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Inventiones mathematicae (2025) 243:33–116
https://doi.org/10.1007/s00222-025-01373-6

Central values of zeta functions of non-Galois cubic fields

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Received: 9 February 2022 / Accepted: 3 September 2025 / Published online: 7 October 2025

Abstract
The Dedekind zeta functions of infinitely many non-Galois cubic fields have negative central values.

Contents

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1 Introduction

Let K be a number field of degree n, and denote its Dedekind zeta function by ζ_K. It was known to Riemann that $\zeta_{\mathbb{Q}}(\frac{1}{2}) = -1.46... < 0$. Hecke proved that $\zeta_K(s)$ has a meromorphic continuation with a simple pole at $s = 1$ and root number $+1$. The generalized Riemann Hypothesis claims that all the nontrivial zeros lie on the line $\Re s = 1/2$, which would imply that $\zeta_K(s)$ takes only negative real values in the open interval $s \in (1/2, 1)$ by the intermediate value theorem. This leads to the question of the *possible vanishing* of $\zeta_K(s)$ at the central point $s = 1/2$. The answer was given by Armitage [1] who showed that a certain number field K of degree 48 constructed by Serre [29, §9] satisfies $\zeta_K(\frac{1}{2}) = 0$, and also by Fröhlich [17] who constructed infinitely many quaternion fields K of degree 8 such that $\zeta_K(\frac{1}{2}) = 0$. In each of these examples, $\zeta_K(s)$ factors into Artin L-functions some of which have root number -1. Such an L-function is forced to vanish at $s = 1/2$ which in turn forces $\zeta_K(\frac{1}{2}) = 0$.

Conversely, which conditions on K can warrant that $\zeta_K(\frac{1}{2})$ is non-vanishing? A conjecture of Serre [19, Conjecture 8.24.1(2)] claims that if ρ is an irreducible representation of $\mathrm{Gal}(M/\mathbb{Q})$ for a finite Galois extension M of $\mathbb{Q}$, then the Artin L-function $L(s, \rho)$ vanishes at the central point $s = 1/2$ if and only if ρ is self-dual and the root number is -1. An *S_n-number field* K is a degree-n extension of $\mathbb{Q}$ such that the normal closure M of K has Galois group S_n over $\mathbb{Q}$. For such a field K,

$\zeta_K(s)$ factors as the product of $\zeta_{\mathbb{Q}}(s)$ and an Artin L-function $L(s, \rho_K)$ which is irreducible because ρ_K is the standard $(n-1)$-dimensional representation of S_n, and whose root number is $+1$ because the root numbers of both ζ_K and $\zeta_{\mathbb{Q}}$ are $+1$. This conjecture of Serre (in conjunction with GRH) would thus imply that $\zeta_K(\frac{1}{2}) < 0$ for every S_n-number field K.

In the case $n = 2$, a classical result of Jutila [23] establishes that $\zeta_K(\frac{1}{2})$ is nonvanishing for infinitely many quadratic number fields K. This was later improved in a landmark result of Soundararajan [32] to a positive proportion of such fields when ordered by discriminant, with this proportion rising to at least 87.5% in some families. In this article, we study the case $n = 3$. Our main result is as follows.

Theorem 1 *The Dedekind zeta functions of infinitely many S_3-fields have negative central values.*

We will in fact prove a stronger version of Theorem 1, in which we restrict ourselves to cubic fields satisfying any finite collection of local specifications. To state this result precisely, we introduce the following notation. Let $\Sigma = (\Sigma_v)$ be a finite collection of *cubic local specifications*. That is, for each place v of $\mathbb{Q}$, Σ_v is a nonempty set of étale cubic extensions of $\mathbb{Q}_v$, such that for large enough primes p, Σ_p contains all étale cubic extensions of $\mathbb{Q}_p$. We let $\mathcal{F}_\Sigma$ denote the set of cubic fields K such that $K \otimes \mathbb{Q}_v \in \Sigma_v$ for each v. Then we have the following result.

Theorem 2 *Let Σ be a finite collection of local specifications. Then there are infinitely many S_3-fields in $\mathcal{F}_\Sigma$ with negative central value.*

Define $\mathcal{F}_\Sigma(X)$ to be the set of fields $K \in \mathcal{F}_\Sigma$ with $|\Delta(K)| < X$. The foundational work of Davenport–Heilbronn [12] determined asymptotics $|\mathcal{F}_\Sigma(X)| \sim \alpha_\Sigma \cdot X$ with an explicit constant $\alpha_\Sigma > 0$.

We prove quantitative versions of our main theorems, where we give lower bounds for the *logarithmic density* $\delta_\Sigma(X)$ of the set of fields arising in Theorem 2 with bounded discriminant:

$$\delta_\Sigma(X) := \log\big|\{K \in \mathcal{F}_\Sigma(X),\ \zeta_K(\tfrac{1}{2}) < 0\}\big| / \log X. \tag{1}$$

Our next result implies that the number of cubic S_3-fields whose Dedekind zeta function is negative at the central point has logarithmic density ≥ 0.67:

Theorem 3 *For any finite collection Σ of local specifications,*

$$\liminf_{X\to\infty} \delta_\Sigma(X) \geq \frac{64}{95} = 0.67368\ldots; \qquad \limsup_{X\to\infty} \delta_\Sigma(X) \geq \frac{97}{128} = 0.75781\ldots$$

Note that Theorem 2 is an immediate consequence of Theorem 3 since we may add a specification Σ_p at an additional prime p that forces all cubic fields $K \in \mathcal{F}_\Sigma$ to be non-Galois. Alternatively, we may observe that the number of Galois cubic fields K, with discriminant less than X, is known to be asymptotic to $cX^{\frac{1}{2}}$ by work of Cohn

[10], where c is an explicit constant. Hence, Theorem 3 implies that most cubic fields $K \in \mathcal{F}_\Sigma(X)$ with $\zeta_K(\frac{1}{2}) < 0$ must be non-Galois.

The above numerical values are established from

$$\liminf_{X\to\infty} \delta_\Sigma(X) \geq \frac{2}{3-4\delta}; \qquad \limsup_{X\to\infty} \delta_\Sigma(X) \geq \frac{3}{4} + \delta,$$

where $\delta = \frac{1}{128}$ is the current record subconvexity exponent due to Blomer–Khan [7], which implies

$$|\zeta_K(\tfrac{1}{2})| \ll_\epsilon |\Delta(K)|^{\frac{1}{4}-\delta+\epsilon}.$$

The *convexity bound* $\delta = 0$ still yields the same kind of asymptotic results for $\delta_\Sigma(X)$, only with the weaker lower bound of $\frac{2}{3}$. The same applies to all other results in this paper so that a reader who wouldn't want to rely on the above recent subconvexity estimate could stay with $\delta = 0$. Other numerical values for $\delta > 0$ have been obtained by Duke–Friedlander–Iwaniec [16], Blomer–Harcos–Michel [8, Corollary 2], and Wu [36].

Conditional on the Lindelöf Hypothesis for all $\zeta_K(\frac{1}{2})$, $K \in \mathcal{F}_\Sigma$, we would have $\lim_{X\to\infty} \delta_\Sigma(X) = 1$. Even this conditional result would not imply that a positive proportion subset of $\mathcal{F}_\Sigma(X)$ is non-vanishing, it does only guarantee the existence of $\gg_\epsilon X^{1-\epsilon}$ cubic fields $K \in \mathcal{F}_\Sigma(X)$ with $\zeta_K(\frac{1}{2}) < 0$ for every $\epsilon > 0$.

A cubic number field is an S_3-field if and only if it is not Galois; hence we refer to non-Galois cubic fields as S_3-fields. Galois cubic fields are cyclic and (as is already noted above) the number of cyclic cubic fields K of discriminant less than X is about $X^{\frac{1}{2}}$. The zeta function of a cyclic cubic field K factors as a product of Dirichlet L-functions of conjugate cubic characters of conductor $|\Delta(K)|^{\frac{1}{2}}$ (see §3.1). It follows from a result of Baier–Young [2, Corollary 1.2] that for $\gg X^{\frac{3}{7}}$ cyclic cubic fields of discriminant less than X the Dedekind zeta function is negative at the central point. Recently, David–Florea–Lalin [14] have studied the analogous problem of cyclic cubic field extensions of the rational function field $\mathbb{F}_q(T)$, where they obtain a positive proportion of non-vanishing. Their results and methods would also yield a positive proportion of non-vanishing (conditional on GRH) for the family of cyclic cubic extensions over $\mathbb{Q}$. See also the papers of David–Güloğlu [13], Güloğlu–Yesilyurt [21], and Güloğlu [20] for analogous results for families of extensions of the Eisenstein field $\mathbb{Q}(\zeta_3)$.

1.1 The first moment of the central values of Artin L-functions of cubic fields

There is an extensive literature on the non-vanishing at special points of L-functions varying in families. The present situation of cubic fields is an important geometric family. Its central values are of GL_2-type and well-studied from an analytic perspective. At the same time, the geometry of the count of cubic number fields with bounded discriminant has a rich history.

Let K be a cubic field. The Dedekind zeta function of K factors as $\zeta_K(s) = \zeta_{\mathbb{Q}}(s)L(s, \rho_K)$, where $L(s, \rho_K)$ denotes the Artin L-function associated with the 2-dimensional Galois representation

$$\rho_K : \mathrm{Gal}(M/\mathbb{Q}) \hookrightarrow S_3 \hookrightarrow \mathrm{GL}_2(\mathbb{C}),$$

where M is the Galois closure of K. It is known from work of Hecke that $L(s, \rho_K)$ is an entire function. It will be more convenient for us to work with the central L-value $L(\frac{1}{2}, \rho_K)$ rather than $\zeta_K(\frac{1}{2})$, which is equivalent since they differ by the non-zero constant $\zeta_{\mathbb{Q}}(\frac{1}{2})$.

In order to prove Theorem 3, the standard approach is to estimate the *first moment* of $L(\frac{1}{2}, \rho_K)$ for $K \in \mathcal{F}_\Sigma$. Thus we ask the question: can one obtain an asymptotic for

$$\sum_{K \in \mathcal{F}_\Sigma(X)} L(\tfrac{1}{2}, \rho_K), \quad \text{as } X \to \infty?$$

This question is still open. Fortunately, we observe that we may weaken the question in the following three ways: First, we shall study the smooth version which is technically much more convenient. Second, we shall impose two local specifications Σ_p, $\Sigma_{p'}$ at additional primes $p \neq p'$. Third, and this is our most important point, we observe that it suffices that the remainder term can be expressed in terms of central values of cubic fields with lower discriminant. Indeed, we then have a dichotomy of either an asymptotic for the first moment or an unusually large remainder term, either of which implies the non-vanishing of many central values.

Theorem 4 *There exists an absolute constant $\mu > 0$ such that the following holds. Suppose that for some prime p, the specification is $\Sigma_p = \{\mathbb{Q}_{p^3}\}$ (i.e., the cubic fields in $\mathcal{F}_\Sigma$ are prescribed to be inert at p), and for another prime p', the specification is $\Sigma_{p'} = \{\mathbb{Q}_{p'} \times \mathbb{Q}_{p'^2}\}$ (i.e., the cubic fields in $\mathcal{F}_\Sigma$ are prescribed to be neither inert nor split at p'). Let $\Psi : \mathbb{R}_{>0} \to \mathbb{C}$ be a smooth compactly supported function and suppose that $\widetilde{\Psi}(1) = \int_0^\infty \Psi = 1$. Then, for every $0 < \nu \leq \mu$, $\epsilon > 0$, and $X \geq 1$,*

$$\begin{aligned}
&\sum_{K \in \mathcal{F}_\Sigma} L\big(\tfrac{1}{2}, \rho_K\big)\Psi\Big(\frac{|\Delta(K)|}{X}\Big) \\
&\quad = C_\Sigma \cdot X \cdot \big(\log X + \widetilde{\Psi}'(1)\big) + C'_\Sigma \cdot X \\
&\qquad + O_{\epsilon,\nu,\Sigma,\Psi}\Big(X^{1+\epsilon-\nu} + X^{\frac{1}{2}+\epsilon} \cdot \sum_{K \in \mathcal{F}_\Sigma\left(X^{\frac{3}{4}+\nu}\right)} \frac{\big|L\big(\frac{1}{2}, \rho_K\big)\big|}{|\Delta(K)|^{\frac{1}{2}}}\Big),
\end{aligned}$$

where $C_\Sigma > 0$ and $C'_\Sigma \in \mathbb{R}$ depend only on Σ.

It is easy to see that Theorem 4 implies that infinitely many fields $K \in \mathcal{F}_\Sigma$ have nonzero central values using an argument by contradiction. If there were finitely many non-vanishing L-values, then the left-hand side would be bounded, and the second term inside $O_{\epsilon,\nu,\Sigma,\Psi}(\cdot)$ of the right-hand side would be bounded by $X^{\frac{1}{2}+\epsilon}$. This is a

contradiction because the term $C_\Sigma X \log X$ would be larger than all the other terms. The fact that Theorem 4 also implies Theorem 3 is established in Sect. 10.

The main term of Theorem 4 is familiar in the study of moments of L-functions. In particular the nature of the constants C_Σ and C'_Σ is transparent, with C_Σ proportional to the Euler product (2). We denote the nth Dirichlet coefficient of $L(s, \rho_K)$ by $\lambda_K(n)$, which is a multiplicative function of n. For a prime power p^k, the coefficient $\lambda_K(p^k)$ depends only on the cubic étale algebra $K \otimes \mathbb{Q}_p$ over $\mathbb{Q}_p$, and is in fact determined by $\mathcal{O}_K \otimes \mathbb{F}_p$, where $\mathcal{O}_K$ denotes the ring of integers of K. Therefore, for a fixed positive integer n, the asymptotic average value of $\lambda_K(n)$ over $K \in \mathcal{F}_\Sigma$ is in fact an average over a finite set (see [27, §2.11] and [30, §2] for a general discussion of this phenomenon in the context of Sato–Tate equidistribution for geometric families). We denote this average by $t_\Sigma(n)$ and note that this is a multiplicative function of n.

We have $t_\Sigma(p) = O_\Sigma(\frac{1}{p})$ as the prime $p \to \infty$, which also is a general feature [30, §2] that implies that the number field family $\mathcal{F}_\Sigma$ is expected [27, Eq. (11)] to have average rank 0. Moreover, $t_\Sigma(p^2) = 1 + O_\Sigma(\frac{1}{p^2})$ for the present family $\mathcal{F}_\Sigma$ which implies that the following normalized Euler product converges:

$$\prod_p \Big[(1-p^{-1}) \sum_{k=0}^{\infty} \frac{t_\Sigma(p^k)}{p^{k/2}}\Big]. \tag{2}$$

This product is shown to be positive and to be proportional to C_Σ (see Sect. 8).

We shall discuss the remainder terms and our proof of Theorem 4 in §1.3. An explicit value of μ is a tenth of a thousandth. This small numerical value arises from the complications in bounding the remainder terms in all of the different ranges in our proof coupled with that the exponent of the secondary term $X^{\frac{5}{6}}$ of the asymptotic count of cubic fields is already by itself close to 1.

1.2 Low-lying zeros of the Dedekind zeta functions of cubic fields

Our equidistribution results in Sect. 6 on the asymptotic average value of $\lambda_K(n)$ over $K \in \mathcal{F}_\Sigma(X)$ with robust remainder terms as $n, X \to \infty$ have applications towards the statistics of low-lying zeros of the Dedekind zeta functions of cubic fields (the Katz–Sarnak heuristics). A conjecture in [27] predicts that for a homogeneous orthogonal family of L-functions, the low-lying zeros of the family should have *symplectic symmetry type*. Given a test function $\Phi : \mathbb{R} \to \mathbb{C}$, let $\mathcal{D}(\mathcal{F}_\Sigma(X), \Phi)$ denote the 1-level density (defined precisely in Sect. 7) of the family of Dedekind zeta functions of the fields in $\mathcal{F}_\Sigma$ with respect to Φ. Then the Katz–Sarnak heuristics predict the equality

$$\lim_{X\to\infty} \mathcal{D}(\mathcal{F}_\Sigma(X), \Phi) = \widehat{\Phi}(0) - \frac{1}{2}\int_{-1}^{1} \widehat{\Phi}(t)dt, \tag{3}$$

for all even functions Φ, whose Fourier transform $\widehat{\Phi}$ has support contained in $(-a, a)$ for a constant a to be determined. Yang [37] verifies (3) for even functions Φ whose Fourier transform has support contained in $(-\frac{1}{50}, \frac{1}{50})$. The constant $\frac{1}{50}$ has been subsequently improved to $\frac{4}{41}$ by work of Cho–Kim [9] and independently [30]. Here, we prove the following result:

Theorem 5 *Let Σ be a finite collection of local specifications and assume that $\Sigma_p = \{\mathbb{Q}_{p^3}\}$ and $\Sigma_{p'} = \{\mathbb{Q}_{p'} \times \mathbb{Q}_{p'^2}\}$ for two primes $p \neq p'$. Then (3) holds for even functions Φ whose Fourier transform has support contained in $(-\frac{2}{5}, \frac{2}{5})$.*

1.3 Overview of the proof of the main theorems

These proofs are carried out in several steps. First, we control the central value $L(\frac{1}{2}, \rho_K)$ using the approximate functional equation. This allows us to approximate $L(\frac{1}{2}, \rho_K)$ in terms of a smooth sum of the Dirichlet coefficients $\lambda_K(n)$, where the sum has length $O_\epsilon(|\Delta(K)|^{1/2+\epsilon})$. More precisely, we have

$$L(\tfrac{1}{2}, \rho_K) = \sum_{n=1}^{\infty} \frac{\lambda_K(n)}{n^{1/2}} V^{\pm}\Big(\frac{n}{\sqrt{|\Delta(K)|}}\Big), \tag{4}$$

where $V^{\pm}$ is a rapidly decaying smooth function depending only on the sign $\pm$ of $\Delta(K)$. Therefore, studying the average value of $L(\frac{1}{2}, \rho_K)$ as K varies over the family $\mathcal{F}_\Sigma(X)$ of cubic fields with discriminant bounded by X necessitates the study of smoothed sums of Dirichlet coefficients $\lambda_K(n)$:

$$\sum_{n \leq X^{1/2+\epsilon}} \frac{1}{n^{1/2}} \sum_{K \in \mathcal{F}_\Sigma} \lambda_K(n) \Psi\Big(\frac{|\Delta(K)|}{X}\Big), \tag{5}$$

where $\Psi : \mathbb{R}_{>0} \to \mathbb{C}$ is a smooth function with compact support. In particular, a basic input for the proof is the determination of the average value $t_\Sigma(n)$ of $\lambda_K(n)$ over $K \in \mathcal{F}_\Sigma(X)$. Moreover, it is necessary to obtain good error terms for this average with an explicit dependence on n.

1.3.1 Expanding the definition of $\lambda_K(n)$ to cubic rings R

In order to compute the average value of $\lambda_K(n)$ over $K \in \mathcal{F}_\Sigma$ with good error terms, it is necessary for us to expand this average to one over cubic orders R. This is because cubic rings can be parametrized by group orbits on a lattice and Poisson summation, applied through the theory of Shintani zeta functions following Taniguchi–Thorne [33] and [34], becomes available as an important tool.[1] It is therefore necessary for us to define a quantity $\lambda_R(n)$, for positive integers n and cubic rings R. There are different natural choices for the value of $\lambda_R(n)$. For example, it is possible to set the Dirichlet coefficients of R to be equal to the corresponding coefficients of $R \otimes \mathbb{Q}$. Another possible choice arises from work of Yun [38], in which Yun defines a natural zeta function $\zeta_R(s)$ associated to orders R in global fields. It is then possible to set the Dirichlet coefficients of R to equal the corresponding coefficients of $\zeta_R(s)/\zeta(s)$.

[1] This is in direct analogy to the quadratic case, in which Pólya–Vinagradov type estimates are used to estimate the sum of Legendre symbols $\left(\frac{n}{D}\right)$, as D varies over all discriminants and not merely the squarefree ones.

However, we require $\lambda_R(n)$ to satisfy the following three conditions:

(a) We require $\lambda_R(n) = \lambda_K(n)$ when R is the ring of integers of K.
(b) We require $\lambda_R(n)$ to be multiplicative in n.
(c) When p is prime, we require the value of $\lambda_R(p^k)$ to be defined modulo p, i.e., $\lambda_R(p^k)$ should be determined by $R \otimes \mathbb{F}_p$.

The above two candidate choices for $\lambda_R(n)$ satisfy the first two properties, but not the third. In fact, the above three conditions uniquely determine the value of $\lambda_R(p^k)$ for rings R such that $R \otimes \mathbb{Z}_p$ is Gorenstein, in the sense that $\mathrm{Hom}(R, \mathbb{Z}_p)$ is free.[2] More precisely, $\lambda_R(n)$ should be defined to be the nth Dirichlet coefficient of $D(s, R)$, where $D(s, R)$ is defined by an Euler product whose pth factor $D_p(s, R)$ is given by

$$D_p(s,R) := \begin{cases} (1-p^{-s})^{-2} & \text{if} \quad R \otimes \mathbb{F}_p = \mathbb{F}_p^3; \\ (1-p^{-2s})^{-1} & \text{if} \quad R \otimes \mathbb{F}_p = \mathbb{F}_p \oplus \mathbb{F}_{p^2}; \\ (1+p^{-s}+p^{-2s})^{-1} & \text{if} \quad R \otimes \mathbb{F}_p = \mathbb{F}_{p^3}; \\ (1-p^{-s})^{-1} & \text{if} \quad R \otimes \mathbb{F}_p = \mathbb{F}_p \oplus \mathbb{F}_p[t]/(t^2); \\ 1 & \text{else.} \end{cases} \tag{6}$$

It is clear from the definition that $\lambda_R(n)$ satisfies the three required properties.

1.3.2 Summing $\lambda_R(n)$ over cubic rings R with bounded discriminant

Next, we need to evaluate a smoothed sum of $\lambda_R(n)$, for R varying over cubic rings having bounded discriminant. Such a result follows immediately from the following three ingredients. First, the Delone–Faddeev parametrization of cubic rings in terms of $\mathrm{GL}_2(\mathbb{Z})$-orbits on $V(\mathbb{Z})$, the space of integral binary cubic forms. Second, results of Shintani [31] on the analytic properties of the Shintani zeta functions associated to $V(\mathbb{Z})$. Third, local Fourier transform computations of Mori [26] on $V(\mathbb{F}_p)$.

Let n be a positive integer, and write $n = mk$, where m is squarefree, k is powerful, and $(m, k) = 1$. Then we have the following result, stated for primes and prime powers as Theorem 5.10, which is a smoothed cubic analogue of the Pólya–Vinogradov inequality: There exist explicit constants $\alpha(n)$ and $\gamma(n)$ such that

$$\sum_{[R:\mathbb{Z}]=3} \lambda_R(n)\Psi\Big(\frac{|\Delta(R)|}{X}\Big) = \alpha(n)X + \gamma(n)X^{5/6} + O_\epsilon\big(n^\epsilon \cdot m \cdot \mathrm{rad}(k)^2\big), \tag{7}$$

where $\mathrm{rad}(k)$ denotes the radical of k, and the sum over rings is weighted by the inverse of the size of the stabilizer, $|\mathrm{Stab}(R)|^{-1}$.

[2] Non Gorenstein rings R over $\mathbb{Z}_p$ are those such that $R \otimes \mathbb{F}_p$ is of the form $\langle 1, x, y\rangle$ with $x^2 = y^2 = xy = 0$ (see [18]).

1.3.3 Sieving to maximal orders

We define the quantity

$$S(R) = \sum_n \frac{\lambda_R(n)}{n^{1/2}} V^{\pm}\Big(\frac{n}{\sqrt{|\Delta(R)|}}\Big).$$

We note that $S(R) = L(\frac{1}{2}, \rho_K)$ when R is the ring of integers of K. However, when R is not maximal, it is *not* necessarily true that $S(R)$ is equal to $D(\frac{1}{2}, R)$. In order to evaluate (5), we need to perform an inclusion-exclusion sieve. Thus, for all squarefree integers q, we need estimates on the sums

$$\sum_{R \in \mathcal{M}_q} S(R)\Psi\Big(\frac{|\Delta(R)|}{X}\Big), \tag{8}$$

where $\mathcal{M}_q$ denotes the space of cubic orders R that have index divisible by q in the ring of integers of $R \otimes \mathbb{Q}$. Estimating sums over $\mathcal{M}_q$ is tricky since the condition of nonmaximality at q is defined modulo q^2 and not modulo q. That is, maximality of R at a prime p cannot be detected from the local algebra $R \otimes \mathbb{F}_p$. To reduce our mod q^2 sum to a mod q sum, we use an idea originating in the work of Davenport–Heilbronn [12] and further developed as a precise switching trick in [5]. Namely, we replace the sum over $\mathcal{M}_q$ with a sum over the set of overorders of $\mathcal{M}_q$ of index-q.

For q in what we call the "small range", i.e., $q \leq X^{1/8-\epsilon}$, the switching trick in conjunction with (7) allows us to estimate each summand in (8) with a sufficiently small error term. Ideally, we would use a tail estimate for large q. This tail estimate requires bounding the value of $S(R)$ for nonmaximal rings R. The convexity bound yields the following estimate for rings $R \in \mathcal{M}_q$ with $\Delta(R) \asymp X$:

$$|S(R)| \ll_\epsilon \frac{X^{1/4+\epsilon}}{q^{1/2}}. \tag{9}$$

Neither the convexity bound nor the best known subconvexity bounds give sufficiently good estimates to cover all squarefree integers $q > X^{1/8-\epsilon}$. However, assuming the generalized Lindelöf Hypothesis (or indeed, a sufficiently strong subconvexity bound) is enough to determine the first moment for $L(\frac{1}{2}, \rho_K)$. Moreover, this method yields unconditional upper bounds on the average value of $L(\frac{1}{2}, \rho_K)$, a slightly stronger version of which is proven in Theorem 8.7:

Theorem 6 *Let Σ be a finite collection of local specifications and assume that $\Sigma_p = \{\mathbb{Q}_{p^3}\}$ and $\Sigma_{p'} = \{\mathbb{Q}_{p'} \times \mathbb{Q}_{p'^2}\}$ for two primes $p \neq p'$. Then for $X \geq 1$, we have*

$$\sum_{K \in \mathcal{F}_\Sigma} L\big(\tfrac{1}{2}, \rho_K\big)\Psi\Big(\frac{|\Delta(K)|}{X}\Big) \ll_{\Sigma,\Psi} X^{29/28}. \tag{10}$$

We note that this average bound is significantly stronger than the bound obtained by simply summing the best known pointwise upper bounds for $L(\frac{1}{2}, \rho_K)$.

1.3.4 The approximate functional equation for cubic rings

The first ingredient required for estimating $S(R)$, when R is a nonmaximal cubic order with index $> X^{1/8-\epsilon}$, is a generalization of the approximate functional equation (4) to the setting of cubic orders. This modification is proved in Proposition 4.11, and expresses $S(R) - D(\frac{1}{2}, R)$ as a sum of arithmetic quantities associated to R. The advantage of expressing $S(R)$ in this way is that this latter sum is much shorter than the original sum defining $S(R)$: of length $\ll_\epsilon X^{1/2+\epsilon}/q$ rather than $\ll_\epsilon X^{1/2+\epsilon}$. However, this shortening comes at a cost. The summands of this new sum involve Dirichlet coefficients from both $D(s, R)$ and $L(s, \rho_{R\otimes\mathbb{Q}})$.

In order to control the coefficients of $L(s, \rho_{R\otimes\mathbb{Q}})$, it is necessary to isolate the exact index of R in the ring of integers of $R \otimes \mathbb{Q}$. Merely knowning that q divides the index is not enough. To precisely control the index, a secondary sieve is necessary. Carrying out this secondary sieve yields the following estimate for $q > X^{1/8-\epsilon}$:

$$\sum_{R\in\mathcal{M}_q} S(R)\Psi\Big(\frac{|\Delta(R)|}{X}\Big) \approx \sum_{R\in\mathcal{M}_q} D(\tfrac{1}{2}, R)\Psi\Big(\frac{|\Delta(R)|}{X}\Big). \tag{11}$$

This estimate is proved in Sect. 9, and is the crucial technical ingredient in the proof of Theorem 4. Equation (11) allows us to exploit the advantages of using $S(R)$ and $D(\frac{1}{2}, R)$ in the original inclusion exclusion sieve. Namely, for small q, the sum of $S(R)$ over $R \in \mathcal{M}_q$, can be well estimated with Equation (7) since $S(R)$ is simply a sum of the coefficients $\lambda_R(n)$. However for large q, it is advantageous to instead sum $D(\frac{1}{2}, R)$ over $R \in \mathcal{M}_q$. This is because the value of $D(\frac{1}{2}, R)$ behaves predictably as R varies over suborders of a fixed cubic field.

1.3.5 Summing $D(\frac{1}{2}, R)$ over $R \in \mathcal{M}_q$ and over large q

We are left to estimate the sum

$$\sum_{q>X^{1/8-\epsilon}} \mu(q) \sum_{R\in\mathcal{M}_q} D(\tfrac{1}{2}, R)\Psi\Big(\frac{|\Delta(R)|}{X}\Big). \tag{12}$$

Expressing $D(\frac{1}{2}, R)$ in terms of $L(\frac{1}{2}, \rho_{R\otimes\mathbb{Q}})$ allows us to repackage (12) into sums of the following form:

$$\begin{aligned}&\sum_{\substack{K\in\mathcal{F}_\Sigma\\ |\Delta(K)|\asymp Y}} \sum_{\substack{R\subset\mathcal{O}_K\\ \mathrm{ind}(R)\asymp\sqrt{X/Y}}} D(\tfrac{1}{2}, R)\\ &\ll_{\epsilon,\Sigma} X^\epsilon \sum_{\substack{K\in\mathcal{F}_\Sigma\\ |\Delta(K)|\asymp Y}} \#\big\{R\subset\mathcal{O}_K : \mathrm{ind}(R)\asymp\sqrt{X/Y}\big\}\cdot|L(\tfrac{1}{2}, \rho_K)|.\end{aligned} \tag{13}$$

Let K be a fixed cubic field. A result of Datskovsky–Wright [11] gives asymptotics for the number of suborders of K having bounded index. This yields Theorem 4.

Our next idea is to assume the nonnegativity of $L(\frac{1}{2}, \rho_K)$. Since the result of Datskovsky–Wright is very precise, it turns out that we can input the unconditional

upper bound on the sums of $L(\frac{1}{2}, \rho_K)$ in (10), to obtain an improved upper bound on the right-hand side of (13). This improved upper bound is enough to obtain asymptotics for the first moment of $L(\frac{1}{2}, \rho_K)$, conditional on its nonnegativity.

Finally, we obtain Theorem 3 by making a version of the following simple idea precise: If $L(\frac{1}{2}, \rho_K)$ does indeed vanish for most fields K, then the right-hand side of (13) is forced to be small, which in turn implies an upper bound on the left-hand side of (13), which in turn allows for the computation of the first moment of $L(\frac{1}{2}, \rho_K)$, which in turn implies non-vanishing for many fields K. This leads to a contradiction, and it follows that $L(\frac{1}{2}, \rho_K)$ does not vanish for many fields K.

Finally, we observe that the same method of proof applies to the values $L(\frac{1}{2} + it, \rho_K)$ for a fixed $t \in \mathbb{R}$ and yield variants of Theorems 1, 2, 3, 4, 6 with suitable modifications.

1.4 Organization of the paper

This paper is organized as follows. In Sect. 2, we collect preliminary results on the space of cubic rings and fields. In particular, we recall the Delone–Faddeev parametrization of cubic rings in terms of $\mathrm{GL}_2(\mathbb{Z})$-orbits on integral binary cubic forms. We also discuss Fourier analysis on the space of binary cubic forms over $\mathbb{F}_p$ and $\mathbb{Z}/n\mathbb{Z}$. In Sect. 3, we introduce the Artin character on cubic fields K that arise as Dirichlet coefficients of $L(s, \rho_K) = \zeta_K(s)/\zeta(s)$. We then define an extension to the space of cubic rings (and thus also the space of binary cubic forms). Next, in Sect. 4, we recall the analytic properties of $L(s, \rho_K)$, for a cubic field K. In particular, we recall the approximate functional equation. We then discuss an unbalanced form of the approximate functional equation for orders within cubic fields.

In Sect. 5, we recall Shintani's theory of the zeta functions associated to the space of binary cubic forms. As a well-known consequence of this theory, we derive estimates for the sums of congruence functions (i.e., functions ϕ on the space of cubic rings R such that ϕ is determined by $R \otimes \mathbb{Z}/n\mathbb{Z}$ for some integer n) over the space of cubic rings with bounded discriminant. Then in Sect. 6, we apply a squarefree sieve to determine the sum of these congruence functions over the space of cubic fields.

In Sect. 7, we use the results from Sect. 6 to prove Theorem 5 on the statistics of the low-lying zeros of the zeta functions of cubic fields. Next, in Sect. 8, we start our analysis of the average central values of $L(s, \rho_K)$, where K ranges over cubic fields. In particular we prove the upper bound Theorem 8.7, obtaining an improved estimate on the average size of $L(\frac{1}{2}, \rho_K)$ compared to the pointwise bound.

In Sect. 9, we complete the most difficult part of the proof, in which we show that for each somewhat large q, the values of $S(R)$ and $D(\frac{1}{2}, R)$ are close to each other, on average over $R \in \mathcal{M}_q$. We use this result in Sect. 10 to first prove Theorem 4, and using this in addition, to prove our main result Theorem 3.

1.5 Notations and conventions

- A positive integer k is said to be *powerful* if $v_p(k) \geq 2$ for every prime $p|k$.
- The *radical*, also called the square-free kernel, of a positive integer k is the product of its prime factors, $\mathrm{rad}(k) := \prod_{p|k} p$.

- We shall always use Σ to refer to the finite set of local conditions imposed on the family of cubic fields.
- We shall always use Ψ to denote a compactly supported Schwartz function that will control the discriminants of binary cubic forms, cubic rings, or cubic fields.

2 Preliminaries on cubic rings and fields

Let $V = \mathrm{Sym}^3(2)$ denote the space of binary cubic forms. The group GL_2 acts on V via the following twisted action:

$$\gamma \cdot f(x, y) := \det(\gamma)^{-1} f((x, y) \cdot \gamma).$$

It is well-known that the representation (GL_2, V) is *prehomogeneous* and that the ring of relative invariants for the action of GL_2 on V is freely generated by the *discriminant* which we denote by Δ. We have that Δ is homogeneous of degree 4 and $\Delta(\gamma \cdot f) = (\det \gamma)^2 \Delta(f)$. In this section, we describe the parametrization of cubic rings and fields in terms of $\mathrm{GL}_2(\mathbb{Z})$-orbits on $V(\mathbb{Z})$. We also discuss Fourier analysis on the space $V(\mathbb{Z}/n\mathbb{Z})$, and in particular describe the Fourier transforms of all $\mathrm{GL}_2(\mathbb{F}_p)$-invariant functions on $V(\mathbb{F}_p)$.

2.1 Binary cubic forms and the parametrization of cubic rings

Levi [25] and Delone–Faddeev [15], further refined by Gan–Gross–Savin [18], prove that there is a bijection between the set of $\mathrm{GL}_2(\mathbb{Z})$-equivalence classes of integral binary cubic forms and isomorphism classes of cubic rings over $\mathbb{Z}$:

Proposition 2.1 *There is a bijection between the set of isomorphism classes of cubic rings and the set of* $\mathrm{GL}_2(\mathbb{Z})$*-orbits on* $V(\mathbb{Z})$*, given as follows. A cubic ring* R *is associated to the* $\mathrm{GL}_2(\mathbb{Z})$*-equivalence class of the integral binary cubic form corresponding to the map*

$$\begin{aligned} R/\mathbb{Z} &\to \wedge^2(R/\mathbb{Z}) \\ \theta &\mapsto \theta \wedge \theta^2. \end{aligned}$$

Throughout this paper, for an integral binary cubic form $f \in V(\mathbb{Z})$, we denote the cubic ring corresponding to f by R_f, the cubic algebra $R_f \otimes \mathbb{Q}$ by K_f, and the ring of integers of K_f by $\mathcal{O}_{K_f}$. We have

$$\Delta(R_f) = \Delta(f) = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd,$$

for $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$, and where we denote by the same letter Δ the discriminants of rings and algebras. Since $\Delta(K_f) = \Delta(\mathcal{O}_{K_f})$ by definition, we have the equality

$$\Delta(f) = \Delta(K_f)[\mathcal{O}_{K_f} : R_f]^2 = \Delta(K_f)\mathrm{ind}(f)^2, \tag{14}$$

where we define the *index* of f, or $\mathrm{ind}(f)$, to be $[\mathcal{O}_{K_f} : R_f]$.

