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On the structure of spectral sets

by

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Abstract. We discuss the convergence in the Fourier algebra $A(G)$ of a locally compact group G and obtain a new characterization of local spectral sets of G.

1. Introduction. The notion of spectral synthesis was introduced by A. Beurling in the late 1930s. Since then it has been a subject of extensive research in harmonic analysis, primarily in the context of the Fourier algebra $A(G)$ for a locally compact group G. If G is abelian then $A(G)$ is isometrically isomorphic to $L^1(\widehat{G})$, the L^1-algebra of the dual group $\widehat{G}$. If G is non-commutative, the Fourier algebra is defined directly on G as the algebra of matrix coefficients of the left regular representation of G. The algebra $A(G)$ is a semisimple, regular, commutative Banach algebra with the Gelfand spectrum G: characters correspond to evaluation at points of G.

The question of spectral synthesis can be framed as a question about the ideals of $A(G)$, specifically whether a given closed ideal $I \subset A(G)$ is the intersection of all maximal ideals containing I. This intersection can be identified with a closed subset of G, known as the hull of I. A closed subset E of G is said to be a set of spectral synthesis if the only closed ideal having E as its hull is the intersection of all maximal ideals associated with the points of E.

The first example demonstrating the failure of spectral synthesis was provided by L. Schwartz in 1948 for $A(\mathbb{R}^n)$, $n \geq 3$. That synthesis fails for any $A(G)$ where G is a non-discrete locally compact abelian group, was proved by P. Malliavin in 1959. This result was later extended to arbitrary locally compact groups, under a mild additional assumption, by E. Kaniuth and A. T.-M. Lau; see e.g. [KL, Theorem 6.2.3]. To classify sets of spectral

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synthesis seems to be a problem out of reach for the moment. Only some special classes of sets of spectral synthesis have been identified; see e.g. [GM]. An outstanding problem in the field, as noted in [GM], is whether the union of two sets of spectral synthesis is itself a set of spectral synthesis. The question was initially posed by C. Herz in [He1] for abelian groups and reiterated three years later by H. Reiter in [Re]. To date, no solution has been found, underscoring the need for new tools to analyze sets of spectral synthesis and advance our understanding of the union problem.

In this paper we present a characterization of sets of spectral synthesis formulated as a Hilbert space approximation. The paper is organized as follows. In Section 2 we recall definitions and fix notations. In Section 3 convergence properties in $A(G)$ are discussed. Theorem 3.6 and Corollary 3.7 provide conditions for a sequence to converge in $A(G)$ which may be of independent interest. These results are used to establish a characterization of local spectral sets in Theorem 3.8. In Section 3.3 a refinement of Theorem 3.8 is presented for the abelian case. The notion of strongly spectral sets is introduced in Section 4, and it is shown that the union of two such sets remains strongly spectral. This class includes Ditkin sets. Finally, in the Appendix some formulas related to the action of the von Neumann algebra of G on the corresponding Fourier algebra are collected.

2. Preliminaries and notations. Let G be a locally compact group with a fixed left Haar measure m. We denote by $C(G)$ the set of continuous complex-valued functions on G, and by $C_c(G)$ those with compact support in $C(G)$. For any $1 \le p \le \infty$ we let $L^p(G)$ denote the usual L^p space with respect to m with norm $\|\cdot\|_p$. For any appropriate pair of complex-valued functions f, g, we denote $(f * g)(t) = \int_G f(s)g(s^{-1}t)\,ds$. We set, as is customary, $\check{\xi}(s) = \xi(s^{-1})$ and $\tilde{\xi}(s) = \overline{\xi(s^{-1})}$, $s \in G$.

The *Fourier algebra* $A(G)$ of G was defined by P. Eymard [Ey]. We recall that $A(G)$ is the algebra of coefficients of the left regular representation $\lambda : G \to B(L^2(G))$, $(\lambda(s)f)(t) = f(s^{-1}t)$, that is, the algebra of functions of the form $s \mapsto \langle \lambda(s)f, g\rangle = (\bar{g} * \check{f})(s)$ for $f, g \in L^2(G)$. It is known that $A(G)$ is a semisimple, regular, commutative Banach algebra of continuous functions on G with respect to the norm

$$\|u\|_{A(G)} = \inf\{\|f\|_2\|g\|_2 : u = \bar{g} * \check{f}\}.$$

The Gelfand spectrum of $A(G)$ is known to be the space G itself.

We also recall the duality relation $A(G)^* \simeq \mathrm{VN}(G)$, where $\mathrm{VN}(G)$ is the von Neumann algebra generated by $\lambda(G)$. The duality is given by the pairing $\langle u, T\rangle = \langle Tf, g\rangle$, where $u \in A(G)$, $u(s) = \langle \lambda(s)f, g\rangle$. We use the same notation, $\langle \cdot, \cdot\rangle$, both for the inner product on a Hilbert space and for the duality pairing; which one is used should be clear from the context.

If $I \subseteq A(G)$ is an ideal, we define the *hull* of I to be the set

$$h(I) = \{s \in G : u(s) = 0 \text{ for all } u \in I\} \subseteq G.$$

On the other hand, for a closed set $E \subseteq G$, we define the *kernel* $k(E)$ of E and the minimal ideal $j(E)$ with hull E by

$$k(E) := \{f \in A(G) : f(s) = 0,\ s \in E\},$$
$$j(E) := \{f \in A(G) : f \text{ has compact support disjoint from } E\}.$$

We observe that $h(\overline{j(E)}) = h(k(E)) = E$ and if $I \subseteq A(G)$ is a closed ideal with $h(I) = E$, then $\overline{j(E)} \subseteq I \subseteq k(E)$.

A closed subset $E \subseteq G$ is said to be *spectral* or a *set of spectral synthesis* if $k(E) = \overline{j(E)}$, equivalently if $k(E)$ is the only closed ideal whose hull is the set E. Furthermore, E is called a *Ditkin set* if $a \in \overline{aj(E)}$ for any $a \in k(E)$. Clearly any Ditkin set is a set of spectral synthesis. The converse is an open problem.

We say that E is *local spectral* or a *set of local spectral synthesis* if for any $u \in A(G) \cap C_c(G)$ which vanishes on E there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset A(G)$ which converges to u and u_n vanishes on a neighborhood of E for every $n \in \mathbb{N}$.

It is well known that every local spectral subset of G is spectral, provided that $A(G)$ has an (unbounded) approximate identity, for instance if G is amenable, weakly amenable [CoH], or more generally if every $u \in A(G)$ is contained in the closed ideal it generates. Many examples of local spectral subsets are known. For example, it was shown by M. Takesaki and N. Tatsuuma in [TT] for $A(G)$ and by C. Herz in [He2] for the Figà-Talamanca–Herz algebras $A_p(G)$ that closed subgroups of any locally compact group are local spectral.

We finish this section by setting basic notation and introducing some notions that will be used subsequently. We write $B(H)$ for the algebra of all bounded operators on a Hilbert space H. For an (unbounded) operator A on H, $\mathcal{D}(A) \subset H$ will stand for the domain of A. For $W \subset G$ we write W^c for its complement. For a function $f : G \to \mathbb{C}$ we let $\mathrm{null}(f) = \{x \in G : f(x) = 0\}$ and $\mathrm{supp}(f) = \overline{\mathrm{null}(f)^c}$.

3. A characterization of local spectral sets. In this section we will present a new characterization of local spectral sets. But first we examine some convergence properties in $A(G)$.

3.1. Convergence in $A(G)$. We recall first the right and the left actions of $\mathrm{VN}(G)$ on $A(G)$: if $u \in A(G)$, $T \in \mathrm{VN}(G)$, then the elements $u \cdot T$ and $T \cdot u$ in $A(G)$ are defined through the following formulas:

$$\langle S, u \cdot T\rangle := \langle TS, u\rangle, \quad \langle S, T \cdot u\rangle := \langle ST, u\rangle \quad \text{for all } S \in \mathrm{VN}(G).$$

If $u \in A(G) \cap L^2(G)$, then we shall also write $T(u)$ for the action of T on $u \in L^2(G)$.

