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ORIGINAL PAPER

Boundary Toeplitz type operators in weighted harmonic Sobolev spaces

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Abstract

We consider the space $\mathrm{D}^{\alpha}_{n}(\Omega)$ consisting of functions $u(x)$, harmonic in a bounded domain $\Omega \subset \mathbb{R}^{d+1}$ with smooth boundary Γ, satisfying $|\nabla^n u(x)|^2 \rho(x)^{\alpha} \in L_1(\Omega)$, $\alpha > -1$, where $\rho(x)$ is the distance from x to the boundary. For a Borel measure μ on Γ and a weight function V, μ-measurable, we study the operator defined by means of the quadratic form $\boldsymbol{\mu}_V[u] = \int V(x)|\gamma u(x)|^2 \mu(dx)$, where γ is, properly defined, operator of restriction of functions $u \in \mathrm{D}^{\alpha}_{n}(\Omega)$ to Γ. Main interest is directed to the case of a singular measure μ possessing some Ahlfors regularity properties. For such operators, we establish two-sided estimates for singular values, and, under some geometrical conditions, Weyl type eigenvalue asymptotics.

Mathematics Subject Classification 47B35

1 Introduction

Some ten years ago, Nikolai and I came up with an approach to the study of Toeplitz operators in Bergman type spaces with strongly singular symbols. The approach was based on not considering the usual Bergman projection but on using bounded sesquilinear forms on the Bergman space, generated by a symbol, possibly rather singular, belonging to a certain class of distributions. Such definition proved to be rather productive, enabling one, in particular, to define and investigate Toeplitz operators in the setting when, for a given Bergman type space of smooth functions, there is no natural enveloping Hilbert space and no Bergman projection; however, natural candidates for being a natural generalization of Toeplitz sesquilinear form still exist and can be effectively studied. One of the examples was considering sesquilinear forms

$$\mathbf{a}_{\Phi}[u, v] \equiv (\Phi, u\bar{v}), \tag{1.1}$$

Dedicated to the memory of my dear friend Nikolai Vasilevski.

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$\Phi \in \mathscr{S}'(\mathbb{R}^d)$, on the Agmon-Hörmander (Herglotz) space of solutions of the Helmholtz equation in $\mathbb{R}^d$, see [20]. Similarly, we had studied sesquilinear forms (1.1) in the Fock space $\mathcal{F}(\mathbb{C}^1)$, see [19], and in the usual Bergman space $\mathcal{B}(\mathbb{D})$ of analytic functions on the disk $\mathbb{D}$, see [15]. In the latter case, a special reasoning caused by the presence of the boundary was required. When $\Phi \in \mathscr{E}'(\mathbb{D})$, this means, when Φ is a distribution with compact support in $\mathbb{D}$, such sesquiliniear form is easily shown to be bounded in $\mathcal{B}(\mathbb{D})$, the corresponding Toeplitz operator is compact and even has exponentially decaying singular values. On the other hand, if the distribution Φ is only known to belong to a much wider space $\mathscr{D}'(\mathbb{D})$, even the definition of the form (1.1) is not automatic, and in [15], a class of distributions was described such that with properly defined limit procedure, the form (1.1) is bounded in the Bergman space and thus determines a bounded operator. The condition for this boundedness is expressed in [15] in terms of the decay, in a certain sense, of the distribution Φ while approaching the boundary. If this decay is sufficiently fast, the operator is shown to be compact and even in the trace class, see [21]. Although written explicitly for the Bergman space on the disk only, this approach can be easily extended to higher dimensions and to weighted spaces—an example is presented in [8]—but generally we did not consider as exciting writing down all these generalizations in detail, meaning to leave it to some future students. Note only that our distributional approach proved to be rather efficient when studying Toeplitz operators in poly-Bergman and poly-Fock spaces, see [21].

In the particular case when $\Phi = \mu$ is, in fact, a finite Borel measure in $\mathbb{D}$, the formula (1.1) becomes

$$\boldsymbol{\mu}[u, v] = \int u\bar{v}d\mu, \tag{1.2}$$

thus defining the 'Toeplitz operator' with measure-symbol, well studied, see, e.g., [26]. This operator can be given the more usual form

$$u \mapsto \mathbf{T}_\mu u = \boldsymbol{\Pi}\mu u,$$

where $\boldsymbol{\Pi}$ is the Bergman projection, initially defined as the orthogonal projection $\boldsymbol{\Pi} : L_2 \to \mathcal{B}$, and then extended to other spaces, using the analytic structure of $\boldsymbol{\Pi}$ as an integral operator with smooth kernel $\boldsymbol{\Pi}(x, y)$, with known boundary singularities, see [7]. A general description of our sesquilinear forms approach is presented in [16]. Further, we will mostly talk here about operators defined by quadratic form; this is equivalent to using sesquilinear forms but more convenient notationally; to (1.2), there corresponds the quadratic form

