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RESEARCH

Boundedness Properties of the Maximal Operator in a Nonsymmetric Inverse Gaussian Setting

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Abstract

We introduce a generalized inverse Gaussian setting and consider the maximal operator associated with the natural analogue of a nonsymmetric Ornstein–Uhlenbeck semigroup. We prove that it is bounded on L^p when $p \in (1, \infty]$ and that it is of weak type $(1, 1)$, with respect to the relevant measure. For small values of the time parameter t, the proof hinges on the "forbidden zones" method previously introduced in the Gaussian context. But for large times the proof requires new tools.

Keywords Maximal operator · Nondoubling measure · Inverse gaussian measure · Ornstein–Uhlenbeck semigroup · Weak type $(1 \cdot 1)$

Mathematics Subject Classification (2010) 42B25 · 47D03

1 Introduction

Alongside with the Gaussian framework, the Euclidean setting endowed with the absolutely continuous measure whose density is the reciprocal of a Gaussian has acquired independent interest in the last decade. It is nowadays known as the *inverse Gaussian* setting. Its

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introduction in the realm of harmonic analysis dates back to F. Salogni [24], who introduced the operator

$$\mathcal{A} = \tfrac{1}{2}\,\Delta + \langle x, \nabla\rangle$$

as an (essentially) self-adjoint operator in $L^2(\mathbb{R}^n, d\gamma_{-1})$. Here $d\gamma_{-1}$ stands for the inverse Gaussian measure

$$d\gamma_{-1}(x) = \pi^{n/2}\, e^{|x|^2}\, dx.$$

In her PhD thesis [24], Salogni proved the weak type (1, 1) of the maximal operator associated to the semigroup generated by $\mathcal{A}$. A few years later, T. Bruno and P. Sjögren [6, 7] (see also [5]) studied Riesz transforms and Hardy spaces for $\mathcal{A}$. Several other contributions to harmonic analysis in this context appeared more recently, see [1–4, 20].

While this development of the inverse Gaussian setting was going on, V. Casarino, P. Ciatti and P. Sjögren started investigating several classical problems related to the semigroup generated by the generalized Ornstein–Uhlenbeck operator

$$\mathcal{L}^{Q,B} = \tfrac{1}{2}\,\mathrm{tr}\big(Q\nabla^2\big) + \langle Bx, \nabla\rangle \tag{1.1}$$

in the Gaussian setting. Here ∇ indicates the gradient, ∇^2 the Hessian, and Q and B are two real $n \times n$ matrices called covariance and drift, respectively, satisfying

(H1) Q is symmetric and positive definite;
(H2) all the eigenvalues of B have negative real parts.

We refer the reader to [8–13, 15] and the brief overview [14]. The semigroup generated by $\mathcal{L}^{Q,B}$ has the Gaussian probability measure

$$d\gamma_\infty(x) = (2\pi)^{-\frac{n}{2}}(\det Q_\infty)^{-\frac{1}{2}} \exp\big(-\tfrac{1}{2}\langle Q_\infty^{-1}x, x\rangle\big)\, dx \tag{1.2}$$

as invariant measure. Here Q_∞ is a certain symmetric and positive definite matrix whose precise definition in terms of Q and B will be given in Section 2.

In some particular cases, for instance when Q and $-B$ coincide with the identity matrix I_n, the operator $\mathcal{L}^{Q,B}$ is essentially self-adjoint in $L^2(d\gamma_\infty)$, but in general it is not even symmetric. Thus many classical problems, like the boundedness of singular integral operators associated with $\mathcal{L}^{Q,B}$, give rise to interesting and nontrivial questions. New techniques and new ideas are required, strong enough to overcome the lack of symmetry.

In this article, we combine the two approaches by considering a generalized version of the inverse Gaussian setting studied in [6, 7, 24] via some of the techniques developed in [8–13, 15] and others which are new. This may be seen as the starting-point of a program whose goal is to develop an analogous theory in a nonsymmetric inverse Gaussian setting.

We shall replace the density of $d\gamma_\infty$ by its reciprocal, i.e., $\mathbb{R}^n$ will be equipped with the inverse Gaussian measure

$$d\gamma_{-\infty}(x) = (2\pi)^{\frac{n}{2}}(\det Q_\infty)^{\frac{1}{2}} \exp\big(\tfrac{1}{2}\langle Q_\infty^{-1}x, x\rangle\big)\, dx.$$

Like $d\gamma_\infty$, this measure is obviously locally doubling but not globally doubling, but in contrast to the probability measure $d\gamma_\infty$, it has superexponential growth at infinity.

In this setting, the role which was played by the Ornstein–Uhlenbeck operator in $(\mathbb{R}^n, d\gamma_\infty)$ is now played by the so-called inverse Ornstein–Uhlenbeck operator

$$\mathcal{A}^{Q,B} = \tfrac{1}{2}\,\mathrm{tr}\big(Q\nabla^2\big) - \langle Bx, \nabla\rangle.$$

Here B and Q are exactly the covariance and drift matrices inherited from the Ornstein–Uhlenbeck setting and satisfying (H1) and (H2). Notice that $d\gamma_{-1}$ and $\mathcal{A}$, considered in [1–4, 6, 7, 20, 24], are a special case of $d\gamma_{-\infty}$ and $\mathcal{A}^{Q,B}$, corresponding to the choice $Q = -B = I_n$.

The semigroup generated by $\mathcal{A}^{Q,B}$ is

$$\mathcal{H}_t^{UO} := e^{t\mathcal{A}^{Q,B}}, \qquad t > 0,$$

(here "UO" has the scope of emphasizing the contrast with respect to "OU" which will be used for Ornstein–Uhlenbeck). While in the Gaussian context $d\gamma_\infty$ is invariant under the action of the Ornstein–Uhlenbeck semigroup, and therefore chosen as the substitute for Lebesgue measure, $d\gamma_{-\infty}$ is not invariant under the action of $(\mathcal{H}_t^{UO})$. In fact, there are no invariant measures for $(\mathcal{H}_t^{UO})$; see, e.g., [21] or [19]. Nonetheless, $d\gamma_{-\infty}$ appears to be the natural measure for $(\mathcal{H}_t^{UO})$, because whenever $QB^* = BQ$ the operators $\mathcal{H}_t^{UO}$ are symmetric in $L^2(\mathbb{R}^n, d\gamma_{-\infty})$ (see Remark 2.3). As we shall see in Proposition 2.2, each operator of this semigroup is an integral operator, with a kernel K_t^{UO} with respect to $d\gamma_{-\infty}$.

In this paper we study the boundedness of the maximal operator associated to the semigroup $(\mathcal{H}_t^{UO})_{t>0}$, proving that it is of weak type $(1, 1)$ and of strong type (p, p) for all $1 < p \leq \infty$, with respect to $d\gamma_{-\infty}$. This extends a similar result in [24], proved under the assumption $Q = -B = I_n$. Let us note that our results appear to be the first of their kind for nonsymmetric operators on manifolds with superexponential volume growth.

As is standard by now, the proof distinguishes between the local and global parts of the kernel K_t^{UO} of $\mathcal{H}_t^{UO}$. Here local and global mean that $|x - u| \leq 1/(1 + |x|)$ and $|x - u| > 1/(1 + |x|)$, respectively, x, u being the two arguments of the kernel. Beyond this distinction that dates back to [17, 23, 25], the techniques in [24] rely on the spectral resolution of the self-adjoint operator $\mathcal{A} = \mathcal{A}^{I_n,-I_n}$ and seem no longer applicable in a nonsymmetric context. Fortunately, large parts of the machinery developed in [8–13, 15] to study Gaussian harmonic analysis in a nonsymmetric setting can be transferred to the inverse setting, and are useful to treat the local part of the maximal function and its global part for $t \in (0, 1]$. The global part for $t \geq 1$ is more delicate and requires new tools.

Structure of the paper In Section 2 we recall some basic facts concerning the Ornstein–Uhlenbeck setting. We also compute an explicit expression for the inverse Mehler kernel K_t^{UO} and discuss its relationship with the Gaussian Mehler kernel. In Section 3 the maximal operator associated to $(\mathcal{H}_t^{UO})_{t>0}$ is introduced, and the main theorem concerning its weak type $(1, 1)$ and strong type (p, p) is stated. We also give a theorem saying that for the global part and $t \geq 1$ the weak type $(1, 1)$ estimate can be enhanced by a logarithmic factor. Section 4 contains some simplifications and reductions that prepare for the proof of the theorems, and Section 5 is focused on the local part of the maximal function. Then in Section 6 some relevant geometric aspects of the problem are considered; in particular, we define a system of polar-like coordinates used already in [9]. Section 7 concerns the global part of the maximal operator for $0 < t \leq 1$. In this case, the weak type $(1, 1)$ is proved as a nontrivial application of the "forbidden zones", a recursive method introduced first by the fourth author in [25]. The arguments for the global part with $t > 1$ are given in Section 8. Finally, in Section 9 we complete the proofs by putting together the various pieces. An argument showing that the enhanced result mentioned above is sharp ends the paper.

Notation We shall denote by $C < \infty$ and $c > 0$ constants that may vary from place to place. They depend only on n, Q and B, unless otherwise explicitly stated. For two non-negative quantities A and B, we write $A \lesssim B$, or equivalently $B \gtrsim A$, if $A \leq CB$ for some C, and $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$. By $\mathbb{N}$ we mean $\{0, 1, \dots\}$. The symbol T^* will denote the adjoint of the operator T.

2 The Inverse Gaussian Framework

In this section we provide explicit expressions for the integral kernel of $\mathcal{H}_t^{UO}$ with respect both to Lebesgue measure and to $d\gamma_{-\infty}$ (see (2.4) and Proposition 2.2, respectively).

In order to prove these formulae, we need some facts from the general Ornstein–Uhlenbeck setting. Throughout the paper, B and Q will be two real matrices satisfying the hypotheses (H1) and (H2) introduced in Section 1.

2.1 Preliminaries

We first recall the definition of the covariance matrices

$$Q_t = \int_0^t e^{sB} Q e^{sB^*} ds, \qquad t \in (0, +\infty].$$

Each Q_t is well defined, symmetric and positive definite. Then we introduce the quadratic form

$$R(x) = \frac{1}{2} \langle Q_\infty^{-1} x, x \rangle, \qquad x \in \mathbb{R}^n.$$

Sometimes, we shall use the norm

$$|x|_Q := |Q_\infty^{-1/2} x|, \qquad x \in \mathbb{R}^n,$$

which satisfies $R(x) = \frac{1}{2}|x|_Q^2$ and $|x|_Q \simeq |x|$.

We also set

$$D_t = Q_\infty e^{-tB^*} Q_\infty^{-1}, \quad t \in \mathbb{R},$$

which is a one-parameter group of matrices.

