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Joint Design of Radar Receive Filter and Unimodular ISAC Waveform with Sidelobe Level Control

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Abstract—Integrated sensing and communication (ISAC) has been considered a key feature of next-generation wireless networks. This paper investigates the joint design of the radar receive filter and dual-functional transmit waveform for the multiple-input multiple-output (MIMO) ISAC system. While optimizing the mean square error (MSE) of the radar receive spatial response and maximizing the achievable rate at the communication receiver, besides the constraints of full-power radar receiving filter and unimodular transmit sequence, we control the maximum range sidelobe level, which is often overlooked in existing ISAC waveform design literature, for better radar imaging performance. To solve the formulated optimization problem with convex and nonconvex constraints, we propose an inexact augmented Lagrangian method (ALM) algorithm. For each subproblem in the proposed inexact ALM algorithm, we custom-design a block successive upper-bound minimization (BSUM) scheme with closed-form solutions for all blocks of variable to enhance the computational efficiency. Convergence analysis shows that the proposed algorithm is guaranteed to provide a stationary and feasible solution. Extensive simulations are performed to investigate the impact of different system parameters on communication and radar imaging performance. Comparison with the existing works shows the superiority of the proposed algorithm.

Index Terms—Augmented Lagrangian method, dual-functional radar communication, unimodular waveform, range sidelobe control.

I. INTRODUCTION

Integrated sensing and communication (ISAC) unifies sensing and communication (S&C) tasks into a single system, improving efficiency and performance by sharing various resources like spectrum and hardware [1], [2]. With its potential to support emerging applications requiring high-quality wireless connections and accurate sensing, such as autonomous driving and smart homes, ISAC is widely regarded as a key

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enabler for next-generation wireless networks [3]. However, S&C subsystems have distinct performance requirements [4]: sensing favors unimodular and deterministic waveforms, while communication relies on waveforms with high degrees of freedom (DoFs) and randomness for efficient information transmission. As a result, the interference between them is inevitable.

Since wireless systems often need to simultaneously serve multiple users and meet their various sensing and communication needs, multiple-input multiple-output (MIMO) technology becomes particularly important in ISAC systems. Early ISAC research focused on the MIMO radar-communication coexistence by mitigating interference through spectrum-sharing techniques like dynamic spectrum access [5] and null-space projection [6]. Although these approaches enabled S&C coexistence, the systems were designed separately and required side-information exchange, leading to additional cooperation costs [7]. As an advancement from the spectrum-sharing scheme, dual-functional waveform design has recently drawn a lot of interest [8] for its ability to simultaneously sense targets and transmit information using a single device, eliminating the need for side-information exchange in S&C cooperation [7].

There have been many studies focusing on dual-functional waveform design; see [1] and the references therein. Nevertheless, the majority of these works have not considered controlling the sidelobes of the correlation function. In current communication systems, such as 5G NR, signals usually have a certain bandwidth [9]. Using these signals for ISAC inevitably affects the sensing performance due to range sidelobes. Large sidelobes are undesirable in sensing scenarios because they could interfere with or even mask the responses from targets that are far away or with a small radar cross-section (RCS), which will lead to missed detections [10]. Additionally, even with an optimized radar receive beampattern, high range sidelobe levels still result in false detections. Therefore, sidelobe control is crucial in ISAC waveform design to ensure sensing performance. It is well-known that the power amplifier works most efficiently when the input signal is unimodular [11], thus it is necessary to design the unimodular ISAC waveform. In this paper, we focus on the dual-functional waveform design for the monostatic downlink transmission in MIMO-ISAC system, aiming for maximizing the achievable rate and minimizing the beampattern mean square error (MSE), as well as controlling the peak side-lobe level.

A. Literature Review

Existing dual-functional waveform design research can be roughly divided into three categories: sensing-centric,

communication-centric, and joint waveform design, which will be detailed below.

Sensing-centric waveform design focuses on embedding the communication information into sensing waveforms [1]. For example, the communication information can be embedded into chirp waveforms [12], spatial beampattern [13], and ambiguity function [14]. While these approaches generally exhibit strong radar sensing capabilities, they often face low communication rates due to the limited number of embedded information bits. In contrast, communication-centric waveform design implements radar sensing using existing communication waveforms, such as orthogonal frequency division multiplexing (OFDM) [15] and orthogonal time frequency space (OTFS) [16]. However, the sensing performance of communication-centric designs is unpredictable due to the inherent randomness of communication signals and potential distortion from high peak-to-average power ratio.

To address the limitations of separate designs and achieve trade-offs between S&C, many works have focused on joint waveform design [1]. This approach constructs waveforms by solving optimization problems under various S&C constraints. More specifically, in [17], the authors jointly maximized a weighted sum rate while minimizing the radar beampattern approximation MSE, constrained by per-antenna power limits, enabling rate-splitting multiple access and interference management in MIMO-ISAC systems. In [18], the authors minimized the Cramér-Rao bound (CRB) for direction-of-arrival (DoA) estimation by designing the beamforming matrix under individual signal-to-interference-plus-noise ratio (SINR) constraints at each communication receiver and the transmit power budget. Additionally, the work [19] proposed to jointly precode communication and radar waveforms to achieve maximum DoFs in waveform design. Recent progress on joint waveform design in MIMO-ISAC systems has been made in [20] and [21], in which [20] optimized a weighted combination of the sum rate and the CRB for target estimation and [21] maximized the system energy efficiency by constraining the transmit power budget, communication SINR, and target estimation CRB.

The works mentioned previously accomplished a balance between S&C to some extent, but they did not address the issue of range sidelobe control. As for now, there have been few studies on sidelobe control for ISAC systems. The work [10] proposed a MIMO-ISAC waveform design framework that realized an integrated side-lobe level (ISL) reduction by minimizing a weighted sum of beampattern MSE, ISL, and MUI. Alternatively, the work [22] focused on maximizing the SINR at the radar output while ensuring communication performance, which also achieves the ISL reduction. However, the communication channels in the two works are assumed to be frequency-flat fading, whereas the ISAC signals capable of distinguishing symbol-level delays correspond to frequency-selective fading communication channels.

In addition to the ISL metric, the peak side-lobe level (PSL) should be more important for sidelobe control. That's because the PSL at radar receiver dictates the false alarm probability [23]. High PSL can lead to false alarms [24], making PSL control significant for achieving a low false alarm rate (FAR).

Despite its importance, the PSL minimization problem has received relatively little attention in both the radar signal [23] and ISAC waveform design. One of the main difficulties in directly minimizing the PSL is that the design metric is not differentiable and the corresponding optimization problem is a minimax problem. To the best of our knowledge, the existing works on PSL control focus on pure radar sensing, and the PSL control in ISAC scenarios remains unexplored. For instance, the work [11] proposed a two-step scheme to control the PSL in MIMO radar. The work [25] approximated PSL suppression by minimizing an ℓ_p norm metric with large p for single-antenna radar systems.

Taking the above factors into consideration, the goal of this paper is to design a downlink MIMO-ISAC waveform that maximizes the achievable rate at the communication receivers, makes the radar receiver's beampattern as close as possible to the desired one, and controls the PSL under the constraints of unimodular transmit sequences. However, as discussed in [11], designing waveforms with good correlation properties under the constraint of unimodular sequences is already a complicated task, and it will be more challenging to further require good communication performance and the desired beampattern. To simplify the optimization problem and obtain more DoFs on waveform design, we consider the joint design of the receive filter and transmit sequence in this paper.

B. Our Contributions

The main contributions of this paper are as follows.

- *Practical problem formulation:* Many practical factors are taken into account in the problem formulation, including beampattern MSE, communication MUI, PSL, and unimodular transmit sequence. An optimization problem is formulated to minimize a weighted sum of the radar receive beampattern MSE and the MUI at the communication receiver. To avoid solving a minimax problem by minimizing the PSL directly, we control the PSL by constraining the level of sidelobe at each range bin. The formulated problem is a large-scale optimization problem with convex and nonconvex constraints.
- *Efficient solution:* An inexact augmented Lagrangian method (ALM) algorithm is proposed to solve the formulated problem. Specifically, a block successive upper-bound minimization (BSUM) scheme is custom-designed to solve the subproblems in the ALM algorithm, and the updates for all blocks of variable in the BSUM scheme admits closed-form solutions, which makes the proposed algorithm efficient.
- *Convergence guarantee:* We analyze the convergence of the proposed inexact ALM algorithm with an adaptive penalty parameter. We show that the proposed algorithm is guaranteed to find a feasible stationary point of the formulated problem. This is the best that one can expect for this nonconvex optimization problem (with many nonconvex constraints).

Extensive simulation results are provided to demonstrate the effectiveness of the proposed algorithm. Specifically, Monte Carlo simulations are performed to evaluate the convergence performance of the proposed algorithm. The impacts of dif-

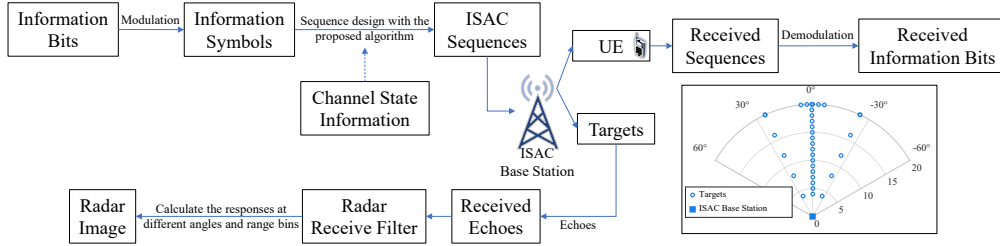


Fig. 1: A dual-functional radar communication system.

ferent system parameters on the system performance are examined. Finally, the proposed algorithm is compared with the ALM algorithm with a fixed penalty parameter and the modified work of [11] to show the superiority of the proposed algorithm.

It is worth mentioning that a similar inexact-ALM framework has been proposed in [26]. In [26], radar sensing beamforming is achieved through quantized constant-envelope waveform design while satisfying the communication performance constraints. However, it does not address how to suppress the sidelobes of the correlation function. Low range bin sidelobes and beampattern are equally important for good radar imaging. Furthermore, after considering sidelobe suppression, the convergence analysis of the optimization problem formulated in this paper becomes more complex compared to [26], since the constraint terms introduced by the auxiliary variables in [26] are linear, whereas those in this paper are nonlinear.

The rest of the paper is organized as follows. In Section II, we introduce the system model and formulate the optimization problem. In Section III, we propose an efficient algorithm for solving the formulated problem and analyze the convergence of the proposed algorithm. We present extensive simulation results in Section IV. Finally, we conclude the paper in Section V.

Notations: We use x , \mathbf{x} , \mathbf{X} , and \mathcal{X} to represent scalar, column vector, matrix, and set, respectively. The notation $\text{vec}(\cdot)$ represents the vectorization of a matrix by stacking its columns. The notations $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_F$ denote correspondingly the ℓ_1 , ℓ_2 , and Frobenius norms of a matrix, respectively. \mathbb{C} and \mathbb{R} denote the sets of complex and real numbers, respectively; $\mathcal{R}\{\cdot\}$ and $\mathcal{I}\{\cdot\}$ are the real and imaginary parts of a complex number, respectively. The superscript $(\cdot)^*$, $(\cdot)^T$ and $(\cdot)^H$ represent the conjugate, the transpose and the conjugate transpose operations, respectively. $\mathbb{I}_{\mathcal{A}}(A)$ denotes an indicator function of \mathcal{A} , and it takes 0 if $A \in \mathcal{A}$ and $+\infty$, otherwise. \otimes represents the Kronecker product. $\mathbf{A}_{[i:j,:]}$ represents the sub-matrix from the i -th row to the j -th row of a matrix \mathbf{A} , and $A[i,j]$ represents the element in the i -th row and j -th column of matrix \mathbf{A} . $\text{tr}(\mathbf{A})$ means the trace of a matrix \mathbf{A} . $\mathbf{A} \succeq \mathbf{B}$ means $\mathbf{A} - \mathbf{B}$ is positive semidefinite. $[n]$ denotes $\{1, 2, \dots, n\}$ for a positive integer n .

II. SYSTEM MODEL AND PROBLEM FORMULATION

As shown in Fig. 1, we consider an ISAC base station (BS) equipped with a uniform linear array (ULA) of N_T antennas. There are N_C single-antenna communication receivers in the downlink transmission. We consider a colocated radar receive

station equipped with a N_R -antenna ULA. Below we introduce the mathematical frameworks used in both S&C functions and formulate the optimization problem.

A. Communication Signal Model

Considering the frequency-selective feature of the channel in the current communication system [9], we implement the channel model proposed in [27], in which the channel between m -th receiver and n -th BS transmitting antenna, denoted as $\mathbf{H}_{m,n}$, is frequency-selective and assumed to be quasi-stationary during one block transmission, and the discrete-time channel gains are defined by a q -ray model. The channel $\mathbf{H}_{m,n}$ can be characterized by a lower triangular Toeplitz matrix with the first column being $[h_0^{(m,n)}, h_1^{(m,n)}, \dots, h_L^{(m,n)}, \mathbf{0}_T]^T$, where $L = \lceil \frac{\tau_{\max}}{T_s} \rceil$, T_s is the duration of one time-slot, τ_{\max} is the maximum delay spread, and T is the length of the block.