$$\boldsymbol{\mu}[u] = \int |u|^2 d\mu. \tag{1.3}$$

When looking at (1.3), a question arises, whether it is possible to define in a reasonable way the quadratic form with measure μ supported only on the boundary of the disk, thus breaking the above-mentioned boundary decay conditions. One might try to define first the form (1.3) on functions in $\mathcal{B} \cap C(\bar{\mathbb{D}})$ and then extending by continuity, provided the latter is established. An unfortunate obstacle here is the fact that (1.3) involves boundary values of functions in $\mathcal{B}$, and it is known that such boundary values belong only to the negative order Sobolev space $H^{-\frac{1}{2}}(\partial\mathbb{D})$, and for any nontrivial measure μ on the boundary, the sesquilinear form (1.3) is not bounded in $H^{-\frac{1}{2}}(\partial\mathbb{D})$. The game cannot be saved by considering forms in the weighted Bergman space $\mathcal{B}^{\alpha}(\mathbb{D})$, $\alpha > \mathbf{-1}$ since for this space, the boundary values belong, again, to a negative order Sobolev space $\mathrm{H}^{\frac{-1-\alpha}{2}}(\partial\mathbb{D})$.

A possible modification of the problem, inspired by (1.1), consists of considering a smaller space instead of $\mathcal{B}^{\alpha}(\mathbb{D})$, for which the boundary values are better than for $\mathcal{B}^{\alpha}(\mathbb{D})$, say, they belong to some Sobolev space of a positive order n. One of natural candidates here is the weighted harmonic Sobolev space, see, e.g., [5, 9, 11, 12]:

$$\mathrm{H}_n^{\alpha}(\mathbb{D}) = \left\{u(x) : \Delta u = 0, (1 - |x|^2)^{\alpha}|\nabla^n u|^2 \in L^1(\mathbb{D})\right\}, \alpha > -1, \tag{1.4}$$

or its analytic counterpart. Another way is to replace the measure μ on $\partial\mathbb{D}$, which is a linear functional on $C(\partial\mathbb{D})$, by some regularization μ_s, a functional on boundary values of $u \in \mathcal{B}^{\alpha}(\mathbb{D})$, such that the regularized quadratic form

$$\boldsymbol{\mu}_s[u] = \langle \mu_s, |u|^2 \rangle,$$

is bounded on the Bergman space $\mathcal{B}^{\alpha}(\mathbb{D})$. Such regularization is considered briefly in Sect. 6.

In this note, we develop the first approach; in the end, we show that the second approach is equivalent. Following [15, 19, 20], we do not need an explicit expression for the Bergman kernel but use general, but still simple, facts about properties of solutions of boundary value problems. This enables us not to restrict ourselves to the spaces of harmonic functions on the disk (or ball), but to extend the considerations to spaces of harmonic functions on bounded domains in any dimension with smooth boundary.

Here, we obtain singular value estimates for Toeplitz type operators defined in weighted harmonic Sobolev spaces by boundary quadratic forms associated with measures, possibly, singular, supported on the boundary of the domain. With an absolutely continuous measure and with trivial weight, this type of spectral problems was considered in 70-s, in [2, 3]; for a particular case of a weighted Sobolev space, again, with absolutely continuous measure on the boundary, some results, for a degenerate elliptic operator, were announced without proof in [25]. Later, in a more general setting, for (non-weighted) Sobolev type spaces associated with the space of solutions of an elliptic equation, again, for an absolutely continuous measure on the boundary, order sharp spectral estimates and asymptotics for Steklov-type problems were obtained by Suslina [22, 23]. Our considerations have these papers as their moral and technical predecessors. Our general construction follows the one developed there, while the results posses the following novelties. First, we consider weighted Sobolev spaces of solutions, with the order of the Sobolev space not related to the order of the operator. Second, we can consider quadratic forms generated by singular measures, possessing Ahlfors regularity; therefore, the resulting decay order of eigenvalues is determined not only by the order of the equation but also by the differential and weight orders of the Sobolev space and, what is most important, on the class of the measure on the boundary. These eigenvalue estimates are order sharp, and under some geometrical conditions, we even find the asymptotics.

For traditional Toeplitz operators in Bergman spaces, generated by a measure supported in the domain, spectral properties have been studied in [4, 10, 11, 13, 26]. In these papers, conditions were found for the operator to belong to the Schatten classes $\mathfrak{S}_p$ (consisting of operators $\mathbf{T}$ possessing singular values $s_j(\mathbf{T})$ satisfying $\sum s_j(\mathbf{T})^p < \infty$). We will see now that for singular measures on the boundary, more natural is the characterization of operators belonging to *weak Schatten classes* $\boldsymbol{\Sigma}_p$ consisting of operators $\mathbf{T}$ with singular numbers satisfying $s_j(\mathbf{T}) = O(j^{-\frac{1}{p}})$.

Our results, at the expense of certain complication of calculations, can be extended to spaces of solutions of rather general elliptic equations and systems, including spaces of holomorphic functions. We mean to return to this topic on a later occasion.

