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RESEARCH

Tensor Hierarchy Algebras and Restricted Associativity

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Abstract
We study local algebras, which are structures similar to $\mathbb{Z}$-graded algebras concentrated in degrees $-1, 0, 1$, but without a product defined for pairs of elements at the same degree ± 1. To any triple consisting of a Kac–Moody algebra $\mathfrak{g}$ with an invertible and symmetrisable Cartan matrix, a dominant integral weight of $\mathfrak{g}$ and an invariant symmetric bilinear form on $\mathfrak{g}$, we associate a local algebra satisfying a restricted version of associativity. From it, we derive a local Lie superalgebra by a commutator construction. Under certain conditions, we identify generators which we show satisfy the relations of the tensor hierarchy algebra W previously defined from the same data. The result suggests that an underlying structure satisfying such a restricted associativity may be useful in applications of tensor hierarchy algebras to extended geometry.

Keywords Integer-graded Lie superalgebras · Local Lie superalgebras · Kac-Moody algebras · Non-associative algebras

Mathematics Subject Classification (2010) 17A30 · 17B60 · 17B67 · 17B70

1 Introduction

The concept of local Lie algebras have played an important role in the classification of simple irreducible $\mathbb{Z}$-graded Lie algebras [1] (and thus to the development of Kac–Moody algebras) by providing a "seed" at degrees $-1, 0, 1$ in the construction. The concept can obviously be generalised from $\mathbb{Z}$-graded Lie algebras to general $\mathbb{Z}$-graded algebras. However, it seems that such "local algebras" have not been studied much in cases other than those where the $\mathbb{Z}$-graded algebra is a Lie algebra or a Lie superalgebra [2]. Still in the context of Lie (super)algebras, it might for example be interesting to consider the commutator in a "local associative algebra". It then turns out that the associative law is relevant only when at least

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one of the three involved elements has degree zero. In the present paper, we introduce the concept of *focal associativity* for local algebras where the associative law is restricted to these cases. We will show that such a structure can be seen as underlying *tensor hierarchy algebras*, which are infinite-dimensional generalisations of Cartan-type Lie superalgebras [3–5]. Tensor hierarchy algebras, originally used in the context of gauged supergravity [6], have proven very useful in the framework of extended geometry, where diffeomorphisms are unified with gauge transformations in supergravity theories [7–12].

The paper is organised as follows. In Section 2 we introduce the concept of local algebras, generalising the concept of local Lie algebras introduced by Kac [1], which we also specialise to *contragredient* local Lie superalgebras. We will show how any contragredient local Lie superalgebra $\mathcal{G}^{\mathbb{L}}$ gives rise to a focally associative local superalgebra $\mathcal{G}^{\ell}$, which in turns gives back a different local Lie superalgebra $\mathcal{G}^{\mathsf{L}}$ with the commutator in $\mathcal{G}^{\ell}$ as the bracket. In Section 3 we show how a contragredient local Lie superalgebra $\mathcal{B}^{\mathbb{L}}$ can be defined from a triple $(\mathfrak{g}, \lambda, \kappa)$, where $\mathfrak{g}$ is a symmetrisable Kac–Moody algebra, λ is a dominant integral weight of $\mathfrak{g}$ and κ is an invariant symmetric bilinear form on $\mathfrak{g}$. This contragredient local Lie superalgebra $\mathcal{B}^{\mathbb{L}}$ is the local part of a contragredient Lie superalgebra $\mathcal{B}$, which is also a Borcherds–Kac–Moody superalgebra [2]. In Section 4 we recall the definition by generators and relations of a tensor hierarchy algebra W from the same data $(\mathfrak{g}, \lambda, \kappa)$, under some further conditions [4]. We then apply the construction in Section 1 to the contragredient Lie superalgebra $\mathcal{B}^{\mathbb{L}}$ defined in Section 2. We identify the generators of W with elements in $\mathcal{B}^{\mathsf{L}}$ and show that they generate a subalgebra where the defining relations of W are satisfied up to an ideal intersecting the degree-zero subspace trivially.

2 Local Algebras

2.1 Definitions

Let the ground field $\mathbb{K}$ be any algebraically closed field of characteristic zero.

We start by recalling that a $\mathbb{Z}$*-graded algebra* is a $\mathbb{Z}$-graded vector space $U = \bigoplus_{k\in\mathbb{Z}} U_k$ together with a degree-preserving linear map $U \otimes U \to U$, where the $\mathbb{Z}$-grading on $U \otimes U$ is given by $(U \otimes U)_k = \bigoplus_{i+j=k} U_i \otimes U_j$. Similarly, we define a *local algebra* as a $\mathbb{Z}$-graded vector space $U = U_{-1} \oplus U_0 \oplus U_1$ together with a degree-preserving linear map $\bigoplus_{k=-1,0,1}(U \otimes U)_k \to U$. The image of a simple tensor $u \otimes v$ is generally called *product* (as well as the map itself) and denoted uv, but in the Lie cases below, it will be called *bracket* and denoted $[u, v]$ or $[\![u, v]\!]$. Note that this notion of "locality" has nothing to do with the usual one for rings and algebras. In particular, a local algebra is actually not an algebra since the product is not defined for any pair of elements.

In a $\mathbb{Z}$-graded or local *superalgebra*, the product is also degree-preserving with respect to an additional $\mathbb{Z}_2$-grading, $U = U_{(0)} \oplus U_{(1)}$. The $\mathbb{Z}$-grading is *consistent* if $U_i \subseteq U_{(j)}$ whenever $i \equiv j \pmod 2$. In powers of -1, we will simplify the notation and write, for example, $(-1)^{uv}$ for homogeneous elements u, v, where the exponent is actually the product of their $\mathbb{Z}_2$-degrees. We will also use subscripts to denote $\mathbb{Z}$-degrees of homogeneous components of elements, for example, $u = \sum_{k\in\mathbb{Z}} u_k$ in a $\mathbb{Z}$-graded superalgebra, and $u = u_{-1} + u_0 + u_1$ in a local superalgebra. Clearly, any $\mathbb{Z}$-graded algebra U gives rise to a local algebra by restricting the vector space to the subspace $U_{-1} \oplus U_0 \oplus U_1$ and the domain of the product to $\bigoplus_{k=-1,0,1}(U \otimes U)_k$. This local algebra is called the *local part* of the $\mathbb{Z}$-graded superalgebra U.

We say that a local algebra is *focally associative* if the degree-zero subspace associates with any element, that is, if the identity $(u_iv_j)w_k = u_i(v_jw_k)$ holds whenever all involved products are defined and at least one of the three indices i, j, k is zero. Thus the following 13 identities are satisfied for any u, v, w in a focally associative local algebra,

$$(u_0v_0)w_0 = u_0(v_0w_0)\,, \tag{2.1}$$

$$(u_{\pm1}v_0)w_0 = u_{\pm1}(v_0w_0)\,, \tag{2.2}$$
$$(u_0v_0)w_{\pm1} = u_0(v_0w_{\pm1})\,, \tag{2.3}$$
$$(u_0v_{\pm1})w_0 = u_0(v_{\pm1}w_0)\,, \tag{2.4}$$

$$(u_0v_{\pm1})w_{\mp1} = u_0(v_{\pm1}w_{\mp1})\,, \tag{2.5}$$
$$(u_{\pm1}v_{\mp1})w_0 = u_{\pm1}(v_{\mp1}w_0)\,, \tag{2.6}$$
$$(u_{\pm1}v_0)w_{\mp1} = u_{\pm1}(v_0w_{\mp1})\,. \tag{2.7}$$

If in addition the two identities

$$(u_{\pm1}v_{\mp1})w_{\pm1} = u_{\pm1}(v_{\mp1}w_{\pm1}) \tag{2.8}$$

are satisfied for any u, v, w, then the local algebra is *associative*.

We define concepts like *subalgebras* and *ideals* of local algebras in the same way as of algebras. For any ideal D of a local algebra U, we also define the *quotient algebra* U/D in the same way as for an ideal of an algebra. (Thus subalgebras and quotient algebras of local algebras are local algebras as well.) We say that the ideal D is *peripheral* if $D = D_{-1} \oplus D_1$, where $D_{\pm1} \subseteq U_{\pm1}$. The sum of all peripheral ideals is again a peripheral ideal, and therefore a unique maximal peripheral ideal.

