THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Boundary-preserving numerical schemes for stochastic ordinary and partial differential equations

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Gothenburg, Sweden 2025

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Gothenburg 2025
ISBN: 978-91-8103-310-6

Acknowledgements, dedications, and similar personal statements in this thesis, reflect the author's own views.

Doktorshavhandlingar vid Chalmers tekniska högskola
Ny serie nr 5767
ISSN 0346-718X
DOI https://doi.org/10.63959/chalmers.dt/5767

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Cover: Comparison of an accepted and a rejected sample path, illustrating the boundary-preserving property.

Typeset with LaTeX
Printed by Chalmers Reproservice
Gothenburg, Sweden 2025

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Abstract

This thesis contributes to the development of boundary-preserving numerical schemes for the strong and weak approximation for stochastic ordinary and partial differential equations (SDEs and SPDEs, respectively). Several of the considered equations model a physical quantity with an inherently restricted range, such as temperature (positive values), stock prices (positive values) or fractions (values in $[0, 1]$), referred to as the invariant domain of the equation. A numerical scheme is said to be boundary-preserving if its numerical approximations are guaranteed to remain within this domain. Boundary preservation is important for the physical interpretability and stability of the numerical approximations. Some established approaches to constructing boundary-preserving schemes are surveyed in the first part of the thesis, and the appended papers explore and develop new methods to guarantee this property.

Paper I combines the Lamperti transform with a Lie–Trotter time splitting to construct a family of boundary-preserving numerical schemes for some scalar SDEs achieving strong convergence of order 1. **Paper II** constructs boundary-preserving numerical schemes for scalar SDEs by introducing auxiliary stochastic processes to convert the considered SDE into an associated reflected SDE. **Paper III** constructs a positivity-preserving temporal numerical scheme for some semilinear stochastic heat equations perturbed by temporal white noise. The proposed scheme employs a Lie-–Trotter time splitting method, allowing the deterministic and stochastic parts of the equation to be treated independently. **Paper IV** combines the ideas from Paper III with a finite difference spatial discretisation to obtain the first positivity-preserving numerical scheme for some semilinear stochastic heat equations perturbed by space-time white noise. **Paper V** combines the ideas from Paper IV with exact simulation for SDEs to obtain the first boundary-preserving numerical scheme for some semilinear SPDEs perturbed by space-time white noise with bounded invariant domain.

Keywords: Stochastic ordinary differential equations, stochastic partial differential equations, geometric numerical integration, boundary-preserving, positivity-preserving, Lie–Trotter time splitting, strong convergence, weak convergence.

List of publications

This thesis is based on the work contained in the following papers:

I. **J. Ulander**. (2023). Boundary-preserving Lamperti-splitting scheme for some stochastic differential equations. *J. Comput. Dyn.* ***11****(3)*.
II. **J. Ulander**. (2024). Artificial Barriers for stochastic differential equations and for construction of boundary-preserving schemes. Preprint available at *arXiv:2410.04850*. Submitted.
III. C.-E. Bréhier, D. Cohen, **J. Ulander**. (2024). Positivity-preserving schemes for some nonlinear stochastic PDEs. Proceeding In *Sixteenth International Conference Zaragoza-Pau on Mathematics and its Applications, volume 43 of Monogr. Mat. García Galdeano, pages 31–40. Prensas Univ. Zaragoza, Zaragoza*.
IV. C.-E. Bréhier, D. Cohen, **J. Ulander**. (2023). Analysis of a positivity-preserving splitting scheme for some semilinear stochastic heat equations. *ESAIM Math. Model. Numer. Anal.* ***58****(4)*.
V. **J. Ulander**. (2024). Boundary-preserving weak approximation for some semilinear stochastic partial differential equations. Preprint available at *arXiv:2412.10800*.

Author contributions

I. The problem was formulated by myself. I carried out the theoretical and numerical analyses, and I wrote the paper with valuable input from my supervisor, David Cohen.

II. The problem was formulated by myself. I carried out the theoretical and numerical analyses, and I wrote the paper.

III. I carried out some numerical experiments and contributed to the theoretical development.

IV. I played a key role in the development and numerical analysis of the proposed scheme. I contributed to the MATLAB implementation, performed several cluster computations, and wrote a substantial part of the manuscript.

V. The problem was formulated by myself. I carried out the theoretical and numerical analyses, and I wrote the paper.

Acknowledgements

First of all, I would like to thank my supervisor David Cohen for his guidance over the years, both mathematically and professionally. Next, I would also like to thank my co-author Charles–Edouard Bréhier and my co-author and co-supervisor Lluís Quer-Sardanyons for their collaboration and guidance throughout these years.

To all my fellow PhD students at the department of Mathematical Sciences at Chalmers University of technology–thank you for all the fun conversations during the lunch breaks. A special thank you to my office mates through the years, Petar and Kasper. To Irina and Marija–thank you for your constant support and for always taking the time to discuss work and non-work matters. The administrators and economists–Marie, Pernilla, and Ai-Linh–also deserve a special mention here for always helping with the small things that make a big difference.

I am grateful to the people that have read and given feedback on preliminary versions of this thesis: David, Ioanna, Kasper, and my dad, Lars. David, thank you for your feedback on multiple iterations. Ioanna, thank you for your continuous input on various aspects and for your help with the stickman drawing in the introduction.

Finally, I would like to thank my family. My dad, Lars, that inspired me to pursue a PhD in the first place. Thank you for all your support and for all the interesting discussions we have had over the past few years, ranging from research to philosophy of everyday life. My mom, Åsa, for you unconditional love and support, for always being there for me. My younger sisters (that are not that small anymore), Sara and Linnéa, for always believing in me. My older sister, Frida, for inspiring me to pursue an engineering degree at Chalmers. Lastly, Ioanna, thank you for being you (and for allowing me to have bikes on the walls at home).

Johan Ulander
Gothenburg, 2025

Contents

1 Introduction

Mathematical models are used to describe phenomena ranging from the grand-scale evolution of the universe to the behaviour of sub-atomic particles (see, e.g., Wald (1984); Elliot and Lira (1999); Meriam et al. (2014, 2020)). Mathematical models are important because they enable us to formulate the dynamics of the studied phenomena in a precise and consistent way. Furthermore, they allow us to analyse different aspects of the phenomena without the need for potentially costly or infeasible physical experiments.

To make the discussion more concrete, imagine that we wish to throw a tennis ball into a basket. See Figure 1.1 for an illustration. If we have several tennis balls at our disposal, then we could make initial guesses (based on, e.g., experience) for the angle to the ground and the speed at which we should throw the first ball to hit the basket. If we hit the basket, then great. If we do not hit the basket, then we adjust the angle or the speed to make the next throw closer to the basket than the previous one. Based on classical physics and mathematics, we can, however, hit the basket without the need to waste a lot of tennis balls.

Newtonian physics can be formalised into a mathematical model that describes and predicts the motion of macroscopic objects, such as tennis balls, where the time evolution is governed by *ordinary differential equations* (see, e.g., Meriam et al. (2014, 2020)). This model enables us to determine the angle and the speed at which the tennis ball should be thrown to hit the basket with the first attempt, assuming the basket's relative position to us is known. This holds provided that the angle and speed of the ball can be adjusted with sufficient precision, and that the model constitutes a reasonable representation of the real world.

Certain types of mathematical models involve quantities with fundamental boundaries. For instance, let us consider the Standard and Poor's 500 (S&P 500) index prices from March 2015 to March 2025 as shown in Figure 1.2 together with a deterministic model of the long-term trend in red. The S&P 500 index is a stock index consisting of 500 stocks of big companies on the

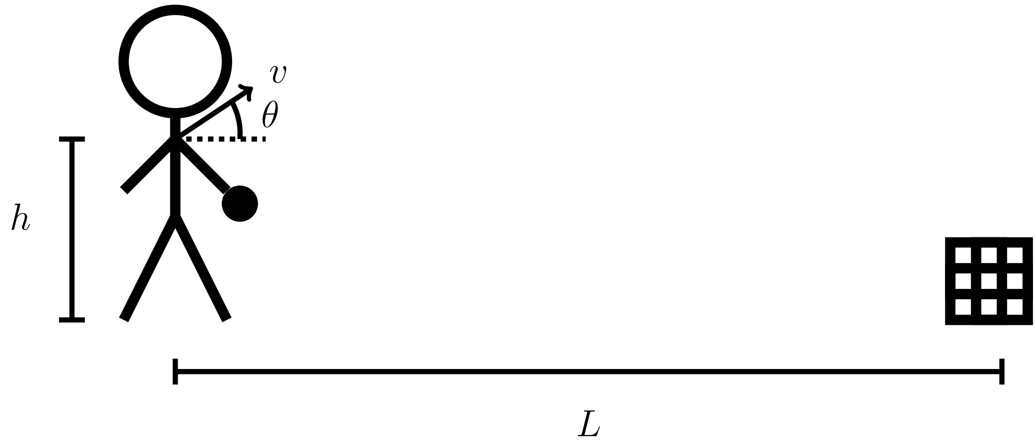

Figure 1.1: Illustration of how to use Newtonian physics to hit a basket when throwing a ball. Some assumptions: The angle θ at which we throw is assumed to be fixed at $\pi/4$ (45 degrees) and we neglect the drag from the air on the ball with mass m. The distance covered is $L = \frac{|v|^2}{2g} + \frac{|v|}{\sqrt{2}}\sqrt{\frac{2h}{g} + \frac{|v|^2}{2g^2}}$, where $g \approx 9.8$ is the free fall acceleration.

New York Stock Exchange (NYSE) or on the National Association of Securities Dealers Automated Quotations (NASDAQ). As the S&P 500 index is a positive

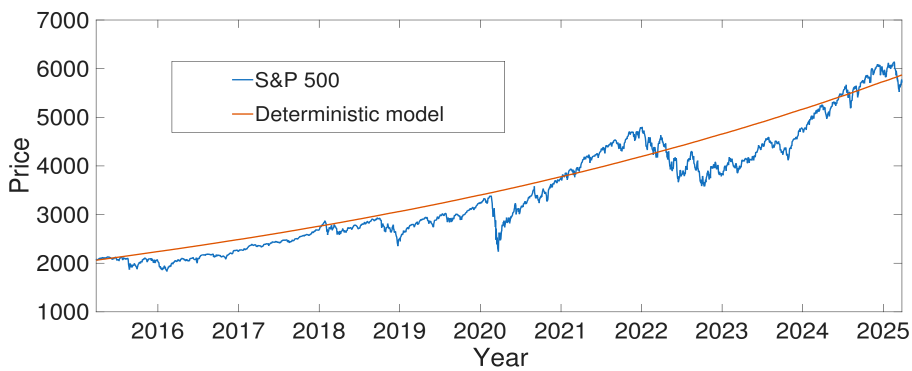

Figure 1.2: Time evolution of the Standard and Poor's 500 (S&P 500) index in blue and a deterministic model of the long-term trend in red.

weighted sum of stock prices that are inherently positive, the S&P 500 index must also be positive. Negative values of a stock index would make investors instantly doubt the index. In other words, positivity is a fundamental boundary of any data set of or model for stock indexes. The red trend line in Figure 1.2 is generated using a deterministic model that tries to capture the long-term growth of the index. This deterministic model is a reasonable model for the long-term evolution of the index, but it cannot capture the short-term variations in the index.

As most realistic mathematical models are too complicated to be solved with pen and paper, we often have to resort to computer simulations. A possible issue with computer simulations is that they may fail to preserve properties of the underlying model. The study of this behaviour in simulations is part of a branch of mathematics called *geometric numerical integration*. We refer to Hairer et al. (2010) for a classical textbook on the topic. Returning to the S&P 500 index example, a computer simulation for future predictions of the index might produce negative values. To address this, we study computer algorithms–sets of instructions that determine a computer simulation–that are guaranteed to produce only physically meaningful results (e.g., positive values in the case of index prices). We call such computer algorithms *boundary-preserving*. We illustrate this terminology using the S&P 500 index. The physically admissible domain of index prices is $[0, \infty)$, and the boundary of this domain is 0 (there is no upper boundary). A boundary-preserving algorithm has the same admissible domain, $[0, \infty)$ in this case, and therefore shares the same boundary, 0 in this case.