2 Harmonic Sobolev spaces

Let $\Omega \subset \mathbb{R}^{d+1}$ be a bounded domain with C^∞-smooth boundary $\Gamma = \partial\Omega$. For a point $x \in \Omega$, we denote by $\rho(x)$ the distance from the point x to Γ. This is a smooth function in some neighborhood of Γ, and we can change it outside this neighborhood so that it is smooth everywhere in Ω. For $\alpha > -1$, we introduce the harmonic Bergman space $\mathrm{D}^\alpha(\Omega)$ consisting of functions $u(x)$, $x \in \Omega$, $\Delta u = 0$, satisfying $I_\alpha[u] := \int_\Omega \rho(x)^\alpha |u(x)|^2 dx < \infty$.

It is known that $C(\bar{\Omega})$ is dense in $\mathrm{D}^\alpha(\Omega)$. Functions in $\mathrm{D}^\alpha(\Omega)$ possess boundary values on Γ in the sense of embedding theorems. Namely, having a Hilbert space $\mathcal{G} \subset \mathscr{D}'(\Gamma)$ of distributions, suppose that for all functions $u \in C(\bar{\Omega}) \cap \mathrm{D}^\alpha(\Omega)$, their boundary values γu on Γ satisfy

$$\|\gamma u\|^2_{\mathcal{G}} \le C I_\alpha[u].$$

Then the mapping $\gamma : C(\bar{\Omega}) \cap \mathrm{D}^\alpha(\Omega) \to \mathcal{G}$ extends by continuity to the *boundary mapping* $\mathrm{D}^\alpha(\Omega) \to \mathcal{G}$, which we also denote by γ. By the standard localization reasoning, one can easily show that the mapping properties of γ are the same as for Ω being the (say, unit) ball $\mathbb{B}^d$ or the half-space. Then, using the spherical functions expansion or the Fourier transform, it is easy to show that the mapping γ is bounded as acting from D^α to the space of distributions $\mathrm{H}^{-\frac{1+\alpha}{2}}(\Gamma)$.

Next, we consider the weighted harmonic Sobolev spaces. For $\alpha > -1$ and integer $n > 0$, we denote by $\mathrm{D}^\alpha_n(\Omega)$ the space of harmonic functions $u(x)$ in Ω satisfying

$$|u|^2_{n,\alpha} := \int \rho(x)^\alpha |\nabla^n u(x)|^2 dx + [u]^2_n < \infty. \tag{2.1}$$

Here, $|\nabla^n u|^2$ in (2.1) is defined by taking the gradient to some power formally,

$$|\nabla^n u(x)|^2 = \sum_{|\nu|=n} \binom{n}{\nu}^2 |D^\nu u(x)|^2.$$

One might of course have chosen some other expressions for the norm in the Sobolev space, producing an equivalent norm. This would lead to the same order spectral estimates but would change slightly the expression for the coefficient in the spectral asymptotics. The second term in (2.1), $[u]_n^2 = \sum_{|\nu|\le n} |D^\nu u(x_0)|^2$ for some $x_0 \in \Omega$, is, as usual, added in order to dispose of the degeneration of the leading term on degree n polynomials; the term $[u]_n^2$ is compact with respect to the norm (2.1) and therefore, does not influence spectral estimates and asymptotics of operators to follow.

As before, the boundary mapping γ can be extended from $C^\infty(\bar{\Omega})$ to D_n^α as acting to a certain Hilbert space of distributions on Γ. Again, a simple calculation involving spherical functions, (or, by straightening the boundary, using the Fourier transform) shows that γ extends by continuity to the bounded mapping from $\mathrm{D}_n^\alpha(\Omega)$ to the Sobolev space $\mathrm{H}^{n-\frac{1+\alpha}{2}}(\Gamma)$. This, in particular, means that for $\alpha < 2n-1$, the boundary mapping γ maps the harmonic Sobolev space $\mathrm{D}_n^\alpha(\Omega)$ into the boundary Sobolev space of positive order $\beta = n - \frac{1+\alpha}{2}$.

An important property is that the mapping γ is invertible as acting in these spaces. Consider the Poisson operator $\mathcal{P}$ associating the harmonic function $u(x)$ with its Dirichlet boundary data. This is an integral operator, by the Boutet de Monvel calculus, mapping $C^\infty(\Gamma)$ to $C^\infty(\bar{\Omega})$. Due to the elliptic regularity property, $\mathcal{P}$ maps the boundary Sobolev space $\mathrm{H}^s(\Gamma)$ to $\mathrm{H}^{s+\frac{1}{2}}(\Omega)$ for any $s \in \mathbb{R}$. In its turn, generally, for solutions of an elliptic equation, the following elliptic regularity estimate is valid, see, e.g., [22], Theorem 3.3, or, in a little bit less generality, [5].

$$\int_\Omega \rho(x)^{2k} |\nabla_{p+k} u(x)|^2 dx \le C\|u\|^2_{\mathrm{H}^p(\Omega)}, \tag{2.2}$$