Let U be a local algebra and let M be a subset of it. With *the subalgebra of U generated by M modulo the maximal peripheral ideal* we mean the quotient algebra V/D, where V is the subalgebra of U generated by the subset M, and D is the maximal peripheral ideal of V.

A *local Lie superalgebra* (the logical ordering of words from our point of view here would rather be *Lie local superalgebra*, but we stick to the established one [2]) is a local superalgebra where the product is a bracket that satisfies the graded antisymmetry

$$[x, y] = -(-1)^{xy}[y, x] \tag{2.9}$$

and the Jacobi identity

$$[[x, y], z] = [x, [y, z]] - (-1)^{xy}[y, [x, z]] \tag{2.10}$$

for any homogeneous elements such that the involved brackets are defined. These two identities can be broken down into the three plus five identities

$$\begin{aligned}
[x_0, y_0] &= -(-1)^{xy}[y_0, x_0]\,,\\
[x_0, y_{\pm1}] &= -(-1)^{xy}[y_{\pm1}, x_0]\,,\\
[x_{\pm1}, y_{\mp1}] &= -(-1)^{xy}[y_{\mp1}, x_{\pm1}]\,,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
[[x_0, y_0], z_0] &= [x_0, [y_0, z_0]] - (-1)^{xy}[y_0, [x_0, z_0]]\,,\\
[[x_0, y_0], z_{\pm1}] &= [x_0, [y_0, z_{\pm1}]] - (-1)^{xy}[y_0, [x_0, z_{\pm1}]]\,,\\
[[x_{\pm1}, y_{\mp1}], z_0] &= [x_{\pm1}, [y_{\mp1}, z_0]] - (-1)^{xy}[y_{\mp1}, [x_{\pm1}, z_0]]
\end{aligned} \tag{2.12}$$

for elements that are homogeneous not only with respect to the $\mathbb{Z}_2$-grading, but also with respect to the $\mathbb{Z}$-grading, in the same way as the associative identity $(uv)w = u(vw)$ can be broken down into the 15 identities (2.1)–(2.8) above.

2.2 From Focally Associative Local Algebras to Local Lie Superalgebras and Back

For any local superalgebra $\mathcal{G}^{\ell}$, we let $\mathcal{G}^{\mathsf{L}}$ be the superalgebra which is the same vector space as $\mathcal{G}^{\ell}$, but with a different product which is a bracket given by the *commutator* $[x, y] = xy - (-1)^{xy} yx$ for homogeneous elements x, y. The following proposition is an immediate consequence of the corresponding fundamental statement for associative and Lie superalgebras, and straightforward to prove.

Proposition 2.1 *If $\mathcal{G}^{\ell}$ is focally associative, then $\mathcal{G}^{\mathsf{L}}$ is a local Lie superalgebra.*

The reason why focal associativity is sufficient is that there is no Jacobi identity involving two elements at degree ± 1 and one element at degree ∓ 1, since such an identity would involve the bracket of the two elements at degree ± 1, which is not defined in a local Lie superalgebra.

We will now go in the opposite direction and associate a focally associative local algebra to a local Lie superalgebra satisfying some further conditions.

Let $\mathcal{G}^{\mathbb{L}} = \mathcal{G}^{\mathbb{L}}{}_{-1} \oplus \mathcal{G}^{\mathbb{L}}{}_0 \oplus \mathcal{G}^{\mathbb{L}}{}_1$ be a local Lie superalgebra with a bracket $[\![-, -]\!]$. We say that $\mathcal{G}^{\mathbb{L}}$ is *contragredient* if there is an element $L \in \mathcal{G}^{\mathbb{L}}{}_0$ such that $[\![L, x_k]\!] = kx_k$ for all $x \in \mathcal{G}^{\mathbb{L}}$ and a bilinear map

$$\mathcal{G}^{\mathbb{L}}{}_{-1} \times \mathcal{G}^{\mathbb{L}}{}_1 \to \mathbb{K}\,, \qquad (x, y) \mapsto \langle x|y\rangle\,, \tag{2.13}$$

which is *invariant* and *homogeneous*. The conditions of invariance and homogeneity mean, respectively, that $\langle [\![x_{-1}, y_0]\!] | z_1\rangle = \langle x_{-1} | [\![y_0, z_1]\!]\rangle$ for all $x, y, z \in \mathcal{G}^{\mathbb{L}}$, and that $\langle x|y\rangle = 0$ whenever x and y are homogeneous with different $\mathbb{Z}_2$-degrees. It is convenient to also define a corresponding bilinear map

$$\mathcal{G}^{\mathbb{L}}{}_1 \times \mathcal{G}^{\mathbb{L}}{}_{-1} \to \mathbb{K}\,, \qquad (x, y) \mapsto \langle x|y\rangle = (-1)^{xy}\langle y|x\rangle\,. \tag{2.14}$$

by graded symmetry.

To any contragredient local Lie superalgebra $\mathcal{G}^{\mathbb{L}} = \mathcal{G}^{\mathbb{L}}{}_{-1} \oplus \mathcal{G}^{\mathbb{L}}{}_0 \oplus \mathcal{G}^{\mathbb{L}}{}_1$, we associate a focally associative local superalgebra $\mathcal{G}^{\ell} = \mathcal{G}^{\ell}{}_{-1} \oplus \mathcal{G}^{\ell}{}_0 \oplus \mathcal{G}^{\ell}{}_1$ in the following way. Let $\mathcal{G}^{\ell}{}_0$ be the universal enveloping algebra of $\mathcal{G}^{\mathbb{L}}{}_0$, set $\mathcal{G}^{\ell}{}_{\pm 1} = \mathcal{G}^{\mathbb{L}}{}_{\pm 1} \otimes \mathcal{G}^{\ell}{}_0$ and write $x \otimes 1 = x$ for any $x \in \mathcal{G}^{\mathbb{L}}{}_{\pm 1}$ (so that we consider $\mathcal{G}^{\mathbb{L}}{}_{\pm 1}$ as a subspace of $\mathcal{G}^{\ell}{}_{\pm 1}$). Accordingly, we consider $\mathcal{G}^{\mathbb{L}}{}_{-1} \oplus \mathbb{K} \oplus \mathcal{G}^{\mathbb{L}}{}_1$ as a subspace of $\mathcal{G}^{\ell}$. For x and y in this subspace, set

$$\begin{aligned} x_{-1}y_1 &= -a[\![x_{-1}, y_1]\!] + b\langle x_{-1}|y_1\rangle L\,,\\ x_1 y_{-1} &= a[\![x_1, y_{-1}]\!] + b\langle x_1|y_{-1}\rangle L + c\langle x_1|y_{-1}\rangle \end{aligned} \tag{2.15}$$

for some constants $a, b, c \in \mathbb{K}$, and let $x_0 y_{\pm 1} = y_{\pm 1} x_0$ be given by the action of $x_0 \in \mathbb{K}$ that $\mathcal{G}^{\mathbb{L}}{}_{-1} \oplus \mathcal{G}^{\mathbb{L}}{}_1$ is equipped with as a vector space over $\mathbb{K}$.

Note that $\mathcal{G}^{\mathbb{L}}{}_{-1} \oplus \mathbb{K} \oplus \mathcal{G}^{\mathbb{L}}{}_1$ is in general not a local algebra with respect to the product defined so far, since the right hand sides of (2.15) in general do not belong to this subspace of $\mathcal{G}^{\ell}$. In order to achieve a local algebra, we will now extend the product to the whole of $\mathcal{G}^{\ell}$. First, we recursively define subspaces $(\mathcal{G}^{\ell}{}_0)^k$ of $\mathcal{G}^{\ell}{}_0$ for any integer $k \geq 0$ by setting $(\mathcal{G}^{\ell}{}_0)^0 = \mathbb{K}$ and letting $(\mathcal{G}^{\ell}{}_0)^{k+1}$ be spanned by all elements ux where $u \in (\mathcal{G}^{\ell}{}_0)^k$ and $x \in \mathcal{G}^{\mathbb{L}}{}_0$. As the universal enveloping algebra of $\mathcal{G}^{\mathbb{L}}{}_0$, the algebra $\mathcal{G}^{\ell}{}_0$ is the sum of all such subspaces. Second, we define the product on $\mathcal{G}^{\ell}$ recursively by

$$x\big(y \otimes v\big) = (xy)v \tag{2.16}$$

and

$$\big(x \otimes (uz)\big)\big(y \otimes v\big) = (x \otimes u)\big([\![z, y]\!] \otimes v\big) + (-1)^{yz}(x \otimes u)\big(y \otimes (zv)\big)\,, \tag{2.17}$$

where

$$x \in \mathbb{K} \oplus \mathcal{G}^{\mathbb{L}}{}_{\pm 1}\,, \qquad y \in \mathbb{K} \oplus \mathcal{G}^{\mathbb{L}}{}_{\mp 1}\,, \qquad z \in \mathcal{G}^{\mathbb{L}}{}_{0}\,, \qquad u \in (\mathcal{G}^{\ell}{}_{0})^{i}\,, \qquad v \in (\mathcal{G}^{\ell}{}_{0})^{j} \tag{2.18}$$

for $i \geq 0$ and where we set $[\![z, y]\!] = 0$ if $y \in \mathbb{K}$.

