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RESEARCH

The versal deformation of elliptic m-fold point curve singularities

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To Juan José Nuño-Ballesteros on his 60th birthday

Abstract

We give explicit, highly symmetric equations for the versal deformation of the singularity L^n_{n+1} consisting of $n + 1$ lines through the origin in $\mathbb{A}^n(k)$ in generic position. These make evident that the base space of the versal deformation of L^n_{n+1} is isomorphic to the total space for L^{n-1}_n, if $n \geq 5$. By induction it follows that the base space is irreducible and Gorenstein. We discuss the known connection to a modular compactification of the moduli space of $(n + 1)$-pointed curves of genus 1. For other elliptic partition curves it seems unfeasible to compute the versal deformation in general. It is doubtful whether the base space is Gorenstein. For rational partition curves we show that the base space in general has components of different dimensions.

Keywords: Versal deformation, Moduli of curves, Partition curve, Elliptic m-fold points

Mathematics Subject Classification: 14B05, 14H10, 32S05

Introduction

All possible deformations of a singularity can be obtained from its versal deformation. In general this object is too complicated and one has to be content with partial information, like the vector space T^1 of infinitesimal deformations or only its dimension. An explicit description of the versal deformation of a whole family of singularities is only possible if the equations of the singularities have a high degree of symmetry. An example is the case of n lines through the origin in the form of the coordinate axes in $\mathbb{A}^n$. The resulting equations are very simple. The ones given in [18] are due to D. S. Rim, but the computation was done independently by various authors [1,8]. Here we consider the next case, the curve L^n_{n+1} consisting of $n+1$ lines in generic position through the origin in $\mathbb{A}^n(k)$, obtained by adding a line to the coordinate axes, which does not lie in any of the coordinate hyperplanes; in the terminology of Smyth [19] this an elliptic m-fold point, with $m = n + 1$ the multiplicity. We assume that k is an algebraically closed field of characteristic zero.

An elliptic $(n + 1)$-fold point is quasi-homogeneous, and has only deformations of negative weight [15]. This means that the degree of the perturbations is lower than the degree of the equations, and the deformation variables have positive weights. Therefore the equations of the total space of the versal deformation are polynomial, of the same degrees as the equations of the singularity. Fibrewise the projective closure can be taken, in a suitable weighted projective space. The base space B_n is then a fine moduli space for reduced projective Gorenstein curves of arithmetic genus one with a hyperplane section defined by a specific function t; this is a special case of a general result of Looijenga [13]. The projective scheme $\mathbb{P}(B_n)$ is a compactification of the moduli space $M_{1,n+1}$ of $(n+1)$-pointed curves of genus 1. It is isomorphic to the compactification $\overline{M}_{1,n+1}(n)$ constructed by Smyth [20]. This is proved by Lekili and Polishchuk [12], who construct the projective family by computing the coordinate ring of the fibres. In their equations the first two coordinates have a special role, leading to a less symmetric form. The connection to the versal deformation of L^n_{n+1} is not made explicitly.

Our symmetric equations were originally obtained in a project with Theo de Jong to extend the results of [7] on rational surface singularities with reduced fundamental cycle (that is, with L^n_n as general hyperplane section) to the case of minimally elliptic singularities with reduced fundamental cycle. The equations for L^n_{n+1} and its deformations are very similar to those for L^n_n (the coordinate axes), but a bit more complicated. Whereas for L^n_n the equations are just $z_iz_j = 0$ for $i < j$, we now get the equations $z_iz_j = z_kz_l$, which as written do not provide a minimal system of generators of the ideal of the curve. To obtain such a system the symmetry of the equations has to be broken. Instead of using the full system of equations we introduce a new variable y of weight 2 and get $\binom{n}{2}$ equations $y = z_iz_j$. This simplifies the computation of the infinitesimal deformations. For the versal deformation we end up with equations in terms of the original full system. Due to the symmetry it suffices to write down only one equation for the total space and one equation for the base space. The obtained equations are again similar to the equations for the versal deformation of the coordinate axes L^n_n, but a bit more involved.

In the case at hand it is possible to analyse the equations, as they turn out to have an inductive structure; in fact, due to the symmetry in several ways. The base space of the versal deformation of L^n_{n+1} is isomorphic to the total space of the versal deformation of L^{n-1}_n, if $n \geq 5$. An equivalent observation is made by Lekili and Polishchuk [12], and used to study the base space. Like them we obtain the following result (see Corollary 4.2).

Theorem *The base space of the versal deformation of L^n_{n+1} is Gorenstein and irreducible.*

The curve L^n_{n+1} is the simplest elliptic partition curve of embedding dimension n. For all such curves the dimension of T^1 and T^2 is known [3]. The other extreme is the Gorenstein monomial curve of minimal multiplicity, the one with semigroup generated by $\langle n+1, n+2, \dots, 2n\rangle$. Its equations have less symmetry, and a general computation of the versal deformation is out of reach. For $n = 6$ the calculation is still doable, but the resulting equations are too long to be given here. The situation can again be compared with rational partition curves. For L^n_n the base space is Cohen-Macaulay [17], with rather simple equations. They become more involved if one starts from the general hyperplane section of the cone over the rational normal curve of degree n (with its standard toric equations). We prove here that the monomial curve with semigroup $\langle n, n+1, \dots, 2n-1\rangle$ deforms into non-smoothable singularities, if $n \geq 14$. In particular, its base space has components of different dimensions. A similar elementary explicit deformation cannot be given for the Gorenstein monomial curve, but we see no reason preventing the base space to be reducible.

1 Gorenstein curve singularities with minimal δ-invariant

The singularity L^n_{n+1} of $n+1$ lines through the origin in $\mathbb{A}^n$ in generic position is the simplest of a whole class of singularities, for which the dimension of T^1 and T^2 is known [3]. We review the classification of this class of curves.

Let (X,p) be a curve singularity with r branches and let $\nu\colon \widetilde{X} \to X$ be the normalisation, with $\nu^{-1}(p) = \{p_1,\dots,p_r\}$. Denote the completion of the local ring of X at p by $\widehat{\mathcal{O}}_X$ and the semi-local ring $\oplus_{i=1}^r \widehat{\mathcal{O}}_{\widetilde{X},p_i}$ by $\widehat{\mathcal{O}}_{\widetilde{X}}$. The δ-*invariant* of (X,p) is $\delta(X) = \dim_k \widehat{\mathcal{O}}_{\widetilde{X}}/\widehat{\mathcal{O}}_X$. For curves $X = X_1 \cup X_2 \subset \mathbb{A}^N$ with singular point p, where the X_i may be reducible, $\delta(X) = \delta(X_1) + \delta(X_2) + (X_1 \cdot X_2)$; here the intersection multiplicity $(X_1 \cdot X_2)$ is $\dim \widehat{\mathcal{O}}_{\mathbb{A}^N}/(\widehat{I}_1 + \widehat{I}_2)$ with $\widehat{I}_1$ and $\widehat{I}_2$ the ideals of X_1 and X_2 in the completed local ring of $\mathbb{A}^N$ at p.