Due to the multi-taps feature of the channel, inter-block interference (IBI) occurs between every two consecutive transmissions (blocks). To eliminate IBI, a cyclic-prefix (CP) of length L is added at each transmission and will be removed at the receiver. The noiseless received signal at the m -th receiver is given by $\mathbf{y}_m = \sum_{n=1}^{N_T} \tilde{\mathbf{H}}_{m,n} \mathbf{x}_n$, where $\tilde{\mathbf{H}}_{m,n} = \mathbf{\Pi}_{\text{cp}} \mathbf{H}_{m,n} \mathbf{\Gamma}_{\text{cp}}$ is a $T \times T$ matrix, $\mathbf{\Gamma}_{\text{cp}} = [\mathbf{I}_{\text{cp}}, \mathbf{I}_T]^T$ and $\mathbf{\Pi}_{\text{cp}} = [\mathbf{0}_{T \times L}, \mathbf{I}_T]$ denote the CP-inducing and CP-removing matrices, respectively, and \mathbf{I}_{cp} contains the last L columns of a T -dimensional identity matrix \mathbf{I}_T . Then, the received signals at all N_C users can be represented as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \quad (1)$$

where $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, \dots, \mathbf{y}_{N_C}^T]^T$ with $\mathbf{y}_i \in \mathbb{C}^{T \times 1}$ being the received signal at the i -th user, $\mathbf{x} = [\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_{N_T}^T]^T$ with $\mathbf{x}_i \in \mathbb{C}^{T \times 1}$ being the i -th row vector of the transmission signal matrix $\mathbf{X} \in \mathbb{C}^{N_T \times T}$, $\mathbf{n} \in \mathbb{C}^{N_C T \times 1}$ is an additive white Gaussian noise (AWGN) vector with power σ_n^2 , and $\mathbf{H} \in \mathbb{C}^{N_C T \times N_T T}$ is defined as

$$\mathbf{H} = \begin{bmatrix} \tilde{\mathbf{H}}_{1,1} & \tilde{\mathbf{H}}_{1,2} & \dots & \tilde{\mathbf{H}}_{1,N_T} \\ \tilde{\mathbf{H}}_{2,1} & \tilde{\mathbf{H}}_{2,2} & \dots & \tilde{\mathbf{H}}_{2,N_T} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathbf{H}}_{N_C,1} & \tilde{\mathbf{H}}_{N_C,2} & \dots & \tilde{\mathbf{H}}_{N_C,N_T} \end{bmatrix}. \quad (2)$$

Let $\mathbf{S} \in \mathbb{C}^{N_C \times T}$ denote the transmitted information symbol matrix, where each entry of \mathbf{S} is randomly drawn from a given constellation. The received signal at the communication users can then be represented in vector form as

$$\mathbf{y} = \mathbf{s} + (\mathbf{H}\mathbf{x} - \mathbf{s}) + \mathbf{n}, \quad (3)$$

where $\mathbf{s} = [\mathbf{s}_1^T, \mathbf{s}_2^T, \dots, \mathbf{s}_{N_C}^T]^T$ with $\mathbf{s}_i \in \mathbb{C}^{T \times 1}$ being the i -th row vector of \mathbf{S} , the term $\mathbf{H}\mathbf{x} - \mathbf{s}$ can be viewed as the MUI [1] that interferes with the symbol detection at the communication

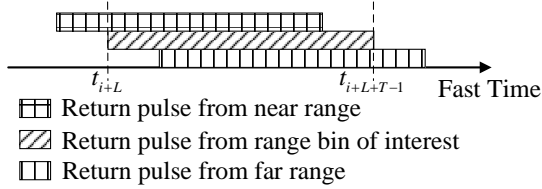


Fig. 2: The overlapped returned pulses.

receiver side. The received SINR at the i -th user is defined as

$$\gamma_i = \frac{\mathbb{E}_{\mathbf{s}_i} \{\|\mathbf{s}_i\|_2^2\}}{\mathbb{E}_{\mathbf{s}_i} \{\|\mathbf{H}_i \mathbf{x} - \mathbf{s}\|_2^2\} + \sigma_n^2}, \quad (4)$$

where $\mathbf{H}_i = [\tilde{\mathbf{H}}_{i,1}, \tilde{\mathbf{H}}_{i,2}, \dots, \tilde{\mathbf{H}}_{i,N_T}]$. The achievable downlink sum rate of the users can be given as

$$R = \sum_{i=1}^{N_C} \log_2(1 + \gamma_i). \quad (5)$$

Given that $\mathbb{E}_{\mathbf{s}_i} \{\|\mathbf{s}_i\|_2^2\}$ is a fixed value for a specific constellation strategy, minimizing $P_{\text{MUI}} = \|\mathbf{H}\mathbf{x} - \mathbf{s}\|_2^2$ leads to an increase in SINR, thereby indicating a higher achievable sum rate¹.

B. Signal Model for Radar Imaging

As shown in Fig. 2, after removing the CP, the received echos for radar imaging at the i -th range bin, $\mathbf{D}^{(i)} \in \mathbb{C}^{N_R \times T}$, are obtained from the t_{i+L} -th time slot to the $t_{i+L+T-1}$ -th time slot, where t_{i+L} is the starting time slot of the i -th range bin, and $\mathbf{D}^{(i)} \in \mathbb{C}^{N_R \times T}$ can be expressed as [11, eq. (13)]

$$\begin{aligned} \mathbf{D}^{(i)} &= \sum_{p^{(i)}=1}^{P^{(i)}} h_{p^{(i)}} \mathbf{a}(\theta_{p^{(i)}}) \mathbf{v}(\theta_{p^{(i)}})^T \mathbf{X} \\ &+ \sum_{k^{(i)} \in \Omega_K} \sum_{p'_{k^{(i)}}=1}^{P'_{k^{(i)}}} h_{p'_{k^{(i)}}} \mathbf{a}(\theta_{p'_{k^{(i)}}}) \mathbf{v}(\theta_{p'_{k^{(i)}}})^T \mathbf{X} \mathbf{J}_k + \mathbf{Z}_R, \end{aligned} \quad (6)$$

where \mathbf{X} is the transmitted ISAC waveform defined above, \mathbf{Z}_R is the additive Gaussian noise matrix with zero mean and covariance matrix \mathbf{R}_s , $\{\mathbf{a}(\theta_{p^{(i)}}) \in \mathbb{C}^{N_R \times 1}\}_{p^{(i)}=1}^{P^{(i)}}$ and $\{\mathbf{v}(\theta_{p^{(i)}}) \in \mathbb{C}^{N_T \times 1}\}_{p^{(i)}=1}^{P^{(i)}}$ are the radar receive and transmit steering vectors for the $P^{(i)}$ targets in the i -th range bin, respectively, and $\{h_{p^{(i)}} \in \mathbb{C}\}_{p^{(i)}=1}^{P^{(i)}}$ are complex amplitudes proportional to the RCS of these $P^{(i)}$ targets. The parameters with subscript $p'_{k^{(i)}}$ are defined for the $P'_{k^{(i)}}$ scatterers at the $k^{(i)}$ -th range bin with the same meanings, the temporal shifting matrix \mathbf{J}_k is defined as

$$\mathbf{J}_k = \mathbf{J}_{-k}^T = \begin{bmatrix} \mathbf{0}_{(T-k) \times k} & \mathbf{I}_{T-k} \\ \mathbf{I}_k & \mathbf{0}_{k \times (T-k)} \end{bmatrix},$$

and $\Omega_K = \{-K, \dots, -1, 1, 2, \dots, K\}$, where K is the maximum difference of arrival times between back scattered signals from the range bin of interest and signals from neighboring range bins.

To obtain more DoFs on the waveform design, we jointly design the radar receive filter and transmit sequence. Denote the receive filter at the radar receiver as $\mathbf{F} \in \mathbb{C}^{N_R \times T}$, the radar image at angle θ and range bin i is given by

$$r_{\theta,i} = |\mathbf{a}(\theta)^H \mathbf{D}^{(i)} \mathbf{F}^H \mathbf{a}(\theta)|. \quad (7)$$

¹Minimizing P_{MUI} is not equivalent to maximizing sum rate. We use this metric to simplify the optimization problem.

Under the signal models (6) and (7), we consider optimizing two metrics to improve radar imaging performance. One is the radar receive beampattern MSE, defined as $\|\mathbf{X}\mathbf{F}^H - \mathbf{R}_d\|_F$, measuring how close the receive beampattern approximates the desired one, where \mathbf{R}_d is the desired spatial response determined by the prior knowledge about the targets [26]. Another metric is the maximum sidelobe level $\max_{k \in \Omega_K} \{\|\mathbf{X}\mathbf{J}_k \mathbf{F}^H\|_F\}$, which is required to be less than a preset level ξ for all $k \in \Omega_K$ to reduce the clutters from neighboring range bins.

C. Problem Formulation

Based on the above discussions, the joint design problem of the radar receive filter and the ISAC waveform is formulated to minimize a weighted sum of the receive radar beampattern MSE and MUI at communication receivers, under the constraints of unimodular transmit sequence and the full power limit of the radar receive filter². The optimization problem is presented below:

$$\min_{\mathbf{F}, \mathbf{X}} f(\mathbf{F}, \mathbf{X}) \quad (8a)$$

$$\text{s.t. } \|\mathbf{F}\|_F^2 = P_F, \quad (8b)$$

$$\|\mathbf{X}\mathbf{J}_k \mathbf{F}^H\|_F \leq \xi, \quad \forall k \in \Omega_K, \quad (8c)$$

$$|x_{ij}| = P_x, \quad \forall x_{ij} \in \mathbf{X}, \quad (8d)$$

where $f(\mathbf{F}, \mathbf{X}) = \alpha \|\mathbf{X}\mathbf{F}^H - \mathbf{R}_d\|_F^2 + (1 - \alpha) \|\mathbf{H}\mathbf{x} - \mathbf{s}\|_2^2$.

Solving problem (8) is not straightforward due to the presence of both convex PSL constraints and nonconvex constraints on the receive filter and transmit sequence. In the next section, we propose an inexact ALM algorithm for solving problem (8) efficiently. ‘‘Inexact’’ here refers to solving the subproblems in the ALM algorithm inexactly. By doing so, we can significantly reduce the computational cost of solving the ALM subproblems without sacrificing the solution quality.

III. PROPOSED APPROACH

In this section, we propose an inexact ALM algorithm for solving problem (8). More specifically, we first introduce the main idea of the proposed inexact ALM algorithm in Section III-A. Then, we propose a scheme for finding a feasible point of problem (8) in Section III-B, which plays an important role in guaranteeing the convergence of the proposed algorithm to a *feasible* stationary point. We custom-design a BSUM algorithm for efficiently solving the ALM subproblem in Section III-C. Finally, we analyze the convergence of the proposed algorithm in Section III-D.

A. Framework of Proposed ALM Algorithm

The inexact ALM algorithm is proposed to solve problem (8). Its basic idea is to decompose the original optimization problem into smaller and more manageable subproblems. At each iteration, the algorithm updates primal variables associated with each ALM subproblem with a penalty term for penalizing the violation of the constraints in the original

²We adopt the full-power radar receive filter to ensure that the mainlobe energy remains as high as possible while suppressing the sidelobe level. Although this can be achieved by maximizing the sidelobe-to-mainlobe ratio, it leads to a fractional programming problem, increasing the complexity of the solution. In future work, we will consider using the sidelobe-to-mainlobe ratio as an optimization objective or constraint.

problem, followed by an update of the dual variables. Unlike the classic ALM algorithm, which solves each subproblem exactly, we solve each subproblem *inexactly* at each iteration for computational efficiency.

To present the inexact ALM algorithm, we first reformulate problem (8) by introducing auxiliary variables $\mathbf{C} = (\mathbf{C}_k)_{k \in \Omega_K}$ with $\mathbf{C}_k \in \mathbb{C}^{N_T \times N_R}$ for each $k \in \Omega_K$. Then problem (8) becomes

$$\begin{aligned} & \min_{\mathbf{F}, \mathbf{X}, \mathbf{C}} f(\mathbf{F}, \mathbf{X}) \\ & \text{s.t. } \|\mathbf{F}\|_{\mathbb{F}}^2 = P_{\mathbb{F}}, \\ & \quad \mathbf{X} \mathbf{J}_k \mathbf{F}^{\text{H}} = \mathbf{C}_k, \\ & \quad \|\mathbf{C}_k\|_{\mathbb{F}} \leq \xi, \quad \forall k \in \Omega_K, \\ & \quad |x_{ij}| = P_x, \quad \forall x_{ij} \in \mathbf{X}. \end{aligned} \quad (9)$$

By denoting the Lagrange multipliers associated with the constraints $\mathbf{X} \mathbf{J}_k \mathbf{F}^{\text{H}} = \mathbf{C}_k$ for all $k \in \Omega_K$ as $\mathbf{U} = (\mathbf{U}_k)_{k \in \Omega_K}$, the corresponding augmented Lagrangian function (ALF) [28] of problem (9) is

$$\begin{aligned} \mathcal{L}_{\rho}(\mathbf{C}, \mathbf{F}, \mathbf{X}; \mathbf{U}) = & f(\mathbf{F}, \mathbf{X}) + \sum_{k \in \Omega_K} \left[\frac{\rho}{2} \|\mathbf{X} \mathbf{J}_k \mathbf{F}^{\text{H}} - \mathbf{C}_k\|_{\mathbb{F}}^2 \right. \\ & \left. + \mathcal{R}\{\text{tr}[\mathbf{U}_k^{\text{H}} (\mathbf{X} \mathbf{J}_k \mathbf{F}^{\text{H}} - \mathbf{C}_k)]\} \right]. \end{aligned} \quad (10)$$

The proposed ALM algorithm is based on the above ALF and mainly consists of four main components, which are the initialization, the solution of the ALM subproblem, the update of the Lagrange variable (also called dual variable), and the update of the penalty parameter. In the next, we introduce them one by one and highlight their important roles in the whole ALM algorithm. To simplify the notations, the point $(\mathbf{C}, \mathbf{F}, \mathbf{X})$ is represented by a multiplet $\mathbf{z} = (\mathbf{C}, \mathbf{F}, \mathbf{X})$.

1) Initialization

To initialize the proposed algorithm, we choose a feasible point $\mathbf{z}^{(\text{feas})} \in \mathcal{S} \cap \mathcal{S}_0$, where

$$\begin{aligned} \mathcal{S}_0 = & \{(\mathbf{C}, \mathbf{F}, \mathbf{X}) \mid \mathbf{X} \mathbf{J}_k \mathbf{F}^{\text{H}} = \mathbf{C}_k, \quad \forall k \in \Omega_K\}, \\ \mathcal{S} = & \{(\mathbf{C}, \mathbf{F}, \mathbf{X}) \mid \mathbf{C}_k \in \mathcal{S}_{\mathbf{C}}, \quad \forall k \in \Omega_K; \mathbf{F} \in \mathcal{S}_{\mathbb{F}}; \mathbf{X} \in \mathcal{S}_{\mathbf{X}}\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}_{\mathbf{C}} = & \{\mathbf{C} \in \mathbb{C}^{N_T \times N_T} \mid \|\mathbf{C}\|_{\mathbb{F}} \leq \xi\}, \\ \mathcal{S}_{\mathbb{F}} = & \{\mathbf{F} \in \mathbb{C}^{N_T \times T} \mid \|\mathbf{F}\|_{\mathbb{F}}^2 = P_{\mathbb{F}}\}, \\ \mathcal{S}_{\mathbf{X}} = & \{\mathbf{X} \in \mathbb{C}^{N_T \times T} \mid |x_{ij}| = P_x, \quad \forall x_{ij} \in \mathbf{X}\}. \end{aligned} \quad (11)$$

The details of finding a feasible point of problem (9) is relegated to Section III-B to maintain smoothness in presenting the proposed inexact ALM algorithm.