Recall that a continuous function $u : G \to \mathbb{C}$ is called *positive definite* if for each $n \in \mathbb{N}$ and $s_1, \ldots, s_n$ in G the matrix $(u(s_i^{-1}s_j))_{j,j=1}^n$ is positive definite.

We say that a locally integrable function φ on G is a function *of positive type* if

$$\int \varphi(s)(f * \tilde{f})(s)\, ds \geq 0 \quad \text{for all } f \in C_c(G).$$

If such a φ is continuous, we obtain the previous definition of positive definiteness [Di2, 13.4.4]. Moreover, if u is positive definite, then there exists a unitary representation π of G on H_π and a vector $\xi \in H_\pi$ such that $u(s) = \langle \pi(s)\xi, \xi\rangle$; we write c_ξ^π for the latter matrix coefficient.

If $f \in C_c(G)$ then for any $b \in L^2(G)$ we have $f * b \in L^2(G)$ and we can consider the linear operator $f \mapsto f * b$, $f \in C_c(G)$, on $L^2(G)$. If b is of positive type then

$$\langle f * b, f\rangle \geq 0;$$

see [Di2, 13.7.6, 13.8.1]. Therefore, the operator $f \mapsto r(b)(f) := f * b$ defined on $C_c(G) \subset L^2(G)$ is positive. Let $\rho(b)$ denote its Friedrichs extension, which is a positive self-adjoint operator. By [Di2, 13.8.3], if h is in $\mathcal{D}(\rho(b))$, then $\rho(b)h = h * b$.

Lemma 3.1. *Let $b \in C(G)$ be a positive definite square integrable function. Then there exists a square integrable function c of positive type such that $b = c * \tilde{c} = c * c$. Moreover, in this case $\rho(c)f = \rho(b)^{1/2}f$ for all $f \in C_c(G)$.*

Proof. The existence of a square integrable function c of positive type satisfying $b = c * \tilde{c} = c * c$ follows from [Di2, 13.8.6]. It is constructed as the $L^2(G)$-limit of a sequence $(c_n)_{n\in\mathbb{N}}$ of continuous positive definite functions, such that $\rho(c_n) = \sqrt{\rho(b)}E_n$, where $\rho(b) = \int_{[0,\infty]} x\, dE(x)$ is the spectral decomposition of $\rho(b)$ and $E_n = E([0, n])$, $n \in \mathbb{N}$.

For $f \in C_c(G)$, one has $\rho(c_n)f = f * c_n$ and, as $c_n \to c$ in $L^2(G)$, this gives $\rho(c_n)f \to \rho(c)f$, since $\|\rho(c_n)f - \rho(c)f\|_2 \leq \|f\|_1\|c_n - c\|_2$ for any n.

On the other hand, $\sqrt{\rho(b)}E_nf \to \sqrt{\rho(b)}f$ for all $f \in \mathcal{D}(\sqrt{\rho(b)})$; in particular, we have the convergence for all $f \in C_c(G)$, since $C_c(G) \subset \mathcal{D}(\rho(c)) \subset \mathcal{D}(\sqrt{\rho(c)})$. Therefore, $\rho(c)f = \sqrt{\rho(b)}f$ for $f \in C_c(G)$. ∎

We call the function c from the lemma the *positive square root* of b. Note that it is unique as an element in $L^2(G)$.

As $A(G)$ is the predual of $\mathrm{VN}(G)$, each $u \in A(G)$ admits polar decomposition: there exists a unique pair (A, p), where p is a positive definite function in $A(G)$ such that $\|u\|_{A(G)} = \|p\|_{A(G)}$, and A is a partial isometry in $\mathrm{VN}(G)$ with final space equal to the support, $s(p)$, of p and such that $u = A \cdot p$ and

$p = A^* \cdot u$ (see [Di1, 1.4, Théorème 4]). The element p is called the *absolute value* of u.

Note that for $\tilde{u}$ and $\check{u}$ as defined above, we have $\tilde{u}, \check{u} \in A(G)$ if $u \in A(G)$ and $u = \tilde{u}$ if u is positive definite. We can define a linear involution $T \mapsto \check{T}$ on $\mathrm{VN}(G)$ by

$$\langle \check{T}, u \rangle := \langle T, \check{u} \rangle, \qquad T \in \mathrm{VN}(G),\ u \in A(G),$$

and an antilinear involution $T \mapsto \bar{T}$ of $\mathrm{VN}(G)$ by

$$\bar{T}(f) = \overline{T(\bar{f})}, \qquad f \in L^2(G).$$

In the Appendix we collect various equalities involving $\check{T}$, $\bar{T}$, $\check{u}$ and $\tilde{u}$.

PROPOSITION 3.2. *Let u be an element of $A(G)$ such that u and $\tilde{u}$ are in $L^2(G)$. Let $\tilde{u} = A \cdot p$ be the polar decomposition of $\tilde{u}$. Then $p \in L^2(G)$ and*

$$p = c * c, \quad p = \check{A}(u) \quad \textit{and} \quad u = \bar{A}(p) = \bar{A}(c) * c,$$

where c is the positive square root of p.

Proof. If $\tilde{u} = A \cdot p$ is the polar decomposition of $\tilde{u} \in A(G) \cap L^2(G)$ then by equalities (5.5) and (5.8) from the Appendix we have

$$p = \tilde{p} = (A^* \cdot \tilde{u})\tilde{} = u \cdot A = \check{A}(u) \in L^2(G). \tag{3.1}$$

Hence, by Lemma 3.1,

$$p = c * c, \tag{3.2}$$

where $c \in L^2(G)$ is the positive square root of p. Using (5.5), (5.8) and (5.1) we see that

$$u = (A \cdot p)\tilde{} = p \cdot A^* = \bar{A}(p) = \bar{A}(c) * c. \ ■$$

Replacing the partial isometry $A \in \mathrm{VN}(G)$ by the partial isometry $\bar{A} \in \mathrm{VN}(G)$ we call the representation $u = A(c)*c = A(c)*\tilde{c}$ from Proposition 3.2 the *canonical representation* of $u \in A(G) \cap L^2(G)$.

Define for $p \in L^2(G)$ the subspace $L^2(G)_p$ of $L^2(G)$ by

$$L^2(G)_p := \{\xi \in L^2(G) : \xi * p \in L^2(G)\}.$$

Then $C_c(G) \subset L^2(G)_p$ and for every $f \in C_c(G)$, a positive definite function $p = c^\pi_\varphi \in L^2(G)$ and $\eta \in L^2(G)$ we deduce (using the Fubini theorem) that

$$\begin{aligned}\langle f * p, \eta \rangle &= \int_G \int_G f(s) \langle \pi(s^{-1}t)\varphi, \varphi \rangle \, ds \, \overline{\eta(t)} \, dt \\ &= \int_G f(s) \int_G \overline{\langle \pi(t^{-1}s)\varphi, \varphi \rangle \eta(t)} \, dt \, ds = \int_G f(s) \overline{(\eta * p)(s)} \, ds.\end{aligned} \tag{3.3}$$

DEFINITION 3.3. Let G be a locally compact group. We call G *right positive* if for any positive definite function p we have

$$\langle \eta * p, \eta \rangle \geq 0 \quad \text{for every } \eta \in L^2(G)_p.$$

LEMMA 3.4. *For any locally compact σ-compact right positive group G and any positive definite function $p \in L^2(G)$ the subspace*

$$I_p := \{f * p + if : f \in C_c(G)\}$$

is dense in $L^2(G)$.