In practical applications, uncertainties should be incorporated into deterministic models. Such uncertainties may arise from data errors, measurement noise or external disturbances, and are collectively referred to as *noise*. Chapter 1 in Øksendal (2003) provides several motivational examples illustrating the need to go beyond deterministic models. We integrate such effects by employing stochastic models; that is, models in which the dynamics evolve according to probabilistic rules. Returning to the example with the S&P 500 index, if we wish to account for the seemingly unpredictable short-term variations of the index, then a stochastic model for the time-evolution of the S&P 500 index may be a reasonable choice. Typically, such models are described by *stochastic differential equations*. See Figure 1.3 for an illustration of how a stochastic model can account for short-term variations in the S&P 500 index price.

Preserving fundamental boundaries of models in computer simulations becomes even more challenging in the presence of noise. A widely used example in practice is that of noise following a normal density, referred to as *Gaussian noise*. See Figure 1.4 for an illustration of the normal density. The value of the normal density function at a point x indicates how likely that value is relative to other values. Gaussian noise can, however, with very small probability, take arbitrary large positive and negative values. Therefore, in traditional computer simulations of index prices, there is a non-zero probability of obtaining negative values at each simulation step. Consequently, according to Murphy's law, such computer simulation will eventually yield negative index prices. In summary, ensuring that computer simulations for stochastic models are boundary-preserving is essential to guarantee meaningful results.

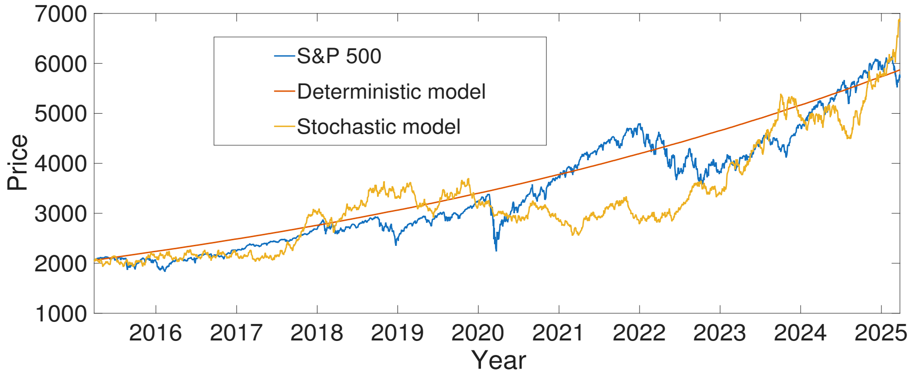

Figure 1.3: Time evolution of the Standard and Poor's 500 (S&P 500) index in blue, a deterministic model of the long-term trend of the S&P 500 index in red, and a stochastic model of the short-term variations of the S&P 500 index in yellow.

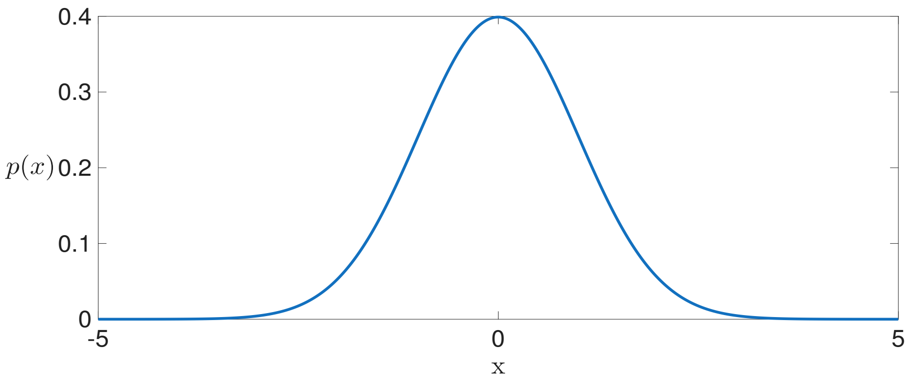

Figure 1.4: Plot of the standard normal density: $p(x) = \frac{1}{\sqrt{2}} e^{-\frac{x^2}{2}}$, $x \in [-5, 5]$. The standard normal density is also known as the bell curve for its shape.

This thesis is a contribution to the research topic of boundary-preserving numerical schemes to approximate the solutions of complicated mathematical models given by stochastic differential equations. This refers to the construction of numerical schemes that are guaranteed to preserve the boundary of the admissible domain of the equation. Classical numerical schemes are, in general, not boundary-preserving.

This thesis consists of a comprehensive introduction to the research topic referred to as a *kappa* and five appended papers. The kappa provides the background necessary to understand the importance of boundary-preserving numerical schemes for stochastic differential equations and the main approaches to their construction. In particular, the intention is that the reader will be able to understand the summaries of the papers in Section 3 based on the material presented up to that point. The focus is not to present all rigorous mathematical details, but to convey my own understanding of the topics in a personal style. References to classical and rigorous works are provided throughout the kappa. The reader is assumed to have mathematical knowledge corresponding to undergraduate studies in mathematics.

The rest of thesis is organised as follows. Section 2.1 introduces stochastic ordinary differential equations (SDEs) and presents the properties of their solutions most relevant for our purposes. Section 2.2 is devoted to time discretisations of SDEs. In this section, we first define and discuss convergence of numerical schemes, and then introduce two classical numerical schemes to approximate solutions of SDEs. We then present the Lie–Trotter time splitting method and some current approaches to constructing boundary-preserving schemes for SDEs. The summaries of Papers I and II can be read after completing Section 2.1 and Section 2.2. Section 2.3 introduces stochastic partial differential equations (SPDEs) and discusses their solutions. Section 2.4 is dedicated to spatial discretisations, with a focus on the finite difference method, of SPDEs. Sections 2.1- 2.5 provide the necessary background to read the summaries of Papers III–V. Lastly, we provide the references and the appended papers.

2 Background

2.1 Stochastic ordinary differential equations

In this section, we introduce the background on stochastic ordinary differential equations (SDEs) needed for the summaries of the appended papers. To avoid repetition, we let $T \in (0, \infty)$ denote a fixed finite end time and $t \in [0, T]$ denote the time variable.

In the following, we introduce and focus on the stochastic calculus known as *Itô calculus*, as this is used for the appended papers. In contrast to standard calculus, there are choices to be made in the set-up for a stochastic calculus resulting in different frameworks. The most common other choice of stochastic calculus is the *Stratonovich calculus*, introduced in Stratonovich (1964). This is used in, for example, physics and in stochastic analysis on manifolds (see, e.g., Hsu (2002)).

Let us start with an illustrative example of an ordinary differential equation (ODE) with solutions that remain in a domain $\mathcal{D} \subset \mathbb{R}$ that we call the *invariant domain* of the ODE. The susceptible–infected–susceptible (SIS) ODEs form a family of ODEs whose solutions represent the *fractions* of populations that are susceptible to diseases or viruses. The simplest SIS ODE can be expressed as

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = y(t)(1 - y(t)), \ t \in (0, T], \\ y(0) = y_0 \in [0, 1]. \end{cases} \tag{2.1}$$

The SIS ODE in (2.1) admits a closed-form solution given by

$$y(t) = \frac{e^t}{e^t - 1 + y_0^{-1}}, \ t \in [0, T], \tag{2.2}$$

for $y_0 \in (0,1]$, and by $y(t) = 0$, $t \in [0,T]$, for $y_0 = 0$. In other contexts, (2.1) is referred to as the logistic equation. As seen from (2.2), the solution $y(t)$ of (2.1) satisfies $y(t) \in [0,1]$ for all $t \in [0,T]$. Hence, we refer to $\mathcal{D} = [0,1]$ as the invariant domain of (2.1). A value outside $\mathcal{D} = [0,1]$ cannot be interpreted as a fraction of a population. We refer to Chapter 1 in López-Flores et al. (2021) for more details on SIS ODEs.

The SIS ODE in (2.1) is an example of an ODE of the form

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = f(y(t)),\ t \in (0,T], \\ y(0) = y_0 \in \mathbb{R}, \end{cases} \tag{2.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is sufficiently regular. A wide range of phenomena can be modelled using ODEs of the form (2.3). Applications are found in epidemiology, chemical reactions, the motion of particles and bodies and many other areas. We refer to Tenenbaum and Pollard (1985); Arnold (2006) for classical textbooks on ODE theory, and to Chapter 1.1 and Chapter 3, respectively, for applications.

ODEs cannot, in certain scenarios, fully capture the behaviour we observe in physical systems. In such cases, it is reasonable to add noise to the ODE, yielding an SDE. The solution of an SDE is given by a *stochastic process* that depends on time $t \in [0,T]$ and an additional parameter, typically denoted by ω, which represents the randomness. For clarity of presentation, we omit this parameter. To this end, we first recall that a Brownian motion $B : [0,T] \to \mathbb{R}$ is a zero-mean continuous Gaussian process with covariance given by

$$\mathbb{E}\left[B(t)B(s)\right] = \min(t,s),\ t,s \in [0,T],$$

where $\mathbb{E}$ denotes the expected value operator. Brownian motions exhibits erratic behaviour–they are nowhere differentiable, with probability one. We can quantify this irregular behaviour using a Hölder continuity condition: A Brownian motion is, with probability one, Hölder continuous with exponent $\gamma = 1/2 - \epsilon$ for every $\epsilon > 0$, meaning that there exists a constant $C > 0$ such that

$$|B(t) - B(s)| \leq C|t-s|^{\gamma},$$

for all $t, s \in [0,T]$, with probability one. This rough behaviour is illustrated by the five sample paths (2-dimensional visualisations for different choices of ω) shown in Figure 2.1. We refer to Chapter 2 in Karatzas and Shreve (1988) for a comprehensive treatment of Brownian motion.

Heuristically, we aim to perturb the ODE in (2.3) by noise having no correlation in time called *temporal white noise*. Temporal white noise $\frac{\mathrm{d}B(t)}{\mathrm{d}t}$ is the formal

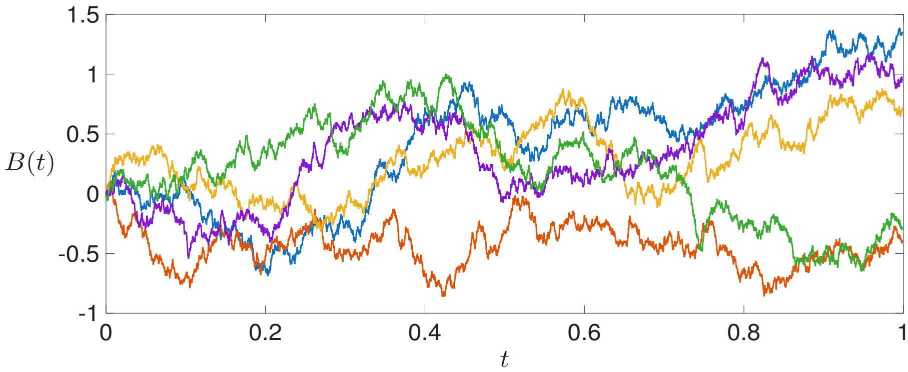

Figure 2.1: Five sample paths of a Brownian motion.

derivative of Brownian motion. However, due to the low regularity of the sample paths of Brownian motion, the integral version of (2.3) that reads

$$y(t) = y_0 + \int_0^t f(y(s))\,\mathrm{d}s,\ t \in [0, T], \tag{2.4}$$

is the preferred starting point to introduce noise driven by Brownian motion to the ODE in (2.3). Therefore, we now instead include a second integral term in (2.4) to obtain

$$X(t) = x_0 + \int_0^t f(X(s))\,\mathrm{d}s + \int_0^t g(X(s))\frac{\mathrm{d}B(s)}{\mathrm{d}s}\,\mathrm{d}s,\ t \in [0, T], \tag{2.5}$$

where $x_0 \in \mathbb{R}$ is the non-random initial value replacing y_0 and $g : \mathbb{R} \to \mathbb{R}$ is sufficiently regular. To make the distinction clear between ODEs and SDEs, we use the notations $y(t)$ and $z(t)$ to denote solutions of ODEs and we use $X(t)$, $Y(t)$ and $Z(t)$ to denote solutions of SDEs.