By choosing the penalty parameter $\rho^{(0)} > 0$, choosing an arbitrary initial point $\mathbf{z}^{(0)} \in \mathcal{S}$, and setting a finite Lagrange multiplier $\mathbf{U}^{(0)}$, we can choose a finite constant number ζ satisfying

$$\zeta \geq \max \left\{ f(\mathbf{F}^{(\text{feas})}, \mathbf{X}^{(\text{feas})}), \mathcal{L}_{\rho^{(0)}}(\mathbf{z}^{(0)}; \mathbf{U}^{(0)}) \right\}. \quad (12)$$

The upperboundness of the objective function and the ALF are important in proving the convergence of the proposed ALM algorithm to a feasible stationary point. In our proposed ALM algorithm, the penalty parameter is adaptively updated. The adaptive update of the penalty parameter enables a fast convergence but also makes the convergence analysis difficult. The upperboundness here can overcome the difficulty in the

convergence analysis and guarantees that any limit point of the sequence generated by the proposed algorithm is always feasible. More details on the convergence analysis will be presented in Section III-D.

2) Solving the ALM Subproblem Inexactly

At the ℓ -th iteration, the classic ALM algorithm updates the primal variables by solving the following ALM subproblem:

$$\mathbf{z}^{(\ell+1)} \in \arg \min_{\mathbf{z} \in \mathcal{S}} \mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}; \mathbf{U}^{(\ell)}). \quad (13)$$

However, problem (13) is nonconvex due to the nonconvex constraints of full power radar receiving filter and unimodular transmit sequence. To improve the computational efficiency, we propose to inexactly solve problem (13) to an $\varepsilon^{(\ell)}$ -stationary point, where the ε -stationary point is defined below:

Definition 1 (ε -stationary point of problem (13)). *For a fixed point $\bar{\mathbf{U}}$, the point $\bar{\mathbf{z}} \in \mathcal{S}$ is called an ε -stationary point of problem (13) if there exist $\bar{\mathbf{A}} \in \partial \mathbb{I}_{\mathcal{S}_{\mathbb{F}}}(\bar{\mathbf{F}})$, $\bar{\mathbf{B}} \in \partial \mathbb{I}_{\mathcal{S}_{\mathbf{X}}}(\bar{\mathbf{X}})$, and $\bar{\mathbf{D}}_k \in \partial \mathbb{I}_{\mathcal{S}_{\mathbf{C}}}(\bar{\mathbf{C}}_k)$ for all $k \in \Omega_K$ such that*

$$\begin{aligned} & \|\bar{\mathbf{A}} + \nabla_{\mathbf{F}} \mathcal{L}_{\rho}(\bar{\mathbf{z}}; \bar{\mathbf{U}})\|_{\mathbb{F}} \leq \varepsilon, \\ & \|\bar{\mathbf{B}} + \nabla_{\mathbf{X}} \mathcal{L}_{\rho}(\bar{\mathbf{z}}; \bar{\mathbf{U}})\|_{\mathbb{F}} \leq \varepsilon, \\ & \|\bar{\mathbf{D}}_k + \nabla_{\mathbf{C}_k} \mathcal{L}_{\rho}(\bar{\mathbf{z}}; \bar{\mathbf{U}})\|_{\mathbb{F}} \leq \varepsilon, \quad \forall k \in \Omega_K. \end{aligned}$$

The notations $\partial \mathbb{I}_{\mathcal{S}_{\mathbb{F}}}(\mathbf{F})$, $\partial \mathbb{I}_{\mathcal{S}_{\mathbf{X}}}(\mathbf{X})$, and $\partial \mathbb{I}_{\mathcal{S}_{\mathbf{C}}}(\mathbf{C}_k)$ are subdifferentials of indicator functions; see [29] and [30, Corollary 8.20] for more details. The ε -stationary point defined in Definition 1 reduces to the standard stationary point when setting $\varepsilon = 0$.

In inexactly solving the ALM subproblem in (13), we also require that the inexact solution $\mathbf{z}^{(\ell+1)}$ also satisfies that the value of the ALF is bounded by

$$\mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}^{(\ell+1)}; \mathbf{U}^{(\ell)}) \leq \zeta, \quad (14)$$

where ζ is defined in (12). To find an ε -stationary point that satisfies (9), we shall develop a BSUM algorithm, which will be elaborated in detail in Section III-C. This algorithm offers closed-form updates for all blocks of variable and is guaranteed to find an ε -stationary point within a finite number of iterations.

3) Updating Lagrange Multipliers

Once the $\varepsilon^{(\ell)}$ -stationary point is obtained, we update the Lagrange multipliers in \mathbf{U} for all $k \in \Omega_K$ according to the following updating rule:

$$\tilde{\mathbf{U}}_k = \mathbf{U}_k^{(\ell)} + \rho^{(\ell)} (\mathbf{X}^{(\ell+1)} \mathbf{J}_k \mathbf{F}^{(\ell+1)\text{H}} - \mathbf{C}_k^{(\ell+1)}), \quad (15a)$$

$$\mathbf{U}_k^{(\ell+1)}[i, j] = \begin{cases} \frac{u_{\max}}{|\tilde{U}[i, j]|} \tilde{U}[i, j], & \text{if } |\tilde{U}[i, j]| > u_{\max}; \\ \tilde{U}[i, j], & \text{otherwise.} \end{cases} \quad (15b)$$

In the above, \mathbf{U}_k is first updated through the standard updating rule [28] in (15a); then, each element in \mathbf{U}_k is projected onto an interval $[-u_{\max}, u_{\max}]$ with $u_{\max} > 0$ being a preset constant. The projection onto the bounded set in (15a) guarantees that the Lagrange multipliers are uniformly bounded and plays a central role in guaranteeing the convergence of the proposed algorithm.

4) Updating the Penalty Parameter

To accelerate the convergence speed, we propose an adaptive update rule for the penalty parameter. Denote the violation

Algorithm 1 An Inexact ALM Algorithm for Problem (8)

Input: Initial point $\mathbf{z}^{(0)}$, $\mathbf{U}^{(0)}$, $\{\varepsilon^{(\ell)}\}_{\ell \geq 0}$ strictly decreasing with $\lim_{\ell \rightarrow \infty} \varepsilon^{(\ell)} = 0$, penalty parameter $\rho^{(0)}$.

- 1: **repeat**
 - 2: Obtain an $\varepsilon^{(\ell)}$ -stationary point $(\mathbf{z}^{(\ell+1)}, \mathbf{U}^{(\ell+1)})$ by Algorithm 2 (details provided in Section III-C);
 - 3: Update the dual variables $\mathbf{U}^{(\ell+1)}$ by (15a);
 - 4: Update the penalty parameter $\rho^{(\ell+1)}$ by (17);
 - 5: $\ell \leftarrow \ell + 1$;
 - 6: **until** certain stopping criteria;
- Output:** $(\mathbf{z}^{(\ell+1)}, \mathbf{U}^{(\ell+1)})$.
-

of the constraints after the ℓ -th iteration as $v^{(\ell+1)}$, i.e.,

$$v^{(\ell+1)} = \sqrt{\sum_{k \in \Omega_K} \|\mathbf{X}^{(\ell+1)} \mathbf{J}_k \mathbf{F}^{(\ell+1)\text{H}} - \mathbf{C}_k^{(\ell+1)}\|_{\mathbb{F}}^2}. \quad (16)$$

The penalty parameter $\rho^{(\ell+1)}$ will be updated through the following rule:

$$\rho^{(\ell+1)} = \begin{cases} \gamma \rho^{(\ell)}, & \text{if } v^{(\ell+1)} > \delta v^{(\ell)}; \\ \rho^{(\ell)}, & \text{otherwise.} \end{cases} \quad (17)$$

where $\delta > 0$ and $\gamma > 1$. The update rule in (17) will increase the penalty parameter if the violation at the current iteration is not reduced sufficiently compared with that in the previous one.

The proposed inexact ALM algorithm for solving problem (9) is summarized in Algorithm 1. By setting a positive sequence $\{\varepsilon^{(\ell)}\}_{\ell \geq 0}$ with $\varepsilon^{(\ell)} \rightarrow 0$ as $\ell \rightarrow +\infty$, the limit point of the sequence generated by Algorithm 1, $(\mathbf{z}^{(\infty)}, \mathbf{U}^{(\infty)})$, is a stationary point of the ALF in (10). Besides, since $\mathbf{X}^{(\infty)} \mathbf{J}_k \mathbf{F}^{(\infty)\text{H}} = \mathbf{C}_k^{(\infty)}$ holds for all $k \in \Omega_K$, which will be proved further ahead, then $\mathbf{z}^{(\infty)}$ is a feasible stationary point of problem (9). According to [31], any feasible stationary point of (9) is also a feasible stationary point of (8), i.e., $(\mathbf{F}^{(\infty)}, \mathbf{X}^{(\infty)})$ is a feasible stationary point of (8). The detailed convergence analysis will be given in Section III-D.

B. A Feasible Point of Problem (9)

In this subsection, we focus on finding a feasible point of problem (9), i.e., a pair of matrices $(\mathbf{F}, \mathbf{X}) \in \mathcal{S}_{\mathbb{F}} \times \mathcal{S}_{\mathbb{X}}$ such that

$$\|\mathbf{X} \mathbf{J}_k \mathbf{F}^{\text{H}}\|_{\mathbb{F}} \leq \xi, \quad \forall k \in \Omega_K. \quad (18)$$

The auxiliary variable \mathbf{C} can be easily obtained by setting $\mathbf{C}_k = \mathbf{X} \mathbf{J}_k \mathbf{F}^{\text{H}}$ for all $k \in \Omega_K$.

We formulate two subproblems for finding $(\mathbf{F}, \mathbf{X}) \in \mathcal{S}_{\mathbb{F}} \times \mathcal{S}_{\mathbb{X}}$ that satisfies (18). To be specific, after randomly initializing (\mathbf{F}, \mathbf{X}) , the first subproblem is formulated to reduce the ISL, i.e., $\sum_{k \in \Omega_K} \|\mathbf{X} \mathbf{J}_k \mathbf{F}^{\text{H}}\|_{\mathbb{F}}^2$, concerning \mathbf{X} . We update \mathbf{X} several times to minimize the ISL: if $\max_{k \in \Omega_K} \{\|\mathbf{X} \mathbf{J}_k \mathbf{F}^{\text{H}}\|_{\mathbb{F}}\}$ is already less than ξ , we then find a feasible point; otherwise, we formulate the second subproblem to minimize the ISL concerning $\bar{\mathbf{F}} = \mathbf{F}^{\text{H}} \mathbf{F}$. When solving for the variable \mathbf{X} , the following lemma is useful.

Lemma 1 [32]. *Given Hermitian $\mathbf{M} \in \mathbb{C}^{n \times n}$ and $\mathbf{Z} \in \mathbb{C}^{m \times m}$ and any $\mathbf{X}^{(t)} \in \mathbb{C}^{m \times n}$, the function $\text{tr}(\mathbf{Z} \mathbf{X} \mathbf{M} \mathbf{X}^{\text{H}})$ can be*

majorized by

$$\lambda \|\mathbf{X}\|_{\mathbb{F}}^2 + 2\mathcal{R} \left\{ \text{tr} \left[(\mathbf{Z} \mathbf{X}^{(t)} \mathbf{M} - \lambda \mathbf{X}^{(t)})^{\text{H}} \mathbf{X} \right] \right\} + C,$$

where $C = \text{tr}[(\mathbf{X}^{(t)})^{\text{H}} (\lambda \mathbf{X}^{(t)} - \mathbf{Z} \mathbf{X}^{(t)} \mathbf{M})]$ is irrelevant to \mathbf{X} and λ satisfies $\lambda \mathbf{I} \succeq \mathbf{M}^{\text{T}} \otimes \mathbf{Z}$.

1) Reducing the ISL Regarding \mathbf{X}

By choosing any $\mathbf{F} \in \mathcal{S}_{\mathbb{F}}$, and denoting $\Theta = \sum_{k \in \Omega_K} \mathbf{J}_k \mathbf{F}^{\text{H}} \mathbf{F} \mathbf{J}_k^{\text{H}}$, the problem of reducing the ISL regarding \mathbf{X} is formulated as

$$\begin{aligned} & \min_{\mathbf{X}} \text{tr}(\mathbf{X} \Theta \mathbf{X}^{\text{H}}) \\ & \text{s.t. } \mathbf{X} \in \mathcal{S}_{\mathbb{X}}. \end{aligned} \quad (19)$$

According to Lemma 1, the tightest upper bound λ to majorize $\text{tr}(\mathbf{X} \Theta \mathbf{X}^{\text{H}})$ is the maximum eigenvalue of Θ , denoted by $\lambda_{\max}(\Theta)$. However, Θ may change after updating \mathbf{F} . To avoid the high computational cost of computing $\lambda_{\max}(\Theta)$, an alternative upper bound λ_{θ} is implemented, where

$$\lambda_{\theta} = \|\Theta\|_1 \geq \lambda_{\max}(\Theta).$$

Then, the quadratic term $\text{tr}(\mathbf{X} \Theta \mathbf{X}^{\text{H}})$ can be majorized as follows:

$$\begin{aligned} \text{tr}(\mathbf{X} \Theta \mathbf{X}^{\text{H}}) & \leq \lambda_{\theta} \|\mathbf{X}\|_{\mathbb{F}}^2 \\ & + 2\mathcal{R} \left\{ \text{tr} \left[(\mathbf{X}^{(t)} \Theta - \lambda_{\theta} \mathbf{X}^{(t)})^{\text{H}} \mathbf{X} \right] \right\} + C_{\mathbb{X}}, \end{aligned} \quad (20)$$

where $C_{\mathbb{X}}$ is the term irrelevant to \mathbf{X} and $\mathbf{X}^{(t)}$ is the result of previous iteration. Thus, the majorized problem of (19) is

$$\begin{aligned} & \min_{\mathbf{X}} \mathcal{R} \left\{ \text{tr} \left[(\mathbf{X}^{(t)} \Theta - \lambda_{\theta} \mathbf{X}^{(t)})^{\text{H}} \mathbf{X} \right] \right\} \\ & \text{s.t. } \mathbf{X} \in \mathcal{S}_{\mathbb{X}}. \end{aligned} \quad (21)$$

It is clear that solving problem (21) is equivalent to solving the following problem:

$$\begin{aligned} & \min_{\mathbf{X}} \|\mathbf{X} - \mathbf{Z}^{(t)}\|_{\mathbb{F}}^2 \\ & \text{s.t. } \mathbf{X} \in \mathcal{S}_{\mathbb{X}}, \end{aligned} \quad (22)$$

where $\mathbf{Z}^{(t)} = \lambda_{\theta} \mathbf{X}^{(t)} - \mathbf{X}^{(t)} \Theta$, whose closed form solution is $\mathbf{X}^{(t+1)} = e^{j \arg(\mathbf{Z}^{(t)})}$.

