \mu\,|_{\{f \neq 0\}} \circ f^{-1}.$$

(3) For every $f \in L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$ for which $I_\mu(f)$ is bounded from below,

$$\ell_f(\mathbb{R}_{<0}) = 0 \text{ and } \int_{\mathbb{R}_{\geqslant 0}} t d\ell_f(t) < \infty.$$

Proof. The stochastic integral I_μ is generally defined on the dense subspace $L^1_{\mathbb{R}}(X, \mathcal{B}, \mu) \cap L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$ as in part (2), and extends to $L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$ by continuity. It is a classical fact that if $f \in L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$ then $I_\mu(f)$ is an IDp random variable and its Lévy measure is ℓ_f as in part (1) (see [42, Lemma 20.6], [38, Proposition 2.10] and note that on the dense subspace $L^1_{\mathbb{R}}(X, \mathcal{B}, \mu) \cap L^2_{\mathbb{R}}(X, \mathcal{B}, \mu)$ the stochastic integral I_μ differs from the stochastic integrals in these references only by a constant, whence they have the same Lévy measure). This establishes parts (1) and (2).

In order to establish part (3) we exploit the general characterization of IDp random variables that are bounded from below as in [42, Theorem 24.7], by which if $I_\mu(f)$ is bounded from below then its Lévy measure ℓ_f satisfies the following two properties. First, $\ell_f(\mathbb{R}_{<0}) = 0$ as in the first property in part (3). Second, one of the following alternatives occurs (*type A* or *type B* in the terminology of [42, Definition 11.9]):

(1) $\ell\left(\mathbb{R}_{\geqslant 0}\right)<\infty$, in which case by the Cauchy-Schwartz inequality

$$\int_{\mathbb{R}_{\geqslant 0}} t d\ell_f(t)=\int_{\{f>0\}} f d\mu \leqslant \ell_f\left(\mathbb{R}_{\geqslant 0}\right) \int_{\{f>0\}} f^2 d\mu<\infty .$$

(2) $\ell_f\left(\mathbb{R}_{\geqslant 0}\right)=\infty$ and $\int_{\{0\leqslant t\leqslant 1\}} t d\ell_f(t)<\infty$, in which case we have

$$\int_{\mathbb{R}_{\geqslant 0}} t d\ell_f(t)=\int_{\{0\leqslant t\leqslant 1\}} t d\ell_f(t)+\int_{\{t>1\}} t d\ell_f(t)<\infty,$$

where the finiteness of the second term follows from

$$\int_{\{t>1\}} t d\ell_f(t)=\int_{\{f>1\}} f d\mu \leqslant \int_{\{f>1\}} f^2 d\mu<\infty .$$

This completes the proof of part (3). □

Appendix B. Ergodicity of Maharam Extensions

A useful criterion for the ergodicity of the Maharam Extension is based on Krieger's theory of orbit equivalence classification of nonsingular transformations (i.e. nonsingular actions of $\mathbb{Z}$). While this theory, by its nature, applies to countable amenable groups, by reviewing its details one may derive a general criterion to the ergodicity of Maharam Extensions for actions of general groups. In the following discussion, an action of a general group G is a homomorphism from G into $\operatorname{Aut}(X,\mathcal{B},[\mu])$, and the measurability or continuity of this homomorphism are irrelevant.

Let G be a group. Suppose we are given a nonsingular action of G on a standard measure space $(X,\mathcal{B},\mu)$, that is a homomorphism

$$\mathbf{T}: G \rightarrow \operatorname{Aut}(X,\mathcal{B},[\mu]), \quad \mathbf{T}:(g,x) \mapsto T_g(x) .$$

Denote its Radon–Nikodym cocycle by

$$\nabla_g(\cdot)=\frac{d\mu \circ T_g}{d\mu}(\cdot) \in L^1(X,\mathcal{B},\mu), \quad g \in G .$$

Recall the Maharam Extension construction as in Proposition 6.1, of the standard infinite measure space $\left(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu}\right)$ given by

$$\widetilde{X}:=X \times \mathbb{R}_{>0}, \quad \widetilde{\mathcal{B}}:=\mathcal{B} \otimes \mathcal{B}\left(\mathbb{R}_{>0}\right), \quad d\widetilde{\mu}(x,t)=d\mu(x)\, dt,$$

so from $\mathbf{T}$ we obtain the infinite measure-preserving action

$$\widetilde{\mathbf{T}}:G\to\mathrm{Aut}\big(\widetilde{X},\widetilde{\mathcal{B}},\widetilde{\mu}\big),\quad \widetilde{T}_g:(x,t)\mapsto\left(T_g\left(x\right),t/\nabla_g\left(x\right)\right).$$