Suppose, for the time being, that $\frac{\mathrm{d}B(t)}{\mathrm{d}t}$ is well-defined. Then we could equate

$$\int_0^t g(X(s))\frac{\mathrm{d}B(s)}{\mathrm{d}s}\,\mathrm{d}s = \int_0^t g(X(s))\,\mathrm{d}B(s),\ t \in [0, T],$$

in the sense of Riemann—Stieltjes integrals. In other words, if we can make sense of the *stochastic integral* $\int_0^t g(X(s)\,\mathrm{d}B(s)$, then we could represent (2.5) as

$$X(t) = x_0 + \int_0^t f(X(s))\,\mathrm{d}s + \int_0^t g(X(s))\,\mathrm{d}B(s),\ t \in [0, T]. \tag{2.6}$$

To make sense of (2.6), Itô (1944) defined the stochastic integral in (2.6) for piecewise constant integrands and then extended to a broader family of stochastic processes using limits. For this, the stochastic integral in (2.6) is now known as the *Itô integral* and (2.6) is known as an *Itô integral equation* first introduced in Ito (1951). On a historical note, the construction of the integral of real-valued functions with respect to Brownian motion was first introduced by Paley and Wiener in 1934 (see Paley and Wiener (1987) for the reprint).

Usually, the stochastic integral equation in (2.6) is written in Itô differential form as

$$\begin{cases} \mathrm{d}X(t) = f(X(t))\,\mathrm{d}t + g(X(t))\,\mathrm{d}B(t),\ t \in (0,T], \\ X(0) = x_0 \in \mathbb{R}. \end{cases} \tag{2.7}$$

This should be understood as the integral equation in (2.6). The expression (2.7) is referred to as an *Itô stochastic ordinary differential equation* (SDE). Intuitively, over a small time interval of length $\mathrm{d}t$, the solution $X(t)$ of (2.7) changes by a random amount that is normally distributed with mean $f(X(t))\,\mathrm{d}t$ and variance $g(X(t))^2\,\mathrm{d}t$.

The noise term $g(X(t))\,\mathrm{d}B(t)$ in (2.7) is called additive if $g(r) = C$ for some constant C and it is called multiplicative otherwise. In general, SDEs with additive noise are known to be easier to handle than SDEs with multiplicative noise.

Commonly, f is called the drift coefficient function and g is called the diffusion coefficient function in (2.7). If f and g are globally Lipschitz continuous functions; meaning that there exists a constant $C > 0$ such that

$$|f(r) - f(s)| + |g(r) - g(s)| \leq C|r - s|,$$

for all $r, s \in \mathbb{R}$, then there exists a unique stochastic process $X(t)$ that satisfies (2.6), with probability one; this process is called the solution of the SDE in (2.7). Such a stochastic process is typically referred to as a strong solution, emphasising that the driving noise $B(t)$ is specified in advance. In contrast, if the driving noise $B(t)$ is part of the solution (that is, not specified a priori), then the solution is called a weak solution. We refer to Chapter 5 in Øksendal (2003) and to Chapter 5 in Karatzas and Shreve (1988) for more details. Moreover, the solution $X(t)$ of (2.7) is Hölder continuous with exponent $1/2-$, with probability one, the same property as for Brownian motion. If f and g are not assumed to be globally Lipschitz continuous, then the situation is more involved. See Chapter 3 of Borodin and Salminen (2002) and Chapter 5 of Hutzenthaler et al. (2011) for collections of such results.

The Itô integral in (2.6) enjoys many useful properties. The presented properties were first introduced in Itô (1944). The Itô integral is a zero-mean martingale satisfying the fundamental properties known as the *Itô isometry*, stating that

$$\mathbb{E}\left[\left(\int_0^t g(X(s))\,\mathrm{d}B(s)\right)^2\right] = \mathbb{E}\left[\int_0^t g(X(s))^2\,\mathrm{d}s\right],\ t \in [0,T], \tag{2.8}$$

and *Itô's formula*, which we discuss next.

The chain rule plays a fundamental role in many areas of mathematics and its applications, but the chain rule assumes that the involved functions are differentiable. Stochastic processes arising as solutions of SDEs are nowhere differentiable, with probability one, and the classical chain rule does therefore not apply. Itô's formula can be seen as a generalisation of the classical chain rule to non-differentiable stochastic processes

In the context of this thesis, Itô's formula is used to derive the *Lamperti transform*, which forms the basis for a class of boundary-preserving numerical schemes in Section (2.2.4). The Lamperti transform is a key component in Papers I and V. We now illustrate this procedure. Let $\mathcal{D} \subset \mathbb{R}$ be the invariant domain of the solution $X(t)$ of (2.7), meaning that

$$\mathbb{P}\left(X(t) \in \mathcal{D},\ \text{for all}\ t \in (0,T]\right) = 1,$$

whenever $x_0 \in \mathcal{D}$, where $\mathbb{P}$ denotes the probability measure. Furthermore, let $\Phi : \mathcal{D} \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ (twice continuously differentiable). Itô's formula states that the transformed process $\Phi(X(t))$ is the solution of the SDE

$$\begin{cases} \mathrm{d}\Phi(X(t)) = \hat{f}(X(t))\,\mathrm{d}t + \hat{g}(X(t))\,\mathrm{d}B(t),\ t \in (0,T], \\ \Phi(X(0)) = \Phi(x_0), \end{cases} \tag{2.9}$$

where the coefficient functions $\hat{f}$ and $\hat{g}$ are given by

$$\hat{f}(r) = \left(f(r)\Phi'(r) + \frac{1}{2}g(r)^2\Phi''(r)\right),\ r \in \mathcal{D},$$

and

$$\hat{g}(r) = g(r)\Phi'(r),\ r \in \mathcal{D}.$$

If Φ is chosen in such a way that $\hat{g}(r) = 1$ for all $r \in \mathcal{D}$, then the noise in the SDE in (2.9) reduces to additive noise. This choice specifies Φ (up to a constant) as

$$\Phi(r) = \int_{w_0}^{r} \frac{1}{g(w)}\,\mathrm{d}w,\ r \in \mathcal{D}, \tag{2.10}$$

and is referred to as the Lamperti transform, named after Lamperti (1962).

Inserting the first and second derivatives of Φ in (2.10) into the SDE (2.9), we see that the transformed process $Y(t) = \Phi(X(t))$ is the solution of the following SDE with additive noise

$$\begin{cases} \mathrm{d}Y(t) = \alpha(Y(t))\,\mathrm{d}t + \mathrm{d}B(t),\ t \in (0, T], \\ Y(0) = \Phi(x_0), \end{cases} \tag{2.11}$$

where the drift coefficient function α is given by

$$\alpha(r) = \frac{f(\Phi^{-1}(r))}{g(\Phi^{-1}(r))} - \frac{1}{2} g'(\Phi^{-1}(r)),\ r \in \mathbb{R}. \tag{2.12}$$

For the above to be well-defined, we require that $g(r) > 0$ for all $r \in \mathcal{D}$ and that Φ is invertible. The latter is, for example, true if g, in addition, is continuous.

To make the discussion more concrete, let us consider an SDE with a closed-form solution that can be obtained using the Lamperti transform. The SDE for geometric Brownian motion is typically written as

$$\begin{cases} \mathrm{d}X(t) = \mu X(t)\,\mathrm{d}t + \sigma X(t)\,\mathrm{d}B(t),\ t \in (0, T], \\ X(0) = x_0 \in \mathcal{D} = (0, \infty), \end{cases} \tag{2.13}$$

where $\mu \in \mathbb{R}$ is called the drift parameter and $\sigma \in \mathbb{R}$ is called the diffusion parameter. The invariant domain of (2.13) is $\mathcal{D} = (0, \infty)$. The solution of (2.13) is often denoted by $S(t)$ in financial contexts–to emphasise that it models a stock price–we use $X(t)$ for consistency with our notation. The Lamperti transform associated with the diffusion term in (2.13) is given by

$$\Phi(r) = \int_{w_0}^{r} \frac{1}{\sigma w}\,\mathrm{d}w = \frac{1}{\sigma}\left(\ln(r) - \ln(w_0)\right),\ r \in (0, \infty),$$

for any choice $w_0 \in (0, \infty)$, with inverse given by

$$\Phi^{-1}(r) = w_0 e^{\sigma r},\ r \in \mathbb{R}.$$

Let us choose $w_0 = 1$ for simplicity. We insert the drift and diffusion coefficient functions from (2.13) into (2.11) to see that the transformed process $\Phi(X(t))$ is the solution of the following SDE

$$\begin{cases} \mathrm{d}\Phi\left(X(t)\right) = \left(\frac{\mu}{\sigma} - \frac{1}{2}\sigma\right)\mathrm{d}t + \mathrm{d}B(t),\ t \in (0, T], \\ \Phi\left(X(0)\right) = \frac{1}{\sigma}\ln(x_0) \in \mathbb{R}. \end{cases} \tag{2.14}$$

The solution of the SDE in (2.14) can be written in closed-form as

$$\Phi(X(t)) = \frac{1}{\sigma}\ln(x_0) + \left(\frac{\mu}{\sigma} - \frac{1}{2}\sigma\right) t + B(t),\ t \in [0,T].$$

The solution of the original SDE in (2.13) can then be obtained by

$$X(t) = \Phi^{-1}\left(\frac{1}{\sigma}\ln(x_0) + \left(\frac{\mu}{\sigma} - \frac{1}{2}\sigma\right) t + B(t)\right) = x_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B(t)}, \quad (2.15)$$

for $t \in [0,T]$.

Let us now come back to the SIS ODE in (2.1). By introducing multiplicative noise with $g(r) = r(1-r)$ to (2.1), we obtain a SIS SDE of the form

$$\begin{cases} \mathrm{d}X(t) = X(t)(1-X(t))\,\mathrm{d}t + X(t)(1-X(t))\,\mathrm{d}B(t),\ t \in (0,T], \\ X(0) = x_0 \in [0,1]. \end{cases} \quad (2.16)$$

With probability one, the solution $X(t)$ of (2.16) remains within the same domain as the solution of the SIS ODE in (2.1):

$$\mathbb{P}\left(X(t) \in \mathcal{D} = [0,1],\ \text{for all } t \in [0,T]\right) = 1.$$

We refer to Gray et al. (2011) for more details on SIS SDEs. In other words, by introducing noise of a certain form, we can make sure that the invariant domain of the SDE coincides with the invariant domain of the ODE. In contrast, if we introduce additive noise to the ODE in (2.1), then the invariant domain would be all of $\mathbb{R}$. Put differently, additive noise is (in many cases) incompatible with $\mathcal{D} \subsetneq \mathbb{R}$ being a strict subset of $\mathbb{R}$. Thus, if the modelled phenomena represents a physical quantity only taking values in an invariant domain $\mathcal{D} \subsetneq \mathbb{R}$, then we have to consider multiplicative noise.

The study of SDEs of the form (2.7) with invariant domains $\mathcal{D} \subsetneq \mathbb{R}$ goes back to the early works Feller (1951, 1952) and is usually referred to as *Feller's boundary classification* for SDEs. We refer the interested reader to Chapter 15 in Karlin and Taylor (1981) for a detailed treatment of Feller's boundary classification, and to Chapter 2 in Borodin and Salminen (2002) for a collection of such results. Without going too much into details, Feller's boundary classification characterises the behaviour of the solution of (2.7) near and at the boundary points of $\mathcal{D}$ in terms of the drift and diffusion coefficient functions f and g. Of particular interest in this thesis is determining whether the boundary points $\partial\mathcal{D}$ of $\mathcal{D}$ are attainable or unattainable for the solution process. This corresponds to $\mathcal{D}$ being closed or open, respectively, as a set.

Typically, the invariant domain of an SDE reflects the domain of a physical quantity, and it is desirable for numerical approximations to preserve this domain. How can we construct numerical schemes with this property? This question is a focus of the next section.

2.2 Time discretisations

This section reviews time discretisations of SDEs, covering *convergence*, classical and modern numerical schemes, and established approaches to construct boundary-preserving numerical schemes.