$u \in \mathrm{H}^p(\Omega)$, for all $u(x)$ satisfying some elliptic equation in Ω, as soon as $p, k \ge 0$ and $p + k$ is an integer (here the constant C, of course, depends on the elliptic equation in question, the domain Ω and the parameters p, k). We apply this result to harmonic functions $u(x)$, with $p = n - \frac{\alpha}{2}$ and $k = \frac{\alpha}{2}$. This gives us the inclusion $\mathrm{H}^{n-\alpha/2}(\Omega) \subset \mathrm{D}_n^\alpha(\Omega)$, therefore for $\alpha > -1$, the Poisson operator $\mathcal{P}$ maps $\mathrm{H}^\beta(\Gamma)$ into $\mathrm{D}_n^\alpha(\Omega)$, $\alpha = 2(n-\beta) - 1$. This reasoning takes care of positive α. For negative $\alpha > -1$, the inequality (2.2) follows from the Hardy inequality.

3 Boundary quadratic forms

3.1 Regular measures

In [15, 19, 20], together with Nikolai, we considered operators defined by means of (now, quadratic) forms $\mathbf{a}[u] = \int |D^\beta u(x)|^2 d\mu$ in the analytical Bergman spaces, with measure μ on Ω. The conditions imposed on the measure μ required its vanishing, in a certain sense, when approaching the boundary and thus excluded measures supported on the boundary. Here we consider this excluded case. We demonstrate that some new effects arise, in particular, we show that measures, singular with respect to the natural measure on Γ can, under some regularity conditions, generate compact operators.

We recall the definition of Ahlfors-regular measures (see details, e.g., in [6]). Let $\mathcal{E} \subset \mathbb{R}^{d+1}$ be a compact set and μ be a finite Borel measure supported on $\mathcal{E}$. Measure μ is called Ahlfors σ-regular if for any $x \in \mathcal{E}$ and $r \in (0, \operatorname{diam} \mathcal{E}]$, for the ball $\mathbf{B}(x, r)$,

$$A_- r^\sigma \le \mu(\mathbf{B}(x, r) \cap \mathcal{E}) \le A_+ r^\sigma, \tag{3.1}$$

with some $A_- > 0$, $A_+ < \infty$. It is known, see [6], that a σ-regular measure on a subset in the Euclidean space is equivalent to the σ-dimensional Hausdorff measure. In particular, a δ-regular measure on a δ-dimensional Lipschitz surface $\Sigma \subset \mathbb{R}^{d+1}$ is equivalent to the induced surface measure; on the other hand, a measure satisfying (3.1) with $\sigma = 0$ is necessarily discrete; such measures are excluded further on. The class of measures satisfying (3.1) is denoted by $\mathcal{A}^\sigma$. If, for a measure μ, only the upper (lower) estimate in (3.1) holds, we call it upper (lower) σ-regular, and the corresponding class of measures is denoted $\mathcal{A}^\sigma_+$, resp., $\mathcal{A}^\sigma_-$.

3.2 Quadratic forms

Let μ be a Borel measure on the smooth boundary Γ of the compact domain $\Omega \subset \mathbb{R}^{d+1}$. With measure μ and a μ-measurable function V (called density), we associate the quadratic form

$$\boldsymbol{\mu}_V[u] = \int V(x)|\gamma u(x)|^2 \mu(dx),$$

defined initially on functions $u \in C^\infty(\bar{\Omega})$. If this form is bounded in $\mathrm{D}^\alpha_n(\Omega)$-norm, it defines an operator which we denote by $\mathbf{T}^{(n,\alpha)}_{V\mu}$ (for α, n fixed, these parameters may be omitted in this notation as long as no confusion arises.) As shown above, the $\mathrm{D}^\alpha_n(\Omega)$-norm is equivalent to the $\mathrm{H}^\beta(\Gamma)$-norm, $\beta = n - \frac{1+\alpha}{2}$, so, as it concerns spectral estimates, we can consider the quadratic form

$$\boldsymbol{\mu}_V[f] = \int V(x)|f(x)|^2 \mu(dx),$$

$f \in \mathrm{H}^\beta(\Gamma)$; for the latter space, we fix the norm inherited from (2.1) (as explained above, it is equivalent to the standard norm in this Sobolev space). We use the notation $\mathbf{S}_{V\mu}$ for the corresponding operator in $\mathrm{H}^\beta(\Gamma)$.

4 Eigenvalue estimates

4.1 Upper estimates

Our first main result is the following theorem.