A positive number $s\in\mathbb{R}_{>0}$ is said to be an **essential value** of $\mathbf{T}$ if:

$$\begin{aligned}&\text{For every } A\in\mathcal{B} \text{ with } \mu(A)>0 \text{ and every } \epsilon>0,\\ &\text{there are } A\supseteq B\in\mathcal{B} \text{ with } \mu(B)>0 \text{ and } g\in G,\\ &\text{such that } T_g(B)\subseteq A \text{ and } \nabla_g(B)\subseteq(s-\epsilon,s+\epsilon).\end{aligned}\tag{B.0.1}$$

The set of all essential values of $\mathbf{T}$ is called the **ratio set** and is denoted

$$\mathrm{r}\left(\mathbf{T},\mu\right).$$

The ratio set is a closed multiplicative subgroup of $\mathbb{R}_{>0}$, depending only on the measure class of μ. As such, it serves as a principal invariant of nonsingular actions [43, §3].

Proposition B.1 *(Schmidt). Suppose* $\mathbf{T}$ *is an ergodic nonsingular action. If* $\mathrm{r}\left(\mathbf{T}\right)=\mathbb{R}_{>0}$ *then the Maharam Extension* $\widetilde{\mathbf{T}}$ *is ergodic.*

Proof. Let $A\in\widetilde{\mathcal{B}}$ be a $\widetilde{\mathbf{T}}$-invariant set of $\widetilde{\mu}$-positive measure. Denote by $\left(S_s\right)_{s\in\mathbb{R}}$ the flow on $\widetilde{X}$ given by $S_s\left(x,t\right)=\left(x,te^s\right)$. Since $\mathrm{r}\left(\mathbf{T}\right)=\mathbb{R}_{>0}$, by [43, Theorem 5.2][6] we have

$$\widetilde{\mu}\left(A\triangle S_s\left(A\right)\right)=0 \text{ for every } s\in\mathbb{R},$$

so that A is $\left(S_s\right)_{s\in\mathbb{R}}$-invariant. By the Fubini Theorem there exists $A'\in\mathcal{B}$ such that $\widetilde{\mu}\left(A\triangle\left(A'\times\mathbb{R}_{>0}\right)\right)=0$. Since A is $\widetilde{\mathbf{T}}$-invariant, A' is $\mathbf{T}$-invariant, and from the ergodicity of $\mathbf{T}$ it follows that $\mu\left(A'\triangle X\right)=0$. Thus, $\widetilde{\mu}\big(A\triangle\widetilde{X}\big)=0$, completing the proof. □

Proposition B.2. *Let* M *be a compact smooth manifold with a volume form* Vol*, let* $1\leqslant r\leqslant\infty$ *and* $\mathbf{D}:\mathrm{Diff}^r\left(M\right)\to\mathrm{Aut}\left(M,[\mathrm{Vol}]\right)$ *the tautological nonsingular action. Then the Maharam Extension* $\widetilde{\mathbf{D}}:\mathrm{Diff}^r\left(M\right)\to\mathrm{Aut}\big(\widetilde{M},\widetilde{\mathrm{Vol}}\big)$ *is ergodic.*

Remark B.3. While this proposition requires a proof which turns out to be somehow technical, it should be regarded as easy. In fact, in many common cases much more is known: when M is compact and the dimension is either $d=1$ (Katznelson) or $d\geqslant 3$ (Herman), there exists a single diffeomorphism whose Maharam Extension is ergodic. See [18, §9] and the references therein.

Proof. It is well known that $\mathbf{D}$ is ergodic. In fact, there always exists a single diffeomorphism in $\mathrm{Diff}^{\infty}(M)$ which acts ergodically (see [9, Théorème 2.2]). Then by Proposition B.1 it suffices to show that $\mathrm{r}\left(\mathbf{D},\mathrm{Vol}\right)=\mathbb{R}_{>0}$.

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[6] Schmidt's theorem is formulated when G is countable and with the Haar measure of the multiplicative group $\mathbb{R}_{>0}$. Nevertheless, the proof of the part being used here is valid for every group. Also, since the usual Lebesgue measure on $\mathbb{R}_{>0}$ is mutually absolutely continuous with the Haar measure of $\mathbb{R}_{>0}$, the choice between those measures does not affect the ergodicity of the Maharam extension.