As most interesting SDEs do not have closed form solutions, simulations or visualisations of solutions of such SDEs have to rely on numerical approximation. In this thesis, we are concerned with numerical time approximations obtained from discretising the time variable, referred to as *time discretisations*.

The study of time discretisations of SDEs dates back to Maruyama (1955) introducing the classical scheme known as the Euler–Maruyama (EM) scheme. Since then, it has remained an active area of research. We also mention the work Milstein (1974) where the author derives the Milstein scheme achieving more accurate approximations than the EM scheme. The presented material can be found in well-known textbooks on numerical analysis for SDEs; we mention Kloeden and Platen (1992); Milstein (1995); Milstein and Tretyakov (2021) as a selective list.

Let us recall the considered SDE

$$\begin{cases} \mathrm{d}X(t) = f(X(t))\,\mathrm{d}t + g(X(t))\,\mathrm{d}B(t),\ t \in (0,T], \\ X(0) = x_0 \in \mathbb{R}, \end{cases} \tag{2.17}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are sufficiently regular. We let $M \in \mathbb{N}$ denote the discretisation parameter determining the number of time grid points and we let $\Delta t = T/M$ denote the (uniform) time step size. The generated time grid points are denoted by $t_m = m\Delta t$ for $m = 0, \ldots, M$.

By a numerical approximation of the solution $X(t)$ of the SDE in (2.17), we mean a sequence of random variables $X_0, X_1, \ldots, X_M$ generated by a numerical scheme such that $X_m \approx X(t_m)$, for all $m = 0, \ldots, M$, in some probabilistic sense. The *convergence order* of a numerical scheme quantifies how rapidly X_m, for $m = 0, \ldots, M$, approaches $X(t_m)$ as M increases, and is the focus of the next section.

2.2.1 Convergence of numerical schemes

In this section we introduce the notions of strong and weak convergence needed for the remaining part of the thesis. We define convergence for the end time point $t = T$, the generalisation to any $t \in [0, T]$ is straightforward.

Other common notions of convergence in stochastic numerics include, but is not limited to, almost sure convergence and convergence in probability. Almost sure convergence requires that a sequence converges to a limit, with probability one. Heuristically speaking, convergence in probability formalises the idea that events with small probability become increasingly rare as the sequence progresses. Further details on the different notions of convergence of random variables can be found in standard textbooks on probability theory and stochastic processes, such as in Chapter 7 of Grimmett and Stirzaker (2020).

Let $X_0, \ldots, X_M$ be a sequence of random variables generated from a numerical scheme to approximate the solution $X(t)$ of the SDE in (2.17) at time $t = T$ corresponding to the time grid $t_0 < \ldots < t_M$ with time step size $\Delta t = T/M$. We say that X_M converges p-strongly (for $p \geq 1$) of order $\gamma_1 > 0$ to $X(T)$ if

$$\left(\mathbb{E}\left[|X_M - X(T)|^p\right]\right)^{1/p} \leq C\Delta t^{\gamma_1}, \tag{2.18}$$

for some constant C that is independent of M. We say that X_M converges weakly of order $\gamma_2 > 0$ to $X(T)$ if

$$\left|\mathbb{E}\left[F(X_M)\right] - \mathbb{E}\left[F(X(T))\right]\right| \leq C\Delta t^{\gamma_2}, \tag{2.19}$$

for every test function F in a suitable test function space and for some constant C that is independent of Δt. Strong convergence requires the approximating sequence and the limit to be defined on the same probability space while this is not required for weak convergence. The regularity of the coefficient functions f and g in the SDE in (2.7) is closely linked to the strong and weak convergence orders of numerical schemes used to approximate the solution $X(t)$.

If X_M converges p-strongly to $X(T)$, then X_M also converges weakly γ_1 to $X(T)$ with respect to bounded continuous test functions. For a globally Lipschitz test function F, we can immediately obtain that the weak order is at least that of the strong order

$$\left|\mathbb{E}\left[F(X_M)\right] - \mathbb{E}\left[F(X(T))\right]\right| \leq C\left(\mathbb{E}\left[|X_M - X(T)|^p\right]\right)^{1/p},$$

where C is a constant that depends on the Lipschitz constant of F. However, the order of weak convergence is sometimes strictly higher than the order of strong convergence. As is true in many cases, the *rule of thumb* of weak convergence states that the order of weak convergence is twice the order of strong convergence. See, for example, Debussche and Printems (2009); Andersson and Larsson (2016) for such results. The order of weak convergence may, however, depend on the regularity of the chosen test functions. For the EM scheme in the SDE case, smooth test functions and merely bounded measurable test functions yield the same order of weak convergence and is twice the order of strong convergence (see, for instance, Talay and Tubaro (1990); Bally and Talay (1995)). As is shown in Bréhier (2020), this is not the case for stochastic partial differential equation (SPDEs), the order of weak convergence for SPDEs for bounded globally Lipschitz continuous test functions is equal to the order of strong convergence while the rule of thumb holds for more regular test functions.

In practice, the expected values in (2.18) and in (2.19) have to be estimated from finitely many samples of $X_M - X(T)$. A classical approach known as the *Monte Carlo method* is to generate $K \in \mathbb{N}$ independent samples $\{X_M^k - X^k(T) : k = 1, \dots, K\}$, each having the same distribution as $X_M - X(T)$, and use

$$\mathbb{E}\left[|X_M - X(T)|^p\right] \approx \frac{1}{K} \sum_{k=1}^{K} \left|X_M^k - X(T)\right|^p$$

to estimate the left hand side of (2.18). The left hand side of (2.19) can be estimated in a similar manner. We refer to Robert and Casella (2004) for a well-known reference on Monte Carlo methods.

2.2.2 Classical numerical schemes

In this section, we first introduce the forward Euler scheme and the backward Euler scheme, two of the most well-know time integrators for ODEs, and then discuss the extensions of these numerical schemes to SDEs of the form (2.17). The presented material can be found in well-known textbooks such as Süli and Mayers (2003); Iserles (2009); Griffiths and Higham (2010) for numerical treatment of ODEs and such as Kloeden and Platen (1992); Milstein (1995); Milstein and Tretyakov (2021) for numerical treatment of SDEs.

The numerical schemes for SDEs presented in this section are sometimes referred to as *lower-order schemes* due to their strong convergence order of $1/2$ and weak convergence order of 1. *Higher-order schemes* include, but is not limited to, higher-order stochastic Itô-Taylor and Runge-Kutta schemes. The reader can find treatments of higher-order numerical schemes in, for example, Parts V-VI of Kloeden and Platen (1992), Chapter 1 of Milstein (1995), and Chapters 1-2 of

Milstein and Tretyakov (2021).

Let us first recall the considered ODE

$$\begin{cases} \frac{\mathrm{d}y(t)}{\mathrm{d}t} = f(y(t)), \ t \in (0,T], \\ y(0) = y_0 \in \mathbb{R}, \end{cases} \tag{2.20}$$

where $f : \mathbb{R} \to \mathbb{R}$ is sufficiently regular. By integrating the ODE in (2.20) over the interval $[t_m, t_{m+1}]$, for some $m = 0, \ldots, M-1$, we obtain the following integral equation that many numerical schemes are constructed from

$$y(t_{m+1}) = y(t_m) + \int_{t_m}^{t_{m+1}} f(y(s))\, \mathrm{d}s. \tag{2.21}$$

We obtain the forward Euler method for (2.20) by approximating $f(y(s)) \approx f(y(t_m))$, for all $s \in [t_m, t_{m+1}]$, in (2.21). The forward Euler method reads

$$y_{m+1} = y_m + f(y_m)\Delta t, \ m = 0, \ldots, M-1,$$

initialised with the initial value $y(0) = y_0$. The value y_m is a numerical approximation of $y(t_m)$. Since the next value y_{m+1} can be explicitly computed from y_m, the forward Euler scheme is a so-called *explicit scheme*. Similarly, the backward Euler method for (2.20) is obtained by approximating $f(y(s)) \approx f(y(t_{m+1}))$, for all $s \in [t_m, t_{m+1}]$, in (2.21). The backward Euler method reads

$$y_{m+1} = y_m + f(y_{m+1})\Delta t, \ m = 0, \ldots, M-1,$$

initialised with the initial value $y(0) = y_0$. In contrast to the forward Euler scheme, the next value y_{m+1} of the backward Euler scheme has to be determined from an implicit equation, and is hence referred to as an *implicit scheme*.

Explicit numerical schemes are usually more easy to implement and are computationally faster, but some explicit numerical schemes suffer from step size restrictions which limits their practical usefulness. Implicit numerical schemes, on the other hand, can be more difficult to implement and require more computational time, but they are often more widely applicable and, in some cases, enjoys useful properties.

We now extend these classical numerical schemes to the SDE in (2.17). To this end, let $\Delta B_m = B(t_{m+1}) - B(t_m)$, for $m = 0, \ldots, M-1$, be the increment of the Brownian motion over $[t_m, t_{m+1}]$. Comparing the considered SDE in (2.17) to the ODE in (2.20), the natural extensions of the forward Euler method to

SDEs would be

$$X_{m+1} = X_m + f(X_m)\Delta t + g(X_m)\Delta B_m,\ m = 0, \ldots, M-1, \tag{2.22}$$

initialised with $X_0 = x_0$. The scheme defined in (2.22) is called the Euler—Maruyama (EM) scheme and is an explicit scheme. The natural extension of the backward Euler method would be

$$X_{m+1} = X_m + f(X_{m+1})\Delta t + g(X_{m+1})\Delta B_m,\ m = 0, \ldots, M-1, \tag{2.23}$$

initialised with $X_0 = x_0$. However, as shown in Milstein et al. (1998), the definition in (2.23) cannot be used due to infinite moments of the scheme. To solve this issue, one usually defines the semi-implicit Euler–Maruyama (SEM) method as

$$X_{m+1} = X_m + f(X_{m+1})\Delta t + g(X_m)\Delta B_m,\ m = 0, \ldots, M-1,$$

and is an implicit scheme.

If f and g are globally Lipschitz continuous functions, then the EM and SEM schemes are 2-strongly convergent of order $1/2$ with respect to Δt. Moreover, if f and g are sufficiently regular, then the EM and SEM schemes are weakly convergent of order 1. We refer to the well-known textbooks Kloeden and Platen (1992); Milstein (1995); Milstein and Tretyakov (2021) for more details. If we relax the assumptions on f and g for strong convergence, then the EM and SEM schemes might not convergence. In fact, it was shown in Hutzenthaler et al. (2011) that the moments of the EM scheme blow up in finite time for some SDEs that violate the globally Lipschitz continuity assumption. The search for convergent numerical schemes for SDEs with non-globally Lipschitz coefficient functions has been an active research topic ever since, and the literature on this topic is now extensive. We mention the works Higham et al. (2002); Milstein and Tretyakov (2005); Tretyakov and Zhang (2013); Hutzenthaler and Jentzen (2015); Mao (2015); Mickel and Neuenkirch (2025) as a selection.

2.2.3 Lie–Trotter time splitting

This section introduces time splitting schemes, which are used in Papers I, III, and IV. Such schemes have a long history for ODEs and have, in recent decades, gained popularity for SDEs and SPDEs. We mention the works Moro and Schurz (2007); Cohen and Vilmart (2022); Berg et al. (2021); Kelly and Lord (2023); Bréhier et al. (2023) in this direction for SDEs and the works Liu (2013); Bréhier et al. (2019); Bréhier and Goudenège (2019); Bréhier and Goudenège

(2020); Bréhier and Cohen (2023) in this direction for SPDEs.

The idea of time splitting schemes is to decompose a "difficult" problem into subproblems that, separately, can be more easily numerically or exactly integrated. Splitting schemes are important in the field of geometric numerical integration where important properties of the equations are preserved in numerical approximation. See, for example, Hairer et al. (2010) for a classical textbook on geometric numerical integration, and McLachlan and Quispel (2002); Blanes et al. (2024) for works focusing specifically on splitting schemes.