The Milnor number $\mu(X)$ has been defined by Buchweitz and Greuel and it is equal to $2\delta - r + 1$ [5, Prop. 1.2.1]. For a smoothable curve singularity over the complex numbers the Milnor number is equal to the first Betti number of the Milnor fibre [5, Cor. 4.2.3], and the genus of the Milnor fibre is equal to $\delta - r + 1$. Therefore we define for all curve singularities the genus as $g(X) = \delta - r + 1$. We have

$$g(X_1 \cup X_2) = g(X_1) + g(X_2) + (X_1 \cdot X_2) - 1\,. \tag{1}$$

A curve singularity X is *decomposable* (into X_1 and X_2), if X is the union of two curves X_1 and X_2, such that the Zariski tangent spaces of X_1 and X_2 intersect each other only in one point, the singular point of X. We write $X = X_1 \vee X_2$. With the exception of an ordinary double point a decomposable curve is never Gorenstein.

Curves singularities with minimal δ-invariant occur in the literature under several names (see e.g. [21]). Here we call them partition curves following [2]. They are wedges of monomial curves. Let X_n be the monomial curve with semigroup generated by $\langle n, n+1, \dots, 2n-1\rangle$. For a partition $\boldsymbol{p} = (n_1,\dots,n_r)$ of n we define the *partition curve*

$$X_{\boldsymbol{p}} = X_{n_1} \vee \dots \vee X_{n_r}\,.$$

In particular $X_{(n)} = X_n$, and $X_{(1,\dots,1)} = L^n_n$, the singularity consisting of the coordinate axes. We include the smooth point L^1_1, which has $\delta = 0$. Partition curves are exactly the curves of multiplicity m equal to the embedding dimension n with $\delta = m - 1 = n - 1$. Given n, they have minimal multiplicity and minimal δ. They occur as hyperplane sections of rational singularities.

For an isolated Gorenstein curve singularity Y of embedding dimension n, $n \geq 3$, the delta invariant has value at least $n+1$. Gorenstein curve singularities with $\delta = n+1, n \geq 2$, are classified in [3] and are called elliptic partition curves. The term elliptic in the name is explained by the fact that such curves occur as hyperplane sections of minimally elliptic singularities. The easiest description is as follows: given a partition $\boldsymbol{p} = (n_1,\dots,n_r)$ of $n+1$, the elliptic partition curve $Y_{\boldsymbol{p}} \subset \mathbb{A}^n$ is the curve obtained by a generic linear projection of the partition curve $X_{\boldsymbol{p}} \subset \mathbb{A}^{n+1}$. In particular $Y_{(n+1)}$ is the monomial curve generated by $\langle n+1, n+2, \dots, 2n\rangle$, and $Y_{(1,\dots,1)} = L^n_{n+1}$, the curve consisting of $n+1$ lines through the origin in generic position. The generic projection might not respect the grading, but elliptic partition curves have no moduli, and there exists a quasi-homogeneous representative.

Remark 1.1 The elliptic partition curves (with $n \geq 2$) have also minimal multiplicity $m = n+1$. So they are the curves with minimal $\delta = m$. This view point allows to extend the classification to $n+1=m=2$. There are two Gorenstein curves with $m=\delta=2$: the tacnode A_3 corresponding to the partition $(1,1)$ and the rhamphoid cusp A_4 corresponding to (2). Because $\dim \mathcal{O}_X/\mathcal{O}_Y = 1$ for the generic projection $X \to Y$, the curves A_3 and A_4 can be considered to be the projections of A_1 and A_2. We remark that A_1 and A_2 are the first blow-ups of A_3 and A_4. The first blow-up of A_2 is a smooth curve, so A_2 can be considered to belong to the partition (1), for $n=0$. With these conventions [3, Prop. 4.1.1] continues to hold for $n=0,1$: the general hypersurface section of a minimally elliptic surface singularity with $-Z\cdot Z = n+1$ (where Z is the fundamental cycle) is an elliptic partition curve for a partition of $n+1$.

The curves L^n_{n+1} are also elliptic in the sense that they satisfy $g=\delta-r+1=1$, so $\delta=r$. The classification of curves singularities with $g=1$ is due to Greuel [10,11]. By (1) such a curve is of the form $X\vee L^s_s$ with X indecomposable of genus 1. For indecomposable X with $r\geq 2$ all branches are smooth and by removing one branch a curve L^{r-1}_{r-1} is left. Therefore such an X is an elliptic partition curve belonging to the partition $(1,\dots,1)$ of $n+1$, including the cases $n=0,1$, and consequently it is Gorenstein. These curves are called elliptic m-fold points by Smyth [19], who comes to the same classification in arbitrary characteristic.

Behnke and Christophersen have determined the dimension of T^1 and T^2 for all elliptic partition curves [3, Prop. 5.4.1]:

Proposition 1.2 *For an elliptic partition curve Y of multiplicity $n+1$, where $n\geq 4$, with r branches*

$$\dim T^1_Y = \frac{n(n+1)}{2} - r + 1\,,$$
$$\dim T^2_Y = \frac{n(n+1)(n-4)}{6}\,.$$

Furthermore T^2_Y is annihilated by the maximal ideal of the local ring.

For $n=2$ the curve is a plane curve and for $n=3$ it is a complete intersection, so the formula for T^1 does not extend to these cases. For $r=n+1$ the formula gives $\binom{n}{2}$, while for $n=3$ the curve is the simple space curve S_5 with $\dim T^1=\mu=5$ and for $n=2$ we have D_4 with $\dim T^1=\mu=4$. The formula for T^2 would give negative dimensions; the correct dimension for complete intersections is 0.

For the special case of L^n_{n+1} it had already been shown by Pinkham [15] that T^1 is negatively graded, and Greuel [11] had computed its dimension: T^1 is concentrated in degree -1, if $n\geq 4$, and has dimension $\binom{n}{2}$.