In [9, Lemma 3.1] it has been proved that

$$e^{ct} |x| \lesssim |D_t x| \lesssim e^{Ct} |x| \qquad \text{and} \qquad e^{-Ct} |x| \lesssim |D_{-t} x| \lesssim e^{-ct} |x|, \tag{2.1}$$

for $t > 0$ and all $x \in \mathbb{R}^n$.

When $x \neq 0$ and $0 < t \leq 1$, [10, Lemma 2.3] says that

$$|x - D_t x| \simeq |t| \, |x|. \tag{2.2}$$

2.2 The Inverse Mehler Kernel

The Ornstein–Uhlenbeck operator $\mathcal{L}^{Q,B}$ given by (1.1) is essentially selfadjoint in $L^2(\gamma_\infty)$; the measure $d\gamma_\infty$ is defined in (1.2). We will sometimes write $\gamma_\infty(x)$ for its density. For each $f \in L^1(\gamma_\infty)$ and all $t > 0$ one has

$$e^{t\mathcal{L}^{Q,B}} f(x) = \int K_t^{OU}(x, u) \, f(u) \, d\gamma_\infty(u), \qquad x \in \mathbb{R}^n,$$

where for $x, u \in \mathbb{R}^n$ and $t > 0$ the Mehler kernel K_t^{OU} (with respect to $d\gamma_\infty$) is given by

$$K_t^{OU}(x, u) = \left(\frac{\det Q_\infty}{\det Q_t} \right)^{\frac{1}{2}} e^{R(x)} \exp\left[-\frac{1}{2} \left\langle \left(Q_t^{-1} - Q_\infty^{-1} \right) (u - D_t x), u - D_t x \right\rangle \right];$$

see [9, (2.6)]. This immediately yields

$$e^{t\mathcal{L}^{Q,B}} f(x) = \int M_t^{OU}(x, u) \, f(u) \, du, \qquad x \in \mathbb{R}^n,$$

where the kernel M_t^{OU} (with respect to Lebesgue measure) fulfills

$$\begin{aligned} M_t^{OU}(x,u) &= K_t^{OU}(x,u)\,\gamma_\infty(u) \\ &= (2\pi)^{-\frac{n}{2}}(\det Q_t)^{-\frac{1}{2}}\,e^{R(x)-R(u)}\exp\left[-\frac{1}{2}\langle (Q_t^{-1}-Q_\infty^{-1})(u-D_t\,x),\ u-D_t\,x\rangle\right]. \end{aligned} \tag{2.3}$$

From this we can deduce the corresponding kernel for the inverse Ornstein–Uhlenbeck setting.

Lemma 2.1 *The kernel of $e^{t\mathcal{A}^{Q,B}}$ with respect to Lebesgue measure is*

$$M_t^{UO}(x,u) = e^{t\,\mathrm{tr}B}\,M_t^{OU}(u,x),$$

for any $x, u \in \mathbb{R}^n$ and $t > 0$.

Proof We first compute the adjoint of $\mathcal{L}^{Q,B}$ in $L^2(\mathbb{R}^n, dx)$, where dx denotes Lebesgue measure. Let f and g be smooth functions with compact supports in $\mathbb{R}^n$. The second-order term in $\mathcal{L}^{Q,B}$ is symmetric, and for the first-order term we integrate by parts, getting

$$\begin{aligned} \langle f, \mathcal{L}^{Q,B}\,g\rangle &= \int f(x)\left(\frac{1}{2}\,\mathrm{tr}(Q\nabla^2 g)(x) + \langle Bx, \nabla g(x)\rangle\right)dx \\ &= \int\left(\frac{1}{2}\,\mathrm{tr}(Q\nabla^2 f)(x) - \langle Bx, \nabla f(x)\rangle - \mathrm{tr}Bf(x)\right)g(x)\,dx = \langle \mathcal{A}^{Q,B}f - \mathrm{tr}Bf, g\rangle. \end{aligned}$$

Thus

$$\mathcal{A}^{Q,B} = \left(\mathcal{L}^{Q,B}\right)^* + \mathrm{tr}B,$$

and

$$e^{t\mathcal{A}^{Q,B}} = e^{t\,\mathrm{tr}B}\,e^{t(\mathcal{L}^{Q,B})^*}, \qquad t > 0.$$

Since $e^{t(\mathcal{L}^{Q,B})^*}$ is the adjoint of $e^{t\mathcal{L}^{Q,B}}$, it has kernel $(M_t^{OU})^*(x,u) = M_t^{OU}(u,x)$, whence the claim. □

From (2.3) and Lemma 2.1 we have

$$\begin{aligned} &M_t^{UO}(x,u) \\ &= (2\pi)^{-\frac{n}{2}}(\det Q_t)^{-\frac{1}{2}}\,e^{t\,\mathrm{tr}B}\,e^{R(u)-R(x)}\exp\left[-\frac{1}{2}\langle (Q_t^{-1}-Q_\infty^{-1})(x-D_t\,u),\ x-D_t\,u\rangle\right]. \end{aligned} \tag{2.4}$$

This kernel is for integration against Lebesgue measure, but the relevant measure in the inverse setting is $d\gamma_{-\infty}$. Dividing M_t^{UO} by the density $\gamma_{-\infty}(u)$, one obtains the kernel of $e^{t\mathcal{A}^{Q,B}}$ for integration against $d\gamma_{-\infty}$, as follows.

Proposition 2.2 *The kernel of $e^{t\mathcal{A}^{Q,B}}$ with respect to $d\gamma_{-\infty}$ is*

$$\begin{aligned} &K_t^{UO}(x,u) \\ &= (2\pi)^{-n}\left(\det Q_\infty\,\det Q_t\right)^{-\frac{1}{2}}\,e^{t\,\mathrm{tr}B}\,e^{-R(x)}\exp\left[-\frac{1}{2}\langle (Q_t^{-1}-Q_\infty^{-1})(x-D_t\,u), x-D_t\,u\rangle\right] \end{aligned}$$

for all $x, u \in \mathbb{R}^n$, and $t > 0$.

It follows that the Mehler kernel K_t^{OU} and its counterpart K_t^{UO} in the inverse Gaussian setting are related by

$$K_t^{UO}(x,u) = (2\pi)^{-n}(\det Q_\infty)^{-1}\, e^{-R(x)}\, e^{-R(u)}\, e^{t\,\mathrm{tr}B}\, K_t^{OU}(u,x). \tag{2.5}$$

Remark 2.3 Because of (2.5), the semigroup $e^{t\mathcal{A}^{Q,B}}$ is symmetric on $L^2(\gamma_{-\infty})$ if and only if $e^{t\mathcal{L}^{Q,B}}$ is symmetric on $L^2(\gamma_\infty)$. Since the latter property holds if and only if $QB^* = BQ$ by [16, Theorem 2.4] (see also [22, Lemma 2.1]), we see that also $e^{t\mathcal{A}^{Q,B}}$ is symmetric on $L^2(\gamma_{-\infty})$ if and only if $QB^* = BQ$.

Formula (2.5) allows us to transfer the upper and lower estimates of K_t^{OU} in [9, formulae (3.4) and (3.5)] to K_t^{UO}, as follows.

Proposition 2.4 *If* $0 < t \le 1$*, one has*

$$\frac{e^{-R(x)}}{t^{n/2}} \exp\left(-C\,\frac{|D_{-t}\,x-u|^2}{t}\right) \lesssim K_t^{UO}(x,u) \lesssim \frac{e^{-R(x)}}{t^{n/2}} \exp\left(-c\,\frac{|D_{-t}\,x-u|^2}{t}\right) \tag{2.6}$$

for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$*. If instead* $t \ge 1$*, then*

$$e^{-|\mathrm{tr}B|t}\, e^{-R(x)} \exp\left(-C|D_{-t}\,x-u|^2\right) \lesssim K_t^{UO}(x,u) \lesssim e^{-|\mathrm{tr}B|\,t}\, e^{-R(x)} \exp\left(-c|D_{-t}\,x-u|^2\right) \tag{2.7}$$

for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$.

2.3 A Kolmogorov-Type Formula

We conclude this section by deducing a Kolmogorov-type formula in the inverse Gaussian setting. Recall that on the space $\mathcal{C}_b(\mathbb{R}^n)$ of bounded continuous functions, the Ornstein–Uhlenbeck semigroup $(\mathcal{H}_t^{OU})_{t>0}$ is explicitly given by the Kolmogorov formula

$$\mathcal{H}_t^{OU} f(x) = (2\pi)^{-\frac{n}{2}}(\det Q_t)^{-\frac12} \int f(e^{tB}x-u)\, e^{-\frac12\langle Q_t^{-1}u,u\rangle}\, du\,, \quad x \in \mathbb{R}^n\,, \tag{2.8}$$

see e.g. [18].

Proposition 2.5 *For all* $f \in \mathcal{C}_b(\mathbb{R}^n)$ *one has*

$$\mathcal{H}_t^{UO} f(x) = (2\pi)^{-\frac{n}{2}}(\det Q_t)^{-\frac12} \int f(e^{-tB}(u+x))\, e^{-\frac12\langle Q_t^{-1}u,u\rangle}\, du\,, \qquad x \in \mathbb{R}^n\,.$$

Proof From (2.8) it follows that

$$\mathcal{H}_t^{OU} f(x) = (2\pi)^{-\frac{n}{2}}(\det Q_t)^{-\frac12} \int f(u)\, e^{-\frac12\langle Q_t^{-1}(e^{tB}x-u),(e^{tB}x-u)\rangle} du\,, \qquad x \in \mathbb{R}^n\,,$$

which means that

$$M_t^{OU}(x,u) = (2\pi)^{-\frac{n}{2}}(\det Q_t)^{-\frac12}\, e^{-\frac12\langle Q_t^{-1}(e^{tB}x-u),(e^{tB}x-u)\rangle}\,, \qquad x,u \in \mathbb{R}^n\,.$$

This and Lemma 2.1 imply that

$$\begin{aligned}
\mathcal{H}_t^{UO} f(x) &= e^{t\,\mathrm{tr}B} \int f(u)\, M_t^{OU}(u,x)\, du \\
&= e^{t\,\mathrm{tr}B}\,(2\pi)^{-\frac{n}{2}}(\det Q_t)^{-\frac12} \int f(u)\; e^{-\frac12\langle Q_t^{-1}(e^{tB}u-x),(e^{tB}u-x)\rangle}\, du \\
&= (2\pi)^{-\frac{n}{2}}(\det Q_t)^{-\frac12} \int f(e^{-tB}(u+x))\, e^{-\frac12\langle Q_t^{-1}u,u\rangle} du\,, \qquad x \in \mathbb{R}^n,
\end{aligned}$$

where the last equality follows by a trivial change of variables. This proves the assertion. □

Remark 2.6 Proposition 2.5 immediately implies that $(\mathcal{H}_t^{UO})_{t>o}$ is conservative, that is, each $\mathcal{H}_t^{UO}$ maps the constant function $\mathbf{1}$ into itself.