2) Minimizing the Sidelobe Level Regarding \mathbf{F}

Suppose that the obtained solution of problem (19) is not feasible, we further minimize the sidelobe level with respect to \mathbf{F} in this part. By denoting $\Phi = \sum_{k \in \Omega_K} \mathbf{J}_k^{\text{H}} \mathbf{X}^{\text{H}} \mathbf{X} \mathbf{J}_k$ and its singular value decomposition (SVD) as $\Phi = \mathbf{U}_{\Phi} \Sigma_{\Phi} \mathbf{U}_{\Phi}^{\text{H}}$ (assuming that the singular values are arranged in descending order), the ISL becomes

$$\text{tr}(\mathbf{Z} \Sigma_{\Phi} \mathbf{Z}^{\text{H}}) = \sum_{i=1}^N \lambda_i(\Phi) \|\mathbf{z}_i\|_2^2, \quad (23)$$

where $\mathbf{Z} = \mathbf{F} \mathbf{U}_{\Phi}$, \mathbf{z}_i is the i -th row vector of \mathbf{Z} , and $\lambda_i(\Phi)$ is the i -th singular value of Φ . Since Φ is positive semidefinite, we obtain the minimum of (23) by setting all rows of \mathbf{Z} as zero except for the row corresponding to the minimum singular value of Φ . That is to say, we obtain an \mathbf{F} that minimizes the ISL by setting $\mathbf{F} = \Sigma_{\mathbb{F}} \mathbf{U}_{\Phi}^{\text{H}}$, where $\Sigma_{\mathbb{F}} = \text{Diag}(\mathbf{d}_{\mathbb{F}})$ with $\mathbf{d}_{\mathbb{F}} = [\mathbf{0}_{N_{\text{T}}-1}^{\text{T}}, \sqrt{P_{\mathbb{F}}}]^{\text{T}}$, and $\text{Diag}(\mathbf{d}_{\mathbb{F}})$ is a diagonal matrix with its main diagonal entries being $\mathbf{d}_{\mathbb{F}}$.

C. BSUM Algorithm for Inexactly Solving Subproblem (13)

In this subsection, we propose a BSUM scheme to solve the ALM subproblem (13) inexactly. In particular, the initial point at the ℓ -th iteration is chosen as follows:

$$\mathbf{z}_{\text{init}}^{(\ell)} = \begin{cases} \mathbf{z}^{(\text{feas})}, & \text{if } f(\mathbf{F}^{(\text{feas})}, \mathbf{X}^{(\text{feas})}) < \mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}^{(\ell)}; \mathbf{U}^{(\ell)}); \\ \mathbf{z}^{(\ell)}, & \text{otherwise,} \end{cases} \quad (24)$$

where $\mathbf{z}^{(\ell)}$ is the approximate solution obtained at the $(\ell-1)$ -th iteration of Algorithm 1, and $\mathbf{z}^{(\text{feas})}$ is a feasible point obtained from Section III-B. It follows from (10) and the definition of ζ in (12) that

$$\mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}^{(\text{feas})}; \mathbf{U}^{(\ell)}) \leq f(\mathbf{F}^{(\text{feas})}, \mathbf{X}^{(\text{feas})}) \leq \zeta. \quad (25)$$

This shows that the choice of the initial point in (24) guarantees $\mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}_{\text{init}}^{(\ell)}; \mathbf{U}^{(\ell)}) \leq \zeta$. Moreover, the proposed BSUM method has a sufficient descent property, which will be proved in Section III-D. Therefore, for any $\ell \geq 0$, we always have

$$\mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}^{(\ell+1)}; \mathbf{U}^{(\ell)}) \leq \mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}_{\text{init}}^{(\ell)}; \mathbf{U}^{(\ell)}) \leq \zeta. \quad (26)$$

Consequently, the uniform upper bound requirement in (14) can always be satisfied under the initial point choice strategy of (24).

Denote $\mathbf{z}^{(t)} = (\mathbf{C}^{(t)}, \mathbf{F}^{(t)}, \mathbf{X}^{(t)})$. Next we present the update of each block of variables in the BSUM method.

1) Solving for Auxiliary Variables

The update of auxiliary variables in \mathbf{C} can be formulated as $|\Omega_K|$ independent optimization problems for each \mathbf{C}_k . Instead of solving (13) with respect to (w.r.t.) each \mathbf{C}_k directly, we consider the following subproblem:

$$\min_{\mathbf{C}_k} \frac{\rho^{(\ell)}}{2} \left\| \mathbf{X}^{(t)} \mathbf{J}_k \mathbf{F}^{(t)\text{H}} - \mathbf{C}_k + \frac{\mathbf{U}_k^{(\ell)}}{\rho^{(\ell)}} \right\|_{\text{F}}^2 + \frac{\beta}{2} \left\| \mathbf{C}_k - \mathbf{C}_k^{(t)} \right\|_{\text{F}}^2, \quad (27)$$

s.t. $\|\mathbf{C}_k\|_{\text{F}} \leq \xi$,

where $\beta > 0$. As mentioned in [29], by setting $\beta > 0$, the distance between two consecutive iterations of each \mathbf{C}_k becomes controllable, and every update of \mathbf{C}_k achieves a sufficient decrease. It is clear that problem (27) has a closed-form solution as follows:

$$\mathbf{C}_k^{(t+1)} = \begin{cases} \tilde{\mathbf{C}}_k, & \text{if } \|\tilde{\mathbf{C}}_k\|_{\text{F}} \leq \xi; \\ \frac{\xi}{\|\tilde{\mathbf{C}}_k\|_{\text{F}}} \tilde{\mathbf{C}}_k, & \text{otherwise,} \end{cases} \quad (28)$$

$$\text{where } \tilde{\mathbf{C}}_k = \frac{1}{\rho^{(\ell)} + \beta} \left(\rho^{(\ell)} \mathbf{X}^{(t)} \mathbf{J}_k \mathbf{F}^{(t)\text{H}} + \mathbf{U}_k^{(\ell)} + \beta \mathbf{C}_k \right).$$

2) Solving for Radar Receive Filter

The problem of solving (13) w.r.t. \mathbf{F} can be rewritten as

$$\min_{\mathbf{F}} \mathcal{L}_{\text{F}}(\mathbf{F}) \quad (29)$$

s.t. $\|\mathbf{F}\|_{\text{F}}^2 = P_{\text{F}}$,

where $\mathcal{L}_{\text{F}}(\mathbf{F})$ is defined as follows:

$$\mathcal{L}_{\text{F}}(\mathbf{F}) = \text{tr}(\mathbf{F} \mathbf{Q} \mathbf{F}^{\text{H}}) - 2\alpha \mathcal{R} \{ \text{tr}(\mathbf{R}_{\text{d}}^{\text{H}} \mathbf{X} \mathbf{F}^{\text{H}}) \} - \sum_{k \in \Omega_K} \mathcal{R} \{ \text{tr}[(\rho \mathbf{C}_k - \mathbf{U}_k)^{\text{H}} \mathbf{X} \mathbf{J}_k \mathbf{F}^{\text{H}}] \}, \quad (30)$$

which is reformulated from (10) by ignoring the terms irrelevant to \mathbf{F} and denoting $\mathbf{Q} = \alpha \mathbf{X}^{\text{H}} \mathbf{X} + \sum_{k \in \Omega_K} \frac{\rho}{2} \mathbf{J}_k^{\text{H}} \mathbf{X}^{\text{H}} \mathbf{X} \mathbf{J}_k$, and the superscripts of $\mathbf{X}^{(t)}$, $\mathbf{C}^{(t+1)}$, $\mathbf{U}^{(\ell)}$, and $\rho^{(\ell)}$ are omitted for notational simplicity in this part. The quadratic

term $\text{tr}(\mathbf{F} \mathbf{Q} \mathbf{F}^{\text{H}})$ can be majorized at $\mathbf{F}^{(t)}$ through Lemma 1 as follows:

$$\text{tr}(\mathbf{F} \mathbf{Q} \mathbf{F}^{\text{H}}) \leq \lambda_q \|\mathbf{F}\|_{\text{F}}^2 + 2\mathcal{R} \{ \text{tr}[(\mathbf{F}^{(t)} (\mathbf{Q} - \lambda_q \mathbf{I}))^{\text{H}} \mathbf{F}] \} + C_{\text{F}}, \quad (31)$$

where $\lambda_q > \|\mathbf{Q}\|_1$, and C_{F} is the term irrelevant to \mathbf{F} .

By substituting (31) into (30), we have $\mathcal{G}_{\text{F}}(\mathbf{F} | \mathbf{F}^{(t)})$ as the tight upper bound of $\mathcal{L}_{\text{F}}(\mathbf{F})$, i.e., $\mathcal{G}_{\text{F}}(\mathbf{F} | \mathbf{F}^{(t)}) \geq \mathcal{L}_{\text{F}}(\mathbf{F})$ for all $\mathbf{F} \in \mathcal{S}_{\text{F}}$ with equality holding when $\mathbf{F} = \mathbf{F}^{(t)}$, and

$$\mathcal{G}_{\text{F}}(\mathbf{F} | \mathbf{F}^{(t)}) = \lambda_q \|\mathbf{F}\|_{\text{F}}^2 - 2\mathcal{R} \left\{ \text{tr} \left(\Xi^{(t)\text{H}} \mathbf{F} \right) \right\} + C_{\text{F}}, \quad (32)$$

where

$$\Xi^{(t)} = \sum_{k \in \Omega_K} \frac{1}{2} (\rho \mathbf{C}_k - \mathbf{U}_k)^{\text{H}} \mathbf{X} \mathbf{J}_k + \alpha \mathbf{R}_{\text{d}}^{\text{H}} \mathbf{X} - \mathbf{F}^{(t)} \mathbf{Q} + \lambda_q \mathbf{F}^{(t)}. \quad (33)$$

Then, the majorized problem of (29) is

$$\min_{\mathbf{F}} \mathcal{G}_{\text{F}}(\mathbf{F} | \mathbf{F}^{(t)}) \quad (34)$$

s.t. $\|\mathbf{F}\|_{\text{F}}^2 = P_{\text{F}}$.

Problem (34) has a closed-form solution as follows:

$$\mathbf{F}^{(t+1)} = \frac{\sqrt{P_{\text{F}}}}{\|\Xi^{(t)}\|_{\text{F}}} \Xi^{(t)}. \quad (35)$$

3) Solving for Transmit Waveform

The problem of solving (13) w.r.t. \mathbf{X} can be rewritten as

$$\min_{\mathbf{X}} \mathcal{L}_{\text{X}}(\mathbf{X}), \quad (36)$$

s.t. $\mathbf{X} \in \mathcal{S}_{\text{X}}$,

where

$$\mathcal{L}_{\text{X}}(\mathbf{X}) = \text{tr}(\mathbf{X} \mathbf{P} \mathbf{X}^{\text{H}}) - 2\alpha \mathcal{R} \{ \text{tr}(\mathbf{X} \mathbf{F}^{\text{H}} \mathbf{R}_{\text{d}}^{\text{H}}) \} - \sum_{k \in \Omega_K} \mathcal{R} \{ \text{tr}[(\rho \mathbf{C}_k - \mathbf{U}_k)^{\text{H}} \mathbf{X} \mathbf{J}_k \mathbf{F}^{\text{H}}] \} + (1 - \alpha) (\mathbf{x}^{\text{H}} \mathbf{H}^{\text{H}} \mathbf{H} \mathbf{x} - 2\mathcal{R} \{ \mathbf{x}^{\text{H}} \mathbf{H}^{\text{H}} \mathbf{s} \}), \quad (37)$$

and $\mathbf{P} = \alpha \mathbf{F}^{(t+1)\text{H}} \mathbf{F}^{(t+1)} + \frac{\rho}{2} \sum_{k \in \Omega_K} \mathbf{J}_k \mathbf{F}^{(t+1)} \mathbf{F}^{(t+1)\text{H}} \mathbf{J}_k^{\text{H}}$.

We consider majorizing the two quadratic terms, $\text{tr}(\mathbf{X} \mathbf{P} \mathbf{X}^{\text{H}})$ and $\mathbf{x}^{\text{H}} \mathbf{H}^{\text{H}} \mathbf{H} \mathbf{x}$ separately. According to Lemma 1, we have

$$\text{tr}(\mathbf{X} \mathbf{P} \mathbf{X}^{\text{H}}) \leq \lambda_p \|\mathbf{X}\|_{\text{F}}^2 + 2\mathcal{R} \{ \text{tr}(\mathbf{X}^{(t)} \mathbf{P} - \lambda_p \mathbf{X}^{(t)})^{\text{H}} \mathbf{X} \} + C_{\text{X}}, \quad (38)$$

where $\lambda_p > \|\mathbf{P}\|_1$, and C_{X} is the term irrelevant to \mathbf{X} . Similarly,

$$\mathbf{x}^{\text{H}} \mathbf{H}^{\text{H}} \mathbf{H} \mathbf{x} \leq \lambda_h \|\mathbf{x}\|_2^2 + 2\mathcal{R} \{ \mathbf{x}^{\text{H}} (\mathbf{H}^{\text{H}} \mathbf{H} - \lambda_h \mathbf{I}) \mathbf{x}^{(t)} \} + C_{\text{x}}, \quad (39)$$

in which $\lambda_h = \lambda_{\max}(\mathbf{H}^{\text{H}} \mathbf{H})$ and C_{x} is irrelevant to \mathbf{x} . By substituting (38) and (39) into (37), the tight upper bound of $\mathcal{L}_{\text{X}}(\mathbf{X})$ is

$$\mathcal{G}_{\text{X}}(\mathbf{X} | \mathbf{X}^{(t)}) = [\lambda_p + (1 - \alpha) \lambda_h] \|\mathbf{X}\|_{\text{F}}^2 - 2\mathcal{R} \{ \text{tr}[\Psi^{(t)\text{H}} \mathbf{X}] \} + \tilde{C}_{\text{x}}, \quad (40)$$

where $\Psi^{(t)} = \alpha \mathbf{R}_{\text{d}} \mathbf{F} + \sum_{k \in \Omega_K} \frac{1}{2} (\rho \mathbf{C}_k - \mathbf{U}_k) \mathbf{F} \mathbf{J}_k^{\text{H}} - (1 - \alpha) \Phi^{(t)\text{T}} - \mathbf{X}^{(t)} (\mathbf{P} - \lambda_p \mathbf{I})$ with $\Phi^{(t)} = \text{mat} \{ (\mathbf{H}^{\text{H}} \mathbf{H} - \lambda_h \mathbf{I}) \mathbf{x}^{(t)} - \mathbf{H}^{\text{H}} \mathbf{s} \}$, $\text{mat} \{ \cdot \}$ means reshaping a column vector into a matrix, and $\tilde{C}_{\text{x}} = (1 - \alpha) C_{\text{x}} + C_{\text{X}}$. Then, the majorized problem of (36) is

$$\min_{\mathbf{X}} \mathcal{G}_{\text{X}}(\mathbf{X} | \mathbf{X}^{(t)}) \quad (41)$$

s.t. $\mathbf{X} \in \mathcal{S}_{\text{X}}$,

whose closed form solution is

$$\mathbf{X}^{(t+1)} = e^{j \arg(\Psi^{(t)})}, \quad (42)$$

Algorithm 2 BSUM for Inexactly Solving Problem (13)

Input: Error tolerance $\varepsilon^{(\ell)}$ and $\rho^{(\ell)}$ from Algorithm 1.