It is straightforward to check that the product is well defined. What we have to check is that the right hand side of (2.17) does not depend on the way of writing an element in $(\mathcal{G}^{\ell}{}_{0})^{i+1}$ as a sum of products uz where $u \in (\mathcal{G}^{\ell}{}_{0})^{i}$ and $z \in \mathcal{G}^{\mathbb{L}}{}_{0}$. This can be shown by induction over i. Setting $u = u'w$, where $u' \in (\mathcal{G}^{\ell}{}_{0})^{i-1}$ (for $i \geq 1$) and $w \in \mathcal{G}^{\mathbb{L}}{}_{0}$, we have

$$uz = u'wz = u'[\![w, z]\!] + (-1)^{wz}u'zw\,. \tag{2.19}$$

and thus applying the recursive definition to

$$\big(x \otimes (u'[\![w, z]\!])\big)\big(y \otimes v\big) + (-1)^{wz}\big(x \otimes (u'z)w\big)\big(y \otimes v\big) \tag{2.20}$$

must give the right hand side of (2.17). Indeed,

$$\begin{aligned}
&\big(x \otimes u'[\![w, z]\!]\big)\big(y \otimes v\big) + (-1)^{wz}(x \otimes u'zw)\big(y \otimes v\big)\\
&\quad = (x \otimes u')\big([\![[\![w, z]\!], y]\!] \otimes v\big) + (-1)^{y(w+z)}(x \otimes u')\big(y \otimes [\![w, z]\!]v\big)\\
&\qquad + (-1)^{wz}(x \otimes u'z)\big([\![w, y]\!] \otimes v\big) + (-1)^{w(y+z)}(x \otimes u'z)\big(y \otimes wv\big)\\
&\quad = (x \otimes u')\big([\![w, [\![z, y]\!]]\!] \otimes v\big) - (-1)^{zw}(x \otimes u')\big([\![z, [\![w, y]\!]]\!] \otimes v\big)\\
&\qquad + (-1)^{y(w+z)}(x \otimes u')(y \otimes wzv) - (-1)^{y(w+z)+wz}(x \otimes u')(y \otimes zwv)\\
&\qquad + (-1)^{wz}(x \otimes u'z)\big([\![w, y]\!] \otimes v\big) + (-1)^{w(y+z)}(x \otimes u'z)\big(y \otimes wv\big)\\
&\quad = (x \otimes u')\big([\![w, [\![z, y]\!]]\!] \otimes v\big) + (-1)^{wz+(w+y)z}(x \otimes u')\big([\![w, y]\!] \otimes zv\big)\\
&\qquad + (-1)^{y(w+z)}(x \otimes u')(y \otimes wzy) + (-1)^{w(y+z)}(x \otimes u')\big([\![z, y]\!] \otimes wv\big)\\
&\quad = (x \otimes u'w)\big([\![z, y]\!] \otimes v\big) + (-1)^{yz}(x \otimes u'w)\big(y \otimes (zv)\big)\,.
\end{aligned} \tag{2.21}$$

By setting $x = 1$ in (2.16), we see that the tensor product symbol $\otimes$ is superfluous (and it will henceforth be omitted). Also, it follows from the two equations that we obtain from (2.17) by setting $x = y = u = 1$ and $x = y = v = 1$ that the product on $\mathcal{G}^{\ell}{}_{0}$ defined by this equation is the same as the one that this vector space is equipped with as the universal enveloping algebra of $\mathcal{G}^{\mathbb{L}}{}_{0}$. The product is thus associative on $\mathcal{G}^{\ell}{}_{0}$.

Let us compute the commutator $[x, y] = xy - (-1)^{yx}yx$ given by the product above for elements in $\mathcal{G}^{\mathbb{L}} \subseteq \mathcal{G}^{\ell}$. It is equal to the original bracket in the following cases,

$$[x_0, y_0] = [\![x_0, y_0]\!]\,, \qquad [x_0, y_{\pm 1}] = [\![x_0, y_{\pm 1}]\!]\,, \qquad [x_{\pm 1}, y_0] = [\![x_{\pm 1}, y_0]\!]\,, \tag{2.22}$$

but not when $x \in \mathcal{G}^{\mathbb{L}}{}_{\pm 1}$ and $y \in \mathcal{G}^{\mathbb{L}}{}_{\mp 1}$. In this case we instead get

$$\begin{aligned}
[x_{-1}, y_1] &= x_{-1}y_1 - (-1)^{xy}y_1x_{-1}\\
&= -a[\![x_{-1}, y_1]\!] + b\langle x_{-1}|y_1\rangle L\\
&\quad - a(-1)^{xy}[\![y_1, x_{-1}]\!] - b(-1)^{xy}\langle y_1|x_{-1}\rangle L - c(-1)^{xy}\langle y_1|x_{-1}\rangle\\
&= -c(-1)^{xy}\langle y_1|x_{-1}\rangle = -c\langle x_{-1}|y_1\rangle\,.
\end{aligned} \tag{2.23}$$

We will now show that the local algebra $\mathcal{G}^{\ell} = \mathcal{G}^{\ell}{}_{-1} \oplus \mathcal{G}^{\ell}{}_{0} \oplus \mathcal{G}^{\ell}{}_{1}$ is indeed focally associative. We already know that the identity (2.1) holds for all $u, v, w \in \mathcal{G}^{\ell}$ since $\mathcal{G}^{\ell}$ is the universal enveloping algebra of $\mathcal{G}^{\mathbb{L}}$ and thus associative. The identities (2.2), (2.3) and (2.7) are consequences of the following proposition.

Proposition 2.2 *The identity*

$$\big((xu)w\big)(yv) = (xu)\big(w(yv)\big)\,, \tag{2.24}$$

where

$$x \in \mathbb{K} \oplus \mathcal{G}^{\mathbb{L}}{}_{\pm 1}\,, \quad y \in \mathbb{K} \oplus \mathcal{G}^{\mathbb{L}}{}_{\mp 1}\,, \quad u \in (\mathcal{G}^{\ell}{}_0)^i\,, \quad v \in (\mathcal{G}^{\ell}{}_0)^j\,, \quad w \in (\mathcal{G}^{\ell}{}_0)^k\,, \tag{2.25}$$

holds for all integers $i, j, k \geq 0$.

Proof We will prove this by induction over $i + k \geq 0$. The base cases are trivial. Suppose the identities hold for $i + k \leq p$ for some $p \geq 0$. For $i = p$ we then have

$$((xu)z)(yv) = (x(uz))(yv) = (xu)([z, y]v) + (-1)^{yz}(xu)(y(zv)) = (xu)(z(yv))\,, \tag{2.26}$$

where $z \in \mathcal{G}^{\mathbb{L}}{}_0$, by the induction hypothesis in the first step, and (2.17) in the other two. Thus the identity (2.24) holds for $k = 1$ and $i = p$. It is now straightforward to proceed by induction over k, and show that it holds for any $k \geq 1$ and $i + k = p + 1$. It suffices to say that the idea in the induction step of this second induction is, as in (2.32) below, to move one element at the time from one pair of parentheses to the other. The proposition then follows by the principle of induction. □

We now turn to the remaining parts (2.4)–(2.6) of the focal associativity.

Lemma 2.3 *The identities*

$$\begin{aligned} (uy)v &= u(yv)\,, \\ (ux)(yv) &= u(xy)v \end{aligned} \tag{2.27}$$

hold for all variables as in (2.25).