Proof. Let $\eta \in I_p^{\perp}$. Hence, using $p = \tilde{p}$ and applying the Fubini theorem, we obtain

$$\begin{aligned} 0 = \langle \eta, f * p + if\rangle &= \int_G (\eta * p)(s)\overline{f(s)}\,ds - i\int_G \eta(s)\overline{f(s)}\,ds \\ &= \int_G ((\eta * p)(s) - i\eta(s))\overline{f(s)}\,ds. \end{aligned}$$

This implies that the measurable function $\eta * p - i\eta$ is 0 almost everywhere on each compact $K \subset G$. As G is σ-compact, we see that $\eta * p - i\eta = 0$ almost everywhere on G. Hence $\eta * p = i\eta$, $\eta * p \in L^2(G)$ and so

$$\langle \eta * p, \eta\rangle = i\langle \eta, \eta\rangle.$$

Since G is right positive, it follows that $i\langle \eta, \eta\rangle \geq 0$, giving $\eta = 0$. ∎

PROPOSITION 3.5. *Any separable, type I, unimodular group is right positive.*

Proof. If G is separable, type I and unimodular, we have a clear Plancherel picture of G as follows. The unitary dual $\widehat{G}$ becomes a standard Borel space and there is a unique Borel measure μ on $\widehat{G}$ with the following property: for a fixed μ-measurable cross-section $\xi \mapsto \pi^\xi$ from $\widehat{G}$ to concrete irreducible unitary representations acting on H_ξ we have

$$\langle f_1, f_2\rangle = \int_{\widehat{G}} \mathrm{Tr}(\widehat{f_1}^G(\xi)\widehat{f_2}^G(\xi)^*)\,d\mu(\xi), \qquad f_1, f_2 \in L^1(G) \cap L^2(G),$$

where

$$\widehat{f}^G(\xi) = \mathcal{F}^G(f)(\xi) := \int_G f(g)\pi^\xi(g)\,dg \in \mathcal{B}(H_\xi), \qquad f \in L^1(G) \cap L^2(G). \tag{3.4}$$

Thus, the group Fourier transform

$$\mathcal{F}^G : L^1(G) \to L^\infty(\widehat{G}, d\mu; \mathcal{B}(H_\xi)), \qquad f \mapsto \mathcal{F}^G(f),$$

with $\mathcal{F}^G(f) = (\mathcal{F}^G(f)(\xi))_{\xi\in\widehat{G}} = (\widehat{f}^G(\xi))_{\xi\in\widehat{G}}$ extends to a unitary

$$\mathcal{F}^G : L^2(G) \to L^2(\widehat{G}, d\mu; S^2(H_\xi)), \qquad f \mapsto \mathcal{F}^G(f).$$

Here, $S^2(H)$ is the space of Hilbert–Schmidt operators on a Hilbert space H. If $\eta, p \in L^2(G)$ are such that $\eta * p \in L^2(G)$, then

$$\mathcal{F}^G(\eta * p)(\xi) = \mathcal{F}^G(\eta)(\xi)\mathcal{F}^G(p)(\xi)$$

and

$$\langle \eta * p, \eta \rangle = \int_{\widehat{G}} \mathrm{Tr}(\mathcal{F}^G(\eta)(\xi)\mathcal{F}^G(p)(\xi)\mathcal{F}^G(\eta)(\xi)^*)\, d\mu(\xi).$$

As $p \in L^2(G)$ is positive definite, by Lemma 3.1, $p = c * \tilde{c}$ for a square integrable function c of positive type, giving $\mathcal{F}^G(p)(\xi) = \mathcal{F}^G(c)(\xi)\mathcal{F}^G(c)(\xi)^*$, and hence $\mathcal{F}^G(p)(\xi) \geq 0$ almost everywhere. Therefore, $\langle \eta * p, \eta \rangle \geq 0$, giving the statement. ∎

THEOREM 3.6. *Let G be a right positive, σ-compact, locally compact group and let $u \in A(G)$ and $u_k \in A(G)$, $k \in \mathbb{N}$, be such that $u, \tilde{u} \in L^2(G)$ and $u_k, \tilde{u}_k \in L^2(G)$ for every k. Let $u = A(c) * c$ and $u_k = A_k(c_k) * c_k$, $k \in \mathbb{N}$, be the canonical representations of $u, u_k \in A(G)$ respectively, and let $p = c * c$, $p_k = c_k * c_k$. Suppose that*

(1) $\|u_k\|_{A(G)} \to \|u\|_{A(G)}$;
(2) $\|u_k - u\|_2 \to 0$;
(3) $p_k \to p$ *weakly in* $L^2(G)$.

Then $\|c_k - c\|_2 \to 0$ and $\|A_k(c_k) - A(c)\|_2 \to 0$. In particular, $\|p_k - p\|_{A(G)} \to 0$ and $\|u_k - u\|_{A(G)} \to 0$.

Proof. We shall use an idea from [CaH, proof of Proposition 5.1] to conclude that $(c_k)_k$ converges to c in $L^2(G)$.

We have

$$\|u_k\|_2 = \|A_k(p_k)\|_2 \leq \|p_k\|_2 = \|A_k^*(u_k)\|_2 \leq \|u_k\|_2 \quad \text{for all } k.$$

Hence, $\|u_k\|_2 = \|p_k\|_2$ for all k and similarly $\|u\|_2 = \|p\|_2$. The same is true for the $A(G)$-norms. Since $\|u_k - u\|_2 \to 0$, we have the convergence $\|p_k\|_2 \to \|p\|_2$. Together with the weak convergence of $(p_k)_k$ to p, this shows that $\|p_k - p\|_2 \to 0$.

We are going to show next that $\rho(c_k)f$ converges to $\rho(c)f$ in $L^2(G)$ for any $f \in C_c(G)$. Since each $\rho(p_k)$ and $\rho(p)$ are self-adjoint and positive, the operators $\rho(p_k) + i\mathbb{I}$ and $\rho(p) + i\mathbb{I}$ have bounded inverses; here we write $\mathbb{I}$ for the identity operator on $L^2(G)$. Furthermore,

$$\|(\rho(p_k) + i\mathbb{I})^{-1}\|_{\mathrm{op}} \leq 1 \quad \text{for all } k, \quad \text{and} \quad \|(\rho(p) + i\mathbb{I})^{-1}\|_{\mathrm{op}} \leq 1.$$

Let $f \in C_c(G)$ and $g = (\rho(p) + i\mathbb{I})f \in L^2(G)$. As $p_k \to p$ in $L^2(G)$, we have

$$\|\rho(p_k)f - \rho(p)f\|_2 = \|f * p_k - f * p\|_2 \leq \|f\|_1 \|p_k - p\|_2 \to 0$$

and

$$[(\rho(p_k) + i\mathbb{I})^{-1} - (\rho(p) + i\mathbb{I})^{-1}]g = (\rho(p_k) + i\mathbb{I})^{-1}(\rho(p) - \rho(p_k))f \to 0.$$

Since the operators $(\rho(p_k) + i\mathbb{I})^{-1}$ are uniformly bounded and since, by Lemma 3.4, the subspace $\{(\rho(p) + i\mathbb{I})f : f \in C_c(G)\}$ is dense in $L^2(G)$, it follows that

$$(\rho(p_k)+i\mathbb{I})^{-1}g \to (\rho(p)+i\mathbb{I})^{-1}g \quad \text{for all } g\in L^2(G). \tag{3.5}$$

A similar proof works for $(\rho(p_k)-i\mathbb{I})^{-1}$.