3 The Main Result

The maximal operator associated to the semigroup $\left(\mathcal{H}_t^{UO}\right)_{t>0}$ is defined as

$$\mathcal{H}_* f(x) = \sup_{t>0} \left|\mathcal{H}_t^{UO} f(x)\right|. \tag{3.1}$$

Notice that we omit indicating that $\mathcal{H}_*$ refers to the inverse Ornstein–Uhlenbeck semigroup. Our main result is the following.

Theorem 3.1 *The inverse Ornstein–Uhlenbeck maximal operator $\mathcal{H}_*$ is bounded from $L^1(\gamma_{-\infty})$ to $L^{1,\infty}(\gamma_{-\infty})$, and on $L^p(\gamma_{-\infty})$ for all $1 < p \leq \infty$.*

We first deal with the strong type (p, p). Remark 2.6 implies that $\mathcal{H}_*$ is bounded on $L^\infty(\gamma_{-\infty})$. Given the weak type $(1, 1)$, one can then interpolate to obtain the boundedness on $L^p(\gamma_{-\infty})$ for $1 < p < \infty$.

All we have to do is thus to prove the weak type $(1, 1)$ of $\mathcal{H}_*$, that is, the estimate

$$\gamma_{-\infty}\{x \in \mathbb{R}^n : \mathcal{H}_* f(x) > \alpha\} \leq \frac{C}{\alpha} \|f\|_{L^1(\gamma_{-\infty})}, \qquad \alpha > 0, \tag{3.2}$$

for all functions $f \in L^1(\gamma_{-\infty})$ and some $C = C(n, Q, B) < \infty$. Let us emphasize that we do not keep track of the precise dependence of the constants on the parameters n, Q and B.

If we consider only the supremum over $t \geq 1$, the estimate (3.2) can be improved for small α, as follows.

Theorem 3.2 *Suppose $f \in L^1(\gamma_{-\infty})$ has norm 1, and take $\alpha \in (0, 1/2)$. Then*

$$\gamma_{-\infty} \left\{x \in \mathbb{R}^n : \sup_{t>1} \left|\mathcal{H}_t^{UO} f(x)\right| > \alpha\right\} \lesssim \frac{1}{\alpha \sqrt{\log(1/\alpha)}}. \tag{3.3}$$

This estimate is sharp in the following sense. If

$$\gamma_{-\infty} \left\{x \in \mathbb{R}^n : \sup_{t>1} \left|\mathcal{H}_t^{UO} f(x)\right| > \alpha\right\} \lesssim \frac{1}{\Phi(\alpha)}, \qquad 0 < \alpha < \frac{1}{2}, \tag{3.4}$$

where Φ is a function defined in $(0, 1/2)$, then

$$\Phi(\alpha) = \mathcal{O}\left(\alpha \sqrt{\log(1/\alpha)}\right) \qquad \text{as} \qquad \alpha \to 0. \tag{3.5}$$

A similar improvement has been observed in the Ornstein-Uhlenbeck setting, see [9].

4 Preparation for the Proof

We introduce some simplifications and reductions which will be useful in the proofs of the theorems, and do away with some simple cases.

First of all, we may assume that f is nonnegative and normalized in $L^1(\gamma_{-\infty})$.

4.1 Splitting of the Operator

We consider separately the supremum in (3.1) for small and large values of t. Further, we divide the operator into a local and a global part, by means of a local region

$$L = \left\{ (x, u) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ |x - u| \le \frac{1}{1 + |x|} \right\}$$

and a global region

$$G = \left\{ (x, u) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ |x - u| > \frac{1}{1 + |x|} \right\}.$$

Clearly, $\mathcal{H}_*$ is dominated by the sum of the following four operators

$$\mathcal{H}_*^{-,L} f(x) = \sup_{t \le 1} \left| \int K_t^{UO}(x, u) \, \mathbf{1}_L(x, u) \, f(u) \, d\gamma_{-\infty}(u) \right|;$$
$$\mathcal{H}_*^{-,G} f(x) = \sup_{t \le 1} \left| \int K_t^{UO}(x, u) \, \mathbf{1}_G(x, u) \, f(u) \, d\gamma_{-\infty}(u) \right|;$$
$$\mathcal{H}_*^{+,L} f(x) = \sup_{t > 1} \left| \int K_t^{UO}(x, u) \, \mathbf{1}_L(x, u) \, f(u) \, d\gamma_{-\infty}(u) \right|;$$
$$\mathcal{H}_*^{+,G} f(x) = \sup_{t > 1} \left| \int K_t^{UO}(x, u) \, \mathbf{1}_G(x, u) \, f(u) \, d\gamma_{-\infty}(u) \right|.$$

We will prove estimates like (3.2) for each of these four operators and (3.3) for the latter two.

4.2 Simple Upper Bounds

Lemma 4.1 *If $f \ge 0$ is normalized in $L^1(\gamma_{-\infty})$, then*

$$\mathcal{H}_*^{+,L} f + \mathcal{H}_*^{+,G} f + \mathcal{H}_*^{-,G} f \lesssim 1.$$

Proof If $t > 1$, we see from (2.7) that $K_t^{UO}(x, u) \lesssim 1$ for all (x, u). This implies the claimed estimate for $\mathcal{H}_*^{+,L}$ and $\mathcal{H}_*^{+,G}$. To deal with $\mathcal{H}_*^{-,G}$, we need another lemma.

Lemma 4.2 *If $(x, u) \in G$ and $0 < t \le 1$, then*

$$K_t^{UO}(x, u) \lesssim e^{-R(x)} \, (1 + |x|)^n.$$

Proof We apply the definition of G and then (2.2), to get

$$\frac{1}{1 + |x|} < |u - x| \le |u - D_{-t} \, x| + |D_{-t} \, x - x| \le |u - D_{-t} \, x| + Ct|x|. \tag{4.1}$$

Thus $|D_{-t} \, x - u| \ge \frac{1}{1+|x|} - Ct|x|$, and

$$\frac{|D_{-t} \, x - u|^2}{t} \ge \frac{1}{t(1 + |x|)^2} - C \frac{|x|}{1 + |x|} \ge \frac{1}{t(1 + |x|)^2} - C.$$

From (2.6) we then see that

$$K_t^{UO}(x, u) \lesssim e^{-R(x)} \, t^{-n/2} \, \exp\left(- \frac{c}{t(1 + |x|)^2} \right) \lesssim e^{-R(x)} \, (1 + |x|)^n,$$

and Lemma 4.2 is proved. □

To complete the proof of Lemma 4.1, it is now enough to observe that Lemma 4.2 implies $K_t^{UO}(x,u) \lesssim 1$ for $(x,u) \in G$ and $0 < t \le 1$, and thus $\mathcal{H}_*^{-,G} f \lesssim 1$.

The following consequence of Lemma 4.1 will be useful.

Let $\alpha_0 = \alpha_0(n, Q, B) \in (0, 1/2]$. To prove the estimates (3.2) and (3.3) for any of the three operators in Lemma 4.1, it is enough to estimate the relevant level set for levels $\alpha < \alpha_0$, with f normalized in $L^1(\gamma_{-\infty})$. Indeed, Lemma 4.1 says that the level set is empty for levels larger than some C. For levels $\alpha \in (\alpha_0, C]$, one can use the estimate corresponding to the level $\alpha_0/2$, since then both sides of the inequalities (3.2) and (3.3) will be of order of magnitude 1.

4.3 Observations for Small Levels α

Assuming $\alpha < 1/2$, we can estimate the set where $R(x)$ is not too large. Indeed,

$$\gamma_{-\infty}\left\{x \in \mathbb{R}^n : R(x) < \frac{3}{4}\log\frac{1}{\alpha}\right\} = \int_{R(x)\le \frac{3}{4}\log\frac{1}{\alpha}} e^{R(x)}\,dx\,,$$

and in the integral here we change variables to $x' = Q_\infty^{-1/2}x$ so that $R(x) = |x'|^2/2$. Then one passes to polar coordinates and finds that

$$\gamma_{-\infty}\left\{x \in \mathbb{R}^n :\; R(x) < \frac{3}{4}\log\frac{1}{\alpha}\right\} \simeq \frac{1}{\alpha^{3/4}}\left(\log\frac{1}{\alpha}\right)^{(n-2)/2} \lesssim \frac{1}{\alpha\sqrt{\log(1/\alpha)}}. \tag{4.2}$$

The last, simple estimate here is stated in view of Theorem 3.2. As soon as $\alpha < 1/2$, we can thus neglect the set where $R(x) < \frac{3}{4}\log(1/\alpha)$ when we prove Theorems 3.1 and 3.2.

Further, if $(x,u) \in G$ and $R(x) > \frac{5}{4}\log(1/\alpha)$ with $\alpha < 1/2$, then Lemma 4.2 implies for $t \le 1$

$$K_t^{UO}(x,u) \lesssim \alpha^{5/4}\left(\log\frac{1}{\alpha}\right)^{n/2} \lesssim \alpha.$$

When $t > 1$, this remains true, since (2.7) then shows that $K_t^{UO}(x,u) \lesssim e^{-R(x)}$.

For the operators $\mathcal{H}_*^{\pm,G}$ we need thus only consider points x in the annulus

$$\mathcal{E}_\alpha = \left\{x \in \mathbb{R}^n : \frac{3}{4}\log\frac{1}{\alpha} \le R(x) \le \frac{5}{4}\log\frac{1}{\alpha}\right\}. \tag{4.3}$$

5 The Local Part of the Maximal Function

This section consists of the proof of the following result.

Proposition 5.1 *The operators $\mathcal{H}_*^{+,L}$ and $\mathcal{H}_*^{-,L}$ are of weak type* $(1,1)$ *with respect to the measure $d\gamma_{-\infty}$. Further, $\mathcal{H}_*^{+,L}$ satisfies also the sharpened estimate of* Theorem 3.2.

We start with $H_*^{+,L}$, which is of strong type (1,1) with respect to $d\gamma_{-\infty}$. Indeed, (2.7) shows that $K_t^{UO}(x,u) \lesssim e^{-R(x)}$ for $t > 1$. With $0 \le f \in L^1(\gamma_{-\infty})$ we then have

$$H_*^{+,L} f(x) \lesssim e^{-R(x)} \int \mathbf{1}_L(x,u)\, f(u)\, d\gamma_{-\infty}(u).$$

Hence,

$$\int H_*^{+,L} f(x)\, d\gamma_{-\infty}(x) \lesssim \int f(u) \int \mathbf{1}_L(x,u)\, dx\, d\gamma_{-\infty}(u),$$

where we swapped the order of integration and passed to Lebesgue measure dx. Now $(x,u) \in L$ implies $1+|x| \simeq 1+|u|$ and thus $|x-u| \lesssim 1/(1+|u|)$, so the inner integral here is no larger than $C(1+|u|)^{-n} \lesssim 1$. The strong type (1, 1) follows. To obtain also (3.3) for $H_*^{+,L}$, we take $\alpha \in (0, 1/2)$ and see from (4.2) that we need only consider points x with $|x| \gtrsim \sqrt{\log(1/\alpha)}$. When restricted to such x, the above inner integral is no larger than $C\,(\log(1/\alpha))^{-n/2}$. Chebyshev's inequality then implies that $H_*^{+,L}$ satisfies (3.3).