For simplicity, we illustrate the idea of time splitting in the context of ODEs, restricting the discussion to two subproblems and, in particular, to the Lie–Trotter splitting scheme. To this end, let us consider an ODE of the form

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = f_1(y(t)) + f_2(y(t)),\ t \in (0,T], \\ y(0) = y_0 \in \mathbb{R}, \end{cases} \tag{2.24}$$

where $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are sufficiently regular. Suppose that, for some $m = 0, \ldots, M-1$, an approximation y_m of $y(t_m)$ is given. The next value y_{m+1} approximating $y(t_{m+1})$ is computed as follows: We decompose (2.24) into two subproblems

$$\begin{cases} \dfrac{\mathrm{d}z_1(t)}{\mathrm{d}t} = f_1(z_1(t)),\ t \in (t_m, t_{m+1}], \\ z_1(t_m) = y_m, \end{cases} \tag{2.25}$$

and

$$\begin{cases} \dfrac{\mathrm{d}z_2(t)}{\mathrm{d}t} = f_2(z_2(t)),\ t \in (t_m, t_{m+1}], \\ z_2(t_m) = z_1(t_{m+1}). \end{cases} \tag{2.26}$$

We let $y_{m+1} = z_2(t_{m+1})$. If either of (2.25) or (2.26) cannot be solved exactly, then numerical discretisations have to be employed. Lie–Trotter time splitting schemes for SDEs and SPDEs are used in Papers I, III, IV, and V.

2.2.4 Boundary-preserving numerical schemes

This thesis concerns the construction of numerical schemes that preserve the invariant domain of the underlying differential equation, referred to as *boundary-preserving* schemes. This section formalises and provides an overview of explicit boundary-preserving schemes for SDEs.

Historically, implicit numerical schemes were the first proposed boundary-preserving schemes. More precisely, it was first noted and explored in Schurz (1996) that certain so-called *balanced implicit methods* are boundary-preserving. We do not cover implicit schemes in this section, but we mention the works Dangerfield et al. (2012); Dereich et al. (2012); Alfonsi (2013); Neuenkirch and Szpruch (2014); Liu et al. (2025a); Jiang et al. (2025) as a selection of boundary-preserving numerical schemes based on implicit techniques.

We are interested in the cases of SDEs with an invariant domain $\mathcal{D} \subsetneq \mathbb{R}$, as any numerical scheme that does not blow-up is boundary-preserving if $\mathcal{D} = \mathbb{R}$. Typically, if the invariant domain is $\mathcal{D} = [0, \infty)$ or $\mathcal{D} = (0, \infty)$ then the solutions are said to be positive and strictly positive, respectively. In this case, the term boundary-preserving is often referred to as *positivity-preserving*.

We call a numerical scheme $X_0, \dots, X_M$ approximating the solution $X(t)$ to the SDE

$$\begin{cases} \mathrm{d}X(t) = f(X(t))\,\mathrm{d}t + g(X(t))\,\mathrm{d}B(t),\ t \in (0, T], \\ X(0) = x_0 \in \mathcal{D}, \end{cases} \tag{2.27}$$

with invariant domain $\mathcal{D}$, boundary-preserving if

$$\mathbb{P}\left(X_m \in \mathcal{D},\ \text{for all } m = 0, \dots, M\right) = 1.$$

The term boundary-preserving is sometimes instead referred to as preserving the invariant domain. For SDEs with invariant domain that represents a physical quantity with physical boundaries (e.g., prices, temperature, concentration or fraction of something), boundary-preserving numerical schemes are crucial to obtain physically interpretable results.

The EM and SEM schemes discussed in Section 2.2.2 are known to not preserve the invariant domain of SDEs, in general. This behaviour is illustrated in, for example, Appleby et al. (2010) and in the appended papers. To illustrate this, consider the SIS SDE given by

$$\begin{cases} \mathrm{d}X(t) = X(t)(1 - X(t))\,\mathrm{d}t + 4X(t)(1 - X(t))\,\mathrm{d}B(t),\ t \in (0, 0.4], \\ X(0) = 0.9, \end{cases} \tag{2.28}$$

with invariant domain $\mathcal{D} = (0, 1)$. That is, we choose $f(r) = r(1-r)$ and $g(r) = 4r(1-r)$ in (2.27). Figure 2.2 shows sample paths generated by the following schemes, applied to (2.28): EM, SEM, tamed Euler (TE) (see Hutzenthaler et al. (2012)), and the Lamperti-splitting (LS) scheme proposed in Paper I. As is seen in Figure 2.2, the EM, SEM and TE schemes produce values outside the invariant domain $\mathcal{D} = (0, 1)$ of (2.28) and are hence not boundary-preserving. In other words, the EM, SEM and TE schemes produce approximations of (2.28) that cannot be interpreted as a fraction of a population that the solution of (2.28) represents. In contrast, the LS scheme is boundary-preserving and produces only values in the invariant domain $\mathcal{D} = (0,1)$ of the SDE in (2.28). The boundary-preserving property of the LS scheme is proved and empirically verified in Paper I.

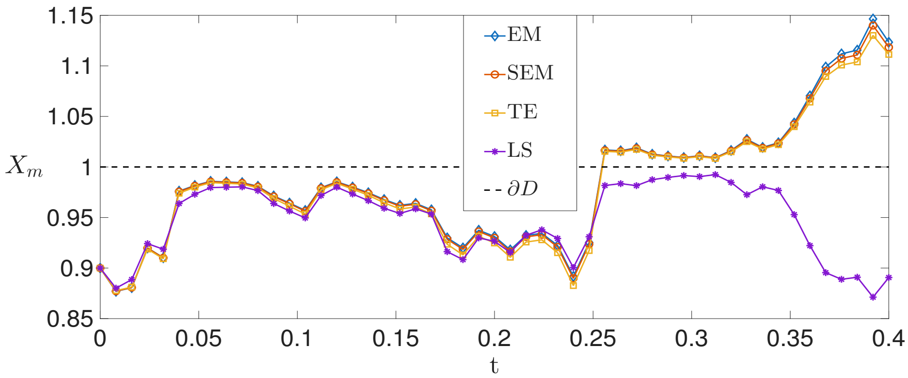

Figure 2.2: Illustration of *boundary-preserving*. Sample paths of the Euler-Maruyama (EM) scheme, semi-implicit Euler-Maruyama (SEM) scheme, tamed Euler (TE) scheme and Lamperti-splitting (LS) scheme applied to the SIS SDE in (2.28). The reference $\partial D = 1$ is the upper boundary of the invariant domain $\mathcal{D} = (0,1)$.

Going one step further, numerical schemes that are not boundary-preserving may fail to remain stable under modifications of the SIS SDE that leave the solution unchanged. To illustrate this, consider the following modification of the SDE in (2.28)

$$\begin{cases} \mathrm{d}X(t) = \overline{f}(X(t))\,\mathrm{d}t + \overline{g}(X(t))\,\mathrm{d}B(t), \ t \in (0,0.4], \\ X(0) = 0.9, \end{cases} \tag{2.29}$$

where

$$\overline{f}(r) = \begin{cases} e^r - e^1, & r \geq 1, \\ r(1-r), & r \in (0,1), \\ e^{-r} - 1, & r \leq 0, \end{cases}$$

and

$$\overline{g}(r) = \begin{cases} 4r(1-r), & r \in (0,1), \\ 0, & r \notin (0,1). \end{cases}$$

In comparison to (2.28), the drift coefficient in (2.29) is set to exponential

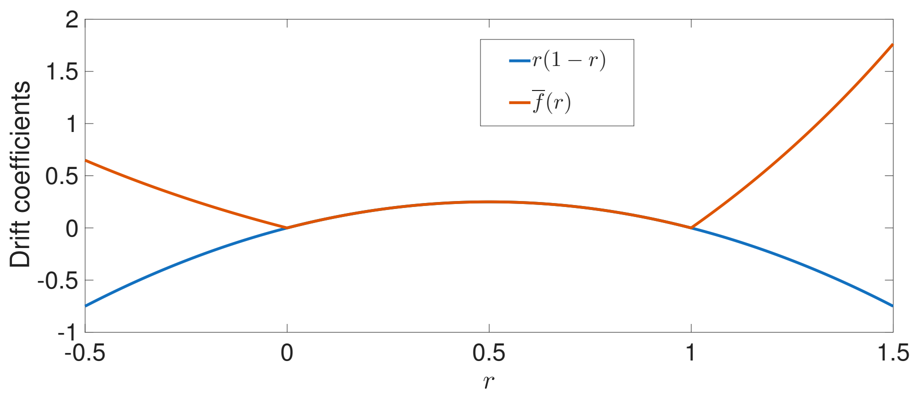

Figure 2.3: Comparison of the drift coefficient functions $r(1-r)$ in (2.28) and $\overline{f}$ in (2.29).

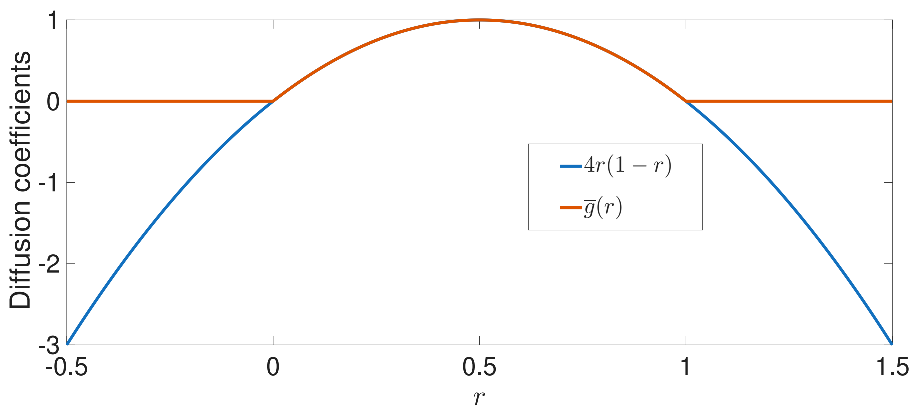

Figure 2.4: Comparison of the diffusion coefficient functions $4r(1-r)$ in (2.28) and $\overline{g}$ in (2.29).

functions outside $\mathcal{D} = (0, 1)$ and the diffusion coefficient in (2.29) is set to 0 outside $\mathcal{D} = (0, 1)$. Figure 2.3 and Figure 2.4 compares the coefficient functions of the modified SIS SDE in (2.29) with the coefficient functions of the original SIS SDE in (2.28).

The coefficient functions in (2.28) and in (2.29), respectively, are equal whenever $r \in [0, 1]$. As $\mathcal{D} = (0, 1)$ is the invariant domain of (2.28), and the coefficient functions agree on this domain, $\mathcal{D} = (0, 1)$ is also the invariant domain of (2.29). This implies that the solution of (2.28) and the solution of (2.29) coincide. See Figure 2.5 for an illustration of sample paths generated by the EM, SEM, TE and LS schemes applied to the SDE (2.29). As is seen in Figure 2.5, the LS scheme produces only values in the invariant domain $\mathcal{D} = (0, 1)$ and the EM, SEM and TE schemes diverges once outside $\mathcal{D} = (0, 1)$. We say that the LS scheme has the same *domain of influence* as the solution of (2.28) and of (2.29). In contrast, the EM, SEM and TE schemes do not have the same domain of influence as the solution of (2.28) and of (2.29).

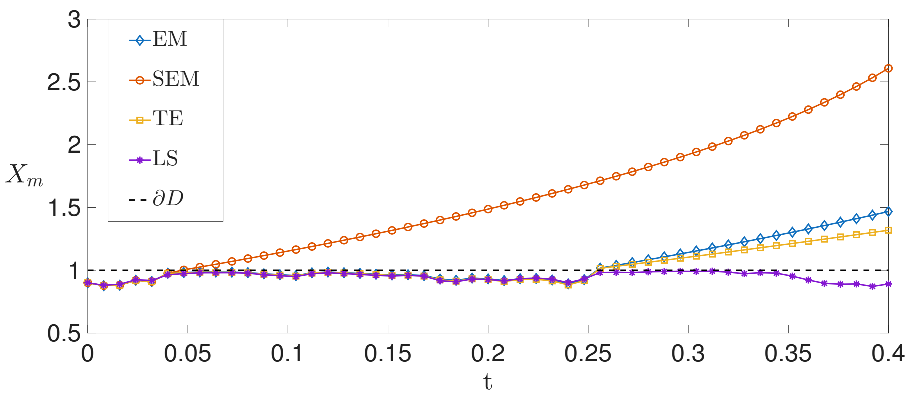

Figure 2.5: Illustration of *domain of influence*. Sample paths of the Euler-Maruyama (EM) scheme, semi-implicit Euler-Maruyama (SEM) scheme, tamed Euler (TE) scheme and Lamperti-splitting (LS) scheme applied to the modified SIS SDE in (2.29). The reference $\partial D = 1$ is the upper boundary of the invariant domain $\mathcal{D} = (0, 1)$.