Define now two continuous functions $h, q : [0,\infty) \to \mathbb{R}$ by letting

$$h(t) := \begin{cases} \sqrt{t}-t & \text{if } t\le 1,\\ 0 & \text{if } t\ge 1,\end{cases} \quad \text{and} \quad q(t) := \begin{cases} 1 & \text{if } t\le 1,\\ \frac{1}{\sqrt{t}} & \text{if } t\ge 1.\end{cases}$$

Then $h, q \in C_0([0,\infty))$, and

$$\sqrt{t} = h(t) + q(t)t, \qquad t\ge 0.$$

By the Stone–Weierstrass theorem, the polynomials in $(x+i)^{-1}$ and $(x-i)^{-1}$ are dense in $C_0([0,\infty))$. Thus, given $\varepsilon>0$, we can find a polynomial $P(s,t)$ such that

$$\left| q(x) - P\Big(\frac{1}{x+i}, \frac{1}{x-i}\Big)\right| < \frac{\varepsilon}{3} \qquad \text{for all } x\ge 0.$$

Therefore,

$$\begin{aligned} \big\|q(\rho(p_k)) - P\big((\rho(p_k)+i\mathbb{I})^{-1}, (\rho(p_k)-i\mathbb{I})^{-1}\big)\big\|_{\mathrm{op}} &< \frac{\varepsilon}{3} \qquad \text{for all } k,\\ \big\|q(\rho(p)) - P\big((\rho(p)+i\mathbb{I})^{-1}, (\rho(p)-i\mathbb{I})^{-1}\big)\big\|_{\mathrm{op}} &< \frac{\varepsilon}{3}. \end{aligned}$$

It follows from (3.5), that

$$P\big((\rho(p_k)+i\mathbb{I})^{-1}, (\rho(p_k)-i\mathbb{I})^{-1}\big)f \to P\big((\rho(p)+i\mathbb{I})^{-1}, (\rho(p)-i\mathbb{I})^{-1}\big)f$$

for all $f\in L^2(G)$. Thus, for any $f\in L^2(G)$, there exists an $N_1(f)\in\mathbb{N}$ such that

$$\|q(\rho(p_k))f - q(\rho(p))f\|_2 \le \varepsilon \qquad \text{for any } k\ge N_1(f). \tag{3.6}$$

Since $\rho(p_k)f\to\rho(p)f$ for all $f\in C_c(G)$, there exists an $N_2(f)\in\mathbb{N}$ with

$$\|\rho(p_k)f - \rho(p)f\|_2 \le \varepsilon \qquad \text{for all } k\ge N_2(f).$$

Finally, for $f\in C_c(G)$ and $k\ge \max\{N_1(\rho(p)f), N_2(f)\}$ we have

$$\begin{aligned} &\|q(\rho(p_k))(\rho(p_k)f) - q(\rho(p))(\rho(p)f)\|_2\\ &\qquad \le \|q(\rho(p_k))(\rho(p_k)f) - q(\rho(p_k))(\rho(p)f)\|_2\\ &\qquad\quad + \|q(\rho(p_k))(\rho(p)f) - q(\rho(p))(\rho(p)f)\|_2\\ &\qquad \le \|\rho(p_k)f-\rho(p)f\|_2 + \|q(\rho(p_k))(\rho(p)f) - q(\rho(p))(\rho(p)f)\|_2\\ &\qquad \le \varepsilon+\varepsilon, \end{aligned}$$

where we use $\|q(\rho(p_k))\|_{\mathrm{op}} \le \|q\|_\infty = 1$ for all k.

Similar arguments applied to h instead of q give us

$$h(\rho(p_k))f \to h(\rho(p))f \qquad \text{for all } f\in C_c(G). \tag{3.7}$$

Together this shows that

$$\sqrt{\rho(p_k)}f \to \sqrt{\rho(p)}f \quad \text{for all } f \in C_c(G).$$

But by (3.2) we have $p_k = c_k * c_k$, $p = c * c$ and $\sqrt{\rho(p_k)}f = \rho(c_k)f$ for all k, and $\sqrt{\rho(p)}f = \rho(c)f$, $f \in C_c(G)$. Hence,

$$f * c_k \to f * c \quad \text{for all } f \in C_c(G).$$

Therefore, for all $f, g \in C_c(G)$, it follows that

$$\langle c_k, f^* * g\rangle = \langle f * c_k, g\rangle \to \langle f * c, g\rangle = \langle c, f^* * g\rangle. \tag{3.8}$$

Here $f^*(s) = \bar{f}(s^{-1})\Delta(s^{-1})$ and Δ is the modular function. Since the functions $f^* * g$, $f, g \in C_c(G)$, generate a dense subspace in $L^2(G)$ and since by assumption

$$\|c_k\|_2^2 = p_k(e) = \|p_k\|_{A(G)} = \|u_k\|_{A(G)} \to \|u\|_{A(G)} = \|p\|_{A(G)} = p(e) = \|c\|_2^2,$$

it follows from (3.8) that $(c_k)_k$ converges to c weakly in $L^2(G)$ and finally also in norm, as

$$\|c_k - c\|_2^2 = \|c_k\|_2^2 + \|c\|_2^2 - 2\Re\langle c, c_k\rangle| \to 0.$$

The sequence $(A_k)_k$, being uniformly bounded by 1, admits a weakly convergent subnet $(A_{k_i})_i$. Let $A_\infty \in \mathrm{VN}(G)$ be its limit. We are going to show that $A_\infty(p) = A(p)$ and $A_\infty^*(u) = A^*(u)$.

Indeed, for $f \in C_c(G)$, we have

$$\begin{aligned} \langle u_{k_i}, f\rangle &= \langle A_{k_i}(p_{k_i}), f\rangle \\ &= \langle p_{k_i}, A_{k_i}^*(f)\rangle \\ \downarrow \quad & \quad \downarrow \quad (\text{since } p_{k_i} \to p, u_{k_i} \to u \in L^2(G)) \\ \langle u, f\rangle &= \langle p, A_\infty^*(f)\rangle = \langle A_\infty(p), f\rangle. \end{aligned}$$

Hence, $u = A(p) = A_\infty(p)$. Similarly,

$$\begin{aligned} \langle A^*(u), f\rangle &= \langle p, f\rangle = \lim_k \langle p_k, f\rangle = \lim_k \langle A_k^*(u_k), f\rangle = \lim_{k_i} \langle u_{k_i}, A_{k_i}(f)\rangle \\ &= \langle u, A_\infty(f)\rangle = \langle A_\infty^*(u), f\rangle \end{aligned}$$

and hence $A^*(u) = A_\infty^*(u)$.

Furthermore, for any $x \in G$,

$$\begin{aligned} \langle A_\infty(c) - A(c), \lambda(x)c\rangle &= ((A_\infty(c) - A(c)) * c)(x) \\ &= (A_\infty(c) * c)(x) - (A(c) * c)(x) \\ &= A_\infty(c * c)(x) - A(c * c)(x) \\ &= A_\infty(p)(x) - A(p)(x) = u(x) - u(x) = 0. \end{aligned}$$

This relation tells us that $A_\infty(c) - A(c)$ is contained in the orthogonal complement to $\lambda(C_c(G))(c)$. On the other hand, since $A_\infty, A \in \mathrm{VN}(G)$, we see that A_∞, A are strong limits of nets contained in $\lambda(C_c(G))$. Consequently, $A(c), A_\infty(c) \in \overline{\lambda(C_c(G))(c)}$ and therefore $A_\infty(c) - A(c) = 0$.

Observe next that $A^*A(p) = p$. As $p = c * c = c * \tilde{c}$, it follows from (5.1) and (5.8) that

$$p(x) = A^*A(p)(x) = (A^*A(c) * \tilde{c})(x) \quad \text{for almost all } x \in G. \tag{3.9}$$

Now, since both p and $A^*A(c) * \tilde{c}$ are continuous, we have the equality everywhere on G.