When we now deal with $H_*^{-,L}$, we can replace $d\gamma_{-\infty}$ by Lebesgue measure. We briefly describe the argument for this, which is based on a covering procedure to be found in [12, Subsection 7.1] and for the standard Gaussian measure also in [24, Lemma 3.2.3]. Notice that in [12] the localization is by means of balls, but the same method works with cubes instead of balls. The idea is to cover $\mathbb{R}^n$ by pairwise disjoint cubes Q_j with centers c_j and of sides roughly $1/(1+|c_j|)$. Then $f = \sum f\mathbf{1}_{Q_j}$, and $\mathcal{H}_*^{-,L}(f\mathbf{1}_{Q_j})$ is supported in a concentrically scaled cube CQ_j. These scaled cubes have bounded overlap, and in each CQ_j the measure $d\gamma_{-\infty}$ is essentially proportional to Lebesgue measure. It is therefore enough to verify the weak type (1, 1) of $\mathcal{H}_*^{-,L}$ with respect to Lebesgue measure in $\mathbb{R}^n$, and then apply this to each $f\mathbf{1}_{Q_j}$ and sum in j.

Observe that $(x,u) \in L$ implies

$$\begin{aligned} |R(u)-R(x)| &= \frac{1}{2}\left||u|_Q^2 - |x|_Q^2\right| = \frac{1}{2}\left||u|_Q - |x|_Q\right|\left(|u|_Q + |x|_Q\right) \\ &\lesssim |u-x|_Q\,\left(|u-x|_Q + 2\,|x|_Q\right) \lesssim \frac{1}{(1+|x|)^2} + \frac{|x|}{1+|x|} \lesssim 1. \end{aligned} \tag{5.1}$$

If $(x,u) \in L$ and $0 < t \le 1$ we have because of (2.2)

$$|D_{-t}\,x - u| = |D_{-t}\,x - x + x - u| \ge |u-x| - |D_{-t}\,x - x| \gtrsim |u-x| - Ct|x|.$$

Squaring and dividing by t, we get

$$\frac{|D_{-t}\,x-u|^2}{t} \gtrsim \frac{|u-x|^2}{t} - C\,|x-u||x| \ge \frac{|u-x|^2}{t} - C.$$

In view of (2.6), this implies that the relevant kernel is no larger than constant times

$$e^{-R(x)}\,t^{-n/2}\exp\Big(-c\,\frac{|u-x|^2}{t}\Big)\,\mathbf{1}_L(x,u)\,.$$

With $f \ge 0$ then

$$\begin{aligned} \mathcal{H}_*^{-,L} f(x) &\lesssim \sup_{0<t\le 1} \frac{e^{-R(x)}}{t^{n/2}} \int \exp\Big(-c\,\frac{|x-u|^2}{t}\Big)\,\mathbf{1}_L(x,u)\,f(u)\,d\gamma_{-\infty}(u) \\ &\simeq \sup_{t>0} \frac{1}{t^{n/2}} \int \exp\Big(-c\,\frac{|x-u|^2}{t}\Big)\,\mathbf{1}_L(x,u)\,f(u)\,du, \end{aligned}$$

where the second step relied on (5.1). The last supremum here defines a maximal operator which is well known to be dominated by the Hardy–Littlewood maximal function and so of weak type (1, 1) with respect to Lebesgue measure in $\mathbb{R}^n$, see e.g. [26, p. 73].

Proposition 5.1 is proved.

6 Some Geometric Background

6.1 Polar Coordinates

For $\beta > 0$ we introduce the ellipsoid

$$E_\beta = \{x \in \mathbb{R}^n : R(x) = \beta\}\,.$$

Then we recall [9, formula (4.3)] saying that

$$\frac{\partial}{\partial s} R\big(D_s\,x\big) = \frac{1}{2}\,\big|Q^{1/2}\,Q_\infty^{-1} D_s\,x\big|^2 \simeq \big|D_s\,x\big|^2 \tag{6.1}$$

for all x in $\mathbb{R}^n$ and $s \in \mathbb{R}$. If $x \neq 0$ the map $s \mapsto R(D_s\,x)$ is thus strictly increasing, and x can be written uniquely as

$$x = D_s\,\tilde{x}$$

for some $\tilde{x} \in E_\beta$ and $s \in \mathbb{R}$. We call s and $\tilde{x}$ the polar coordinates of x.

From [9, Proposition 4.2] we recall that the Lebesgue measure in $\mathbb{R}^n$ is given in terms of polar coordinates $(s, \tilde{x})$ by

$$dx = e^{-s\,\mathrm{tr}\,B}\,\frac{|Q^{1/2}\,Q_\infty^{-1}\tilde{x}|^2}{2\,|Q_\infty^{-1}\tilde{x}|}\,dS(\tilde{x})\,ds\,, \tag{6.2}$$

where dS denotes the area measure of E_β.

We shall use the estimates for the distance between two points expressed in terms of polar coordinates given in [9, Lemma 4.3]. Let $x^{(0)},\ x^{(1)} \in \mathbb{R}^n \setminus \{0\}$ be given by

$$x^{(0)} = D_{s_0}\,\tilde{x}^{(0)} \qquad \text{and} \qquad x^{(1)} = D_{s_1}\,\tilde{x}^{(1)},$$

with $s_0, s_1 \in \mathbb{R}$ and $\tilde{x}^{(0)},\ \tilde{x}^{(1)} \in E_\beta$. If $R(x^{(0)}) > \beta/2$, then

$$\big|x^{(0)} - x^{(1)}\big| \gtrsim \big|\tilde{x}^{(0)} - \tilde{x}^{(1)}\big|. \tag{6.3}$$

We observe here that the assumption $R(x^{(0)}) > \beta/2$ may be weakened to $R(x^{(0)}) > c\,\beta$ with $c = c(n, Q, B) > 0$, as can be seen from the proof in [9].

If $R(x^{(0)}) > \beta/2$ and also $s_1 \geq 0$, the same lemma says that

$$\big|x^{(0)} - x^{(1)}\big| \gtrsim \sqrt{\beta}\,|s_0 - s_1|. \tag{6.4}$$

6.2 The Inverse Gaussian Measure of a Tube

We fix a large $\beta > 0$. Define for $y \in E_\beta$ and $a > 0$ the set

$$\Omega = \big\{x \in E_\beta :\ |x - y| < a\big\}\,.$$

This is a spherical cap of the ellipsoid E_β, centered at y. The area of Ω is $S(\Omega) \simeq a^{n-1} \wedge \beta^{(n-1)/2}$. Then consider the tube

$$Z = \{D_s\,\tilde{x} : s \leq 0,\ \tilde{x} \in \Omega\}. \tag{6.5}$$

Lemma 6.1 *There exists a constant C such that for $\beta > C$ the inverse Gaussian measure of the tube Z fulfills*

$$\gamma_{-\infty}(Z) \lesssim \frac{a^{n-1}}{\sqrt{\beta}}\,e^{\beta}.$$

Proof From (6.2) we obtain, since $|\tilde{x}| \simeq \sqrt{\beta}$ and $R(\tilde{x}) = \beta$,

$$\gamma_{-\infty}(Z) \simeq \int_0^\infty e^{-s|\operatorname{tr} B|} \int_\Omega e^{R(D_{-s}\tilde{x})} |\tilde{x}|\, dS(\tilde{x})\, ds \lesssim \sqrt{\beta}\, e^\beta \int_0^\infty e^{-s|\operatorname{tr} B|} \int_\Omega e^{-(R(\tilde{x})-R(D_{-s}\tilde{x}))}\, dS(\tilde{x})\, ds.$$

By Eqs. 6.1 and 2.1 we have

$$R(\tilde{x}) - R(D_{-s}\tilde{x}) \simeq \int_0^s \left|D_{-s'}\tilde{x}\right|^2 ds' \gtrsim \int_0^s e^{-Cs'} |\tilde{x}|^2\, ds' \simeq (1 - e^{-Cs})\, |\tilde{x}|^2 \simeq (s \wedge 1)\, \beta,$$

which implies

$$\gamma_{-\infty}(Z) \lesssim \sqrt{\beta}\, e^\beta\, S(\Omega) \left[\int_0^1 e^{-cs\beta}\, ds + e^{-c\beta} \int_1^\infty e^{-s|\operatorname{tr} B|}\, ds\right].$$

Since $S(\Omega) \lesssim a^{n-1}$ and both terms in the bracket are bounded by C/β, this proves the lemma. □

6.3 Decomposing the Global Region

In the following two sections, we will use a decomposition of the global region into annuli. More precisely, for $t > 0$ and $m = 1, 2, \ldots$ one sets

$$\mathcal{T}_t^m := \left\{(x, u) \in G\colon 2^{m-1}\big(1 \wedge \sqrt{t}\big) < |u - D_{-t}\, x| \le 2^m \big(1 \wedge \sqrt{t}\big)\right\}. \tag{6.6}$$

But if $m = 0$, we have only the upper estimate, setting

$$\mathcal{T}_t^0 := \left\{(x, u) \in G\colon |u - D_{-t}\, x| \le 1 \wedge \sqrt{t}\right\}. \tag{6.7}$$

Note that for any fixed $t > 0$ these sets form a partition of G.

7 The Global Case for Small *t*

Proposition 7.1 *The maximal operator* $\mathcal{H}_*^{-,G}$ *is of weak type* $(1, 1)$ *with respect to the measure* $d\gamma_{-\infty}$.

We will prove this result in a way that follows the proof of [9, Proposition 8.1], but several adjustments are necessary to pass from the Ornstein–Uhlenbeck framework to the inverse setting. Here $0 < t \le 1$, and as before we let the function f be nonnegative and normalized in $L^1(\gamma_{-\infty})$. It is enough to consider the level sets $\{\mathcal{H}_*^{-,G} f > \alpha\}$ for $\alpha < \alpha_0$ with some small α_0; see the end of Section 4.2. Further, we need only consider points x in the annulus $\mathcal{E}_\alpha$ defined in (4.3).