We now briefly discuss the most common approaches to constructing explicit boundary-preserving numerical schemes for SDEs; they are based on transformations, time splittings, approximation by geometric Brownian motion (gBm), and truncation (or projection). Note that some numerical schemes combine two or more of the mentioned approaches. We also provide examples of boundary-preserving numerical schemes based on the different approaches.

The transformation-based approach

Transformation-based numerical schemes aim to transform the SDE in (2.27) with an invariant domain $\mathcal{D} \subsetneq \mathbb{R}$ into another SDE (2.9) with an invariant domain that is easier to construct boundary-preserving numerical schemes for. Typically, $\mathcal{D} = \mathbb{R}$ for the transformed SDE, as any numerical scheme that does not blow-up in finite time is boundary-preserving in this case. Since transformation-based numerical schemes require the use of Itô's formula in (2.9), the transformation has to satisfy some regularity assumptions. Two commonly used transformations for this approach are the Lamperti transform in (2.10) and the logarithmic function. For instance, combining the Lamperti transform Φ in (2.10) with the EM scheme in (2.22) to approximate the SDE in (2.11) yields the boundary-preserving numerical scheme defined by

$$X_{m+1} = \Phi^{-1}\left(\Phi(X_m) + \alpha\left(\Phi\left(X_m\right)\right)\Delta t + \Delta B_m\right),$$

for $m = 0, \ldots, M-1$, where $\alpha : \mathbb{R} \to \mathbb{R}$ is given in (2.12). We mention the works by Alfonsi (2013); Neuenkirch and Szpruch (2014); Chen et al. (2021); Yang and Huang (2021); Kelly and Lord (2023); Liu and Wang (2023); Liu et al. (2025b) as a selective, but non-exhaustive, list of boundary-preserving numerical schemes that are transformation-based. The schemes proposed in Papers I and V belong to this class.

The time splitting approach

The idea of using time splitting schemes to construct boundary-preserving numerical schemes for SDEs is to decompose the SDE in (2.27) into sub-problems consisting of SDEs to be treated separately. Each sub-problem has the same invariant domain as the original SDE in (2.27). If each sub-problem can be solved or numerically integrated while preserving the invariant domain, then the full time splitting scheme will also preserve the invariant domain.

In Section 2.2.3, the Lie–Trotter time splitting scheme was discussed for ODEs. Let us apply the same idea to the considered SDE in (2.27), and split the SDE into its deterministic and stochastic parts. Let the previous value $X_m \in \mathcal{D}$ (with probability one), for some $m = 0, \ldots, M-1$, be given. Suppose, for ease of presentation, that the ODE part of the SDE in (2.27); that is,

$$\begin{cases} \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = f(y(t)),\ t \in (t_m, t_{m+1}], \\ y(t_m) = X_m, \end{cases}$$

with invariant domain $\mathcal{D}$ admits a closed-form solution $y : [t_m, t_{m+1}] \to \mathcal{D}$, and that the diffusion part of the SDE in (2.27); that is,

$$\begin{cases} \mathrm{d}Y(t) = g(Y(t))\,\mathrm{d}B(t),\ t \in (t_m, t_{m+1}], \\ Y(t_m) = y(t_m), \end{cases}$$

with invariant domain $\mathcal{D}$ admits a closed-form $Y : [t_m, t_{m+1}] \to \mathbb{R}$ with

$$\mathbb{P}\left(Y(t) \in \mathcal{D},\ \text{for all } t \in [t_m, t_{m+1}]\right) = 1.$$

We compute the next value as $X_{m+1} = Y(t_{m+1})$, and this yields a boundary-preserving numerical scheme $X_0, \ldots, X_M$. We mention Moro and Schurz (2007); Halidias (2016); Kelly and Lord (2023) for works on boundary-preserving numerical schemes for SDEs based on time splitting. The schemes in Papers I, III, IV, and V all belong to this class.

The gBM-based approach

The idea behind the gBm-based approach to construct boundary-preserving numerical schemes for SDEs is to rewrite the SDE under consideration in (2.27) as

$$\mathrm{d}X(t) = X(t)\frac{f(X(t))}{X(t)}\,\mathrm{d}t + X(t)\frac{g(X(t))}{X(t)}\,\mathrm{d}B(t),\ t \in (0, T].$$

Under suitable assumptions on f and g, with the quotients $f(r)/r$ and $g(r)/r$ interpreted as limits for $r = 0$, this makes the considered SDE resemble the SDE for gBm in (2.13). Typically, in this case, the invariant domain is $\mathcal{D} = (0, \infty)$ and we will therefore also refer to boundary-preserving as positivity-preserving. Exponential Euler schemes is another term used in the literature to refer to gBm-based schemes. See Bossy et al. (2021); Bossy and Martínez (2024); Erdogan and Lord (2025) for some recent uses of this approach.

Suppose now that $f(0) = g(0) = 0$ and $f, g \in \mathcal{C}^1$ (continuously differentiable) with bounded derivatives. Then the quotients $f(r)/r$ and $g(r)/r$ are continuous and bounded functions, and we can regard $f(X(t))/X(t) \approx f(X(t_m))/X(t_m)$ and $g(X(t))/X(t) \approx g(X(t_m))/X(t_m)$ as approximately fixed during a small interval $[t_m, t_{m+1}]$, for some $m = 0, \ldots, M-1$. We say that we *freeze* the quotients at the previous time step t_m. Next, let the previous value $X_m \in (0, \infty)$, for some $m = 0, \ldots, M-1$, be given. Taking inspiration from the closed-form solution of gBm in (2.15), we compute the next value as

$$X_{m+1} = X_m e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma \Delta B_m}.$$

where $\mu = f(X_m)/X_m$ and $\sigma = g(X_m)/X_m$ are introduced to resemble the drift and diffusion constants in the SDE for geometric Brownian motion in (2.13). This yields a positivity-preserving numerical scheme $X_0, \ldots, X_M$. Recently, Erdogan and Lord (2025) applied the above approach twice to obtain numerical approximation guaranteed to preserve hypercubes $\mathcal{D} \subset \mathbb{R}^d$. Papers III and IV uses this approach to construct positivity-preserving schemes.

The truncation-based approach

Lastly, we mention boundary-preserving numerical schemes based on truncation or projection. The first use of truncation or projection schemes was to avoid blow-up of numerical approximations in Mao (2015). This is particularly important for SDEs with drift and diffusion coefficient functions that do not satisfy the globally Lipschitz condition. In some sense, this could be viewed as preserving the "boundary" at infinity of numerical approximations. Further on, in e.g. Mao et al. (2021), truncation was then used to preserve the boundary at 0. The idea of truncation-based schemes is to combine a truncation mapping of the form

$$\Pi_{\Delta t}(r) = \min\left(\Delta t^{-1}, \max\left(\Delta t, r\right)\right), \ r \in \mathbb{R}, \tag{2.30}$$

with a classical numerical scheme to ensure that $X_m \in (0, \infty)$, for all $m = 0, \ldots, M$, with probability one. For example, combining the truncation mapping in (2.30) with the EM scheme in (2.22) yields a positivity-preserving numerical scheme defined by

$$X_{m+1} = \Pi_{\Delta t}\left(X_m + f(X_m)\Delta t + g(X_m)\Delta B_m\right) \in (0, \infty), \tag{2.31}$$

for $m = 0, \ldots, M-1$. Note that the exact form of the truncation mapping $\Pi_{\Delta t}$ is often a variation of the form (2.30). If $\mathcal{D} = (a, b) \subset \mathbb{R}$, then we can guarantee that $X_{m+1} \in \mathcal{D} = (a, b)$ in (2.31), for all $m = 0, \ldots, M-1$, with probability one, by replacing the right hand side of (2.30) with $\min\left(b - \Delta t, \max\left(a + \Delta t, r\right)\right)$. Lastly, we also mention *symmetrised* schemes, studied for example in Berkaoui et al. (2008); Bossy and Olivero (2018), where the truncation map in (2.30) is defined as $\Pi_{\Delta t}(r) = |r| = \max(r, 0)$. We mention Mao et al. (2021); Deng et al. (2024) for works on boundary-preserving numerical schemes based on the truncation approach. Paper II develops a framework that includes certain truncation-based boundary-preserving numerical schemes.

2.3 Stochastic partial differential equations

This section provides the background on stochastic partial differential equations (SPDEs) for understanding the summaries of the appended papers. The aim is to gradually build up to the definition of *mild solutions* to SPDEs driven by space-time white noise.

There are different approaches to defining and studying SPDEs. Two of the main approaches are the so-called random field setting with well-known references such as Walsh (1986); Dalang (1999); Dalang et al. (2009); Khoshnevisan (2014); Dalang and Sanz-Solé (2024) and the so-called Hilbert space formulation with well-known references such as Prévôt and Röckner (2007); Da Prato and Zabczyk (2014). We also mention Walsh (1981); Funaki (1983) for early works on the former and Cabaña (1966); Curtain and Falb (1971); Kuo (1972) for early works on the latter. Intuitively speaking, the random field setting can be viewed as perturbing a PDE by noise and the Hilbert space formulation can be regarded as extending finite-dimensional Itô SDEs discussed in Section 2.1 to SDEs taking values in an infinite-dimensional space. These two approaches have similarities and differences and they are useful in their own regard. As it turns out, these two approaches yield the same objects in many situations. We refer to the early paper Jetschke (1986) and the more recent paper Dalang and Quer-Sardanyons (2011) for more details on this connection. We focus on the random field approach in this thesis, as it is well-suited for constructing boundary-preserving numerical schemes for SPDEs.

To keep the presentation simple, we first consider a semilinear reaction-diffusion PDE (reducing to the classical heat equation when $f = 0$)

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \dfrac{\partial^2 v}{\partial x^2}(t,x) + f(v(t,x)), \\ \quad v(t,0) = v(t,1) = 0, \\ \quad v(0,x) = v_0(x), \end{cases} \tag{2.32}$$

for $(t,x) \in (0,T] \times (0,1)$, where $f : \mathbb{R} \to \mathbb{R}$, $v_0 : [0,1] \to \mathbb{R}$ are sufficiently regular and $v_0(0) = v_0(1) = 0$. The requirement that $v(t,0) = v(t,1) = 0$, for every $t \in [0,T]$, in (2.32) is known as homogeneous Dirichlet boundary conditions. For simplicity, the presentation is for one spatial dimension.

A classical solution to (2.32) is a function $v \in \mathcal{C}^{1,2}((0,T] \times [0,1])$ (continuously differentiable in $t \in (0,T]$ and twice continuously differentiable in $x \in [0,1]$) such that the differential equation in (2.32) is satisfied for every $(t,x) \in (0,T] \times [0,1]$ and such that the initial condition is satisfied as a limit as $t \to 0$. The notion of a classical solution is too restrictive in some cases, as it requires differentiability. To resolve this, we introduce the notion of a mild solution of the PDE (2.32).