Theorem 4.1 *Let μ be a measure on Γ.*

(1) Let $d > 2\beta$, $\mu \in \mathcal{A}_+^\sigma$, $\sigma > d - 2\beta$. Let $V \in L_{\theta,\mu}$, $\theta = \frac{\sigma}{2\beta-d+\sigma}$. Then $\mathbf{T}_{V\mu} \in \mathbf{\Sigma}_\theta$,

$$\limsup_{j\to\infty} j^{\frac{1}{\theta}} s_j(\mathbf{T}_{V\mu}) \le C\|V\|_{L_{\theta,\mu}}. \tag{4.1}$$

(2) Let $d = 2\beta$, $\mu \in \mathcal{A}^\sigma$, $\sigma > 0$, $\theta = 1$, $V \in L_{\Psi,\mu}$, where $L_{\Psi,\mu}$ is the Orlicz space corresponding to the function $\Psi(t) = (t+1)\log(t+1) - t$: $\int \Psi(|V(x)|)\mu(dx) < \infty$, then $\mathbf{T}_{V\mu} \in \mathbf{\Sigma}_1$,

$$\limsup_{j\to\infty} j s_j(\mathbf{T}_{V\mu}) \le C\|V\|_{L_{\Psi,\mu}}. \tag{4.2}$$

(3) Let $2\beta > d$, $\sigma > 0$, $\mu \in \mathcal{A}_-^\sigma$, $V \in L_{1,\mu}$, $\theta = \frac{\sigma}{2\beta-d+\sigma}$. Then $\mathbf{T}_{V\mu} \in \mathbf{\Sigma}_\theta$,

$$\limsup_{j\to\infty} j^{\frac{1}{\theta}} s_j(\mathbf{T}_{V\mu}) \le C\|V\|_{L_{1,\mu}}. \tag{4.3}$$

The constant in (4.1)*,* (4.2)*,* (4.3) *does not depend on the density V.*

Proof As explained in Sect. 2, the norm in the weighted harmonic Sobolev space $\mathrm{D}_n^\alpha(\Omega)$ is equivalent to the Sobolev norm in the space $\mathrm{H}^\beta(\Gamma)$, the trace operator γ and the Poisson operator $\mathcal{P}$ realizing this equivalence; thus, the singular values of the operator $\mathbf{T}_{V\mu}$ are estimated, both from above and from below, by the singular values of the operator $\mathbf{S}_{V\mu}$. After the usual localization, we arrive at a finite sum of operators defined by quadratic forms of the same kind, in the Sobolev space $\mathrm{H}^\beta(\mathbb{R}^d)$, with compactly supported V, μ.

To the study of the spectrum of such operators, we apply results of the papers [1, 18]. In these papers, the case of a real-valued function V is considered and for operators of the form $\mathbf{S}_{V\mu}$, eigenvalue estimates are obtained. To pass to the case of a general complex-valued function V, we consider separately the real and imaginary components V_R, V_I of the density V and the operators corresponding to $\mathbf{S}_{V_R\mu}$, $\mathbf{S}_{V_I\mu}$. Then we derive the singular numbers estimates using Ky Fan's inequalities.

So, in [1], we considered the 'critical' case, namely, where the 'differential order' β of the space and the dimension d are related as $2\beta = d$. Here, the measure is required to belong to the class $\mathcal{A}^\sigma$, this means, satisfy the two-sided estimate. The density V should belong to the Orlicz space, and this means to be just a tiny little bit better than $L_{1,\mu}$. The order of the spectral estimate is $\theta = 1$, thus not depending explicitly on σ. The resulting singular numbers estimate has, according to [1], the form

$$\limsup j s_j(\mathbf{S}_{V\mu}) \le C\|V\|_{\Psi,\mu}, \tag{4.4}$$

where the norm in (4.4) is the Orlicz norm associated with Ψ.

The cases 1 and 3 in the theorem, where $2\beta < d$ or $2\beta > d$, considered in [18], are called, respectively, the subcritical and supercritical ones. There, the conditions imposed on the measure differ for $2\beta < d$ and for $2\beta > d$. In the former case, the main theorem in [18] requires $\mu \in \mathcal{A}^{\sigma}_{+}$, while in the latter case, it requires the condition $\mu \in \mathcal{A}^{\sigma}_{-}$. In this case, the order of eigenvalue estimate proved in [18], $\theta = \frac{\sigma}{2\beta - d + \sigma}$ gives

$$\limsup_{j\to\infty} j^{\frac{1}{\theta}} s_j(\mathbf{S}_{V\mu}) \le C\|V\|_{L_{\theta',\mu}}, \tag{4.5}$$

where $\theta' = \theta$ for $2\beta < d$ and $\theta' = 1$ for $2\beta > d$. □

4.2 Sharpness. Lower estimates

A natural question arising about the above results concerns the sharpness of the singular numbers decay order. The result to follow, concerning general measures $\mu \in \mathcal{A}^{\sigma}$, requires the density V to be sign-definite and gives a lower estimate for the eigenvalues of the operator $\mathbf{T}_{V\mu}$.