Similarly, $p_k(x) = (A_k^*A_k(c_k) * c_k)(x)$ for all $x \in G$, $k \in \mathbb{N}$. Hence,

$$\begin{aligned}\|A_k(c_k)\|_2^2 &= \langle A_k(c_k), A_k(c_k)\rangle = \langle A_k^*A_k(c_k), c_k\rangle \\ &= (A_k^*A_k(c_k) * c_k)(e) = p_k(e) = \|p_k\|_{A(G)} \\ &\to \|p\|_{A(G)} = p(e) \quad (\text{as } k \to \infty) \\ &= (A^*A(c) * c)(e) = \|A(c)\|_2^2.\end{aligned}$$

Therefore, the weakly convergent net $(A_{k_i}(c_{k_i}))_i$ converges in fact in norm to $A_\infty(c) = A(c)$, from which we can conclude that the convergence holds for the entire sequence $(A_k(c_k))_k$. In fact, otherwise, there exist $\varepsilon > 0$ and a subsequence $(A_{k(n)}(c_{k(n)}))_n$ such that $\|A_{k(n)}(c_{k(n)}) - A(c)\| > \varepsilon$. Repeating the previous arguments, we find a subsequence of $(A_{k(n)}(c_{k(n)}))_n$ that converges to $A(c)$ in norm, contradicting the choice of $(A_{k(n)}(c_{k(n)}))_n$. ∎

Corollary 3.7. *Suppose $u, u_k \in A(G)$, $k \in \mathbb{N}$, satisfy $\|u_k - u\|_{A(G)} \to 0$ and $\operatorname{supp}(u_k) \subset K$ for all $k \in \mathbb{N}$ and some compact set $K \subset G$. Let $u_k = A_k(c_k) * c_k$ and $u = A(c) * c$ be the canonical representations of u_k and u respectively. Then $\|c_k - c\|_2 \to 0$ and $\|A_k(c_k) - A(c)\|_2 \to 0$.*

Proof. As $\|\tilde{w}\|_{A(G)} = \|w\|_{A(G)}$ for all $w \in A(G)$ (see [Ey, Remark 2.10]), we have $\|\tilde{u}_k - \tilde{u}\|_{A(G)} \to 0$, and if p_k and p are the absolute values of $\tilde{u}_k$ and $\tilde{u}$ respectively, applying [Ta, III, Proposition 4.10] we obtain $\|p_k - p\|_{A(G)} \to 0$. In particular, p_k tends to p and u_k to u uniformly on G. Now, by the assumption, we can find a common compact subset that contains the supports of u, $\tilde{u}$ and $u_k, \tilde{u}_k$, $k \in \mathbb{N}$. Therefore, $\|u_k - u\|_2 \to 0$. As $p_k = A_k^*(u_k)$, the sequence $(p_k)_k$ is bounded in $L^2(G)$. The uniform convergence $p_k \to p$ on G gives $\langle p_k - p, f\rangle \to 0$ for any $f \in C_c(G)$. As $C_c(G)$ is dense in $L^2(G)$ and $(p_k)_k$ is bounded, this implies that $p_k \to p$ weakly in $L^2(G)$. We have thus verified all the conditions of Theorem 3.6, which gives us the statement. ∎

3.2. A characterization of local spectral sets. Let $t \in G$, $\xi \in L^2(G)$ and $T \subset G$. For simplicity of notation, write

$$t \cdot \xi := \lambda(t)\xi,$$

and set

$$\Lambda(T \cdot \xi) := \overline{\operatorname{span}(T \cdot \xi)}.$$

For a closed subspace E of $L^2(G)$, let P_E be the orthogonal projection onto E and write $P_{T\cdot\alpha}$ for the projection $P_{\Lambda(T\cdot\alpha)}$.

We are now ready to prove the main result.

THEOREM 3.8. *Let G be a locally compact, σ-compact, right positive group and let S be a closed subset of G. Then S is a local spectral set if and only if for any $u \in k(S) \cap C_c(G)$ there exist a representation $u(s) = \langle d, \lambda(s)c\rangle$, $c, d \in L^2(G)$, a sequence $(c_k)_k$ in $L^2(G)$ and a sequence $(S_k)_k$ of closed neighborhoods of S such that*

$$\lim_{k\to\infty} c_k = c \quad \text{and} \quad \lim_{k\to\infty} P_{S_k\cdot c_k}(d) = 0. \tag{3.10}$$

Moreover, if S is a set of local spectral synthesis, then there is a sequence $(c_k)_k \subset L^2(G)$ of functions of positive type satisfying (3.10), *with $c, d = A(c) \in L^2(G)$ from the canonical representation $u(s) = \langle d, \lambda(s)c\rangle = (A(c) * \tilde{c})(s)$.*

Proof. Assume $u \in k(S) \cap C_c(G)$ has a representation $u(s) = \langle d, \lambda(s)c\rangle$, $c, d \in L^2(G)$. Let $(c_k)_k \subset L^2(G)$ and let $(S_k)_k$ be a sequence of neighborhoods of S which satisfy the conditions of the theorem. Set $d_k = d - P_{S_k\cdot c_k}(d)$. Then $d_k \in \Lambda(S_k \cdot c_k)^{\perp}$ and hence $u_k(s) := \langle d_k, \lambda(s)c_k\rangle$ vanishes on S_k. Moreover,

$$\begin{aligned}\|u - u_k\|_{A(G)} &\le \|\langle d_k, \lambda(s)(c - c_k)\rangle\|_{A(G)} + \|\langle d - d_k, \lambda(s)c_k\rangle\|_{A(G)}\\ &\le \|c - c_k\|_2\|d_k\|_2 + \|c_k\|_2\|d - d_k\|_2 \to 0\end{aligned}$$

showing that S is a set of local spectral synthesis.

Suppose now that S is a set of local spectral synthesis and take $u \in k(S) \cap C_c(G)$. Let K be a compact neighborhood of the support of u. Since S is local spectral, there exists a sequence $(u_k)_k$ in $A(G)$ such that, for every k, $\mathrm{supp}(u_k) \subset K$, u_k vanishes on a neighborhood S_k of S and u_k converges to u in $A(G)$.

Consider the canonical representation $u_k = A_k(c_k) * c_k$ and set $p_k = c_k * c_k$ and $d_k = A_k(c_k)$. Then $d_k \in \Lambda(S_k \cdot c_k)^{\perp}$, since u_k vanishes on S_k. Therefore, $P_{S_k\cdot c_k}(d_k) = 0$ for every k. By Corollary 3.7, $\lim_k c_k = c, \lim_k d_k = d$ and hence $\lim_k P_{S_k\cdot c_k}(d) = 0$. ∎

3.3. The abelian case. The goal of this section is to provide a refinement of the characterization of local spectral sets from Section 3.2 in the case of abelian groups. For an abelian locally compact group G let $\widehat{G}$ be the dual of G. We write $\hat{a}$ for the Fourier transform of $a \in L^2(G)$.

LEMMA 3.9. *Let G be an abelian locally compact group. Let $c, d \in L^2(G)$ be such that $\hat{d} = \psi\hat{c}$ for some $\psi \in L^{\infty}(\widehat{G})$. Then for any subset S of G we have*

$$\|P_{S\cdot c}(d)\|_2 \le \|\psi\|_{\infty}\|P_{S^{-1}\cdot d}(c)\|_2.$$

Proof. We shall use the Plancherel theorem and consider therefore the action of G on the Hilbert space $L^2(\widehat{G})$:

$$t\cdot\xi(x)=\chi_t(x)\xi(x),\qquad x\in\widehat{G},\ \xi\in L^2(\widehat{G}),\ t\in G,$$

where χ_t denotes the character of $\widehat{G}$ defined by $t\in G$.