- 1: **repeat**
- 2: **for** $k \in \Omega_K$ **do**
- 3: Update $\mathbf{C}_k^{(t+1)}$ by (28);
- 4: **end for**
- 5: Update $\mathbf{F}^{(t+1)}$ by (35);
- 6: Update $\mathbf{X}^{(t+1)}$ by (42)
- 7: $t \leftarrow t + 1$;
- 8: **until** $(\mathbf{C}^{(t)}, \mathbf{F}^{(t)}, \mathbf{X}^{(t)})$ is an $\varepsilon^{(\ell)}$ -stationary point of problem (13);

Output: $(\mathbf{C}^{(t)}, \mathbf{F}^{(t)}, \mathbf{X}^{(t)})$.

The proposed BSUM algorithm for solving the ALM problem (13) inexactly is summarized in Algorithm 2. It is worth mentioning that the SQUAREM scheme [33] is implemented when updating \mathbf{F} and each row of \mathbf{X} in Algorithm 2 to accelerate the convergence. The BSUM algorithm performs efficiently since every variable admits closed-form updates. The computational cost of updating \mathbf{F} , \mathbf{X} , \mathbf{C} , and \mathbf{U} is dominated by calculating Ξ , Ψ , and $\mathbf{X}\mathbf{J}_k\mathbf{F}^H$ for each $k \in \Omega_K$, respectively, whose computational complexities are $\mathcal{O}(|\Omega_K| \cdot [(N_T + T)N_R T])$, $\mathcal{O}(|\Omega_K| \cdot [(N_R + T)N_T T])$, $\mathcal{O}(|\Omega_K| \cdot [(N_R + T)N_T T])$, and $\mathcal{O}(|\Omega_K| \cdot [(N_R + T)N_T T])$, respectively. Therefore, the total per-iteration complexity of Algorithm 2 is $\mathcal{O}(|\Omega_K| \cdot [(N_T T + N_R T + N_T N_R) T])$, which scales linearly with the number of range bins for sidelobe suppression $|\Omega_K|$, the number of transmit antennas N_T , and the number of receive antennas N_R .

D. Convergence Analysis

The convergence analysis of the proposed inexact ALM algorithm (i.e., Algorithm 1) consists of four parts. We first show the sequence of the variables generated by Algorithm 2 enjoys a sufficient descent property in Lemma 2; then, we show that the subgradient of $\mathcal{L}_\rho(\mathbf{C}, \mathbf{F}, \mathbf{X}; \mathbf{U})$ is bounded in Lemma 3; based on the results of Lemma 2 and Lemma 3, we show that Algorithm 2 achieves an ε -stationary point within a finite number of iterations in Theorem 1; finally, we prove that any limit point generated by Algorithm 1 is a feasible stationary point in Theorem 2.

The convergence analysis relies significantly on the uniform boundness of the primal variables $(\mathbf{C}, \mathbf{F}, \mathbf{X})$ and the multiplier \mathbf{U} generated by Algorithm 2. The boundness of \mathbf{U} can be guaranteed by the updating rule proposed in (15b), and the boundness of $(\mathbf{C}, \mathbf{F}, \mathbf{X})$ can be ensured by the updating rules proposed in (28), (35), and (42).

Lemma 2 (Sufficient descent property). *At the ℓ -th iteration of Algorithm 1, there exist positive τ_f and τ_x such that*

$$\begin{aligned} & \mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}^{(t)}; \mathbf{U}^{(\ell)}) - \mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}^{(t+1)}; \mathbf{U}^{(\ell)}) \geq \tau_x \|\mathbf{X}^{(t)} - \mathbf{X}^{(t+1)}\|_F^2 \\ & + \tau_f \|\mathbf{F}^{(t)} - \mathbf{F}^{(t+1)}\|_F^2 + \frac{\beta}{2} \sum_{k \in \Omega_K} \|\mathbf{C}_k^{(t)} - \mathbf{C}_k^{(t+1)}\|_F^2. \end{aligned} \quad (43)$$

Proof. See Appendix A. \square

Lemma 3 (Subgradient boundness). *At the ℓ -th iteration of Algorithm 1, there exist a subgradient $\mathbf{J}^{(t+1)} = (\mathbf{J}_C^{(t+1)}, \mathbf{J}_F^{(t+1)}, \mathbf{J}_X^{(t+1)})$ with $\mathbf{J}^{(t+1)} \in \partial[\mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}^{(t+1)}; \mathbf{U}^{(\ell)}) + \mathbb{I}_S(\mathbf{z}^{(t+1)})]$ and a constant $M > 0$ such that*

$$\begin{aligned} & \|\mathbf{J}^{(t+1)}\|_F \leq M (\|\mathbf{F}^{(t)} - \mathbf{F}^{(t+1)}\|_F + \|\mathbf{X}^{(t)} - \mathbf{X}^{(t+1)}\|_F \\ & + \sum_{k \in \Omega_K} \|\mathbf{C}_k^{(t)} - \mathbf{C}_k^{(t+1)}\|_F). \end{aligned} \quad (44)$$

Proof. See Appendix B. \square

Theorem 1 (Iteration complexity to obtain an $\varepsilon^{(\ell)}$ -stationary point). *Given any positive $\varepsilon^{(\ell)}$ and $\rho^{(\ell)}$, Algorithm 2 returns an $\varepsilon^{(\ell)}$ -stationary point in $\mathcal{O}\left(\frac{\rho^{(\ell)2}}{\varepsilon^{(\ell)2}}\right)$ iterations.*

Proof. See Appendix C. \square

Before presenting the convergence results of the proposed Algorithm to the stationary point of problem (8), we first formally define its stationary point. The idea is to first transform problem (8) into an equivalent problem defined in the real space and then use the classic definition in the real space. Specifically, by denoting $\tilde{\mathbf{f}} = [\mathcal{R}\{\text{vec}\{\mathbf{F}^H\}\}^T, \mathcal{I}\{\text{vec}\{\mathbf{F}^H\}\}^T]^T$, $\mathcal{S}_{\tilde{\mathbf{f}}} = \{\tilde{\mathbf{f}} \in \mathbb{R}^{2N_T T \times 1} \mid \|\tilde{\mathbf{f}}\|_2^2 = P_F\}$, $\mathcal{S}_{\tilde{\mathbf{c}}} = \{\tilde{\mathbf{c}} \in \mathbb{R}^{2N_T^2 \times 1} \mid \|\tilde{\mathbf{c}}\|_2 \leq \xi\}$, and $\mathbf{D}_k = \mathbf{I} \otimes (\mathbf{X}\mathbf{J}_k)$ with $\tilde{\mathbf{D}}_k = \begin{bmatrix} \mathcal{R}\{\mathbf{D}_k\} & -\mathcal{I}\{\mathbf{D}_k\} \\ \mathcal{I}\{\mathbf{D}_k\} & \mathcal{R}\{\mathbf{D}_k\} \end{bmatrix}$, problem (8) w.r.t. \mathbf{F} is equivalent to the following real-valued vector minimization problem:

$$\min_{\tilde{\mathbf{f}}} g(\tilde{\mathbf{f}}) + \mathbb{I}_{\mathcal{S}_{\tilde{\mathbf{f}}}}(\tilde{\mathbf{f}}) + \sum_{k \in \Omega_K} \mathbb{I}_{\mathcal{S}_{\tilde{\mathbf{c}}}}(\tilde{\mathbf{D}}_k \tilde{\mathbf{f}}), \quad (45)$$

where $g(\tilde{\mathbf{f}})$ denotes the reformulated objective function $f(\mathbf{F}, \mathbf{X})$ in problem (8) w.r.t. the real-valued vector $\tilde{\mathbf{f}}$. Similarly, by denoting $\tilde{\mathbf{x}} = [\mathcal{R}\{\text{vec}\{\mathbf{X}\}\}^T, \mathcal{I}\{\text{vec}\{\mathbf{X}\}\}^T]^T$,

$\mathcal{S}_{\tilde{\mathbf{x}}} = \{\tilde{\mathbf{x}} \in \mathbb{R}^{2N_T T \times 1} \mid \|\tilde{\mathbf{x}}_{[i]}\|_2 = P_i, \|\tilde{\mathbf{x}}_{[i]}\|_\infty \leq \eta', \forall i \in [1, N_T]\}$,

$$\tilde{\mathbf{x}}_{[i]} = \left[[\tilde{x}_i, \dots, \tilde{x}_{TN_T+i}]^T, [\tilde{x}_{(T+1)N_T+i}, \dots, \tilde{x}_{(2T-1)N_T+i}]^T \right]^T,$$

and $\mathbf{E}_k = (\mathbf{J}_k \mathbf{F}^H)^T \otimes \mathbf{I}$ with $\tilde{\mathbf{E}}_k = \begin{bmatrix} \mathcal{R}\{\mathbf{E}_k\} & -\mathcal{I}\{\mathbf{E}_k\} \\ \mathcal{I}\{\mathbf{E}_k\} & \mathcal{R}\{\mathbf{E}_k\} \end{bmatrix}$, problem (8) w.r.t. \mathbf{X} is equivalent to

$$\min_{\tilde{\mathbf{x}}} h(\tilde{\mathbf{x}}) + \mathbb{I}_{\mathcal{S}_{\tilde{\mathbf{x}}}}(\tilde{\mathbf{x}}) + \sum_{k \in \Omega_K} \mathbb{I}_{\mathcal{S}_{\tilde{\mathbf{c}}}}(\tilde{\mathbf{E}}_k \tilde{\mathbf{x}}), \quad (46)$$

where $h(\tilde{\mathbf{x}})$ denotes the reformulated objective function in problem (8) w.r.t. the real-valued vector $\tilde{\mathbf{x}}$.

Based on the equivalence between problem (8) w.r.t. \mathbf{F} and \mathbf{X} and problems (45) and (46), we have the following definition of the stationary point of problem (8).

Definition 2 (Stationary point of problem (8)). *The point $(\tilde{\mathbf{f}}, \tilde{\mathbf{x}})$ is a stationary point of problem (8), if for $\tilde{\mathbf{f}} = [\mathcal{R}\{\text{vec}\{\mathbf{F}^H\}\}^T, \mathcal{I}\{\text{vec}\{\mathbf{F}^H\}\}^T]^T$, $\tilde{\mathbf{x}} = [\mathcal{R}\{\text{vec}\{\mathbf{X}\}\}^T, \mathcal{I}\{\text{vec}\{\mathbf{X}\}\}^T]^T$, there exist $\tilde{\mathbf{a}} \in \partial\mathbb{I}_{\mathcal{S}_{\tilde{\mathbf{f}}}}(\tilde{\mathbf{f}})$, $\tilde{\mathbf{b}} \in \partial\mathbb{I}_{\mathcal{S}_{\tilde{\mathbf{x}}}}(\tilde{\mathbf{x}})$, and $\tilde{\mathbf{p}}_k \in \partial\mathbb{I}_{\mathcal{S}_{\tilde{\mathbf{c}}}}(\tilde{\mathbf{D}}_k \tilde{\mathbf{f}})$ and $\tilde{\mathbf{q}}_k \in \partial\mathbb{I}_{\mathcal{S}_{\tilde{\mathbf{c}}}}(\tilde{\mathbf{E}}_k \tilde{\mathbf{x}})$ for all $k \in \Omega_K$, such that*

$$\begin{aligned} \mathbf{0} &= \tilde{\mathbf{a}} + \nabla_{\tilde{\mathbf{f}}} g(\tilde{\mathbf{f}}) + \sum_{k \in \Omega_K} \tilde{\mathbf{D}}_k^T \tilde{\mathbf{p}}_k, \\ \mathbf{0} &= \tilde{\mathbf{b}} + \nabla_{\tilde{\mathbf{x}}} h(\tilde{\mathbf{x}}) + \sum_{k \in \Omega_K} \tilde{\mathbf{E}}_k^T \tilde{\mathbf{q}}_k. \end{aligned}$$

Now we can present our main theoretical result.

TABLE I: Parameters setting in simulations.

Parameter	Definition	Value
N_T	Number of transmit antennas	8
N_R	Number of radar receive antennas	8
N_C	Number of users	4
L	CP length	6
T	Block length	64
K	Maximum index of range bin of interest	6
ξ'	Desired PSLR, $\xi = \sqrt{TN_T} \cdot 10^{(-\frac{\xi'}{20})}$	30 dB
P_F	Power of the receive filter	64
δ	Violation of constraints descent criterion in (17)	0.965
γ	Penalty parameter update factor in (17)	1.1
β	Parameter in (27)	1
u_{\max}	Upper bound of Lagrange multiplier	10^3
$\rho^{(0)}$	Initial penalty parameter	10^{-3}
—	Maximum inner iterations	50

Theorem 2 (Convergence to the stationary point). *Any limit point of $\{(\mathbf{F}^{(\ell+1)}, \mathbf{X}^{(\ell+1)})\}_{\ell \geq 0}$ generated by Algorithm 1 is a stationary point of problem (8).*

Proof. See Appendix D. \square

IV. NUMERICAL RESULTS

In this section, we present simulation results to evaluate the performance of the proposed algorithms under various parameter configurations. The communication channel \mathbf{H} is generated by following [27] with Extended Pedestrian A (EPA) fading profile [9]. The information symbol matrix \mathbf{S} is modulated using a unit-power QPSK alphabet, with each entry in \mathbf{S} having unit power. The desired radar receive spatial response, \mathbf{R}_d , is considered to be a 3 dB beamwidth of 20° focusing at 0° , generated by the algorithm in [34].