What we must prove is that for all $0 < \alpha < \alpha_0$

$$\gamma_{-\infty}\left\{x \in \mathcal{E}_\alpha\colon \sup_{0<t\le 1} \int K_t^{UO}(x, u)\, \mathbf{1}_G(x, u)\, f(u)\, d\gamma_{-\infty}(u) > \alpha\right\} \lesssim \frac{1}{\alpha}. \tag{7.1}$$

The global region G is covered by the sets $\mathcal{T}_t^m$, $m \in \mathbb{N}$, defined in (6.6) and (6.7) and now given by

$$\mathcal{T}_t^m = \left\{(x, u) \in G\colon 2^{m-1}\sqrt{t} < |u - D_{-t}\, x| \le 2^m \sqrt{t}\right\},$$

where the lower bound is to be suppressed for $m = 0$.

From (2.6) we then see that for $(x, u) \in \mathcal{T}_t^m$

$$K_t^{UO}(x, u) \lesssim \frac{e^{-R(x)}}{t^{n/2}} \exp\left(-c2^{2m}\right).$$

Setting

$$K_t^{-,m}(x, u) = \frac{e^{-R(x)}}{t^{n/2}} \mathbf{1}_{\mathcal{T}_t^m}(x, u), \tag{7.2}$$

one has, for all $(x, u) \in G$ and $0 < t \le 1$,

$$K_t^{UO}(x, u) \lesssim \sum_{m=0}^{\infty} \exp\left(-c2^{2m}\right) K_t^{-,m}(x, u). \tag{7.3}$$

We define the operator

$$\mathcal{M}_{m,\alpha}^{-} h(x) = \mathbf{1}_{\mathcal{E}_\alpha}(x) \sup_{0<t\le 1} \int K_t^{-,m}(x, u)\, |h(u)|\, d\gamma_{-\infty}(u)$$

for $h \in L^1(\gamma_{-\infty})$, and observe that (7.3) implies

$$\mathbf{1}_{\mathcal{E}_\alpha}(x) \sup_{0<t\le 1} \int K_t^{UO}(x, u)\, \mathbf{1}_G(x, u)\, f(u)\, d\gamma_{-\infty}(u) \le \sum_{m=0}^{\infty} \exp\left(-c2^{2m}\right) \mathcal{M}_{m,\alpha}^{-} f(x). \tag{7.4}$$

Lemma 7.2 *For $0 < \alpha < \alpha_0$ with a suitably small $\alpha_0 > 0$ and any $m \in \mathbb{N}$, the operator $\mathcal{M}_{m,\alpha}^{-}$ maps $L^1(\gamma_{-\infty})$ into $L^{1,\infty}(\gamma_{-\infty})$, with operator quasinorm at most $C\, 2^{Cm}$.*

Given this lemma, [27, Lemma 2.3] will imply that the $L^{1,\infty}(\mathcal{E}_\alpha; \gamma_{-\infty})$ quasinorm of the right-hand side of (7.4) is bounded, uniformly in $\alpha \in (0, \alpha_0)$. Then (7.1) and Proposition 7.1 will follow.

Proof of Lemma 7.2 Let $h \ge 0$ be normalized in $L^1(\gamma_{-\infty})$. Fixing $m \in \mathbb{N}$, we must show that for any $\alpha' > 0$

$$\gamma_{-\infty} \left\{ x \in \mathcal{E}_\alpha : \sup_{0<t\le 1} \int K_t^{-,m}(x, u)\, h(u)\, d\gamma_{-\infty}(u) > \alpha' \right\} \lesssim \frac{2^{Cm}}{\alpha'}. \tag{7.5}$$

If $(x, u) \in \mathcal{T}_t^m$ for some $t \in (0, 1]$, then $|u - D_{-t}\, x| \lesssim 2^m \sqrt{t}$, and (4.1) implies

$$1 \lesssim 2^m \sqrt{t}\, (1 + |x|) + t|x|(1 + |x|) \le 2^m \sqrt{t}\, (1 + |x|) + \left(2^m \sqrt{t}\, (1 + |x|)\right)^2.$$

It follows that

$$2^m \sqrt{t}\, (1 + |x|) \gtrsim 1, \tag{7.6}$$

and thus $t \gtrsim 2^{-2m}(1 + |x|)^{-2}$. If also $x \in \mathcal{E}_\alpha$ so that $|x| \simeq \sqrt{\log(1/\alpha)}$, we conclude that $t \ge \delta > 0$ for some $\delta = \delta(\alpha, m) > 0$. Hence, (7.5) can be replaced by

$$\gamma_{-\infty}\left(\mathcal{A}_1(\alpha')\right) \lesssim \frac{2^{Cm}}{\alpha'}, \tag{7.7}$$

where

$$\mathcal{A}_1(\alpha') = \left\{ x \in \mathcal{E}_\alpha : \sup_{\delta\le t\le 1} \int K_t^{-,m}(x, u)\, h(u)\, d\gamma_{-\infty}(u) \ge \alpha' \right\}.$$

A benefit with this is that the supremum is now a continuous function of $x \in \mathcal{E}_\alpha$, and the set $\mathcal{A}_1(\alpha')$ is compact.

Using the method from [9, Proposition 8.1], we shall prove (7.7) by building a finite sequence of pairwise disjoint balls $\left(\mathcal{B}^{(\ell)}\right)_{\ell=1}^{\ell_0}$ in $\mathbb{R}^n$ and at the same time a finite sequence of tubes $\left(\mathcal{Z}^{(\ell)}\right)_{\ell=1}^{\ell_0}$ covering $\mathcal{A}_1(\alpha')$ and called forbidden zones.

We will then verify the following three items:

(1) the $\mathcal{B}^{(\ell)}$ are pairwise disjoint;

(2)
$$\mathcal{A}_1(\alpha') \subset \bigcup_{\ell=1}^{\ell_0} \mathcal{Z}^{(\ell)};$$

(3) for each ℓ
$$\gamma_{-\infty}(\mathcal{Z}^{(\ell)}) \lesssim \frac{2^{Cm}}{\alpha'} \int_{\mathcal{B}^{(\ell)}} h(u)\, d\gamma_{-\infty}(u).$$

This would imply (7.7) and Lemma 7.2, as follows:
$$\gamma_{-\infty}\left(\mathcal{A}_1(\alpha')\right) \le \gamma_{-\infty}\left(\bigcup_{\ell=1}^{\ell_0} \mathcal{Z}^{(\ell)}\right) \lesssim \frac{2^{Cm}}{\alpha'} \sum_{\ell=1}^{\ell_0} \int_{\mathcal{B}^{(\ell)}} h(u)\, d\gamma_{-\infty}(u) \lesssim \frac{2^{Cm}}{\alpha'}.$$

To construct the sets $\mathcal{B}^{(\ell)}$ and $\mathcal{Z}^{(\ell)}$, we define by recursion a sequence of points $x^{(\ell)}$, $\ell = 1, \dots, \ell_0$.

Let $x^{(1)}$ be a maximum point for the quadratic form $R(x)$ in the compact set $\mathcal{A}_1(\alpha')$. Notice that if this set is empty, (7.7) is immediate. Then by continuity we choose $t_1 \in [\delta, 1]$ such that
$$\int K_{t_1}^{-,m}(x^{(1)}, u)\, h(u)\, d\gamma_{-\infty}(u) \ge \alpha'.$$

Using this t_1, we associate with $x^{(1)}$ the tube
$$\mathcal{Z}^{(1)} = \left\{D_{-s}\,\eta \in \mathbb{R}^n : s \ge 0, \quad R(\eta) = R(x^{(1)}), \quad |\eta - x^{(1)}| < A\, 2^{3m} \sqrt{t_1}\right\}.$$

The positive constant A here will depend only on n, Q and B and will be determined later.

Recursively, suppose $x^{(\ell')}$, $t_{\ell'}$ and $\mathcal{Z}^{(\ell')}$ have been defined for all $\ell' \le \ell$, where $\ell \ge 1$. We choose $x^{(\ell+1)}$ as a maximizing point of $R(x)$ in the set
$$\mathcal{A}_{\ell+1}(\alpha') := \left\{x \in \mathcal{E}_\alpha \setminus \bigcup_{\ell'=1}^{\ell} \mathcal{Z}^{(\ell')} : \sup_{\delta \le t \le 1} \int K_t^{-,m}(x, u)\, h(u)\, d\gamma_{-\infty}(u) \ge \alpha'\right\}, \tag{7.8}$$

which is compact as shown later, provided this set is nonempty. But if $\mathcal{A}_{\ell+1}(\alpha')$ is empty, the process stops with $\ell_0 = \ell$ and item (2) follows. We will see that this actually occurs for some finite ℓ.

Assume now that $\mathcal{A}_{\ell+1}(\alpha') \ne \emptyset$. By continuity, we can select a $t_{\ell+1} \in [\delta, 1]$ such that
$$\int K_{t_{\ell+1}}^{-,m}\left(x^{(\ell+1)}, u\right) h(u)\, d\gamma_{-\infty}(u) \ge \alpha'. \tag{7.9}$$

Then we define the tube
$$\mathcal{Z}^{(\ell+1)} = \left\{D_{-s}\,\eta \in \mathbb{R}^n : s \ge 0, \quad R(\eta) = R\left(x^{(\ell+1)}\right), \quad \left|\eta - x^{(\ell+1)}\right| < A\, 2^{3m} \sqrt{t_{\ell+1}}\right\}.$$

It must be proved that $\mathcal{A}_{\ell+1}(\alpha')$ is closed and thus compact, even though the $\mathcal{Z}^{(\ell')}$ are not open. This will guarantee the existence of a maximizing point. We use induction, and observe that $\mathcal{A}_1(\alpha')$ is closed. Assume that $\mathcal{A}_{\ell'}(\alpha')$ is closed for $1 \le \ell' \le \ell$. For these ℓ', the maximizing property of $x^{(\ell')}$ shows that there is no point x in $\mathcal{A}_{\ell'}(\alpha')$ with $R(x) > R(x^{(\ell')})$. Hence,

$$\mathcal{A}_{\ell+1}(\alpha') \subset \mathcal{A}_{\ell'}(\alpha') \subset \left\{x : R(x) \le R(x^{(\ell')})\right\}, \qquad 1 \le \ell' \le \ell,$$

and so

$$\begin{aligned}\mathcal{A}_{\ell+1}(\alpha') &= \bigcap_{1\le\ell'\le\ell} \left(\mathcal{A}_{\ell+1}(\alpha') \cap \left\{x\colon R(x) \le R(x^{(\ell')})\right\}\right) \\ &= \bigcap_{1\le\ell'\le\ell} \left\{x \in \mathcal{E}_\alpha \setminus \mathcal{Z}^{(\ell')}\colon R(x) \le R(x^{(\ell')}),\ \sup_{\delta\le t\le 1} \int K_t^{-,m}(x,u)\, h(u)\, d\gamma_{-\infty}(u) \ge \alpha'\right\}.\end{aligned} \tag{7.10}$$

For each $\ell' = 1, \dots, \ell$ one has

$$\begin{aligned}&\{x \in \mathcal{E}_\alpha \setminus \mathcal{Z}^{(\ell')} : R(x) \le R(x^{(\ell')})\} \\ &\qquad = \left\{D_{-s}\, \eta \in \mathcal{E}_\alpha : s \ge 0,\ R(\eta) = R(x^{(\ell')}),\ |\eta - x^{(\ell')}| \ge A\, 2^{3m} \sqrt{t_{\ell'}}\right\},\end{aligned}$$

and this set is closed. Now (7.10) shows that $\mathcal{A}_{\ell+1}(\alpha')$ is closed, a maximizing point $x^{(\ell+1)}$ can be chosen, and the recursion is well defined.