Proof We suppose that $x \in \mathcal{G}^{\mathbb{L}}{}_{\pm 1}$ and $y \in \mathcal{G}^{\mathbb{L}}{}_{\mp 1}$, since this is sufficient, and prove the lemma by induction over i. The base case $i = 0$ is either trivial or given by (2.16). Suppose the identities hold for $i \leq p$ for some $p \geq 0$. Let $z \in \mathcal{G}^{\mathbb{L}}{}_0$. For $i = p$ we then have

$$\begin{aligned} (uzy)v &= (u[z, y])v + (-1)^{yz}(uyz)v \\ &= (u[z, y])v + (-1)^{yz}(uy)(zv) \\ &= u([z, y]v) + (-1)^{yz}u(y(zv)) = (uz)(yv)\,, \end{aligned} \tag{2.28}$$

where we have used (2.24) in the second step, the induction hypothesis in the third and (2.17) in the fourth. For $i = p$ we furthermore have

$$\begin{aligned} (uzx)(yv) &= (u[z, x])(yv) + (-1)^{zx}(uxz)(yv) \\ &= u([z, x]y)v + (-1)^{zx}(ux)(zyv) \\ &= u([z, x]y)v + (-1)^{zx}(ux)([z, y]v) + (-1)^{zx+zy}(ux)(yzv) \\ &= u([z, x]y)v + (-1)^{zx}u(x[z, y])v + (-1)^{zx+zy}u(xy)zv \\ &= u\big([z, x]y + (-1)^{zx}x[z, y] + (-1)^{zx+zy}(xy)z\big)v \end{aligned} \tag{2.29}$$

using the induction hypothesis in the second and fourth step. If $(x, y) = (x_{-1}, y_1)$, the expression between u and v equals

$$\begin{aligned}
&[z, x]y + (-1)^{zx}x[z, y] + (-1)^{zx+zy}(xy)z \\
&\quad = -a[\![[z, x], y]\!] + b\langle [z, x]|y\rangle L \\
&\qquad - a(-1)^{zx}[\![x, [\![z, y]\!]]\!] + b(-1)^{zx}\langle x|[\![z, y]\!]\rangle L \\
&\qquad - a(-1)^{z(x+y)}[\![x, y]\!]z + b(-1)^{z(x+z)}\langle x|y\rangle zL \\
&\quad = -a[\![z, [\![x, y]\!]]\!] - a(-1)^{z(x+y)}[\![x, y]\!]z + b(-1)^{z(x+z)}\langle x|y\rangle zL \\
&\quad = -az[\![x, y]\!] + b\langle x|y\rangle zL = z(xy)
\end{aligned} \tag{2.30}$$

Thus $(uzx)(yv) = uz(xy)v$. The case $(x, y) = (x_1, y_{-1})$ is similar. We have thus shown that the identities (2.27) hold when $i = p + 1$ as well, and the lemma follows by the principle of induction. □

Note that all products of three elements written without parentheses in (2.28) and (2.29) are well defined, either by the induction hypothesis or by Proposition 2.2.

Proposition 2.4 *The identities*

$$\begin{aligned}
(u(yw))v &= u((yw)v)\,, \\
((ux)(yv))w &= (ux)((yv)w)\,, \\
(w(ux))(yv) &= w((xu)(yv))\,,
\end{aligned} \tag{2.31}$$

hold for all variables as in (2.25).

Proof We have

$$\begin{aligned}
(u(yw))v &= ((uy)w)v = (uy)(wv) = u(y(wv)) = u((yw)v)\,, \\
((ux)(yv))w &= u(xy)vw = (ux)(y(vw)) = (ux)((yv)w)\,, \\
(w(ux))(yv) &= ((wu)x)(yv) = wu(xy)v = w\big((ux)(yv)\big)\,,
\end{aligned} \tag{2.32}$$

by Proposition 2.2 and Lemma 2.3. □

Theorem 2.5 *The local algebra $\mathcal{G}^\ell$ is focally associative.*

Proof This follows directly from Propositions 2.2 and 2.4, and the fact that any element in ${\mathcal{G}^\ell}_{\pm 1}$ can be written as a sum of elements ux, where $u \in {\mathcal{G}^\ell}_0$ and $x \in {\mathcal{G}^{\mathbb{L}}}_{\pm 1}$, which is easily shown by induction. □

We have shown that $\mathcal{G}^\ell$ is a focally associative local superalgebra, and thus, by Proposition 2.1, it gives rise to a new local Lie superalgebra where the bracket is given by the commutator in $\mathcal{G}^\ell$. We denote this local Lie superalgebra by $\mathcal{G}^{\mathsf{L}}$, and the bracket in it by $[-, -]$, to be distinguished from the original bracket $[\![-, -]\!]$ on $\mathcal{G}^{\mathbb{L}}$. This is particularly important when one of the elements belong to ${\mathcal{G}^{\mathbb{L}}}_1$ and the other to ${\mathcal{G}^{\mathbb{L}}}_{-1}$ since both brackets are defined in this case, but disagree according to (2.23).

Note that it was only in the second part of the proof of Lemma 2.3 that we used the form (2.15) of the product xy as an element in $\mathbb{K} \oplus {\mathcal{G}^{\mathbb{L}}}_0$, and that the values of the constants a, b, c did not matter. We will henceforth assume that $b = c$ since this condition turns out be important for the relation to the tensor hierarchy algebras (more precisely, it is crucial in the proof of Lemma 4.2 below).

In order to include all possible focally associative superalgebras obtained from contragredient local Lie superalgebras in this way (with $b = c$), we may then fix a and $b = c$ without loss of generality since we can always rescale the bracket $[\![-,-]\!]$ and the invariant form $\langle -|-\rangle$. It turns out that a natural choice is $a = 1$ when the subspaces $\mathcal{G}^{\mathbb{L}}{}_{\pm 1}$ are odd, and $a = -1$ when they are even, along with $b = c = 1$. Accordingly, we then have

$$x_{-1}y_1 = -[\![y_1, x_{-1}]\!] + \langle x_{-1}|y_1\rangle L\,,$$
$$x_1 y_{-1} = [\![y_{-1}, x_1]\!] + \langle x_1|y_{-1}\rangle L + \langle x_1|y_{-1}\rangle\,, \tag{2.33}$$

so that $[x_{\pm 1}, y_{\mp 1}] = \pm\langle x_{\pm 1}|y_{\mp 1}\rangle$.

For any contragredient local Lie superalgebra $\mathcal{G}^{\mathbb{L}}$ we thus let $\mathcal{G}^{\ell}$ be the focally associative local algebra constructed in the way above with the products (2.33), and $\mathcal{G}^{\mathsf{L}}$ the local Lie superalgebra obtained from $\mathcal{G}^{\ell}$ with the commutator $[-,-]$ as the bracket, to be distinguished from the original one $[\![-,-]\!]$. Note that $\mathcal{G}^{\ell}$ (and thus also $\mathcal{G}^{\mathsf{L}}$) is in general infinite-dimensional even when $\mathcal{G}^{\mathbb{L}}$ is finite-dimensional.

2.3 An Example

Before applying our construction to the setting motivated by tensor hierarchy algebras, let us demonstrate it in a simple example. Let $\mathcal{G}^{\mathbb{L}}$ be the local part of the Lie algebra $\mathfrak{sl}(2,\mathbb{K})$, with one-dimensional subspaces $\mathcal{G}^{\mathbb{L}}{}_{-1}, \mathcal{G}^{\mathbb{L}}{}_{0}, \mathcal{G}^{\mathbb{L}}{}_{1}$ spanned by elements f, h, e, respectively, and the relations

$$[\![h, e]\!] = 2e\,, \qquad [\![h, f]\!] = -2f\,, \qquad [\![e, f]\!] = h\,. \tag{2.34}$$

This is an (even) contragredient local Lie superalgebra with grading element $L = \frac{1}{2}h$ and $\langle e|f\rangle = \langle f|e\rangle = 1$. Since the subalgebra $\mathcal{G}^{\mathbb{L}}{}_0$ is an abelian Lie algebra spanned by L, its universal enveloping algebra is the commutative algebra $\mathbb{K}[L]$ of polynomials in L. Thus a basis of $\mathcal{G}^{\ell}{}_0$ is formed by all powers L^n for $n = 0, 1, \ldots$. By multiplying from the left with f and e, we obtain bases of $\mathcal{G}^{\ell}{}_{-1}$ and $\mathcal{G}^{\ell}{}_1$, respectively, consisting of elements eL^n and fL^n.