In particular, we see that $|\Delta(K_f)| \leq |\Delta(f)|$, and that the signs of $\Delta(f)$ and $\Delta(K_f)$ coincide. If $\Delta(f) \neq 0$, then the algebra K_f is étale. If $f \in V(\mathbb{Z})^{\mathrm{irr}}$ is irreducible, then K_f is a field. Furthermore, $\Delta(f) > 0$ when K_f is totally real, and $\Delta(f) < 0$ when K_f is complex.

We say that a ring R has *rank* n if it is free of rank n as a $\mathbb{Z}$-module. We say that a rank n ring R is *maximal* if it is not a proper subring of any other ring of rank n. For a prime p, we say that a rank n ring R is *maximal at* p if $R \otimes \mathbb{Z}_p$ is maximal in the sense that it is not a proper subring of any other ring that is free of rank n as a $\mathbb{Z}_p$-module. We have that R is maximal if and only if it is maximal at p for every prime p.

We say that an integral binary cubic form f is *maximal* (resp. *maximal at* p) if the corresponding cubic ring R_f is maximal (resp. maximal at p). We have the following result [5, §3] characterizing binary cubic forms that are maximal at p.

Proposition 2.2 *An integral binary cubic form $f \in V(\mathbb{Z})$ is maximal at a prime p if and only if both of the following two properties hold*:

(i) *f is not a multiple of p, and*
(ii) *f is not $\mathrm{GL}_2(\mathbb{Z})$-equivalent to a form $ax^3 + bx^2y + cxy^2 + dy^3$, with $p^2 \mid a$ and $p \mid b$.*

We will also need the following result, proved in [5, Props.15-16], that determines the number of index-p subrings and index-p overrings of a cubic ring.

Proposition 2.3 *For an integral binary cubic form $f \in V(\mathbb{Z})$, the number of cubic rings in K_f containing R_f with index p is equal to the number of double zeros $\alpha \in \mathbb{P}^1(\mathbb{F}_p)$ of f modulo p such that $p^2 | f(\alpha')$ for all $\alpha' \in \mathbb{P}^1(\mathbb{Z})$ with $\alpha' \equiv \alpha \bmod p$.*

For an integral binary cubic form $g \in V(\mathbb{Z})$, there is a bijection between index-p subrings of R_g and zeros in $\mathbb{P}^1(\mathbb{F}_p)$ of g modulo p, whose number we denote by $\omega_p(g)$.

Example 2.4 Consider a form $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 \in V(\mathbb{Z})$, with $p^2 \mid a$ and $p \mid b$ which is nonmaximal by Proposition 2.2.(ii). Then $\alpha = [1 : 0] \in \mathbb{P}^1(\mathbb{F}_p)$ is a double root of f modulo p. The form $\left(\begin{smallmatrix} \frac{1}{p} & \\ & 1 \end{smallmatrix}\right) \cdot f(x, y) = (a/p^2)x^3 + (b/p)x^2y + cxy^2 + pdy^3$ corresponds to an index-p overring of R_f. This is consistent with Proposition 2.3 which implies that the number of cubic rings in K_f containing R_f with index p is at least one.

2.2 Binary cubic forms over $\mathbb{F}_p$ and $\mathbb{Z}/n\mathbb{Z}$

Let $V^* = \mathrm{Sym}_3(2)$ denote the *dual* of V, and denote by $[,]$ the duality pairing. The GL_2-action on V^* is defined by the rule that $[,]$ is relatively invariant:

$$[\gamma \cdot f, \gamma \cdot f_*] = \det(\gamma)[f, f_*], \quad \forall \gamma \in \mathrm{GL}_2,\ f \in V,\ f_* \in V^*. \tag{15}$$

The scalar matrices in $Z(\mathrm{GL}_2)$ act by scalar multiplication on both V and V^*.

Let $a_* := [y^3, f_*]$, $b_* := -[xy^2, f_*]$, $c_* := [x^2y, f_*]$, $d_* := -[x^3, f_*]$, and

$$\Delta_*(f_*) := 3b_*^2c_*^2 + 6a_*b_*c_*d_* - 4a_*c_*^3 - 4b_*^3d_* - a_*^2d_*^2.$$

Both Δ and Δ_* are homogeneous of degree 4 and satisfy $\Delta(\gamma \cdot f) = (\det\gamma)^2\Delta(f)$ and $\Delta_*(\gamma \cdot f_*) = (\det\gamma)^2\Delta_*(f_*)$.

Following [31, §3] and [4, Table 1], the lattice $V^*(\mathbb{Z})$ is isomorphic to the sublattice

$$V^*(\mathbb{Z}) \simeq \{a_*x^3 + 3b_*x^2y + 3c_*xy^2 + d_*y^3 : a_*, b_*, c_*, d_* \in \mathbb{Z}\} \subset V(\mathbb{Z}), \quad (16)$$

with compatible $\mathrm{GL}_2(\mathbb{Z})$-action. The restriction of Δ to $V^*(\mathbb{Z})$ coincides with $27\Delta_*$ as a direct calculation shows. We also see that the pairing $[,] : V(\mathbb{Z}) \times V^*(\mathbb{Z}) \to \mathbb{Z}$ coincides with the restriction of the antisymmetric bilinear form

$$\begin{aligned} V(\mathbb{Z}) \times V(\mathbb{Z}) &\to \tfrac{1}{3}\mathbb{Z} \\ (f_1, f_2) &\mapsto d_1a_2 - \tfrac{c_1b_2}{3} + \tfrac{b_1c_2}{3} - a_1d_2. \end{aligned}$$

For an integer $n \geq 1$, the $\mathbb{Z}/n\mathbb{Z}$ points of V, which we denote by $V(\mathbb{Z}/n\mathbb{Z})$, form a finite abelian group which can be identified with the quotient $V(\mathbb{Z})/nV(\mathbb{Z})$. The same holds for $V^*(\mathbb{Z}/n\mathbb{Z}) \simeq V^*(\mathbb{Z})/nV^*(\mathbb{Z})$. We obtain a perfect pairing $[,] : V(\mathbb{Z}/n\mathbb{Z}) \times V^*(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$.

The finite abelian group $V^*(\mathbb{Z}/n\mathbb{Z})$ is in natural bijection with the *group of characters* $V(\mathbb{Z}/n\mathbb{Z}) \to S^1$, where S^1 denotes the unit circle in $\mathbb{C}^\times$. Indeed, given $f_* \in V^*(\mathbb{Z}/n\mathbb{Z})$, we associate the character

$$\begin{aligned} \chi_{f_*} : V(\mathbb{Z}/n\mathbb{Z}) &\to S^1 \\ f &\mapsto e\Big(\tfrac{[f, f_*]}{n}\Big), \end{aligned}$$

where $e(\alpha) := e^{2\pi i\alpha}$.

Given a function $\phi : V(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$, we have the notion of its *Fourier transform* $\widehat{\phi}$ given by

$$\begin{aligned} \widehat{\phi} : V^*(\mathbb{Z}/n\mathbb{Z}) &\to \mathbb{C} \\ \widehat{\phi}(f_*) &:= \frac{1}{n^4} \sum_{f \in V(\mathbb{Z}/n\mathbb{Z})} e\Big(\frac{[f, f_*]}{n}\Big)\phi(f). \end{aligned}$$

In this paper, we will be concerned with the Fourier transforms of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant functions. Regarding this, we have the following result which is probably known although we couldn't find the statement in the literature.

Lemma 2.5 *The Fourier transform $\widehat{\phi}$ of a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant function ϕ is $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant.*

Proof Let $\gamma \in \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$, $f_* \in V^*(\mathbb{Z}/n\mathbb{Z})$ and the function ϕ be given. We have

$$\begin{aligned}\widehat{\phi}(\gamma \cdot f_*) &= \frac{1}{n^4} \sum_{f \in V(\mathbb{Z}/n\mathbb{Z})} e\Big(\frac{[f, \gamma \cdot f_*]}{n}\Big)\phi(f) \\ &= \frac{1}{n^4} \sum_{f \in V(\mathbb{Z}/n\mathbb{Z})} e\Big(\frac{\det(\gamma)[\gamma^{-1} \cdot f, f_*]}{n}\Big)\phi(f) \\ &= \frac{1}{n^4} \sum_{f \in V(\mathbb{Z}/n\mathbb{Z})} e\Big(\frac{\det(\gamma)[f, f_*]}{n}\Big)\phi(f),\end{aligned} \tag{17}$$

where the first equality is by definition, the second equality follows from (15), and the third equality follows from the $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariance of ϕ and the bijective change of variable f by $\gamma \cdot f$. To finish the proof of the lemma, we absorb the $\det(\gamma)$ factor into the sum over f since $\phi(uf) = \phi(f)$ for every $u \in (\mathbb{Z}/n\mathbb{Z})^\times$ and $f \in V(\mathbb{Z}/n\mathbb{Z})$ because $Z(\mathrm{GL}_2)$ acts by scalar multiplication on V. □

2.3 Fourier transforms of GL_2-orbits

We now consider a prime number $p \neq 3$. The *orbits* for the action of $\mathrm{GL}_2(\mathbb{F}_p)$ on $V(\mathbb{F}_p)$ and $V^*(\mathbb{F}_p)$ are characterized as follows [33, §5]. There are six $\mathrm{GL}_2(\mathbb{F}_p)$-orbits on $V(\mathbb{F}_p)$ depending on how a binary cubic form factors over $\mathbb{F}_p$. Using (16), we may identify $V^*(\mathbb{F}_p) = V^*(\mathbb{Z}) \otimes \mathbb{F}_p$ with $V(\mathbb{F}_p)$. There are thus also six $\mathrm{GL}_2(\mathbb{F}_p)$-orbits on $V^*(\mathbb{F}_p)$. We denote the orbits on $V(\mathbb{F}_p)$ by

$$\mathcal{O}_{(111)}, \mathcal{O}_{(12)}, \mathcal{O}_{(3)}, \mathcal{O}_{(1^21)}, \mathcal{O}_{(1^3)}, \mathcal{O}_{(0)}, \tag{18}$$

and the orbits on $V^*(\mathbb{F}_p)$ by

$$\mathcal{O}^*_{(111)}, \mathcal{O}^*_{(12)}, \mathcal{O}^*_{(3)}, \mathcal{O}^*_{(1^21)}, \mathcal{O}^*_{(1^3)}, \mathcal{O}^*_{(0)}, \tag{19}$$

respectively, where $\mathcal{O}_{(111)}$, $\mathcal{O}^*_{(111)}$ denote the sets of forms having three distinct rational roots in $\mathbb{P}^1(\mathbb{F}_p)$, the sets $\mathcal{O}_{(12)}$, $\mathcal{O}^*_{(12)}$ consist of forms having one root in $\mathbb{P}^1(\mathbb{F}_p)$ and one pair of conjugate roots defined over the quadratic extension of $\mathbb{F}_p$, the sets $\mathcal{O}_{(3)}$, $\mathcal{O}^*_{(3)}$ consist of forms irreducible over $\mathbb{F}_p$, the sets $\mathcal{O}_{(1^21)}$, $\mathcal{O}^*_{(1^21)}$ (resp. $\mathcal{O}_{(1^3)}$, $\mathcal{O}^*_{(1^3)}$) consist of forms having a root in $\mathbb{P}^1(\mathbb{F}_p)$ of multiplicity 2 (resp. 3), and $\mathcal{O}_{(0)}$, $\mathcal{O}^*_{(0)}$ is the singleton set containing the zero form. Given a subset S of $V(\mathbb{F}_p)$ or $V^*(\mathbb{F}_p)$, let C_S denote its characteristic function. Every $\mathrm{GL}_2(\mathbb{F}_p)$-invariant function on $V(\mathbb{F}_p)$ (resp. $V^*(\mathbb{F}_p)$) is a linear combination of the six functions

$$\begin{aligned}&C_{\mathcal{O}_{(0)}}, C_{\mathcal{O}_{(1^3)}}, C_{\mathcal{O}_{(1^21)}}, C_{\mathcal{O}_{(111)}}, C_{\mathcal{O}_{(12)}}, C_{\mathcal{O}_{(3)}} \\ &\quad(\text{resp. } C_{\mathcal{O}^*_{(0)}}, C_{\mathcal{O}^*_{(1^3)}}, C_{\mathcal{O}^*_{(1^21)}}, C_{\mathcal{O}^*_{(111)}}, C_{\mathcal{O}^*_{(12)}}, C_{\mathcal{O}^*_{(3)}}).\end{aligned}$$

Therefore, the Fourier transforms of the first six of the above functions determine the Fourier transforms of every $\mathrm{GL}_2(\mathbb{F}_p)$-invariant function on $V(\mathbb{F}_p)$.

Proposition 2.6 (Mori [26]) *Let $p \neq 3$ be a prime number, and $M = (m_{ij})$ be the following 6×6 matrix*

$$M := \frac{1}{p^4}\begin{bmatrix} 1 & (p+1)(p-1) & p(p+1)(p-1) & p(p+1)(p-1)^2/6 & p(p+1)(p-1)^2/2 & p(p+1)(p-1)^2/3 \\ 1 & -1 & p(p-1) & p(p-1)(2p-1)/6 & -p(p-1)/2 & -p(p+1)(p-1)/3 \\ 1 & p-1 & p(p-2) & -p(p-1)/2 & -p(p-1)/2 & 0 \\ 1 & 2p-1 & -3p & p(\pm p+5)/6 & -p(\pm p-1)/2 & p(\pm p-1)/3 \\ 1 & -1 & -p & -p(\pm p-1)/6 & p(\pm p+1)/2 & -p(\pm p-1)/3 \\ 1 & -p-1 & 0 & p(\pm p-1)/6 & -p(\pm p-1)/2 & p(\pm p+2)/3 \end{bmatrix},$$

where the signs $\pm$ appearing in the bottom-right 3×3 corner are according as $p \equiv \pm 1 \pmod 3$. Then

$$\widehat{C_j} = \sum_{i=1}^{6} m_{ij} C_i^*, \quad 1 \leq j \leq 6,$$

where we have set

$$\begin{aligned} (C_1, C_2, C_3, C_4, C_5, C_6) &:= (C_{\mathcal{O}_{(0)}}, C_{\mathcal{O}_{(1^3)}}, C_{\mathcal{O}_{(1^2 1)}}, C_{\mathcal{O}_{(111)}}, C_{\mathcal{O}_{(12)}}, C_{\mathcal{O}_{(3)}}); \\ (C_1^*, C_2^*, C_3^*, C_4^*, C_5^*, C_6^*) &:= (C_{\mathcal{O}^*_{(0)}}, C_{\mathcal{O}^*_{(1^3)}}, C_{\mathcal{O}^*_{(1^2 1)}}, C_{\mathcal{O}^*_{(111)}}, C_{\mathcal{O}^*_{(12)}}, C_{\mathcal{O}^*_{(3)}}). \end{aligned}$$

Proof The result was announced in [26], and a proof appears in the work of Taniguchi–Thorne [35, Thm.11] and [33, Rem.6.8]. □

Remarks (i) For $j = 1$, that is for the first column of M, Proposition 2.6 says that the Fourier transform of $C_{\mathcal{O}_{(0)}}$, which is the Dirac function of the origin, is equal to the constant function $1/p^4$ as should be.

(ii) For $i = 1$, the first row of M in Proposition 2.6 provides the respective sizes of each of the 6 conjugacy classes, because

$$\sum_{f \in V(\mathbb{F}_p)} C_j(f) = p^4 \widehat{C_j}(0) = p^4 m_{1j}.$$

They add up to $m_{11} + m_{12} + \cdots + m_{16} = 1$ as should be.

(iii) For every j, k, we have $\sum_{f \in V(\mathbb{F}_p)} C_j(f) C_k(f) = p^4 \delta_{jk} m_{1j}$, because the characteristic functions are pairwise orthogonal since the orbits are pairwise disjoint. This implies, by the Plancherel formula, $\sum_{f_* \in V^*(\mathbb{F}_p)} \widehat{C_j}(f_*) \widehat{C_k}(f_*) = \delta_{jk} m_{1j}$. Hence, Proposition 2.6 implies

$$p^4 \sum_{i=1}^{6} m_{ij} m_{ik} m_{1i} = \delta_{jk} m_{1j}, \quad 1 \leq j, k \leq 6, \tag{20}$$

which indeed holds true as a direct verification shows. Because of the symmetry between j, k, verifying (20) entails to verifying 21 equalities.

Proposition 2.6 has the following important consequence.

Corollary 2.7 *Let $p \neq 3$ be a prime number, and let $\phi : V(\mathbb{F}_p) \to \mathbb{C}$ be a $\mathrm{GL}_2(\mathbb{F}_p)$-invariant function such that $|\phi(f)| \leq 1$ for every $f \in V(\mathbb{F}_p)$. Then we have*

$$\widehat{\phi}(f_*) \ll \begin{cases} p^{-2} & \text{if} \quad f_* \in \mathcal{O}^*_{(111)} \cup \mathcal{O}^*_{(12)} \cup \mathcal{O}^*_{(3)} \cup \mathcal{O}^*_{(1^21)}; \\ p^{-1} & \text{if} \quad f_* \in \mathcal{O}^*_{(1^3)}; \\ 1 & \text{if} \quad f_* \in \mathcal{O}^*_{(0)}. \end{cases}$$

The absolute constant in $\ll$ can be taken to be 4.

Proof The rows of M are bounded by $m_{1\bullet} = O(1)$, $m_{2\bullet} = O(p^{-1})$ and $m_{i\bullet} = O(p^{-2})$ for $3 \leq i \leq 6$, or equivalently $M = \left[O(1), O(p^{-1}), O(p^{-2}), O(p^{-2}), O(p^{-2}), O(p^{-2})\right]^T$. For example, we can make the absolute constant explicit as follows: $\sum_{j=1}^6 m_{1j} = 1$, $\sum_{j=1}^6 |m_{2j}| \leq 1/p$, $\sum_{j=1}^6 |m_{3j}| \leq 2/p^2$, $\sum_{j=1}^6 |m_{4j}| \leq 4/p^2$, $\sum_{j=1}^6 |m_{5j}| \leq 2/p^2$, $\sum_{j=1}^6 |m_{6j}| \leq 2/p^2$.

By assumption, $\phi = \sum_{j=1}^{6} a_j C_j$ with $|a_j| \leq 1$. Proposition 2.6 implies that

$$|\widehat{\phi}(f_*)| \leq \sum_{i=1}^{6} C_i^*(f_*) \sum_{j=1}^{6} |m_{ij}|.$$

We deduce

$$|\widehat{\phi}(f_*)| \ll C_1^*(f_*) + p^{-1} C_2^*(f_*) + p^{-2}\left(C_3^*(f_*) + C_4^*(f_*) + C_5^*(f_*) + C_6^*(f_*)\right),$$

from which the corollary follows. □

3 The Artin character of cubic fields and rings

Let K be a cubic field extension of $\mathbb{Q}$, with normal closure M. The Dedekind zeta function $\zeta_K(s)$ of K factors as

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s) L(s, \rho_K),$$

where $\zeta_{\mathbb{Q}}(s)$ denotes the Riemann zeta function and $L(s, \rho_K)$ is an Artin L-function associated to the two-dimensional representation ρ_K of $\mathrm{Gal}(M/\mathbb{Q})$,

$$\rho_K : \mathrm{Gal}(M/\mathbb{Q}) \hookrightarrow S_3 \to \mathrm{GL}_2(\mathbb{C}).$$

In this section, we first begin by collecting some well-known properties of $L(s, \rho_K)$. We denote the Dirichlet coefficients of $L(s, \rho_K)$ by $\lambda_K(n)$. Then we extend the definition of $\lambda_K(n)$ to the set of all cubic rings R. We do this by defining $\lambda_n(f)$ for all binary cubic forms f. Finally, for primes $p \neq 3$, we compute the Fourier transform of the function λ_p.

3.1 Standard properties of $L(s, \rho_K)$

We denote the Euler factors of $L(s, \rho_K)$ at primes p by $L_p(s, \rho_K)$, and the nth Dirichlet coefficient of $L(s, \rho_K)$ by $\lambda_K(n)$. We have that λ_K is multiplicative. We write the p^kth Dirichlet coefficient of the logarithmic derivative of $L(s, \rho_K)$ as $\theta_K(p^k)\log p$. That is, we have for $\Re(s) > 1$,

$$\begin{aligned} L(s, \rho_K) &= \prod_{p \text{ prime}} L_p(s, \rho_K) = \sum_{n=1}^{\infty} \frac{\lambda_K(n)}{n^s}, \\ -\frac{L'(s, \rho_K)}{L(s, \rho_K)} &= -\sum_{p \text{ prime}} \frac{L_p'(s, \rho_K)}{L_p(s, \rho_K)} = \sum_{n=1}^{\infty} \frac{\theta_K(n)\Lambda(n)}{n^s}. \end{aligned} \tag{21}$$

Note that θ_K is supported on prime powers.

Next, we recall some classical facts about $L(s, \rho_K)$. Let $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(\frac{s}{2})$ and $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$. Hecke proved that the *completed Dedekind zeta function*

$$\xi_K(s) := |\Delta(K)|^{s/2}\zeta_K(s) \cdot \begin{cases} \Gamma_{\mathbb{R}}(s)^3, & \text{if } \Delta(K) > 0, \\ \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s), & \text{if } \Delta(K) < 0, \end{cases}$$

has a meromorphic continuation to $s \in \mathbb{C}$ with simple poles at $s = 0, 1$ and satisfies the functional equation $\xi_K(s) = \xi_K(1 - s)$. We introduce the following notation:

$$\begin{aligned} \gamma^+(s) &:= \Gamma_{\mathbb{R}}(s)^2 = \pi^{-s}\Gamma(\tfrac{s}{2})^2; \\ \gamma^-(s) &:= \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s). \end{aligned}$$

Proposition 3.1 (Hecke) *$L(s, \rho_K)$ is entire and satisfies the functional equation $\Lambda(s, \rho_K) = \Lambda(1 - s, \rho_K)$, where $\Lambda(s, \rho_K) := |\Delta(K)|^{s/2}L_\infty(s, \rho_K)L(s, \rho_K)$ is the completed L-function, and*

$$L_\infty(s, \rho_K) := \gamma^{\operatorname{sgn}(\Delta(K))}(s) = \begin{cases} \Gamma_{\mathbb{R}}(s)^2, & \text{if } \Delta(K) > 0, \\ \Gamma_{\mathbb{C}}(s), & \text{if } \Delta(K) < 0. \end{cases}$$

Proof The functional equation of $L(s, \rho_K)$ follows from the functional equations of $\zeta_K(s)$ and $\zeta_{\mathbb{Q}}(s)$. It remains to show that $L(s, \rho_K)$ is entire and there are two cases to distinguish: If K is non-Galois, then $M/\mathbb{Q}$ is Galois with Galois group isomorphic to S_3, whereas if K is Galois, then $M = K$ with Galois group isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

(i) If $K = M$ is Galois, then the Artin representation

$$\rho_K : \operatorname{Gal}(M/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z} \hookrightarrow S_3 \to \mathrm{GL}_2(\mathbb{C})$$

is the direct sum of the two nontrivial characters of $\mathbb{Z}/3\mathbb{Z}$. Hence $L(s, \rho_K) = L(s, \chi_K)L(s, \overline{\chi_K})$ for two conjugate Dirichlet characters χ_K and $\overline{\chi_K}$ of order 3 and conductor $|\Delta(K)|^{\frac{1}{2}}$. Dirichlet proved that $L(s, \chi_K)$ and $L(s, \overline{\chi_K})$ are entire.

(ii) If K is non-Galois, then the Artin representation

$$\rho_K : \operatorname{Gal}(M/\mathbb{Q}) \cong S_3 \to \mathrm{GL}_2(\mathbb{C})$$

obtained from the standard representation of S_3 is irreducible. In this case, the sextic field M has a unique quadratic subfield denoted L. We have an exact sequence

$$\mathrm{Gal}(M/L) \hookrightarrow \mathrm{Gal}(M/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(L/\mathbb{Q}),$$

and the representation ρ_K of $\mathrm{Gal}(M/\mathbb{Q}) \simeq S_3$ is induced from a character χ_K of $\mathrm{Gal}(M/L) \simeq A_3 = \mathbb{Z}/3\mathbb{Z}$:

$$\rho_K \simeq \mathrm{Ind}_{\mathrm{Gal}(M/L)}^{\mathrm{Gal}(M/\mathbb{Q})}(\chi_K).$$

Thus we have $L(s, \rho_K) = L(s, \chi_K)$. Via class field theory, χ_K corresponds to a ring-class character of L of order 3. We have that $L(s, \chi_K)$ is entire by work of Hecke on the L-functions attached to Grössencharacters. □

The following standard result isn't directly used in the rest of the paper, except that the second case of the proposition when K is an S_3-field is relevant to Theorem 4.1 below. The reader can safely skip it.

Proposition 3.2 (Hecke, Maass) *The representation ρ_K is modular. That is, there exists a unique automorphic representation π_K of GL_2 such that $L(s, \rho_K)$ is equal to the principal L-function $L(s, \pi_K)$.*

- *If $K/\mathbb{Q}$ is cyclic, then π_K is an Eisenstein series with trivial central character.*
- *If K is an S_3-field, then π_K is cuspidal and its central character is the quadratic Dirichlet character associated to the quadratic resolvent of K. Moreover,*
  - *if $\Delta(K) < 0$ then $\pi_{K,\infty}$ is holomorphic of weight 1,*
  - *if $\Delta(K) > 0$ then $\pi_{K,\infty}$ is spherical of weight 0.*

Sketch of proof The construction of π_K is due to Hecke and Maass and comes from the theory of theta series. The unicity of π_K follows from the strong multiplicity-one theorem for GL_2. The central character of π_K corresponds under class field theory to the determinant character

$$\det \rho_K : \mathrm{Gal}(M/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(M/\mathbb{Q})^{ab} \to \mathbb{C}^\times.$$

If K is Galois, then the permutations in $\mathbb{Z}/3\mathbb{Z}$ have trivial determinant. If K is non-Galois with quadratic resolvent L, then the transposition permutations in S_3 have non-trivial determinant, and since $\mathrm{Gal}(M/\mathbb{Q})^{ab} = \mathrm{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ we obtain that $\det \rho_K$ is the quadratic Dirichlet character associated with $L/\mathbb{Q}$. □

3.2 Definition and properties of $\lambda_n(f)$

Let K be a cubic field with ring of integers $\mathcal{O}_K$. We say that K has *splitting type* $\sigma_p(K)$ to be (111), (12), (3), $(1^2 1)$ or (1^3) at p if p factors as $\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$, $\mathfrak{p}_1\mathfrak{p}_2$, p, $\mathfrak{p}_1^2\mathfrak{p}_2$, or $\mathfrak{p}^3$, respectively. Recall that $L(s, \rho_K)$ has an Euler factor decomposition, where

it may be checked that the pth Euler factor $L_p(s, \rho_K)$ only depends on the splitting type of K at p, and is as follows:

$$L_p(s,\rho_K) = \begin{cases} (1-p^{-s})^{-2} = \sum_{m=0}^{\infty} (m+1)p^{-ms} & \text{if } \sigma_p(K) = (111); \\ (1-p^{-2s})^{-1} = \sum_{m=0}^{\infty} p^{-2ms} & \text{if } \sigma_p(K) = (12); \\ (1+p^{-s}+p^{-2s})^{-1} = \sum_{m=0}^{\infty} (p^{-3ms} - p^{-(3m+1)s}) & \text{if } \sigma_p(K) = (3); \\ (1-p^{-s})^{-1} = \sum_{m=0}^{\infty} p^{-ms} & \text{if } \sigma_p(K) = (1^2 1); \\ 1 & \text{if } \sigma_p(K) = (1^3). \end{cases} \tag{22}$$

For a prime p, recall the six $\mathrm{GL}_2(\mathbb{F}_p)$-orbits $\mathcal{O}_\sigma$ on $V(\mathbb{F}_p)$ defined in (18).

Definition 3.3 Given an element $f \in V(\mathbb{F}_p)$, we define the *splitting type* $\sigma_p(f)$ of f to be σ if $f \in \mathcal{O}_\sigma$. For $m \geq 1$, we define the function $\lambda_{p^m} : V(\mathbb{F}_p) \to \mathbb{Z}$ as follows:

Let $f \in V(\mathbb{F}_p)$ have splitting type σ. Let K be any field also having splitting type σ at p. Then we define $\lambda_{p^m}(f) := \lambda_K(p^m)$. This serves as a definition for all nonzero f. For the zero form, we simply define $\lambda_{p^m}(0) := 0$.

Explicitly, we compute

$$\lambda_{p^m}(f) := \begin{cases} (m+1) & \text{if } \sigma_p(f) = (111); \\ 1 & \text{if } \sigma_p(f) = (12) \text{ and } m \equiv 0 \pmod 2; \\ 0 & \text{if } \sigma_p(f) = (12) \text{ and } m \equiv 1 \pmod 2; \\ 1 & \text{if } \sigma_p(f) = (3) \text{ and } m \equiv 0 \pmod 3; \\ -1 & \text{if } \sigma_p(f) = (3) \text{ and } m \equiv 1 \pmod 3; \\ 0 & \text{if } \sigma_p(f) = (3) \text{ and } m \equiv 2 \pmod 3; \\ 1 & \text{if } \sigma_p(f) = (1^2 1); \\ 0 & \text{if } \sigma_p(f) = (1^3); \\ 0 & \text{if } \sigma_p(f) = (0). \end{cases} \tag{23}$$

Extending notation, we set $\lambda_{p^m} : V(\mathbb{Z}) \to \mathbb{Z}$ by defining $\lambda_{p^m}(f) := \lambda_{p^m}(f \pmod p)$, where on the right-hand side we have the reduction of f modulo p. We also write $\sigma_p(f) = \sigma_p(f \pmod p)$ for the splitting type of f at p. For a positive integer $n \geq 1$, we define $\lambda_n : V(\mathbb{Z}) \to \mathbb{Z}$ multiplicatively in n, i.e., we set

$$\lambda_n(f) := \prod_{p^m \| n} \lambda_{p^m}(f).$$

The function $\lambda_n(f)$ is $\mathrm{GL}_2(\mathbb{Z})$-invariant and only depends on the reduction of f modulo $\mathrm{rad}(n)$, where $\mathrm{rad}(n)$ is the radical of n, that is the largest square-free divisor of n.

Next, given a binary cubic form $f \in V(\mathbb{Z})$, we define the following Dirichlet series:

$$D(s, f) := \sum_{n \geq 1} \frac{\lambda_n(f)}{n^s} = \prod_p D_p(s, f), \tag{24}$$

where the function $D_p(s, f)$ depends only on the splitting type of f at p. In fact, if a cubic field K has the same splitting type as f at p, then $D_p(s, f) = L_p(s, \rho_K)$, where $L_p(s, \rho_K)$ is given explicitly in (22). When f is a multiple of p, we have $D_p(s, f) = 1$.

For an irreducible integral binary cubic form f, with associated number field K_f as in Proposition 2.1, the relationship between $D(s, f)$ and $L(s, \rho_{K_f})$ is given by the following.

Lemma 3.4 *Let $f \in V(\mathbb{Z})^{\mathrm{irr}}$ be irreducible. Assume that f is maximal at the prime p. Then $\sigma_p(f) = \sigma_p(K_f)$, and therefore*

$$D_p(s, f) = L_p(s, \rho_{K_f}). \tag{25}$$

Proof Since f is maximal at p, we have $R_f \otimes \mathbb{Z}_p \cong \mathcal{O}_{K_f} \otimes \mathbb{Z}_p$, where R_f denotes the cubic ring corresponding to f and $\mathcal{O}_{K_f}$ denotes the ring of integers of K_f. Further tensoring with $\mathbb{F}_p$, we obtain $R_f \otimes \mathbb{F}_p \cong \mathcal{O}_{K_f} \otimes \mathbb{F}_p$. The former determines $\sigma_p(f)$ while the latter determines the splitting of K_f at p. Thus, the claim follows. □

Corollary 3.5 *If $f \in V(\mathbb{Z})^{\mathrm{irr,max}}$ is irreducible and maximal, that is if R_f is the ring of integers of the number field K_f, then $L(s, \rho_{K_f}) = D(s, f)$, and $\lambda_{K_f}(n) = \lambda_n(f)$ for all $n \geq 1$.*

Proof This is immediate from Definition 3.3 and the previous Lemma 3.4. □

Corollary 3.6 *Let $f \in V(\mathbb{Z})^{\mathrm{irr}}$ be irreducible. Then the function $D(s, f)$ converges absolutely for $\Re(s) > 1$.*

Proof This is immediate since $D(s, f)$ and $L(s, \rho_{K_f})$ can differ only at the finitely many Euler factors at p, where f is nonmaximal at p. □

For every $f \in V(\mathbb{Z})$, and prime power $n = p^m$, define $\theta_{p^m}(f)$ from the p^mth-coefficient of the logarithmic derivative,

$$-\frac{D'(s, f)}{D(s, f)} = -\sum_p \frac{D_p'(s, f)}{D_p(s, f)} = \sum_{n=1}^{\infty} \frac{\theta_n(f)\Lambda(n)}{n^s}, \quad \Re(s) > 1.$$

Lemma 3.7 *For every prime p and $f \in V(\mathbb{Z})$, we have $\theta_p(f) = \lambda_p(f)$ and $\theta_{p^2}(f) = 2\lambda_{p^2}(f) - \lambda_p(f)^2$. Furthermore, we have the bound $|\theta_{p^m}(f)| \leq 2$ for every prime p, integer $m \geq 1$ and $f \in V(\mathbb{Z})$.*

Proof The first two claims follow from $D_p(s, f) = 1 + \lambda_p(f)p^{-s} + \lambda_{p^2}(f)p^{-2s} + O(p^{-3s})$ and expanding its logarithmic derivative. The third claim is the case $n = 3$

of [30, Lem.2.2], of which we now repeat the argument for completeness. We have $D_p(s, f) = (1 - \alpha_1 p^{-s})^{-1}(1 - \alpha_2 p^{-s})^{-1}$, where $|\alpha_1|, |\alpha_2| \leq 1$ as can be seen by inspecting each case of (22). Then $\theta_{p^m}(f) = \alpha_1^m + \alpha_2^m$, which implies the desired inequality $|\theta_{p^m}(f)| \leq 2$. □

We conclude this section with certain Fourier transform computations. First, we have the following result, which will be useful in the sequel when we sum λ_p and θ_{p^2} over $\mathrm{GL}_2(\mathbb{Z})$-orbits on integral binary cubic forms having bounded discriminant.

Proposition 3.8 *Let $p \neq 3$ be a prime. Then*

$$\widehat{\lambda_p}(f_*) \;=\; \begin{cases} \dfrac{-1}{p^3} & \text{if} \quad f_* \in \mathcal{O}^*_{(111)} \cup \mathcal{O}^*_{(12)} \cup \mathcal{O}^*_{(3)} \cup \mathcal{O}^*_{(1^21)}; \\ \dfrac{p^2-1}{p^3} & \text{if} \quad f_* \in \mathcal{O}^*_{(1^3)} \cup \mathcal{O}^*_{(0)}. \end{cases}$$

Moreover, $\widehat{\theta_{p^2}}(0) \;=\; 1 - \dfrac{1}{p^2}$.

Proof A beautiful proof of a related result can be found in [35, Prop.1]. However, for the sake of completeness, we explain how we can recover this result (and indeed can compute the Fourier transform of any $\mathrm{GL}_2(\mathbb{F}_p)$-invariant function) from a simple application of Proposition 2.6. When $f_* \in \mathcal{O}^*_{(111)}$, we compute

$$\begin{aligned} \widehat{\lambda_p}(f_*) &= \frac{1}{p^4}\Big(\lambda_p(0) + \lambda_p(1^3)(2p-1) + \lambda_p(1^21)(-3p) + \lambda_p(111)(p(5 \pm p)/6) \\ &\quad + \lambda_p(12)(-p(-1 \pm p)/2) + \lambda_p(3)(p(-1 \pm p)/3)\Big) \\ &= \frac{1}{p^4}\Big(0 + 0 - 3p + p(5 \pm p)/3 - 0 - p(-1 \pm p)/3\Big) \\ &= \frac{-1}{p^3}, \end{aligned}$$

as claimed. The computation when f_* is in the other orbits is similar.