By applying (7.6) to t_ℓ and $x^{(\ell)}$, one obtains, since $|x^{(\ell)}|$ is large,

$$|x^{(\ell)}|^2\, 2^{2m}\, t_\ell \gtrsim 1. \tag{7.11}$$

Then set

$$\mathcal{B}^{(\ell)} = \left\{u \in \mathbb{R}^n : |u - D_{-t_\ell}\, x^{(\ell)}| \le 2^m \sqrt{t_\ell}\right\}.$$

Combining (7.2) and (7.9), with $\ell + 1$ replaced by ℓ, we see that

$$\alpha' \le \frac{\exp\left(-R(x^{(\ell)})\right)}{t_\ell^{n/2}} \int_{\mathcal{B}^{(\ell)}} h(u)\, d\gamma_{-\infty}(u). \tag{7.12}$$

We will verify (1), (2), (3) and start with (1).

The balls $\mathcal{B}^{(\ell)}$ and $\mathcal{B}^{(\ell')}$, with $\ell' < \ell$, will be disjoint if

$$\left|D_{-t_{\ell'}}\, x^{(\ell')} - D_{-t_\ell}\, x^{(\ell)}\right| > 2^m \left(\sqrt{t_\ell} + \sqrt{t_{\ell'}}\right). \tag{7.13}$$

By means of our polar coordinates with $\beta = R(x^{(\ell')})$, we write

$$x^{(\ell)} = D_s\, \tilde{x}^{(\ell)}$$

for some $\tilde{x}^{(\ell)}$ with $R(\tilde{x}^{(\ell)}) = R(x^{(\ell')})$ and some $s \in \mathbb{R}$. Note that $s \le 0$, since $R(x^{(\ell)}) \le R(x^{(\ell')})$. The point $x^{(\ell)}$ cannot belong to the forbidden zone $\mathcal{Z}^{(\ell')}$, so

$$|\tilde{x}^{(\ell)} - x^{(\ell')}| \ge A\, 2^{3m} \sqrt{t_{\ell'}}\,. \tag{7.14}$$

Since $t_{\ell'} \le 1$, (2.1) implies $R(D_{-t_{\ell'}}\, x^{(\ell')}) > c\, \beta$. This allows us to apply (6.3) and the observation following it, to obtain

$$\left|D_{-t_{\ell'}}\, x^{(\ell')} - D_{-t_\ell}\, x^{(\ell)}\right| \gtrsim A\, 2^{3m} \sqrt{t_{\ell'}}. \tag{7.15}$$

If $\sqrt{A}\,2^{2m}\sqrt{t_{\ell'}} \geq \sqrt{t_\ell}$, we will have

$$A2^{3m}\sqrt{t_{\ell'}} \geq \frac{1}{2}A2^{3m}\sqrt{t_{\ell'}} + \frac{1}{2}\sqrt{A}\,2^m\sqrt{t_\ell}\,,$$

and (7.13) follows from (7.15), provided A is large enough.

It remains to make the contrary assumption $\sqrt{t_\ell} > \sqrt{A}\,2^{2m}\sqrt{t_{\ell'}}$, which implies in particular that $t_\ell > t_{\ell'}$. Observe that

$$\left|D_{-t_{\ell'}}\,x^{(\ell')} - D_{-t_\ell}\,x^{(\ell)}\right| = \left|D_{-t_\ell}\big(D_{t_\ell - t_{\ell'}}\,x^{(\ell')} - x^{(\ell)}\big)\right| \simeq \left|D_{t_\ell - t_{\ell'}}\,x^{(\ell')} - D_s\,\tilde{x}^{(\ell)}\right|. \quad (7.16)$$

Both $x^{(\ell)}$ and $x^{(\ell')}$ are in $\mathcal{E}_\alpha$, so they satisfy

$$R(x^{(\ell)}) \geq \frac{3}{4}\,\log\frac{1}{\alpha} \geq \frac{3/4}{5/4}\,R(x^{(\ell')}) > \frac{1}{2}\,R(x^{(\ell')}) = \frac{1}{2}\,\beta\,.$$

Since also $t_\ell - t_{\ell'} \geq 0$, we can apply (6.4) to the last expression in (7.16) and get

$$\left|D_{-t_{\ell'}}\,x^{(\ell')} - D_{-t_\ell}\,x^{(\ell)}\right| \gtrsim |t_\ell - t_{\ell'} - s|\left|x^{(\ell')}\right| \gtrsim t_\ell\left|x^{(\ell')}\right|, \quad (7.17)$$

the second step because our assumption implies $t_\ell - t_{\ell'} \simeq t_\ell$, and $s < 0$. By means again of this assumption and then (7.11), we find

$$t_\ell\left|x^{(\ell')}\right| \gtrsim \sqrt{t_\ell}\left|x^{(\ell')}\right|\sqrt{A}\,2^{2m}\sqrt{t_{\ell'}} \gtrsim \sqrt{A}\,2^m\sqrt{t_\ell} \simeq \sqrt{A}\,2^m\,\left(\sqrt{t_\ell} + \sqrt{t_{\ell'}}\,\right).$$

With A large enough, (7.13) now follows from this and (7.17). Item (1) is verified.

Next, we will prove item (2). For $\ell' < \ell$, we can apply (6.3) and then (7.14), to get

$$\left|x^{(\ell')} - x^{(\ell)}\right| \gtrsim \left|x^{(\ell')} - \tilde{x}^{(\ell)}\right| \gtrsim A\,2^{3m}\sqrt{t_{\ell'}}\,.$$

Since $t_{\ell'} \geq \delta$, the distances $\left|x^{(\ell')} - x^{(\ell)}\right|$ are thus bounded below by a positive constant. This implies that the sequence $(x^{(\ell)})$ is finite, since all the $x^{(\ell)}$ are contained in the bounded set $\mathcal{E}_\alpha$. Thus the set $\mathcal{A}_{\ell+1}(\alpha)$ defined in (7.8) will be empty for some ℓ, say $\ell = \ell_0$, and the recursion stops. This implies item (2).

We are left with the proof of item (3). The forbidden zone $\mathcal{Z}^{(\ell)}$ is a tube as defined in (6.5), with $a = A\,2^{3m}\sqrt{t_\ell}$ and $\beta = R(x^{(\ell)})$. This value of β will be large since $x^{(\ell)} \in \mathcal{E}_\alpha$ and $\alpha < \alpha_0$ for some small α_0. Thus we can apply Lemma 6.1 to obtain

$$\gamma_{-\infty}(\mathcal{Z}^{(\ell)}) \lesssim \frac{\big(A2^{3m}\sqrt{t_\ell}\big)^{n-1}}{\sqrt{R(x^{(\ell)})}}\,\exp\left(R\left(x^{(\ell)}\right)\right).$$

We bound the exponential here by means of (7.12) and observe that $R(x^{(\ell)}) \simeq |x^{(\ell)}|^2$, getting

$$\gamma_{-\infty}\left(\mathcal{Z}^{(\ell)}\right) \lesssim \frac{1}{\alpha'\,|x^{(\ell)}|\,\sqrt{t_\ell}}\,(A2^{3m})^{n-1}\int_{\mathcal{B}^{(\ell)}} h(u)\,d\gamma_{-\infty}(u)\,.$$

As a consequence of this and (7.11), we obtain

$$\gamma_{-\infty}(\mathcal{Z}^{(\ell)}) \lesssim \frac{2^m}{\alpha'}\,\big(A2^{3m}\big)^{n-1}\int_{\mathcal{B}^{(\ell)}} h(u)\,d\gamma_{-\infty}(u) \lesssim \frac{2^{Cm}}{\alpha'}\int_{\mathcal{B}^{(\ell)}} h(u)\,d\gamma_{-\infty}(u)\,,$$

which proves item (3). This completes the proof of Lemma 7.2 and that of Proposition 7.1.

□

8 The Global Case for Large *t*

This section consists of the proof of the following result.

Proposition 8.1 *For all functions $f \in L^1(\gamma_{-\infty})$ with $\|f\|_{L^1(\gamma_{-\infty})} = 1$,*

$$\gamma_{-\infty}\left\{x\colon \left|\mathcal{H}_*^{+,G} f(x)\right| > \alpha\right\} \lesssim \frac{1}{\alpha\sqrt{\log(1/\alpha)}}, \qquad \alpha \in (0, 1/2).$$

In particular, the maximal operator $\mathcal{H}_*^{+,G}$ maps $L^1(\gamma_{-\infty})$ into $L^{1,\infty}(\gamma_{-\infty})$.

Observe first that the second statement follows from the first together with the observation at the end of Section 4.2. The same observation allows us to reduce the range of α in the first statement to $\alpha < e^{-2}$, in the proof that follows.

The proof of the first statement runs at first like that of Proposition 7.1, although now $t > 1$. In particular, $f \geq 0$ is normalized in $L^1(\gamma_{-\infty})$, and from Section 4.3 we know that it is enough to consider the values of $\mathcal{H}_*^{+,G} f$ in the set $\mathcal{E}_\alpha$ defined in (4.3). The annuli $\mathcal{T}_t^m$ are now

$$\mathcal{T}_t^m = \left\{(x,u) \in G\colon 2^{m-1} \leq |u - D_{-t}\, x| < 2^m\right\}, \qquad m \in \mathbb{N},$$

without the lower bound when $m = 0$.