From (2.33) we get $fe = -L$ and $ef = -(L-1)$. It is straightforward to generalise the relations $[L, e] = e$ and $[L, f] = -f$ to

$$[L^n, e] = e\big((L+1)^n - L^n\big)\,, \qquad [L^n, f] = f\big((L-1)^n - L^n\big)\,. \tag{2.35}$$

Using this, we get

$$L^m(eL^n) = [L^m, e]L^n + eL^{m+n} = e(L+1)^m L^n\,,$$
$$L^m(fL^n) = [L^m, f]L^n + fL^{m+n} = f(L-1)^m L^n\,. \tag{2.36}$$

and we can then in turn generalise the relations $fe = -L$ and $ef = -(L-1)$ to

$$(eL^m)(fL^n) = e(L^m f)L^n = e[L^m, f]L^n + efL^{m+n} = -(L-1)^{m+1}L^n\,,$$
$$(fL^n)(eL^m) = f(L^n e)L^m = f[L^n, e]L^m + feL^{m+n} = -(L+1)^n L^{m+1}\,. \tag{2.37}$$

We have thus described all the products in the local algebra $\mathcal{G}^{\ell}$ in this simple case. It turns out that, unlike the general case, it is in fact associative. In order to see this, let us first notice that

$$(feL^m)fL^n = -L^{m+1}fL^n = -f(L-1)^{m+1}L^n$$
$$= f\big(ef(L-1)^m L^n\big) = f(eL^m fL^n)\,. \tag{2.38}$$

Assuming that $(fL^peL^m)fL^n = fL^p(eL^mfL^n)$, we get

$$(fL^{p+1}eL^m)fL^n = \big([f,L]L^peL^m\big)fL^n + L(fL^peL^m)fL^n$$
$$= [f,L]L^p(eL^mfL^n) + Lf(L^peL^mfL^n) = fL^{p+1}(eL^mfL^n)\,. \quad (2.39)$$

It then follows by induction that we have $(fL^peL^m)fL^n = fL^p(eL^mfL^n)$, and similarly $(eL^pfL^m)eL^n = eL^p(fL^meL^n)$, for all non-negative integers m, n, p.

Since the local algebra $\mathcal{G}^\ell$ is associative, one may ask whether it is the local part of some well known associative algebra. Indeed, consider the Weyl algebra $\mathcal{W}$ which is the associative algebra generated by e and f modulo the relations $ef - fe = 1$, with a $\mathbb{Z}$-grading where $e \in \mathcal{W}_1$ and $f \in \mathcal{W}_{-1}$. An element in $\mathcal{W}_0$ is then a linear combination of monomials in e and f with equally many factors e as factors f. If needed, we can then use the relations $ef - fe = 1$ to rewrite it as a linear combination of monomials of the form $fefe\cdots fe$. Setting $L = -fe$, we see that $\mathcal{W}_0$ consists of all polynomials in L. Similarly, any element in $\mathcal{W}_1$ or $\mathcal{W}_{-1}$ can be written as a polynomial in L multiplied with e or f, respectively. Thus $\mathcal{W}_k = \mathcal{G}^\ell{}_k$ for $k = 0, \pm 1$ and it is easy to see that also the products are the same, so the local part of this Weyl algebra is in fact isomorphic to $\mathcal{G}^\ell$ when $\mathcal{G}^{\mathbb{L}}$ is the local part of $\mathfrak{sl}(2, \mathbb{K})$.

3 Contragredient Lie Superalgebras

Let $\mathfrak{g}$ be a Kac–Moody algebra of rank r with an invertible and symmetrisable Cartan matrix A, let λ be a dominant integral weight of $\mathfrak{g}$ and let κ be a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$. In this section we will associate a contragredient local Lie superalgebra $\mathcal{B}^{\mathbb{L}}$ to the triple $(\mathfrak{g}, \lambda, \kappa)$, from which we in turn can construct a focally associative local superalgebra $\mathcal{B}^\ell$ and a local Lie superalgebra $\mathcal{B}^{\mathsf{L}}$ as above.

3.1 Kac–Moody Algebras and Pseudo-Minuscule Weights

We recall that $\mathfrak{g}$ is generated by $3r$ elements e_k, f_k, h_k, where $k = 1, 2, \ldots, r$, modulo the Chevalley–Serre relations [19]. We also recall that the invariant symmetric bilinear form κ on $\mathfrak{g}$ is unique up to an overall normalisation, that it satisfies $\kappa(e_k, f_k) \neq 0$ for any $k = 1, 2, \ldots, r$, and that it induces a symmetric bilinear form on the vector space $\mathfrak{h}^*$ dual to the Cartan subalgebra $\mathfrak{h}$ (spanned by the generators h_k) by the relation $(\alpha_i{}^\vee, \alpha_j{}^\vee) = \kappa(h_i, h_j)$, where the simple coroots are defined by $\alpha_k{}^\vee = \kappa(e_k, f_k)\alpha_k$. It then follows that $(\alpha_k, \alpha_k) = 2/\kappa(e_k, f_k)$ so that $\alpha_k{}^\vee = 2\alpha_k/(\alpha_k, \alpha_k)$. These well known results will be re-derived below for the contragredient Lie superalgebra $\mathcal{B}$ with Cartan matrix B obtained by adding a row and column to the Cartan matrix A.

Let $\lambda_k = (\lambda, \alpha_k{}^\vee)$ be the Dynkin labels of the dominant integral weight λ, so that $\lambda_k \in \mathbb{Z}$ and $\lambda_k \geq 0$ for any $k = 1, 2, \ldots, r$ (not all zero). The Dynkin labels are the components of λ in the basis of fundamental weights Λ_k, defined by $(\Lambda_i, \alpha_j{}^\vee) = \delta_{ij}$. Let $\lambda^\triangle$ be the weight with Dynkin labels $\lambda^\triangle{}_k = \lambda_k/\kappa(e_k, f_k)$. We will be interested in cases where $\mathfrak{g}$ is finite and where λ and κ are such that $\lambda^\triangle$ is a fundamental weight Λ_k for which the corresponding Coxeter label (the component of the highest root θ in the basis of simple roots) is equal to 1. We say that such a weight $\lambda^\triangle$ is a *pseudo-minuscule* weight. The reason for choosing this term (although it has been used in a different meaning [13]) is that the pseudo-minuscule weights coincide with the *minuscule* weights (highest weights of representations on which the Weyl group acts transitively [14]) for all $\mathfrak{g}$ other than $\mathfrak{g} = B_r$ and $\mathfrak{g} = C_r$. Moreover, the isomorphism between the weight spaces of B_r and C_r given by transposing the Cartan matrix (or flipping the arrow in the Dynkin diagram) maps a minuscule weight of one algebra to a pseudo-minuscule weight of the other, and vice versa. (This in fact holds for any Cartan matrix of a finite Kac–Moody algebra $\mathfrak{g}$, but for other $\mathfrak{g}$ it just says that the minuscule and pseudo-minuscule weights coincide.) Below follows the complete list of pseudo-minuscule weights in the numbering of Bourbaki [15] (with some additional information about the corresponding highest weight representations). There are no pseudo-minuscule weights of E_8, F_4 or G_2.

- $A_r : \Lambda_1, \ldots, \Lambda_r$
- $B_r : \Lambda_1$ (vector representation)
- $C_r : \Lambda_r$
- $D_r : \Lambda_1, \Lambda_{r-1}, \Lambda_r$ (vector and spinor representations)
- $E_6 : \Lambda_1, \Lambda_6$ (27-dimensional)
- $E_7 : \Lambda_7$ (56-dimensional)

In extended geometry with extended structure algebra $\mathfrak{g}$ and extended coordinate representation with highest weight λ, it is precisely when $\lambda^\triangle$ is a pseudo-minuscule weight that additional "ancillary" transformations are not needed for closure and covariance of the generalised diffeomorphisms [16]. (In [16], the normalisation was chosen such that $\lambda = \lambda^\triangle$, if possible. Accordingly, the conclusion there was that ancillary transformations are absent precisely when λ is a pseudo-minuscule weight. However, with a different normalisation they would presumably be absent also when λ is an integer multiple of a pseudo-minuscule weight.)