Finally, note that $\theta_{p^2}(f)$ is equal to 2 when $\sigma_p(f) \in \{(111), (12)\}$, equal to -1 when $\sigma_p(f) = (3)$, equal to 1 when $\sigma_p(f) = (1^21)$, and equal to 0 otherwise. Therefore, from the first row of the table in Proposition 2.6, we have

$$\begin{aligned} \widehat{\theta_{p^2}}(0) &= \Big(\frac{2}{6} + 1 - \frac{1}{3}\Big)\frac{p(p+1)(p-1)^2}{p^4} + \frac{p(p+1)(p-1)}{p^4} \\ &= (p - 1 + 1)\Big(\frac{(p+1)(p-1)}{p^3}\Big) \\ &= 1 - \frac{1}{p^2}, \end{aligned}$$

as necessary. □

Remark Requiring that equality (25) of Lemma 3.4 holds is enough to force the value of $\lambda_{p^m}(f)$ for every non-zero element $f \in V(\mathbb{F}_p) - \{0\}$ to be as in (23). We have then chosen $\lambda_{p^m}(0) := 0$ specifically so that the identities of Proposition 3.8 hold.

Let $u_p : V(\mathbb{Z}/p^2\mathbb{Z}) \to \{0, 1\}$ denote the characteristic function of the set of elements that lift to binary cubic forms in $V(\mathbb{Z}_p)$ that are maximal at p. We then have the following result.

Proposition 3.9 *We have*

$$\begin{aligned}
\widehat{u_p \cdot \lambda_p}(0) &= \frac{(p-1)(p^2-1)}{p^4};\\
\widehat{u_p \cdot \lambda_{p^2}}(0) &= \frac{(p^2-1)^2}{p^4};\\
\widehat{u_p \cdot \theta_{p^2}}(0) &= \frac{(p^2-1)^2}{p^4}.
\end{aligned}$$

Proof The Fourier transform at 0 can be evaluated by a density computation. That it so say, for any function $\phi : V(\mathbb{Z}/p^2\mathbb{Z}) \to \mathbb{R}$, we have

$$\widehat{\phi}(0) = \frac{1}{p^8} \sum_{f \in V(\mathbb{Z}/p^2\mathbb{Z})} \phi(f).$$

In [5, Lem.18], the densities of u_p are listed for each splitting type, as $\mu(\mathcal{U}_p(111))$, $\mu(\mathcal{U}_p(12))$, and so on, which we will abbreviate simply as $\mu(111)$, $\mu(12)$, and so on. And so we may calculate:

$$\begin{aligned}
\widehat{u_p \cdot \lambda_p}(0) &= \mu(111)\lambda_p(111) + \mu(12)\lambda_p(12)\\
&\quad + \mu(3)\lambda_p(3) + \mu(1^21)\lambda_p(1^21) + \mu(1^3)\lambda_p(1^3)\\
&= \frac{1}{p^4}\Big(\frac{1}{6}(p-1)^2p(p+1)\cdot 2 + \mu(12)\cdot 0\\
&\quad + \frac{1}{3}(p-1)^2p(p+1)\cdot(-1) + (p-1)^2(p+1)\cdot 1\Big)\\
&= \frac{(p-1)(p^2-1)}{p^4},
\end{aligned}$$

as necessary. Similarly, we have

$$\begin{aligned}
\widehat{u_p \cdot \lambda_{p^2}}(0) &= \mu(111)\lambda_{p^2}(111) + \mu(12)\lambda_{p^2}(12) + \mu(3)\lambda_{p^2}(3)\\
&\quad + \mu(1^21)\lambda_{p^2}(1^21) + \mu(1^3)\lambda_{p^2}(1^3)\\
&= \frac{1}{p^4}\Big(\frac{1}{6}(p-1)^2p(p+1)\cdot 3 + \frac{1}{2}(p-1)^2p(p+1)
\end{aligned}$$

$$+\mu(3)\cdot 0+(p-1)^2(p+1)\cdot 1\Big)$$
$$=\frac{(p^2-1)(p-1)(p+1)}{p^4},$$

as necessary. Finally, we have

$$\begin{aligned}\widehat{u_p\cdot\theta_{p^2}}(0)&=\mu(111)\theta_{p^2}(111)+\mu(12)\theta_{p^2}(12)+\mu(3)\theta_{p^2}(3)\\&\quad+\mu(1^21)\theta_{p^2}(1^21)+\mu(1^3)\theta_{p^2}(1^3)\\&=\frac{1}{p^4}\Big(\Big(\frac{1}{6}+\frac{1}{2}\Big)(p-1)^2p(p+1)\cdot 2\\&\quad+\frac{1}{3}(p-1)^2p(p+1)\cdot(-1)+(p-1)^2(p+1)\cdot 1\Big)\\&=\frac{1}{p^4}\big((p-1)^2p(p+1)+(p-1)^2(p+1)\big)\\&=\frac{(p^2-1)^2}{p^4},\end{aligned}$$

as necessary. □

4 Estimates on partial sums of Dirichlet coefficients of cubic fields and rings

In this section, we compute smoothed partial sums of the coefficients $\lambda_K(n)$ as well as of $\lambda_n(f)$. This section is organized as follows. First we collect some preliminary facts about Mellin inversion. Then, we recall the convexity bounds as well as current records towards the Lindelöf Hypothesis for principal GL(2) L-functions. We use these estimates to obtain bounds on smooth sums of the Dirichlet coefficients $\lambda_K(n)$ in terms of $|\Delta(K)|$, where K is a cubic field. Finally in §4.2, we prove analogous bounds on smooth sums of $\lambda_n(f)$ in terms of $|\Delta(f)|$, where $f\in V(\mathbb{Z})^{\mathrm{irr}}$ is an irreducible integral binary cubic form.

4.1 Upper bounds on smooth sums of $\lambda_K(n)$

We begin with a discussion of Mellin inversion, which will be used throughout this paper. Let $\Phi:\mathbb{R}_{\geq 0}\to\mathbb{C}$ be a smooth function that is rapidly decaying at infinity. We recall the definition of the *Mellin transform*

$$\widetilde{\Phi}(s):=\int_0^\infty x^s\Phi(x)\frac{dx}{x}.$$

The integral converges absolutely for $\Re(s)>0$. Integrating by parts yields the functional equation $\widetilde{\Phi}(s)=-\widetilde{\Phi}'(s+1)/s$. Hence, it follows that $\widetilde{\Phi}$ has a meromorphic continuation to $\mathbb{C}$, with possible simple poles at non-positive integers. Furthermore,

$\widetilde{\Phi}(s)$ has superpolynomial decay on vertical strips. *Mellin inversion* states that we have, for every $x \in \mathbb{R}_{>0}$,

$$\Phi(x) = \int_{\Re(s)=2} x^{-s} \widetilde{\Phi}(s) \frac{ds}{2\pi i}.$$

Consider a general Dirichlet series $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ which converges absolutely for $\Re(s) > 1$. We can then express the smoothed sums of the Dirichlet coefficients a_n as line integrals. For every positive real number $X \in \mathbb{R}_{>0}$, we have

$$\sum_{n\geq 1} a_n \Phi\Big(\frac{n}{X}\Big) = \int_{\Re(s)=2} D(s) X^s \widetilde{\Phi}(s) \frac{ds}{2\pi i}.$$

Consider the function $L(s, \rho_K)$ for a cubic field K. The *convexity bound* obtained from the Phragmén–Lindelöf principle,

$$L\big(\tfrac{1}{2} + it, \rho_K\big) \ll_\epsilon (1+|t|)^{\frac{1}{2}+\epsilon} |\Delta(K)|^{\frac{1}{4}+\epsilon},$$

will suffice for our purpose of establishing the main Theorem 4. We shall also use the current best bound for $L(\frac{1}{2}+it, \rho_K)$ due to Blomer–Khan [7] to achieve an improved numerical quality of the exponents in Theorem 3 and in the other results.

Theorem 4.1 (Bound for GL(2) L-functions in the level aspect) *For every $\epsilon > 0$, $t \in \mathbb{R}$ and cubic number field K,*

$$L\big(\tfrac{1}{2} + it, \rho_K\big) \ll_\epsilon (1+|t|)^{O(1)} |\Delta(K)|^{\theta+\epsilon},$$

where $\theta := \frac{1}{4} - \delta$ and $\delta := \frac{1}{128}$.

Proof In the proof of Proposition 3.1, we have seen that if K is cyclic, then $L(s, \rho_K) = L(s, \chi_K) L(s, \overline{\chi}_K)$. We then apply the Burgess estimate for Dirichlet characters, which yields the upper bound

$$L\big(\tfrac{1}{2} + it, \rho_K\big) \ll_\epsilon (1+|t|)^{O(1)} |\Delta(K)|^{\frac{1}{4}-\frac{1}{16}+\epsilon}.$$

If K is an S_3-field, then $L(s, \rho_K) = L(s, \pi_K)$ is the L-function of a GL(2) form of level $|\Delta(K)|$, unitary central character and weight 0 or 1. We then apply the estimate of Blomer–Khan [7, Thm.1], which yields the desired bound. □

The above result allows us to bound smoothed weighted partial sums of the Dirichlet coefficients of $L(s, \rho_K)$.

Corollary 4.2 *For every smooth function with compact support $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{C}$, $\epsilon > 0$, $T \geq 1$ and cubic number field K,*

$$\sum_{n\geq 1} \frac{\lambda_K(n)}{n^{1/2}} \Phi\left(\frac{n}{T}\right) \ll_{\epsilon,\Phi} T^\epsilon |\Delta(K)|^{\theta+\epsilon}.$$

Proof Applying Mellin inversion, we obtain

$$\begin{aligned}\sum_{n\geq 1}\frac{\lambda_K(n)}{n^{1/2}}\Phi\left(\frac{n}{T}\right) &= \frac{1}{2\pi i}\int_{\Re(s)=2} L\left(\tfrac{1}{2}+s,\rho_K\right)\widetilde{\Phi}(s)T^s ds\\ &\ll_{\epsilon,N} T^{\epsilon}\max_{|t|\leq T^{\epsilon}}\left|L\left(\tfrac{1}{2}+\epsilon+it,\rho_K\right)\right|\\ &\quad+\max_{|t|>T^{\epsilon}}|t|^{-N}\left|L\left(\tfrac{1}{2}+\epsilon+it,\rho_K\right)\right|,\end{aligned}$$

where the bound follows by shifting the integral contour to the line $\Re(s)=\epsilon$, and using the rapid decay of the Mellin transform $\widetilde{\Phi}(\epsilon+it)\ll_{\epsilon,N}|t|^{-N}$ for $|t|\geq 1$. The corollary now follows from Theorem 4.1 and the Phragmén–Lindelöf principle, the upper-bound on the vertical line $\frac{1}{2}+it$ being transported to the vertical line $\frac{1}{2}+\epsilon+it$. □

We continue with the *approximate functional equation* which gives the value of $L(\frac{1}{2},\rho_K)$ as a sum of its Dirichlet coefficients λ_K. Let $G(u)$ be an even, bounded and holomorphic function in the strip $|\Re(u)|<A$, where A is sufficiently large, and normalized by $G(0)=1$. For example [22, p.99], we could fix $G(u):=(\cos\frac{\pi u}{3A})^{-1}$. Define for $y\in\mathbb{R}_{>0}$

$$V^{\pm}(y):=\frac{1}{2\pi i}\int_{\Re(u)=3} y^{-u}G(u)\frac{\gamma^{\pm}(1/2+u)}{\gamma^{\pm}(1/2)}\frac{du}{u}. \tag{26}$$

We have that $V^{\pm}(y)$ is a rapidly decaying function as $y\to\infty$ that extends continuously at the origin with $V^{\pm}(0)=1$.

Proposition 4.3 *For every cubic number field K with $\pm\Delta(K)\in\mathbb{R}_{>0}$, we have*

$$L\left(\tfrac{1}{2},\rho_K\right)=2\sum_{n=1}^{\infty}\frac{\lambda_K(n)}{n^{1/2}}V^{\pm}\left(\frac{n}{|\Delta(K)|^{1/2}}\right). \tag{27}$$

Proof In view of the functional equation of Proposition 3.1, this is [22, Thm.5.3]. □

4.2 Upper bounds on smooth sums of $\lambda_n(f)$

Let $f\in V(\mathbb{Z})^{\mathrm{irr}}$ be an irreducible binary cubic form and recall the Dirichlet series $D(s,f)$ with Dirichlet coefficients $\lambda_n(f)$ defined in Sect. 3.

Definition 4.4 For $f\in V(\mathbb{Z})^{\mathrm{irr}}$ and a prime p, define $E_p(s,f)$ by

$$D_p(s,f)=L_p(s,\rho_{K_f})E_p(s,f).$$

Let $E(s,f)=\prod_p E_p(s,f)$, hence we have $D(s,f)=L(s,\rho_{K_f})E(s,f)$.

It follows from Lemma 3.4 that $E_p(s,f)=1$ if f is maximal at p, thus $E(s,f)=\prod_{p|\mathrm{ind}(f)}E_p(s,f)$.

We next list the different possible values taken by $E_p(s,f)$.

Lemma 4.5 *Let $f \in V(\mathbb{Z})^{\mathrm{irr}}$ be an irreducible binary cubic form. For every prime p, we have that $E_p(s, f)$ is a polynomial in p^{-s} of degree at most two. In fact, it is one of*

$$1, \quad 1 - p^{-s}, \quad 1 + p^{-s}, \quad (1 - p^{-s})^2, \quad 1 - p^{-2s}, \quad 1 + p^{-s} + p^{-2s}.$$

Moreover, if $p \parallel \mathrm{ind}(f)$, or if the splitting type of f at p is $(1^2 1)$, then $E_p(s, f)$ is of degree at most one, hence it is one of

$$1, \quad 1 - p^{-s}, \quad 1 + p^{-s}.$$

Proof We consider each possible splitting type of f seperately.

If $\sigma_p(f) = (0)$, then $D_p(s, f) = 1$ and $p^2 | \mathrm{ind}(f)$, hence the lemma follows from (22).

If $\sigma_p(f) = (111), (12)$, or (3), then f is maximal at p, thus $E_p(s, f) = 1$ by Lemma 3.4, and the lemma follows.

Suppose next that $\sigma_p(f) = (1^2 1)$. Then we claim that the splitting type of $\mathcal{O}_{K_f}$ at p is either (111), (12), or $(1^2 1)$, which implies the lemma by (22) because then either $E_p(s, f) = 1 - p^{-s}$, $E_p(s, f) = 1 + p^{-s}$, or $E_p(s, f) = 1$, respectively. Indeed, when f is nonmaximal at p, Proposition 2.2 implies that by replacing f with a $\mathrm{GL}_2(\mathbb{Z})$-translate, we may assume that $f(x, y) = ax^3 + bx^2y + pcxy^2 + p^2dy^3$, where $p \nmid b$. The overorder S of R_f having index $[S : R_f] = p$ corresponds to the form $g(x, y) = pax^3 + bx^2y + cxy^2 + dy^3$. Now the splitting type $\sigma_p(g)$ is either (111), (12), or $(1^2 1)$. In the former two cases, S is maximal at p and the claim is proved. In the last case, the claim follows by induction on the index, by repeating the argument with g instead of f.

Suppose finally that $\sigma_p(f) = (1^3)$, then $D_p(s, f) = 1$, hence $E_p(s, f) = L(s, \rho_{K_f})^{-1}$ is a polynomial in p^{-s} of degree at most two given by (22). Suppose moreover that $p \parallel \mathrm{ind}(f)$. We need to show that $E_p(s, f)$ is of degree at most one. From Proposition 2.2, we may assume that $f(x, y)$ is of the form $ax^3 + pbx^2y + pcxy^2 + p^2dy^3$. The index-$p$ overorder S of R_f must be maximal at p, which implies that the binary cubic form corresponding to $\mathcal{O}_{K_f} \otimes \mathbb{Z}_p$ is $pax^3 + pbx^2y + cxy^2 + dy^3$. Clearly, the splitting type of $\mathcal{O}_{K_f}$ at p is $(1^2 1)$ or (1^3). Thus $E_p(s, f) = 1 - p^{-s}$ or $E_p(s, f) = 1$, respectively. □

We obtain the following result analogous to Corollary 4.2 for the coefficients $\lambda_n(f)$ where f is an irreducible (not necessarily maximal) binary cubic form.

Proposition 4.6 *Let $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{C}$ be a smooth function rapidly decaying at infinity. For every $f \in V(\mathbb{Z})^{\mathrm{irr}}$, $\epsilon > 0$ and $T \geq 1$,*

$$\sum_{n \geq 1} \frac{\lambda_n(f)}{n^{1/2}} \Phi\left(\frac{n}{T}\right) \ll_{\epsilon, \Phi} \mathrm{ind}(f)^{-2\theta} |\Delta(f)|^{\theta + \epsilon} T^{\epsilon}, \tag{28}$$

where $\theta = \frac{1}{4} - \delta$ is as in Theorem 4.1.

Proof The proof is similar to that of Corollary 4.2. We have that the left-hand side is equal to

$$\frac{1}{2\pi i}\int_{\Re(s)=2} T^s \sum_{n\geq 1}\frac{\lambda_n(f)}{n^{\frac{1}{2}+s}}\widetilde{\Phi}(s)ds$$
$$= \frac{1}{2\pi i}\int_{\Re(s)=2} T^s L\left(\tfrac{1}{2}+s,\rho_K\right)\prod_{p|\mathrm{ind}(f)} E_p\left(\tfrac{1}{2}+s,f\right)\widetilde{\Phi}(s)ds.$$

For $\Re(s)\geq 0$, these local factors $E_p(\frac{1}{2}+s,f)$ are absolutely bounded, (indeed by the number 4). We have the elementary estimate

$$\prod_{p|\mathrm{ind}(f)} E_p\left(\tfrac{1}{2}+s,f\right) \leq \prod_{p|\mathrm{ind}(f)} 4 \ll_\epsilon |\mathrm{ind}(f)|^\epsilon.$$

As before, pulling the line of integration to $\Re(s)=\epsilon$, we deduce that

$$\sum_{n\geq 1}\frac{\lambda_n(f)}{n^{1/2}}\Phi\left(\frac{n}{T}\right) \ll_{\epsilon,\Phi} T^\epsilon |\Delta(K_f)|^\theta |\Delta(f)|^\epsilon,$$

from which the assertion follows since $\Delta(f)=\mathrm{ind}(f)^2\Delta(K_f)$. □

In our next result below (Theorem 4.11), we give a more precise estimate of the smoothed partial sums of $\lambda_n(f)$ when we use $\Phi = V^\pm$ as a smoothing function. We start by defining, for an irreducible binary cubic form $f\in V(\mathbb{Z})^{\mathrm{irr}}$, such that $\pm\Delta(f)\in\mathbb{R}_{>0}$, the quantity $S(f)$:

$$S(f) := \sum_{n\geq 1}\frac{\lambda_n(f)}{n^{1/2}}V^\pm\left(\frac{n}{|\Delta(f)|^{1/2}}\right). \tag{29}$$

If $f\in V(\mathbb{Z})^{\mathrm{irr,max}}$ is irreducible and maximal, then $2S(f)=L\left(\frac{1}{2},\rho_{K_f}\right)$ by Corollary 3.5 and Proposition 4.3.

For general irreducible $f\in V(\mathbb{Z})^{\mathrm{irr}}$, Proposition 4.6 yields the bound

$$S(f) \ll_\epsilon \mathrm{ind}(f)^{-2\theta}|\Delta(f)|^{\theta+\epsilon}. \tag{30}$$

Moreover, we have $D(\frac{1}{2},f)=L(\frac{1}{2},\rho_{K_f})E(\frac{1}{2},f)$ and

$$E\left(\tfrac{1}{2},f\right) = \prod_{p|\mathrm{ind}(f)}\left(1+O\left(p^{-\frac{1}{2}}\right)\right) = |\mathrm{ind}(f)|^{o(1)}, \tag{31}$$

which implies that the same upper bound as (30) holds for $D(\frac{1}{2},f)\ll_\epsilon \mathrm{ind}(f)^{-2\theta}\times |\Delta(f)|^{\theta+\epsilon}$.

Definition 4.7 For $f \in V(\mathbb{Z})^{\mathrm{irr}}$, a prime $p \mid \mathrm{ind}(f)$, and an integer $m \geq 0$, define $e_{p,m}(f)$ from the following power series expansion:

$$\frac{E_p\left(\frac{1}{2}-s, f\right)}{E_p\left(\frac{1}{2}+s, f\right)} = p^{2s-1} \sum_{m=0}^{\infty} e_{p,m}(f) p^{m(1/2-s)}.$$

Recall from Lemma 4.5 that $E_p(s, f)$ is a polynomial in p^{-s} of degree at most two. If $p \nmid \mathrm{ind}(f)$, let $e_{p,m}(f) = 0$ for every $m \geq 0$.

Examples **(a)** $E_p(s, f) = 1 - p^{-s}$: In this case, we have

$$\begin{aligned}
\frac{p}{p^{2s}} \frac{E_p\left(\frac{1}{2}-s, f\right)}{E_p\left(\frac{1}{2}+s, f\right)} &= \frac{p}{p^{2s}}\left(1 - \frac{p^s}{p^{1/2}}\right)\left(1 - \frac{1}{p^{1/2+s}}\right)^{-1} \\
&= \left(\frac{p}{p^{2s}} - \frac{p^{1/2}}{p^s}\right)\left(\sum_{n\geq 0} \frac{1}{p^{n/2+ns}}\right) \\
&= 0 - \frac{p^{1/2}}{p^s} + \frac{p-1}{p^{2s}} + \frac{p^{1/2} - p^{-1/2}}{p^{3s}} + \cdots \\
&\quad + \frac{p^{-(m-4)/2} - p^{-(m-2)/2}}{p^{ms}} + \cdots
\end{aligned}$$

It therefore follows that we have

$$e_{p,0}(f) = 0, \quad e_{p,1}(f) = -1, \quad e_{p,2}(f) = 1 - \frac{1}{p}, \quad e_{p,m}(f) = (p^{-m+2} - p^{-m+1}),$$

for all $m \geq 3$. If $E_p(s, f) = 1 + p^{-s}$, we obtain similar formulas.

(b) $E_p(s, f) = (1 - p^{-s})^2$: In this case, we have

$$\begin{aligned}
\frac{p}{p^{2s}} \frac{E_p\left(\frac{1}{2}-s, f\right)}{E_p\left(\frac{1}{2}+s, f\right)} &= \frac{p}{p^{2s}}\left(1 - \frac{p^s}{p^{1/2}}\right)^2\left(1 - \frac{1}{p^{1/2+s}}\right)^{-2} \\
&= \left(\frac{\sqrt{p}}{p^s} - 1\right)^2\left(\sum_{n\geq 0} \frac{1}{p^{n/2+ns}}\right)^2 \\
&= \left(1 - 2\frac{p^{1/2}}{p^s} + \frac{p}{p^{2s}}\right) \\
&\quad \times \left(1 + \frac{2}{p^{1/2+s}} + \frac{3}{p^{1+2s}} + \frac{4}{p^{3/2+3s}} + \cdots\right) \\
&= 1 + \left(\frac{2}{p^{1/2}} - 2p^{1/2}\right)\frac{1}{p^s} + \left(p + \frac{3}{p} - 4\right)\frac{1}{p^{2s}} \\
&\quad + \left(2p^{1/2} - \frac{6}{p^{1/2}} + \frac{4}{p^{3/2}}\right)\frac{1}{p^{3s}} + \cdots,
\end{aligned}$$

where the coefficient of $1/p^{ms}$ is $\ll m/p^{(m-4)/2}$. It therefore follows that we have

$$e_{p,0}(f) = 1, \quad e_{p,1}(f) = -2 + \frac{2}{p}, \quad e_{p,2}(f) = 1 - \frac{4}{p} + \frac{3}{p^2}, \quad e_{p,m}(f) \ll \frac{m}{p^{m-2}},$$

for all $m \geq 3$.

(c) $E_p(s, f) = 1 + p^{-s} + p^{-2s}$: In this case, we have

$$\begin{aligned}\frac{p}{p^{2s}} \frac{E_p\left(\frac{1}{2} - s, f\right)}{E_p\left(\frac{1}{2} + s, f\right)} &= \left(1 + \frac{p^{1/2}}{p^s} + \frac{p}{p^{2s}}\right)\left(1 + \frac{1}{p^{1/2+s}} + \frac{1}{p^{1+2s}}\right)^{-1} \\ &= \left(1 + \frac{p^{1/2}}{p^s} + \frac{p}{p^{2s}}\right)\left(1 - \frac{1}{p^{1/2+s}} + \frac{1}{p^{3/2+3s}} + \cdots\right) \\ &= 1 + \left(p^{1/2} - \frac{1}{p^{1/2}}\right)\frac{1}{p^s} + (p-1)\frac{1}{p^{2s}} \\ &\quad + \left(\frac{1}{p^{3/2}} - p^{1/2}\right)\frac{1}{p^{3s}} + \cdots,\end{aligned}$$

where the coefficient of $1/p^{ms}$ is $\ll m/p^{(m-4)/2}$. It therefore follows that once again we have

$$e_{p,0}(f) = 1, \quad e_{p,1}(f) = 1 - \frac{1}{p}, \quad e_{p,2}(f) = 1 - \frac{1}{p}, \quad e_{p,m}(f) \ll \frac{m}{p^{m-2}},$$

for all $m \geq 3$.

For every integer $k \geq 1$, define $e_k(f) := 0$ if there exists a prime $p|k$ at which f is maximal, and otherwise

$$e_k(f) := \prod_{p|\mathrm{ind}(f)} e_{p,v_p(k)}(f).$$

In other words, $e_k(f)$ is supported on the integers k all of whose prime factors divide $\mathrm{ind}(f)$.

Proposition 4.8 *For every $f \in V(\mathbb{Z})^{\mathrm{irr}}$, and $\Re(s) > -\frac{1}{2}$,*

$$\frac{E\left(\frac{1}{2} - s, f\right)}{E\left(\frac{1}{2} + s, f\right)} = \mathrm{rad}(\mathrm{ind}(f))^{2s-1} \sum_{k=1}^{\infty} e_k(f) k^{1/2-s}.$$

Proof Since $E(s, f) = \prod_{p|\mathrm{ind}(f)} E_p(s, f)$, the proposition follows from Definition 4.7, and from Lemma 4.5 which implies that $E_p(\frac{1}{2} + s, f)$ has no zero for $\Re(s) > -\frac{1}{2}$. □

We will need the following result, bounding the absolute value $|e_k(f)|$.

Proposition 4.9 *For every $f \in V(\mathbb{Z})^{\mathrm{irr}}$, $\epsilon > 0$, and $k \geq 1$,*

$$e_k(f) \ll_\epsilon k^\epsilon,$$

where the multiplicative constant depends only on ϵ. If k is powerful, then we have the improved bound

$$e_k(f) \ll_\epsilon \frac{\mathrm{rad}(k)^2}{k} k^\epsilon.$$

Proof The first claim of the proposition will follow from $e_{p,0}(f) \in \{-1, 0, 1\}$ and the estimate $e_{p,m}(f) \ll 1$ uniformly for every prime p and $m \geq 0$. The second claim will follow from $e_{p,0}(f) \in \{-1, 0, 1\}$, $e_{p,2}(f) \ll 1$, together with the estimate $e_{p,m}(f) \ll \frac{m}{p^{m-2}}$ uniformly for every prime p and $m \geq 3$.

These estimates have been verified in Examples **(a)**, **(b)**, and **(c)** above (in fact we have the absolute constant $|e_{p,m}(f)| \leq 2$, although this numerical value won't be needed). Note that Example **(a)** implies the estimates for the case $E_p(s, f) = 1 - p^{-2s}$, and also that the case of $E_p(s, f) = 1 + p^{-s}$ is identical to that of Example **(a)**. This concludes the proof of the proposition. □

Next, we fix a single form f, and analyze the coefficients $e_k(f)$.

Proposition 4.10 *Let $f \in V(\mathbb{Z})^{\mathrm{irr}}$, and write $\mathrm{ind}(f) = q_1 q_2$, where q_1 is squarefree, $(q_1, q_2) = 1$, and q_2 is powerful. Then $e_{(\cdot)}(f) : \mathbb{Z}_{\geq 1} \to \mathbb{R}$ is supported on multiples of q_1. Namely $q_1 \nmid k$ implies $e_k(f) = 0$.*

Proof Since q_1 is squarefree, it follows from Lemma 4.5 that for every prime $p \mid q_1$, we have $E_p(s, f)$ is one of 1, or $1 \pm p^{-s}$. Observe from Example **(a)** above that $e_{p,0}(f) = 0$. The proposition follows immediately. □

The following is an *unbalanced* approximate function equation for $D(s, f)$ analogous to Proposition 4.3 for $L(s, \rho_K)$.

Theorem 4.11 *For every $f \in V(\mathbb{Z})^{\mathrm{irr}}$,*

$$S(f) = D\big(\tfrac{1}{2}, f\big) - \sum_{k=1}^{\infty} \frac{e_k(f) k^{1/2}}{\mathrm{rad}(\mathrm{ind}(f))} \sum_{n=1}^{\infty} \frac{\lambda_n(f)}{n^{1/2}} V^{\mathrm{sgn}(\Delta(f))}\Big(\frac{\mathrm{ind}(f)^2 kn}{\mathrm{rad}(\mathrm{ind}(f))^2 |\Delta(f)|^{\frac{1}{2}}}\Big).$$

Proof To ease notation for the proof, we let $\pm := \mathrm{sgn}(\Delta(f))$ and $K := K_f$. We begin by noting that Mellin inversion yields

$$\widetilde{V^{\pm}}(s) = \frac{G(s)}{s} \frac{\gamma^{\pm}(\frac{1}{2} + s)}{\gamma^{\pm}(\frac{1}{2})}, \tag{32}$$

implying that $\widetilde{V^{\pm}}(s)$ decays rapidly and has a pole at $s = 0$ with residue 1. Hence, by shifting the line of integration, we obtain

$$\begin{aligned} S(f) &= \int_{\Re(s)=2} D\big(\tfrac{1}{2}+s, f\big)|\Delta(f)|^{s/2}\widetilde{V^{\pm}}(s)\frac{ds}{2\pi i} \\ &= D\big(\tfrac{1}{2}, f\big) + \int_{\Re(s)=-1/4} D\big(\tfrac{1}{2}+s, f\big)|\Delta(f)|^{s/2}\widetilde{V^{\pm}}(s)\frac{ds}{2\pi i}. \end{aligned}$$

By Proposition 3.1, the functional equation for $L(\frac{1}{2}+s, \rho_K)$ is

$$L(\tfrac{1}{2}+s, \rho_K)\gamma^{\pm}(\tfrac{1}{2}+s)|\Delta(K)|^{\frac{s}{2}} = L(\tfrac{1}{2}-s, \rho_K)\gamma^{\pm}(\tfrac{1}{2}-s)|\Delta(K)|^{-\frac{s}{2}}.$$

Therefore, we have

$$\begin{aligned} &S(f) - D(\tfrac{1}{2}, f) \\ &= \int_{\Re(s)=-1/4} L\big(\tfrac{1}{2}+s, \rho_K\big)E(\tfrac{1}{2}+s, f)|\Delta(f)|^{s/2}\widetilde{V^{\pm}}(s)\frac{ds}{2\pi i} \\ &= \int_{\Re(s)=-1/4} L\big(\tfrac{1}{2}-s, \rho_K\big)\frac{\gamma^{\pm}(\frac{1}{2}-s)}{\gamma^{\pm}(\frac{1}{2}+s)} \\ &\quad \times E(\tfrac{1}{2}+s, f)|\Delta(K)|^{-s}|\Delta(f)|^{s/2}\widetilde{V^{\pm}}(s)\frac{ds}{2\pi i} \\ &= \int_{\Re(s)=1/4} D\big(\tfrac{1}{2}+s, f\big)\frac{E(\frac{1}{2}-s, f)}{E(\frac{1}{2}+s, f)}|\Delta(f)/q^4|^{s/2}\frac{\gamma^{\pm}(\frac{1}{2}+s)}{\gamma^{\pm}(\frac{1}{2}-s)}\widetilde{V^{\pm}}(-s)\frac{ds}{2\pi i}, \end{aligned}$$

where the final equality follows since $\Delta(f) = q^2\Delta(K)$, where we have set $q := \mathrm{ind}(f)$. As a consequence of the above and (32), we have

$$\frac{\gamma^{\pm}(\frac{1}{2}+s)}{\gamma^{\pm}(\frac{1}{2}-s)}\widetilde{V^{\pm}}(-s) = -\frac{G(s)}{s}\frac{\gamma^{\pm}(\frac{1}{2}+s)}{\gamma^{\pm}(\frac{1}{2})} = -\widetilde{V^{\pm}}(s),$$

which we inject in the previous equality:

$$\begin{aligned} &D(\tfrac{1}{2}, f) - S(f) \\ &= \int_{\Re(s)=1/4} D\big(\tfrac{1}{2}+s, f\big)\frac{E(\frac{1}{2}-s, f)}{E(\frac{1}{2}+s, f)}|\Delta(f)/q^4|^{s/2}\widetilde{V^{\pm}}(s)\frac{ds}{2\pi i} \\ &= \int_{\Re(s)=1/4} D\big(\tfrac{1}{2}+s, f\big)\Big(\mathrm{rad}(q)^{2s-1}\sum_{k=1}^{\infty} e_k(f)k^{1/2-s}\Big) \\ &\quad \times |\Delta(f)/q^4|^{s/2}\widetilde{V^{\pm}}(s)\frac{ds}{2\pi i}, \end{aligned} \tag{33}$$

where the final equality follows from Proposition 4.8. The summand corresponding to k in the second line of (33) yields $\mathrm{rad}(q)^{-1}e_k(f)k^{1/2}$ times the integral

$$\int_{\Re(s)=1/4} D\left(\tfrac{1}{2}+s, f\right)\left(\frac{|\Delta(f)|^{\frac{1}{2}}\mathrm{rad}(q)^2}{kq^2}\right)^s \widetilde{V^{\pm}}(s)\frac{ds}{2\pi i}$$
$$= \sum_{n\geq 1}\frac{\lambda_n(f)}{n^{1/2}} V^{\pm}\left(\frac{nkq^2}{\mathrm{rad}(q)^2|\Delta(f)|^{\frac{1}{2}}}\right).$$

Theorem 4.11 follows by summing over $k \geq 1$. □

We end this section with the following remark.

Remark 4.12 When we consider sums weighted by the function $V^{\pm}(\cdot/X)$, which is rapidly decaying, we say that the *length of the sum* is at most $X^{1+\epsilon}$ (since we have that $V^{\pm}(y)$ is negligible for $y > X^{\epsilon}$).

Suppose $f \in V(\mathbb{Z})^{\mathrm{irr}}$ has large index $q = \mathrm{ind}(f)$, then all of the inner sums arising in Theorem 4.11 to express $S(f) - D(\frac{1}{2}, f)$ are always significantly shorter than the sum defining $S(f)$. Indeed, the sum defining $S(f)$ has length $|\Delta(f)|^{1/2+\epsilon}$. The length of any inner sum arising in Theorem 4.11 is easily computed. Let $q = q_1q_2$, where q_1 is squarefree, $(q_1, q_2) = 1$, and q_2 is powerful. Then note that we have

$$\frac{q^2}{\mathrm{rad}(q)^2} = \frac{q_2^2}{\mathrm{rad}(q_2)^2} \geq q_2,$$

with equality if and only if the exponent of every prime dividing q_2 is 2. Also note that we have $q_1|k$ from Proposition 4.10. Therefore, the length of the inner sum is at most $|\Delta(f)|^{1/2+\epsilon}/\mathrm{ind}(f)$.

5 Counting binary cubic forms using Shintani zeta functions

In this section we recall the asymptotics for the number of $\mathrm{GL}_2(\mathbb{Z})$-orbits of integral binary cubic forms ordered by discriminant. We will impose congruence conditions modulo positive integers n and study how the resulting error terms depend on n. This section is organized as follows: first, in §5.1, we collect results from the theory of Shintani zeta functions corresponding to the representation of GL_2 on V. Next, we use standard counting methods to determine the required asymptotics in §5.3, and moreover give an explicit bound on the error terms. Finally, in §5.4, we prove a smoothed analogue of the Pólya–Vinogradov inequality in the setting of cubic rings.