Step 1: $\mathrm{r}\left(\mathrm{Diff}^r(\Omega),\lambda\right)=\mathbb{R}_{>0}$ for every bounded hyperrectangle $\Omega\subset\mathbb{R}^d$ and λ being the Lebesgue measure on Ω. In this discussion, the boundary of the hyperrectangle is irrelevant, so it may be open, closed, or neither.

To see this, we will use a well-known criterion for essential values, which allows one to check a slightly stronger property than (B.0.1), but only for A that comes from a generating algebra (see e.g. [8, Fact 2.7]). Consider the algebra $\mathcal{R}$ that is generated by the hyperrectangles contained in Ω. Since Ω is itself a hyperrectangle, $\mathcal{R}$ consists of all finite unions of hyperrectangles within Ω. Therefore, to verify that $s\in\mathrm{r}\left(\mathrm{Diff}^r(\Omega),\lambda\right)$, it is sufficient to check the following condition:

For every hyperrectangle $R\subseteq\Omega$ and $\epsilon>0$, there are $\delta>0$, a Borel set $B\subseteq R$ and $f\in\mathrm{Diff}^r(\Omega)$, such that

$$\lambda(B)\geqslant\delta\cdot\lambda(R),\ f(B)\subseteq R,\text{ and }\tfrac{d\lambda\circ f}{d\lambda}(B)\subseteq(s-\epsilon,s+\epsilon).$$

Since the ratio set is in general a closed subgroup of $\mathbb{R}_{>0}$, it is sufficient to assume that $s>1$. Now given a hyperrectangle $R=R_p(u_0)$ centered at $u_0\in\Omega$ with side lengths $p=(p_1,\dots,p_d)$, then irrespectively of ϵ, consider the open hyperrectangle $B=R_{p/s}(u_0)$, where $p/s=(p_1/s,\dots,p_d/s)$, and let $f\in\mathrm{Diff}^r(\Omega)$ be any diffeomorphism such that

$$f(u)=s(u-u_0)+u_0\text{ for every }u\in B.$$

It is then evident that $\lambda(B)=s^{-d}\cdot\lambda(R)$, $f(B)\subset R$, and $\frac{d\lambda\circ f}{d\lambda}\mid_B\equiv s$. This establishes the aforementioned condition, hence $s\in\mathrm{r}\left(\mathrm{Diff}^r(\Omega),\lambda\right)$.

Step 2: $\mathrm{r}\left(\mathrm{Diff}^r(\Omega),\lambda\right)=\mathbb{R}_{>0}$ for every open domain $\Omega\subset\mathbb{R}^d$ and λ being the Lebesgue measure on Ω.

To deduce this from the previous case, we use that Ω is a countable union of (open) bounded hyperrectangles, hence for every Borel set $A\subset\Omega$ with $\lambda(A)>0$ there can be found an open bounded hyperrectangle $\Omega'\subset\Omega$ such that $\lambda(A\cap\Omega')>0$. Then for an arbitrary $s\in\mathbb{R}_{>0}$, by the previous case we have $s\in\mathrm{r}\left(\mathrm{Diff}^r(\Omega'),\lambda'\right)$, where $\lambda'=\lambda\mid_{\Omega'}$. By the definition (B.0.1), this means that for every $\epsilon>0$ there are $B\subset A\cap\Omega'$ and $f'\in\mathrm{Diff}^r(\Omega')$ such that $\lambda'(B)>0$, $f'(B)\subset A\cap\Omega'\subset A$, and $\frac{d\lambda'\circ f'}{d\lambda'}(B)\subseteq(s-\epsilon,s+\epsilon)$. Since $\lambda(B)>0$ and since one can always find $f\in\mathrm{Diff}^r(\Omega)$ with $f\mid_{\Omega'}=f'$, this readily implies that s satisfies (B.0.1) for A and ϵ with the same B and any $f\in\mathrm{Diff}^r(\Omega)$ extending f'. Thus, $s\in\mathrm{r}\left(\mathrm{Diff}^r(\Omega),\lambda\right)$.

Step 3: $\mathrm{r}\left(\mathrm{Diff}^r(U),\mathrm{Vol}\right)=\mathbb{R}_{>0}$ for every local chart $\varphi:U\to\varphi(U)\subseteq\mathbb{R}^d$ on M and Vol being the volume form of M restricted to U.