2 Deformations of negative weight

By a result of Pinkham [15] the curve L^n_{n+1} has only deformations of negative weight. This allows to take the projective closure of the fibres of the versal family and obtain in this way also the versal deformation of the projective cone over $n+1$ points in $\mathbb{P}^{n-1}$ in general position. We recall the general construction, following Looijenga, who makes Pinkham's results [15,16] more precise in the Appendix of [13].

Let $X\subset \mathbb{A}^n$ be an isolated quasi-homogeneous singularity, over a fixed algebraically closed field k of characteristic 0. This means that the ideal of X is generated by quasi-homogeneous polynomials. Then $X=\operatorname{Spec} R$, where $R=\oplus_{l=0}^{\infty} R_l$ is a reduced $\mathbb{Z}_+$-graded

k-algebra, with $R_0 = k$. The grading on R corresponds to a $\mathbb{G}_m$-action on X, defined by $\lambda \cdot \varphi = \lambda^l \varphi$ for $\lambda \in \mathbb{G}_m$ and $\varphi \in R_l$. As $R_0 = k$ and $R_l = 0$ for $l < 0$, this action is good, meaning that the unique fixed point, defined by the maximal ideal $R_+ = \oplus_{l=1}^{\infty} R_l$, is in the closure of every orbit. The standard projective closure $X \subset \overline{X}$ is defined as follows: let $\bar{R}_k = R_0 \oplus \cdots \oplus R_k$, then $\bar{R} := \oplus_{k=0}^{\infty} \bar{R}_k$ is a graded $\mathbb{Z}_+$-algebra and $\overline{X} = \operatorname{Proj} \bar{R}$. If $t = (1, 0) \in \bar{R}_1 = R_0 \oplus R_1$, then $\bar{R} = R[t]$ and X becomes a subscheme of $\overline{X}$ by making t invertible. The complement $\operatorname{Proj} R = \overline{X}_\infty = \overline{X} \backslash X$ is the divisor defined by t.

A deformation (π, i), where $\pi \colon (\mathcal{X}, X_s) \to (S, s)$ and $i \colon X \cong X_s$, has (good) $\mathbb{G}_m$-action if both $(\mathcal{X}, X_s)$ and (S, s) have (good) $\mathbb{G}_m$-actions making π and i $\mathbb{G}_m$-equivariant. In case of a good $\mathbb{G}_m$-action the (formal) schemes $(\mathcal{X}, X_s)$ and (S, s) can be taken to be affine schemes $\mathcal{X} = \operatorname{Spec} \mathcal{R}$ and $S = \operatorname{Spec} A$. By the same construction as for X the projective closure of the deformation can fibrewise be formed. Put $\overline{\mathcal{R}}_k = A\mathcal{R}_0 \oplus \ldots A\mathcal{R}_k$ and define $\overline{\mathcal{X}} = \operatorname{Proj}_A \overline{\mathcal{R}}$. If $t = (1, 0) \in \overline{\mathcal{R}}_1$, then $\overline{\mathcal{R}}/t\overline{\mathcal{R}}$ is naturally isomorphic to $A \otimes_k R$ and $\overline{\mathcal{X}}_\infty$ is naturally isomorphic to $\overline{X}_\infty \times S$.

By [16] there exists a miniversal object (π, i) for deformations of X with $\mathbb{G}_m$-action, which also induces a miniversal deformation of the isolated singularity (X, p). The part π_- of π of negative weight is miniversal for deformations of X with good $\mathbb{G}_m$-action. Looijenga proves that π_- is actually universal for this property [13, Thm A.2]:

Theorem 2.1 *Let X be a reduced affine scheme with good $\mathbb{G}_m$-action with as only singular point the vertex. The negative weight part π_- of the miniversal deformation of X is a final object in the category of deformations with good $\mathbb{G}_m$-action. The group G of automorphisms of X commuting with the $\mathbb{G}_m$-action acts on π_- and its projective closure $\bar{\pi}_-$. Any isomorphism between two fibres of the $\bar{\pi}_-$ preserving the pieces at infinity is induced by a unique element of G.*

If the isomorphism $(\bar{\sigma}, \bar{\sigma}_\infty)$, induced by $g \in G$, satisfies $\bar{\sigma}_\infty = 1$, then $g \in \mathbb{G}_m$ [13, Lemma A.4].

The morphism $\bar{\pi}_-$ has also a moduli interpretation. Looijenga defines an R-polarised scheme as a triple $(\bar{Z}, \bar{Z}_\infty, \varphi^*)$ consisting of a projective scheme $\bar{Z}$, an ample reduced Weil divisor $\bar{Z}_\infty$ on $\bar{Z}$ and an isomorphism $\varphi^* \colon R_{\bar{Z}}/tR_{\bar{Z}} \to R$ of graded k-algebras, where $R_{\bar{Z}} = \bigoplus_{l=0}^{\infty} H^0(\mathcal{O}_{\bar{Z}}(l\bar{Z}_\infty))$ with $t \in H^0(\mathcal{O}_{\bar{Z}}(\bar{Z}_\infty))$ the element corresponding to 1 [13, A.5]. He shows:

Proposition 2.2 *The morphism $\bar{\pi}_- \colon (\overline{\mathcal{X}}, \overline{\mathcal{X}}_\infty) \to S_-$ is a fine moduli space for R-polarised schemes.*

We specialise to the case L^n_{n+1}. This curve has only deformations of negative weight, so $\pi_- = \pi$. Each geometric fibre of $\bar{\pi}$ is a reduced Gorenstein curve of arithmetic genus 1 with $n + 1$ marked points (the points at infinity).

The approach of Lekili and Polishchuk [12] to determining explicit equations for $\bar{\pi}$ in this case is to construct the ring $R_{\bar{Z}}$. They start from a reduced, connected projective curve C of arithmetic genus 1 over an arbitrary algebraically closed field with $n+1$ distinct smooth marked points $p_0, \ldots, p_n$ such that $\mathcal{O}_C(p_0 + \cdots + p_n)$ is ample and $h^0(\mathcal{O}_C(p_i)) = 1$ for all i. Generators of $R_C = \bigoplus_{l=0}^{\infty} H^0(\mathcal{O}_C(l(p_0 + \cdots + p_n))$ are 1 and functions $h_{0i} \in H^0(\mathcal{O}_C(p_0+p_i))$ with $-\operatorname{res}_{p_i}(h_{0i}\omega) = \operatorname{res}_{p_0}(h_{0i}\omega) = 1$ for a fixed generator ω of $H^0(C, \omega_C)$. Under certain normalisations they determine the ring structure. In their equations the indices 1 and 2 have a special role. This approach does not extend to the computation of the versal deformation of other elliptic partition curves.