5.1 Functional equations, poles, and residues of Shintani zeta functions

Let n be a positive integer and let $\phi : V(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$ be a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant function. Let $\xi(\phi, s)$ denote the *Shintani zeta function* defined by

$$\xi^{\pm}(\phi, s) := \sum_{f \in \frac{V(\mathbb{Z})^{\pm}}{\mathrm{GL}_2(\mathbb{Z})}} \phi(f)\frac{|\Delta(f)|^{-s}}{|\mathrm{Stab}(f)|}, \tag{34}$$

where we abuse notation and also denote the composition of ϕ with the reduction modulo n map $V(\mathbb{Z}) \twoheadrightarrow V(\mathbb{Z}/n\mathbb{Z})$ by ϕ. For a function $\psi : V^*(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$, let $\xi^{*\pm}(\psi, s)$ denote the *dual* Shintani zeta function defined in [33, Def.4.2].

Theorem 5.1 (F. Sato–Shintani) *The functions $\xi^{\pm}$ and $\xi^{*\pm}$ have a meromorphic continuation to the whole complex plane, and satisfy the functional equations*

$$\begin{pmatrix} \xi^+(\phi, 1-s) \\ \xi^-(\phi, 1-s) \end{pmatrix} = n^{4s} \frac{(3^6\pi^{-4})^s}{18} \Gamma\Big(s - \frac{1}{6}\Big)\Gamma(s)^2\Gamma\Big(s + \frac{1}{6}\Big) \times \begin{pmatrix} \sin 2\pi s & \sin \pi s \\ 3\sin \pi s & \sin 2\pi s \end{pmatrix} \begin{pmatrix} \xi^{*+}(\hat{\phi}, s) \\ \xi^{*-}(\hat{\phi}, s) \end{pmatrix},$$

where $\hat{\phi} : V^*(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$ is the Fourier transform of ϕ as in §2.2.

Proof This is due to Shintani [31] for $n = 1$ and Sato [28] for general n. See also [33, Thm.4.3] for a modern exposition. In fact the above theorem is a special case because the congruence function ϕ in [28, 33] is not necessarily $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant. In the more general case of an arbitrary congruence function $\phi : V(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$, the Shintani zeta functions, respectively its dual, are defined using the principal subgroup $\Gamma(n)$ and summing f over the quotient $V(\mathbb{Z})^{\pm}/\Gamma(n)$, respectively $V^*(\mathbb{Z})^{\pm}/\Gamma(n)$. Assuming that ϕ is $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant, the general definition reduces to (34). □

The possible poles of $\xi^{\pm}(\phi, s)$ occur at 1 and 5/6, and the residues shall be given in Proposition 5.2 below. First we define

$$\begin{aligned} \alpha^+ &:= \frac{\pi^2}{36}; \quad \beta^+ := \frac{\pi^2}{12}; \quad \gamma^+ := \zeta(1/3)\frac{2\pi^2}{9\Gamma(2/3)^3}; \\ \alpha^- &:= \frac{\pi^2}{12}; \quad \beta^- := \frac{\pi^2}{12}; \quad \gamma^- := \zeta(1/3)\frac{2\sqrt{3}\pi^2}{9\Gamma(2/3)^3}. \end{aligned}$$

Then the functions $\xi^{\pm}(s) = \xi^{\pm}(1, s)$, corresponding to the constant function $\phi = 1$, have residues $\alpha^{\pm} + \beta^{\pm}$ at $s = 1$ and $\gamma^{\pm}$ at $s = 5/6$. Moreover, the pole at 1 has the following interpretation: the term $\alpha^{\pm}$ comes from the contribution of irreducible cubic forms and the term $\beta^{\pm}$ comes from the contribution of reducible cubic forms.

As before, let n be a positive integer. Let $\phi : V(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$ be a function of the form $\phi = \prod_{p^\beta \| n} \phi_{p^\beta}$, where $\phi_{p^\beta} : V(\mathbb{Z}/p^\beta\mathbb{Z}) \to \mathbb{C}$ and $\beta := v_p(n)$. We define the linear functionals $\mathcal{A}_{p^\beta}$, $\mathcal{B}_{p^\beta}$, and $\mathcal{C}_{p^\beta}$ to be

$$\begin{aligned} &\mathcal{A}_{p^\beta}(\phi_{p^\beta}) := \widehat{\phi_{p^\beta}}(0), \qquad \mathcal{B}_{p^\beta}(\phi_{p^\beta}) := \widehat{\phi_{p^\beta} \cdot b_p}(0), \\ &\mathcal{C}_{p^\beta}(\phi_{p^\beta}) := \widehat{\phi_{p^\beta} \cdot c_p}(0), \end{aligned} \tag{35}$$

where $\phi_{p^\beta} \mapsto \widehat{\phi_{p^\beta}}$ is the Fourier transform of functions on $V(\mathbb{Z}/p^\beta\mathbb{Z})$ from §2 and where the functions

$$b_p, c_p : V(\mathbb{Z}/p^\beta\mathbb{Z}) \twoheadrightarrow V(\mathbb{Z}/p\mathbb{Z}) \to \mathbb{R}_{\geq 0}$$

Table 1 Densities of splitting types

Splitting type of f at p	$b_p(f)$	$(1-p^{-2})c_p(f)$
(111)	3	$(1-p^{-2/3})(1+p^{-1/3})^2$
(12)	1	$(1-p^{-4/3})$
(3)	0	$(1-p^{-1/3})(1+p^{-1})$
$(1^2 1)$	$\frac{p+2}{p+1}$	$(1+p^{-1/3})(1-p^{-1})$
(1^3)	$\frac{1}{p+1}$	$(1-p^{-4/3})$
(0)	1	$(1-p^{-2})p^{2/3}$

are $\mathrm{GL}_2(\mathbb{Z}/p^\beta\mathbb{Z})$-invariant and defined in Table 1. We define $\mathcal{A}_n(\phi)$, $\mathcal{B}_n(\phi)$, and $\mathcal{C}_n(\phi)$ multiplicatively as the product over $p^\beta \parallel n$ of $\mathcal{A}_{p^\beta}(\phi_{p^\beta})$, $\mathcal{B}_{p^\beta}(\phi_{p^\beta})$, and $\mathcal{C}_{p^\beta}(\phi_{p^\beta})$, respectively. By multilinearity, the domain of definition of the functionals $\mathcal{A}_n$, $\mathcal{B}_n$, and $\mathcal{C}_n$ extends to all functions $\phi : V(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$. Abusing notation, we denote the lift of ϕ (resp. ϕ_{p^β}) to $V(\widehat{\mathbb{Z}})$ (resp. $V(\mathbb{Z}_p)$) also by ϕ (resp. ϕ_{p^β}). Note that $\mathcal{A}_n(\phi)$ can be interpreted as the integral

$$\mathcal{A}_n(\phi) = \int_{V(\widehat{\mathbb{Z}})} \phi(f)df = \prod_p \int_{V(\mathbb{Z}_p)} \phi_{p^\beta}(f)df,$$

where ϕ_{p^β} is simply defined to be the function 1 when $p \nmid n$. This is true because, under our normalizations $\mathrm{Vol}(V(\mathbb{Z}_p)) = 1$.

We then have the following expressions for the residues of Shintani zeta functions, see [11, 28, 33].

Proposition 5.2 *The functions $\xi^{\pm}(\phi, s)$ are holomorphic on $\mathbb{C} - \{1, 5/6\}$ with at worst simple poles at $s = 1, 5/6$ and the residues are given by*

$$\begin{aligned} \operatorname*{Res}_{s=1} \xi^{\pm}(\phi, s) &= \alpha^{\pm} \cdot \mathcal{A}_n(\phi) + \beta^{\pm} \cdot \mathcal{B}_n(\phi), \\ \operatorname*{Res}_{s=5/6} \xi^{\pm}(\phi, s) &= \gamma^{\pm} \cdot \mathcal{C}_n(\phi). \end{aligned}$$

The interpretation of these residues is that the term $\alpha^{\pm} \cdot \mathcal{A}_n(\phi)$ is the main term contribution from counting irreducible binary cubic forms, the term $\beta^{\pm} \cdot \mathcal{B}_n(\phi)$ is the main term contribution from counting reducible binary cubic forms, and the term $\gamma^{\pm} \cdot \mathcal{C}_n(\phi)$ is the secondary term contribution from counting irreducible binary cubic forms, particularly arising from cubic rings that are close to being *monogenic*, i.e., that have an element which generates a subring of small index.

5.2 Uniform bound for Shintani zeta functions near the abscissa of convergence

We recall the following *tail estimate* due to Davenport–Heilbronn [12]. See also [3] for a streamlined proof.

Proposition 5.3 (Davenport–Heilbronn) *Let n and m be positive squarefree integers. The number of $\mathrm{GL}_2(\mathbb{Z})$-orbits on the set of binary cubic forms having discriminant*

bounded by X and splitting type (1^3) at every prime dividing n and splitting type (0) at every prime dividing m is bounded by $O_\epsilon(X/(m^4n^{2-\epsilon}))$, where the implied constant is independent of X, m, and n.

Let p be a prime. Recall that for $p \neq 3$ the set of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$-orbits on $V^*(\mathbb{Z}/p\mathbb{Z})$ (resp. $V(\mathbb{Z}/p\mathbb{Z})$) is classified by the possible splitting types, namely, (111), (12), (3), $(1^2 1)$, (1^3), and (0). For $p = 3$, one could extend this classification, or, more simply, define $E_3(\psi_3) := ||\psi_3||_\infty$, which will only affect the multiplicative constants in this paper.

Definition 5.4 For a prime p and a $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$-invariant function ψ_p on $V^*(\mathbb{Z}/p\mathbb{Z})$ (resp. ϕ_p on $V(\mathbb{Z}/p\mathbb{Z})$), we define

$$E_p(\psi_p) := |\psi_p(111)| + |\psi_p(12)| + |\psi_p(3)| + |\psi_p(1^2 1)| + |\psi_p(1^3)|p^{-2} + |\psi_p(0)|p^{-4},$$

and similarly for $E_p(\phi_p)$.

Let n be a positive integer, and let $\psi : V^*(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$ (resp. $\phi : V(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$) be a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant function. If ψ factors as $\psi = \prod_{p^\beta \| n} \psi_{p^\beta}$, where $\psi_{p^\beta} : V^*(\mathbb{Z}/p^\beta\mathbb{Z}) \to \mathbb{C}$ are $\mathrm{GL}_2(\mathbb{Z}/p^\beta\mathbb{Z})$-invariant functions, then we define

$$E_n(\psi) \quad := \quad \prod_{p \| n} E_p(\psi_p) \cdot \prod_{\substack{p^\beta \| n \\ \beta \geq 2}} \|\psi_{p^\beta}\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the L^∞-norm. We have a similar definition for $E_n(\phi)$.

Proposition 5.5 *Let n be a positive integer. Let ψ be a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant function on $V^*(\mathbb{Z}/n\mathbb{Z})$. For every $\epsilon > 0$ and $t \in \mathbb{R}$, we have*

$$\xi^{*\pm}(\psi, 1 + \epsilon + it) \ll_\epsilon n^\epsilon E_n(\psi). \tag{36}$$

The same bound holds for $\xi^{\pm}(\phi, 1 + \epsilon + it)$ for a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant function ϕ on $V(\mathbb{Z}/n\mathbb{Z})$.

Proof Let q be a positive squarefree integer. We say that τ is a *splitting type modulo q* if $\tau = (\tau_p)_{p|q}$ is a collection of splitting types τ_p for each prime p dividing q. Let $q(\tau, 1^3)$ (resp. $q(\tau, 0)$) denote the product of primes p dividing q, such that $\tau_p = (1^3)$ (resp. $\tau_p = (0)$). That is,

$$q(\tau, 1^3) := \prod_{\substack{p|q \\ \tau_p = (1^3)}} p, \quad q(\tau, 0) := \prod_{\substack{p|q \\ \tau_p = (0)}} p.$$

We write $n = q\ell$, where q is squarefree, ℓ is powerful, and $(q, \ell) = 1$. Given an integral binary cubic form f, we have the factorization $\psi(f) = \psi_q(f)\psi_\ell(f)$,

where $\psi_q : V(\mathbb{Z}/q\mathbb{Z}) \to \mathbb{C}$ and $\psi_\ell : V(\mathbb{Z}/\ell\mathbb{Z}) \to \mathbb{C}$ are $\mathrm{GL}_2(\mathbb{Z}/q\mathbb{Z})$-invariant and $\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$-invariant functions, respectively, and as usual, we are denoting the lifts of ψ_q and ψ_ℓ to $V^*(\mathbb{Z})$ also by ψ_q and ψ_ℓ, respectively. Let $S(q)$ denote the set of splitting types modulo q. For $f \in V^*(\mathbb{Z})$, the value of $\psi_q(f)$ is determined by the splitting type τ modulo q of f. For such a splitting type $\tau \in S(q)$, we accordingly define $\psi_q(\tau) := \psi_q(f)$, where $f \in V^*(\mathbb{Z})$ is any element with splitting type τ modulo q.

Let $s = 1 + \epsilon + it$. We have

$$|\xi^{*\pm}(\psi, s)| \leq \|\psi_\ell\|_\infty \cdot \sum_{\tau \in S(q)} |\psi_q(\tau)| \sum_{m=1}^{\infty} \frac{c_\tau(m)}{m^{1+\epsilon}},$$

where $c_\tau(m)$ denotes the number of $\mathrm{GL}_2(\mathbb{Z})$-orbits on the set of elements in $V^*(\mathbb{Z})$ having discriminant m and splitting type τ modulo q. From partial summation, we obtain

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{c_\tau(m)}{m^{1+\epsilon}} &= \sum_{k=1}^{\infty} \Big(\frac{1}{k^{1+\epsilon}} - \frac{1}{(k+1)^{1+\epsilon}} \Big) \sum_{m=1}^{k} c_\tau(m) \\ &\ll_\epsilon \sum_{k=1}^{\infty} \frac{1}{k^{2+\epsilon}} \sum_{m=1}^{k} c_\tau(m). \end{aligned}$$

From Proposition 5.3, it follows that for every splitting type τ modulo q and every $k \geq 1$, we have

$$\sum_{m=1}^{k} c_\tau(m) \ll_\epsilon k \cdot q(\tau, 1^3)^{-2+\epsilon} \cdot q(\tau, 0)^{-4},$$

where the multiplicative constant is independent of q and k. Therefore, we have

$$\begin{aligned} \xi^{*\pm}(\psi, s) &\ll_\epsilon \|\psi_\ell\|_\infty \cdot \sum_{\tau \in S(q)} |\psi_q(\tau)| q(\tau, 1^3)^{-2+\epsilon} \cdot q(\tau, 0)^{-4} \Big(\sum_{k=1}^{\infty} \frac{1}{k^{1+\epsilon}} \Big) \\ &\ll_\epsilon n^\epsilon E_n(\psi). \end{aligned}$$

In the last equation, we used that

$$E_n(\psi) = \|\psi_\ell\|_\infty \cdot \prod_{p|q} E_p(\psi_p) = \|\psi_\ell\|_\infty \cdot \sum_{\tau \in S(q)} |\psi_q(\tau)| q(\tau, 1^3)^{-2} q(\tau, 0)^{-4}.$$

□

5.3 Smooth counts of binary cubic forms satisfying congruence conditions

As in the previous subsection, let n be a positive integer, and let $\phi : V(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$ be a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant function. Let $\Psi : \mathbb{R}_{>0} \to \mathbb{C}$ be a smooth function of compact support. For a real number $X \geq 1$, define the *counting function* $N_\Psi^\pm(\phi; X)$ to be

$$N_\Psi^\pm(\phi; X) := \sum_{f \in \frac{V(\mathbb{Z})^\pm}{\mathrm{GL}_2(\mathbb{Z})}} \frac{\phi(f)}{|\mathrm{Stab}(f)|} \Psi\Big(\frac{|\Delta(f)|}{X} \Big).$$

Note that the notation differs from the notation $N(\Sigma; X)$ and $N(S; X)$ from [5] in the presence of stabilizers.

Applying the Mellin transform results from Sect. 4, and shifting the line of integration from $\Re(s) = 2$ to $\Re(s) = -\epsilon$, with $0 < \epsilon < 1$, we obtain

$$\begin{aligned} N_\Psi^\pm(\phi; X) &= \frac{1}{2\pi i} \int_{\Re(s)=2} X^s \xi^\pm(\phi, s) \widetilde{\Psi}(s) ds \\ &= \mathrm{Res}_{s=1} \xi^\pm(\phi, s) \cdot \widetilde{\Psi}(1) \cdot X \\ &\quad + \mathrm{Res}_{s=5/6} \xi^\pm(\phi, s) \cdot \widetilde{\Psi}(\frac{5}{6}) \cdot X^{5/6} + \mathcal{E}_\epsilon(\phi, \Psi) \\ &= (\alpha^\pm \mathcal{A}_n(\phi) + \beta^\pm \mathcal{B}_n(\phi)) \cdot \widetilde{\Psi}(1) \cdot X \\ &\quad + \gamma^\pm \mathcal{C}_n(\phi) \cdot \widetilde{\Psi}(\frac{5}{6}) \cdot X^{5/6} + \mathcal{E}_\epsilon(\phi, \Psi). \end{aligned} \tag{37}$$

The *error term* $\mathcal{E}_\epsilon(\phi, \Psi)$ is defined below, and bounded using the functional equation in Theorem 5.1 and Stirling's asymptotic formula in the form $\Gamma(\sigma + it) \ll_\sigma (1 + |t|)^{\sigma - \frac{1}{2}} e^{\frac{-\pi|t|}{2}}$ for every $\sigma \notin \mathbb{Z}_{\leq 0}$ and $t \in \mathbb{R}$:

$$\begin{aligned} \mathcal{E}_\epsilon(\phi, \Psi) &:= \int_{\Re(s)=-\epsilon} X^s \xi^\pm(\phi, s) \widetilde{\Psi}(s) \frac{ds}{2\pi i} \\ &\ll_\epsilon n^{4+\epsilon} \max_{t \in \mathbb{R}} |\xi^{*\pm}(\widehat{\phi}, 1 + \epsilon + it)| E_\infty(\widetilde{\Psi}; \epsilon), \end{aligned} \tag{38}$$

where we define $E_\infty(\widetilde{\Psi}; \epsilon) := \int_{-\infty}^{\infty} \left|\widetilde{\Psi}(-\epsilon + it)\right| (1 + |t|)^{2+4\epsilon} dt$.

Theorem 5.6 *Let $\Psi : \mathbb{R}_{>0} \to \mathbb{C}$ be a smooth function with compact support and let $\epsilon > 0$. Let n be a positive integer, and write $n = qm$, where q is squarefree, $(q, m) = 1$, and m is powerful. For every real $X \geq 1$, and $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant function $\phi : V(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$, we have*

$$\begin{aligned} N_\Psi^\pm(\phi; X) &= \left(\alpha^\pm \mathcal{A}_n(\phi) + \beta^\pm \mathcal{B}_n(\phi)\right) \cdot \widetilde{\Psi}(1) \cdot X \\ &\quad + \gamma^\pm \mathcal{C}_n(\phi) \cdot \widetilde{\Psi}(\frac{5}{6}) \cdot X^{5/6} + O_\epsilon\left(n^{4+\epsilon} E_n(\widehat{\phi}) E_\infty(\widetilde{\Psi}; \epsilon)\right). \end{aligned}$$

Proof This follows from (37), (38), and Proposition 5.5. □

The following lemmas bound $E_n(\widehat{\phi})$ for various functions ϕ.

Lemma 5.7 *Let n be a positive integer and ϕ be a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant function on $V(\mathbb{Z}/n\mathbb{Z})$. Then we have, for every $\epsilon > 0$,*

$$E_n(\widehat{\phi}) \ll_\epsilon n^\epsilon \Big(\prod_{p \| n} p\Big)^{-2} \|\phi\|_\infty.$$

Proof This follows from the definitions of E_n and E_p, along with Corollary 2.7. □

Recall from §3.2 the function λ_n, which is a $\mathrm{GL}_2(\mathbb{Z}/\mathrm{rad}(n)\mathbb{Z})$-invariant function on $V(\mathbb{Z}/\mathrm{rad}(n)\mathbb{Z})$.

Lemma 5.8 *For every positive integer n and every $\epsilon > 0$,*

$$E_n(\widehat{\lambda_n}) \ll_\epsilon n^\epsilon \Big(\prod_{p\|n} p\Big)^{-3} \Big(\prod_{p^2|n} p\Big)^{-2}.$$

Proof Recall that the functions λ_{p^k} are defined modulo p irrespective of k. Hence the claimed saving from the factors p with $p^2 \mid n$ follows from Lemma 5.7. The additional saving from the factors p with $p \| n$ is a consequence of Proposition 3.8. □

Lemma 5.9 *For every prime $p \neq 3$,*

$$\mathcal{A}_p(\lambda_p) = \widehat{\lambda_p}(0) = \frac{p^2-1}{p^3}, \qquad \mathcal{B}_p(\lambda_p) = \widehat{\lambda_p b_p}(0) = \frac{p^3-1}{p^3},$$
$$\mathcal{C}_p(\lambda_p) = \widehat{\lambda_p c_p}(0) \ll \frac{1}{p^{1/3}}.$$

Proof The first equation is derived in Proposition 3.8. The second equation is derived similarly: we have

$$\widehat{\lambda_p b_p}(0) = 6 \cdot \frac{p(p+1)(p-1)^2}{6p^4} + \frac{p+2}{p+1} \cdot \frac{p(p+1)(p-1)}{p^4} = \frac{p^3-1}{p^3}.$$

To prove the final inequality, we write

$$\begin{aligned} c_p(111) &= (1-p^{-1/3})(1+p^{-1/3})^3\Big(1-\frac{1}{p^2}\Big)^{-1}; \\ c_p(3) &= (1-p^{-1/3})(1+p^{-1})\Big(1-\frac{1}{p^2}\Big)^{-1}; \\ c_p(1^21) &= (1+p^{-1/3})(1-p^{-1})\Big(1-\frac{1}{p^2}\Big)^{-1}. \end{aligned}$$

We compute $\widehat{\lambda_p c_p}(0)$ using Proposition 2.6 and obtain

$$\mathcal{C}_p(\lambda_p) = \frac{1}{3}\Big(1-\frac{1}{p}\Big)(1-p^{-1/3})\big((1+p^{-1/3})^3-(1+p^{-1})\big) + \frac{1}{p}\Big(1-\frac{1}{p}\Big)(1+p^{-1/3}),$$

which concludes the proof of the lemma. □

5.4 Application to cubic analogues of Pólya–Vinogradov

We sum the Artin character over isomorphism classes of cubic rings. This is a cubic analogue of the Pólya–Vinogradov inequality [22, Thm.12.5], which sums Artin characters over quadratic rings. There are some substantial differences between quadratic

and cubic cases: first, in the cubic case we see the presence of second order terms which do not occur in the quadratic case. Second, since the parameter space of cubic rings is four dimensional (as opposed to one dimensional), the trivial range for summing the Artin character λ_n over cubic rings with discriminant bounded by X is $X \gg n^4$ (as opposed to $X \gg n$ in the quadratic case).

Theorem 5.10 (Cubic analogue of Pólya–Vinogradov) *Let p be a prime and let $k \geq 2$ be an integer. Let $\Psi : \mathbb{R}_{>0} \to \mathbb{C}$ be a smooth function with compact support such that $\int_0^\infty \Psi(x)dx = 1$. Then we have*

$$\begin{aligned}
&\sum_{f \in \frac{V(\mathbb{Z})^{\pm}}{\mathrm{GL}_2(\mathbb{Z})}} \frac{\lambda_p(f)}{|\mathrm{Stab}(f)|} \Psi\Big(\frac{|\Delta(f)|}{X}\Big) \\
&= \Big(\alpha^{\pm}\frac{p^2-1}{p^3} + \beta^{\pm}\frac{p^3-1}{p^3}\Big)X \\
&\quad + \gamma^{\pm}\widehat{\lambda_p c_p}(0)\widetilde{\Psi}(\frac{5}{6}) \cdot X^{5/6} + O_{\epsilon,\Psi}(p^{1+\epsilon}); \\
&\sum_{f \in \frac{V(\mathbb{Z})^{\pm}}{\mathrm{GL}_2(\mathbb{Z})}} \frac{\lambda_{p^k}(f)}{|\mathrm{Stab}(f)|} \Psi\Big(\frac{|\Delta(f)|}{X}\Big) \\
&= \Big(\alpha^{\pm}\widehat{\lambda_{p^k}}(0) + \beta^{\pm}\widehat{\lambda_{p^k} b_p}(0)\Big)X \\
&\quad + \gamma^{\pm}\widehat{\lambda_{p^k} c_p}(0)\widetilde{\Psi}(\frac{5}{6}) \cdot X^{5/6} + O_{\epsilon,\Psi}(kp^{2+\epsilon}).
\end{aligned}$$

Proof This is a consequence of Theorem 5.6 in conjunction with Propositions 5.3 and 5.5 and Lemma 5.8. □

6 Sieving to the space of maximal binary cubic forms

In this section, we employ an inclusion-exclusion sieve to sum over *maximal* binary cubic forms. To set up this sieve, we need the following notation. Denote the set of maximal integral binary cubic forms by $V(\mathbb{Z})^{\max}$. For a squarefree positive integer q, we let $\mathcal{W}_q$ denote the set of elements in $V(\mathbb{Z})$ that are *nonmaximal at every prime dividing q*. Given a set S with a $\mathrm{GL}_2(\mathbb{Z})$-action, we let $\overline{S} := \frac{S}{\mathrm{GL}_2(\mathbb{Z})}$ denote the set of $\mathrm{GL}_2(\mathbb{Z})$-orbits on S. Let $\Psi : \mathbb{R}_{>0} \to \mathbb{C}$ be a smooth function with compact support, and let $\phi : V(\mathbb{Z}) \to \mathbb{C}$ be a $\mathrm{GL}_2(\mathbb{Z})$-invariant function defined by congruence conditions modulo some fixed M. Then we have

$$\sum_{f \in \overline{V(\mathbb{Z})^{\pm,\max}}} \frac{\phi(f)}{|\mathrm{Stab}(f)|}\Psi(|\Delta(f)|) = \sum_{q \geq 1} \mu(q) \sum_{f \in \overline{\mathcal{W}_q^{\pm}}} \frac{\phi(f)}{|\mathrm{Stab}(f)|}\Psi\big(|\Delta(f)|\big). \tag{39}$$

In practice, and starting from §6.2 and in subsequent sections, we will assume that the support of ϕ is contained within the set of binary cubic forms corresponding to orders in S_3-fields (this will imply that the stabilizers are trivial, that is $|\mathrm{Stab}(f)| = 1$).

The difficulty in obtaining good estimates for the right-hand side of (39) is that ϕ restricted to $\mathcal{W}_q$ is defined by congruence conditions modulo q^2M, and a direct application of the results of Sect. 5 yields not sufficiently precise error terms for sums over such sets. We overcome this difficulty in §6.1 by using a "switching trick", developed in [5], which transforms the sum over $\mathcal{W}_p$ to a weighted sum over $V(\mathbb{Z})$, where the weights are defined modulo p. We then combine the results of Sect. 5 and §6.1 to carry out the sieve and obtain improved bounds for the error term. Finally, in §6.3, we derive several applications; notably, we obtain a smoothed version of Roberts' conjecture for certain (S_3-) families of cubic fields, and sum the Artin character $\lambda_K(n)$ over cubic fields K.

For a positive squarefree integer m and an integral binary cubic form $f \in V(\mathbb{Z})$, denote the number of roots (resp. simple roots) in $\mathbb{P}^1(\mathbb{Z}/m\mathbb{Z})$ of the reduction of f modulo m by $\omega_m(f)$ (resp. $\omega_m^{(1)}(f)$). By the Chinese remainder theorem, $\omega_m(f)$ and $\omega_m^{(1)}(f)$ are multiplicative in m.

Proposition 6.1 ([5, Eq. (70)]) *For every positive squarefree integer q and every function $\Psi : \mathbb{R}_{>0} \to \mathbb{C}$ of compact support,*

$$\sum_{f \in \overline{\mathcal{W}_q^{\pm}}} \Psi\big(|\Delta(f)|\big) = \sum_{k\ell|q} \mu(\ell) \sum_{f \in \overline{V(\mathbb{Z})^{\pm}}} \omega_{k\ell}(f) \Psi\Big(\frac{q^4|\Delta(f)|}{k^2}\Big).$$

The above identity was proved using the following procedure in [5, §9]. Every element $f \in \mathcal{W}_q$ corresponds to a ring R_f that is nonmaximal at every prime dividing q, hence R_f is contained in a certain ring R', such that the *index* $\mathrm{ind}(f) := [R' : R_f]$ satisfies $q \mid \mathrm{ind}(f)$ and $\mathrm{ind}(f) \mid q^2$. In particular, the discriminant of R' is smaller than that of R_f. Then elements in $\overline{\mathcal{W}_q}$ can be counted by counting the rings R' instead of R_f. In what follows, we formalize this procedure, and adapt it so that we may sum congruence functions over $\overline{\mathcal{W}_q}$ (Theorem 6.5 which is a strenghtening of Proposition 6.1).

6.1 Switching to overrings

We begin with a bijection which allows us to replace sums over $\overline{\mathcal{W}_q}$ with sums over $\overline{\mathcal{W}_{q_1}}$, for various integers $q_1 \mid q$ with $q_1 < q$. Given a set $S \subset V(\mathbb{Z})$ and an element $\alpha \in \mathbb{P}^1(\mathbb{F}_p)$, let $S^{(\alpha)}$ denote the set of elements $f \in S$ such that $f(\alpha) \equiv 0 \pmod p$. Then we have the following result.

Lemma 6.2 *Let q be a positive squarefree number, and let p be a prime such that $p \mid q$. Then there is a bijection between the following two sets*:

$$\begin{aligned} &\Big\{ f \in \overline{\mathcal{W}_q} \backslash \overline{p\mathcal{W}_{q/p}} \Big\} \bigcup \overline{\Big\{ (f,\gamma) : f \in p\mathcal{W}_{q/p}^{(\gamma)},\ \gamma \in \mathbb{P}^1(\mathbb{F}_p) \Big\}} \\ &\qquad \longleftrightarrow \overline{\Big\{ (g,\alpha) : g \in \mathcal{W}_{q/p}^{(\alpha)},\ \alpha \in \mathbb{P}^1(\mathbb{F}_p) \Big\}}, \end{aligned} \tag{40}$$

uniquely characterized as follows. Both sets are in natural bijection with the set of isomorphism classes of pairs (R, R') with $R \subset R'$, where R is an index-p subring of

the cubic ring R'. The two bijections are given via $(R_f = R, R') \mapsto f$ and $(R, R' = R_g) \mapsto g$.

Proof The set $\overline{\mathcal{W}_q}$ is in bijection with the set of cubic rings that are nonmaximal at every prime p dividing q. As in [5, §9], we consider the set of pairs of cubic rings $R \subset R'$, such that R is nonmaximal at every prime dividing q, and the index of R in R' is p. Let f and g be representatives for the $\mathrm{GL}_2(\mathbb{Z})$-orbits on $V(\mathbb{Z})$ corresponding to R and R', respectively. If $f \in W_q$ is not a multiple of p, then there exists a unique index-p overring R' of R by Proposition 2.3. On the other hand, if f is a multiple of p, then the set of index-p overrings R' of R are in natural bijection with the roots of f/p in $\mathbb{P}^1(\mathbb{F}_p)$ (also by Proposition 2.3). Therefore, the set of pairs (R, R') is in natural bijection with $\mathrm{GL}_2(\mathbb{Z})$-orbits on the following set:

$$\left\{f \in \mathcal{W}_q \backslash p\mathcal{W}_{q/p})\right\} \bigcup \left\{(f, \gamma) : f \in p\mathcal{W}_{q/p}^{(\gamma)},\ \gamma \in \mathbb{P}^1(\mathbb{F}_p)\right\}, \tag{41}$$

and every form f in the above set corresponds to the ring $R = R_f$.

On the other hand, the set of index-p subrings of the ring R_g is in natural bijection with the set of roots of g in $\mathbb{P}^1(\mathbb{F}_p)$ by Proposition 2.3. Therefore, the set of pairs (R, R') is also in natural bijection with $\mathrm{GL}_2(\mathbb{Z})$-orbits on the set

$$\left\{(g, \alpha) : \alpha \in \mathbb{P}^1(\mathbb{F}_p),\ g \in \mathcal{W}_{q/p}^{(\alpha)}\right\}, \tag{42}$$

and every form g in the above set corresponds to the ring $R' = R_g$. It follows that $\mathrm{GL}_2(\mathbb{Z})$-orbits on the sets (41) and (42) are in natural bijection. □

We will also need the following lemma determining how the above bijection changes the splitting types of the binary cubic forms.

Lemma 6.3 *Let $g \in \mathcal{W}_{q/p}$ and $\alpha \in \mathbb{P}^1(\mathbb{F}_p)$ be a root of g modulo p. Let $f \in \overline{\mathcal{W}_q}$ correspond to the $\mathrm{GL}_2(\mathbb{Z})$-orbit of (g, α) under the bijection of Lemma 6.2. Then*

$$\sigma_p(f) = \begin{cases} (1^2 1) & \text{if } \alpha \text{ is a simple root of } g; \\ (1^3) \text{ or } (0) & \text{otherwise.} \end{cases}$$

Moreover, for every prime $\ell \neq p$, we have $\sigma_\ell(f) = \sigma_\ell(g)$. And more generally, for every integer n coprime with p, the reduction of f modulo n and the reduction of g modulo n are $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-conjugates.

Proof By translating g with an element of $\mathrm{GL}_2(\mathbb{Z})$ if necessary, we can assume that $\alpha = [0 : 1]$. In that case, we have $g(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$, where $p \mid d$. Furthermore, we have $p \nmid c$ if and only if α is a simple root. Then, the element $f(x, y)$ is given by $f(x, y) = p^2ax^3 + pbx^2y + cxy^2 + d/py^3$, and has splitting type $(1^2 1)$ if and only if $p \nmid c$. The first part of the lemma follows.

To prove the second part of the lemma, note that by tensoring the exact sequence $0 \to R_f \to R_g \to \mathbb{Z}/p\mathbb{Z} \to 0$ by the flat $\mathbb{Z}$-module $\mathbb{Z}_\ell$, we obtain the isomorphism $R_f \otimes \mathbb{Z}_\ell \cong R_g \otimes \mathbb{Z}_\ell$. Reducing modulo ℓ yields $R_f \otimes \mathbb{F}_\ell \cong R_g \otimes \mathbb{F}_\ell$ (and also $R_f \otimes \mathbb{Z}/n\mathbb{Z} \cong R_g \otimes \mathbb{Z}/n\mathbb{Z}$) which implies the desired conclusion. □

Let n be a positive integer, and let $\phi : V(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$ be a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant function such that ϕ is given by

$$\phi = \prod_{p^\beta \| n} \phi_{p^\beta},$$

where $f \mapsto \phi_{p^\beta}(f)$ is $\mathrm{GL}_2(\mathbb{Z}/p^\beta\mathbb{Z})$-invariant. When $\beta = 1$, we have that $\phi_p(f)$ is determined by the splitting type of f at p. For any positive integer k dividing n, such that $(k, n/k) = 1$, we denote $\prod_{p^\beta \| k} \phi_{p^\beta}$ by ϕ_k. Let $d \geq 1$ be a squarefree integer dividing n such that $(d, n/d) = 1$.

Definition 6.4 We say that such a function ϕ_n is *simple* at d, if for all $p \mid d$, we have $\phi_p(f) = \phi_p(0)$ when the splitting type of f at p is (1^3).

Note that the functions of interest in the rest of the paper, namely λ_{p^k} and θ_{p^k} for primes p and positive integers k, are all simple.

We are now ready to prove the main result of this subsection.