This fact can be seen through the equalities

$$\mathrm{r}\left(\mathrm{Diff}^r(U),\mathrm{Vol}\right)=\mathrm{r}\left(\mathrm{Diff}^r(\varphi(U)),\varphi_*\mathrm{Vol}\right)=\mathrm{r}\left(\mathrm{Diff}^r(\varphi(U)),\lambda\right)=\mathbb{R}_{>0};$$

the first equality can be verified routinely using that φ is a measure-preserving diffeomorphism from (U, Vol) to $(\varphi(U), \varphi_*\mathrm{Vol})$, the second equality follows from that $\varphi_*\mathrm{Vol}$ and λ, the Lebesgue measure on $\varphi(U)$, are mutually absolutely continuous while the ratio set generally depends only on the measure class, and the third equality follows from Step 2.

Final step: $\mathrm{r}(\mathrm{Diff}^r(M), \mathrm{Vol}) = \mathbb{R}_{>0}$ for all M as in the proposition.

Fix a countable atlas of local charts $\varphi_i : U_i \to \varphi_i(U_i) \subseteq \mathbb{R}^d$ on M. Following the argument in Step 2, one can deduce that $\mathrm{r}(\mathrm{Diff}^r(M), \mathrm{Vol}) = \mathbb{R}_{>0}$ using the following: (i) for every Borel set $A \subset M$ with $\mathrm{Vol}(A) > 0$ there exists i such that $\mathrm{Vol}(A \cap U_i) > 0$, (ii) $\mathrm{r}(\mathrm{Diff}^r(U_i), \mathrm{Vol}) = \mathbb{R}_{>0}$ for every i by Step 3, and (iii) every $f \in \mathrm{Diff}^r(U_i)$ can be extended to $\mathrm{Diff}^r(M)$. $\square$

References

[1] J. Aaronson, An Introduction to Infinite Ergodic Theory, No. 50, American Mathematical Soc., 1997.

[2] A. Banyaga, The Structure of Classical Diffeomorphism Groups, vol. 400, Springer Science & Business Media, 1997.

[3] H. Becker, Cocycles and continuity, Trans. Am. Math. Soc. 365 (2) (2013) 671–719.

[4] H. Becker, A.S. Kechris, The Descriptive Set Theory of Polish Group Actions, vol. 232, Cambridge University Press, 1996.

[5] D.J. Daley, D. Vere-Jones, An Introduction to the Theory of Point Processes: Volume I: Elementary Theory and Methods, Springer, 2003.

[6] D.J. Daley, D. Vere-Jones, An Introduction to the Theory of Point Processes. Volume II: General Theory and Structure, Springer, 2008.

[7] A.I. Danilenko, Point realization of Boolean actions of countable inductive limits of locally compact groups, J. Math. Phys. Anal. Geom. 7 (2000) 35–48.

[8] A.I. Danilenko, Krieger's type for ergodic non-singular Poisson actions of non-(T) locally compact groups, Ergod. Theory Dyn. Syst. 43 (7) (2023) 2317–2353.

[9] A. Fathi, M.R. Herman, Existence de difféomorphismes minimaux, in: Système dynamique I - Varsovie, Astérisque 49 (1977) 37–59.

[10] J. Feldman, A. Ramsay, Countable sections for free actions of groups, Adv. Math. 55 (3) (1985) 224–227.

[11] G.B. Folland, Real Analysis: Modern Techniques and Their Applications, vol. 40, John Wiley & Sons, 1999.

[12] D.H. Fremlin, Measure Theory, vol. 4, 2008.

[13] T. Giordano, V. Pestov, Some extremely amenable groups related to operator algebras and ergodic theory, J. Inst. Math. Jussieu 6 (2) (2007) 279–315.

[14] E. Glasner, B. Tsirelson, B. Weiss, The automorphism group of the Gaussian measure cannot act pointwise, Isr. J. Math. 148 (1) (2005) 305–329.

[15] E. Glasner, B. Weiss, Spatial and non-spatial actions of Polish groups, Ergod. Theory Dyn. Syst. 25 (5) (2005) 1521–1538.

[16] E. Glasner, B. Weiss, Weak mixing properties for non-singular actions, Ergod. Theory Dyn. Syst. 36 (7) (2016) 2203–2217.

[17] G. Grabinsky, Poisson process over σ-finite Markov chains, Pac. J. Math. 111 (2) (1984) 301–315.