3 Computation of the versal deformation

In this section we compute the versal deformation for L^n_{n+1} for $n \geq 4$ using generators and relations. We recall the main steps of the computation, see also [22, Ch. 3]. Let X be a variety with $\mathbb{G}_m$-action with isolated singularity at the origin in $\mathbb{A}^n$. Let $f = (f_1, \ldots, f_r)$ generate the ideal $I(X)$ of X in the polynomial ring $S = k[X_1, \ldots, X_n]$. The first few terms of the resolution of $k[X] = S/I(X)$ are

$$0 \longleftarrow k[X] \longleftarrow S \xleftarrow{f} S^k \xleftarrow{r} S^l \,,$$

where the columns of the matrix r generate the module of relations. Let $\mathcal{X} \to \operatorname{Spec} A$ be a deformation of X. The flatness of this map translates into the existence of a lifting of the resolution to

$$0 \longleftarrow k[\mathcal{X}] \longleftarrow S \otimes A \xleftarrow{F} (S \otimes A)^k \xleftarrow{R} (S \otimes A)^l \,.$$

To find the versal deformation we must find a lift $FR = 0$ in the most general way. The first step is to compute infinitesimal deformations. We write $F = f + \varepsilon f'$ and $R = r + \varepsilon r'$. As $\varepsilon^2 = 0$, the condition $FR = 0$ gives

$$FR = (f + \varepsilon f')(r + \varepsilon r') = fr + \varepsilon (fr' + f'r) = 0 \,.$$

We solve the equation $f'r = 0$ modulo f and then determine r'. After this we lift to second order. Obstructions to do this may come up, leading to equations in the deformation parameters. In our case the computation terminates at this point. It constructs the versal deformation as the zero fibre of the quadratic obstruction map $\operatorname{ob}\colon T^1 \to T^2$.

3.1

For Gorenstein singularities of minimal multiplicity m it is known [24] in general that the local ring has a free resolution with Betti numbers

$$\beta_i = \frac{i(m-2-i)}{m-1}\binom{m}{i+1}, \qquad i = 1, \ldots, m-3 \,,$$

while $\beta_0 = \beta_{m-2} = 1$.

We consider L^n_{n+1}, with multiplicity $m = n+1$. We can take n of the lines to be the coordinate axes and as last line the line through $(1, \ldots, 1)$. Then the ideal $I_0 \subset P_0 = k[z_1, \ldots, z_n]$ of C_n is minimally generated by $\frac{(n+1)(n-2)}{2} = \binom{n}{2} - 1$ polynomials. We start from a non-minimal system of generators

$$f_{i,j;k,l} = z_i z_j - z_k z_l, \qquad i \neq j, k \neq l, \quad 1 \leq i,j,k,l \leq n \,.$$

A minimal system can be chosen to consist of the $F_{ij;1,2} = z_i z_j - z_1 z_2$, $i < j$, but this choice gives the first two variables a special role.

To write symmetric equations and at the same time minimise the number of equations it is convenient to embed the singularity in $\mathbb{A}^{n+1}$. We introduce a new variable y of weight 2 with $\mathbb{A}^n = \{y = 0\}$ and after a coordinate transformation, say replacing y by $y - z_1 z_2$, which transforms the line through $(1, \ldots, 1)$ into a parabola, the ideal I of the curve C_n in $\mathbb{A}^{n+1}$ is generated by the $\binom{n}{2}$ polynomials

$$g_{ij} = z_i z_j - y, \qquad 1 \le i < j \le n\,.$$

Next we describe the relations between these generators. Write $P = k[z_1, \ldots, z_n, y]$ for the (graded) polynomial ring in $n+1$ variables. Denote by I_0 the ideal in P generated by the polynomials $f_{i,j;k,l}$. There is an exact sequence

$$0 \longrightarrow P/I_0 \xrightarrow{f} P/I_0 \longrightarrow P/I \longrightarrow 0$$

where $f = z_1 z_2 - y$. Let F_0 be a free resolution of P_0/I_0. The same matrices yield a free resolution F of P/I_0. One obtains a resolution of P/I as a mapping cone of a homomorphism of complexes $F \to F$ which is a lift of $f : P/I_0 \to P/I_0$. In particular the number of relations is $(n+1)(n-2)/2 + (n^2-1)(n-3)/3$. The first summand gives the number of Koszul relations, which we can ignore in deformation computations. The other relations are linear relations, which we now describe.

We start by forming $(n-2)\binom{n}{2} = 3\binom{n}{3}$ expressions of the form $z_i g_{jk}$. There are $\binom{n}{3}$ monomials $z_i z_j z_k$. As $z_i g_{jk} - z_j g_{ik} = (z_i - z_j)y$ and $z_i g_{jk} - z_k g_{ij} = (z_i - z_k)y$ we get $n-1$ additional conditions on linear combinations of the $z_i g_{jk}$ to be syzygies. This gives the required number of $2\binom{n}{3} - (n-1) = (n^2-1)(n-3)/3$ linear independent relations of the form

$$R_{ik;jl} = z_k(g_{ij} - g_{il}) - z_i(g_{kj} - g_{kl})\,.$$

This computation also shows that the relations between the polynomials $f_{i,j;k,l}$ are generated by the linear dependencies and

$$R_{ik;jl} = z_k f_{ij;il} - z_i f_{kj;kl}\,.$$

3.2

We determine an explicit basis for the module of infinitesimal deformations T^1, which by [11] is concentrated in degree -1. We perturb the generators g_{ij} to

$$G_{ij} = z_i z_j - y + \sum a_{ij}^m z_m$$

and insert these expressions in the relation $R_{ik;jl}$:

$$z_k(G_{ij} - G_{il}) - z_i(G_{kj} - G_{kl}) \\ = z_k\left(\sum a_{ij}^m z_m - \sum a_{il}^m z_m\right) - z_i\left(\sum a_{kj}^m z_m - \sum a_{kl}^m z_m\right)\,.$$