Theorem 4.2 *Let $\mu \in \mathcal{A}^{\sigma}$, σ satisfies one of the conditions in Theorem 4.1 and $V > c > 0$ μ-almost everywhere. Then*

$$\liminf_{j\to\infty} j^{\frac{1}{\theta}} s_j(\mathbf{T}_{V\mu}) > 0. \tag{4.6}$$

Proof Again, we use the equivalence of the weighted harmonic Sobolev spaces and Sobolev spaces on the boundary. It is sufficient to establish the estimate of type (4.6) for eigenvalues of the operator $\mathbf{S}_{V\mu}$. The latter estimate was proved in [1], Theorem 8.1, in the critical case. As for the sub-critical and super-critical cases, lower spectral estimates for operators of the form $\mathbf{S}_{V\mu}$, with positive weight function V were established in the book [24], Theorem 28.6. An explanation of the relation of the terms used in [24] and our terms is presented in [17, 18]. □

Note that, generally, for V of variable sign, it might happen (more exactly, is not excluded yet) that the positive and negative components of V cancel each other as it concerns their contribution to the eigenvalue estimates. Such cancelation effect, for measures supported inside the domain, may actually take place, as shown in [14]. This, however, cannot happen when the support of the measure μ on Γ possesses a nice geometrical structure, see the next section.

5 Eigenvalue asymptotics

5.1 The boundary mapping

In describing the weighted harmonic space, we have defined the norm in this space by (2.1). It is quite possible to use another expression, producing an equivalent norm, and this would not lead to a different order in eigenvalue estimates. On the other hand, when studying eigenvalue asymptotics, the concrete choice of this norm influences the expression for the coefficient in the asymptotics.

There is a standard way of reducing the norm (the quadratic form) on solutions of an elliptic equation to a norm on its boundary values, see, e.g., [2, 3, 22, 23] (however, weighted spaces have not been considered there). The procedure, in our case, consists of constructing an order 2β elliptic pseudodifferential operator $\mathbf{L}_0$ on the boundary such that

$$\|u\|_{n,\alpha}^2 = (\mathbf{L}_0\gamma u, \gamma u)_{L_2(\Gamma)} + R[\gamma u],$$

where $R[\gamma u]$ is a quadratic form of differential order less than β. According to the general rule, see, e.g., [3], to find the principal symbol of the pseudodifferential operator $\mathbf{L}_0$, one should perform the following calculations. Fix a point x^0 at the boundary Γ, fix local co-ordinates with center at $x^0 = \mathbf{0}$ and place the co-ordinate $t = x_{d+1}$ along the (inner) normal at $\mathbf{0}$ and co-ordinates $x' = (x_1, \dots, x_d)$ along the tangent plane to Γ at $\mathbf{0}$. For the Poisson operator $\mathcal{P}$ solving the Dirichlet problem $\Delta u = 0$, $\gamma u = f$, we should consider the pseudodifferential boundary quadratic form $\boldsymbol{\ell}[f]$ determined by

$$\boldsymbol{\ell}[f] = ((\mathcal{P}_n^\alpha)^*(\mathcal{P}_n^\alpha)f, f) = \int_\Omega \rho(x)^\alpha \langle \nabla^n(\mathcal{P}f)(x), \nabla^n\mathcal{P}f(x)\rangle dx, \tag{5.1}$$

where $\mathcal{P}_n^\alpha = \rho(x)^{\alpha/2}\nabla^n\mathcal{P}$. This form defines an elliptic order 2β pseudodifferential operator $\mathbf{L}_0$, $\boldsymbol{\ell}[f] = (\mathbf{L}_0 f, f)_{L_2(\Gamma)}$. To calculate its principal symbol $\mathbf{l}_{2\beta}$, we use (5.1). After making the Fourier transform along the tangent plane $T_{\mathbf{0}}\Gamma$, we obtain, as the representation of the Poisson kernel, in the leading term, the operator in $\mathbb{R}^d$ with principal symbol

$$\widehat{\mathcal{P}}(t, \xi') = e^{-t|\xi'|},$$

where $\xi = (\tau, \xi')$, $\xi' \in \mathbb{R}^d$. Next, we calculate $\widehat{\nabla^n\mathcal{P}}(t, x')$. Since

$$\nabla^n = \sum_{|\nu|=n} \binom{n}{\nu} \partial_t^{\nu_{d+1}} \nabla_{x'}^{\nu'},$$

we have, in the Fourier representation,

$$\left|\widehat{\nabla^n\mathcal{P}}(t, \xi')\right|^2$$

$$= \sum_{|\nu|=n} \binom{n}{\nu}^2 \left(|\xi'|^{\nu_{d+1}}(\xi')^{\nu'}\right)^2 e^{-2t|\xi'|} = |\xi'|^{2n} R_n(\omega)^2 e^{-2t|\xi'|},\ \omega = \xi'|\xi'|^{-1},$$

where $R_n(\omega)$ is a positive smooth function on the unit sphere $S^{d-1} \subset T_{\mathbf{0}}\Gamma$. When calculating the expression in (5.1), we can, as usual, integrate in t variable over the whole semi-axis $t \in (0, \infty)$ since for $t_0 > 0$, the integral over (t_0, ∞) produces a symbol with fast decay as $|\xi'| \to \infty$. Thus,

$$\mathbf{l}_{2\beta}(\xi') = \int_0^\infty t^\alpha e^{-2t|\xi'|} dt |\xi'|^{2n} (R_n(\omega))^2 = 2^{-\alpha-1}\mathbf{\Gamma}(\alpha+1)|\xi'|^{2n-\alpha-1}(R_n(\omega))^2. \tag{5.2}$$