To this end, we introduce the Green's function (also known as the heat kernel) corresponding to the homogeneous part (that is, $f = 0$) of (2.32). The Green's function is given by

$$G(t, x, y) = 2 \sum_{j=1}^{\infty} e^{-j^2 \pi^2 t} \sin(j\pi x) \sin(j\pi y), \; x, y \in [0, 1], \; t \in (0, T]. \tag{2.33}$$

We call a jointly continuous, in time and space, function $v \in \mathcal{C}((0, T] \times [0, 1])$ a mild solution to (2.32) if $v(t, x)$ satisfies the following integral equation (also known as *Duhamel's principle* or the *variation of constants formula*)

$$v(t, x) = \int_0^1 G(t, x, y) v_0(y) \, \mathrm{d}y + \int_0^t \int_0^1 G(t - s, x, y) f(v(s, y)) \, \mathrm{d}y \, \mathrm{d}s, \tag{2.34}$$

for every $(t, x) \in (0, T] \times [0, 1]$. If f is globally Lipschitz continuous, then there exists a unique mild solution to (2.32) that is also a classical solution of (2.32). We refer to Evans (2010); Brezis (2011) for classical textbooks on PDE theory.

As an intermediate step towards an SPDE perturbed by space-time white noise, we first add temporal white noise introduced in Section 2.1 into the PDE in (2.32). More precisely, we aim to define solutions to the following SPDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = \dfrac{\partial^2 u}{\partial x^2}(t, x) + f(u(t, x)) + g(u(t, x)) \dfrac{\mathrm{d}B}{\mathrm{d}t}(t), \\ \quad u(t, 0) = u(t, 1) = 0, \\ \quad u(0, x) = u_0(x), \end{cases} \tag{2.35}$$

for $(t, x) \in (0, T] \times (0, 1)$. Here $g : \mathbb{R} \to \mathbb{R}, \; u_0 : [0, 1] \to \mathbb{R}$ are sufficiently regular, $u_0(0) = u_0(1) = 0$, and $B(t)$ is a Brownian motion as introduced in Section 2.1. To make the distinction clear between PDEs and SPDEs, we use the notation $v(t, x)$ to denote solutions of PDEs and we use $u(t, x)$ to denote solutions of SPDEs.

Similarly to the deterministic case in (2.32), we say that $u \in \mathcal{C}((0, T] \times [0, 1])$ is a mild solution to (2.35) if $u(t, x)$ satisfies the following stochastic integral equation

$$
\begin{aligned}
u(t,x) = &\int_0^1 G(t,x,y)u_0(y)\,\mathrm{d}y + \int_0^t \int_0^1 G(t-s,x,y) f(u(s,y))\,\mathrm{d}s\,\mathrm{d}y \\
&+ \int_0^t \int_0^1 G(t-s,x,y) g(u(s,y))\,\mathrm{d}y\,\mathrm{d}B(s),
\end{aligned} \tag{2.36}
$$

for every $(t,x) \in (0,T] \times [0,1]$, almost surely. The integration with respect to $\mathrm{d}B$ in (2.36) is in the Itô sense discussed in Section 2.1. If f and g are globally Lipschitz continuous, then there exists a unique mild solution to (2.35). We refer to, for example, Krylov (1999) for more details.

A key difference between the deterministic problem in (2.32) and the stochastic problem in (2.35) is that Itô calculus is needed to treat and analyse the SPDE in (2.35). The mild solution of (2.35) has lower regularity (merely Hölder continuous with exponent $1/2-$ in time and with exponent $1-$ in space) compared to the mild solution of (2.32). This is caused by the irregularity of the noise. The setting for Paper III is (2.35) with $f = 0$ generalised to $d \geq 1$ spatial dimensions.

We next aim to perturb the PDE in (2.32) by noise having no correlation in time and no correlation in space. Such noise is called *space-time white noise*. To this end, we first recall that a Wiener sheet $W : [0,T] \times [0,1] \to \mathbb{R}$ is a zero-mean continuous Gaussian random field with covariance given by

$$
\mathbb{E}\left[W(t,x)W(s,y)\right] = \min(t,s)\min(x,y), \; t,s \in [0,T], \; x,y \in [0,1].
$$

Wiener sheet $W(t,x)$ is the generalisation of Brownian motion $B(t)$ to space-time $\{(t,x) \in [0,T] \times [0,1]\}$. Similarly to Brownian motions, Wiener sheets are, with probability one, nowhere differentiable (except along the coordinate axes). We refer to Chapter 5 in Khoshnevisan (2002) for more details on the Wiener sheet.

Let us now consider the following semilinear reaction-diffusion SPDE

$$
\begin{cases}
\dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + f(u(t,x)) + g(u(t,x))\dfrac{\partial^2 W}{\partial t \partial x}(t,x), \\
u(t,0) = u(t,1) = 0, \\
u(0,x) = u_0(x),
\end{cases} \tag{2.37}
$$

for $(t,x) \in (0,T] \times (0,1)$. The term $\frac{\partial^2 W}{\partial t \partial x}(t,x)$ represents space-time white noise and is the formal mixed space-time derivative of the Wiener sheet W.

As before, we say that a function $u \in \mathcal{C}((0,T] \times [0,1])$ is a mild solution of (2.37) if $u(t,x)$ satisfies the following stochastic integral equation

$$\begin{aligned} u(t,x) = \int_0^1 G(t,x,y)u_0(y)\,\mathrm{d}y + \int_0^t \int_0^1 G(t-s,x,y)f(u(s,y))\,\mathrm{d}s\,\mathrm{d}y \\ + \int_0^t \int_0^1 G(t-s,x,y)g(u(s,y))\,\mathrm{d}W(s,y), \end{aligned} \tag{2.38}$$

for every $(t,x) \in (0,T] \times [0,1]$, almost surely. The last integral in (2.38) is a stochastic integral with respect to a so-called *worthy martingale measure* in the Walsh-Dalang sense. For a rigorous construction of the Walsh-Dalang integral, see, for example, Chapter 2 in Walsh (1986) or part 1 in Dalang et al. (2009). See also Walsh (1981); Dalang (1999) for its early development. For the purpose of this thesis, it is enough that the reader has in mind the Itô integral in (2.6) but generalised to space-time.

If f and g are globally Lipschitz continuous, then there exists a unique mild solution of (2.37). We refer to Funaki (1983); Walsh (1986) for such results. Moreover, the solution of (2.37) is Hölder continuous with exponent $1/4-$ in time and $1/2-$ in space (see, e.g., Chapter 5.2 in Khoshnevisan (2014)). Note the difference in regularity between the solution of (2.35) perturbed by temporal white noise and the solution of (2.37) perturbed by space-time white noise. This difference in regularity of solutions implies that the order of convergence of numerical schemes is expected to be lower for (2.37) compared to (2.35). The setting for Paper IV is (2.38) with $f = 0$ and the setting for Paper V is (2.38).

Similarly to invariant domains for SDEs defined in Section 2.2.4, we call $\mathcal{D} \subset \mathbb{R}$ an invariant domain of the SPDE in (2.37) if

$$\mathbb{P}\left(u(t,x) \in \mathcal{D},\ \text{for all } (t,x) \in [0,T] \times [0,1]\right) = 1,$$

whenever $u_0(x) \in \mathcal{D}$ for every $x \in [0,1]$. As before, we are interested in the case $\mathcal{D} \subsetneq \mathbb{R}$.

Examples of SPDEs with invariant domains arise naturally from stochastic versions of PDEs with invariant domains. Consider, for example, the well-known Allen–Cahn PDE

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \dfrac{\partial^2 v}{\partial x^2}(t,x) + v(t,x) - v(t,x)^3, \\ v(t,0) = v(t,1) = 0, \\ v(0,x) = v_0(x) \in [-1,1], \end{cases} \tag{2.39}$$

for $(t,x) \in (0,T] \times (0,1)$, with invariant domain $\mathcal{D} = [-1,1]$. The Allen–Cahn PDE (2.39) was introduced in Allen and Cahn (1979) as a model for phase separation in multi-component alloys. If we perturb the Allen–Cahn PDE (2.39) by additive noise, then the solution will not remain in the invariant domain $\mathcal{D} = [-1, 1]$ of the PDE in (2.39). Let us, instead, perturb (2.39) by multiplicative noise with $g(r) = 1 - r^2$. Then we obtain the following Allen–Cahn SPDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) = \dfrac{\partial^2 u}{\partial x^2}(t, x) + u(t, x) - u(t, x)^3 + (1 - u(t, x)^2)\dfrac{\partial^2 W}{\partial t \partial x}(t, x), \\ u(t, 0) = u(t, 1) = 0, \\ u(0, x) = u_0(x) \in [-1, 1], \end{cases} \tag{2.40}$$

for $(t, x) \in (0, T] \times (0, 1)$. This SPDE has the invariant domain $\mathcal{D} = [-1, 1]$, which is the same as for the Allen–Cahn PDE in (2.39). In other words, the invariant domain of the PDE carries over to the SPDE when perturbed by particular types of noise.

The analogue of Feller's complete boundary classification for SDEs is not developed for SPDEs. There are, however, some results in this direction that we now briefly discuss. One of the first results on invariant domains for SPDEs was Mueller (1991) where strict positivity for times $t > 0$ of $(1 + 1)$-dimensional stochastic heat equations such as (2.37) (but defined for $x \in \mathbb{R}$) was shown for $f(r) = 0$ and $g(r) = |r|^\gamma$ for $\gamma \geq 1$. Strict positivity means that the invariant domain is $(0, \infty)$. This was subsequently extended to (2.37) for some globally Lipschitz continuous g and for some non-zero source term f in Shiga (1994) (also for unbounded domain $x \in \mathbb{R}$). Since then, many extensions of these results have been established for stochastic heat equations. We first mention Mueller and Sowers (1995) considering the stochastic KPP equation with invariant domain $[0, 1]$. Next, Tessitore and Zabczyk (1998), Moreno Flores (2014) and Han et al. (2024) generalises Mueller's result to d dimensions with spatially correlated noise, to initial Dirac delta measures and to more general diffusion coefficient functions, respectively. Lastly, we mention Chen and Kim (2020) establishing a strict comparison principle, an equivalent concept to strict positivity, in d dimensions with spatially correlated noise.

2.4 Space discretisations

This section provides an overview of the finite difference (FD) method used for space discretisation in Papers III–V. For concreteness, we illustrate the method by applying it to the SPDE in (2.37).

As most SPDEs do not have closed-form solutions, we have to resort to numerical approximation. In contrast to SDEs seen in Section 2.1, we have one time variable t and one space variable x for the SPDEs in (2.35) and in (2.37). There are many different space discretisation methods available in the literature and we mention the finite difference (FD) method, the finite element method (FEM) and spectral methods. Which space discretisation method to choose depends on the problem at hand and what types of questions are to be studied. After any of these space discretisation methods have been employed, the resulting equations form a finite-dimensional system of SDEs. The form and characteristics of the system of SDEs depend, however, on the chosen space discretisation. As Section 2.2 covered time discretisations of SDEs, the remaining task is to discretise the SPDE in space. Since Papers III–V rely on the FD method for the space discretisation, we restrict our attention here to this approach. We mention the well-known references Lord et al. (2014); Kruse (2014) and some selective works Debussche and Printems (2009); Andersson and Larsson (2016) on FEM and spectral methods to approximate solutions of SPDEs similar to (2.37).

Convergence of space-discrete and fully discrete schemes for SPDEs are defined in the analogous way as convergence of time-discrete schemes for SDEs in Section 2.2.1. More details can be found in the aboved-mentioned references.

2.4.1 Finite difference discretisations

Finite difference (FD) methods approximate derivatives with finite difference quotients and goes back to works by L. Euler. We mention the seminal paper by Courant et al. (1967) (the English version, the original is from 1928) as the starting point of the numerical analysis of FD methods for problems coming from mathematical physics. We also mention Thomée (2001) that surveys the historical development of space discretisation procedures with a special focus on FD methods and FEM. Nowadays, a classical book that contains the content presented below is LeVeque (2007) to which we refer the interested reader for proofs and more details. As we here are concerned with space discretisations, we replace spatial derivatives in the SPDE with FD quotients.

The space discretisation of SPDEs by FD methods started in the beginning of the 1990s. We mention Jetschke (1991) for an early work and Gyöngy (1998); Davie and Gaines (2001); Millet and Morien (2005) for a later works. We present the idea of FD methods for the SPDE in (2.37) perturbed by space-time white noise. The SPDE in (2.37) is second order in space, which means that we need to consider finite difference quotients up to order 2.