The coefficient of z_k^2 is $a_{ij}^k - a_{il}^k$ and has to vanish. From this it follows that a_{ij}^k for $i, j \neq k$ have a common value, which we call a^k. Then $z_k(G_{ij} - G_{il}) - z_i(G_{kj} - G_{kl})$ is modulo I equal to

$$y\left((a_{ij}^i - a_{il}^i) - (a_{kj}^k - a_{kl}^k) + (a_{ij}^j - a_{kj}^j) - (a_{il}^l - a_{kl}^l)\right) \tag{2}$$

and the coefficient of y has to vanish. We put

$$b_{ij} = a^i_{ij} + a^j_{ij} \, .$$

Then the condition that (2) vanishes can be written as

$$b_{ij} - b_{il} = b_{kj} - b_{kl} \, . \tag{3}$$

This system of equations allows all b_{ij} to be expressed in terms of n of them, say b_{1k}, $2 \le k \le n$ and b_{23}. A more symmetric solution is to introduce variables b_i, $1 \le i \le n$, and put

$$b_{ij} = b_i + b_j \, .$$

This solves the equations (3) and the b_i are determined by the b_{ij}: we have $b_{12} = b_1 + b_2$, $b_{13} = b_1 + b_3$ and $b_{23} = b_2 + b_3$ so

$$\begin{aligned} 2b_1 &= b_{12} + b_{13} - b_{23} \\ 2b_2 &= b_{12} + b_{23} - b_{13} \\ 2b_3 &= b_{13} + b_{23} - b_{12} \end{aligned}$$

and

$$b_k = b_{1k} - b_1 = b_{1k} + \tfrac{1}{2} b_{23} - \tfrac{1}{2} b_{13} - \tfrac{1}{2} b_{12} \, .$$

We apply coordinate transformations to the

$$G_{ij} = z_i z_j - y + a^i_{ij} z_i + a^j_{ij} z_j + \sum_{s \neq i,j} a^s z_s$$

by transforming $y \mapsto y + \sum_s a^s z_s$ and $z_i \mapsto z_i + a^i - b_i$. Taking only first-order terms into account the result is

$$G_{ij} = z_i z_j - y + (a^i_{ij} - a^i + a^j - b_j) z_i + (a^j_{ij} - a^j + a^i - b_i) z_j \, .$$

The coefficients of z_i and z_j sum to zero:

$$(a^i_{ij} - a^i + a^j - b_j) + (a^j_{ij} - a^j + a^i - b_i) = a^i_{ij} + a^j_{ij} - b_i - b_j = 0 \, .$$

Finally we put $a_{ij} = a^i_{ij} - a^i + a^j - b_j$ and $a_{ji} = a^j_{ij} - a^j + a^i - b_i$. Then $a_{ij} + a_{ji} = 0$. As deformation variables we can take the a_{ij} with $i < j$, but it will be convenient to allow also a_{ij} with $i > j$, and we will freely use that $a_{ij} = -a_{ji}$. The result is the following.

Lemma 3.1 *A basis for T^1 is represented by the $\binom{n}{2}$ first-order deformations*

$$G_{ij} = z_i z_j - y + a_{ij}(z_i - z_j)$$

with $i < j$.

3.3

To compute the versal deformation we have to lift the relations $R_{ik;jl}$, so that they include the deformation variables. We consider an arbitrary relation $R_{ik;jl}$ and compute a lift of this relation up to first order:

$$\begin{aligned}
&z_k(G_{ij}-G_{il})-z_i(G_{kj}-G_{kl})\\
&\qquad -a_{ij}(G_{ik}-G_{jk})+a_{il}(G_{ik}-G_{lk})+a_{kj}(G_{ik}-G_{ij})-a_{kl}(G_{ik}-G_{il})\\
&\quad = z_k(a_{ij}z_i+a_{ji}z_j-a_{il}z_i+a_{li}z_l)-z_i(a_{kj}z_k+a_{jk}z_j-a_{kl}z_k+a_{lk}z_l)\\
&\qquad -a_{ij}(z_iz_k-z_jz_k+a_{ik}z_i+a_{ki}z_k-a_{jk}z_j-a_{kj}z_k)\\
&\qquad +a_{il}(z_iz_k-z_lz_k+a_{ik}z_i+a_{ki}z_k-a_{lk}z_l-a_{kl}z_k)\\
&\qquad +a_{kj}(z_iz_k-z_iz_j+a_{ik}z_i+a_{ki}z_k-a_{ij}z_i-a_{ji}z_j)\\
&\qquad -a_{kl}(z_iz_k-z_iz_l+a_{ik}z_i+a_{ki}z_k-a_{il}z_i-a_{li}z_l)\\
&\quad = (z_k-z_i)(a_{ij}a_{ik}+a_{kj}a_{ki}+a_{jk}a_{ji}-a_{il}a_{ik}-a_{kl}a_{ki}-a_{lk}a_{li})\,.
\end{aligned}$$

We abbreviate

$$\varphi_{ijk}=a_{ij}a_{ik}+a_{ji}a_{jk}+a_{ki}a_{kj}$$

so that the term in parentheses on the right-hand side of the equation above becomes $\varphi_{ijk}-\varphi_{ilk}$. The right-hand side can be made to vanish by subtracting this term from $F_{ij;il}=G_{ij}-G_{il}$, for all choices of three out of the four indices i, j, k and l; this means subtracting $\varphi_{kji}-\varphi_{kli}=\varphi_{ijk}-\varphi_{ilk}$ from $F_{kj;kl}=G_{kj}-G_{kl}$. To check that this indeed gives a lift of the relation we have to compute that

$$-a_{ij}(\varphi_{ikl}-\varphi_{jkl})+a_{il}(\varphi_{ijk}-\varphi_{jlk})+a_{kj}(\varphi_{ikl}-\varphi_{ijl})-a_{kl}(\varphi_{ijk}-\varphi_{ijl})=0\,.$$

For this it suffices to find for every term in the first summand a term which cancels it; the computation is not difficult.

We have now lifted one specific but arbitrary relation $R_{ik;jl}$, with a second-order perturbation of $F_{ij;il}$ depending on k. We need to lift all relations in a consistent way. The polynomial $F_{ij;il}$ occurs also in the relation $R_{im;jl}$ for $m\neq k$, which we can lift in a similar way. In order that the resulting $F_{ij;il}$ is independent of choices we need that $\varphi_{ijk}-\varphi_{ilk}=\varphi_{ijm}-\varphi_{ilm}$, for all k and m. If all these equations are satisfied we indeed have a lift of all relations. So these equations describe the base space of the versal deformation.

We formulate the result of our computation directly in terms of the a_{ij}.