Set $\xi:=\hat{c}$, $\eta:=\hat{d}$. We have

$$\eta=\psi\xi.$$

By the definition of $\Lambda(S\cdot\xi)$ the elements of the space $P_{\Lambda(S\cdot\xi)}(L^2(\hat{G}))$ are of the form $\varphi\xi$ for some measurable function $\varphi:\widehat{G}\to\mathbb{C}$ and $\varphi\xi$ is an L^2-limit of functions $\varphi_k\xi$, where

$$\varphi_k=\sum_{j=1}^{m_k}c_j^k\chi_{s_j^k}$$

for some constants $c_j^k\in\mathbb{C}$ and $s_j^k\in S$. Hence, for such a $\varphi\xi\in\Lambda(S\cdot\xi)$ we have

$$\begin{aligned}\text{(3.11)}\qquad \overline{\varphi}\eta&=\overline{\varphi}\psi\xi=\psi\overline{\varphi}\xi=\psi\Big(\lim_{k\to\infty}\overline{\varphi_k}\xi\Big)=\lim_{k\to\infty}\overline{\varphi_k}\psi\xi\\&=\lim_{k\to\infty}\overline{\varphi_k}\,\eta\in\Lambda(S^{-1}\cdot\eta).\end{aligned}$$

Since

$$\text{(3.12)}\qquad \|P_{\Lambda(S\cdot\xi)}(\eta)\|_2=\sup\{|\langle\eta,\varphi\xi\rangle|:\varphi\xi\in\Lambda(S\cdot\xi),\ \|\varphi\xi\|_2=1\},$$

it follows that for any $\varphi\xi$ of norm 1,

$$\begin{aligned}|\langle\eta,\varphi\xi\rangle|&=|\langle\overline{\varphi}\eta,\xi\rangle|=|\langle P_{\Lambda(S^{-1}\cdot\eta)}(\overline{\varphi}\eta),\xi\rangle|=|\langle\overline{\varphi}\eta,P_{\Lambda(S^{-1}\cdot\eta)}(\xi)\rangle|\\&\le\|\overline{\varphi}\psi\xi\|_2\|P_{\Lambda(S^{-1}\cdot\eta)}(\xi)\|_2\le\|\psi\|_\infty\|\overline{\varphi}\xi\|_2\|P_{\Lambda(S^{-1}\cdot\eta)}(\xi)\|_2\\&=\|\psi\|_\infty\|P_{\Lambda(S^{-1}\cdot\eta)}(\xi)\|_2.\end{aligned}$$

Hence

$$\|P_{\Lambda(S\cdot\xi)}(\eta)\|_2\le\|\psi\|_\infty\|P_{\Lambda(S^{-1}\cdot\eta)}(\xi)\|_2.\ \blacksquare$$

COROLLARY 3.10. *Let G be an abelian locally compact group. Let $c,d\in L^2(G)$ be such that $\hat{d}=\psi\hat{c}$, where $\psi:\widehat{G}\to\mathbb{C}$ is a measurable function of absolute value equal to 1 on the support of $\hat{c}$. Then for any subset S of G we have*

$$\|P_{\Lambda(S\cdot c)}(d)\|_2=\|P_{\Lambda(S^{-1}\cdot d)}(c)\|_2.$$

Proof. As $\hat{d}=\psi\hat{c}$ and $\hat{c}=\overline{\psi}\hat{d}$, Lemma 3.9 gives us

$$\|P_{\Lambda(S\cdot c)}(d)\|_2\le\|P_{\Lambda(S^{-1}\cdot d)}(c)\|_2\le\|P_{\Lambda(S\cdot c)}(d)\|_2,$$

which implies

$$\|P_{\Lambda(S\cdot c)}(d)\|_2=\|P_{\Lambda(S^{-1}\cdot d)}(c)\|_2.\ \blacksquare$$

LEMMA 3.11. *Let G be a locally compact, σ-compact, abelian group. Let $(S_k)_k$ be a sequence of closed subsets of G. Let $(u_k)_k$ be a converging sequence in $A(G)$ with limit $u \in A(G) \cap L^2(G)$, such that $u_k(S_k) = \{0\}$ for every k. Let $u = d*c$ be the canonical form of $u \in L^2(G)$. Then $\lim_{k\to\infty} P_{S_k\cdot c}(d) = 0$.*

Proof. Choose a sequence $(v_i)_i \in C_c(G) \cap A(G)$ such that $\|v_i\|_{A(G)} \le 1$, $i \in \mathbb{N}$, and such that $\lim_{i\to\infty} v_i u = u$ in $A(G)$. Then the sequence $(v_i u_k)_k \in \mathbb{N}$ converges in $A(G)$ and in $L^2(G)$ to $v_i u$. We write the elements $u_k^i := v_i u_k$ in the canonical form $u_k^i = d_k^i * c_k^i$, with $p_k^i = \widehat{c_k^i} \ge 0$ and $\widehat{d_k^i} = \varphi_k^i \widehat{c_k^i}$ and $|\varphi_k^i| = 1_{p_k^i}$, $k \in \mathbb{N}$. Similarly for $v_i u = d^i * c^i$. Then Theorem 3.6 tells us that the sequence $(c_k^i)_k$ converges in $L^2(G)$ to c^i and the sequence $(d_k^i)_k$ to d^i. Since $u_k^i(S_k) = \{0\}$ for any $k, i \in \mathbb{N}$, we have

$$P_{S_k\cdot c_k^i}(d_k^i) = 0, \qquad k, i \in \mathbb{N}.$$

Therefore, since $\lim_{k\to\infty} d_k^i = d^i$ for any i, it follows that

$$\lim_{k\to\infty} P_{S_k\cdot c_k^i}(d^i) = 0.$$

Hence, by Corollary 3.10,

$$\lim_{k\to\infty} \|P_{S_k^{-1}\cdot d^i}(c_k^i)\|_2 = \lim_{k\to\infty} \|P_{S_k\cdot c_k^i}(d^i)\|_2 = 0, \qquad i \in \mathbb{N}.$$

Finally,

$$\lim_{k\to\infty} \|P_{S_k\cdot c^i}(d^i)\|_2 = \lim_{k\to\infty} \|P_{S_k^{-1}\cdot d^i}(c^i)\|_2 = \lim_{k\to\infty} \|P_{S_k^{-1}\cdot d^i}(c_k^i)\|_2 = 0.$$

Now, since $u \in L^2(G)$, again by Theorem 3.6, $\lim_{i\to\infty} c^i = c$, $\lim_{i\to\infty} d^i = d$. Therefore, it follows as before that

$$\begin{aligned}
\|P_{S_k\cdot c}(d)\|_2 &\le \|P_{S_k\cdot c}(d^i - d)\|_2 + \|P_{S_k\cdot c}(d^i)\|_2 \\
&= \|P_{S_k\cdot c}(d^i - d)\|_2 + \|P_{S_k^{-1}\cdot d^i}(c)\|_2 \\
&\le \|P_{S_k\cdot c}(d^i - d)\|_2 + \|P_{S_k^{-1}\cdot d^i}(c - c^i)\|_2 + \|P_{S_k^{-1}\cdot d^i}(c^i)\|_2 \\
&= \|P_{S_k\cdot c}(d^i - d)\|_2 + \|P_{S_k^{-1}\cdot d^i}(c - c^i)\|_2 + \|P_{S_k\cdot c^i}(d^i)\|_2.
\end{aligned}$$

Consequently,

$$\|P_{S_k\cdot c}(d)\|_2 \le \|c - c^i\|_2 + \|d - d^i\|_2 + \|P_{S_k\cdot c^i}(d^i)\|_2$$

and so, for every $i \in \mathbb{N}$,

$$\begin{aligned}
\lim_{k\to\infty} \|P_{S_k\cdot c}(d)\|_2 &\le \lim_{k\to\infty} \|P_{S_k\cdot c^i}(d^i)\|_2 + \|c - c^i\|_2 + \|d - d^i\|_2 \\
&= \|c - c^i\|_2 + \|d - d^i\|_2.
\end{aligned}$$

This shows that $\lim_{k\to\infty} P_{S_k\cdot c}(d) = 0$. ∎

COROLLARY 3.12. *Let G be a locally compact, σ-compact, abelian group and let S be a closed subset of G. Then S is a spectral set if and only if for every $u \in k(S) \cap C_c(G)$ (with $u = d * c$ being its canonical expression)* *we have a decreasing sequence* $(S_k)_k$ *of closed neighborhoods of* S *such that* $\lim_{k\to\infty} P_{S_k\cdot c}(d) = 0$.