We must show that for $0 < \alpha < e^{-2}$

$$\gamma_{-\infty}\left\{x \in \mathcal{E}_\alpha\colon \sup_{t>1} \int K_t^{UO}(x,u)\, f(u)\, d\gamma_{-\infty}(u) > \alpha\right\} \lesssim \frac{1}{\alpha\sqrt{\log(1/\alpha)}}. \tag{8.1}$$

Setting $T = |\mathrm{tr} B| > 0$, we obtain from (2.7) that for any $(x,u) \in \mathcal{T}_t^m$

$$K_t^{UO}(x,u) \lesssim e^{-T\,t}\, e^{-R(x)} \exp\left(-c2^{2m}\right).$$

Observing that $(x,u) \in \mathcal{T}_t^m$ implies $u \in B(D_{-t}\, x,\ 2^m)$, we set

$$K_t^{+,m}(x,u) = e^{-T\,t}\, e^{-R(x)}\, \mathbf{1}_{B(D_{-t}\, x,\, 2^m)}(u),$$

and conclude that for any $(x,u) \in G$ and $t > 1$

$$K_t^{UO}(x,u) \lesssim \sum_{m=0}^{\infty} \exp\left(-c2^{2m}\right) K_t^{+,m}(x,u).$$

Almost as in the preceding section, we introduce the operators

$$\mathcal{M}_{m,\alpha}^{+}\, h(x) = \mathbf{1}_{\mathcal{E}_\alpha}(x) \sup_{t>1} \int K_t^{+,m}(x,u)\, |h(u)|\, d\gamma_{-\infty}(u),$$

so that

$$\mathbf{1}_{\mathcal{E}_\alpha}(x) \sup_{t>1} \int K_t^{UO}(x,u)\, f(u)\, d\gamma_{-\infty}(u) \lesssim \sum_{m=0}^{\infty} \exp\left(-c2^{2m}\right) \mathcal{M}_{m,\alpha}^{+} f(x).$$

Lemma 8.2 *For $0 < \alpha < e^{-2}$ and $m \in \mathbb{N}$, the operator $\mathcal{M}_{m,\alpha}^{+}$ maps $L^1(\gamma_{-\infty})$ into $L^{1,\infty}(\gamma_{-\infty})$, with operator quasinorm at most $C\, 2^{Cm}/\sqrt{\log(1/\alpha)}$.*

In analogy with Lemma 7.2, this lemma implies (8.1) and thus also Proposition 8.1.

Proof With m and α fixed, we will estimate $\mathcal{M}^+_{m,\alpha}\, h$ for a function $h \ge 0$ which is normalized in $L^1(\gamma_{-\infty})$. But we prefer to work with the function $g(u) = (2\pi)^{\frac{n}{2}}(\det Q_\infty)^{\frac{1}{2}}\, e^{R(u)}\, h(u)$, which is normalized in $L^1(\mathbb{R}^n, du)$, and nonnegative. Then

$$\begin{aligned}\mathcal{M}^+_{m,\alpha}\, h(x) &= \mathbf{1}_{\mathcal{E}_\alpha}(x) \sup_{t>1} e^{-Tt}\, e^{-R(x)} \int_{B(D_{-t}\, x,\, 2^m)} h(u)\, d\gamma_{-\infty}(u)\\ &= \mathbf{1}_{\mathcal{E}_\alpha}(x) \sup_{t>1} e^{-Tt}\, e^{-R(x)} \int_{B(D_{-t}\, x,\, 2^m)} g(u)\, du.\end{aligned}$$

With $r > 0$ we write $g_r = g * \mathbf{1}_{B(0,r)}$, which is for each $r > 0$ a continuous function in $L^\infty \cap L^1(\mathbb{R}^n, du)$. Then

$$\mathcal{M}^+_{m,\alpha}\, h(x) = \mathbf{1}_{\mathcal{E}_\alpha}(x) \sup_{t>1} e^{-Tt}\, e^{-R(x)}\, g_{2^m}(D_{-t}\, x),$$

and as a supremum of continuous functions, $M^+_{m,\alpha}\, h$ is lower semicontinuous when restricted to $\mathcal{E}_\alpha$.

Let $\alpha' > 0$. Clearly, $\mathcal{M}^+_{m,\alpha}\, h(x) > \alpha'$ if and only if $x \in \mathcal{E}_\alpha$ and there exists a $t > 1$ such that $e^{-R(x)}\, e^{-Tt}\, g_{2^m}(D_{-t}\, x) > \alpha'$. We use polar coordinates, writing points $x \in \mathcal{E}_\alpha$ as $x = D_\varrho\, \tilde{x}$, where $\varrho \in \mathbb{R}$ and $\tilde{x}$ is on the ellipsoid $E_1 = \{y : R(y) = 1\}$. Notice that actually $\varrho > 0$ here, since $\alpha < e^{-2}$ implies that $R(x) > 1$ for $x \in \mathcal{E}_\alpha$.

Let $A_{\alpha'}$ be the set of points $\tilde{x} \in E_1$ for which there exists a $\varrho > 0$ such that $D_\varrho\, \tilde{x} \in \mathcal{E}_\alpha$ and $\mathcal{M}^+_{m,\alpha}\, f(D_\varrho\, \tilde{x}) > \alpha'$. The lower semicontinuity of $\mathcal{M}^+_{m,\alpha}\, f$ shows that $A_{\alpha'}$ is a relatively open subset of E_1. For $\tilde{x} \in A_{\alpha'}$ we define

$$\begin{aligned}\varrho(\tilde{x}) &= \sup\{\varrho > 0\colon \mathcal{M}^+_{m,\alpha}\, h(D_\varrho\, \tilde{x}) > \alpha'\}\\ &= \sup\{\varrho > 0\colon \exists\, t > 1 \quad \text{with} \quad \mathbf{1}_{\mathcal{E}_\alpha}(D_\varrho\, \tilde{x})\, e^{-Tt}\, e^{-R(D_\varrho\, \tilde{x})}\, g_{2^m}(D_{-t}\, D_\varrho\, \tilde{x}) > \alpha'\}.\end{aligned}$$

For notational convenience, we do not indicate that $\varrho(\tilde{x})$ also depends on α'.

We claim that $\varrho(\tilde{x})$ is a lower semicontinuous function on $A_{\alpha'}$, and thus measurable. Indeed, if $\varrho(\tilde{x}) > \varrho_0$ for some $\tilde{x} \in A_{\alpha'}$ and some $\varrho_0 > 0$, then there exists a $\varrho' > \varrho_0$ such that $\mathcal{M}^+_{m,\alpha}\, h(D_{\varrho'}\, \tilde{x}) > \alpha'$. By the lower semicontinuity of $\mathcal{M}^+_{m,\alpha}\, h$, the same inequality holds at $D_{\varrho'}\, \tilde{x}'$ for points $\tilde{x}' \in A_{\alpha'}$ in a small neighborhood of $\tilde{x}$. The claim follows.

For each $\tilde{x} \in A_{\alpha'}$ we can choose a sequence $\varrho_i \nearrow \varrho(\tilde{x})$ and another sequence (t_i) in $(1, \infty)$ satisfying

$$e^{-Tt_i}\, e^{-R(D_{\varrho_i}\, \tilde{x})}\, g_{2^m}(D_{-t_i}\, D_{\varrho_i}\, \tilde{x}) > \alpha'\,. \tag{8.2}$$

This inequality implies that $e^{R(D_{\varrho_i}\, \tilde{x})} < \|g\|_{L^\infty}/\alpha'$ and $e^{Tt_i} < \|g\|_{L^\infty}/\alpha'$, from which we conclude that $\varrho(\tilde{x}) < \infty$ and that the t_i stay bounded. A suitable subsequence of (t_i) will then converge, say to $t(\tilde{x}) \ge 1$.

From (8.2) in the limit we get

$$e^{-R(D_{\varrho(\tilde{x})}\, \tilde{x})}\, e^{-Tt(\tilde{x})}\, g_{2^m}(D_{-t(\tilde{x})}\, D_{\varrho(\tilde{x})}\, \tilde{x}) \ge \alpha'. \tag{8.3}$$

The level set $\{x : \mathcal{M}^+_{m,\alpha}\, h(x) > \alpha'\}$ is contained in the set

$$\mathcal{F}_{\alpha'} = \{D_\varrho\, \tilde{x}\colon\ \tilde{x} \in A_{\alpha'}\,,\quad 0 < \varrho \le \varrho(\tilde{x})\}\,.$$

To estimate the measure of this set, we switch to polar coordinates with $\beta = 1$ and use (6.2); thus

$$\gamma_{-\infty}(\mathcal{F}_{\alpha'}) = \int_{\mathcal{F}_{\alpha'}} e^{R(x)}\, dx \lesssim \int_{A_{\alpha'}} \int_0^{\varrho(\tilde{x})} e^{T\varrho}\, e^{R(D_\varrho\, \tilde{x})}\, d\varrho\, dS(\tilde{x})\,. \tag{8.4}$$

Here $dS(\tilde{x})$ is the area measure on E_1. We now estimate the inner integral above.

Lemma 8.3 *One has for* $\tilde{x} \in A_{\alpha'}$

$$\int_0^{\varrho(\tilde{x})} e^{T\varrho}\, e^{R(D_\varrho\, \tilde{x})}\, d\varrho \lesssim e^{T\varrho(\tilde{x})}\, e^{R(D_{\varrho(\tilde{x})}\, \tilde{x})}\, |D_{\varrho(\tilde{x})}\, \tilde{x}|^{-2}\,.$$

Proof When proving this, we can delete the factors $e^{T\varrho}$ and $e^{T\varrho(\tilde{x})}$.

If $0 < \varrho < \varrho(\tilde{x}) - A$ for some $A > 0$, (2.1) implies

$$R(D_\varrho\, \tilde{x}) = R(D_{\varrho-\varrho(\tilde{x})}\, D_{\varrho(\tilde{x})}\, \tilde{x}) \lesssim e^{-c(\varrho(\tilde{x})-\varrho)}\, R(D_{\varrho(\tilde{x})}\, \tilde{x}) \le e^{-cA}\, R(D_{\varrho(\tilde{x})}\, \tilde{x})\,.$$

Choosing $A = A(n, Q, B)$ large enough, we conclude that

$$R(D_\varrho\, \tilde{x}) \le \frac{1}{2}\, R(D_{\varrho(\tilde{x})}\, \tilde{x})\,, \qquad \varrho < \varrho(\tilde{x}) - A\,,$$

and thus

$$\begin{aligned}\int_0^{\varrho(\tilde{x})-A} e^{R(D_\varrho\, \tilde{x})}\, d\varrho &\lesssim \exp\left(\frac{1}{2}\, R(D_{\varrho(\tilde{x})}\, \tilde{x})\right) \varrho(\tilde{x}) \\ &= \exp\left(R(D_{\varrho(\tilde{x})}\, \tilde{x})\right) \left[\exp\left(-\frac{1}{4}\, R(D_{\varrho(\tilde{x})}\, \tilde{x})\right)\right]^2 \varrho(\tilde{x})\,.\end{aligned}$$

Here one of the factors $\exp\left(-R(D_{\varrho(\tilde{x})}\, \tilde{x})/4\right)$ takes care of $\varrho(\tilde{x})$ in view of (2.1), and the other is no larger than $C|D_{\varrho(\tilde{x})}\, \tilde{x}|^{-2}$. Hence, this part of the integral in the lemma satisfies the desired estimate.