Theorem 3.2 *The versal deformation of L^n_{n+1}, where $n\geq 4$, is given by the vanishing of the polynomials*

$$\begin{aligned}
F_{ij;il}&=z_iz_j-z_iz_l+(a_{ij}-a_{il})z_i+a_{ji}z_j-a_{li}z_l\\
&\qquad -a_{ij}a_{ik}-a_{ji}a_{jk}-a_{ki}a_{kj}+a_{il}a_{ik}+a_{li}a_{lk}+a_{ki}a_{kl}\\
&=(z_i-a_{ij})(z_j-a_{ji})-(a_{ik}-a_{ij})(a_{jk}-a_{ji})\\
&\qquad -(z_i-a_{il})(z_l-a_{li})+(a_{ik}-a_{il})(a_{lk}-a_{li})\,,
\end{aligned}$$

where $a_{ij} + a_{ji} = 0$ for all $1 \leq i, j \leq n$, with base space B_n given by the vanishing of

$$\Phi^i_{jl;km} = (a_{ik} - a_{ij})(a_{jk} - a_{ji}) - (a_{ik} - a_{il})(a_{lk} - a_{li}) - (a_{im} - a_{ij})(a_{jm} - a_{ji}) + (a_{im} - a_{il})(a_{lm} - a_{li})$$

for all pairwise distinct i, j, k, l, m.

It can be checked that the formulas (1.1) – (1.4) in [12] only differ from the above ones by a coordinate transformation. Their formulas are less symmetric, as they choose to give the first two coordinates a special role.

Only in the case $n = 4$, where there are no equations for the base space, which therefore is smooth, there are nice formulas for the total space in terms of the G_{ij}:

$$G_{ij} = z_i z_j - y + a_{ij} z_i + a_{ji} z_j - \varphi_{ijk} - \varphi_{ijl} \,.$$

For general n one can take the same formulas for $1 \leq i < j \leq 4$ and then find the other G_{kl} as all $G_{ij} - G_{il}$ are known.

3.4

It is well known that the curve L^n_{n+1} is smoothable. One reason is that it is a general hyperplane section of the cone over an elliptic normal curve of degree $n + 1$, so 'sweeping out the cone' defines a smoothing [15, (7.6)]. Pinkham also describes an inductive procedure, where two lines are smoothed to a quadric with given tangent direction, forming an L^{n-1}_n [15, (11.13)]. Such a partial smoothing occurs along the a_{ij}-axis in the base space. To be specific, we choose $a_{n-1,n} = t$ and $a_{ij} = 0$ for $(i, j) \neq (n - 1, n)$. This solves the equations for the base space as $\varphi_{ijk} = 0$ for all (i, j, k). The equations for the total space of this 1-parameter deformation are

$$z_i z_j - y = 0, \qquad z_{n-1} z_n - y + t(z_{n-1} - z_n) = 0 \,.$$

For $t \neq 0$ this is a curve consisting of n smooth branches. Among these are the z_i-axes for $1 \leq i \leq n - 2$ and the curve $(t, \ldots, t, t^2)$. Together they form a curve consisting of $n - 1$ smooth branches in general position and the linear space spanned by their tangents at the origin intersects the (z_{n-1}, z_n)-plane in the diagonal. The last branch of the deformed curve is a hyperbola in the (z_{n-1}, z_n)-plane with the diagonal as tangent line at the origin. This is indeed a singularity of type L^{n-1}_n.

It is also possible to smooth one coordinate axis and the parabola with tangent through $(1, \ldots, 1, 0)$ (in (z_i, y)-coordinates). If the axis is the z_n-axis, then we have the following deformation:

$$\begin{cases} (z_i - t)(z_n + t) - y = 0, & 1 \leq i \leq n - 1, \\ z_i z_j - y = 0, & 1 \leq i < j \leq n - 1. \end{cases}$$

4 The base space

4.1

The base space is smooth for $n = 4$. In that case the curve is codimension three Gorenstein so it and its deformations can be given as Pfaffians of a skew 5×5 matrix. For $n \geq 5$ we have to analyse the polynomials

$$\Phi^i_{jl;km} = \varphi_{ijk} - \varphi_{ilk} - \varphi_{ijm} + \varphi_{ilm} .$$

We note that $\Phi^i_{jl;km}$ is antisymmetric in j, l and k, m and symmetric in the pairs (j, l), (k, m). Furthermore

$$\Phi^i_{jl;km} - \Phi^n_{jl;km} - \Phi^k_{jl;in} + \Phi^m_{jl;in} = 0 . \tag{4}$$

Without explicitly identifying their equations as the versal deformation of L^n_{n+1}, Lekili and Polishchuk observe that the total space of the family for $n - 1$ is isomorphic to the base space for n [12, Prop. 1.1.5 (ii)]. This can be seen easily from our equations.

Theorem 4.1 *For $n \geq 5$ the base space B_n of the versal deformation of L^n_{n+1} is isomorphic to the total space of the versal deformation of L^{n-1}_n.*

Proof We start from the equations of the total space and the base space in Theorem 3.2. By substituting $z_i = a_{in}$ the polynomial $F_{ij;il}$ (written with the index k) becomes the polynomial $\Phi^i_{jl;nk}$. We establish by induction that this procedure gives (together with the equations for the base space of L^{n-1}_n) equations describing the base space of L^n_{n+1}, by showing that the number of independent equations is equal to the dimension of T^2, which is $\frac{n(n+1)(n-4)}{6}$.

The polynomials $F_{ij;il}$ give $\frac{n(n-3)}{2}$ linearly independent equations. The polynomials $\Phi^i_{jl;mk}$ not involving the index n describe the base space B_{n-1}, and by the induction hypothesis $\frac{n(n-1)(n-5)}{6}$ of them are linearly independent (the base case is $n = 5$, where no such polynomials exist). This gives $\frac{n(n-3)}{2} + \frac{n(n-1)(n-5)}{6} = \frac{n(n+1)(n-4)}{6}$ linear independent quadratic equations for B_n. We remark that this procedure does not give polynomials of the form $\Phi^n_{jl;km}$, but equation (4) shows that such polynomials are linear combinations of the other ones. Therefore we obtain all equations of Theorem 3.2 for B_n. □

Corollary 4.2 *The base space B_n is Gorenstein, has dimension $n + 2$ and multiplicity $\frac{n!}{24}$, and is smooth in codimension 6.*

Proof By induction on n. For $n = 4$ the result holds, as B_4 is smooth. We view B_n, $n \geq 5$, as total space of the versal deformation of L^{n-1}_n. By homogeneity it suffices to show the Gorenstein property for the local ring at the origin. For a flat local morphism $\varphi\colon A \to B$ of local Noetherian rings B is Gorenstein if and only if A and $B/\mathfrak{m}_A B$ both are Gorenstein [14, Thm. 23.4]. As both the base and the special fibre are Gorenstein, the same therefore holds for the total space.