Proposition 3.1 *Let $\mathfrak{g}$ be finite with highest root θ. A necessary condition for $\lambda^\triangle$ to be a pseudo-minuscule weight is that $(\lambda, \theta) = 1$. If $\lambda^\triangle$ is a dominant integral weight, then this condition is also sufficient.*

Proof If $\lambda^\triangle$ is a pseudo-minuscule weight and c_k are the components of θ in the basis of simple roots α_k, then

$$1 = \sum_{k=1}^{r} \lambda^\triangle{}_k c_k = \sum_{k=1}^{r} \frac{(\alpha_k, \alpha_k)}{2} \lambda_k c_k = \sum_{i,j=1}^{r} \frac{(\alpha_j, \alpha_j)}{2} \lambda_i c_j \delta_{ij} \tag{3.1}$$

which equals

$$\sum_{i,j=1}^{r} \frac{(\alpha_j, \alpha_j)}{2} \lambda_i c_j (\Lambda_i, \alpha_j{}^\vee) = \sum_{i,j=1}^{r} \lambda_i c_j (\Lambda_i, \alpha_j) = (\lambda, \theta)\,. \tag{3.2}$$

Conversely, if $(\lambda, \theta) = 1$, then the same calculation shows that $\sum_{k=1}^{r} \lambda^\triangle{}_k c_k = 1$. If in addition the Dynkin labels $\lambda^\triangle{}_k$ are non-negative integers, then the only possibility is that all are zero except for one of them which is equal to 1, and that the corresponding Coxeter label c_k is equal to 1 too. □

3.2 Extended Cartan Matrices and Contragredient Lie Superalgebras

Given the triple $(\mathfrak{g}, \lambda, \kappa)$, let B be the square matrix of order $r + 1$ with entries

$$B_{00} = 0\,, \qquad B_{i0} = -\lambda_i\,, \qquad B_{0j} = -\lambda^\triangle{}_j = -\frac{\lambda_j}{\kappa(e_j, f_j)} \qquad B_{ij} = A_{ij}\,, \tag{3.3}$$

where $i, j = 1, 2, \ldots, r$. Then B is symmetrisable. We also assume that λ is such that B is invertible.

The *contragredient Lie superalgebra* $\mathcal{B}$ associated to the Cartan matrix B is defined from a set of $3r$ generators $M_{\mathcal{B}} = \{e_K, f_K, h_K\}$ for $K = 0, 1, \ldots, r$, where e_0 and f_0 are odd, whereas h_0 and e_k, f_k, h_k are even for $k = 1, 2, \ldots, r$. Let $\tilde{\mathcal{B}}$ be the $\mathbb{Z}$-graded Lie superalgebra generated by this set $M_{\mathcal{B}}$ modulo the relations

$$[h_I, e_J] = B_{IJ} e_J\,, \qquad [h_I, f_J] = -B_{IJ} f_J\,, \qquad [e_I, f_J] = \delta_{IJ} h_J \tag{3.4}$$

with the (non-consistent) $\mathbb{Z}$-grading where e_K and f_K have degree 1 and -1, respectively, for any $K = 0, 1, \ldots, r$. Then $\mathcal{B} = \tilde{\mathcal{B}}/D$, where D is the maximal graded ideal of $\tilde{\mathcal{B}}$ intersecting the local part of $\tilde{\mathcal{B}}$ trivially [2]. Since B here satisfies the conditions of a Cartan matrix of a *Borcherds–Kac–Moody algebra*, a generalisation [17] of the Gabber–Kac theorem [18, 19] holds, which in this case says that the ideal D is generated by the Serre relations

$$(\mathrm{ad}\, e_I)^{1-B_{IJ}}(e_J) = (\mathrm{ad}\, f_I)^{1-B_{IJ}}(f_J) = 0\,. \tag{3.5}$$

We refer to [20] for details about Borcherds–Kac–Moody superalgebras. We also note that different overall normalisations of the bilinear form $\langle -|-\rangle$ give isomorphic Lie superalgebras $\mathcal{B}$ with an isomorphism given by a rescaling of h_0 and e_0. Thus $\mathcal{B}$ in fact only depends on $\mathfrak{g}$ and λ, not κ. This is however not true for the contragredient *local* Lie superalgebras $\mathcal{B}^{\mathbb{L}}$ that we will associate to the triple $(\mathfrak{g}, \lambda, \kappa)$ below, since the bilinear form $\langle -, -\rangle$ is part of the data defining it.

Consider the consistent $\mathbb{Z}$-grading of $\mathcal{B}$ where e_0 and f_0 has degree 1 and -1, respectively, whereas all the even generators have degree 0. Assigning these degrees to the generators clearly induces a $\mathbb{Z}$-grading $\mathcal{B} = \bigoplus_{k\in\mathbb{Z}} \mathcal{B}_k$, where $\mathcal{B}_k$ is defined as the subspace spanned by all expressions

$$[g_1, [g_2, \ldots, [g_{m-1}, g_m]\cdots]] \tag{3.6}$$

for positive integers m, where $g_i \in M$ for $i = 1, 2, \ldots, m$ and the number of e_0 minus the number of f_0 among these generators $g_1, \ldots, g_m$ is equal to k. Because of the triangular decomposition that a Borcherds–Kac–Moody algebra admits, we may assume that one of these two numbers (the number of generators equal to e_0 and the number of generators equal to f_0) is zero. (The corresponding statement is not true for the tensor hierarchy algebras that we will come to in the next section.) Furthermore, when (3.6) is an element in $\mathcal{B}_{\pm 1}$ we may assume that generator equal to e_0 or f_0 is the innermost one, g_m. Thus $\mathcal{B}_0$ is in fact the subalgebra generated by $M_{\mathcal{B}} \setminus \{e_0, f_0\}$, and the subspaces $\mathcal{B}_1$ and $\mathcal{B}_{-1}$ are modules over $\mathcal{B}_0$ with respect to the adjoint action, generated by e_0 and f_0 as lowest and highest weight vectors, respectively.

Let $\mathcal{B}^{\mathbb{L}} = \mathcal{B}_{-1} \oplus \mathcal{B}_0 \oplus \mathcal{B}_1$ be the local part of $\mathcal{B}$, together with the unique invariant symmetric bilinear form such that $\langle x|y\rangle = \kappa(x, y)$ for $x, y \in \mathfrak{g}$. It then follows from (3.3) that $\langle e_0|f_0\rangle = -\langle f_0|e_0\rangle = 1$. Since it is invariant, this form satisfies

$$B_{IJ}\langle e_J|f_J\rangle = \langle [h_I, e_J]|f_J\rangle = \langle h_I|[e_J, f_J]\rangle = \langle h_I|h_J\rangle\,, \tag{3.7}$$

and because of the graded symmetry, this is also equal to $B_{JI}\langle e_I|f_I\rangle$.

Roots are defined for $\mathcal{B}$ in the same way as for $\mathfrak{g}$. In particular, the generators h_K and the simple roots α_K, where $K = 0, 1, \ldots, r$, constitute bases for the Cartan subalgebra $\mathcal{H}$ and the dual space $\mathcal{H}^*$, respectively. Let $\varphi : \mathcal{H} \to \mathcal{H}^*$ be the linear map given by $\varphi(h_K) = {\alpha_K}^\vee = \langle e_K|f_K\rangle\alpha_K$. In particular ${\alpha_0}^\vee = \alpha_0$. It then follows from (3.7), and the definition $\alpha_J(h_I) = B_{IJ}$ of the simple roots α_J, that we have

$$\varphi(h_I)(h_J) = \langle h_I | h_J \rangle \, . \tag{3.8}$$

We note that φ is a vector space isomorphism, and $\langle e_K | f_K \rangle \neq 0$ for all $K = 0, 1, \ldots, r$. We may then introduce an inner product on $\mathcal{H}^*$ given by

$$(\alpha_I, \alpha_J) = \langle \varphi^{-1}(\alpha_I) | \varphi^{-1}(\alpha_J) \rangle = \frac{1}{\langle e_I | f_I \rangle} \frac{1}{\langle e_J | f_J \rangle} \langle h_I | h_J \rangle = \frac{B_{IJ}}{\langle e_I | f_I \rangle} \, . \tag{3.9}$$

In particular,

$$(\alpha_0, \alpha_0) = 0 \, , \qquad (\alpha_k, \alpha_k) = \frac{2}{\langle e_k | f_k \rangle} = \frac{2}{\kappa(e_k, f_k)} \tag{3.10}$$

and it follows that $\alpha_k{}^\vee = 2\alpha_k/(\alpha_k, \alpha_k)$, as already stated above.