The stopping criterion for Algorithm 2 at the ℓ -th outer loop is set as $\varepsilon^{(\ell)} = \sup_{\mathbf{z}^{(\ell)}} \|\mathbf{J}^{(0)}\|_F / \ell$, where $\sup_{\mathbf{z}^{(\ell)}} \|\mathbf{J}^{(0)}\|_F$ is the subgradient upper bound in (44). Unless otherwise specified, the parameter configurations are listed in Table I. Algorithm 1 terminates after 500 iterations or when the following optimality violation criterion is met

$$\max\{e^{(\ell)}, v^{(\ell)}\} \leq \sqrt{T} \times 10^{-3}, \quad (47)$$

where $v^{(\ell)}$ is defined in (16), and

$$e^{(\ell)} = \sup_{\mathbf{z}^{(\ell)}} \|\mathbf{J}_F^{(\ell)}\|_F + \sup_{\mathbf{z}^{(\ell)}} \|\mathbf{J}_X^{(\ell)}\|_F + \sum_{k \in \Omega_K} \sup_{\mathbf{z}^{(\ell)}} \|\mathbf{J}_{C_k}^{(\ell)}\|_F,$$

which are the summation of the upper bounds in (54), (59), and (61).

A. Convergence Performance

In this subsection, we evaluate the convergence behavior of the proposed inexact ALM algorithm using 1000 Monte Carlo (MC) simulations. In each simulation, the channel matrix \mathbf{H} and the information symbol matrix \mathbf{S} are generated independently and randomly, with the variables $(\mathbf{C}, \mathbf{F}, \mathbf{X})$, \mathbf{U} also randomly initialized. The predetermined feasible points are consistent across all MC simulations. The shaded areas in the figures represent the standard deviation of the results from the 1000 simulations.

Fig. 3 shows the average behavior of Algorithm 1. Fig. 3(a) plots the optimality violation in (47) alongside the stopping threshold (dashed line) for Algorithm 1. On average, the algorithm reaches the stopping criterion after approximately 480 iterations. Due to random initialization of variables, the

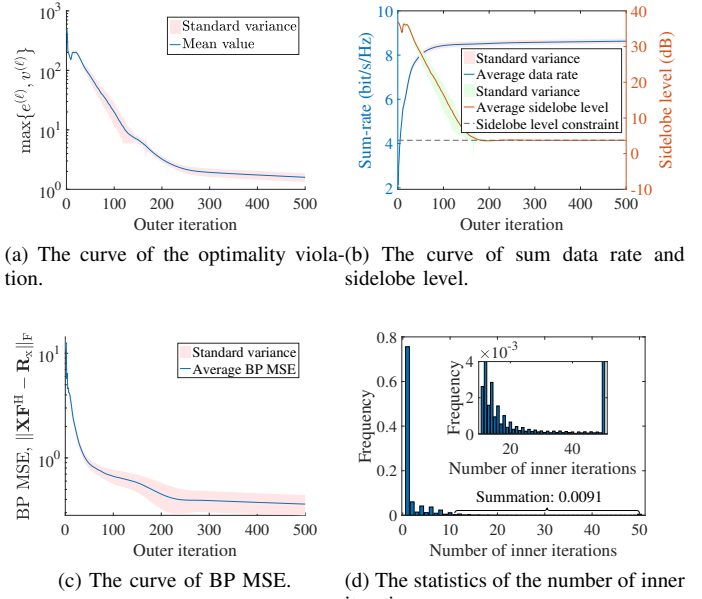


Fig. 3: The average iteration curves of 1000 Monte-Carlo simulations on Algorithm 1.

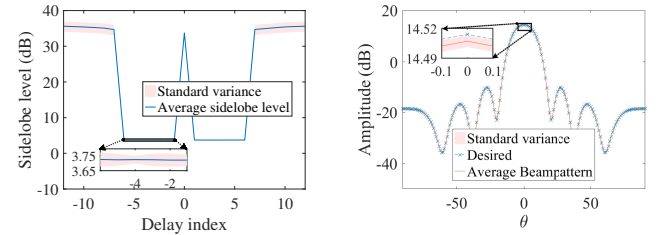


Fig. 4: The average results of 1000 Monte-Carlo simulations on Algorithm 1 after meeting the stopping criteria.

optimized sum rate and beampattern MSE vary. Fig. 3(b) and Fig. 3(c) illustrate the average curves of sum rate, beampattern MSE, and sidelobe level, showing that the final results perform well on average, with the maximum sidelobe level approaching the constraint. Fig. 3(d) shows the number of inner iterations required to find the $\varepsilon^{(\ell)}$ -stationary point. Although the maximum allowed inner iterations is set to 50, over 99.5% of the inner loops converge within 10 iterations. Combined with closed-form updates and the “nearest-vector” algorithm, the proposed BSUM algorithm efficiently solves the ALM subproblems.

Fig. 4 shows the average cross-correlation level and radar receive beampattern of the sequences generated by Algorithm 1 after convergence. The results indicate that all sidelobes in the interested area are well suppressed, and the optimized radar beampattern closely matches the desired one.

B. Communication and Radar Performance under Different System Configurations

In this subsection, we evaluate the communication and radar performance of the proposed algorithm under different system configurations. Specifically, we examine the impact of the

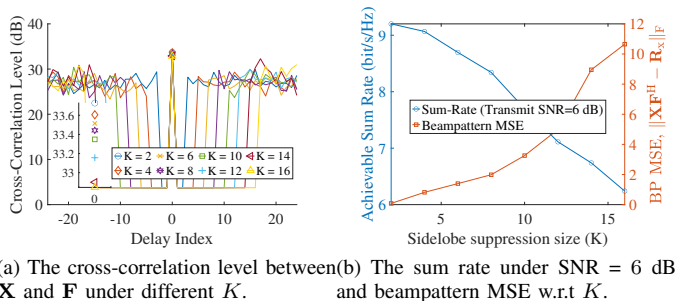


Fig. 5: Properties of optimized ISAC waveform w.r.t. K under $\alpha = 0.2$.

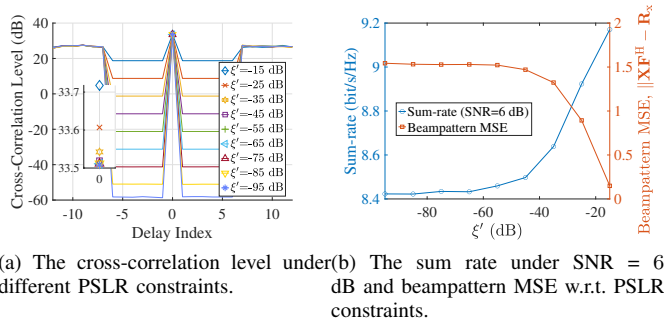


Fig. 6: Properties of optimized ISAC waveform with different PSLR under $\alpha = 0.2$.

range sidelobe suppression area size determined by K , and the maximum sidelobe level constraints ξ' (dB).

Fig. 5 illustrates the impact of the sidelobe suppression area sizes determined by K on the optimized results. As shown in Fig. 5(a), even with the increment of K , the optimized sidelobe levels can still approach the predetermined threshold. In Fig. 5(b), the achievable sum rate decreases and the beampattern MSE increases as K grows, since a larger suppression area imposes more constraints, reducing the feasible region and degrading the performance of optimized results. This higher beampattern MSE leads to a slight decrease in the mainlobe level.

Fig. 6 shows the impact of maximum sidelobe level constraints on the optimized results. As seen in Fig. 6(a), the sidelobe constraints can always be met. Lower sidelobe constraints require \mathbf{X} and \mathbf{F} to be less correlated, making the problem more complex and resulting in worse performance in both S&C. This can be observed in Fig. 6(b), in which a lower maximum sidelobe level constraint leads to a worse achievable sum rate and a larger beampattern MSE.

C. Comparison with Existing SOTA Approaches

In this subsection, we compare the radar and communication performance of the proposed approach with the modified work [11] and the ALM algorithm with fixed penalty parameter. In [11], the joint design of the radar receive filter and transmit waveform focuses on PSL control for pure radar sensing. We adapt the algorithm in [11] by first solving $\min_{\mathbf{X} \in \mathcal{S}_X} \|\mathbf{H}\mathbf{x} - \mathbf{s}\|_2^2$ using the BSUM method and then following the approach in [11] to obtain the radar receive filter \mathbf{F} . We refer to the modified algorithm of [11] as the ‘‘Exact Design’’ scheme. Furthermore, we also implemented the BSUM-based ALM

algorithm with fixed penalty parameter to show the superiority of the proposed adaptive penalty parameter scheme. We refer to this as ‘‘Fixed Penalty’’ scheme. Without loss of generality, the penalty parameters are fixed to be the final value of the proposed ALM algorithm with adaptive penalty parameter scheme. In the legends of the result figures, ‘‘Direc’’ refers to the receive beampattern focused at 0° with a 3 dB beamwidth of 20° , generated by the algorithm in [34], and ‘‘Omni’’ denotes the omnidirectional desired beampattern, i.e., $\mathbf{R}_d = (P_X/N_T)\mathbf{I}_{N_T}$.

Fig. 7 presents the optimized results of different algorithms. In Fig. 7(a), the proposed ALM algorithm achieves the highest sum rate under the directional beampattern case. For an omnidirectional beam, we need to set a larger α to make the final beam pattern closer to omnidirectional because an omnidirectional beam requires that \mathbf{X} and \mathbf{F} be orthogonal in each row and column, which significantly shrinks the feasible region. However, this degrades the minimization on P_{MUI} and lead to a lower rate. Fig. 7(b) shows that the Exact Design scheme perfectly matches the desired beampatterns in both the directional and omnidirectional cases while the two ALM algorithms have slight differences from the omnidirectional beampattern. Fig. 7(c) shows the proposed ALM algorithm and ALM algorithm with fixed penalty parameter have similar sidelobe control performance, but the latter one has a much lower rate due to the inappropriate penalty parameter.

Fig. 8 shows the MIMO radar images formed by the results of different algorithms. Both the proposed ALM algorithm and the ALM algorithm with fixed penalty parameter achieve similar imaging performance but the achievable sum rate is much lower under the fixed penalty parameter scheme. Although the Exact Design scheme meets the desired beampattern perfectly, it has a high sidelobe level, which results in a poor MIMO radar image.

V. CONCLUSION

This paper addressed the joint design of the receive filter and transmit waveform for MIMO-ISAC systems. We formulated an optimization problem to minimize a weighted sum of radar beampattern MSE and MUI at communication receivers, subject to practical constraints in convex and nonconvex forms. An inexact ALM algorithm was developed to solve the problem by iteratively minimizing tight upper bounds for each variable block. We proved the algorithm’s convergence to a feasible stationary point, which is the best that one can expect for this optimization problem with many nonconvex constraints. The trade-offs between sum rate, beampattern MSE, sidelobe suppression size, and maximum sidelobe level were analyzed. Simulation results demonstrated that the proposed algorithm outperforms others across various metrics.

APPENDIX A PROOF OF LEMMA 2

From (27), it is clear that

$$\begin{aligned} \mathcal{L}_{\rho^{(\ell)}}(\mathbf{C}^{(t)}, \mathbf{F}^{(t)}, \mathbf{X}^{(t)}; \mathbf{U}^{(\ell)}) - \mathcal{L}_{\rho^{(\ell)}}(\mathbf{C}^{(t+1)}, \mathbf{F}^{(t)}, \mathbf{X}^{(t)}; \mathbf{U}^{(\ell)}) \\ \geq \frac{\beta}{2} \sum_{k \in \Omega_K} \|\mathbf{C}_k^{(t)} - \mathbf{C}_k^{(t+1)}\|_{\mathbf{F}}^2. \end{aligned} \quad (48)$$

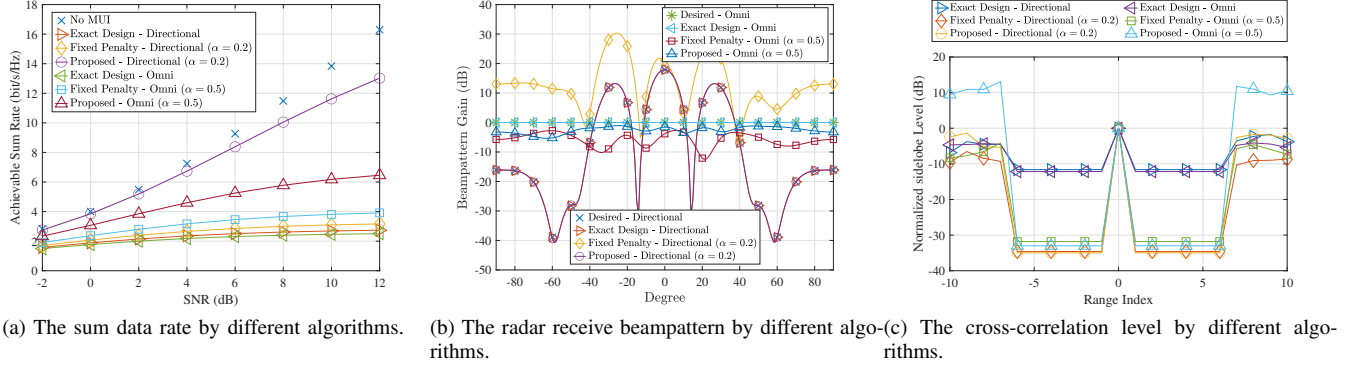


Fig. 7: The performance comparison of different algorithms.

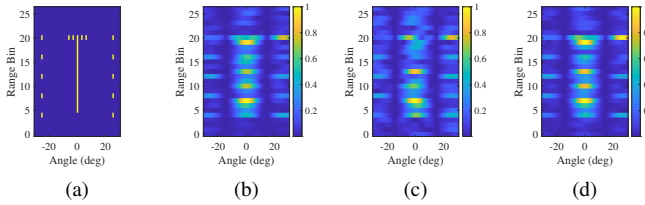


Fig. 8: The MIMO radar image under the results of different algorithms when SNR = 15 dB: (a) Original image of “T”. (b) The image formed by the proposed ALM algorithm. (c) The image formed by the Exact Design scheme. (d) The image formed by the fixed penalty ALM algorithm.