Theorem 6.5 *Let $\Psi : \mathbb{R}_{>0} \to \mathbb{C}$ be a compactly supported function, n be a positive integer and q be a positive squarefree integer. Let ϕ be a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant function on $V(\mathbb{Z}/n\mathbb{Z})$. Denote (q, n) by de, where d is the product of primes dividing (q, n) at which ϕ is simple, and assume that $\phi_p(0) = 0$ for every prime $p|d$. Write $n = dm$ and $\phi = \phi_d\phi_m$. Then*

$$\sum_{f \in \overline{\mathcal{W}_q^{\pm}}} \phi(f)\Psi\big(|\Delta(f)|\big)$$

$$= \phi_d(1^2 1) \sum_{k\ell \mid \frac{q}{de}} \mu(\ell) \sum_{g \in \overline{\mathcal{W}_e^{\pm}}} \omega_d^{(1)}(g)\omega_{k\ell}(g)\phi_m(g)\Psi\Big(\frac{q^4|\Delta(g)|}{e^4 d^2 k^2}\Big).$$

Proof We prove Theorem 6.5 by induction on the number of prime factors of d. First we consider the case $d = 1$ which we establish by induction on the number of prime factors of q/e. Let p be a prime dividing q/e. We again use the bijection of Lemma 6.2 to relate the sum over $f \in \overline{\mathcal{W}_q}$ to a sum over $g \in \overline{\mathcal{W}_{q/p}}$. If $f \in \overline{p\mathcal{W}_{q/p}}$, then $\phi(f/p) = \phi(f)$ because ϕ is $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant and $(p, n) = 1$ implies $1/p \in Z(\overline{\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})})$ which acts by scalar multiplication on $V(\mathbb{Z}/n\mathbb{Z})$. Suppose $f \in \overline{\mathcal{W}_q} \backslash \overline{p\mathcal{W}_{q/p}}$ corresponds to the $\mathrm{GL}_2(\mathbb{Z})$-orbit of $g \in \mathcal{W}_{q/p}$ and a root $\alpha \in \mathbb{P}^1(\mathbb{F}_p)$ of g modulo p under the surjection of Lemma 6.2. Then since $(n, p) = 1$, we have $\phi(f) = \phi(g)$ from Lemma 6.3. Thus,

$$\sum_{f \in \overline{\mathcal{W}_q^{\pm}}} \phi(f)\Psi\big(|\Delta(f)|\big)$$

$$= \sum_{k_1\ell_1 \mid p} \mu(\ell_1) \sum_{f \in \overline{\mathcal{W}_{q/p}^{\pm}}} \omega_{k_1\ell_1}(f)\phi(f)\Psi\Big(\frac{p^4|\Delta(f)|}{k_1^2}\Big)$$

$$= \sum_{k_1\ell_1|p} \mu(\ell_1) \sum_{k_2\ell_2|\frac{q}{pe}} \mu(\ell_2) \sum_{f\in\overline{\mathcal{W}_e^{\pm}}} \omega_{k_1\ell_1}(f)\omega_{k_2\ell_2}(f)\phi(f)\Psi\Big(\frac{q^4|\Delta(f)|}{e^4k_1^2k_2^2}\Big)$$

$$= \sum_{k\ell|\frac{q}{e}} \mu(\ell) \sum_{f\in\overline{\mathcal{W}_e^{\pm}}} \omega_{k\ell}(f)\phi(f)\Psi\Big(\frac{q^4|\Delta(f)|}{e^4k^2}\Big),$$

where the second equality follows by induction on the sum over $\overline{\mathcal{W}_{q/p}^{\pm}}$ of the $\mathrm{GL}_2(\mathbb{Z}/pn\mathbb{Z})$-invariant function $\omega_{k_1\ell_1}\cdot\phi$.

It remains to prove the inductive step on the number of prime factors of d. Let p be a prime dividing d. We use the bijection of Lemma 6.2 to relate the sum over $f\in\overline{\mathcal{W}_q}$ to sums over $f\in\overline{\mathcal{W}_{q/p}}$. Suppose $f\in\overline{\mathcal{W}_q}$ corresponds to $g\in\overline{\mathcal{W}_{q/p}^{(\alpha)}}$ under the bijection of Lemma 6.2, then by Lemma 6.3, we have $\phi_p(g)=\phi_p(1^21)$ if α is a simple root and $\phi_p(g)=\phi_p(1^3)=0$ otherwise. Also, we have $\phi_{n/p}(g)=\phi_{n/p}(f)$. Therefore, we have

$$\begin{aligned}\sum_{f\in\overline{\mathcal{W}_q^{\pm}}} \phi(f)\Psi\big(|\Delta(f)|\big) &= \sum_{g\in\overline{\mathcal{W}_{q/p}^{\pm}}} \omega_p^{(1)}(g)\phi_p(1^21)\phi_{n/p}(g)\Psi\Big(\frac{|\Delta(g)|}{p^2}\Big)\\ &= \phi_d(1^21)\sum_{g\in\overline{\mathcal{W}_{q/d}^{\pm}}} \omega_p^{(1)}(g)\omega_{d/p}^{(1)}(g)\phi_{n/d}(g)\Psi\Big(\frac{|\Delta(g)|}{d^2}\Big),\end{aligned}$$

where the second equation follows by induction on the sum over $\overline{\mathcal{W}_{q/p}^{\pm}}$ of the (simple at d/p) function $\phi_{n/p}\cdot\omega_p^{(1)}$. The result now follows since $\omega_k^{(1)}$ is multiplicative in k. □

6.2 Summing congruence functions over $\overline{\mathcal{W}_q^{\pm}}$

Let n be a positive integer and let $\phi: V(\mathbb{Z}/n\mathbb{Z})\to\mathbb{C}$ be of the form $\phi=\prod_{p^\beta\|n}\phi_{p^\beta}$, where $\phi_{p^\beta}: V(\mathbb{Z}/p^\beta\mathbb{Z})\to\mathbb{C}$ and $\beta:=v_p(n)$. Let $V(\mathbb{Z}_p)^{\mathrm{nm}}$ be the subset of $V(\mathbb{Z}_p)$ of nonmaximal cubic forms. It is the closure of $\mathcal{W}_p^{\pm}$ inside $V(\mathbb{Z}_p)$. Given a positive squarefree integer q, we define

$$\begin{aligned}\mathcal{A}_n^{(q)}(\phi) &:= \prod_{p|q}\int_{V(\mathbb{Z}_p)^{\mathrm{nm}}}\phi_{p^\beta}(f)df\cdot\prod_{\substack{p|n\\ p\nmid q}}\mathcal{A}_{p^\beta}(\phi_{p^\beta}),\\ \mathcal{C}_n^{(q)}(\phi) &:= \prod_{p|q}\int_{V(\mathbb{Z}_p)^{\mathrm{nm}}}\phi_{p^\beta}(f)c_p(f)df\cdot\prod_{\substack{p|n\\ p\nmid q}}\mathcal{C}_{p^\beta}(\phi_{p^\beta}),\end{aligned}$$

where df denotes the probability Haar measure on $V(\mathbb{Z}_p)$, and the values of $c_p(f)$ are given in Table 1. When $p|q$ but $p\nmid n$, we assume by convention that $\phi_{p^\beta}=1$ in the integral above. Note that if $q=1$ then $\mathcal{A}_n^{(q)}=\mathcal{A}_n$ and $\mathcal{C}_n^{(q)}=\mathcal{C}_n$, and more generally if $(n,q)=1$, then $\mathcal{A}_n^{(q)}$ is equal to $\mathcal{A}_n$ times the probability that f is nonmaximal at every prime dividing q (with something similar holding for $\mathcal{C}_n^{(q)}$).

Let $\phi = \prod_{p^\beta \| n} \phi_{p^\beta}$ be a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant function on $V(\mathbb{Z}/n\mathbb{Z})$. We say that ϕ is *an S_3-function* if there exist two prime powers $p_1^{\beta_1} \| n$ and $p_2^{\beta_2} \| n$ such that $\phi_{p_1^{\beta_1}}$ is supported on elements with splitting type (3) and $\phi_{p_2^{\beta_2}}$ is supported on elements with splitting type (12). Note that the support of an S_3-function in $V(\mathbb{Z})$ is contained within the set of integral binary cubic forms corresponding to orders in cubic S_3-fields.

Theorem 6.6 *Let $\Psi : \mathbb{R}_{>0} \to \mathbb{C}$ be a smooth function with compact support, let n be a positive integer, let q be a positive squarefree integer, and let $d := (q, n)$. Let $\phi = \prod_{p^\beta \| n} \phi_{p^\beta}$ be a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant S_3-function on $V(\mathbb{Z}/n\mathbb{Z})$, such that ϕ is simple at d and $\phi_p(0) = 0$ for every prime $p \mid d$. Then for every $X \geq 1$,*

$$\begin{aligned}
&\sum_{f \in \overline{\mathcal{W}_q^{\pm}}} \frac{\phi(f)}{|\mathrm{Stab}(f)|} \Psi\Big(\frac{|\Delta(f)|}{X}\Big) \\
&\quad = \alpha^{\pm} \cdot \mathcal{A}_n^{(q)}(\phi) \cdot \widetilde{\Psi}(1) \cdot X + \gamma^{\pm} \cdot \mathcal{C}_n^{(q)}(\phi) \cdot \widetilde{\Psi}(\frac{5}{6}) \cdot X^{5/6} \\
&\qquad + O_\epsilon\Big(d^{1+\epsilon} q^{1+\epsilon} \cdot \big(\frac{n}{d}\big)^{4+\epsilon} E_{n/d}\big(\widehat{\phi_{n/d}}\big) \cdot E_\infty(\widetilde{\Psi}, \epsilon)\Big).
\end{aligned}$$

Remark Note that ϕ being an S_3-function implies that the elements f in the support of ϕ have trivial stabilizer, and similarly below for the elements $f \in \mathcal{W}_q(\Sigma)$. We keep $\mathrm{Stab}(f)$ in the theorem statement and its Corollary 6.7 below as a reminder that the Shintani zeta function counting methods of Sect. 5 are used to estimate the sum.

Proof The values of the constants in front of the primary and secondary main terms follow from Theorem 5.6. The term $\mathcal{B}_n(\phi)$ vanishes because there exists a prime p dividing n such that ϕ_{p^β} is supported on elements in $V(\mathbb{Z}/p^\beta\mathbb{Z})$ with splitting type (3), which implies $\mathcal{B}_{p^\beta}(\phi_{p^\beta}) = 0$ because $\phi_{p^\beta} \cdot b_p$ vanishes on $V(\mathbb{Z}/p^\beta\mathbb{Z})$ in view of Table 1.

It remains to justify the size of the error term. For this, we first use Theorem 6.5 to write

$$\begin{aligned}
&\sum_{f \in \overline{\mathcal{W}_q^{\pm}}} \frac{\phi_n(f)}{|\mathrm{Stab}(f)|} \Psi\Big(\frac{|\Delta(f)|}{X}\Big) \\
&\quad = \phi_d(1^2 1) \sum_{k\ell \mid \frac{q}{d}} \mu(\ell) \sum_{g \in \overline{V(\mathbb{Z})^{\pm}}} \omega_d^{(1)}(g) \omega_{k\ell}(g) \frac{\phi_{n/d}(g)}{|\mathrm{Stab}(g)|} \Psi\Big(\frac{q^4 |\Delta(g)|}{X d^2 k^2}\Big),
\end{aligned}$$

where we have used that the elements g in the support of $\phi_{n/d}$ have trivial stabilizer.

For each k and ℓ, we apply Theorem 5.6 to the inner sum, obtaining

$$\sum_{g \in \overline{V(\mathbb{Z})^{\pm}}} \omega_d^{(1)}(g) \omega_{k\ell}(g) \frac{\phi_{n/d}(g)}{|\mathrm{Stab}(g)|} \Psi\Big(\frac{q^4 |\Delta(g)|}{X d^2 k^2}\Big)$$

$$
\begin{aligned}
&= c_{k,\ell}^{(1)} X + c_{k,\ell}^{(2)} X^{5/6} \\
&\quad + O_\epsilon\Big((nkl)^{4+\epsilon} \cdot E_d(\widehat{\omega_d^{(1)}}) E_{k\ell}(\widehat{\omega_{k\ell}}) E_{n/d}(\widehat{\phi_{n/d}}) E_\infty(\widetilde{\Psi}, \epsilon)\Big) \\
&= c_{k,\ell}^{(1)} X + c_{k,\ell}^{(2)} X^{5/6} \\
&\quad + O_\epsilon\Big(d^{2+\epsilon}(k\ell)^{1+\epsilon} \cdot \Big(\frac{n}{d}\Big)^{4+\epsilon} E_{n/d}(\widehat{\phi_{n/d}}) E_\infty(\widetilde{\Psi}, \epsilon)\Big).
\end{aligned}
$$

The second estimate above follows since we have the bounds

$$
E_d(\widehat{\omega_d^{(1)}}) \ll \frac{1}{d^{2-\epsilon}}, \qquad E_{k\ell}(\widehat{\omega_{k\ell}}) \ll \frac{1}{k^{3-\epsilon}\ell^{3-\epsilon}}.
$$

Indeed, we have $\omega_{k\ell} = \prod_{p|k\ell}(\lambda_p + 1)$ (with a slight modification when the splitting at p is (0)), and then the bounds on the Fourier transform follow from Lemmas 5.7 and 5.8. Summing over $k\ell$ dividing q/d, we obtain

$$
\sum_{k\ell|\frac{q}{d}} d^{2+\epsilon}(k\ell)^{1+\epsilon} \ll (dq)^{1+\epsilon},
$$

which yields the result. □

Recall that for a finite collection Σ of local specifications, we defined a *family of fields* $\mathcal{F}_\Sigma$. The finite collection Σ can also be used to cut out subsets of binary cubic forms. Namely, for a set S of integral binary cubic forms, let $S(\Sigma)$ denote the subset of elements $f \in S$ such that $R_f \otimes \mathbb{Q}_v \in \Sigma_v$ for each place v associated to Σ. Here, as usual, R_f denotes the cubic ring associated to f. Henceforth, we will always assume that Σ_∞ is a singleton set. That is, it is either $\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$, corresponding to cubic fields and forms with positive discriminant, or it is $\mathbb{R}\oplus\mathbb{C}$, corresponding to cubic fields and forms with negative discriminant. Accordingly the sign $\pm$ in $\alpha^\pm$, $\gamma^\pm$, $V(\mathbb{Z})^\pm$, $\mathcal{W}_q^\pm$, and so on, will be $+$ for the former case and $-$ for the latter case.

Let χ_Σ be the characteristic function of the set of elements $(f_p) \in \prod_p V(\mathbb{Z}_p)$ such that $R_{f_p} \otimes \mathbb{Q}_p \in \Sigma_p$ for every prime p. We have that χ_Σ factors through the quotient $\prod_p V(\mathbb{Z}_p) \twoheadrightarrow V(\mathbb{Z}/r_\Sigma\mathbb{Z})$ to a $\mathrm{GL}_2(\mathbb{Z}/r_\Sigma\mathbb{Z})$-invariant function which we also denote by the same letter χ_Σ. Here r_Σ is a positive integer which is the product of p over all primes $p \neq 2, 3$ such that Σ_p is specified at p and of 16 (resp. 27) for the prime 2 (resp. 3).

We say that Σ is an S_3-*collection*, and that $\mathcal{F}_\Sigma$ is an S_3-*family* if there exist two primes p and p' such that $\Sigma_p = \{\mathbb{Q}_{p^3}\}$ and $\Sigma_{p'} = \{\mathbb{Q}_{p'} \times \mathbb{Q}_{p'^2}\}$. Note that when Σ is an S_3-collection, it automatically follows that χ_Σ is an S_3-function.

Corollary 6.7 *Let Σ be a finite S_3-collection of local specifications. For every positive integer n, positive squarefree integer q and $X \geq 1$, we have*

$$\begin{aligned}\sum_{f \in \overline{\mathcal{W}_q(\Sigma)}} \frac{\lambda_n(f)}{|\mathrm{Stab}(f)|} \Psi\Big(\frac{|\Delta(f)|}{X}\Big) \\ = \alpha^{\pm} \mathcal{A}^{(q)}_{[n,r_\Sigma]}(\lambda_n \chi_\Sigma) \widetilde{\Psi}(1) \cdot X + \gamma^{\pm} \mathcal{C}^{(q)}_{[n,r_\Sigma]}(\lambda_n \chi_\Sigma) \cdot \widetilde{\Psi}(\frac{5}{6}) \cdot X^{5/6} \\ + O_{\epsilon,\Sigma}\big((nq)^{1+\epsilon} \cdot E_\infty(\widetilde{\Psi}, \epsilon)\big).\end{aligned} \tag{43}$$

Proof The two main terms of the asymptotic follow from Theorem 6.6, and it is only necessary to analyze the size of the error term. We write $n = n_1 n_2$, where n_1 is squarefree, n_2 is powerful and $(n_1, n_2) = 1$. Let m denote the radical of n_2. Recall that λ_n is defined modulo $n_1 m$, the radical of n. (Indeed, λ_{p^k} only depends on the reduction of f modulo p.) Thus, Theorem 6.6 yields an error term of

$$O_{\epsilon,\Sigma}\Big(\frac{(n_1 m)^{4+\epsilon} q^{1+\epsilon}}{(n,q)^3} \cdot E_{\frac{n}{(n,q)}}\big(\widehat{\lambda_{\frac{n}{(n,q)}}}\big) E_\infty(\widetilde{\Psi}, \epsilon)\Big).$$

For a prime p and integer $k \geq 2$, it follows from Lemma 5.8 that we have

$$E_p(\widehat{\lambda_p}) \ll \frac{1}{p^3}; \qquad E_{p^k}(\widehat{\lambda_{p^k}}) \ll \frac{k}{p^2}.$$

Therefore, we obtain

$$E_{\frac{n}{(n,q)}}\big(\widehat{\lambda_{\frac{n}{(n,q)}}}\big) \ll_\epsilon \frac{(n,q)^3 n^\epsilon}{n_1^3 m^2}.$$

The theorem now follows since $n_1 m^2 \leq n$. □

We end with two results. The first is a uniform estimate, proved in [12], on the number of elements in $\overline{\mathcal{W}_q}$ with bounded discriminant. This estimate will be used to bound the tail of the sum in the right-hand side of (39).

Proposition 6.8 (Davenport [12]) *For every $\epsilon > 0$, $X \geq 1$, and squarefree integer q,*

$$\#\big\{f \in \overline{\mathcal{W}_q^{\pm}} : |\Delta(f)| < X\big\} \ll_\epsilon \frac{X}{q^{2-\epsilon}}.$$

The multiplicative constant is independent of q and X (it depends only on ϵ).

Proof With the notation we have set up, Davenport's proof can be expressed as follows: We use Proposition 6.1 with Ψ the indicator function of the interval $[\frac{1}{2}, X]$. Then, instead of applying Theorem 5.6 as above, we apply the more direct upper bound $\omega_{k\ell}(f) \ll q^\epsilon$ and estimate from above the sum over $f \in \overline{V(\mathbb{Z})^{\pm}}$ by $\frac{Xk^2}{q^4}$. □

Second, we add up the functionals of Theorem 6.6 over squarefree numbers q. Let $\phi : V(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$ be a function of the form $\phi = \prod_{p^\beta \| n} \phi_{p^\beta}$, where $\phi_{p^\beta} : V(\mathbb{Z}/p^\beta\mathbb{Z}) \to \mathbb{C}$ and $\beta = v_p(n)$. For every prime $p \nmid n$, we define $\phi_{p^\beta} : V(\mathbb{Z}_p) \to \mathbb{C}$ to simply be the constant 1 function. We now define the functionals

$$\begin{aligned} \mathcal{A}^{\max}(\phi) &:= \prod_p \int_{f \in V(\mathbb{Z}_p)^{\max}} \phi_{p^\beta}(f) df; \\ \mathcal{C}^{\max}(\phi) &:= \prod_p \int_{f \in V(\mathbb{Z}_p)^{\max}} c_p(f) \phi_{p^\beta}(f) df, \end{aligned}$$

where the values of $c_p(f)$ are given in Table 1. By multilinearity, the domain of definition of the functionals $\mathcal{A}^{\max}$ and $\mathcal{C}^{\max}$ extends to all functions $\phi : V(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$.

Lemma 6.9 *For every integer n, the following identity between functionals defined on functions from $V(\mathbb{Z}/n\mathbb{Z})$ holds*:

$$\begin{aligned} \sum_{q \geq 1} \mu(q) \mathcal{A}_n^{(q)} &= \mathcal{A}^{\max}; \\ \sum_{q \geq 1} \mu(q) \mathcal{C}_n^{(q)} &= \mathcal{C}^{\max}. \end{aligned}$$

Proof This follows from the partition

$$V(\mathbb{Z}_p) = V(\mathbb{Z}_p)^{\max} \sqcup V(\mathbb{Z}_p)^{\mathrm{nm}}$$

for every prime p, and the inclusion-exclusion principle. □

6.3 Application to smooth counts of cubic fields with prescribed local specifications

In this subsection, we use (39), Theorem 6.6, Proposition 6.8 and Lemma 6.9 to sum congruence functions over the space of cubic fields. We denote the set of all cubic fields K with $\pm\Delta(K) > 0$ by $\mathcal{F}^{\pm}$. We say that $\theta : \mathcal{F}^{\pm} \to \mathbb{C}$ is a *simple function defined modulo n* if there exists a simple $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant function $\phi : V(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$ such that for every cubic field K, whose ring of integers corresponds to a maximal binary cubic form f, we have $\theta(K) = \phi(\bar{f})$, where $\bar{f}$ denotes the reduction of f modulo n. For example $\lambda_K(n)$ is a simple function defined modulo n corresponding to the function $\lambda_n(f)$.

Theorem 6.10 *Let $\Psi : \mathbb{R}_{>0} \to \mathbb{C}$ be a smooth function with compact support such that $\int \Psi = 1$. Let Σ be a finite S_3-collection of local specifications. For every real $X \geq 1$ and integer $n \geq 1$, we have*

$$\sum_{K \in \mathcal{F}_\Sigma} \lambda_K(n) \Psi\Big(\frac{|\Delta(K)|}{X}\Big) = \alpha^{\pm} \mathcal{A}^{\max}(\lambda_n \chi_\Sigma) X + \gamma^{\pm} \mathcal{C}^{\max}(\lambda_n \chi_\Sigma) \widetilde{\Psi}\Big(\frac{5}{6}\Big) X^{5/6} + O_{\epsilon,\Sigma,\Psi}\big(X^{2/3+\epsilon} n^{1/3}\big).$$

Before we turn to the proof of Theorem 6.10, we make the following two observations. First, the quadratic analogue of the above result is the question of summing the Legendre symbol $\left(\frac{\cdot}{n}\right)$ over the set of fundamental discriminants (or squarefree integers).

Second, the case $n = 1$ of the above result (with the simplifying assumption that Σ is an S_3-collection) is a smoothed version (instead of a sharp version counting $K \in \mathcal{F}_\Sigma(X)$ without the Ψ-smoothing) of the refined Roberts' conjecture, proved independently in [5] and [34]. Those works obtain the error terms $O_\epsilon(X^{5/6-1/48+\epsilon})$ and $O_\epsilon(X^{7/9+\epsilon})$ for the sharp version of the refined Roberts' conjecture. Independently from the present article, the recent work [6] obtains an improved error term of $O_\epsilon(X^{2/3+\epsilon})$ for the sharp count. This indicates that $X^{2/3+\epsilon}$ is the natural exponent both for our present purposes of smoothly summing the Artin character of cubic fields and for the problem of sharp counting of cubic fields.

Proof of Theorem 6.10 We start with the sieve (39) to write

$$\sum_{K\in\mathcal{F}_\Sigma} \lambda_K(n)\Psi\Big(\frac{|\Delta(K)|}{X}\Big) = \sum_{q\geq 1}\mu(q)\sum_{f\in\overline{\mathcal{W}_q^{\pm}}}\frac{\lambda_n(f)}{|\mathrm{Stab}(f)|}\chi_\Sigma(f)\Psi\Big(\frac{|\Delta(f)|}{X}\Big).$$

Note that the sum over K is *not* weighted by the size of the automorphism group because every $K \in \mathcal{F}_\Sigma$ is an S_3-field, hence $\mathrm{Aut}(K)$ is trivial.

Pick a real number Q to be optimized later. Using Corollary 6.7 for $q \leq Q$, Proposition 6.8 for $q > Q$, and Lemma 6.9 to evaluate the main terms, we obtain

$$\sum_{K\in\mathcal{F}_\Sigma} \lambda_K(n)\Psi\Big(\frac{|\Delta(K)|}{X}\Big) = \alpha^{\pm}\mathcal{A}^{\max}(\lambda_n\chi_\Sigma)X + \gamma^{\pm}\mathcal{C}^{\max}(\lambda_n\chi_\Sigma)\widetilde{\Psi}\big(\frac{5}{6}\big)X^{5/6}$$
$$+ O_{\epsilon,\Sigma,\Psi}\big((nQ^2)^{1+\epsilon}\big) + O_{\epsilon,\Psi}\Big(\frac{X}{Q^{1-\epsilon}}\Big).$$

Optimizing, we pick $Q = (X/n)^{1/3}$ which yields Theorem 6.10. □

Finally, we have a result estimating smoothed sums over cubic fields, where we sum over arbitrary congruence functions defined modulo a squarefree integer. (We could allow more general specifications, but this situation seems to be the most common in applications).

Theorem 6.11 *Let $\Psi : \mathbb{R}_{>0} \to \mathbb{C}$ be a smooth function with compact support such that $\int \Psi = 1$. Let n be a positive squarefree integer, and let θ be a simple function on the family $\mathcal{F}^+$ (resp. $\mathcal{F}^-$) of totally real cubic fields (resp. complex cubic fields) corresponding to a $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$-invariant congruence S_3-function $\phi : V(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$, which is simple at n and such that $\phi_p(0) = 0$ for every prime $p|n$. Namely $\theta(K_f) = \phi(\overline{f})$ for every $f \in V(\mathbb{Z})^{\pm,\mathrm{irr},\max}$. Then we have*

$$\sum_{K\in\mathcal{F}^{\pm}} \theta(K)\Psi\Big(\frac{|\Delta(K)|}{X}\Big) = \alpha^{\pm}\mathcal{A}^{\max}(\phi)X + \gamma^{\pm}\mathcal{C}^{\max}(\phi)\widetilde{\Psi}\big(\frac{5}{6}\big)X^{5/6}$$
$$+ O_{\epsilon,\Psi}\big(X^{2/3+\epsilon}n^{2/3+\epsilon}||\theta||_\infty\big).$$

Proof As before, we begin with the inclusion-exclusion sieve. Pick $Q > 1$ to be optimized and write

$$\sum_{K\in\mathcal{F}^{\pm}} \theta(K)\Psi\Big(\frac{|\Delta(K)|}{X}\Big) = \sum_{q\le Q}\mu(q)\sum_{f\in\overline{\mathcal{W}_q^{\pm}}}\frac{\phi(f)}{|\mathrm{Stab}(f)|}\Psi\Big(\frac{|\Delta(K)|}{X}\Big) + O_{\epsilon,\Psi}\Big(\frac{X}{Q^{1-\epsilon}}\Big) + O\big(X^{1/2}||\theta||_\infty\big).$$

For $q \le Q$, we use Theorem 6.6 to write

$$\sum_{f\in\overline{\mathcal{W}_q^{\pm}}}\frac{\phi(f)}{|\mathrm{Stab}(f)|}\Psi\Big(\frac{|\Delta(K)|}{X}\Big) = \alpha^{\pm}\mathcal{A}_n^{(q)}(\phi)X + \gamma^{\pm}\mathcal{C}_n^{(q)}(\phi)\widetilde{\Psi}\big(\frac{5}{6}\big)X^{5/6} + O_{\epsilon,\Psi}\Big(\frac{n^{4+\epsilon}q^{1+\epsilon}}{(n,q)^3}\cdot E_{\frac{n}{(n,q)}}\big(\widehat{\phi_{\frac{n}{(n,q)}}}\big)\Big).$$

From the definition of the error term $E_{\frac{n}{(n,q)}}$ and Corollary 2.7, we obtain the bound

$$E_{\frac{n}{(n,q)}}\big(\widehat{\phi_{\frac{n}{(n,q)}}}\big) \ll_\epsilon \frac{(n,q)^2}{n^{2-\epsilon}}||\theta||_\infty.$$

Using Lemma 6.9 to evaluate the main term, we therefore obtain

$$\sum_{K\in\mathcal{F}^{\pm}} \theta(K)\Psi\Big(\frac{|\Delta(K)|}{X}\Big) = \alpha^{\pm}\mathcal{A}^{\max}(\phi)X + \gamma^{\pm}\mathcal{C}^{\max}(\phi)\widetilde{\Psi}\big(\frac{5}{6}\big)X^{5/6} + O_{\epsilon,\Psi}\Big(\frac{X}{Q^{1-\epsilon}}\Big) + O_{\epsilon,\Psi}\big(n^{2+\epsilon}Q^{2+\epsilon}||\theta||_\infty\big).$$

Optimizing, we pick $Q = X^{1/3}/n^{2/3}$, which yields the result. □

7 Low-lying zeros of Dedekind zeta functions of cubic fields

We follow the setup of [30, §2.4] and of the previous Sect. 6. Namely, we pick an S_3-collection of local specifications Σ, and let $\mathcal{F}_\Sigma$ be the associated S_3-family of cubic fields. We will simplify notation throughout this section by letting $\ll_\epsilon$ denote $\ll_{\epsilon,\Sigma}$ - all error terms depend on Σ. For every function $\eta : \mathcal{F}_\Sigma \to \mathbb{C}$, we define

$$\mathcal{S}_\Sigma(\eta, X) := \sum_{K\in\mathcal{F}_\Sigma}\eta(K)\Psi\Big(\frac{|\Delta(K)|}{X}\Big)$$

to be the smoothed average of $\eta(K)$ over fields K in $\mathcal{F}_\Sigma$ with discriminant close to X. Note in particular that $\mathcal{S}_\Sigma(1, X)$ denotes a smooth count of elements in $\mathcal{F}_\Sigma$.

For a cubic field K, recall from Proposition 3.1 that the function $L(s, \rho_K)$ is known to be entire and that the Artin conductor of $L(s, \rho_K)$ is equal to $|\Delta(K)|$.

We define the quantity $\mathcal{L}_X$ to be the average value of $\log|\Delta(K)|$ over $K \in \mathcal{F}_\Sigma(X)$, i.e., we define

$$\mathcal{L}_X := \frac{\mathcal{S}_\Sigma(\log|\Delta(K)|, X)}{\mathcal{S}_\Sigma(1, X)}.$$

The Davenport–Heilbronn theorem implies that we have

$$\mathcal{L}_X = \log X + O(1).$$

We write the *nontrivial zeros* of $L(s, \rho_K)$ as $1/2 + i\gamma_K^{(j)}$, where the imaginary part of $\gamma_K^{(j)}$ is bounded in absolute value by $1/2$. We pick $\Phi : \mathbb{R} \to \mathbb{C}$ to be a smooth and even function such that the Fourier transform $\widehat{\Phi} : \mathbb{R} \to \mathbb{C}$ has compact support contained in the open interval $(-a, a)$. It is then known that Φ can be extended to an entire function of exponential type a. Define $Z_K(X)$ by

$$Z_K(X) := \sum_j \Phi\Big(\frac{\gamma_K^{(j)}\mathcal{L}_X}{2\pi}\Big).$$

We work with the following variant of the 1-*level density* $\mathcal{D}(\mathcal{F}_\Sigma(X), \Phi)$ of the family of Artin L-functions $L(s, \rho_K)$ (equivalently, of the family of Dedekind zeta functions $\zeta_K(s)$) of $K \in \mathcal{F}_\Sigma$:

$$\mathcal{D}(\mathcal{F}_\Sigma(X), \Phi) := \frac{\mathcal{S}_\Sigma\big(Z_K(X), X\big)}{\mathcal{S}_\Sigma(1, X)}.$$

Recall that $\theta_K(n)$ was defined in (21) so that the nth Dirichlet coefficient of the logarithmic derivative of $L(s, \rho_K)$ is $\theta_K(n)\Lambda(n)$. We use the explicit formula [30, Prop.2.1] to evaluate $Z_K(X)$:

$$\sum_j \Phi\big(\gamma_K^{(j)}\big) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi(t)\big(\log|\Delta(K)| + O(1)\big)dt - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\theta_K(n)\Lambda(n)}{n^{1/2}}\widehat{\Phi}\Big(\frac{\log n}{2\pi}\Big).$$

It yields $Z_K(X) = Z_K^{(1)}(X) + Z_K^{(2)}(X)$, where

$$\begin{aligned} Z_K^{(1)}(X) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi\Big(\frac{t\mathcal{L}_X}{2\pi}\Big)\big(\log|\Delta(K)| + O(1)\big)dt; \\ Z_K^{(2)}(X) &= -\frac{2}{\mathcal{L}_X}\sum_{n=1}^{\infty}\frac{\theta_K(n)\Lambda(n)}{\sqrt{n}}\widehat{\Phi}\Big(\frac{\log n}{\mathcal{L}_X}\Big). \end{aligned}$$

A computation identical to [30, Eq. (17)] gives

$$\lim_{X\to\infty}\frac{\mathcal{S}_\Sigma\big(Z_K^{(1)}(X), X\big)}{\mathcal{S}_\Sigma(1, X)} = \widehat{\Phi}(0). \tag{44}$$

To compute the 1-level density, we need to compute the asymptotics of $\mathcal{S}_\Sigma(Z_K^{(2)}(X), X)$. We write

$$\begin{aligned}\mathcal{S}_\Sigma(Z_K^{(2)}(X), X) &= -\frac{2}{\mathcal{L}_X}\mathcal{S}_\Sigma\Big(\sum_{n=1}^{\infty}\frac{\theta_K(n)\Lambda(n)}{\sqrt{n}}\widehat{\Phi}\Big(\frac{\log n}{\mathcal{L}_X}\Big), X\Big)\\ &= -\frac{2}{\mathcal{L}_X}\sum_{p,m}\frac{\log p}{p^{m/2}}\widehat{\Phi}\Big(\frac{m\log p}{\mathcal{L}_X}\Big)\mathcal{S}_\Sigma\big(\theta_K(p^m), X\big).\end{aligned} \tag{45}$$

We now have the following result estimating the ratios $\mathcal{S}_\Sigma\big(\theta_K(p^m), X\big)/\mathcal{S}_\Sigma(1, X)$.

Proposition 7.1 *Let p be a prime number, and let $X \geq 1$ be a real number. Then, for integers $m \geq 3$, we have*

$$\begin{aligned}\frac{\mathcal{S}_\Sigma(\theta_K(p), X)}{\mathcal{S}_\Sigma(1, X)} &\ll_\epsilon \frac{1}{p} + \frac{1}{p^{1/3}X^{1/6}} + \frac{p^{1/3}}{X^{1/3-\epsilon}};\\ \frac{\mathcal{S}_\Sigma(\theta_K(p^2), X)}{\mathcal{S}_\Sigma(1, X)} - 1 &\ll_\epsilon \frac{1}{p^2} + \frac{1}{X^{1/6}} + \frac{p^{2/3}}{X^{1/3-\epsilon}};\\ \frac{\mathcal{S}_\Sigma(\theta_K(p^m), X)}{\mathcal{S}_\Sigma(1, X)} &\ll 1.\end{aligned} \tag{46}$$

Proof From Lemma 3.7 we have that $\theta_K(p) = \lambda_K(p)$. The left-hand side of the first line of (46) can be computed from Theorem 6.10, yielding

$$\frac{\mathcal{S}_\Sigma(\theta_K(p), X)}{\mathcal{S}_\Sigma(1, X)} \ll_\epsilon \mathcal{A}^{\max}(\lambda_p\chi_\Sigma) + X^{-1/6}\mathcal{C}^{\max}(\lambda_p\chi_\Sigma) + X^{-1/3+\epsilon}p^{1/3}.$$

Note that the first summand in the right-hand side is bounded by $O(\widehat{u_p\cdot\lambda_p}(0))$, where u_p (defined in Sect. 3) is the characteristic function of the set of elements in $V(\mathbb{Z}/p^2\mathbb{Z})$ that lift to binary cubic forms which are maximal at p. The required bound then follows from the first part of Proposition 3.9. Similarly, the second term is bounded by $O(X^{-1/6}\widehat{u_p\cdot c_p\lambda_p}(0))$. We prove in Lemma 5.9 that $\widehat{c_p\lambda_p}(0) \ll p^{-1/3}$. The same bound holds for $\widehat{u_p\cdot c_p\lambda_p}(0)$ since u_p differs from 1 only at a density $1/p^2$ subset.