For $\varrho(\tilde{x}) - A < \varrho < \varrho(\tilde{x})$ we use (6.1), the fact that the quantity $|D_s\, \tilde{x}|$ is increasing in s and finally (2.1), to get

$$\begin{aligned}R(D_{\varrho(\tilde{x})}\, \tilde{x}) - R(D_\varrho\, \tilde{x}) &\simeq \int_\varrho^{\varrho(\tilde{x})} |D_s\, \tilde{x}|^2\, ds \ge (\varrho(\tilde{x}) - \varrho)\, |D_\varrho\, \tilde{x}|^2 = (\varrho(\tilde{x}) - \varrho)\, |D_{\varrho-\varrho(\tilde{x})}\, D_{\varrho(\tilde{x})}\, \tilde{x}|^2 \\ &\gtrsim (\varrho(\tilde{x}) - \varrho)\, e^{-CA}\, |D_{\varrho(\tilde{x})}\, \tilde{x}|^2 \simeq (\varrho(\tilde{x}) - \varrho)\, |D_{\varrho(\tilde{x})}\, \tilde{x}|^2\,.\end{aligned}$$

Thus we can write

$$\begin{aligned}\int_{\varrho(\tilde{x})-A}^{\varrho(\tilde{x})} e^{R(D_\varrho\, \tilde{x})}\, d\varrho &\lesssim e^{R(D_{\varrho(\tilde{x})}\, \tilde{x})} \int_{\varrho(\tilde{x})-A}^{\varrho(\tilde{x})} e^{-c(\varrho(\tilde{x})-\varrho)\, |D_{\varrho(\tilde{x})}\, \tilde{x}|^2}\, d\varrho \\ &= e^{R(D_{\varrho(\tilde{x})}\, \tilde{x})} \int_0^A e^{-c\sigma\, |D_{\varrho(\tilde{x})}\, \tilde{x}|^2}\, d\sigma \lesssim e^{R(D_{\varrho(\tilde{x})}\, \tilde{x})}\, |D_{\varrho(\tilde{x})}\, \tilde{x}|^{-2}\,.\end{aligned}$$

The proof of Lemma 8.3 is complete. □

To continue the proof of Lemma 8.2, we now insert in (8.4) the expression from Lemma 8.3, and obtain

$$\gamma_{-\infty}(\mathcal{F}_{\alpha'}) \lesssim \int_{A_{\alpha'}} e^{T\varrho(\tilde{x})}\, e^{R(D_{\varrho(\tilde{x})}\, \tilde{x})}\, |D_{\varrho(\tilde{x})}\, \tilde{x}|^{-2}\, dS(\tilde{x})\,. \tag{8.5}$$

Then we apply the upper estimate of $e^{R(D_{\varrho(\tilde{x})}\, \tilde{x})}$ that follows from (8.3), to conclude that

$$e^{T\varrho(\tilde{x})}\, e^{R(D_{\varrho(\tilde{x})}\, \tilde{x})}\, |D_{\varrho(\tilde{x})}\, \tilde{x}|^{-2} \lesssim \frac{1}{\alpha'}\, \psi(\tilde{x}) \tag{8.6}$$

where

$$\psi(\tilde{x}) = e^{T\varrho(\tilde{x})}\, e^{-Tt(\tilde{x})}\, |D_{\varrho(\tilde{x})}\, \tilde{x}|^{-2}\, g_{2^m}(D_{-t(\tilde{x})} D_{\varrho(\tilde{x})}\, \tilde{x})\,.$$

Let $|\sigma| \le 1 \wedge |D_{-t(\tilde{x})}\, D_{\varrho(\tilde{x})}\, \tilde{x}|^{-1}$. Then (2.2) leads to

$$|D_\sigma\, D_{-t(\tilde{x})}\, D_{\varrho(\tilde{x})}\, \tilde{x} - D_{-t(\tilde{x})}\, D_{\varrho(\tilde{x})}\, \tilde{x}| \lesssim |\sigma|\, |D_{-t(\tilde{x})}\, D_{\varrho(\tilde{x})}\, \tilde{x}| \le 1,$$

and the inclusion $B(D_{-t(\tilde{x})}\, D_{\varrho(\tilde{x})}\, \tilde{x}, 2^m) \subset B(D_\sigma\, D_{-t(\tilde{x})}\, D_{\varrho(\tilde{x})}\, \tilde{x}, 2^m + C)$ yields

$$g_{2^m}(D_{-t(\tilde{x})}\, D_{\varrho(\tilde{x})}\, \tilde{x}) \le g_{2^m+C}(D_\sigma\, D_{-t(\tilde{x})}\, D_{\varrho(\tilde{x})}\, \tilde{x}).$$

Thus we can write for $\tilde{x} \in A_{\alpha'}$

$$\psi(\tilde{x}) \le |D_{\varrho(\tilde{x})}\, \tilde{x}|^{-2}\, e^{-T\sigma}\, e^{T(\sigma - t(\tilde{x}) + \varrho(\tilde{x}))}\, g_{2^m+C}(D_{\sigma - t(\tilde{x}) + \varrho(\tilde{x})}\, \tilde{x}).$$

Here we replace $e^{-T\sigma}$ by C and then take the mean of both sides with respect to σ in the indicated interval, to get

$$\begin{aligned}\psi(\tilde{x}) &\lesssim 1 \vee |D_{-t(\tilde{x})}\, D_{\varrho(\tilde{x})}\, \tilde{x}| \\ &\quad\times \int_{|\sigma| \le 1 \wedge |D_{-t(\tilde{x})} D_{\varrho(\tilde{x})}\, \tilde{x}|^{-1}} |D_{\varrho(\tilde{x})}\, \tilde{x}|^{-2}\, e^{T(\sigma - t(\tilde{x}) + \varrho(\tilde{x}))}\, g_{2^m+C}(D_{\sigma - t(\tilde{x}) + \varrho(\tilde{x})}\, \tilde{x})\, d\sigma \\ &\lesssim |D_{\varrho(\tilde{x})}\, \tilde{x}|^{-1} \int_{|\sigma| \le 1 \wedge |D_{\varrho(\tilde{x})}\, \tilde{x}|^{-1}} e^{T(\sigma - t(\tilde{x}) + \varrho(\tilde{x}))}\, g_{2^m+C}(D_{\sigma - t(\tilde{x}) + \varrho(\tilde{x})}\, \tilde{x})\, d\sigma,\end{aligned}$$

where we used the fact that $\varrho(\tilde{x}) > 0$ and $\tilde{x} \in \mathcal{E}_1$ so that $1 \vee |D_{-t(\tilde{x})}\, D_{\varrho(\tilde{x})}\, \tilde{x}| \le |D_{\varrho(\tilde{x})}\, \tilde{x}|$. Since $D_{\varrho(\tilde{x})}\, \tilde{x} \in \mathcal{E}_\alpha$, we have $|D_{\varrho(\tilde{x})}\, \tilde{x}|^{-1} \lesssim 1/\sqrt{\log(1/\alpha)}$. By replacing σ by $s = \sigma - t(\tilde{x}) + \varrho(\tilde{x})$ and extending the integral to all of $\mathbb{R}$, we obtain

$$\psi(\tilde{x}) \lesssim \frac{1}{\sqrt{\log(1/\alpha)}} \int_{\mathbb{R}} e^{Ts}\, g_{2^m+C}(D_s\, \tilde{x})\, ds.$$

Inserting this in (8.5) combined with (8.6), we find that

$$\gamma_{-\infty}(\mathcal{F}_{\alpha'}) \lesssim \frac{1}{\alpha'} \frac{1}{\sqrt{\log(1/\alpha)}} \int_{A'_\alpha} \int_{\mathbb{R}} e^{Ts}\, g_{2^m+C}(D_s\, \tilde{x})\, ds\, dS(\tilde{x}).$$

Now we use (6.2) to go back to Lebesgue measure dx with $x = D_s\, \tilde{x}$. Since $|\tilde{x}| \simeq 1$, this yields

$$\begin{aligned}\gamma_{-\infty}(\mathcal{F}_{\alpha'}) &\lesssim \frac{1}{\alpha'} \frac{1}{\sqrt{\log(1/\alpha)}} \int_{\{x = D_s\, \tilde{x}:\, s \in \mathbb{R},\, \tilde{x} \in A_{\alpha'}\}} g_{2^m+C}(x)\, dx \\ &\le \frac{1}{\alpha'} \frac{1}{\sqrt{\log(1/\alpha)}} \int_{\mathbb{R}^n} g_{2^m+C}(x)\, dx \\ &\lesssim \frac{1}{\alpha'} \frac{1}{\sqrt{\log(1/\alpha)}} \left(2^m + C\right)^n \|g\|_{L^1(\mathbb{R}^n)} \simeq \frac{1}{\alpha'} \frac{2^{mn}}{\sqrt{\log(1/\alpha)}}.\end{aligned}$$

This ends the proof of Lemma 8.2 and that of Proposition 8.1.

9 Completion of Proofs

Theorem 3.1 is an immediate consequence of Propositions 5.1, 7.1 and 8.1. Further, Propositions 5.1 and 8.1 together imply the positive part of Theorem 3.2. It remains for us to prove the sharpness assertion in Theorem 3.2.

To this end, we take a point z with $R(z)$ large. Let B_1 denote the ball $B(z, 1)$ and set $B_2 = B(D_{-2}\, z, 1)$. If $x \in B_1$ and $u \in B_2$, we will have

$$|D_{-2}\, x - u| \le |D_{-2}\, x - D_{-2}\, z| + |D_{-2}\, z - u| = |D_{-2}\, (x - z)| + |D_{-2}\, z - u| \lesssim 2,$$

so that (2.7) yields $K_2^{UO}(x, u) \simeq e^{-R(x)}$. With $f = \mathbf{1}_{B_2}/\gamma_{-\infty}(B_2)$ it follows that $\mathcal{H}_2^{UO} f(x) \simeq e^{-R(x)}$ for $x \in B_1$.

Then $\mathcal{H}_2^{UO} f(x) \gtrsim e^{-R(z)}$ if x is in the set $B^* = \{x \in B_1 : R(x) < R(z)\}$. For any small $\alpha > 0$ we can choose z satisfying $R(z) = \log(1/\alpha) - A$ for a suitable, large constant $A = A(n, Q, B)$, and conclude that $\mathcal{H}_2^{UO} f(x) > \alpha$ for $x \in B^*$. With our polar coordinates, one can verify that

$$\gamma_{-\infty}(B^*) \gtrsim \frac{e^{R(z)}}{\sqrt{R(z)}} \simeq \frac{1}{\alpha\, \sqrt{\log(1/\alpha)}}$$

if α is small enough. This means that (3.4) implies (3.5).

Theorem 3.2 is completely proved.

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