Proof. Suppose that S is spectral. Let $u \in k(S) \cap C_c(G)$, and $u = d * c$ be its canonical expression. Since S is spectral, there exists a sequence $(S_k)_k$ of neighborhoods of S, which can be chosen to be decreasing by taking intersections, and a sequence $(u_k)_k \subset A(G)$ such that $\lim_{k\to\infty} u_k = u$ in $A(G)$ and u_k vanishes on $S_k, k \in \mathbb{N}$. By Lemma 3.11, $\lim_{k\to\infty} P_{S_k\cdot c}(d) = 0$.

Conversely, if $u = d * c$ vanishes on S and if $\lim_{k\to\infty} P_{S_k\cdot c}(d) = 0$, then the sequence $(u_k := (d - P_{S_k\cdot c}(d)) * c)_k$ converges in $A(G)$ to u, and $u_k(S_k) = \langle (d - P_{S_k\cdot c}(d), S_k \cdot c\rangle = \{0\}$ for any $k \in \mathbb{N}$. Therefore, S is a local spectral set. As G is abelian, S is a spectral set. ■

4. Strongly spectral sets and the union problem. In this section we introduce a new class of sets which includes Ditkin sets, and show that it is closed under the operation of forming finite unions.

It is easy to see that the union of two disjoint (local) spectral sets, and similarly of two Ditkin sets, is a (local) spectral set. The question about the union of any two non-disjoint (local) spectral sets was raised in the paper [He1] of C. Herz (for abelian groups) and three years later again by H. Reiter in [Re]. The problem remains open.

DEFINITION 4.1. We say that a closed subset S of G is *strongly* (*local*) *spectral* if for every compact subset T of G and any $f \in k(S \cup T)$ ($f \in k(S \cup T) \cap C_c(G)$) any $\varepsilon > 0$ there exists an element g_ε in $j(S) \cap k(T)$ such that $\|f - g_\varepsilon\|_{A(G)} < \varepsilon$.

REMARK 4.2. Any Ditkin set is obviously strongly spectral, but we do not know whether the converse is true.

We note that if the group G is such that $u \in \overline{uA(G)}$ for each $u \in A(G)$ (for instance, G is amenable or, more generally, $A(G)$ has an (unbounded) approximate unit), then $S \subset G$ is a strongly spectral set if and only if, for any $f \in k(S)$ and any $\varepsilon > 0$, there exists a function $g_\varepsilon \in j(S)$ vanishing on $\mathrm{null}(f)$ and satisfying $\|f - g_\varepsilon\|_{A(G)} < \varepsilon$. In fact, in that case, if $T \subset G$ is compact and $f \in k(S \cup T)$ then, as $T \subset \mathrm{null}(f)$, the function g_ε vanishes on T. To see the converse, first note that the set of compactly supported functions in $A(G)$ is dense in $A(G)$. Assume that S is strongly spectral and let $f \in k(S)$. Given $\varepsilon > 0$, there exist a compactly supported $h \in A(G)$ and $\tilde g_\varepsilon \in j(S) \cap k(\mathrm{null}(f) \cap \mathrm{supp}(h))$ such that $\|f - fh\|_{A(G)} < \varepsilon$ and $\|f - \tilde g_\varepsilon\|_{A(G)} < \varepsilon/\|h\|_{A(G)}$. Setting $g_\varepsilon = \tilde g_\varepsilon h$, we find that g_ε vanishes on $\mathrm{null}(f) \subset \mathrm{null}(h) \cup (\mathrm{null}(f) \cap \mathrm{supp}(h))$ and satisfies $\|f - g_\varepsilon\|_{A(G)} \le \|f - fh\|_{A(G)} + \|fh - \tilde g_\varepsilon h\|_{A(G)} \le 2\varepsilon$. This establishes the statement. A similar result holds for the local version. If S is compact then the equivalence is clear without the additional assumption on G.

The next statement is a union result for strongly spectral sets, where we assume that either G has the property that $u \in \overline{uA(G)}$, or S_1 and S_2 are compact. The proof is similar to the one for Ditkin sets [Wa].

THEOREM 4.3. *Let S_1, S_2 be two strongly (local) spectral subsets of a locally compact group G. Then $S := S_1 \cup S_2$ is also strongly (local) spectral.*

If S_1 and S_2 are closed subsets such that $S_1 \cap S_2$ is strongly (local) spectral then $S_1 \cup S_2$ is strongly (local) spectral if and only if so are S_1 and S_2.

Proof. We show the statement for strongly spectral sets. Let $u \in k(S)$. Then $u \in k(S_1)$ and therefore for any $\varepsilon > 0$ there exists $u_1 \in j(S_1) \cap k(\mathrm{null}(u))$ such that $\|u - u_1\|_{A(G)} < \varepsilon$. As $\mathrm{null}(u) \supset S_2$, we have $u_1 \in k(S_2)$, and since S_2 is strongly spectral, there exists $u_2 \in j(S_2)$ such that $\|u_2 - u_1\|_{A(G)} < \varepsilon$ and $\mathrm{null}(u_2) \supset \mathrm{null}(u_1)$. We then see that u_2 vanishes on $W_1 \cup W_2$, where W_1 and W_2 are some neighborhoods of S_1 and S_2 respectively and hence $u_2 \in j(S_1 \cup S_2) = j(S)$.

Furthermore,

$$\|u_2 - u\|_{A(G)} \le \|u_2 - u_1\|_{A(G)} + \|u - u_1\|_{A(G)} < 2\varepsilon.$$

This shows that S is strongly spectral.

Assume now that S_1 and S_2 are closed subsets such that $S_1 \cap S_2$ is strongly spectral. Suppose that $S_1 \cup S_2$ is strongly spectral and let $u \in k(S_1)$ and $\varepsilon > 0$. Then $u \in k(S_1 \cap S_2)$ and there exists $g \in j(S_1 \cap S_2) \cap k(\mathrm{null}(u))$ such that $\|u - g\|_{A(G)} < \varepsilon$. Let $C = S_2 \cap \mathrm{supp}(g)$. It is disjoint from S_1 and hence there exists $w \in A(G)$ such that $w = 1$ on C and $w = 0$ on a neighborhood of S_1. Set $h = g - gw$. Then h vanishes on C and hence on $S_2 \subset (S_2 \cap \mathrm{supp}(g)) \cup \mathrm{null}(g)$. As g vanishes on $\mathrm{null}(u) \supset S_1$, we obtain $h \in k(S_1 \cup S_2)$. Therefore, there exists $g' \in j(S_1 \cup S_2) \cap k(\mathrm{null}(h))$ such that $\|h - g'\|_{A(G)} < \varepsilon$. We have $g' + wg \in j(S_1) \cap k(\mathrm{null}(u))$ and

$$\|u - g' - wg\|_{A(G)} \le \|u - g\|_{A(G)} + \|g - wg - g'\|_{A(G)} < 2\varepsilon,$$

that is, S_1 is strongly spectral. The proof for S_2 follows by symmetry. ■

Introducing the class of strongly spectral sets, we hoped that, using the technique developed in the previous section, we could prove that any set of local spectral synthesis is strongly spectral. Let T be a compact subset of an abelian, locally compact, σ-compact group G. Suppose that the function $u = d * c$ (in its canonical form) also vanishes on T, that is, $d \in ((S \cup T) \cdot c)^{\perp}$. Let

$$r_k := P_{(U_k S \cup T) \cdot c}(d), \qquad k \in \mathbb{N},$$

where $(U_k)_k$ is a sequence of decreasing neighborhoods of e such that the sequence $(P_{U_k S \cdot c}(d))_k$ converges to 0. Since the sequence $(U_k)_k$ of compact subsets is decreasing, the sequence of closed subspaces $((U_k S \cup T) \cdot c)_k$ is also decreasing. Consequently, the limit

$$r_\infty := \lim_{k\to\infty} r_k$$

exists in $L^2(G)$. The statement would follow if one could show that $r_\infty = 0$.