The proof of the second inequality is similar: this time, we use Theorem 6.11 to deduce the estimate

$$\frac{\mathcal{S}_\Sigma(\theta_K(p^2), X)}{\mathcal{S}_\Sigma(1, X)} - 1 \ll_\epsilon \mathcal{A}^{\max}((\theta_{p^2}-1)\chi_\Sigma) + X^{-1/6}\mathcal{C}^{\max}(\theta_{p^2}\chi_\Sigma) + X^{-1/3+\epsilon}p^{2/3}.$$

The third part of Proposition 3.9 implies the required bound on the first summand on the right-hand side above, while the required bound on the second summand follows immediately since θ_{p^2} is absolutely bounded. Finally, Lemma 3.7 states that $|\theta_K(p^m)| \leq 2$, from which the third inequality follows immediately. □

We are now ready to prove the main result of this section.

Proof of Theorem 5 From (45) and Proposition 7.1, we obtain

$$\begin{aligned}
&-\frac{\mathcal{S}_\Sigma(Z_K^{(2)}(X),X)}{\mathcal{S}_\Sigma(1,X)}\\
&=\frac{2}{\mathcal{L}_X}\sum_p\frac{\log p}{p}\widehat{\Phi}\Big(\frac{2\log p}{\mathcal{L}_X}\Big)\frac{\mathcal{S}_\Sigma(\theta_K(p^2),X)}{\mathcal{S}_\Sigma(1,X)}\\
&\quad+O\Big(\frac{1}{\log X}\sum_{\substack{p^m\ll X^a\\ m\neq 2}}\frac{\log p}{p^{m/2}}\frac{\mathcal{S}_\Sigma(\theta_K(p^m),X)}{\mathcal{S}_\Sigma(1,X)}\Big)\\
&=\frac{2}{\mathcal{L}_X}\sum_p\frac{\log p}{p}\widehat{\Phi}\Big(\frac{2\log p}{\mathcal{L}_X}\Big)+O_\epsilon\Big(\frac{1}{\log X}+X^{\frac{a-1}{6}}+X^{\frac{5a-2}{6}+\epsilon}\Big)\\
&\quad+O_\epsilon\Big(\frac{1}{\log X}+X^{-\frac{1}{6}+\epsilon}+X^{\frac{a-1}{3}+\epsilon}\Big)+O\Big(\frac{1}{\log X}\Big),
\end{aligned}$$

where the three error terms respectively arise from the three estimates of Proposition 7.1. Assuming that $a<\frac{2}{5}$, and using the above computation in conjunction with (44), gives

$$\begin{aligned}
\lim_{X\to\infty}\mathcal{D}(\mathcal{F}_\Sigma(X),\Phi) &= \widehat{\Phi}(0)-\lim_{X\to\infty}\frac{2}{\mathcal{L}_X}\sum_p\frac{\log p}{p}\widehat{\Phi}\Big(\frac{2\log p}{\mathcal{L}_X}\Big)\\
&= \widehat{\Phi}(0)-\frac{1}{2}\int_{-1}^{1}\widehat{\Phi}(t)dt,
\end{aligned}$$

where the final equality follows from the prime number theorem. This concludes the proof of Theorem 5. □

8 Main term for the average central values

Let $\Sigma=(\Sigma_v)$ be a finite collection of local specifications. Without loss of generality we assume that Σ_∞ is a singleton set, which is to say that either the cubic fields prescribed by $\Sigma_\infty=\{\mathbb{R}\times\mathbb{R}\times\mathbb{R}\}$ are all totally real, or the cubic fields prescribed by $\Sigma_\infty=\{\mathbb{R}\times\mathbb{C}\}$ are all complex. We also assume (by imposing additional constraints at two different primes, if necessary) that Σ is an S_3-collection. Let $\mathcal{F}_\Sigma$ denote the family of cubic fields K prescribed by the set Σ of specifications, namely such that for each place v we have $K\otimes_{\mathbb{Q}}\mathbb{Q}_v\in\Sigma_v$.

We let $V(\mathbb{Z})(\Sigma)$ denote the set of elements $f\in V(\mathbb{Z})$ such that $\chi_\Sigma(f)=1$ and such that $\Delta(f)>0$ if $\Sigma_\infty=\{\mathbb{R}\times\mathbb{R}\times\mathbb{R}\}$ (resp. $\Delta(f)<0$ if $\Sigma_\infty=\{\mathbb{R}\times\mathbb{C}\}$). For each prime p, let $\mathcal{W}_p(\Sigma)$ denote the set of elements in $V(\mathbb{Z})(\Sigma)$ that are nonmaximal at p. If q is a squarefree positive integer, we set $\mathcal{W}_q(\Sigma)=\cap_{p|q}\mathcal{W}_p(\Sigma)$. In particular $\mathcal{W}_1(\Sigma)=V(\mathbb{Z})(\Sigma)$.

Since we have assumed that Σ is an S_3-collection, it follows that every form $f\in V(\mathbb{Z})(\Sigma)$ corresponds to an order in an S_3-field K_f, in particular $V(\mathbb{Z})(\Sigma)\subset$

$V(\mathbb{Z})^{\mathrm{irr}}$. This implies that the set $\overline{V(\mathbb{Z})(\Sigma)^{\max}}$ of $\mathrm{GL}_2(\mathbb{Z})$-orbits parametrizes under the Delone–Faddeev correspondence the S_3-family $\mathcal{F}_\Sigma$ of cubic fields prescribed by the finite collection Σ of specifications.

Let $\Psi : \mathbb{R}_{>0} \to \mathbb{C}$ be a smooth function of compact support with $\int \Psi = 1$.

For a real number $X \geq 1$, the inclusion-exclusion principle in conjunction with Proposition 4.3 yields:

$$\begin{aligned} A_\Sigma(X) &:= \sum_{K \in \mathcal{F}_\Sigma} L(\tfrac{1}{2}, \rho_K)\Psi\Big(\frac{|\Delta(K)|}{X}\Big) \\ &= 2\sum_{q \geq 1} \mu(q) \sum_{f \in \overline{\mathcal{W}_q(\Sigma)}} S(f)\Psi\Big(\frac{|\Delta(f)|}{X}\Big), \end{aligned} \tag{47}$$

where $S(f)$ was defined in (29) to be

$$S(f) = \sum_{n=1}^{\infty} \frac{\lambda_n(f)}{n^{1/2}} V^{\pm}\Big(\frac{n}{\sqrt{|\Delta(f)|}}\Big), \tag{48}$$

with $V^{\pm}$ as in Proposition 4.3 and where the sign is $+$ if $\Sigma_\infty = \{\mathbb{R} \times \mathbb{R} \times \mathbb{R}\}$ and $-$ if $\Sigma_\infty = \{\mathbb{R} \times \mathbb{C}\}$. The identity holds because for a maximal irreducible binary cubic form $f \in V(\mathbb{Z})^{\mathrm{irr,max}}$ corresponding to the ring of integers R_f of a cubic field K_f, we have $2S(f) = L(\frac{1}{2}, \rho_{K_f})$ by Corollary 3.5 and Proposition 4.3.

In this section, we will prove two results. First, we will prove an upper bound on $A_\Sigma(X)$, which improves on the pointwise bound coming from summing the best known upper bounds on $|L(\frac{1}{2}, \rho_K)|$ over the associated fields K. Second, assuming a sufficiently strong upper bound on $|L(\frac{1}{2}, \rho_K)|$, we obtain asymptotics for $A_\Sigma(X)$.

8.1 Asymptotics for the terms with $q < Q$

For $Q \in \mathbb{R}_{\geq 1}$ to be chosen later, we split the right-hand side of (47) into two parts,

$$\sum_{q < Q} \quad \text{and} \quad \sum_{q \geq Q}.$$

This section is concerned with the first part:

$$2\sum_{q < Q} \mu(q) \sum_{f \in \overline{\mathcal{W}_q(\Sigma)}} \sum_{n=1}^{\infty} \frac{\lambda_n(f)}{n^{1/2}} \Psi\Big(\frac{|\Delta(f)|}{X}\Big) V^{\pm}\Big(\frac{n}{\sqrt{|\Delta(f)|}}\Big). \tag{49}$$

It will be convenient for us to set some notation surrounding the smooth functions above and their Mellin transforms. For any positive real number $y \in \mathbb{R}_{>0}$, let $\mathcal{H}_y : \mathbb{R}_{>0} \to \mathbb{C}$ denote the compactly supported function

$$\mathcal{H}_y(t) := \Psi(t) \cdot V^{\pm}\Big(\frac{y}{\sqrt{t}}\Big). \tag{50}$$

The relevance of $\mathcal{H}_y(t)$ is that we have the equality

$$\Psi\Big(\frac{|\Delta(f)|}{X}\Big)V^{\pm}\Big(\frac{n}{\sqrt{|\Delta(f)|}}\Big) = \mathcal{H}_{\frac{n}{\sqrt{X}}}\Big(\frac{|\Delta(f)|}{X}\Big).$$

Lemma 8.1 (*i*) *There exists a constant $C > 0$ depending only on Ψ such that for every $\epsilon \in [-1, 1]$ and $y \in \mathbb{R}_{>0}$,*

$$E_\infty(\widetilde{\mathcal{H}_y}; \epsilon) = \int_{-\infty}^{\infty} |\widetilde{\mathcal{H}_y}(-\epsilon + ir)|(1 + |r|)^{2+4\epsilon} dr \le C.$$

(*ii*) *There exists a constant $C > 0$ depending only on Ψ such that for every $y \in \mathbb{R}_{>0}$, $|\widetilde{\mathcal{H}_y}(\frac{5}{6})| \le C$.*

Proof We have by definition (26),

$$\widetilde{V^{\pm}}(s) = \frac{G(s)}{s}\frac{\gamma^{\pm}(\frac{1}{2}+s)}{\gamma^{\pm}(\frac{1}{2})}.$$

We deduce that the Mellin transform of $t \mapsto V^{\pm}(\frac{y}{\sqrt{t}})$ is equal to

$$2y^{2s}\widetilde{V^{\pm}}(-2s) = -y^{2s}\frac{G(-2s)}{s}\frac{\gamma^{\pm}(\frac{1}{2}-2s)}{\gamma^{\pm}(\frac{1}{2})}.$$

Since $\mathcal{H}_y$ is the product of the two functions Ψ and $t \mapsto V^{\pm}(\frac{y}{\sqrt{t}})$, its Mellin transform is the convolution of the Mellin transforms of the respective functions:

$$\widetilde{\mathcal{H}_y}(\sigma + ir) = 2\int_{\Re(u)=\eta} \widetilde{\Psi}(\sigma + ir + u)y^{-2u}\widetilde{V^{\pm}}(2u)\frac{du}{2\pi i}, \tag{51}$$

where $0 < \eta < \frac{1}{2}$ is fixed. Indeed, to establish (51) it suffices to compute the inverse Mellin transform of the right-hand side with a translation of the integration of the v-variable:

$$\begin{aligned} &2\int_{\Re(v)=0} t^{-v}\int_{\Re(u)=\eta} \widetilde{\Psi}(v+u)y^{-2u}\widetilde{V^{\pm}}(2u)\frac{du}{2\pi i}\frac{dv}{2\pi i} \\ &= \int_{\Re(v)=\eta} t^{-v}\widetilde{\Psi}(v)\frac{dv}{2\pi i}\int_{\Re(u)=\eta} 2t^u y^{-2u}\widetilde{V^{\pm}}(2u)\frac{du}{2\pi i} \\ &= \Psi(t)V^{\pm}\Big(\frac{y}{\sqrt{t}}\Big) = \mathcal{H}_y(t), \end{aligned}$$

which coincides with the inverse Mellin transform of the left-hand side.

We deduce from (51) the following inequality:

$$|\widetilde{\mathcal{H}_y}(\sigma + ir)| \le \frac{y^{-2\eta}}{\pi}\int_{-\infty}^{\infty} |\widetilde{\Psi}(\sigma + ir + \eta + i\tau)| \cdot |\widetilde{V^{\pm}}(2\eta + 2i\tau)|d\tau.$$

We shall use this inequality for $y \in [1, +\infty)$, in which case $y^{-2\eta} \le 1$.

On the other hand, if we shift the contour of (51) to $\Re(u) = -\eta$, picking up a simple pole at $u = 0$ of $\widetilde{V^{\pm}}(2u)$, we then obtain the following inequality:

$$|\widetilde{\mathcal{H}_y}(\sigma + ir)| \leq \frac{y^{2\eta}}{\pi} \int_{-\infty}^{\infty} |\widetilde{\Psi}(\sigma + ir - \eta + i\tau)| \cdot |\widetilde{V^{\pm}}(-2\eta + 2i\tau)| d\tau + |\widetilde{\Psi}(\sigma + ir)| \cdot |G(0)|.$$

We shall use this inequality for the other interval $y \in (0, 1]$, in which case $y^{2\eta} \leq 1$.

Assertion (ii) follows immediately by inserting $\sigma = \frac{5}{6}$ and $r = 0$. Assertion (i) follows by inserting $\sigma = -\epsilon$ and integrating over r because $E_{\infty}(\widetilde{\mathcal{H}_y}; \epsilon)$ for $y \geq 1$ is bounded by

$$\frac{1}{\pi} \int_{-\infty}^{\infty} |\widetilde{\Psi}(-\epsilon + \eta + ir)|(1 + |r|)^{2+4\epsilon} dr \cdot \int_{-\infty}^{\infty} |\widetilde{V^{\pm}}(2\eta + 2i\tau)|(1 + |\tau|)^{2+4\epsilon} d\tau \leq C,$$

where C depends only on Ψ. The estimate for $y \leq 1$ is similar. □

We are now ready to prove the main result of this subsection.

Proposition 8.2 *For every $\epsilon > 0$ and $Q, X \geq 1$, the sum (49) is asymptotic to*

$$C_{\Sigma} \cdot X \cdot \big(\log X + \widetilde{\Psi}'(1)\big) + C'_{\Sigma} \cdot X + O_{\epsilon,\Sigma,\Psi}\Big(\frac{X^{1+\epsilon}}{Q} + X^{11/12+\epsilon} + Q^{2+\epsilon} X^{3/4+\epsilon}\Big),$$

where $C_{\Sigma} > 0$ and $C'_{\Sigma} \in \mathbb{R}$ depend only on the finite collection Σ of local specifications.

Proof Since $V^{\pm}$ is a function rapidly decaying at infinity, we may truncate the n-sum in the definition of $S(f)$ to $n < X^{1/2+\epsilon}$ with negligible error term. To estimate (49) we switch order of summation and consider

$$2 \sum_{n < X^{1/2+\epsilon}} \sum_{q < Q} \mu(q) \sum_{f \in \overline{\mathcal{W}_q(\Sigma)}} \frac{\lambda_n(f)}{n^{1/2}} \mathcal{H}_{\frac{n}{\sqrt{X}}}\Big(\frac{|\Delta(f)|}{X}\Big).$$

We may then use Corollary 6.7, to estimate the inner sum over f:

$$\begin{aligned} &2 \sum_{n < X^{1/2+\epsilon}} \frac{1}{\sqrt{n}} \sum_{q < Q} \mu(q) \\ &\quad \times \left(\alpha^{\pm} \mathcal{A}^{(q)}_{[n,r_{\Sigma}]}(\lambda_n \chi_{\Sigma}) \cdot \widetilde{\mathcal{H}_{\frac{n}{\sqrt{X}}}}(1) X + \gamma^{\pm} \mathcal{C}^{(q)}_{[n,r_{\Sigma}]}(\lambda_n \chi_{\Sigma}) \cdot \widetilde{\mathcal{H}_{\frac{n}{\sqrt{X}}}}(\tfrac{5}{6}) \cdot X^{5/6}\right) \quad (52) \\ &\quad + O_{\epsilon,\Sigma,\Psi}\Big(\sum_{n < X^{1/2+\epsilon}} \frac{1}{\sqrt{n}} \sum_{q < Q} (nq)^{1+\epsilon} \cdot E_{\infty}(\widetilde{\mathcal{H}_{\frac{n}{\sqrt{X}}}}, \epsilon)\Big). \end{aligned}$$

The error term above is seen to be bounded by $O_{\epsilon,\Sigma,\Psi}(Q^{2+\epsilon} X^{3/4+\epsilon})$ thanks to Lemma 8.1.

Next, we bound the secondary term in (52). Since r_Σ is fixed, the contribution to $\mathcal{C}^{(q)}_{[n,r_\Sigma]}(\lambda_n\chi_\Sigma)$ from primes $p \mid r_\Sigma$ is bounded. Therefore, we consider without further mention in the remainder of this paragraph only the primes $p \nmid r_\Sigma$. We begin with the primes p dividing q. The contribution to $\mathcal{C}^{(q)}_{[n,r_\Sigma]}(\lambda_n\chi_\Sigma)$ from primes $p \mid q$ and $p \nmid n$ is given in [34, Thm.2.2] and [33, Cor.8.15] to be $O(p^{-5/3})$. (Note that our quantity $\mathcal{C}^{(p)}_1(1)$ defined in §6.2 corresponds to the quantity denoted $\mathcal{C}_{p^2}(\Phi_p, 1)$ in [33].) The contribution to $\mathcal{C}^{(q)}_{[n,r_\Sigma]}(\lambda_n\chi_\Sigma)$ from primes $p \mid q$ and $p \mid n$ is estimated from [33, Prop.8.16] to also be $O(p^{-5/3})$. (If $a = (1^2 1_*)$, then $\mathcal{C}_{p^2}(a, 1) \asymp p^{1/3}$, and the cardinality of the orbit $\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z}) \cdot a$ inside $V(\mathbb{Z}/p^2\mathbb{Z})$ is equal to $p^4(p^2-1)$ by [33, Lem.5.6], which yields $p^{1/3}p^6/p^8 = p^{-5/3}$, whereas the other nonmaximal types $a = (1^3_*)$, (1^3_{**}), (0) have a smaller contribution.)

We turn to the primes p not dividing q. The contribution to $\mathcal{C}^{(q)}_{[n,r_\Sigma]}(\lambda_n\chi_\Sigma)$ from primes $p \nmid q$ and $p \nmid n$ is a convergent infinite product that is uniformly bounded. The contribution to $\mathcal{C}^{(q)}_{[n,r_\Sigma]}(\lambda_n\chi_\Sigma)$ from primes $p \nmid q$ and $p \parallel n$ is computed from (23) and Table 1 to be $O(p^{-1/3})$ (see also Lemma 5.9). The contribution to $\mathcal{C}^{(q)}_{[n,r_\Sigma]}(\lambda_n\chi_\Sigma)$ from primes $p \nmid q$ and $p^2 \mid n$ is bounded by $O_\epsilon(n^\epsilon)$ since c_p is absolutely bounded and $|\lambda_n| \ll_\epsilon n^\epsilon$. Therefore, letting $n_1 := \prod_{p\|n} p$ and writing $n = n_1 n_2$, we see that the secondary term in (52) is $\ll_{\epsilon,\Sigma,\Psi}$

$$X^{\frac{5}{6}} \sum_{n<X^{1/2+\epsilon}} \frac{1}{\sqrt{n}} \sum_{q<Q} \frac{(n,q)^{1/3}}{q^{5/3-\epsilon} n_1^{1/3-\epsilon}} \ll_{\epsilon,\Sigma,\Psi} X^{\frac{5}{6}+\epsilon} \sum_{\substack{n_1<X^{1/2+\epsilon} \\ |\mu(n_1)|=1}} \frac{1}{n_1^{5/6}} \sum_{n_2<X^{1/2+\epsilon}} \frac{1}{\sqrt{n_2}}$$

$$\ll_{\epsilon,\Sigma,\Psi} X^{\frac{11}{12}+\epsilon},$$

where the final estimate follows since the inner sum is over powerful integers n_2 and hence is $\ll_\epsilon X^\epsilon$.

Finally, to express the first main term in a more convenient form, we define the function $g(y)$ to be

$$g(y) := \widetilde{\mathcal{H}_y}(1) = \int_0^\infty \mathcal{H}_y(t)dt. \tag{53}$$

From Lemma 6.9, we see that for a fixed n, we have

$$\begin{aligned} \sum_{q<Q} \mu(q)\mathcal{A}^{(q)}_{[n,r_\Sigma]}(\lambda_n\chi_\Sigma) &= \mathcal{A}^{\max}(\lambda_n\chi_\Sigma) + O\Big(\sum_{q\geq Q} \mathcal{A}^{(q)}_{[n,r_\Sigma]}(\lambda_n\chi_\Sigma)\Big) \\ &= \mathcal{A}^{\max}(\lambda_n\chi_\Sigma) + O_\epsilon\Big(\sum_{q\geq Q} \frac{(n_1,q)}{q^2 n_1^{1-\epsilon}}\Big), \end{aligned}$$

where as before $n_1 := \prod_{p\|n} p$. We omit the details of the bound on $\mathcal{A}^{(q)}_{[n,r_\Sigma]}(\lambda_n\chi_\Sigma)$, since it is similar (and simpler) to the bound on $\mathcal{C}^{(q)}_{[n,r_\Sigma]}(\lambda_n\chi_\Sigma)$. Thus, writing $n =$

$n_1 n_2$, the first term in (52) is equal to

$$2\alpha^{\pm} \cdot X \cdot \sum_{n<X^{1/2+\epsilon}} \frac{g(\frac{n}{\sqrt{X}})}{\sqrt{n}} \mathcal{A}^{\max}(\lambda_n \chi_\Sigma) + O_{\epsilon,\Sigma,\Psi}\Big(\sum_{n<X^{1/2+\epsilon}} \frac{X}{Q n_1^{3/2-\epsilon} n_2^{1/2}} \Big).$$

The result now follows with the values of the constants being

$$C_\Sigma := \alpha^{\pm} \mathrm{Res}_{s=\frac{1}{2}} T_\Sigma(s), \quad C'_\Sigma := 2\alpha^{\pm} C', \tag{54}$$

as is shown in Proposition 8.5 below, and where T_Σ is defined in (55). □

8.2 Computing the leading constants

We compute the constants C_Σ, C'_Σ arising in Proposition 8.2. We begin with the following lemma.

Lemma 8.3 *The Mellin transform of the function g in (53) is*

$$\widetilde{g}(s) = \widetilde{\Psi}(1+s/2) \frac{G(s)}{s} \frac{\gamma^{\pm}(1/2+s)}{\gamma^{\pm}(1/2)},$$

where G is as in (26). In particular, $\widetilde{g}(s)$ is meromorphic on the half-plane $\Re(s) > -1/2$ with only a simple pole at $s = 0$.

Proof Unwinding definitions (50) and (53), we see that

$$\begin{aligned} \widetilde{g}(s) &= \int_0^\infty \Psi(t) \int_0^\infty V^{\pm}\Big(\frac{y}{\sqrt{t}}\Big) y^s \frac{dy}{y} dt \\ &= \int_0^\infty t^{s/2+1} \Psi(t) \frac{dt}{t} \int_0^\infty V^{\pm}(u) u^s \frac{du}{u} \\ &= \widetilde{\Psi}(1+s/2) \widetilde{V^{\pm}}(s). \end{aligned}$$

The lemma follows from the expression (32) for $\widetilde{V^{\pm}}(s)$. □

Define the Dirichlet series

$$T_\Sigma(s) := \sum_{n=1}^{\infty} \frac{t_\Sigma(n)}{n^s}, \tag{55}$$

where $t_\Sigma(n) = \mathcal{A}^{\max}(\lambda_n \chi_\Sigma)$ is the average of $\lambda_K(n)$ over K in $\mathcal{F}_\Sigma$ (note that this is actually a finite average, since the value of $\lambda_K(n)$ is determined by the splitting type of K at the primes dividing n.)

Proposition 8.4 *The Dirichlet series $T_\Sigma(s)$ has a meromorphic continuation to the half-plane $\Re(s) > 1/3$ with a simple pole at $s = \frac{1}{2}$. Moreover, this simple pole has a positive residue.*

Proof For every integer $n \geq 1$, we have

$$t_\Sigma(n) = \prod_{p^k \| n} \sum_\sigma \frac{\lambda_{p^k}(\sigma)}{\#\mathcal{O}_\sigma},$$

where $\mathcal{O}_\sigma \subset V(\mathbb{F}_p)$ is the $\mathrm{GL}_2(\mathbb{F}_p)$-orbit attached to σ, and σ ranges over all splitting types that are compatible with Σ_p. The quantity $t_\Sigma(n)$ is clearly multiplicative, and so $T_\Sigma(s)$ has an Euler product decomposition

$$T_\Sigma(s) := \prod_p \sum_{k=0}^\infty \frac{t_\Sigma(p^k)}{p^{ks}}.$$

If $p \neq 3$ and there is no specification Σ_p at p, then Proposition 3.9 asserts that $t_\Sigma(p) = \frac{(p-1)(p^2-1)}{p^4}$ and that $t_\Sigma(p^2) = \frac{(p^2-1)^2}{p^4}$. Therefore, the Dirichlet series $T_\Sigma(s)\zeta(2s)^{-1}$ converges absolutely for $\Re(s) > 1/3$.

It follows that the residue at $s = \frac{1}{2}$ is given by the following convergent product

$$\mathrm{Res}_{s=\frac{1}{2}} T_\Sigma(s) = \frac{1}{2} \prod_p (1 - p^{-1}) \sum_{k=0}^\infty \frac{t_\Sigma(p^k)}{p^{k/2}}.$$

We claim that each factor in the product is positive:

$$\sum_{k=0}^\infty \frac{t_\Sigma(p^k)}{p^{k/2}} > 0 \quad \text{for every prime } p.$$

Indeed, $\lambda_{p^m}(f)$ is only negative if $\sigma_p(f) = (3)$ and $m \equiv 1 \pmod 3$, in which case $\lambda_{p^m}(f) = -1$. Therefore, the minimum possible value of $\sum_{k=0}^\infty \frac{t_\Sigma(p^k)}{p^{k/2}}$ occurs when $\Sigma_p = \{(3)\}$. In this case

$$\sum_{k=0}^\infty \frac{t_\Sigma(p^k)}{p^{k/2}} = \sum_{k \equiv 0 \pmod 3} \frac{1}{p^{k/2}} - \sum_{k \equiv 1 \pmod 3} \frac{1}{p^{k/2}}, \tag{56}$$

which is clearly positive since the nth term of the sum on the left is greater than the nth term of the sum on the right. □

Proposition 8.5 *As $X \to \infty$, we have the asymptotic*

$$\sum_{n=1}^\infty \frac{t_\Sigma(n)}{\sqrt{n}} g\left(\frac{n}{\sqrt{X}}\right) = \frac{1}{2} \mathrm{Res}_{s=\frac{1}{2}} T_\Sigma(s) \cdot \left(\log X + \widetilde{\Psi}'(1)\right) + C' + O_{\epsilon,\Sigma,\Psi}\left(X^{-\frac{1}{12}+\frac{\epsilon}{2}}\right),$$

where

$$C' := \frac{d}{ds}\Big|_{s=0} s T_\Sigma(\tfrac{1}{2}+s) \frac{\gamma^\pm\left(\frac{1}{2}+s\right)}{\gamma^\pm(1/2)}.$$

Proof From Lemma 8.3, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{t_\Sigma(n)}{\sqrt{n}} g\left(\frac{n}{\sqrt{X}}\right) \\
&\quad = \frac{1}{2\pi i} \int_{\Re(s)=2} T_\Sigma\left(\tfrac{1}{2}+s\right) \widetilde{g}(s) X^{s/2} ds \\
&\quad = \frac{1}{2\pi i} \int_{\Re(s)=2} s T_\Sigma\left(\tfrac{1}{2}+s\right) \widetilde{\Psi}(1+s/2) G(s) \frac{\gamma^{\pm}\left(\frac{1}{2}+s\right)}{\gamma^{\pm}(1/2)} X^{s/2} \frac{ds}{s^2} \\
&\quad = \frac{1}{2\pi i} \int_{\Re(s)=2} J(s) X^{s/2} \frac{ds}{s^2},
\end{aligned} \tag{57}$$

where the above equation serves as a definition of $J(s)$.

Since $\widetilde{\Psi}(1) = G(0) = 1$, it follows that $J(s)$ is holomorphic in $\Re(s) > -\frac{1}{6}$, and $J(0) = \mathrm{Res}_{s=\frac{1}{2}} T_\Sigma(s)$. Expanding in Taylor series, we write

$$J(s) X^{s/2} = J(0) + \Big(\frac{J(0)\log X}{2} + J'(0)\Big)s + \cdots$$

Shifting the integral to $\Re(s) = -\frac{1}{6} + \epsilon$ for some $0 < \epsilon < \frac{1}{6}$, we therefore obtain

$$\sum_{n=1}^{\infty} \frac{t_\Sigma(n)}{\sqrt{n}} g\left(\frac{n}{\sqrt{X}}\right) = \frac{1}{2}\mathrm{Res}_{s=1/2} T_\Sigma(s) \cdot \log X + J'(0) + O_{\epsilon,\Sigma,\Psi}\big(X^{-\frac{1}{12}+\frac{\epsilon}{2}}\big).$$

Calculating $J'(0)$, we obtain, using that $G(s)$ is even:

$$J'(0) = \frac{1}{2}\mathrm{Res}_{s=\frac{1}{2}} T_\Sigma(s) \cdot \widetilde{\Psi}'(1) + C'.$$

This concludes the proof of the proposition. □

8.3 Upper bound for the first moment

In this subsection we investigate pointwise bounds for the tail of the sieve when $q \geq Q$.

Proposition 8.6 *For every $Q, X \geq 1$ and $\epsilon > 0$,*

$$\sum_{q \geq Q} \sum_{\substack{f \in \overline{\mathcal{W}_q^{\mathrm{irr}}} \\ |\Delta(f)| < X}} |S(f)| = O_\epsilon\Big(\frac{X^{5/4-\delta+\epsilon}}{Q^{3/2-2\delta}}\Big),$$

for $\delta = \frac{1}{128}$ as in Theorem 4.1.

Proof Let $f \in V(\mathbb{Z})^{\mathrm{irr}}$ be an irreducible binary cubic form, and denote the field of fractions of the ring associated to f by K_f.

Note that for $f \in \mathcal{W}_q$ with $|\Delta(f)| < X$, we have $|\Delta(K_f)| < X/q^2$, and recall from Proposition 6.8 that

$$\#\big\{f \in \overline{\mathcal{W}_q} : |\Delta(f)| < X\big\} \ll_\epsilon \frac{X}{q^{2-\epsilon}}.$$

Therefore, we deduce from (30) the estimate

$$\sum_{q \geq Q} \sum_{\substack{f \in \overline{\mathcal{W}_q^{\mathrm{irr}}} \\ |\Delta(f)| < X}} |S(f)| \ll_\epsilon \sum_{q \geq Q} (X/q^2)^{\theta+\epsilon} \cdot X/q^{2-\epsilon},$$

where we recall that $\theta = 1/4 - \delta$. The result follows. □

Optimizing, we pick $Q = X^{\frac{1-2\delta}{7-4\delta}}$ in (49). We have now established the following by combining the two Propositions 8.2 and 8.6.

Theorem 8.7 *For every $X \geq 1$ and $\epsilon > 0$,*

$$A_\Sigma(X) \ll_{\epsilon,\Sigma,\Psi} X^{\frac{29-28\delta}{28-16\delta}+\epsilon}. \tag{58}$$

Numerically,

$$\frac{29-28\delta}{28-16\delta} = \frac{921}{892} = 1.0325\ldots$$

for the best known value of $\delta = \frac{1}{128}$ of Theorem 4.1.

The exponent is smaller than $5/4 - \delta = \frac{159}{128} = 1.2421875$, thus (58) is an improvement on the exponent arising from summing the pointwise bound on $|L(\frac{1}{2}, \rho_K)|$ over cubic fields K with discriminant bounded by X.

9 Conditional computation of the first moment of $L(\frac{1}{2}, \rho_K)$

In this section, we shall compute the first moment of $L(\frac{1}{2}, \rho_K)$ assuming one of two hypotheses. More precisely, we prove the following result.

Theorem 9.1 *Assume* one *of the following two hypotheses*:

(S) ***Strong Subconvexity*:** *For every $K \in \mathcal{F}_\Sigma$, we have $|L(\frac{1}{2}, \rho_K)| \ll |\Delta(K)|^{\frac{1}{6}-\vartheta}$ for some $\vartheta > 0$.*

(N) ***Nonnegativity*:** *For every $K \in \mathcal{F}_\Sigma$, we have $L(\frac{1}{2}, \rho_K) \geq 0$.*

Then we have for small enough $\epsilon > 0$,

$$\sum_{K \in \mathcal{F}_\Sigma} L\big(\tfrac{1}{2}, \rho_K\big)\Psi\Big(\frac{|\Delta(K)|}{X}\Big) = C_\Sigma \cdot X \cdot \big(\log X + \widetilde{\Psi}'(1)\big) + C'_\Sigma \cdot X + O_{\epsilon,\Sigma,\Psi}(X^{1-\epsilon}),$$

where C_Σ and C'_Σ are the constants arising in Proposition 8.5.

Compared to Sect. 8, the proof is significantly more difficult, and will require several new inputs. Indeed, recall that we have

$$A_\Sigma(X) = 2\sum_{q\geq 1}\mu(q)\sum_{f\in\overline{\mathcal{W}_q(\Sigma)}} S(f)\Psi\Big(\frac{|\Delta(f)|}{X}\Big). \tag{59}$$

Pick a small $\kappa_\downarrow > 0$. Proposition 8.2 provided an estimate for the above sum with q in the range $[1, X^{1/8-\kappa_\downarrow}]$.

For $q \geq X^{1/8-\kappa_\downarrow}$, our approach is to approximate the smoothed sum of $S(f)$ with a smoothed sum of $D(\frac{1}{2}, f)$. We do this by breaking up these q into two ranges: the "large range" and the "border range". Namely, we pick a small $\kappa_\uparrow > 0$. Then the range $q \geq X^{1/8+\kappa_\uparrow}$ is the large range while the range $[X^{1/8-\kappa_\downarrow}, X^{1/8+\kappa_\uparrow}]$ is the border range. For q in both of these ranges we want to prove

$$\sum_{f\in\overline{\mathcal{W}_q(\Sigma)}} S(f)\Psi\Big(\frac{|\Delta(f)|}{X}\Big) \approx \sum_{f\in\overline{\mathcal{W}_q(\Sigma)}} D(\tfrac{1}{2}, f)\Psi\Big(\frac{|\Delta(f)|}{X}\Big). \tag{60}$$

On average over $f \in \overline{\mathcal{W}_q(\Sigma)}$, this is an unbalanced approximation of the central value $D(\frac{1}{2}, f)$ by the Dirichlet sum $S(f)$ of the coefficients $\lambda_n(f)$.

In §9.1, we establish (60) with q in the large range, which is straightforward. The bulk of the section is devoted to proving (60) in the border range. This is proved in §9.2 and §9.3 using the unbalanced approximate functional equation of Proposition 4.11. The crux of the proof is to estimate the average of the coefficients $e_k(f)$ of the unbalanced Euler factors $E_p(s, f)$ over the forms $f \in \overline{\mathcal{W}_q(\Sigma)}$. Finally, in §9.4, we compute the average of $D(\frac{1}{2}, f)$ (assuming either nonnegativity or strong subconvexity of $L(\frac{1}{2}, \rho_K)$), thereby obtaining the average of $S(f)$ and finishing the proof of Theorem 9.1.

9.1 Estimates for the large range

We begin by estimating $S(f)$ for integral binary cubic forms with large index.

Lemma 9.2 *For every integral binary cubic form $f \in V(\mathbb{Z})^{\mathrm{irr}}$ and every $\epsilon > 0$, we have*

$$S(f) = D(\tfrac{1}{2}, f) + O_\epsilon\Big(\frac{|\Delta(f)|^{1/4+\epsilon}}{\mathrm{ind}(f)}\Big).$$

Proof Recall the computation of $\widetilde{V^\pm}(s)$ in (32), and note that by definition, we have

$$S(f) = \frac{1}{2\pi i}\int_{\Re(s)=2} D(\tfrac{1}{2}+s, f)\widetilde{V^\pm}(s)|\Delta(f)|^{s/2}ds.$$

Shifting to the line $s = -1/2 + \epsilon$, we pick up the pole of $\widetilde{V^{\pm}}(s)$ at 0 (with residue 1), to obtain

$$\begin{aligned} S(f) &= D(\tfrac{1}{2}, f) + \frac{1}{2\pi i}\int_{\Re(s)=-1/2+\epsilon} D(\tfrac{1}{2}+s, f)\widetilde{V^{\pm}}(s)|\Delta(f)|^{s/2}ds \\ &= D(\tfrac{1}{2}, f) + O_{\epsilon}\big(|\Delta(f)|^{-1/4+\epsilon}|\Delta(K)|^{1/2+\epsilon}\big), \end{aligned}$$

where the final estimate follows since $D(s, f)$ is within $|\Delta(f)|^{\epsilon}$ of $L(s, \rho_K)$ for $\Re(s)$ close to 0. The lemma now follows since $\Delta(f) = \mathrm{ind}(f)^2\Delta(K)$. □

Adding up the above estimate for $f \in \overline{\mathcal{W}_q(\Sigma)}$, we immediately obtain the following result.