We note that $(\alpha_0{}^\vee, \mu) = -(\lambda, \mu)$ for any $\mu \in \mathfrak{h}^*$, since if $\mu = \sum_{i=1}^r m_i \alpha_i$, then

$$\begin{aligned}(\alpha_0{}^\vee, \mu) &= \sum_{i=1}^r m_i (\alpha_0{}^\vee, \alpha_i) = \sum_{i=1}^r m_i B_{0i} = - \sum_{i,j=1}^r \frac{\lambda_i}{\kappa(e_j, f_j)} m_j \delta_{ij} \\ &= - \sum_{i,j=1}^r \frac{\lambda_i}{\kappa(e_j, f_j)} m_j (\Lambda_i, \alpha_j{}^\vee) = - \sum_{i,j=1}^r \lambda_i m_j (\Lambda_i, \alpha_j) = -(\lambda, \mu) \, . \end{aligned} \tag{3.11}$$

Proposition 3.2 *The local part $\mathcal{B}^{\mathbb{L}}$ of $\mathcal{B}$ (with respect to the consistent $\mathbb{Z}$-grading) is a contragredient local Lie superalgebra where the element L is given by $L = \sum_{I=0}^r (B^{-1})_{0I} h_I$ and satisfies $\langle L | L \rangle = -1/(\lambda, \lambda)$.*

Proof [16, 21] We have $[L, e_J] = \alpha_J(L) e_J$, and with $L = \sum_{I=0}^r (B^{-1})_{0I} h_I$ we get

$$\alpha_J(L) = \sum_{I=0}^r (B^{-1})_{0I} \alpha_J(h_I) = \sum_{I=0}^r (B^{-1})_{0I} B_{IJ} = \delta_{0J} \tag{3.12}$$

as we should. Furthermore,

$$\begin{aligned}\langle L | L \rangle &= \sum_{I=0}^r \sum_{J=0}^r (B^{-1})_{0I} (B^{-1})_{0J} \langle h_I | h_J \rangle \\ &= \sum_{I=0}^r \sum_{J=0}^r (B^{-1})_{0I} (B^{-1})_{0J} B_{IJ} \langle e_J | f_J \rangle \\ &= \sum_{J=0}^r \delta_{0J} (B^{-1})_{0J} \langle e_J | f_J \rangle = (B^{-1})_{00} \langle e_0 | f_0 \rangle = (B^{-1})_{00} \, . \end{aligned} \tag{3.13}$$

Since A is invertible,

$$(B^{-1})_{00} = \frac{\det A}{\det B} \neq 0 \, , \tag{3.14}$$

and in $\mathfrak{h}^*$ we can set

$$\lambda = \sum_{j=1}^r \frac{(B^{-1})_{j0}}{(B^{-1})_{00}} \alpha_j \, . \tag{3.15}$$

We then get

$$\lambda_i = (\alpha_i{}^\vee, \lambda) = \sum_{j=1}^{r} B_{ij} \frac{(B^{-1})_{j0}}{(B^{-1})_{00}} = \sum_{J=0}^{r} B_{iJ} \frac{(B^{-1})_{J0}}{(B^{-1})_{00}} - B_{i0} = -B_{i0}\,, \qquad (3.16)$$

as we should, and

$$\begin{aligned}(\lambda, \lambda) &= \sum_{i=1}^{r}\sum_{j=1}^{r} \frac{(B^{-1})_{i0}}{(B^{-1})_{00}} \frac{(B^{-1})_{j0}}{(B^{-1})_{00}} (\alpha_i, \alpha_j) = \sum_{i=1}^{r}\sum_{j=1}^{r} \frac{B_{ij}}{\langle e_i | f_i \rangle} \frac{(B^{-1})_{i0}}{(B^{-1})_{00}} \frac{(B^{-1})_{j0}}{(B^{-1})_{00}} \\ &= -\sum_{i=1}^{r} \frac{B_{i0}}{\kappa(e_i, f_i)} \frac{(B^{-1})_{i0}}{(B^{-1})_{00}} = -\sum_{i=1}^{r} B_{0i} \frac{(B^{-1})_{i0}}{(B^{-1})_{00}} \\ &= -\sum_{I=0}^{r} B_{0I} \frac{(B^{-1})_{I0}}{(B^{-1})_{00}} = -\frac{1}{(B^{-1})_{00}} = -\frac{1}{\langle L|L \rangle}\,. \end{aligned} \qquad (3.17)$$

Thus $\langle L|L\rangle = -1/(\lambda, \lambda)$. □

4 Tensor Hierarchy Algebras

In [4], a Lie superalgebra called *tensor hierarchy algebra* and denoted W was associated to any simple and simply laced Kac–Moody algebra $\mathfrak{g}$ of rank r and any fundamental weight λ of $\mathfrak{g}$. The numbering of the fundamental weights of $\mathfrak{g}$ was chosen such that $\lambda = \Lambda_1$. The construction of W given in [4] starts with the Cartan matrix B of the Lie superalgebra $\mathcal{B}$ associated to the triple $(\mathfrak{g}, \lambda, \kappa)$ as described in the preceding section. The set of generators $M_{\mathcal{B}} = \{e_K, f_K, h_K\}$ of $\mathcal{B}$ (for $K = 0, 1, \ldots, r$) is then modified to a set M_W by replacing the odd generator f_0 by r odd generators f_{0k}, where $k = 0$ or $k = 2, 3, \ldots, r$. From this set M_W of generators, and the Cartan matrix B, an auxiliary Lie superalgebra algebra $\tilde{W}$ is first constructed as the one generated by M_W modulo the relations

$$[h_I, e_J] = B_{IJ} e_J\,, \qquad [h_I, f_J] = -B_{IJ} f_J\,, \qquad [e_I, f_J] = \delta_{IJ} h_J\,, \qquad (4.1)$$

$$(\text{ad}\, e_I)^{1-B_{IJ}}(e_J) = (\text{ad}\, f_I)^{1-B_{IJ}}(f_J) = 0\,, \qquad (4.2)$$

$$[e_0, f_{0I}] = h_I\,, \qquad [h_I, f_{0J}] = -B_{I0} f_{0J}\,, \qquad [e_i, [f_j, f_{0K}]] = \delta_{ij} B_{Kj} f_{0j}\,, \qquad (4.3)$$

$$[e_1, f_{0K}] = [e_I, [e_I, f_{0J}]] = [f_I, [f_I, f_{0J}]] = 0\,, \qquad (4.4)$$

where $I, J, K = 0, 1, \ldots, r$ and $i, j, k = 2, \ldots, r$. (Whenever f_K appears, we assume $K \neq 0$, and whenever f_{0k} appears, we assume $k \neq 1$.) Then W is obtained from $\tilde{W}$ by factoring out the maximal ideal intersecting the local part trivially, with respect to the consistent $\mathbb{Z}$-grading. By modifying the set of generators further to $M_S = M_W \backslash \{h_0, f_{00}\}$ a Lie superalgebra S (called tensor hierarchy algebra as well) can be defined in the same way (with the relations involving h_0 and f_{00} removed).

It was shown in [4] that S coincides with the original tensor hierarchy algebras introduced in [3] in the cases where $\mathfrak{g}$ is finite. It was also shown that when $\mathfrak{g} = A_{n-1}$ $(n \geq 2)$ and $\lambda = \Lambda_1$ with the usual numbering of fundamental weights W and S coincide with the Cartan-type Lie superalgebras $W(n)$ and $S(n)$ of vector fields (for which generators and relations have also been given in [22]). In [9] the tensor hierarchy algebras were defined in a similar way, but with the additional relation $[f_{0I}, f_{0J}] = 0$ in the definition of $\tilde{W}$, and W obtained from $\tilde{W}$ by considering a different (non-consistent) $\mathbb{Z}$-grading.

We will now see that the relations (4.1)–(4.4) arise naturally in the context of focally associative local algebras. We consider $\mathcal{B}^{\mathbb{L}}$, the local part of the contragredient Lie superalgebra $\mathcal{B}$ in the preceding section. We will then investigate the subalgebra of $\mathcal{B}^{\mathbb{L}}$ generated by $\mathcal{B}_1$ and $\mathcal{B}_{-1}\mathcal{B}_0$ modulo the maximal peripheral ideal (where $\mathcal{B}_k = \mathcal{B}^{\mathbb{L}}{}_k$ for $k = \pm 1$ and $\mathcal{B}_{-1}\mathcal{B}_0$ consists of all products $x_{-1}y_0$ for $x, y \in \mathcal{B}^{\mathbb{L}}$) and end the paper with a theorem relating it to the local part of the tensor hierarchy algebra W.