It is simple to verify that updating $\mathbf{F}^{(t+1)}$ by solving problem (34) is equivalent to

$$\mathbf{F}^{(t+1)} \in \arg \min_{\mathbf{F} \in \mathbb{C}^{N_T \times L}} \left\{ \mathcal{R} \left\{ \text{tr} \left[\nabla_{\mathbf{F}} \mathcal{L}_{\mathbf{F}}(\mathbf{F}^{(t)})^H (\mathbf{F} - \mathbf{F}^{(t)}) \right] \right\} + \frac{\lambda_q^{(t)}}{2} \|\mathbf{F} - \mathbf{F}^{(t)}\|_{\mathbf{F}}^2 + \mathbb{I}_{\mathcal{S}_{\mathbf{F}}}(\mathbf{F}) \right\}, \quad (49)$$

where $\lambda_q^{(t)}$ is defined in (31) that relies on $\mathbf{X}^{(t)}$. From Lemma 2 in [35], we have

$$\mathcal{L}_{\mathbf{F}}(\mathbf{F}^{(t)}) - \mathcal{L}_{\mathbf{F}}(\mathbf{F}^{(t+1)}) \geq \frac{\lambda_q^{(t)} - L_f^{(t)}}{2} \|\mathbf{F}^{(t+1)} - \mathbf{F}^{(t)}\|_{\mathbf{F}}^2,$$

where $L_f^{(t)}$ denotes the Lipschitz constant of $\nabla_{\mathbf{F}}^* \mathcal{L}_{\mathbf{F}}(\mathbf{F})$ at the t -th iteration of Algorithm 2, and $\nabla_{\mathbf{F}}^* \mathcal{L}_{\mathbf{F}}(\mathbf{F})$ is the conjugate gradient of $\mathcal{L}_{\mathbf{F}}(\mathbf{F})$. We always have $\lambda_q^{(t)} - L_f^{(t)} > 0$, since $L_f^{(t)}$ is the maximum eigenvalue of $\mathbf{Q}^{(t)}$ according to the results in [36], where $\mathbf{Q}^{(t)}$ is defined in (31) and its superscript (t) means it relies on $\mathbf{X}^{(t)}$. Then $\lambda_q^{(t)} > L_f^{(t)}$ for $\forall t > 0$ according to (31). Therefore, there exists a $\tau_f > 0$, such that the following holds for all $t > 0$:

$$\mathcal{L}_{\rho^{(\ell)}}(\mathbf{C}^{(t+1)}, \mathbf{F}^{(t)}, \mathbf{X}^{(t)}; \mathbf{U}^{(\ell)}) - \mathcal{L}_{\rho^{(\ell)}}(\mathbf{C}^{(t+1)}, \mathbf{F}^{(t+1)}, \mathbf{X}^{(t)}; \mathbf{U}^{(\ell)}) \geq \tau_f \|\mathbf{F}^{(t)} - \mathbf{F}^{(t+1)}\|_{\mathbf{F}}^2. \quad (50)$$

By applying the above analysis procedure to (36), we obtain

$$\mathcal{L}_{\rho^{(\ell)}}(\mathbf{C}^{(t+1)}, \mathbf{F}^{(t+1)}, \mathbf{X}^{(t)}; \mathbf{U}^{(\ell)}) - \mathcal{L}_{\rho^{(\ell)}}(\mathbf{C}^{(t+1)}, \mathbf{F}^{(t+1)}, \mathbf{X}^{(t+1)}; \mathbf{U}^{(\ell)}) \geq \tau_x \|\mathbf{X}^{(t)} - \mathbf{X}^{(t+1)}\|_{\mathbf{F}}^2, \quad (51)$$

where $\tau_x > 0$. Combining (48), (50), and (51), we have

$$\begin{aligned} \mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}^{(t)}; \mathbf{U}^{(\ell)}) - \mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}^{(t+1)}; \mathbf{U}^{(\ell)}) &\geq \tau_f \|\mathbf{F}^{(t)} - \mathbf{F}^{(t+1)}\|_{\mathbf{F}}^2 \\ &+ \tau_x \|\mathbf{X}^{(t)} - \mathbf{X}^{(t+1)}\|_{\mathbf{F}}^2 + \frac{\beta}{2} \sum_{k \in \Omega_K} \|\mathbf{C}_k^{(t)} - \mathbf{C}_k^{(t+1)}\|_{\mathbf{F}}^2. \end{aligned} \quad (52)$$

Therefore, Lemma 2 holds.

APPENDIX B PROOF OF LEMMA 3

1) Upper Bound of $\mathbf{J}_{\mathbf{F}}^{(t+1)}$

By solving problem (34), we have $\mathbf{0} \in \nabla_{\mathbf{F}} \mathcal{G}_{\mathbf{F}}(\mathbf{F}^{(t+1)} | \mathbf{F}^{(t)}) + \partial \mathbb{I}_{\mathcal{S}_{\mathbf{F}}}(\mathbf{F}^{(t+1)})$. Hence, there exists a subgradient $\mathbf{A}^{(t+1)} \in \partial \mathbb{I}_{\mathcal{S}_{\mathbf{F}}}(\mathbf{F}^{(t+1)})$ such that

$$\nabla_{\mathbf{F}} \mathcal{G}_{\mathbf{F}}(\mathbf{F}^{(t+1)} | \mathbf{F}^{(t)}) + \mathbf{A}^{(t+1)} = \mathbf{0}, \quad (53)$$

which further implies that there exists a subgradient $\mathbf{J}_{\mathbf{F}}^{(t+1)} \in \partial_{\mathbf{F}} [\mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}; \mathbf{U}^{(\ell)}) + \mathbb{I}_{\mathcal{S}_{\mathbf{F}}}(\mathbf{F})] |_{\mathbf{F}=\mathbf{F}^{(t)}}$ such that

$$\mathbf{J}_{\mathbf{F}}^{(t+1)} = \nabla_{\mathbf{F}} \mathcal{L}_{\mathbf{F}}(\mathbf{F}^{(t+1)}) + \mathbf{A}^{(t+1)}. \quad (54)$$

By combining (53) and (54), we have

$$\begin{aligned} \|\mathbf{J}_{\mathbf{F}}^{(t+1)}\|_{\mathbf{F}} &= \|\nabla_{\mathbf{F}} \mathcal{L}_{\mathbf{F}}(\mathbf{F}^{(t+1)}) - \nabla_{\mathbf{F}} \mathcal{G}_{\mathbf{F}}(\mathbf{F}^{(t+1)} | \mathbf{F}^{(t)})\|_{\mathbf{F}} \\ &\stackrel{(a)}{\leq} \|\nabla_{\mathbf{F}} \mathcal{L}_{\mathbf{F}}(\mathbf{F}^{(t+1)}) - \nabla_{\mathbf{F}} \mathcal{L}_{\mathbf{F}}(\mathbf{F}^{(t)})\|_{\mathbf{F}} \\ &+ \|\nabla_{\mathbf{F}} \mathcal{G}_{\mathbf{F}}(\mathbf{F}^{(t+1)} | \mathbf{F}^{(t)}) - \nabla_{\mathbf{F}} \mathcal{G}_{\mathbf{F}}(\mathbf{F}^{(t)} | \mathbf{F}^{(t)})\|_{\mathbf{F}}, \end{aligned} \quad (55)$$

where (a) holds due to $\nabla_{\mathbf{F}} \mathcal{L}_{\mathbf{F}}(\mathbf{F}^{(t)}) = \nabla_{\mathbf{F}} \mathcal{G}_{\mathbf{F}}(\mathbf{F}^{(t)} | \mathbf{F}^{(t)})$. Since

$$\begin{aligned} \|\nabla_{\mathbf{F}} \mathcal{L}_{\mathbf{F}}(\mathbf{F}^{(t+1)}) - \nabla_{\mathbf{F}} \mathcal{L}_{\mathbf{F}}(\mathbf{F}^{(t)})\|_{\mathbf{F}} &= \|\mathbf{Q}(\mathbf{F}^{(t)} - \mathbf{F}^{(t+1)})\|_{\mathbf{F}} \leq \|\mathbf{Q}\|_{\mathbf{F}} \|\mathbf{F}^{(t)} - \mathbf{F}^{(t+1)}\|_{\mathbf{F}}, \end{aligned} \quad (56)$$

and

$$\begin{aligned} \|\nabla_{\mathbf{F}} \mathcal{G}_{\mathbf{F}}(\mathbf{F}^{(t+1)} | \mathbf{F}^{(t)}) - \nabla_{\mathbf{F}} \mathcal{G}_{\mathbf{F}}(\mathbf{F}^{(t)} | \mathbf{F}^{(t)})\|_{\mathbf{F}} &= \lambda_q \|\mathbf{F}^{(t)} - \mathbf{F}^{(t+1)}\|_{\mathbf{F}} \leq \|\mathbf{Q}\|_{\mathbf{F}} \|\mathbf{F}^{(t)} - \mathbf{F}^{(t+1)}\|_{\mathbf{F}}, \end{aligned} \quad (57)$$

it follows that

$$\|\mathbf{J}_{\mathbf{F}}^{(t+1)}\|_{\mathbf{F}} \leq \bar{L}_{\mathbf{F}} \|\mathbf{F}^{(t)} - \mathbf{F}^{(t+1)}\|_{\mathbf{F}}, \quad (58)$$

where the inequality holds due to

$$\begin{aligned} \|\mathbf{Q}\|_{\mathbf{F}} &\leq \alpha \|\mathbf{X}\|_{\mathbf{F}} + \sum_{k \in \Omega_K} \frac{\rho^{(\ell)}}{2} \|\mathbf{X} \mathbf{J}_k\|_{\mathbf{F}} \\ &\leq \sqrt{P_X} \left(\alpha + \frac{1}{2} \rho^{(\ell)} \sqrt{T} |\Omega_K| \right) = \frac{\bar{L}_{\mathbf{F}}}{2}. \end{aligned}$$

2) Upper Bound of $\mathbf{J}_X^{(t+1)}$

By following the similar analysis procedure in the previous part, we have

$$\|\mathbf{J}_X^{(t+1)}\|_{\mathbb{F}} \leq \bar{L}_X \|\mathbf{X}^{(t)} - \mathbf{X}^{(t+1)}\|_{\mathbb{F}}, \quad (59)$$

where $\bar{L}_X = 2[\sqrt{P_{\mathbb{F}}}(\alpha + \frac{1}{2}\rho^{(\ell)}\sqrt{T}|\boldsymbol{\Omega}_K|) + (1 - \alpha)\|\mathbf{H}^H\mathbf{H}\|_{\mathbb{F}}]$.

3) Upper Bound of $\mathbf{J}_C^{(t+1)}$

We now consider calculating the upper bound of the sub-gradient vector

$$\mathbf{J}_C^{(t+1)} = \left[(\mathbf{J}_{C_1}^{(t+1)})^T, (\mathbf{J}_{C_2}^{(t+1)})^T, \dots, (\mathbf{J}_{C_{|\Omega_K|}}^{(t+1)})^T \right]^T,$$

where $\mathbf{J}_{C_k}^{(t+1)}$ denotes the subgradient of $\mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}; \mathbf{U}^{(\ell)}) + \mathbb{I}_{\mathcal{S}_C}(\mathbf{C}_k)$ w.r.t. \mathbf{C}_k at $\mathbf{C}_k^{(t+1)}$. Denoting the objective function in (27) as $\mathcal{H}_{\rho^{(\ell)}, k}(\mathbf{C}, \mathbf{F}^{(t)}, \mathbf{X}^{(t)})$ for each $k \in \Omega_K$, we have

$$\mathbf{0} = \nabla_{\mathbf{C}_k} \mathcal{H}_{\rho^{(\ell)}, k}(\mathbf{C}_k^{(t+1)}, \mathbf{F}^{(t)}, \mathbf{X}^{(t)}) + \mathbf{D}_k^{(t+1)}, \quad (60)$$

where $\mathbf{D}_k^{(t+1)} \in \partial \mathbb{I}_{\mathcal{S}_C}(\mathbf{C}_k^{(t+1)})$. Then there exists a subgradient such that

$$\mathbf{J}_{C_k}^{(t+1)} = \nabla_{\mathbf{C}_k} \mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}; \mathbf{U}^{(\ell)})|_{\mathbf{z}=\mathbf{z}^{(t+1)}} + \mathbf{D}_k^{(t+1)}.$$

Following the similar derivation procedure regarding $\mathbf{J}_F^{(t+1)}$, we have the upper bound of $\|\mathbf{J}_{C_k}\|_{\mathbb{F}}$. By summing $\mathbf{J}_{C_k}^{(t+1)}$ over $k \in \Omega_K$, we finally get

$$\begin{aligned} \|\mathbf{J}_C^{(t+1)}\|_{\mathbb{F}} &\leq \frac{1}{2}\rho^{(\ell)}\sqrt{T}|\boldsymbol{\Omega}_K| \left(\sqrt{P_{\mathbb{F}}}\|\mathbf{X}^{(t)} - \mathbf{X}^{(t+1)}\|_{\mathbb{F}} \right. \\ &\quad \left. + \sqrt{P_X}\|\mathbf{F}^{(t)} - \mathbf{F}^{(t+1)}\|_{\mathbb{F}} \right) + \beta \sum_{k \in \Omega_K} \|\mathbf{C}_k^{(t+1)} - \mathbf{C}_k^{(t)}\|_{\mathbb{F}}. \end{aligned} \quad (61)$$

Combining (58), (59), and (61) together, we arrive at Lemma 3.