Proposition 9.3 *For every square-free q, and $X \geq 1$, we have*

$$\sum_{f\in\overline{\mathcal{W}_q(\Sigma)}} S(f)\Psi\Big(\frac{|\Delta(f)|}{X}\Big) = \sum_{f\in\overline{\mathcal{W}_q(\Sigma)}} D(\tfrac{1}{2}, f)\Psi\Big(\frac{|\Delta(f)|}{X}\Big) + O_{\epsilon,\Sigma,\Psi}\Big(\frac{X^{5/4+\epsilon}}{q^3}\Big).$$

Proof The proposition follows from Lemma 9.2 and the tail estimate in Proposition 6.8. □

An immediate consequence of the previous result is the following estimate for q in the large range.

Corollary 9.4 *For every small $\kappa_{\uparrow} > 0$, square-free $q > X^{1/8+\kappa_{\uparrow}}$ and $X \geq 1$, we have*

$$\sum_{f\in\overline{\mathcal{W}_q(\Sigma)}} \big(S(f) - D(\tfrac{1}{2}, f)\big)\Psi\Big(\frac{|\Delta(f)|}{X}\Big) \ll_{\epsilon,\kappa_{\uparrow},\Sigma,\Psi} \frac{X^{1-2\kappa_{\uparrow}+\epsilon}}{q}.$$

9.2 Preparations and strategy for the border range

In this subsection, we shall introduce spaces, notation, and some preliminary results that will be useful subsequently in handling the border range. One of the key tools in comparing $S(f)$ and $D(\frac{1}{2}, f)$ is the unbalanced approximate functional equation of Proposition 4.11. To apply this result, it is not possible to only work with the information that forms $f \in \mathcal{W}_q(\Sigma)$ are nonmaximal at primes dividing q. Rather, we shall work with the additional information of the index of f, including at primes not dividing q.

To this end, for a positive (not necessarily squarefree) integer b, let $\mathcal{U}_b(\Sigma)$ denote the set of binary cubic forms $f \in V(\mathbb{Z})(\Sigma)$ such that $\mathrm{ind}(f) = b$. Note the inclusion $\mathcal{U}_b(\Sigma) \subset \mathcal{W}_{\mathrm{rad}(b)}(\Sigma)$, and in fact we have

$$\mathcal{W}_q(\Sigma) = \bigsqcup_{m\geq 1} \mathcal{U}_{mq}(\Sigma),$$

where the union is disjoint and q is square-free. Let $\overline{\mathcal{U}_b(\Sigma)}$ denote the set of $\mathrm{GL}_2(\mathbb{Z})$-orbits on $\mathcal{U}_b(\Sigma)$.

Let b be a positive integer, and let r be a positive squarefree integer such that $(b, r) = 1$. Finally, we define the set $\mathcal{Y}_{b,r}(\Sigma)$ to be the subset of elements in $\mathcal{W}_r(\Sigma)$ whose index at primes p dividing b is exactly $p^{v_p(b)}$. As usual we let $\overline{\mathcal{Y}_{b,r}(\Sigma)}$ denote the set of $\mathrm{GL}_2(\mathbb{Z})$-orbits on $\mathcal{Y}_{b,r}(\Sigma)$. The significance of these subsets $\mathcal{Y}_{b,r}(\Sigma)$ is the following disjoint union

$$\mathcal{Y}_{b,r}(\Sigma) = \bigsqcup_{(b,s)=1} \mathcal{U}_{brs}(\Sigma),$$

hence for any function $\phi : \overline{\mathcal{U}_b(\Sigma)} \to \mathbb{C}$, we have

$$\sum_{f \in \overline{\mathcal{U}_b(\Sigma)}} \phi(f)\Psi\Big(\frac{|\Delta(f)|}{X}\Big) = \sum_{(b,r)=1} \mu(r) \sum_{f \in \overline{\mathcal{Y}_{b,r}(\Sigma)}} \phi(f)\Psi\Big(\frac{|\Delta(f)|}{X}\Big).$$

Recall that the border range is what we are calling $q \in [X^{1/8-\kappa_\downarrow}, X^{1/8+\kappa_\uparrow}]$, where $\kappa_\downarrow, \kappa_\uparrow$ are positive constants that can eventually be taken to be arbitrarily small. We next estimate the sum of $S(f) - D(\frac{1}{2}, f)$ over f in $\overline{\mathcal{U}_{mq}(\Sigma)}$, where m is somewhat large.

We begin by bounding the number of elements in $\overline{\mathcal{U}_{mq}(\Sigma)} \subset \overline{\mathcal{U}^{\mathrm{irr}}_{mq}}$ that have discriminant less than X.

Lemma 9.5 *For every positive integer m and square-free q, write $mq = m_1 q_1$, where m_1 is powerful, $(m_1, q_1) = 1$, and q_1 is squarefree. Then for every $X \geq 1$,*

$$|\{f \in \overline{\mathcal{U}^{\mathrm{irr}}_{mq}} : |\Delta(f)| < X\}| \ll_\epsilon \frac{X^{1+\epsilon}}{m_1^{5/3} q_1^2}. \tag{61}$$

The multiplicative constant depends only on ϵ (it is independent of m, q, X).

Proof Elements f in the left-hand side of (61) are in bijection with rings R_f that have index $mq = m_1 q_1$ in the maximal orders $\mathcal{O}_{K_f}$ of their fields of fractions K_f. It follows that the discriminants of these fields K_f are less than $X/(m_1^2 q_1^2)$. It follows that the total number of such fields that can arise is bounded by $O(X/m_1^2 q_1^2)$.

To estimate the total number of rings R_f that can arise, it suffices to estimate the number of such rings R_f within a single K_f. This can be done prime by prime, for each prime dividing the index $m_1 q_1$. Let p be a prime dividing q_1. Since q_1 is squarefree, it follows that the index of R_f, at the prime p, is p. Given the index p overorder R of R_f, it follows from Proposition 2.3, that the number of index p suborders of R is bounded by 3.

For primes dividing m_1, this procedure is more complicated since there can be many more subrings with prime power index. However, this question is completely answered by work of Shintani [31] and Datskovsky–Wright [11] (see [24, §1.2]), who give an explicit formula for the counting function of suborders R of a fixed cubic field

K, which we state as Proposition 9.18. They show that the number of suborders of index m, for $m \geq 1$, is the mth Dirichlet coefficient of

$$\frac{\zeta_K(s)}{\zeta_K(2s)}\zeta_{\mathbb{Q}}(3s)\zeta_{\mathbb{Q}}(3s-1).$$

To verify the lemma, it suffices to bound the Dirichlet coefficients of the Euler factor of primes p having splitting type (111), since these coefficients majorize those of primes with all other splitting types. For such a prime, the pth Euler factor of the above Dirichlet series is:

$$\begin{aligned}&(1-p^{-s})^{-3}(1-p^{-2s})^3(1-p^{-3s})^{-1}(1-p^{-3s+1})^{-1}\\ &\quad=(1+p^{-s})^3\Big(\sum_{k=0}^{\infty}p^{-3ks}\Big)\Big(\sum_{k=0}^{\infty}p^{k-3ks}\Big).\end{aligned}$$

It is thus clear that the kth Dirichlet coefficient is bounded by $O(p^{k/3})$. Therefore, the number of possible suborders of index p^k is bounded by $O(p^{k/3})$.

Putting this together, it follows that the number of suborders of K, having index q_1m_1 is bounded by $O(q_1^{\epsilon}m_1^{1/3})$. Multiplying this quantity by $X/(m_1q_1)^2$ yields the result. □

Lemma 9.6 *For $X \geq 1$, square-free q, and small enough $\eta > 0$*

$$\sum_{m>X^{\eta}}\sum_{f\in\overline{\mathcal{U}_{mq}(\Sigma)}}\big(S(f)-D(\tfrac{1}{2},f)\big)\Psi\Big(\frac{|\Delta(f)|}{X}\Big)=O_{\epsilon,\Sigma,\Psi}\Big(\frac{X^{5/4-\eta+\epsilon}}{q^3}\Big).$$

Proof From Lemma 9.2, it follows that for $f\in\mathcal{U}_{mq}(\Sigma)$ with $|\Delta(f)|\asymp X$, we have $S(f)-D(\frac{1}{2},f)=O(\frac{X^{1/4+\epsilon}}{mq})$. We write mq as m_1q_1, where q_1 is squarefree with $(q_1,m_1)=1$, and m_1 is powerful. We now have

$$\sum_{m>X^{\eta}}\sum_{f\in\overline{\mathcal{U}_{mq}(\Sigma)}}\big(S(f)-D(\tfrac{1}{2},f)\big)\Psi\Big(\frac{|\Delta(f)|}{X}\Big)\ll_{\epsilon,\Sigma,\Psi}\sum_{m>X^{\eta}}\frac{X^{1/4+\epsilon}}{mq}\cdot\frac{X^{1+\epsilon}}{m_1^{5/3}q_1^2},$$

where the final estimate follows from Lemma 9.5. □

We then have the following corollary.

Corollary 9.7 *Let $X \geq 1$, squarefree $q > X^{1/8-\kappa_{\downarrow}}$, and $\eta > 0$ be such that $\eta - 2\kappa_{\downarrow} > 0$. Then we have*

$$\sum_{m>X^{\eta}}\sum_{f\in\overline{\mathcal{U}_{mq}(\Sigma)}}\big(S(f)-D(\tfrac{1}{2},f)\big)\Psi\Big(\frac{|\Delta(f)|}{X}\Big)=O_{\epsilon,\kappa_{\downarrow},\Sigma,\Psi}\Big(\frac{X^{1+2\kappa_{\downarrow}-\eta+\epsilon}}{q}\Big).$$

Furthermore, $\kappa_\downarrow$ and hence η can be taken to be arbitrarily small. Therefore, a consequence of the above lemma is that when q is in the border range, sums over $\overline{\mathcal{U}_{mq}(\Sigma)}$ only have to be considered for m less than arbitrarily small powers of X.

Let $q \in [X^{1/8-\kappa_\downarrow}, X^{1/8+\kappa_\uparrow}]$ be fixed for the rest of this subsection. For a positive integer m, we write $mq = m_1 q_1$, where m_1 is powerful, $(m_1, q_1) = 1$, and q_1 is squarefree. Note that since m will be taken to be very small ($\ll X^\eta$), q_1 will be quite close in size to q. We restate Proposition 4.11 for convenience: for $f \in \overline{\mathcal{U}_{m_1 q_1}(\Sigma)}$, we have

$$S(f) = D(\tfrac{1}{2}, f) - \sum_{k=1}^{\infty} \frac{e_k(f)k^{1/2}}{q_1 \mathrm{rad}(m_1)} \sum_{n=1}^{\infty} \frac{\lambda_n(f)}{n^{1/2}} V^{\pm}\left(\frac{m_1^2 kn}{\mathrm{rad}(m_1)^2 |\Delta(f)|^{1/2}}\right), \tag{62}$$

where $e_k(f)k^{1/2}$ is the kth Dirichlet coefficient of the series

$$\sum_{k=1}^{\infty} \frac{e_k(f)k^{1/2}}{k^s} = q_1^{1-2s} \mathrm{rad}(m_1)^{1-2s} \frac{E(\frac{1}{2} - s, f)}{E(\frac{1}{2} + s, f)}.$$

Our next and final goal of this subsection is to perform a switching trick, analogous to Theorem 6.5, in which our sums over $\overline{\mathcal{U}_{m_1 q_1}(\Sigma)}$ are replaced with sums over $\overline{\mathcal{U}_{m_1}(\Sigma)}$. We thus need to understand how the quantity $e_k(f)$ behaves under such a switch. The next lemma does just that: more precisely, if f is nonmaximal and switches to the pair (g, α) with prime index p, then the next lemma determines $e_{p,m}(f)$ in terms of (g, α).

As recalled in Proposition 2.3, the proof of [5, Prop.16] implies that there is a bijection between the zeros in $\mathbb{P}^1(\mathbb{F}_p)$ of the reduction modulo p of $g(x, y)$ and the set of cubic rings that are index-p subrings of R_g. Thus, f corresponds uniquely to a pair (g, α), where $\alpha \in \mathbb{P}^1(\mathbb{F}_p)$ is a root of $g(x, y)$ modulo p. Then the following lemma determines $E_p(s, f)$ given this pair (g, α).

Lemma 9.8 *Let $g \in V(\mathbb{Z})^{\mathrm{irr}}$ be an irreducible binary cubic form that is maximal at a prime p. Let $\alpha \in \mathbb{P}^1(\mathbb{F}_p)$ be a root of the reduction of g modulo p. Let $f \in V(\mathbb{Z})^{\mathrm{irr}}$ be a binary cubic form corresponding to the index-p subring of R_g associated to the pair (g, α). Then $E_p(s, f)$, and hence $e_{p,m}(f)$ for $m \geq 0$, is determined by the pair (g, α). More precisely, we have*

(a) *If $\sigma_p(g) = (111)$, then $\sigma_p(f) = (1^21)$ and $E_p(s, f) = 1 - p^{-s}$;*
(b) *If $\sigma_p(g) = (12)$, then $\sigma_p(f) = (1^21)$ and $E_p(s, f) = 1 + p^{-s}$;*
(c) *If $\sigma_p(g) = (1^21)$ and α is the single root, then $\sigma_p(f) = (1^21)$ and $E_p(s, f) = 1$;*
(d) *If $\sigma_p(g) = (1^21)$ and α is the double root, then $\sigma_p(f) = (1^3)$ and $E_p(s, f) = 1 - p^{-s}$;*
(e) *If $\sigma_p(g) = (1^3)$, then $\sigma_p(f) = (1^3)$ and $E_p(s, f) = 1$.*

Proof As in the proof of the second part of Lemma 6.3, the procedure to compute $f(x, y)$ given the pair (g, α) is as follows: use the action of $\mathrm{GL}_2(\mathbb{Z})$ to move α to the point $[1 : 0] \in \mathbb{P}^1(\mathbb{F}_p)$. This yields the binary cubic form $ax^3 + bx^2y + cxy^2 + dy^3$, where $p \mid a$. Moreover, since g is maximal at p, we see that $p \mid b$ implies that $p^2 \nmid a$.

Then $f(x, y)$ can be taken to be $(a/p)x^3 + bx^2y + pcxy^2 + p^2dy^3$. Running this procedure for the different splitting types of g immediately shows that the corresponding f has the splitting type listed in the lemma.

For example, if g has splitting type (111) or (12), then we may bring one of the simple roots (using a $\mathrm{GL}_2(\mathbb{Z})$-transformation) to infinity. Then we may write $g(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$, where $p \mid a$ and $p \nmid b$ since g is unramified. Then the procedure gives $f(x, y) = (a/p)x^3 + bx^2y + pcxy^2 + p^2dy^3$. Since $p \nmid b$, the splitting type of $f(x, y)$ is $(1^2 1)$ as claimed. The other cases are similar, and we omit them. □

Lemma 9.9 *Suppose that $f, g \in V(\mathbb{Z})^{\mathrm{irr}}$ are $\mathrm{GL}_2(\mathbb{Z}_\ell)$-equivalent for some prime ℓ. Then we have $E_\ell(s, f) = E_\ell(s, g)$, and hence $e_{\ell,m}(f) = e_{\ell,m}(g)$ for all $m \geq 0$.*

Proof As in the proof of the second part of Lemma 6.3, we have $\sigma_\ell(f) = \sigma_\ell(g)$. In view of Definition 3.3 and (24) this implies $D_\ell(s, f) = D_\ell(s, g)$. We also have $\sigma_\ell(K_f) = \sigma_\ell(K_g)$, and hence $L_\ell(s, \rho_{K_f}) = L_\ell(s, \rho_{K_g})$. In view of Definition 4.4 this implies $E_\ell(s, f) = E_\ell(s, g)$. □

The final result of this subsection is to determine what happens to the quantity $e_k(f)\lambda_n(f)$ after applications of switches.

Lemma 9.10 *Let m_1 and q_1 be positive integers, where m_1 is powerful, $(m_1, q_1) = 1$, and q_1 is squarefree. Let k be a positive integer divisible only by primes dividing m_1q_1. Let n be a positive integer and write $n = n_1\ell_1$ where $(\ell_1, q_1) = 1$ and n_1 is divisible only by primes dividing q_1. Then we have*

$$\sum_{f \in \overline{\mathcal{U}_{m_1q_1}(\Sigma)}} e_k(f)\lambda_n(f)\Psi(|\Delta(f)|) = \sum_{g \in \overline{\mathcal{U}_{m_1}(\Sigma)}} c_{q_1}(g)d_{m_1}(g)\lambda_{\ell_1}(g)\Psi(q_1^2|\Delta(g)|),$$

where c_{q_1} and d_{m_1} are congruence functions on $V(\mathbb{Z})$ defined modulo q_1 and m_1^3, respectively. Furthermore, we have $c_{q_1}(g) \ll_\epsilon q_1^\epsilon$ and $d_{m_1}(g) \ll_\epsilon m_1^\epsilon$ uniformly for every $g \in V(\mathbb{Z})$.

Proof As in Sect. 6, we will write sums over $\overline{\mathcal{U}_{m_1q_1}(\Sigma)}$ in terms of sums over $\overline{\mathcal{U}_{m_1}(\Sigma)}$. In this case, we have the simple bijection

$$\mathcal{U}_{m_1q_1}(\Sigma) \leftrightarrow \big\{(g, \alpha) : g \in \mathcal{U}_{m_1}(\Sigma), \alpha \in \mathbb{P}^1(\mathbb{Z}/q_1\mathbb{Z}), g(\alpha) \equiv 0 \pmod{q_1}\big\},$$

which follows by an argument similar to that of Lemma 6.2.

Since every prime factor of k also divides $\mathrm{ind}(f) = m_1q_1$, we have

$$e_k(f) = \prod_{p \mid m_1q_1} e_{p,v_p(k)}(f) = \Big(\prod_{p \mid q_1} e_{p,v_p(k)}(f)\Big)\Big(\prod_{\ell \mid m_1} e_{\ell,v_\ell(k)}(f)\Big).$$

We denote the two terms between the parentheticals in the right-hand side above $e'_{q_1}(f)$ and $d_{m_1}(f)$, respectively. Since the function $n \mapsto \lambda_n(f)$ is multiplicative, we have $\lambda_n(f) = \lambda_{n_1}(f)\lambda_{\ell_1}(f)$.

Therefore, to prove the lemma, we need to express $e'_{q_1}(f)$, $d_{m_1}(f)$, $\lambda_{n_1}(f)$, and $\lambda_{\ell_1}(f)$ in terms of congruence functions on the (g,α) corresponding to f under the above bijection. We begin by noting that we have $d_{m_1}(f) = d_{m_1}(g)$ by Lemma 9.9 and $\lambda_{\ell_1}(f) = \lambda_{\ell_1}(g)$, since f and g are $\mathrm{GL}_2(\mathbb{Z}_\ell)$-equivalent for any $\ell \nmid q_1$ (in particular for every $\ell | m_1$ and for every $\ell | \ell_1$). Moreover, the function $g \mapsto d_{m_1}(g)$ on $\mathcal{U}_{m_1}(\Sigma)$ can be extended to a congruence function on $V(\mathbb{Z})$ defined modulo m_1^3 because $\sigma_\ell(K_g)$ depends only on g modulo $\ell^{3v_\ell(m_1)}$ for every prime $\ell | m_1$ and we apply Definition 4.4 and Definition 4.7 to find also that $e_{\ell, v_\ell(k)}(g)$ depends only on g modulo $\ell^{3v_\ell(m_1)}$.

Next, we have that $\lambda_{n_1}(f) = 0$ if $\alpha \in \mathbb{P}^1(\mathbb{Z}/q_1\mathbb{Z})$ reduces modulo p to a double root of g for some prime $p \mid n_1$, and $\lambda_{n_1}(f) = 1$ otherwise. We have seen in Lemma 9.8 that the value $e'_{q_1}(f)$ depends only on the splitting type of g modulo all the primes dividing q_1, and on whether α is a simple or double root modulo all the primes dividing q_1. It is thus clear that the sum of $e'_{q_1}(f)\lambda_{n_1}(f)$ over the roots $\alpha \in \mathbb{P}^1(\mathbb{Z}/q_1\mathbb{Z})$ of g, where $f \leftrightarrow (g,\alpha)$ under the above bijection, is a function of $g \in \mathcal{U}_{m_1}(\Sigma)$ that can be extended to a congruence function on $V(\mathbb{Z})$ defined modulo q_1. The first claim of the lemma now follows.

The estimates in the second claim of the lemma are immediate since $\lambda_{n_1}(f) \ll_\epsilon q_1^\epsilon$ because the radical of n_1 divides q_1, and since $e'_{q_1}(f)$ and $d_{m_1}(f)$, each are bounded by $\ll_\epsilon q_1^\epsilon$ and $\ll_\epsilon m_1^\epsilon$, respectively, because of the estimate $e_{p,m}(f) \ll 1$ in the proof of Proposition 4.9. □

9.3 Estimates for the border range

In this subsection, we assume that our integers q lie in the border range $[X^{1/8-\kappa_\downarrow}, X^{1/8+\kappa_\uparrow}]$ with small enough $\kappa_\downarrow, \kappa_\uparrow > 0$. Our goal is to bound

$$\sum_{f \in \overline{\mathcal{W}_q(\Sigma)}} \big(S(f) - D(\tfrac{1}{2}, f)\big)\Psi\Big(\frac{|\Delta(f)|}{X}\Big),$$

for q in this range. Recall that we have a disjoint union

$$\overline{\mathcal{W}_q(\Sigma)} = \bigsqcup_{m \geq 1} \overline{\mathcal{U}_{mq}(\Sigma)},$$

and that we will be summing $S(f) - D(\frac{1}{2}, f)$ over $\overline{\mathcal{U}_{mq}(\Sigma)}$ (and then summing over m) rather than simply summing over $\overline{\mathcal{W}_q(\Sigma)}$. From Lemma 9.6, it follows that we may restrict the sum to $m \leq X^\eta$, where η may be taken to be arbitrarily small. All multiplicative constants are understood to depend on the initial choices of $\kappa_\downarrow, \kappa_\uparrow, \eta > 0$.

We write $mq = m_1 q_1$, where m_1 is powerful, $(m_1, q_1) = 1$ and q_1 is squarefree. Note then that $m_1 \leq m^2 \leq X^{2\eta}$, and thus $q_1 \geq q/m \geq X^{1/8-\eta-\kappa_\downarrow}$. We begin by fixing k and n in (62), and bounding the sum over $f \in \overline{\mathcal{U}_{m_1 q_1}(\Sigma)}$.

Proposition 9.11 *For every small enough $\kappa_1 > 0$, the following estimate holds. Let m_1, q_1, k, and n be positive integers and $X \geq 1$. Assume that m_1 is powerful,*

$(m_1, q_1) = 1$, and q_1 is squarefree. Write $n = n_1\ell_1$ where $(\ell_1, q_1) = 1$ and n_1 is divisible only by primes dividing q_1. Denote the radical of ℓ_1 by ℓ. Then

$$\sum_{f\in\overline{\mathcal{U}_{m_1q_1}(\Sigma)}} e_k(f)\lambda_n(f)V^{\pm}\Big(\frac{nkm_1^2}{\mathrm{rad}(m_1)^2|\Delta(f)|^{1/2}}\Big)\Psi\Big(\frac{|\Delta(f)|}{X}\Big)$$
$$\ll_{\epsilon,\Sigma,\Psi} X^{\epsilon}\cdot H(n, m_1, q_1; X),$$

where

$$H(n, m_1, q_1; X) = \frac{X}{q_1^2 m_1^{5/3}\ell} + \frac{X^{5/6+\kappa_1/3}}{q_1^{5/3}\ell^{1/3}} + \ell q_1^2 m_1^{12} X^{9\kappa_1} + \frac{X^{1-\kappa_1}}{q_1^2 m_1^{5/3}}.$$

Proof Without loss of generality, we may assume that every prime factor of k also divides m_1q_1, since otherwise $e_k(f) = 0$ by definition and hence the sum is zero. Applying the preceding Lemma 9.10, we obtain

$$\sum_{f\in\overline{\mathcal{U}_{m_1q_1}(\Sigma)}} e_k(f)\lambda_n(f)V\Big(\frac{nkm_1^2}{\mathrm{rad}(m_1)^2\sqrt{|\Delta(f)|}}\Big)\Psi\Big(\frac{|\Delta(f)|}{X}\Big)$$
$$= \sum_{f\in\overline{\mathcal{U}_{m_1}(\Sigma)}} c_{q_1}(f)d_{m_1}(f)\lambda_{\ell_1}(f)\Psi_1\Big(\frac{q_1^2|\Delta(f)|}{X}\Big),$$

where c_{q_1} is defined modulo q_1, d_{m_1} is defined modulo m_1^3, and $\Psi_1 = \mathcal{H}_{\frac{nkm_1^2}{\sqrt{X}\mathrm{rad}(m_1)^2}}$. Recall that in Corollary 8.1, we bound $E_\infty(\widetilde{\Psi_1}; -\epsilon)$ by an absolute constant. For brevity in this proof, we will write $\ll$ as a shorthand for $\ll_{\epsilon,\Sigma,\Psi}$.

We perform an inclusion-exclusion principle to write the sum over $\overline{\mathcal{U}_{m_1}(\Sigma)}$ in terms of sums over $\overline{\mathcal{Y}_{m_1,r}(\Sigma)}$. This yields

$$\sum_{f\in\overline{\mathcal{U}_{m_1}(\Sigma)}} c_{q_1}(f)d_{m_1}(f)\lambda_{\ell_1}(f)\Psi_1\Big(\frac{q_1^2|\Delta(f)|}{X}\Big)$$
$$= \sum_{(m_1,r)=1}\mu(r)\sum_{f\in\overline{\mathcal{Y}_{m_1,r}(\Sigma)}} c_{q_1}(f)d_{m_1}(f)\lambda_{\ell_1}(f)\Psi_1\Big(\frac{q_1^2|\Delta(f)|}{X}\Big).$$

We split up the above sum into two sums, corresponding to the ranges $r < B$ and $r \geq B$, for some $B > 1$.

We estimate each summand in the range $r < B$ using Theorem 5.6, and each summand in the range $r \geq B$ using Lemma 9.5, to respectively obtain

$$\sum_{f\in\overline{\mathcal{Y}_{m_1,r}(\Sigma)}} c_{q_1}(f)d_{m_1}(f)\lambda_{\ell_1}(f)\Psi_1\Big(\frac{q_1^2|\Delta(f)|}{X}\Big)$$

$$\ll \frac{X^{1+\epsilon}(\ell,r)}{q_1^2 m_1^{5/3} r^2 \ell} + \frac{X^{5/6+\epsilon}(\ell,r)}{q_1^{5/3} r^{5/3} \ell^{1/3}} + \ell q_1^2 m_1^{12} r^8 X^{\epsilon}$$
$$\ll \frac{X^{1+\epsilon}}{q_1^2 m_1^{5/3} r \ell} + \frac{X^{5/6+\epsilon}}{q_1^{5/3} r^{2/3} \ell^{1/3}} + \ell q_1^2 m_1^{12} r^8 X^{\epsilon};$$

$$\sum_{f \in \overline{\mathcal{Y}_{m_1,r}(\Sigma)}} c_{q_1}(f) d_{m_1}(f) \lambda_{\ell_1}(f) \Psi_1\Big(\frac{q_1^2|\Delta(f)|}{X}\Big) \ll \frac{X^{1+\epsilon}}{q_1^2 m_1^{5/3} r^2}.$$

The second bound is simply an application of the tail estimate of Lemma 9.5. The first bound is more complicated, and we explain how it is derived. Summing over $\overline{\mathcal{Y}_{m_1,r}(\Sigma)}$ can be replaced by summing a function $\phi\chi_\Sigma$ over $\overline{V(\mathbb{Z})}$, where ϕ is defined modulo $m_1^2 r^2$ and χ_Σ is the indicator function defined in §6.2 before Corollary 6.7. In the above equation, we are therefore summing a function defined modulo $r^2 m_1^3 q_1 \ell r_\Sigma$ (here, we also use Lemma 9.10). Moreover q_1 is squarefree, and the function defined modulo ℓ is λ_{ℓ_1}. Therefore, the error term with applying Theorem 5.6 is bounded by $\ll \ell q_1^2 m_1^{12} r^8 X^{\epsilon}$.

We now estimate the first and second main terms. The density of the first main term follows from the uniformity estimates and the bound $\mathcal{A}_{\ell_1}(\lambda_{\ell_1}) \ll \frac{1}{\ell}$ from Lemma 5.9. The second main term computation follows similarly using the bound $\mathcal{C}_{\ell_1}(\lambda_{\ell_1}) \ll \frac{1}{\ell^{1/3}}$ from Lemma 5.9.

Adding the above bounds over the appropriate ranges of r yields

$$\sum_{f \in \overline{\mathcal{U}_{m_1 q_1}(\Sigma)}} e_k(f)\lambda_n(f) V\Big(\frac{nkm_1^2}{\mathrm{rad}(m_1)^2\sqrt{|\Delta(f)|}}\Big)\Psi\Big(\frac{|\Delta(f)|}{X}\Big)$$
$$\ll \frac{X^{1+\epsilon}\log B}{q_1^2 m_1^{5/3}\ell} + \frac{X^{5/6+\epsilon}B^{1/3}}{q_1^{5/3}\ell^{1/3}} + \ell q_1^2 m_1^{12} B^9 X^{\epsilon} + \frac{X^{1+\epsilon}}{q_1^2 m_1^{5/3} B}.$$

Choosing $B = X^{\kappa_1}$ concludes the proof of the proposition. □

Let notation be as in the beginning of this section. We have

$$\sum_{f \in \overline{\mathcal{W}_q(\Sigma)}} \big(D(\tfrac{1}{2}, f) - S(f)\big)\Psi\Big(\frac{\Delta(f)}{X}\Big)$$
$$= \sum_{f \in \overline{\mathcal{W}_q(\Sigma)}} \Psi\Big(\frac{\Delta(f)}{X}\Big)\sum_{k=1}^{\infty}\frac{e_k(f)k^{1/2}}{\mathrm{rad}(\mathrm{ind}(f))}$$
$$\times \sum_{n=1}^{\infty}\frac{\lambda_n(f)}{n^{1/2}} V^{\mathrm{sgn}(\Delta(f))}\Big(\frac{\mathrm{ind}(f)^2 kn}{\mathrm{rad}(\mathrm{ind}(f))^2|\Delta(f)|^{1/2}}\Big)$$
$$= \sum_{m=1}^{\infty}\sum_{f \in \overline{\mathcal{U}_{mq}(\Sigma)}} \Psi\Big(\frac{\Delta(f)}{X}\Big)\sum_{k\geq 1}^{\flat}\frac{e_k(f)k^{1/2}}{q_1\mathrm{rad}(m_1)}$$

$$\times \sum_{n=1}^{\infty} \frac{\lambda_n(f)}{n^{1/2}} V^{\mathrm{sgn}(\Delta(f))}\Big(\frac{m_1^2 kn}{\mathrm{rad}(m_1)^2|\Delta(f)|^{1/2}}\Big)$$

$$= \sum_{m=1}^{X^\eta} \sum_{f\in\overline{\mathcal{U}_{mq}(\Sigma)}} \Psi\Big(\frac{\Delta(f)}{X}\Big) \sum_{k\geq 1}^{\flat} \frac{e_k(f)k^{1/2}}{q_1\mathrm{rad}(m_1)}$$

$$\times \sum_{n\leq \frac{X^{1/2+\epsilon}}{k}} \frac{\lambda_n(f)}{n^{1/2}} V^{\mathrm{sgn}(\Delta(f))}\Big(\frac{m_1^2 kn}{\mathrm{rad}(m_1)^2|\Delta(f)|^{1/2}}\Big)$$

$$+ O_{\epsilon,\kappa_\downarrow,\Sigma,\Psi}\Big(\frac{X^{1-\eta+2\kappa_\downarrow+\epsilon}}{q}\Big),$$

where the final estimate follows from Corollary 9.7, and the rapid decay of $V^{\pm}$ to truncate the n-sum, and where the $\flat$ above indicates that the sum over k is supported on multiples of q_1 and ranges only over integers whose prime factors are all divisors of mq.

Next, we truncate the sum over k as follows. For the next two results, we will write $k = k_1k_2$, where k_1 is cubefree, k_2 is cubeful, and $(k_1, k_2) = 1$.

Lemma 9.12 *For every small enough $\kappa_2 > 0$, $X \geq 1$, and q_1, m_1 as above (i.e., satisfying $m_1 \leq X^{2\eta}$ and $q_1 \geq X^{1/8-\eta-\kappa_\downarrow}$), we have*

$$\sum_{m=1}^{X^\eta} \sum_{\substack{f\in\overline{\mathcal{U}_{mq}^{\mathrm{irr}}}\\ |\Delta(f)|<X}} \sum_{\substack{k\\ k_2>X^{3\kappa_2}}}^{\flat} \frac{|e_k(f)|k^{1/2}}{q_1\mathrm{rad}(m_1)} \sum_{n\leq \frac{X^{1/2+\epsilon}}{k}} \frac{|\lambda_n(f)|}{n^{1/2}} \ll_{\epsilon,\kappa_2} \frac{X^{1-\kappa_2+4\eta+2\kappa_\downarrow+\epsilon}}{q}. \tag{63}$$

Proof The integers k that arise range over products of powers of primes dividing mq. It follows from Proposition 4.9 that

$$e_k(f) \ll_\epsilon \frac{\mathrm{rad}(k_2)^2}{k_2} X^\epsilon \leq k_2^{-1/3} X^\epsilon < X^{-\kappa_2+\epsilon}.$$

Hence the sums over n and k can be bounded as follows: we have

$$\sum_{\substack{k\\ k_2>X^{3\kappa_2}}}^{\flat} \frac{|e_k(f)|k^{1/2}}{q_1\mathrm{rad}(m_1)} \sum_{n\leq \frac{X^{1/2+\epsilon}}{k}} \frac{|\lambda_n(f)|}{n^{1/2}} \ll_\epsilon \frac{X^{1/4+\epsilon}}{q_1\mathrm{rad}(m_1)} \sum_{\substack{k\\ k_2>X^{3\kappa_2}}} |e_k(f)|$$

$$\ll_{\epsilon,\kappa_2} \frac{X^{1/4-\kappa_2+2\epsilon}}{q_1\mathrm{rad}(m_1)}.$$

We already know from Lemma 9.5 that

$$\sum_{\substack{f\in\overline{\mathcal{U}_{mq}^{\mathrm{irr}}}\\ |\Delta(f)|<X}} 1 \ll_\epsilon \frac{X^{1+\epsilon}}{m_1^{5/3}q_1^2}.$$

Therefore, the left-hand side of (63) is bounded by

$$\ll_{\epsilon,\kappa_2} X^{5/4-\kappa_2+3\epsilon}\cdot\sum_{m=1}^{X^\eta}\frac{1}{m_1^{5/3}\mathrm{rad}(m_1)q_1^3} \ll_{\epsilon,\kappa_2} \frac{X^{5/4-\kappa_2+\eta+3\epsilon}}{q_1^3} \le \frac{X^{1-\kappa_2+3\eta+2\kappa_\downarrow+3\epsilon}}{q_1},$$

which is sufficient because $q_1 \ge q/m$ and $m \le X^\eta$. □

We input Proposition 9.11, which bounds the sum over f, and obtain with Corollary 9.7 and (63):

$$\begin{aligned}&\sum_{f\in\overline{\mathcal{W}_q(\Sigma)}}\big(D(\tfrac12,f)-S(f)\big)\Psi\Big(\frac{\Delta(f)}{X}\Big)\\ &\ll_{\epsilon,\kappa_2,\Sigma,\Psi}\sum_{m_1=1}^{X^{2\eta}}\sideset{}{^\flat}\sum_{k\le X^{3\kappa_2}}\sum_{n\le\frac{X^{1/2+\epsilon}}{k}}\frac{k^{1/2}}{n^{1/2}q_1\mathrm{rad}(m_1)}X^\epsilon H(n,m_1,q_1;X) \qquad (64)\\ &\quad+\frac{X^{1-\eta+2\kappa_\downarrow+\epsilon}}{q}+\frac{X^{1-\kappa_2+4\eta+2\kappa_\downarrow+\epsilon}}{q}.\end{aligned}$$

In our next result, we estimate the triple sum in (64):

Proposition 9.13 *For every square-free $q\in[X^{1/8-\kappa_\downarrow},X^{1/8+\kappa_\uparrow}]$ and $X\ge1$, we have*

$$\sum_{m_1=1}^{X^{2\eta}}\sideset{}{^\flat}\sum_{k\le X^{3\kappa_2}}\sum_{n\le\frac{X^{1/2+\epsilon}}{k}}\frac{k^{1/2}}{n^{1/2}q_1\mathrm{rad}(m_1)}H(n,m_1,q_1;X)\ll_{\epsilon,\kappa_1,\kappa_2}H(q;X),$$

where $H(q;X)$ is the sum of the final terms in Equations (65), (66), (67), and (68).