5. Appendix. In this section we collect some of the properties of $A(G)$ as a $\mathrm{VN}(G)$-module.

Following [Ta, III.2] we define the left and right actions of a von Neumann algebra $\mathcal{M}$ on its predual space $\mathcal{M}_*$ in the following way:

$$\langle S, T\cdot u\rangle := \langle ST, u\rangle, \quad \langle S, u\cdot T\rangle := \langle TS, u\rangle, \qquad S,T\in\mathcal{M},\ u\in\mathcal{M}_*.$$

Let us write $u = \gamma_{f,g}\in A(G)$ for the coefficient of the left regular representation defined by $f,g\in L^2(G)$, i.e. $\gamma_{f,g} = \langle\lambda(t)f, g\rangle$. From the definition of the right and left actions of $\mathrm{VN}(G)$ on $A(G)$ it follows that

$$T\cdot\gamma_{f,g} = \gamma_{T(f),g}, \quad \gamma_{f,g}\cdot T = \gamma_{f,T^*(g)}, \qquad f,g\in L^2(G), \tag{5.1}$$

since for $T\in\mathrm{VN}(G)$ and $u=\gamma_{f,g}\in A(G)$, $f,g\in L^2(G)$, we have

$$\langle S, T\cdot u\rangle = \langle ST, u\rangle = \langle ST(f), g\rangle = \langle S(T(f)), g\rangle = \langle S, \gamma_{T(f),g}\rangle,$$
$$\langle S, u\cdot T\rangle = \langle TS, u\rangle = \langle TS(f), g\rangle = \langle S(f), T^*(g)\rangle = \langle S, \gamma_{f,T^*(g)}\rangle.$$

Let also

$$\check{u}(t) = u(t^{-1}), \qquad u\in A(G),\ t\in G.$$

We define an antilinear map $T\mapsto\overline{T}$ of $\mathrm{VN}(G)$ by

$$\overline{T}(f) := \overline{T(\overline{f})}, \qquad f\in L^2(G),$$

and a linear involution $T\mapsto\check{T}$ on $\mathrm{VN}(G)$ by

$$\langle\check{T}, u\rangle := \langle T, \check{u}\rangle, \qquad T\in\mathrm{VN}(G),\ u\in A(G).$$

For $u=\gamma_{f,g}$ we have

$$\check{u}(t) = u(t^{-1}) = \langle\lambda(t^{-1})f, g\rangle = \overline{\langle\lambda(t)g, f\rangle} = \langle\lambda(t)\overline{g}, \overline{f}\rangle = \gamma_{\overline{g},\overline{f}}(t),$$

that is,

$$\check{\gamma}_{f,g} = \gamma_{\overline{g},\overline{f}}, \qquad f,g\in L^2(G). \tag{5.2}$$

Similarly,

$$\tilde{\gamma}_{f,g} = \gamma_{g,f}, \qquad f,g\in L^2(G). \tag{5.3}$$

Hence, for $u=\gamma_{f,g}\in A(G)$ we see that

$$\begin{aligned}\langle\check{T}, u\rangle &= \langle T, \check{u}\rangle = \langle T, \gamma_{\overline{g},\overline{f}}\rangle = \langle T(\overline{g}), \overline{f}\rangle \\ &= \langle\overline{g}, T^*(\overline{f})\rangle = \langle\overline{g}, \overline{(\overline{T}^*(f))}\rangle = \langle\overline{T}^*(f), g\rangle,\end{aligned}$$

whence

$$\check{T} = \overline{T}^*, \qquad T\in\mathrm{VN}(G). \tag{5.4}$$

We also have the following identities:

$$(u\cdot T)\tilde{}= T^*\cdot\tilde u,\quad (T\cdot u)\tilde{}=\tilde u\cdot T^*,\qquad u\in A(G),\ T\in \mathrm{VN}(G). \tag{5.5}$$

Indeed, for $u=\gamma_{f,g}\in A(G)$, by (5.1) and (5.3) we see that

$$(u\cdot T)\tilde{}=\tilde\gamma_{f,T^*(g)}=\gamma_{T^*(g),f}=T^*\cdot\gamma_{g,f}=T^*\cdot\tilde u.$$

The other equality is obtained in a similar way.

In [Ey] Eymard considered another action $(T,u)\mapsto T\circ u$ of $\mathrm{VN}(G)$ on $A(G)$ which is defined through the following formula:

$$\langle S, T\circ u\rangle := \langle \check T S, u\rangle,\qquad u\in A(G).$$

It follows from the relations above that

$$T\circ(\gamma_{f,g})=\gamma_{f,\overline{T}(g)}. \tag{5.6}$$

Indeed, for $u=\gamma_{f,g}\in A(G)$ we get

$$\langle S,T\circ u\rangle=\langle\check TS,u\rangle=\langle\check TS(f),g\rangle=\langle S(f),\overline{T}(g)\rangle=\langle S,\gamma_{f,\overline{T}(g)}\rangle.$$

Hence, by (5.1) and (5.4),

$$T\circ u=u\cdot\check T,\qquad u\in A(G),\ T\in\mathrm{VN}(G). \tag{5.7}$$

In particular, it follows from [Ey, Proposition 3.17] that for $u\in A(G)\cap L^2(G)$ and $T\in\mathrm{VN}(G)$, we have $T(u)\in A(G)\cap L^2(G)$, and

$$T(u)=T\circ u=u\cdot\check T. \tag{5.8}$$

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References

[CaH] J. de Cannière and U. Haagerup, *Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups*, Amer. J. Math. 107 (1985), 455–500.

[CoH] M. Cowling and U. Haagerup, *Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one*, Invent. Math. 96 (1989), 507–549.

[Di1] J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien* (*algèbres de von Neumann*), 2nd ed., Cahiers Sci. 25, Gauthier-Villars, Paris, 1969.

[Di2] J. Dixmier, *Les C^*-algèbres et leurs représentations*, Cahiers Sci. 29, Gauthier-Villars, Paris, 1969.

[Ey] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France 92 (1964), 181–236.

[GM] C. C. Graham and O. C. McGehee, *Essays in Commutative Harmonic Analysis*, Grundlehren Math. Wiss. 238, Springer, New York, 1979.

[He1] C. S. Herz, *The spectral theory of bounded functions*, Trans. Amer. Math. Soc. 94 (1960), 181–232.

[He2] C. Herz, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier (Grenoble) 23 (1973), no. 3, 91–123.

[KL] E. Kaniuth and A. T.-M. Lau, *Fourier and Fourier–Stieltjes Algebras on Locally Compact Groups*, Math. Surveys Monogr. 231, Amer. Math. Soc., Providence, RI, 2018.

[Re] H. Reiter, *Contributions to harmonic analysis: VI*, Ann. of Math. (2) 77 (1963), 552–562.

[Ta] M. Takesaki, *Theory of Operator Algebras I*, Springer, New York, 1979.

[TT] M. Takesaki and N. Tatsuuma, *Duality and subgroups, II*, J. Funct. Anal. 11 (1972), 184–190.

[Wa] C. R. Warner, *A class of spectral sets*, Proc. Amer. Math. Soc. 57 (1976), 99–102.

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