The total space is Gorenstein and therefore Cohen-Macaulay, so all irreducible components have the same dimension, given by the formula for the dimension of a smoothing component, which yields $n + 2$. The dimension statement follows also by induction, the base being that the dimension of B_4 is 6. The curve L^n_{n+1} deforms (over reduced bases) only to other elliptic m-fold points or ordinary double points [11, Prop. 3.6]. As B_4 is smooth for $n \leq 4$, it follows that B_n is smooth in codimension 6. Therefore B_n is normal, and there is only one component.

The multiplicity follows again by induction. We can also use the formula for the Hilbert series of the graded ring in [12, Cor. 1.1.7] □

The inductive construction of the base space makes it possible to give a minimal system of equations. There are $\binom{n}{3}$ expressions φ_{ijk} and the $\frac{n(n+1)(n-4)}{6}$ equations allow to express exactly so many in terms of n of them, for which we take the four φ_{ijk} with $1 \le i < j < k \le 4$ and the $n-4$ expressions φ_{12k} with $5 \le k \le n$.

Proposition 4.3 *A minimal system of equations for the base space B_n is*

$$\begin{aligned}
\varphi_{1ij} &= \varphi_{12i} + \varphi_{12j} + \varphi_{134} - \varphi_{123} - \varphi_{124} \\
\varphi_{2ij} &= \varphi_{12i} + \varphi_{12j} + \varphi_{234} - \varphi_{123} - \varphi_{124} \\
\varphi_{ijk} &= \varphi_{12i} + \varphi_{12j} + \varphi_{12k} + \varphi_{134} + \varphi_{234} - 2\varphi_{123} - 2\varphi_{124}
\end{aligned}$$

where $3 \le i < j \le n$, $(i,j) \ne (3,4)$ in the first two lines and $3 \le i < j < k \le n$.

Proof We reduce φ_{ijk} using $\Phi^k_{i1;j2} = \varphi_{ijk} - \varphi_{1jk} - \varphi_{2ik} + \varphi_{12k}$ for $i \ge 3$ and $\Phi^1_{i2;j3} = \varphi_{1ij} - \varphi_{12j} - \varphi_{13i} + \varphi_{123}$, $\Phi^2_{1;j3}$, and finally $\Phi^1_{32;i4} = \varphi_{13i} - \varphi_{12i} - \varphi_{134} + \varphi_{124}$ and $\Phi^2_{31;i4}$. In the last two expressions all terms cancel if $i = 4$. Therefore the formulas in the statement continue to hold if an index has the value 3 or 4; some terms then cancel. □

Conversely, we also get equations for the total space, which deform the minimal generating set $z_iz_j - z_1z_2$, by substituting $a_{i,n+1} = z_i$ in the equations for B_{n+1} just given. This gives rather complicated formulas, as the general expression for φ_{ijk} shows.

For $n = 5$ the resulting equations of B_5 can be written as the Pfaffians of the skew symmetric 5×5 matrix

$$\begin{bmatrix}
a_{24} - a_{25} & -a_{14} + a_{15} & -a_{23} + a_{25} & a_{13} - a_{15} \\
 & a_{34} - a_{35} & -a_{12} - a_{25} & a_{12} - a_{13} + a_{24} - a_{34} \\
 & & a_{12} - a_{14} + a_{23} + a_{34} & -a_{12} + a_{15} \\
 & & & a_{34} + a_{45}
\end{bmatrix}$$

where we only write the part of the matrix above the diagonal.

4.2

The interpretation of the base space B_n as fine moduli space for R-polarised curves shows that the projectivisation $\mathbb{P}(B_n)$ is a compactification of the moduli space $M_{1,n+1}$ of $(n+1)$-pointed curves of genus 1 by curves with Gorenstein singularities; by forgetting the choice of t all curves $(C, p_0, \dots, p_n)$ above a line in the base space are isomorphic.

We compare this compactification with the compactifications constructed by Smyth [19,20].

Definition 4.4 Let C be a connected, reduced, complete curve of arithmetic genus one, with $n+1$ distinct smooth marked points $p_0, \dots, p_n$. Let $m < n+1$. The curve $(C, p_0, \dots, p_n)$ is m-stable if

1. the curve C has only nodes and Gorenstein singularities of genus one with $r \le m$ branches as singularities,
2. if $E \subset C$ is any connected subcurve with $p_a = 1$, then

$$|E \cap \overline{C\setminus E}| + |\{p_i \in E\}| > m\,,$$

3. $H^0(C, \Omega^\vee_C(-\sum p_i)) = 0$.

Smyth proves that the moduli stack $\overline{\mathcal{M}}_{1,n+1}(m)$ of m-stable curves is a proper irreducible Deligne-Mumford stack over Spec $\mathbb{Z}[1/6]$. In [20], working over a fixed algebraically closed field of characteristic zero, he proves that the corresponding coarse moduli space $\overline{M}_{1,n+1}(m)$ is projective.

Proposition 4.5 *The moduli space $\overline{M}_{1,n+1}(n)$ is isomorphic to the projectivisation $\mathbb{P}(B_n)$ of the base space of the versal deformation of L^n_{n+1}, for $n \geq 4$.*

Proof As L^n_{n+1} deforms into L^{n-1}_n (see Section 3.4), all Gorenstein genus 1 singularities with at most n branches occur over $\mathbb{P}(B_n)$. There cannot be a proper subcurve with $p_a = 1$: if there would be one with degree $k+1 < n+1$, then the cone over the hyperplane section at infinity would be of type $L^k_{k+1} \vee L^{n-k}_{n-k}$, which is not Gorenstein. For $m = n$ the condition (2) in Definition 4.4 excludes the case of a proper subcurve with $p_a = 1$.

By the definition of a coarse moduli space there is a map to $\mathbb{P}(B_n)$, which is bijective on closed points. As $\mathbb{P}(B_n)$ is normal, it is an isomorphism. □

For a discussion of the identification as stacks over Spec $\mathbb{Z}[1/6]$, see [12].

5 Elliptic partition curves

In this section we discuss the deformation theory of general partition curves.