Proof In this proof we shall write $\ll$ as a shorthand for $\ll_{\epsilon,\kappa_1,\kappa_2}$. As before, we write $n=n_1\ell_1$, where n_1 is only divisible by primes dividing q and $(\ell_1,q)=1$, and denote the radical of ℓ_1 by ℓ. For convenience, we recall the definition of $H(n,m_1,q_1;X)$:

$$H(n,m_1,q_1;X)=\frac{X}{q_1^2m_1^{5/3}\ell}+\frac{X^{5/6+\kappa_1/3}}{q_1^{5/3}\ell^{1/3}}+\ell q_1^2m_1^{12}X^{9\kappa_1}+\frac{X^{1-\kappa_1}}{q_1^2m_1^{5/3}}.$$

To prove the proposition, we take each term in $H(n,m_1,q_1;X)$ by turn, and sum it over n, k, and m_1. The sum over n is broken up into sums over n_1 and ℓ_1. Note that since n_1 is only divisible by primes dividing q, the presence of $1/n^{1/2}$ in the sum

(and no n_1's in $H(n, m_1, q_1; X)$) means that the sum over n_1 can be ignored, at the cost of the harmless factor $O(X^\epsilon)$. Indeed, we have

$$\sum_{n_1} \frac{1}{n_1^{1/2}} \le \prod_{p|q} \Big(1 + \frac{1}{p^{1/2}} + \frac{1}{p} + \cdots\Big) \ll 2^{\omega(q)} \ll_\epsilon X^\epsilon.$$

Next note that $k = k_1 k_2$, where k_1 is cubefree, and k is only divisible by primes dividing $mq = m_1 q_1$. Hence, we have $k_1 \le q_1^2 \mathrm{rad}(m_1)^2$, and in conjunction with $k_2 \le X^{3\kappa_2}$, we also have $k \le q_1^2 X^{2\eta+3\kappa_2}$. We begin with the first term: in this case, the sums over ℓ_1 and m_1 converge, and so we have

$$\begin{aligned}
&\frac{X^{1+\epsilon}}{q_1^2} \sum_{m_1=1}^{X^{2\eta}} \sideset{}{^\flat}\sum_{\substack{k \\ k_2 \le X^{3\kappa_2}}} \sum_{\ell_1 \le \frac{X^{1/2+\epsilon}}{k}} \frac{k^{1/2}}{\ell_1^{1/2} q_1 \mathrm{rad}(m_1)} \frac{1}{m_1^{5/3}\ell} \\
&\ll \frac{X^{1+\epsilon}}{q_1^3} \sideset{}{^\flat}\sum_{\substack{k \\ k_2 \le X^{3\kappa_2}}} k^{1/2} \\
&\ll \frac{X^{1+\epsilon}}{q_1^3} \cdot q_1 X^{\eta+3\kappa_2/2} \sum_{k_2 \le X^{3\kappa_2}} 1 \\
&\ll \frac{X^{7/8+3\eta+(9/2)\kappa_2+\kappa_\downarrow+\epsilon}}{q},
\end{aligned} \tag{65}$$

where the final estimate follows because $q_1 \ge q X^{-\eta}$ and $q \ge X^{1/8-\kappa_\downarrow}$. Similarly, for the second term, we have

$$\begin{aligned}
&\frac{X^{5/6+\kappa_1/3+\epsilon}}{q_1^{5/3}} \sum_{m_1=1}^{X^{2\eta}} \sideset{}{^\flat}\sum_{\substack{k \\ k_2 \le X^{3\kappa_2}}} \sum_{\ell_1 \le \frac{X^{1/2+\epsilon}}{k}} \frac{k^{1/2}}{\ell_1^{1/2} q_1 \mathrm{rad}(m_1)} \frac{1}{\ell^{1/3}} \\
&\ll \frac{X^{11/12+(8/3)\eta+\kappa_1/3+4\kappa_2+\epsilon}}{q^2}.
\end{aligned} \tag{66}$$

To estimate the third term, we write

$$\begin{aligned}
&q_1^2 X^{9\kappa_1+\epsilon} \sum_{m_1=1}^{X^{2\eta}} \sideset{}{^\flat}\sum_{\substack{k \\ k_2 \le X^{3\kappa_2}}} \sum_{\ell_1 \le \frac{X^{1/2+\epsilon}}{k}} \frac{k^{1/2}}{\ell_1^{1/2} q_1 \mathrm{rad}(m_1)} \ell m_1^{12} \\
&\ll q_1 X^{3/4+9\kappa_1+26\eta+\epsilon} \sideset{}{^\flat}\sum_{\substack{k \\ k_2 \le X^{3\kappa_2}}} \frac{1}{k} \\
&\ll \frac{X^{7/8+9\kappa_1+26\eta+\kappa_\uparrow+\epsilon}}{q},
\end{aligned} \tag{67}$$

where the final estimate follows because non-zero values of k are all multiples of the squarefree q_1; see Proposition 4.10. Finally, we have

$$\frac{X^{1-\kappa_1+\epsilon}}{q_1^2}\sum_{m_1=1}^{X^{2\eta}}\sum_{k_2\le X^{3\kappa_2}}^{\flat}\sum_{\ell_1\le\frac{X^{1/2+\epsilon}}{k}}\frac{k^{1/2}}{\ell_1^{1/2}q_1\mathrm{rad}(m_1)}\frac{1}{m_1^{5/3}}\ll\frac{X^{1-\kappa_1+3\eta+3\kappa_2+2\kappa_\downarrow+\epsilon}}{q}. \quad (68)$$

This concludes the proof of Proposition 9.13. □

We are now ready to prove the main result of this subsection.

Proposition 9.14 *There exist positive constants $\kappa_\uparrow$, $\kappa_\downarrow$, κ_3 such that the following holds. For every $X \ge 1$ and every squarefree $q \in [X^{1/8-\kappa_\downarrow}, X^{1/8+\kappa_\uparrow}]$, we have*

$$\sum_{f\in\overline{\mathcal{W}_q(\Sigma)}}\Big(S(f)-D(\tfrac12,f)\Big)\Psi\Big(\frac{|\Delta(f)|}{X}\Big)=O_{\Sigma,\Psi}\Big(\frac{X^{1-\kappa_3}}{q}\Big).$$

Proof We apply (64) and then apply Proposition 9.13. It is only necessary to ensure that the exponent of X is less than 1 for each of the 6 different error terms. This is easily done. First, we temporarily pick any positive $\kappa_\uparrow$ and $\kappa_\downarrow$. Next we pick $\eta > 2\kappa_\downarrow$. Then we pick $\kappa_2 > 4\eta + 2\kappa_\downarrow$ and $\kappa_1 > 3\eta + 2\kappa_\downarrow + 3\kappa_2$. This takes care of (68) and of the last two terms of (64).

Finally, to ensure that the exponents of X in the final terms of (65), (66), and (67) are less than 1, we simply divide our constants $\kappa_\uparrow$, $\kappa_\downarrow$, η, κ_1, κ_2 by the same sufficiently large number. □

We now put together our results for the border range and the large range.

Theorem 9.15 *There exists an absolute constant $\varkappa > 0$ such that for every $X \ge 1$ and every squarefree $q \ge X^{1/8-\varkappa}$, we have*

$$\sum_{f\in\overline{\mathcal{W}_q(\Sigma)}}\Big(S(f)-D(\tfrac12,f)\Big)\Psi\Big(\frac{|\Delta(f)|}{X}\Big)=O_{\Sigma,\Psi}\Big(\frac{X^{1-\varkappa}}{q}\Big).$$

Proof We combine Corollary 9.4 and Proposition 9.14, where we choose $\varkappa = \min(\kappa_\uparrow, \kappa_3)$. □

Corollary 9.16 *There exists an absolute constant $\mu > 0$ such that for every $X \ge 1$, we have*

$$\sum_{\substack{q\text{ squarefree}\\ q\ge X^{1/8-\mu}}}\left|\sum_{f\in\overline{\mathcal{W}_q(\Sigma)}}\Big(S(f)-D(\tfrac12,f)\Big)\Psi\Big(\frac{|\Delta(f)|}{X}\Big)\right|=O_{\Sigma,\Psi}\big(X^{1-\mu}\big). \quad (69)$$

Proof Adding up the above result for $q \geq X^{1/8-\varkappa}$, we note that $\{f \in \overline{\mathcal{W}_q(\Sigma)} : |\Delta(f)| < X\}$ is empty for $q \geq X^{1/2}$ because $\Delta(f) = \mathrm{ind}(f)^2\Delta(K_f) \geq q^2\Delta(K_f) \geq q^2$ for $f \in \mathcal{W}_q$. □

Remark 9.17 An admissible set of values of the constants is as follows: $\kappa_\downarrow = \frac{1}{9000}$, $\kappa_\uparrow = \frac{1}{300}$, $\eta = \frac{1}{3000}$ $\kappa_1 = \frac{1}{100}$, $\kappa_2 = \frac{1}{600}$, $\varkappa = \frac{1}{10{,}000}$. To verify the admissibility of these numerical values, it suffices to insert them in each of the remainder terms of Proposition 8.2, Corollary 9.4, Corollary 9.7, (63), (65), (66), (67), and (68).

9.4 Counting suborders

In this subsection we prove Theorem 9.1 by conditionally bounding

$$\sum_{q > X^{1/8-\varkappa}} \sum_{f \in \overline{\mathcal{W}_q(\Sigma)}} S(f).$$

Note that by Corollary 9.16, we may replace $S(f)$ in the above sum by $D(\frac{1}{2}, f)$. The advantage of using $D(\frac{1}{2}, f)$ over $S(f)$ is that the values of $D(\frac{1}{2}, f)$ for binary cubic forms f corresponding to suborders of a fixed cubic field K can be simultaneously controlled in terms of $L(\frac{1}{2}, \rho_K)$. To this end, we start by recalling the following result, due to works of Shintani [31] and Datskovsky–Wright [11] (see [24, §1.2]), giving an explicit formula for the counting function of suborders R of a fixed cubic field K.

Proposition 9.18 *Let K be a cubic field with ring of integers $\mathcal{O}_K$. For an order $R \subset \mathcal{O}_K$, let $\mathrm{ind}(R)$ denote the index of R in $\mathcal{O}_K$. Then*

$$\sum_{R \subset \mathcal{O}_K} \frac{1}{\mathrm{ind}(R)^s} = \frac{\zeta_K(s)}{\zeta_K(2s)} \zeta_{\mathbb{Q}}(3s)\zeta_{\mathbb{Q}}(3s-1).$$

We thus obtain the following corollary regarding the number $N_K(Z)$ of orders of $\mathcal{O}_K$ with index less than Z for a cubic field K.

Corollary 9.19 *For every $\epsilon > 0$, $Z \geq 1$ and cubic field K, we have*

$$N_K(Z) \ll_\epsilon Z^{1+\epsilon}|\Delta(K)|^\epsilon.$$

The implied constant is independent of K and Z.

Proof This follows from Perron's formula integrating along the vertical line $\Re(s) = 1 + \epsilon$. □

The above result can be used to give a very useful bound on the sum of $D(\frac{1}{2}, f)$, over $f \in \overline{\mathcal{W}_q(\Sigma)}$ for q greater than some positive Q.

Lemma 9.20 *For every $Q, X \geq 1$ and $\epsilon > 0$,*

$$\sum_{q \geq Q} \sum_{\substack{f \in \overline{\mathcal{W}_q(\Sigma)} \\ |\Delta(f)| < X}} |D(\tfrac{1}{2}, f)| \ll_{\epsilon,\Sigma} X^{\frac{1}{2}+\epsilon} \sum_{2^{\mathbb{N}} \ni Y \leq X/Q^2} Y^{-\frac{1}{2}} \sum_{\substack{K \in \mathcal{F}_\Sigma \\ Y \leq |\Delta(K)| < 2Y}} |L(\tfrac{1}{2}, \rho_K)|. \quad (70)$$

Proof Consider a real number Y with $Y \ll X/Q^2$ and a cubic field K such that $Y \leq |\Delta(K)| < 2Y$. Then the number of binary cubic forms $f \in \cup_{q \geq Q} \overline{W_q(\Sigma)}$ such that $|\Delta(f)| < X$ and $K_f = K$ is bounded by

$$N_K\Big(\frac{X^{\frac{1}{2}}}{Y^{\frac{1}{2}}}\Big) = O_\epsilon\big(X^{1/2+\epsilon}/Y^{1/2}\big),$$

using Corollary 9.19.

Summing over all K in the discrimant range $Y \leq |\Delta(K)| < 2Y$, and then summing over $Y \in 2^{\mathbb{N}}$ such that the dyadic ranges $[Y, 2Y)$ cover (more than) the interval $[1, X/Q^2]$, we capture the sum over $f \in \overline{\mathcal{W}_q(\Sigma)}$, for all $q > Q$, such that $|\Delta(f)| < X$.

Recall from (31) that we have $D(\frac{1}{2}, f) = L(\frac{1}{2}, \rho_{K_f})E(\frac{1}{2}, f)$ and $E(\frac{1}{2}, f) = \prod_{p|\Delta(f)}(1 + O(p^{-\frac{1}{2}})) = |\Delta(f)|^{o(1)}$, which concludes the proof of the lemma. □

The above lemma yields the following consequence, which clarifies how nonnegativity is used by us.

Corollary 9.21 *For every cubic field $K \in \mathcal{F}_\Sigma$, assume that $L(\frac{1}{2}, \rho_K) \geq 0$. Then for $Q, X \geq 1$, we have*

$$\sum_{q \geq Q} \sum_{\substack{f \in \overline{\mathcal{W}_q(\Sigma)} \\ |\Delta(f)| < X}} D(\tfrac{1}{2}, f) \ll_{\epsilon,\Sigma} X^{29/28+\epsilon} Q^{-15/14}. \tag{71}$$

Proof First note that the assumption $L(\frac{1}{2}, \rho_K) \geq 0$ for all cubic fields K implies that $D(\frac{1}{2}, f) \geq 0$ for all irreducible integral binary cubic forms. Thus, we may apply the previous lemma to estimate the left-hand side of (71).

From Theorem 8.7 (using a smooth function which dominates the characteristic function of $[1, 2]$), we obtain

$$\sum_{\substack{K \in \mathcal{F}_\Sigma \\ Y \leq |\Delta(K)| < 2Y}} |L(\tfrac{1}{2}, \rho_K)| \ll_{\epsilon,\Sigma} Y^{\frac{29-28\delta}{28-16\delta}+\epsilon},$$

for $\delta = 1/128$. Even the bound with $\delta = 0$ in conjunction with (70), yields the result. □

We are now ready to prove Theorem 9.1.

Proof of Theorem 9.1 Proof assuming strong subconvexity: The hypothesis (S) would imply that the central value in the right-hand side of (70) is bounded by $Y^{\frac{1}{6}-\vartheta}$. Hence the bound in (70) becomes $X^{\frac{1}{2}+\epsilon}(X/Q^2)^{\frac{2}{3}-\vartheta}$. We pick $Q = X^{\frac{1}{8}-\kappa_\downarrow}$ with $\epsilon, \kappa_\downarrow > 0$ sufficiently small such that $\frac{1}{2} + \epsilon + (\frac{3}{4} + 2\kappa_\downarrow)(\frac{2}{3} - \vartheta) < 1$. Proposition 8.2, together with Corollary 9.4 and Corollary 9.16, now yield the result.

Proof assuming nonnegativity: We pick $Q = X^{1/8-\varkappa}$, with $\varkappa$ as in Theorem 9.15. It follows that we have

$$\sum_{q\asymp Q}\sum_{f\in\overline{\mathcal{W}_q(\Sigma)}} S(f)\Psi\Big(\frac{|\Delta(K)|}{X}\Big) = \sum_{q\asymp Q}\sum_{f\in\overline{\mathcal{W}_q(\Sigma)}} D(\tfrac{1}{2},f)\Psi\Big(\frac{|\Delta(K)|}{X}\Big) + O_{\epsilon,\Sigma,\Psi}(X^{1-\varkappa+\epsilon}).$$

Since we are assuming hypothesis (N), Corollary 9.21 implies that we have

$$\sum_{q\asymp Q}\sum_{\substack{f\in\overline{\mathcal{W}_q(\Sigma)}\\ |\Delta(f)|\ll X}} D(\tfrac{1}{2},f) \ll_{\epsilon,\Sigma} X^{101/112+\epsilon},$$

which is sufficiently small. The result now follows from Proposition 8.2. □

10 Proofs of Theorems 3 and 4

In addition to the quantity $A_\Sigma(X)$, that we defined in (47), we also define

$$\begin{aligned} MA_\Sigma(X) &:= \sum_{\substack{K\in\mathcal{F}_\Sigma\\ X\le|\Delta(K)|<2X}} |L(\tfrac{1}{2},\rho_K)|;\\ PA_\Sigma(X) &:= \sum_{\substack{K\in\mathcal{F}_\Sigma\\ X/2\le|\Delta(K)|<3X\\ L(\frac{1}{2},\rho_K)\ge 0}} L(\tfrac{1}{2},\rho_K). \end{aligned}$$

The letter M stands for *maximal* and the letter P for *positive*.

Proposition 10.1 *For every $\epsilon>0$ and $X\ge 1$, we have the asymptotic inequality*

$$MA_\Sigma(X) \le 2PA_\Sigma(X) + O_{\epsilon,\Sigma}\big(X^{\frac{29-28\delta}{28-16\delta}+\epsilon}\big).$$

Proof We let $\Psi_1:\mathbb{R}_{>0}\to[0,1]$ be a smooth function compactly supported on the interval $[\frac{1}{2},3]$ such that $\Psi_1(t)=1$ for $t\in[1,2]$. We have an inequality followed by a basic identity

$$\begin{aligned} MA_\Sigma(X) &\le \sum_{K\in\mathcal{F}_\Sigma} |L(\tfrac{1}{2},\rho_K)|\Psi_1\Big(\frac{|\Delta(K)|}{X}\Big)\\ &= 2\sum_{\substack{K\in\mathcal{F}_\Sigma\\ L(\frac{1}{2},\rho_K)\ge 0}} L(\tfrac{1}{2},\rho_K)\Psi_1\Big(\frac{|\Delta(K)|}{X}\Big)\\ &\quad - \sum_{K\in\mathcal{F}_\Sigma} L(\tfrac{1}{2},\rho_K)\Psi_1\Big(\frac{|\Delta(K)|}{X}\Big), \end{aligned} \tag{72}$$

which follows from $|x| = 2\max(x,0) - x$ for every $x \in \mathbb{R}$. The first sum is $\leq 2PA_\Sigma(X)$. (Note that in the respective definitions of $MA_\Sigma(X)$ and $PA_\Sigma(X)$, the discriminant range has increased from $X \leq |\Delta(K)| < 2X$ to $X/2 \leq |\Delta(K)| < 3X$ for this purpose). The second sum is equal to $A_\Sigma(X)$ (up to negligible error) for which we have established the estimate (58). This concludes the proof. □

We finally arrive at the proof of our main result of this paper. In Sect. 8, we have estimated the terms $q < Q$ of the first moment $A_\Sigma(X)$. In Sect. 9, we have estimated for the other terms $q \geq Q$ the difference $S(f) - D(\frac{1}{2}, f)$. The conclusion of all these results is summarized in the following which was stated in the introduction as Theorem 4:

Theorem 10.2 *There is an absolute constant $\mu > 0$ such that the following holds. For every $0 < \nu \leq \mu$, $\epsilon > 0$, and $X \geq 1$,*

$$\begin{aligned} &A_\Sigma(X) - C_\Sigma \cdot X\big(\log X + \widetilde{\Psi}'(1)\big) - C'_\Sigma \cdot X \\ &\ll_{\epsilon,\nu,\Sigma,\Psi} X^{1+\epsilon-\nu} + X^{1/2+\epsilon} \cdot \sum_{2^{\mathbb{N}} \ni Y \leq X^{3/4+\nu}} \frac{MA_\Sigma(Y)}{Y^{1/2}}, \end{aligned} \tag{73}$$

where the sum over Y is dyadic, namely $Y \in 2^{\mathbb{N}}$ is constrained to be a power of 2.

Proof The result will follow from Proposition 8.2, Corollary 9.16 and (70). It follows from Proposition 8.2 that

$$\begin{aligned} &A_\Sigma(X) - C_\Sigma \cdot X\big(\log X + \widetilde{\Psi}'(1)\big) - C'_\Sigma \cdot X \\ &\ll_{\epsilon,\Sigma,\Psi} \frac{X^{1+\epsilon}}{Q} + X^{\frac{11}{12}+\epsilon} + Q^{2+\epsilon}X^{\frac{3}{4}+\epsilon} + \sum_{q \geq Q} \left| \sum_{f \in \overline{W_q(\Sigma)}} S(f)\Psi\left(\frac{|\Delta(f)|}{X}\right) \right|. \end{aligned}$$

Let $a > 0$ be sufficiently small such that $\Psi(t) = 0$ whenever $a^2 t \geq 1$. Choose $Q = a^{-1}X^{1/8-\nu/2}$. Using that $Q \gg_\Psi X^{1/8-\mu}$, we can apply Corollary 9.16 to obtain the bound

$$\sum_{q \geq Q} \left| \sum_{f \in \overline{W_q(\Sigma)}} \Big(S(f) - D(\tfrac{1}{2}, f)\Big)\Psi\left(\frac{|\Delta(f)|}{X}\right) \right| \ll_{\Sigma,\Psi} X^{1-\mu} \leq X^{1-\nu}.$$

The estimate (70) yields

$$\begin{aligned} \sum_{q \geq Q} \sum_{f \in \overline{W_q(\Sigma)}} \left| D(\tfrac{1}{2}, f) \right| \Psi\left(\frac{|\Delta(f)|}{X}\right) &\ll_\Psi \sum_{q \geq Q} \sum_{\substack{f \in \overline{W_q(\Sigma)} \\ |\Delta(f)| < X/a^2}} \left| D(\tfrac{1}{2}, f) \right| \\ &\ll_{\epsilon,\Sigma,\Psi} X^{\frac{1}{2}+\epsilon} \cdot \sum_{2^{\mathbb{N}} \ni Y \leq (X/a^2)/Q^2} \frac{MA_\Sigma(Y)}{Y^{1/2}}. \end{aligned}$$

It remains to observe that $(X/a^2)/Q^2 = X^{3/4+\nu}$ to conclude the proof. □

We are now ready to prove our main Theorem 3. Recall that the qualitative version in Theorem 2 follows from Theorem 3.

Proof of Theorem 3 Recall that $C_\Sigma > 0$ in Proposition 8.2. We are going to use (73), Theorem 4.1, Proposition 10.1 which each have their own arbitrarily small $\epsilon > 0$, and without loss of generality we arrange that ϵ be the same in each. (An alternative approach commonly used would be to assume that ϵ may vary from line to line.)

We distinguish *two cases* depending on the size of the sum of $MA_\Sigma(Y)$ in the right-hand side of (73).

In the *first case*, if the right-hand side of (73) is $< X$, then we have $A_\Sigma(X) \sim C_\Sigma \cdot X \cdot \log X$. In combination with Theorem 4.1, we obtain that $\gg_{\epsilon,\Sigma} X^{\frac{3}{4}+\delta-\epsilon}$ cubic fields $K \in \mathcal{F}_\Sigma$ with $|\Delta(K)| < X$ satisfy $L(\frac{1}{2}, \rho_K) > 0$. Hence

$$\delta_\Sigma(X) \geq \frac{3}{4} + \delta - \epsilon - O_\epsilon\big(\frac{1}{\log X}\big). \tag{74}$$

Assume in the *second case* that the right-hand side of (73) is $\geq X$, namely

$$\sum_{2^{\mathbb{N}} \ni Y \leq X^{3/4+\nu}} \frac{MA_\Sigma(Y)}{Y^{1/2}} \geq X^{1/2-\epsilon}.$$

This implies that there exists $Y \in 2^{\mathbb{N}}$ with $Y \leq X^{3/4+\nu}$ such that $MA_\Sigma(Y) \geq X^{1/2-\epsilon} \log_2(X)^{-1} Y^{1/2}$. It follows from Proposition 10.1 that

$$2PA_\Sigma(Y) \geq X^{1/2-\epsilon} \log_2(X)^{-1} Y^{1/2} + O_{\epsilon,\Sigma}\big(Y^{\frac{29-28\delta}{28-16\delta}+\epsilon}\big).$$

Since $Y \leq X^{3/4+\nu}$, the error term is negligible. (The convexity bound $\delta = 0$ suffices for this). We deduce

$$PA_\Sigma(Y) \gg_\epsilon X^{1/2-2\epsilon} Y^{1/2}. \tag{75}$$

Theorem 4.1 and (75) imply that $\gg_\epsilon X^{1/2-2\epsilon} Y^{1/4+\delta-\epsilon}$ cubic fields $K \in \mathcal{F}_\Sigma$ with $|\Delta(K)| < Y$ satisfy the inequality $L(\frac{1}{2}, \rho_K) > 0$. Hence

$$\delta_\Sigma(Y) \geq \big(\frac{1}{2} - 2\epsilon\big)\frac{\log X}{\log Y} + \big(\frac{1}{4} + \delta - \epsilon\big) - O_\epsilon\big(\frac{1}{\log Y}\big). \tag{76}$$

We are now ready to conclude the lower bound on the lim sup in Theorem 3 by showing that

$$\limsup_{X\to\infty} \delta_\Sigma(X) \geq \frac{3}{4} + \delta.$$

Indeed, the inequality is satisfied either in the first case by $\delta_\Sigma(X)$ in (74), or in the second case by $\delta_\Sigma(Y)$ in (76), since $\frac{\log X}{\log Y} \geq \frac{4}{3+4\nu} \geq 1$ and since (75) implies that $Y \to \infty$.

To establish the lower bound on the lim inf in Theorem 3, we need a lower bound on Y in the second case. Theorem 4.1 implies $PA_\Sigma(Y) = O_\epsilon(Y^{\frac{5}{4}-\delta+\epsilon})$. Together with (75), this yields the following lower bound:

$$Y \gg_\epsilon X^{\frac{2}{3-4\delta}-\epsilon}. \tag{77}$$

This implies

$$\delta_\Sigma(X) \geq \frac{1}{2} + \left(\frac{1}{4}+\delta\right)\cdot\frac{2}{3-4\delta} - \epsilon - O_\epsilon\left(\frac{1}{\log X}\right). \tag{78}$$

The first two terms of (78) simplify to $\frac{2}{3-4\delta}$, hence

$$\liminf_{X\to\infty} \delta_\Sigma(X) \geq \frac{2}{3-4\delta}.$$

This concludes the proof of Theorem 3. □

The same argument implies something slightly stronger than an *Omega* result $MA_\Sigma(X) = \Omega_\Sigma(X)$ as $X \to \infty$. Namely, that there is a sequence $X_k \to \infty$ such that $MA_\Sigma(X_k)/X_k \to \infty$. Indeed, in the first case of the proof of Theorem 3, we have $A_\Sigma(X) \sim C_\Sigma \cdot X\log X$. In the second case, we have

$$MA_\Sigma(Y) \gg X^{1/2-o(1)}Y^{1/2} \gg Y^{7/6-o(1)},$$

in view of $Y \leq X^{3/4+\nu}$. Moreover we have seen that (75) implies $Y \to \infty$, which enables us to extract a sequence $X_k = Y \to \infty$ such that $MA_\Sigma(Y)/Y \to \infty$.

For completeness, we also record the following lower bound for the first moment:

Proposition 10.3 *For every $\epsilon > 0$ and $X \geq 1$,*

$$\sum_{K\in\mathcal{F}_\Sigma(X)} \left|L\left(\tfrac{1}{2},\rho_K\right)\right| \gg_{\epsilon,\Sigma} X^{\frac{5-4\delta}{6-8\delta}-\epsilon}.$$

Proof Suppose first that we are in the first case of the proof of Theorem 3. Then we have $A_\Sigma(X) \sim C_\Sigma \cdot X\log X$, implying that the left-hand side of the above equation is $\gg_\Sigma X\log X$. Suppose instead that we are in the second case. Then the lower bound (77) for Y implies the lower bound in Proposition 10.3 as follows:

$$\sum_{K\in\mathcal{F}_\Sigma(X)} |L(\tfrac{1}{2},\rho_K)| \geq \sum_{K\in\mathcal{F}_\Sigma(Y)} |L(\tfrac{1}{2},\rho_K)| \gg_{\epsilon,\Sigma} X^{\frac{1}{2}-\epsilon}Y^{\frac{1}{2}},$$

and $\frac{1}{2} + \frac{1}{3-4\delta} = \frac{5-4\delta}{6-8\delta}$. □

Notation

$A_\Sigma(X)$	smoothed first moment of $L(\frac{1}{2},\rho_K)$
C_Σ, C'_Σ	main terms for the first moment

$D(s, f)$	Dirichlet series of $\lambda_n(f)$
$E_\infty(\widetilde{\Psi}; \epsilon)$	archimedean norm of $\widetilde{\Psi}$
$E_n(\psi)$	norm of ψ weighted by splitting types
$E_p(s, f)$	Euler factor of the form f nonmaximal at p
$G(s)$	choice of an even holomorphic function
$K_f = R_f \otimes \mathbb{Q}$	cubic field corresponding to the form $f \in V(\mathbb{Z})^{\mathrm{irr}}$
M	matrix of the Fourier transform of $\mathrm{GL}_2(\mathbb{F}_p)$-orbits on $V(\mathbb{F}_p)$
$MA_\Sigma(X)$	sum of $\lvert L(\frac{1}{2}, \rho_K)\rvert$ for $K \in \mathcal{F}_\Sigma(X)$
$PA_\Sigma(X)$	sum of $L(\frac{1}{2}, \rho_K) \geq 0$ for $K \in \mathcal{F}_\Sigma(X)$
R_f	cubic ring corresponding to a form $f \in V(\mathbb{Z})$
$T_\Sigma(s)$	Dirichlet series of $t_\Sigma(n)$
V	space of binary cubic forms with twisted action by GL_2
$V(\mathbb{Z})^{\max}$	subset of maximal binary cubic forms
$V(\mathbb{Z})^{\mathrm{irr}}$	subset of irreducible binary cubic forms
$V(\mathbb{Z}_p)^{\mathrm{nm}}$	subset of $V(\mathbb{Z}_p)$ of nonmaximal cubic forms
V^*	dual of V with compatible action by GL_2
$V^\pm$	test function in the approximate functional equation
$\Delta(K)$	discriminant of the cubic field K
$\Delta(R)$	discriminant of the cubic ring R
$\Delta(f)$	discriminant of the binary cubic form f
$\mathcal{F}_\Sigma$	family of cubic fields prescribed by Σ
$\mathcal{H}_y$	compactly supported function on $\mathbb{R}_{>0}$
$\mathcal{O}_\sigma^*$	orbits for the action of $\mathrm{GL}_2(\mathbb{F}_p)$ on $V^*(\mathbb{F}_p)$
$\mathcal{O}_\sigma$	orbits for the action of $\mathrm{GL}_2(\mathbb{F}_p)$ on $V(\mathbb{F}_p)$
$\Sigma = (\Sigma_v)$	finite collection of local specifications
$\mathcal{U}_b$	set of cubic forms f with $\mathrm{ind}(f) = b$
$\mathcal{W}_q$	elements in $V(\mathbb{Z})$ nonmaximal at every prime dividing q
$\mathcal{Y}_{b,r}$	subset of cubic forms $f \in \mathcal{W}_r$ with $b \parallel \mathrm{ind}(f)$
$\alpha^\pm, \beta^\pm, \gamma^\pm$	residues of Shintani zeta function
$\mathcal{A}_n^{(q)}, \mathcal{C}_n^{(q)}$	residue functionals with nonmaximality condition at q
$\mathcal{A}^{\max}, \mathcal{C}^{\max}$	residue functionals with maximality condition
$\mathcal{A}_n(\phi), \mathcal{B}_n(\phi), \mathcal{C}_n(\phi)$	linear functionals for residues of $\xi^\pm(\phi, s)$
χ_Σ	characteristic function of forms with specification Σ
$\delta > 0$	subconvexity exponent for $\zeta_K(\frac{1}{2})$
$\delta_\Sigma(X)$	logarithmic density of fields $K \in \mathcal{F}_\Sigma(X)$ with $\zeta_K(\frac{1}{2}) < 0$
$\gamma^\pm(s)$	Gamma factor in the functional equation of $L(s, \rho_K)$
$\mathrm{ind}(f)$	index of R_f in $\mathcal{O}_{K_f}$
$\lambda_K(n)$	nth Dirichlet coefficient of $L(s, \rho_K)$
$\lambda_n(f)$	Artin character on the space of cubic forms
$\omega_m^{(1)}(f)$	number of simple roots of f modulo m
$\omega_p(g)$	number of zeros in $\mathbb{P}^1(\mathbb{F}_p)$ of g modulo p
$\overline{T}$	set of $\mathrm{GL}_2(\mathbb{Z})$-orbits on T
$\phi_p(1^3) = \phi_p(0)$	simple congruence function at p
$\pm$	$+$ is for totally real fields and $-$ is for complex fields
$\mathrm{rad}(k)$	radical of the positive integer k

ρ_K	two-dimensional Galois representation
$\sigma_p(f)$	splitting type of f at p
$\theta_K(n)$	coefficient of the logarithmic derivative of $L(s, \rho_K)$
$\theta_n(f)$	coefficients of the logarithmic derivative of $D(s, f)$
$\widehat{\phi}: V(\mathbb{Z}/n\mathbb{Z}) \to \mathbb{C}$	Fourier transform of function ϕ on $V(\mathbb{Z}/n\mathbb{Z})$
$\widetilde{\Phi}, \widetilde{\Psi}$	Mellin transforms of Φ, Ψ
$\xi^{\pm}(\phi, s)$	Shintani zeta function with congruence function ϕ
$\xi^{*\pm}(\psi, s)$	dual Shintani zeta function with congruence function ψ
$b_p(f), c_p(f)$	densities of splitting types
$e_{p,m}(f)$	coefficients of Euler factor of f nonmaximal at p
$f \leftrightarrow (g, \alpha)$ switch	R_f is an index-p subring of R_g
$g(y)$	equal to $\widetilde{H_y}(1)$
q	square-free integer entering into the sieve
$q \geq X^{1/8+\kappa_\uparrow}$	large range of the sieve
$q \in [X^{1/8-\kappa_\downarrow}, X^{1/8+\kappa_\uparrow}]$	border range of the sieve
r_Σ	product of primes p such that Σ_p is specified at p
$t_\Sigma(n)$	average of $\lambda_K(n)$ over K in $\mathcal{F}_\Sigma$
$v_p(k) \geq 2$ for every $p \mid k$	powerful integer
$S(f)$	truncated Dirichlet sum of $\lambda_n(f)$

Acknowledgements We are very grateful to the anonymous referee for a thorough reading and several valuable comments and suggestions. We are also happy to thank Chantal David, Mehmet Durlanik, and Jacob Tsimerman for many helpful conversations.

Funding information Open access funding provided by Chalmers University of Technology. ASh was supported by an NSERC discovery grant and a Sloan fellowship. ASo was supported by the grant 2016-03759 from the Swedish Research Council. NT acknowledges support from the National Science Foundation under grants DMS-1454893 and DMS-2001071.

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