[18] J. Hawkins, Ergodic Dynamics, Springer, 2021.

[19] F. Hoareau, F.L. Maître, Spatial models for Boolean actions in the infinite measure-preserving setup, arXiv preprint, arXiv:2406.02401, 2024.

[20] A. Kechris, Classical Descriptive Set Theory, vol. 156, Springer Science & Business Media, 2012.

[21] A. Kechris, Dynamics of non-Archimedean Polish groups, in: R. Latala, A. Rucinski (Eds.), European Congress of Mathematics, Krakow, Poland, July 2–7, 2013, European Mathematical Society, 2013, pp. 375–397.

[22] A.S. Kechris, Countable sections for locally compact group actions, Ergod. Theory Dyn. Syst. 12 (2) (1992) 283–295.
[23] A.S. Kechris, The Theory of Countable Borel Equivalence Relations, Cambridge Tracts in Mathematics, vol. 234, Cambridge University Press, 2024.
[24] A.S. Kechris, M. Malicki, A. Panagiotopoulos, J. Zielinski, On Polish groups admitting non-essentially countable actions, Ergod. Theory Dyn. Syst. 42 (1) (2022) 180–194.
[25] A. Kwiatkowska, S. Solecki, Spatial models of Boolean actions and groups of isometries, Ergod. Theory Dyn. Syst. 31 (2) (2011) 405–421.
[26] G. Last, M. Penrose, Lectures on the Poisson Process, vol. 7, Cambridge University Press, 2017.
[27] F. Le Maître, Polish topologies on groups of non-singular transformations, J. Log. Anal. 14 (2022).
[28] V. Liebscher, On the isomorphism of Poisson space and symmetric Fock space, in: Quantum Probability and Related Topics: QP-PQ (Volume IX), World Scientific, 1994, pp. 295–300.
[29] D. Maharam, Incompressible transformations, Fundam. Math. 56 (1) (1964) 35–50.
[30] F. Marchat, A class of measure-preserving transformations arising by the Poisson process, PhD thesis, Berkeley, 1978.
[31] M.G. Megrelishvili, Every semitopological semigroup compactification of the group $\mathrm{H}_+[0,1]$ is trivial, in: Semigroup Forum, vol. 63, Springer, 2001, pp. 357–370.
[32] I. Molchanov, Theory of Random Sets, 2nd ed., Probability Theory and Stochastic Modelling (PTSM), vol. 87, 2017.
[33] J.T. Moore, S. Solecki, A Boolean action of $\mathrm{C}(\mathrm{M},\mathrm{U}(1))$ without a spatial model and a re-examination of the Cameron–Martin theorem, J. Funct. Anal. 263 (10) (2012) 3224–3234.
[34] F. Parreau, E. Roy, Prime Poisson suspensions, Ergod. Theory Dyn. Syst. 35 (7) (2015) 2216–2230.
[35] V. Pestov, Dynamics of Infinite Dimensional Groups: The Ramsey–Dvoretzky–Milman Phenomenon, University Lecture Series, vol. 40, American Mathematical Society, 2006.
[36] A. Ramsay, Virtual groups and group actions, Adv. Math. 6 (3) (1971) 253–322.
[37] A. Rényi, Remarks on the Poisson process, in: Symposium on Probability Methods in Analysis, Springer, 1967, pp. 280–286.
[38] J. Rosiński, Representations and isomorphism identities for infinitely divisible processes, Ann. Probab. 46 (6) (2018) 3229–3274.
[39] E. Roy, Ergodic properties of Poissonian ID processes, Ann. Probab. 35 (2) (2007) 551–576.
[40] E. Roy, Poisson suspensions and infinite ergodic theory, Ergod. Theory Dyn. Syst. 29 (2) (2009) 667–683.
[41] E. Roy, Quelques résultats sur les Suspensions de Poisson, Habilitation à diriger des recherches, Université Paris 13, 2021.
[42] K.-I. Sato, Lévy Processes and Infinitely Divisible Distributions, vol. 68, Cambridge University Press, 1999.
[43] K. Schmidt, Lectures on Cocycles of Ergodic Transformation Groups, Macmillan Co. of India, January 1976.
[44] S.M. Srivastava, A Course on Borel Sets, vol. 180, Springer Science & Business Media, 2008.
[45] A.I. Tulcea, Ergodic properties of isometries in L^p spaces $1 < p < \infty$, Bull. Am. Math. Soc. 70 (3) (1964) 366–371.