5.1

The equations for the monomial curve with parametrisation $z_1 = t^{n+1}, z_2 = t^{n+2}, \ldots, z_{n-1} = t^{2n-1}, z_n = t^{2n}$ are

$$z_i z_j = \begin{cases} z_1 z_{i+j-1}, & i+j \leq n+1, \\ z_2 z_n, & i+j = n+2, \\ z_1^2 z_{i+j-n-2}, & i+j \geq n+3, \end{cases}$$

where $2 \leq i \leq j$. The curve deforms into other elliptic partition curves of the same multiplicity $n+1$. An explicit deformation is

$$z_i z_j = \begin{cases} z_1 z_{i+j-1}, & i+j \leq n+1, \\ z_2 z_n, & i+j = n+2, \\ (z_1^2 + a_2 z_n + \cdots + a_n z_2 + a_{n+1} z_1) z_{i+j-n-2}, & i+j \geq n+3. \end{cases}$$

The projection on the (z_1, z_2)-plane is given by $z_2^{n+1} = z_1^{n+2} + a_2 z_1^2 z_2^{n-1} + \cdots + a_{n+1} z_1^{n+1}$ and the factorisation of $z_2^{n+1} - a_2 z_1^2 z_2^{n-1} - \cdots - a_{n+1} z_1^{n+1}$ determines the partition of $n+1$. In particular we find a non-rational form of L^n_{n+1}: the (quadratic) equations, similar to the above ones but with $z_i z_j = z_1 z_{i+j-n-2}$ for $i+j \geq n+3$, define $n+1$ lines through the origin, passing through the points $(\varepsilon, \varepsilon^2, \ldots, \varepsilon^n)$ with ε running through the $(n+1)$-st roots of unity.

Knowing the base space for the monomial curve implies knowing the base space for all elliptic partition curves. However, it seems unfeasible to compute the base space in general. For the monomial curve of multiplicity 6 the computation of the versal deformation is described in some detail in [6]. The base space is just as for L^5_6 a cone over the Grassmannian $G(2,5)$, more precisely it is $B_5 \times \mathbb{A}^5$. By openness of versality the base space for a partition curve with r branches is $B_5 \times \mathbb{A}^{6-r}$. We have computed equations for the base space of the monomial curve of multiplicity 7. The equations contain a very large number of monomials, and the result is too complicated to be given here. Even for the non-rational form of seven lines the equations are rather complicated and not very useful; in particular, the inductive structure given by Theorem 4.1 is not visible.

5.2

The same phenomenon with complicated equations occurs for the versal deformation of (rational) partition curves. For L_n^n in the form of the coordinate axes the result is very easy, see [18] or [21]. The dimension of T^1 is $n(n-2)$, but we use $n(n-1)$ deformation parameters a_{ij}: the coordinate transformations $z_i \mapsto z_i - \delta_i$ induce $a_{ij} \mapsto a_{ij} + \delta_i$, for all j. The versal family is given by

$$F_{ij} = (z_i - a_{ij})(z_j - a_{ji}) - (a_{ik} - a_{ij})(a_{jk} - a_{ji})$$

with as equations for the base space

$$(a_{ik} - a_{ij})(a_{jk} - a_{ji}) - (a_{il} - a_{ij})(a_{jl} - a_{ji}) = 0\,.$$

For the irrational form of L_n^n, which is the hyperplane section $z_0 = z_n$ of the standard form of the cone over the rational normal curve of degree n, the equations become much more complicated. For the monomial curve the computation is only done up to multiplicity 5 [21]. The quadratic part of the equations of the base space involves as many variables as the base space of L_5^5 and it defines a degeneration of that space.

The (reduced) base space for L_n^n has been studied by Goryonov and Lando [9]. In particular, its degree in n^{n-3}. By a recent result of Polishchuk and Rains [17] this space is Cohen-Macaulay; they identify its coordinate ring with an algebra of global sections.

Cohen-Macaulayness does in general not hold for the base space of the monomial partition curve X_n.

Theorem 5.1 *The base space of the versal deformation of the monomial partition curve X_n with $n \geq 14$ has components which are not smoothing components; there are components of different dimensions.*

The idea behind this statement is that the curves X_n are the most singular curves, in the sense that that any curve singularity degenerates to a partition curve (in a δ-constant degeneration) [4].

Pinkham's examples of non-smoothable curves are among the curves L_r^n consisting of r lines in general position through the origin in $\mathbb{A}^n$ [11,15], with $n+1 \leq r \leq \binom{n+1}{2}$. One has $\delta(L_r^n) = 2r - n - 1$, so the genus of the curve is $g = r - n$. We can write $\delta = n + 2g - 1$. The general such curve is not smoothable if $r > n + 2 + \frac{6}{n-5}$ (where $n > 5$), or in terms of g if $r > g + 5 + \frac{6}{g-2}$. For $g = 4$ and $g = 6$ it suffices that $r > g + 5$, see e.g. [23]. The smallest example of a non-smoothable curve is L_{10}^6. The dimension of its base space is 15. A general smoothable L_{10}^6 has a reducible base space, with a smoothing component of dimension 20 and the 15-dimensional equisingular component. For the general smoothable L_r^n the smoothing component has dimension smaller than the number of moduli.

A partition curve with $r = n + g$ branches and $\delta = n + 2g - 1$ is $L_n^n \vee gA_2$, where gA_2 stands for $A_2 \vee \cdots \vee A_2$; it belongs to the partition $(1, \ldots, 1, 2 \ldots, 2)$ of $n + 2g$.

Lemma 5.2 *The curve $L_{r-1}^n \vee A_2$ deforms into L_r^n, if $n + 1 \leq r \leq \binom{n+1}{2}$.*

Proof Let the r-th line in L^n_r be parametrised by $(a_1 t, \dots, a_n t)$. We may suppose that $a_1 = 1$. Consider the deformation of the parametrisation of $L^n_{r-1} \vee A_2 \subset \mathbb{A}^n \times \mathbb{A}^2$, where only the paramatrisation of the cusp A_2 is changed: $(a_1 st, \dots, a_n st, t^2, t^3)$; here s is the deformation parameter. As the r lines impose independent conditions on quadrics, there exists a quadric q in the variables $(z_1, \dots, z_n)$ which vanishes on the first $r-1$ lines, and restricts to t^2 on the line $(a_1 t, \dots, a_n t)$. The coordinate transformation $z_{n+1} \mapsto z_{n+1} - q/s^2$, $z_{n+2} \mapsto z_{n+2} - z_1 q/s^3$ transforms $(a_1 st, \dots, a_n st, t^2, t^3)$ into $(a_1 st, \dots, a_n st, 0, 0)$ and leaves the first $r-1$ lines unchanged. Therefore the resulting curve is for $s \neq 0$ isomorphic to L^n_r. The deformation is flat as it is δ-constant. □

As the monomial curve X_{n+2g} deforms into $L^n_n \vee gA_2$, the proof of this lemma shows that it deforms into L^n_{n+g} for any (general) position of the lines, including smoothable L^n_{n+g}, which have components of different dimensions. We note also that X_N deforms into X_{N-1}. Therefore X_N with $N \geq 14$ has a base space with components of different dimensions.

Such a simple argument is not available for elliptic partition curves. It would be interesting to know into which type of singularities the Gorenstein monomial curve can deform.

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