Formula (5.2) shows that $\mathbf{L}_0$ in (5.1) is an order $2n - \alpha - 1$ elliptic pseudodifferential operator with principal symbol $\mathbf{l}_{2\beta}(\xi') = 2^{-\alpha-1}\mathbf{\Gamma}(\alpha+1)|\xi'|^{2n-\alpha-1}(R_n(\omega))^2$, $\omega = \xi'|\xi'|^{-1}$. Therefore, the weighted Sobolev norm $|u|^2_{n,\alpha}$ in (2.1) equals, up to lower order terms, $\|\mathbf{M}\gamma u\|^2$, where $\mathbf{M} = \mathbf{L}_0^{\frac{1}{2}}$ is an elliptic order $\beta = n - \frac{1+\alpha}{2}$ pseudodifferential operator on Γ with principal symbol

$$\mathbf{m}(\xi') = (\mathbf{l}_{2\beta}(\xi'))^{\frac{1}{2}} = 2^{-\frac{\alpha+1}{2}}(\mathbf{\Gamma}(\alpha+1))^{\frac{1}{2}} R_n(\omega)|\xi'|^\beta.$$

Further on, we will consider the Sobolev space $\mathrm{H}^\beta(\Gamma)$ equipped with the norm $\|\mathbf{M}f\|^2 + \|f\|^2$ and use the notation $\mathrm{H}^\beta_{\mathbf{M}}(\Gamma)$. The lower-order terms mentioned above produce a relatively compact perturbation and therefore do not influence the eigenvalue asymptotics. Note that the principal symbol of $\mathbf{M}$ does not depend on $x \in \Gamma$.

5.2 Eigenvalue asymptotics formulas

We suppose here that the measure μ coincides with the natural Hausdorff measure on a Lipschitz surface $\Sigma \subset \Gamma$, of dimension $\delta > d - 2\beta$. For the study of the spectral asymptotics of the operator $\mathbf{T}_{V\mu}$ generated by the quadratic form $\boldsymbol{\mu}_V[f]$ in $\mathrm{H}^\beta_{\mathbf{M}}(\Gamma)$, we apply the results of the papers [1], Theorem 2.4, and [18], Theorem 6.2. The setting there corresponds to the study of the spectrum of the operator, $\mathfrak{A}^* V\mu\mathfrak{A}$ in $L_2(\mathbb{R}^d)$, where, in our case, $\mathfrak{A}$ is the inverse of the pseudodifferential operator $\mathbf{M}$, i.e., the operator with principal symbol, locally,

$$\mathbf{a}(\xi') = \mathbf{m}(\xi')^{-1} = 2^{\frac{\alpha+1}{2}}(\mathbf{\Gamma}(\alpha+1))^{-\frac{1}{2}} R_n(\omega)^{-1}|\xi'|^{-\beta}.$$

We describe briefly the construction in [1, 18]. With each point $x \in \Sigma$, where the tangent plane to Σ (and the conormal plane) exists, this means, μ-almost everywhere on Σ, due to the Rademacher theorem, a density $\boldsymbol{\rho}_{\mathfrak{a}}(x)$ is associated in the following way, see formulas (6.2), (6.3) in [18] and (2.6), (2.7) in [1]. On the tangent space to Γ at x, $\mathrm{T}_x(\Gamma)$, we introduce co-ordinates $\xi' = (\zeta, \eta)$, where $\zeta \in \mathrm{T}_x(\Sigma)$ and η is the normal vector, $\eta \in \mathrm{N}_x\Sigma \cap \mathrm{T}_x(\Gamma) \equiv \mathbf{N}_x$.

Next, we define the auxiliary symbol $\mathbf{r}_{-m}(x,\zeta)$, $\zeta \in \mathrm{T}_x(\Sigma)$, of order $-m = \delta - 2\beta < 0$, by

$$\mathbf{r}_{-m}(x,\zeta) = (2\pi)^{-(d-\delta)} \int_{\mathbf{N}_x} |\mathbf{a}(x,\zeta,\eta)|^2 d\eta, \, (x,\zeta) \in \mathrm{T}(\Sigma),$$

and the density

$$\mathbf{h}(x) = \int_{\mathrm{S}_x\Sigma} \mathbf{r}_{-m}(x,\zeta)^{\theta} d\zeta,$$

where the last integration is performed over the sphere in $\mathrm{T}_x(\Sigma)$.

With application of the above theorems, this leads to the following asymptotic formulas for eigenvalues of the operator $\mathbf{T}_{V\mu}$.