Proposition 4.1 *Let λ and κ be such that $\lambda^{\triangle}$ is a dominant integral weight. Then*

$$\big((\alpha_0{}^{\vee}, \alpha) + 1\big)[\![f_0, e_\alpha]\!] = 0 \tag{4.5}$$

for all roots $\alpha \neq \alpha_0$ of $\mathcal{B}$ with corresponding root vectors $e_\alpha \in \mathcal{B}_1$ if and only if $\mathfrak{g}$ is finite and $(\lambda, \theta) = 1$ (so that $\lambda^{\triangle}$ is a pseudo-minuscule weight).

Proof Suppose that $\mathfrak{g}$ is infinite-dimensional or that $(\lambda, \theta) \neq 1$. In either case it is possible to find a root ζ of $\mathfrak{g}$ such that $\alpha_0 + \zeta$ is a root of $\mathcal{B}$ and $(\alpha_0{}^{\vee}, \zeta) \neq -1$, for example $\zeta = \theta$ if $\mathfrak{g}$ is finite, since then

$$(\alpha_0{}^{\vee}, \zeta) = (\alpha_0{}^{\vee}, \theta) = -(\lambda, \theta) \neq -1\,. \tag{4.6}$$

We can then set $\alpha = \alpha_0 + \zeta$ and $e_\alpha = [\![e_0, e_\zeta]\!]$, where e_ζ is a root vector corresponding to ζ, and it follows that $\big((\alpha_0{}^{\vee}, \alpha) + 1\big)[\![f_0, e_\alpha]\!] \neq 0$.

On the other hand, suppose that $\mathfrak{g}$ is finite and that $\lambda^{\triangle}$ is a pseudo-minuscule weight, say $\lambda^{\triangle} = \Lambda_j$ for some j such that $c_j = 1$ and let α be a root of $\mathcal{B}$ such that $e_\alpha \in \mathcal{B}_1$ and $[\![f_0, e_\alpha]\!] \neq 0$. Then $\alpha - \alpha_0 = \sum_{k=1}^{r} b_k \alpha_k$ is a root of $\mathfrak{g}$ and

$$(\alpha_0{}^{\vee}, \alpha) = (\alpha_0{}^{\vee}, \alpha - \alpha_0) = \sum_{i=1}^{r} b_i (\alpha_0{}^{\vee}, \alpha_i) = -\sum_{i=1}^{r} b_i \lambda^{\triangle}{}_i = -\sum_{i=1}^{r} b_i \delta_{ij} = -b_j\,. \tag{4.7}$$

Since θ is the highest root, $b_j \leq c_j = 1$. But b_j cannot be zero since then $(\alpha_0{}^{\vee}, \alpha - \alpha_0)$ would be zero as well, and $\alpha = \alpha_0 + (\alpha - \alpha_0)$ would not be a root. Thus $b_j = 1$ and $(\alpha_0{}^{\vee}, \alpha) + 1 = 0$. □

Lemma 4.2 *Let $(\mathfrak{g}, \lambda, \kappa)$ be such that $\mathfrak{g}$ is finite and $\lambda^{\triangle}$ is a pseudo-minuscule weight. Then $f_0(h_0 + L)$ generates a peripheral ideal of the subalgebra generated by $\mathcal{B}_1$ and $\mathcal{B}_{-1}\mathcal{B}_0$.*

Proof Let e_α be a root vector corresponding to a root α of $\mathcal{B}$ such that $e_\alpha \in \mathcal{B}_1$ but $\alpha \neq \alpha_0$. We then have

$$\begin{aligned}
[e_0, f_0 h_0] + [e_0, f_0 L] &= [e_0, f_0]h_0 - f_0[e_0, h_0] + [e_0, f_0]L - f_0[e_0, L]\\
&= h_0 + L + f_0 e_0 = h_0 + L - h_0 - L = 0\,,
\end{aligned}$$

$$\begin{aligned}
[e_\alpha, f_0 h_0] + [e_\alpha, f_0 L] &= [e_\alpha, f_0]h_0 - f_0[e_\alpha, h_0] + [e_\alpha, f_0]L - f_0[e_\alpha, L]\\
&= f_0[h_0, e_\alpha] + f_0[L, e_\alpha]\\
&= \big((\alpha_0{}^{\vee}, \alpha) + 1\big) f_0 e_\alpha = -\big((\alpha_0{}^{\vee}, \alpha) + 1\big)[\![f_0, e_\alpha]\!] = 0\,,
\end{aligned} \tag{4.8}$$

where the last equation follows from Proposition 4.1. □

Theorem 4.3 *Let $(\mathfrak{g}, \lambda, \kappa)$ be such that $\mathfrak{g}$ is finite and $\lambda = \lambda^{\triangle}$ is a pseudo-minuscule weight. Choose a numbering of the fundamental weights where $\lambda = \lambda^{\triangle} = \Lambda_1$, so that κ is given* *by* $\kappa(e_1, f_1) = 1$. *Then there is a local Lie superalgebra homomorphism from the local part of* W *to the subalgebra of* $\mathcal{B}^{\llcorner}$ *generated by* $\mathcal{B}_1$ *and* $\mathcal{B}_{-1}\mathcal{B}_0$ *modulo the maximal peripheral ideal, given by* $f_{0K} \mapsto f_0 h_K$ *(and leaving the other generators unchanged).*

Proof Let V be the subalgebra of $\mathcal{B}^{\llcorner}$ generated by $\mathcal{B}_1$ and $\mathcal{B}_{-1}\mathcal{B}_0$, and let D be the maximal peripheral ideal of V. We will show that the relations (4.1)–(4.4) are satisfied in V/D with $f_{0K} = f_0 h_K$. We will first show that $[f_0 h, e_1] = [f_1, [f_1, f_0 h]] = 0$ for any $h \in \mathfrak{h}$. From Lemma 4.2 we know that $f_0(h_0 + L) = 0$ in V/D, and we also have $B_{01} = -1$ since $\lambda^{\triangle} = \Lambda_1$. We get

$$\begin{aligned}
[f_0 h, e_1] &= f_0[h, e_1] = \alpha_1(h) f_0 e_1 \\
&= -\alpha_1(h) f_0[h_0 + L, e_1] = -\alpha_1(h)[f_0(h_0 + L), e_1] = 0\,.
\end{aligned} \tag{4.9}$$

Similarly,

$$\begin{aligned}
[f_1, [f_1, f_0 h]] &= [f_1, f_0[f_1, h]] + [f_1, [f_1, f_0]h] \\
&= f_0[f_1, [f_1, h]] + 2[f_1, f_0][f_1, h] + [f_1, [f_1, f_0]]h \\
&= 2[f_1, f_0][f_1, h] = 2\alpha_1(h)[f_1, f_0] f_1 \\
&= -2\alpha_1(h)[f_1, f_0][f_1, h_0 + L]\,.
\end{aligned} \tag{4.10}$$

We can then perform the first three steps in (4.10) backwards, but with h replaced by $h_0 + L$, and find that

$$[f_1, f_0][f_1, h_0 + L] = [f_1, [f_1, f_0(h_0 + L)]] = 0\,. \tag{4.11}$$

We have thus shown the relations (4.4). The other relations involving f_{0K} are straightforward to show,

$$\begin{aligned}
[e_0, f_{0K}] &= [e_0, f_0 h_K] = [e_0, f_0]h_K - f_0[e_0, h_K] = h_K\,, \\
[h_I, f_{0J}] &= [h_I, f_0 h_J] = [h_I, f_0]h_J + f_0[h_I, h_J] = -B_{I0} f_0 h_J = -B_{I0} f_{0J}\,, \\
[e_i, [f_j, f_{0K}]] &= [e_i, [f_j, f_0 h_K]] = f_0[e_i, [f_j, h_K]] = \delta_{ij} B_{Kj} f_0 h_j = \delta_{ij} B_{Kj} f_{0j}\,,
\end{aligned} \tag{4.12}$$

and those not involving f_{0K} automatically satisfied. □

We conjecture that this homomorphism is in fact an isomorphism, but leave the proof for future work.

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