APPENDIX C

PROOF OF THEOREM 1

From the results of Lemma 2 and Lemma 3, with penalty parameter $\rho^{(\ell)}$ at the ℓ -th iteration of Algorithm 1, there exists a positive real number $M > 0$ such that

$$\begin{aligned} M[\mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}^{(t)}; \mathbf{U}^{(\ell)}) - \mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}^{(t+1)}; \mathbf{U}^{(\ell)})] &\geq M[\|\mathbf{X}^{(t)} - \mathbf{X}^{(t+1)}\|_{\mathbb{F}}^2 + \|\mathbf{F}^{(t)} - \mathbf{F}^{(t+1)}\|_{\mathbb{F}}^2 + \frac{\beta}{2} \sum_{k \in \Omega_K} \|\mathbf{C}_k^{(t)} - \mathbf{C}_k^{(t+1)}\|_{\mathbb{F}}^2] \\ &\geq \frac{1}{\rho^{(\ell)2}(\|\mathbf{J}_F^{(t+1)}\|_{\mathbb{F}}^2 + \|\mathbf{J}_X^{(t+1)}\|_{\mathbb{F}}^2 + \|\mathbf{J}_C^{(t+1)}\|_{\mathbb{F}}^2)} = \frac{\|\mathbf{J}^{(t+1)}\|_{\mathbb{F}}^2}{\rho^{(\ell)2}}, \end{aligned} \quad (62)$$

where $\mathbf{J}^{(t+1)}$ is defined in Lemma 3. The sufficient descent property in Lemma 3 guarantees that the left-hand side of (62) is always nonnegative. By summing (62) from 1 to T , we have

$$\begin{aligned} M\rho^{(\ell)2}[\mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}^{(1)}; \mathbf{U}^{(\ell)}) - \mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}^{(T)}; \mathbf{U}^{(\ell)})] \\ \geq \sum_{t=1}^T \|\mathbf{J}^{(t)}\|_{\mathbb{F}}^2 \geq T \min_{t \in [1, T]} \|\mathbf{J}^{(t)}\|_{\mathbb{F}}^2. \end{aligned} \quad (63)$$

The value of ALF is lower bounded, i.e., $\mathcal{L}_{\rho}(\mathbf{z}; \mathbf{U}) > -\infty$, since the summation of Frobenius norms in the ALF is no less than zero and the term $\|\mathbf{U}_k\|_{\mathbb{F}}^2/(2\rho^{(\ell)})$ is finite for all $k \in \Omega_K$ due to the boundness of multipliers. The $\varepsilon^{(\ell)}$ -stationary point means $\min_{t \in [1, T]} \|\mathbf{J}^{(t)}\|_{\mathbb{F}} \leq \varepsilon^{(\ell)}$. Then, (63) yields that

$$T \leq \frac{Mu\rho^{(\ell)2}}{\varepsilon^{(\ell)2}},$$

where $u = \mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}^{(1)}; \mathbf{U}^{(\ell)}) - \mathcal{L}_{\rho^{(\ell)}}(\mathbf{z}^{(T)}; \mathbf{U}^{(\ell)})$ is a positive number. Therefore, the sequences generated by Algorithm 2 will achieve an $\varepsilon^{(\ell)}$ -stationary point after $\mathcal{O}\left(\frac{\rho^{(\ell)2}}{\varepsilon^{(\ell)2}}\right)$ iterations.

APPENDIX D

PROOF OF THEOREM 2

The convergence proof consists of two main steps, where the first step shows the feasibility of the limit point of the sequence generated by Algorithm 1 and the second step shows that any limit point is a feasible stationary point of problem (8). We present these two steps in two subsections separately.

A. Feasibility of the Limit Point

According to the updating rule of the penalty parameter in (17), either the penalty parameter keeps fixed after a finite number of iterations or $\lim_{\ell \rightarrow \infty} \rho^{(\ell+1)} = +\infty$. The violation of the constraints will be decreased by a factor of $\delta < 1$ after each iteration and thus tends to zero as $\ell \rightarrow \infty$ if it is the former case.

Now we consider the condition where $\rho^{(\ell)} \rightarrow \infty$. Denote $(\mathbf{C}^{(\infty)}, \mathbf{F}^{(\infty)}, \mathbf{X}^{(\infty)})$ as any limit point of the sequence generated by Algorithm 1. Based on the choice of the initial point in (24) and the previous analysis, we can see that condition (14) holds for any $\ell \geq 0$. That is

$$\begin{aligned} f^{(\ell+1)} + \sum_{k \in \Omega_K} \left[\frac{\rho^{(\ell)}}{2} \|\mathbf{X}^{(\ell+1)} \mathbf{J}_k \mathbf{F}^{(\ell+1)H} - \mathbf{C}_k^{(\ell+1)}\|_{\mathbb{F}}^2 \right. \\ \left. + \mathcal{R} \left\{ \text{tr} \left[\mathbf{U}_k^{(\ell)H} \left(\mathbf{X}^{(\ell+1)} \mathbf{J}_k \mathbf{F}^{(\ell+1)H} - \mathbf{C}_k^{(\ell+1)} \right) \right] \right\} \right] \leq \zeta, \end{aligned}$$

where $f^{(\ell+1)} = f(\mathbf{F}^{(\ell+1)}, \mathbf{X}^{(\ell+1)})$. Dividing both sides of the above inequality by $\rho^{(\ell)}$ yields

$$\begin{aligned} \sum_{k \in \Omega_K} \frac{1}{2} \|\mathbf{X}^{(\ell+1)} \mathbf{J}_k \mathbf{F}^{(\ell+1)H} - \mathbf{C}_k^{(\ell+1)}\|_{\mathbb{F}}^2 \leq \frac{1}{\rho^{(\ell)}} \left[\zeta - f^{(\ell+1)} \right. \\ \left. - \sum_{k \in \Omega_K} \mathcal{R} \left\{ \text{tr} \left[\mathbf{U}_k^{(\ell)H} \left(\mathbf{X}^{(\ell+1)} \mathbf{J}_k \mathbf{F}^{(\ell+1)H} - \mathbf{C}_k^{(\ell+1)} \right) \right] \right\} \right]. \end{aligned}$$

Since ζ is a finite constant given in (12), $f(\mathbf{F}, \mathbf{X})$ is lower bounded by zero, and \mathbf{U}_k is also bounded according to (15a), it follows that the right-hand side of the above inequality tends to zero when $\rho^{(\ell)} \rightarrow \infty$. Therefore,

$$\lim_{\ell \rightarrow \infty} \sum_{k \in \Omega_K} \|\mathbf{X}^{(\ell+1)} \mathbf{J}_k \mathbf{F}^{(\ell+1)H} - \mathbf{C}_k^{(\ell+1)}\|_{\mathbb{F}}^2 \leq 0,$$

which implies $\|\mathbf{X}^{(\infty)} \mathbf{J}_k \mathbf{F}^{(\infty)H} - \mathbf{C}_k^{(\infty)}\|_{\mathbb{F}} = 0$ for all $k \in \Omega_K$. Therefore, any limit point of the sequence generated by Algorithm 1 is feasible.

B. Stationarity of the Limit Point

By reformulating problem (13) w.r.t. \mathbf{F} , \mathbf{X} , and \mathbf{C} into real-valued vectors $\tilde{\mathbf{f}}$, $\tilde{\mathbf{x}}$, and $\tilde{\mathbf{c}}_k = [\mathcal{R}\{\mathbf{c}_k\}^T, \mathcal{I}\{\mathbf{c}_k\}^T]^T$ with $\mathbf{c}_k = \text{vec}\{\mathbf{C}_k\}$ for all $k \in \Omega_K$, and $\tilde{\mathbf{u}}_k = [\mathcal{R}\{\mathbf{u}_k\}^T, \mathcal{I}\{\mathbf{u}_k\}^T]^T$ with $\mathbf{u}_k = \text{vec}\{\mathbf{U}_k\}$ for all $k \in \Omega_K$, we have the following equivalent real-valued vector problems:

$$\begin{aligned} \min_{\tilde{\mathbf{f}}} g(\tilde{\mathbf{f}}) + \mathbb{I}_{\mathcal{S}_{\tilde{\mathbf{f}}}}(\tilde{\mathbf{f}}) + \sum_{k \in \Omega_K} \left[\tilde{\mathbf{u}}_k^T (\tilde{\mathbf{D}}_k \tilde{\mathbf{f}} - \tilde{\mathbf{c}}_k) + \frac{\rho}{2} \|\tilde{\mathbf{D}}_k \tilde{\mathbf{f}} - \tilde{\mathbf{c}}_k\|_2^2 \right], \\ \min_{\tilde{\mathbf{x}}} h(\tilde{\mathbf{x}}) + \mathbb{I}_{\mathcal{S}_{\tilde{\mathbf{x}}}}(\tilde{\mathbf{x}}) + \sum_{k \in \Omega_K} \left[\tilde{\mathbf{u}}_k^T (\tilde{\mathbf{E}}_k \tilde{\mathbf{x}} - \tilde{\mathbf{c}}_k) + \frac{\rho}{2} \|\tilde{\mathbf{E}}_k \tilde{\mathbf{x}} - \tilde{\mathbf{c}}_k\|_2^2 \right], \\ \min_{\tilde{\mathbf{c}}_k} \mathbb{I}_{\mathcal{E}}(\tilde{\mathbf{c}}_k) + \tilde{\mathbf{u}}_k^T (\tilde{\mathbf{v}}_k - \tilde{\mathbf{c}}_k) + \frac{\rho}{2} \|\tilde{\mathbf{v}}_k - \tilde{\mathbf{c}}_k\|_2^2, \end{aligned}$$

where $g(\tilde{\mathbf{f}})$ and $h(\tilde{\mathbf{x}})$ are defined in (45) and (46), respectively, $\tilde{\mathbf{v}}_k = [\mathcal{R}\{\mathbf{v}_k\}^T, \mathcal{I}\{\mathbf{v}_k\}^T]^T$ with $\mathbf{v}_k = \text{vec}\{\mathbf{X}\mathbf{J}_k\mathbf{F}^H\}$. It is simple to verify that $\tilde{\mathbf{v}}_k = \tilde{\mathbf{D}}_k\tilde{\mathbf{f}} = \tilde{\mathbf{E}}_k\tilde{\mathbf{x}}$. Then, from the definition of the ε -stationary point of (13), we immediately get the following result: If the point $(\tilde{\mathbf{C}}, \tilde{\mathbf{F}}, \tilde{\mathbf{X}})$ is an ε -stationary point of problem (13), then for $\tilde{\mathbf{f}} = [\mathcal{R}\{\text{vec}\{\tilde{\mathbf{F}}^H\}\}^T, \mathcal{I}\{\text{vec}\{\tilde{\mathbf{F}}^H\}\}^T]^T$, $\tilde{\mathbf{x}} = [\mathcal{R}\{\text{vec}\{\tilde{\mathbf{X}}\}\}^T, \mathcal{I}\{\text{vec}\{\tilde{\mathbf{X}}\}\}^T]^T$ and $\tilde{\mathbf{c}}_k = [\mathcal{R}\{\text{vec}\{\tilde{\mathbf{C}}_k\}\}^T, \mathcal{I}\{\text{vec}\{\tilde{\mathbf{C}}_k\}\}^T]^T$ for all $k \in \Omega_K$, there exist $\tilde{\mathbf{a}} \in \partial\mathbb{I}_{S_{\tilde{\mathbf{f}}}}(\tilde{\mathbf{f}})$, $\tilde{\mathbf{b}} \in \partial\mathbb{I}_{S_{\tilde{\mathbf{x}}}}(\tilde{\mathbf{x}})$, $\tilde{\mathbf{w}}_k \in \partial\mathbb{I}_{S_{\tilde{\mathbf{c}}_k}}(\tilde{\mathbf{c}}_k)$, and $\{\tilde{\mathbf{u}}_k\}_{k \in \Omega_K}$, such that

$$\begin{aligned} \|\tilde{\mathbf{a}} + \nabla_{\tilde{\mathbf{f}}}g(\tilde{\mathbf{f}}, \tilde{\mathbf{x}}) + \sum_{k \in \Omega_K} \tilde{\mathbf{D}}_k^T[\tilde{\mathbf{u}}_k + \rho(\tilde{\mathbf{D}}_k\tilde{\mathbf{f}} - \tilde{\mathbf{c}}_k)]\|_2 &\leq \varepsilon, \\ \|\tilde{\mathbf{b}} + \nabla_{\tilde{\mathbf{x}}}h(\tilde{\mathbf{f}}, \tilde{\mathbf{x}}) + \sum_{k \in \Omega_K} \tilde{\mathbf{E}}_k^T[\tilde{\mathbf{u}}_k + \rho(\tilde{\mathbf{E}}_k\tilde{\mathbf{x}} - \tilde{\mathbf{c}}_k)]\|_2 &\leq \varepsilon, \\ \|\tilde{\mathbf{w}}_k - [\tilde{\mathbf{u}}_k + \rho(\tilde{\mathbf{v}}_k - \tilde{\mathbf{c}}_k)]\|_2 &\leq \varepsilon, \quad \forall k \in \Omega_K. \end{aligned}$$

From the results in Lemma 3 and Theorem 1, Algorithm 2 returns an $\varepsilon^{(\ell)}$ -stationary point after a finite number of (inner) iterations. Taking $\ell \rightarrow \infty$ with $\varepsilon^{(\ell)} \rightarrow 0$ yields

$$\begin{aligned} \mathbf{0} &= \mathbf{a}^{(\infty)} + \nabla_{\tilde{\mathbf{f}}}g(\tilde{\mathbf{f}}^{(\infty)}) + \sum_{k \in \Omega_K} \tilde{\mathbf{D}}_k^T \mathbf{w}_k^{(\infty)}, \\ \mathbf{0} &= \mathbf{b}^{(\infty)} + \nabla_{\tilde{\mathbf{x}}}h(\tilde{\mathbf{x}}^{(\infty)}) + \sum_{k \in \Omega_K} \tilde{\mathbf{E}}_k^T \mathbf{w}_k^{(\infty)}. \end{aligned}$$

Together with the feasibility proved in the previous subsection, we can verify that $\tilde{\mathbf{v}}_k^{(\infty)} = \tilde{\mathbf{c}}_k^{(\infty)}$ for all $k \in \Omega_K$. According to the limiting subdifferential defined in [29], we have $\mathbf{w}_k^{(\infty)} \in \partial\mathbb{I}_{S_{\tilde{\mathbf{c}}_k}}(\tilde{\mathbf{D}}_k\tilde{\mathbf{f}}^{(\infty)})$ and $\mathbf{w}_k^{(\infty)} \in \partial\mathbb{I}_{S_{\tilde{\mathbf{c}}_k}}(\tilde{\mathbf{E}}_k\tilde{\mathbf{x}}^{(\infty)})$ since $\tilde{\mathbf{v}}_k = \tilde{\mathbf{D}}_k\tilde{\mathbf{f}} = \tilde{\mathbf{E}}_k\tilde{\mathbf{x}}$ for all $k \in \Omega_K$. Therefore, any limit point of the sequence generated by Algorithm 1, $(\mathbf{F}^{(\infty)}, \mathbf{X}^{(\infty)})$, is a stationary point of problem (8).

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