Theorem 5.1 *Let the conditions of Theorem 4.1 be satisfied. Moreover, let μ be the Hausdorff measure on the δ-dimensional Lipschitz surface $\Sigma \subset \Gamma$ of dimension $\delta > d - 2\beta$. Let the function $V(x)$, $x \in \Sigma$, be real-valued. Denote by $n_\pm(\lambda, \mathbf{T}_{V\mu})$ the distribution function for positive (negative) eigenvalues of the operator $\mathbf{T}_{V\mu}$. Then the following asymptotic formula holds*

$$n_\pm(\lambda, \mathbf{T}_{V\mu}) \sim \lambda^{-\theta} \frac{1}{d} (2\pi)^{d-1} \int_\Sigma \mathbf{h}(x) V_\pm(x)^\theta \mu(dx), \; \lambda \to 0, \tag{5.3}$$

where $V_\pm(x)$ are the positive, resp., negative components of $V(x)$.

Note that although in the conditions of Theorem 4.1, three different cases are distinguished, the resulting asymptotics is given by the same expression in all these cases. One should also keep in mind that unlike the eigenvalue estimates, the coefficients in the asymptotic formulas depend on the concrete explicit choice of the expression for the norm in the harmonic Sobolev space $\mathbb{D}_n^\alpha(\Omega)$.

5.3 Extensions

As found in [1, 18], by means of eigenvalue estimates in Theorem 4.1, the eigenvalue asymptotics can be extended from Lipschitz surfaces to sets of more complicated structure. We recall (see, e.g., [6]) that a δ-regular Borel set $\mathcal{X}$ is called *uniformly rectifiable* if, up to a set of δ-Hausdorff measure zero, it can be represented as a union of a countable family of Lipschitz surfaces Σ_j of dimension δ. Following the reasoning in [1, 18], we obtain that if μ is the δ-dimensional Hausdorff measure on a *uniformly rectifiable* set $\mathcal{X} \subset \Gamma$ of dimension δ, the asymptotic formula

$$n_\pm(\lambda, \mathbf{T}_{V,\mu}) \sim \lambda^{-\theta} C(d,\delta,n,\alpha) \sum_j \int_{\mathcal{X}} \mathbf{h}_{\Sigma_j}(x) V_\pm(x)^\theta \mu(dx), \; \lambda \to 0, \tag{5.4}$$

with density $\mathbf{h}_{\Sigma_j}(x)$ determined by the geometry of Σ_j and the particularly chosen norm in the weighted harmonic Sobolev space. Note here that for $2\beta = d$, Lipschitz surfaces in Γ give to the eigenvalue asymptotics contribution of the same order $\theta = 1$ for all dimensions δ; therefore, the formula (5.4) is valid for the union of rectifiable sets of different dimensions, and the asymptotic formula should contain all such surfaces.

6 An alternative regularization

There can be several approaches to regularizing quadratic forms generated by boundary measures. We discuss here shortly one of such methods, close to the one discussed above. Let, again, $\mathrm{D}^\alpha(\Omega)$ be the standard weighted harmonic Bergman space in a domain $\Omega \subset \mathbb{R}^{d+1}$, let μ be a Borel measure on $\Gamma = \partial\Omega$ and let V be a density, a μ-measurable function. Let further $\mathfrak{A}$ be an order $-k < 0$ pseudodifferential operator on Γ. We associate with μ, V and $\mathfrak{A}$ the quadratic form

$$\boldsymbol{\mu}_{V,\mathfrak{A}}[u] = \int_\Gamma V(x)|\mathfrak{A}(\gamma u)(x)|^2 d\mu(x),\ u \in \mathrm{D}^\alpha(\Omega). \tag{6.1}$$

This quadratic form, defined initially on functions $u \in \mathrm{D}^\alpha(\Omega) \cap C(\overline{\Omega})$, under conditions specified below, extends by continuity to the whole of $\mathrm{D}^\alpha(\Omega)$.

As explained in Sect. 2, for $n = 0$, the boundary operator γ maps $\mathrm{D}^\alpha(\Omega)$ to $\mathrm{H}^\vartheta(\Gamma)$, $\vartheta = -(1+\alpha)/2$. Therefore, the composition $\mathfrak{A} \circ \gamma$ maps $\mathrm{D}^\alpha(\Omega)$ to $\mathrm{H}^\beta(\Gamma)$, with $\beta = k + \vartheta = k - \frac{1+\alpha}{2}$.

We suppose now that $k > \frac{1+\alpha}{2}$, therefore, $\beta > 0$. Then for the analysis of the operator defined by the quadratic form (6.1), we can repeat the reasoning leading to eigenvalue estimates in Sect. 4. Thus, depending on the relation of 2β and d, we suppose that the measure μ belongs to the class $\mathcal{A}_+^\sigma$, $\mathcal{A}^\sigma$, or $\mathcal{A}_-^\sigma$, and for V belonging to the corresponding space of integrable functions, $L_{\theta,\mu}$, $L_{\Psi,\mu}$, or $L_{1,\mu}$, with $\theta = \frac{\sigma}{\sigma+2\beta-d}$, the estimates (4.4) or (4.5) hold. Under similar conditions, Theorem 4.2, the lower estimate, is valid.

Finally, if the measure μ is the Hausdorff measure on a δ-dimensional Lipschitz surface, the asymptotics of eigenvalues, as in Theorem 5.1, holds, as well as its generalization to uniformly rectifiable sets.

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