Suppose that the mild solution u of (2.37) is sufficiently regular in the spatial variable and let $\Delta x > 0$ be the space grid size. The spatial derivative of u at

$x \in [\Delta x, 1-\Delta x]$ can be approximated by the finite difference quotient

$$\frac{\partial u}{\partial x}(t,x) \approx \frac{u(t,x+\Delta x)-u(t,x)}{\Delta x},$$

which is commonly known as the forward difference quotient. In a similar manner, the backward difference quotient

$$\frac{u(t,x)-u(t,x-\Delta x)}{\Delta x} \approx \frac{\partial u}{\partial x}(t,x),$$

and the central difference quotient

$$\frac{u(t,x+\Delta x)-u(t,x-\Delta x)}{2\Delta x} \approx \frac{\partial u}{\partial x}(t,x).$$

Higher order derivatives can be approximated in a similar manner. For example, the second derivative of u at x can be approximated by two repeated (each with space grid size $\Delta x/2$) central difference quotients as

$$\frac{\partial^2 u}{\partial x^2}(t,x) \approx \frac{u(t,x+\Delta x)-2u(t,x)+u(t,x-\Delta x)}{\Delta x^2}.$$

Let us now use the finite difference quotients introduced above to spatially discretise the SPDE (2.37). To this end, we let $N \in \mathbb{N}$ denote the discretisation parameter determining the number of space grid points and we let $\Delta x = 1/N$ denote the space grid size. This yields the space grid points given by $x_n = n\Delta x$ for $n = 0, \ldots, N$. There are two spatial derivatives in (2.37) to be approximated: $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 W}{\partial t \partial x}$. Thus, for $t \in [0,T]$ and for $x = x_n$, with $n = 1, \ldots, N-1$, we approximate

$$\frac{\partial^2 u}{\partial x^2}(t,x_n) \approx \frac{u(t,x_n+\Delta x)-2u(t,x_n)+u(t,x_n-\Delta x)}{\Delta x^2}$$

and

$$\frac{\partial^2 W}{\partial t \partial x}(t,x_n) \approx \frac{\partial}{\partial t}\left(\frac{W(t,x_n+\Delta x)-W(t,x_n)}{\Delta x}\right),$$

ignoring regularity issues for the time being. By inserting the two approximations above into (2.37), we obtain the following

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x_n) \approx{}& \frac{u(t,x_n+\Delta x)-2u(t,x_n)+u(t,x_n-\Delta x)}{\Delta x^2} + f(u(t,x_n)) \\ &+ g(u(t,x_n))\frac{\partial}{\partial t}\left(\frac{W(t,x_n+\Delta x)-W(t,x_n)}{\Delta x}\right),\end{aligned}$$

for $n = 1, \dots, N-1$. This motivates the following definition for the FD approximation $u^N : [0,T] \times \{x_1, \dots, x_{N-1}\} \to \mathbb{R}$. The process u^N satisfies the equations

$$\begin{aligned} \frac{\partial u^N}{\partial t}(t, x_n) = &\frac{u^N(t,x_{n+1}) - 2u^N(t,x_n) + u^N(t,x_{n-1})}{\Delta x^2} + f(u^N(t,x_n)) \\ &+ g(u^N(t,x_n)) \frac{\partial}{\partial t}\left(\frac{W(t,x_{n+1}) - W(t,x_n)}{\Delta x}\right), \end{aligned} \tag{2.41}$$

for $n = 1, \dots, N-1$, where we also replaced $x_n + \Delta x$ with x_{n+1} and $x_n - \Delta x$ with x_{n-1}. Note that the values of u at $x = 0$ ($n = 0$) and $x = 1$ ($n = N$) need not be approximated, as the boundary conditions $u(0) = u(1) = 0$ are prescribed.

We insert the values $u^N(t, x_n)$, for $n = 1, \dots, N-1$, into a vector that we denote by $u^N(t) \in \mathbb{R}^{N-1}$, we define the matrix

$$D^N = \begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 1 & \ddots & 0 & 0 & 0 \\ 0 & 1 & -2 & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & -2 & 1 & 0 \\ 0 & 0 & 0 & \ddots & 1 & -2 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & -2 \end{pmatrix} \tag{2.42}$$

and we let $W^N(t) \in \mathbb{R}^{N-1}$ be the vector with elements

$$\left(W^N(t)\right)_n(t) = \sqrt{N}\left(W(t,x_{n+1}) - W(t,x_n)\right), \; n = 1, \dots, N-1.$$

Note that this implies, in particular, that $W^N(t)$ is a vector of $N-1$ independent Brownian motions. Thus, we may rewrite (2.41) as the following $(N-1)$-dimensional system of SDEs

$$\begin{cases} \mathrm{d}u^N(t) = N^2 D^N u^N(t)\,\mathrm{d}t + f(u^N(t))\,\mathrm{d}t + \sqrt{N} g(u^N(t))\,\mathrm{d}W^N(t), \\ u^N(0) = u_0^N, \end{cases} \tag{2.43}$$

for $t \in (0, T]$, where u_0^N is the vector of size $N-1$ with the discretised initial value of the SPDE (2.37) with elements $\left(u_0^N\right)_n = u_0(x_n)$ for $n = 1, \dots, N-1$.

If f and g are globally Lipschitz continuous, then the system of SDEs in (2.43) has a unique solution u^N that converges in the p-strong sense of order $1/2$ (for every $p \geq 2$) to the mild solution of the considered SPDE in (2.37). Weak convergence holds true also for weaker regularity assumptions on f and g. We refer to Gyöngy (1998) for more details.

2.5 Full discretisations

The final step to constructing a fully discrete approximation $u^{M,N} \in \mathbb{R}^{(M+1)\times(N-1)}$ of the SPDE in (2.37) is to discretise the time variable in the $(N-1)-$dimensional system of SDEs in (2.43). We emphasise that $u^{M,N}$ is a $(M+1)\times(N-1)$ matrix with elements $u^{M,N}_{m,n} \approx u(t_m, x_n)$ for $m = 0, \dots, M$ and for $n = 1, \dots, N-1$. Classical choices of time discretisation procedures include the multidimensional generalisations of the EM and SEM schemes discussed in Section 2.2.2.

If f and g are globally Lipschitz continuous, then the fully discrete approximation $u^{M,N}$ obtained from either the EM or SEM time discretisations converges in the p-strong sense of order $1/4$ in time and of order $1/2$ in space (for every $p \geq 2$) to the mild solution u of the SPDE (2.37). We refer to Gyöngy (1999) for more details. We also mention Anton et al. (2020) proving p-strong convergence of in time (for every $p \geq 2$) for the fully discrete numerical scheme obtained from discretising (2.43) using an *exponential integrator*. Weak convergence of the above-mentioned schemes can be proved without the global Lipschitz condition, for this we also refer to Gyöngy (1999); Anton et al. (2020)

We say that a fully discrete numerical scheme $u^{M,N}$ for the SPDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + f(u(t,x)) + g(u(t,x))\dfrac{\partial^2 W}{\partial t \partial x}(t,x), \\ \quad u(t,0) = u(t,1), \\ \quad u(0,x) = u_0(x), \end{cases}$$

for $(t,x) \in (0,T] \times (0,1)$, with invariant domain $\mathcal{D}$ is boundary-preserving if $u^{M,N}_{m,n} \in \mathcal{D}$ for all $m = 0, \dots, M$ and for all $n = 1, \dots, N-1$, almost surely. The classical EM and SEM schemes to approximate the solution of (2.43) are not boundary-preserving in general and can therefore yield non-physical results.

In contrast to the SDE case in Section 2.2.4, the research topic of boundary-preserving numerical schemes for SPDEs is underdeveloped. Part of this thesis is dedicated to sparking interest in this by providing first contributions to the development and analysis of boundary-preserving numerical schemes for SPDEs.

3 Summaries of the included papers

In this section, we provide summaries of the appended papers. The central research question of the papers is the construction and analysis of boundary-preserving numerical schemes, as defined in Section 2.2.4 for SDEs and as defined in Section 2.5 for SPDEs. The focus is on boundary-preserving time discretisations of SDEs and SPDEs.

3.1 Paper I: Boundary-preserving Lamperti-splitting scheme for some stochastic differential equations

Short summary

We propose and study a family of boundary-preserving numerical schemes to approximate the solutions to some scalar Itô SDEs with open and bounded invariant domain. The proposed numerical schemes combine the Lamperti transform defined in (2.10) and a Lie–Trotter time splitting discussed in Section 2.2.3. The Lie–Trotter time splitting decomposes the Lamperti-transformed SDE in (2.11) into an ODE, that is exactly or numerically integrated, and Brownian motion. This approach achieves p-strong convergence of order 1 (for every $p \geq 1$), while preserving the invariant domain of the considered SDE.

Motivation

The Lamperti transform is a common tool for constructing boundary-preserving numerical schemes that achieve a strong order of convergence higher than classical schemes such as the EM and SEM schemes (see Section 2.2.2). Under suitable assumptions, the convergence order obtained for SDEs with additive noise can be transferred to more general SDEs via the Lamperti transform.

Previously proposed boundary-preserving schemes based on the Lamperti transform have typically been problem-specific (e.g., tailored to particular choices of f and g) or have relied on indirect assumptions on the coefficient functions. Such indirect assumptions may, for instance, concern the drift term of the transformed process in (2.12).

The purpose of this paper is to develop a family of boundary-preserving schemes to approximate the solutions to a wide range of SDEs while requiring only easily verifiable assumptions.

Contributions

The contributions of Paper I are:

- Construction of a family of boundary-preserving numerical schemes for some SDEs with open and bounded invariant domain.
- Proof of p-strong convergence of order 1 (for every $p \geq 1$) of the proposed boundary-preserving numerical schemes.
- Numerical verification of boundary-preservation and of 2-strong convergence of order 1.

Detailed summary

Paper I studies scalar SDEs of the form

$$\begin{cases} \mathrm{d}X(t) = f(X(t))\,\mathrm{d}t + g(X(t))\,\mathrm{d}B(t),\ t \in (0, T], \\ X(0) = x_0 \in \mathcal{D}, \end{cases} \tag{3.1}$$

where $f : \operatorname{cl}\mathcal{D} \to \mathbb{R}$ is $\mathcal{C}^2$, $g : \operatorname{cl}\mathcal{D} \to (0, \infty)$ is $\mathcal{C}^3$, and f and g satisfy some decay conditions near the boundary points of the open and bounded invariant domain $\mathcal{D} \subsetneq \mathbb{R}$ of the SDE. Here, $\operatorname{cl}\mathcal{D}$ denotes the closure of the invariant domain $\mathcal{D}$. The decay conditions ensure that the Lamperti transform Φ in (2.10) is well-defined with a globally Lipschitz continuous inverse and that

$$\mathbb{P}\left(X(t) \in \mathcal{D}, \text{ for all } t \in (0,T]\right) = 1.$$

We emphasise that the solution process $X(t)$ cannot reach the boundary points $\partial\mathcal{D}$ of $\mathcal{D}$, since $\mathcal{D}$ is open. Note that we do not impose any assumptions on f and g outside $\operatorname{cl}\mathcal{D}$. For instance, f and g may exhibit superlinear growth outside $\operatorname{cl}\mathcal{D}$.

The proposed numerical schemes to approximate the solution $X(t)$ of (3.1) are built in three steps. Step 1 is to transform the solution $X(t)$ of the SDE in (3.1) using the Lamperti transform Φ to obtain the process $Y(t) = \Phi(X(t))$ that solves the following Itô SDE with additive noise (see also (2.11))

$$\begin{cases} \mathrm{d}Y(t) = \alpha(Y(t))\,\mathrm{d}t + \mathrm{d}B(t),\ t \in (0,T], \\ Y(0) = \Phi(x_0) \in \mathbb{R}, \end{cases} \tag{3.2}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is given by (2.12). The assumptions on f and g implies, in particular, that α is bounded, $\mathcal{C}^1$, and has bounded derivative.

In step 2, we apply a Lie–Trotter time splitting as follows. Suppose Y_m, for some $m = 0, \ldots, M-1$, approximating $Y(t_m)$ is given. We first numerically or exactly integrate the ODE

$$\begin{cases} \mathrm{d}z_1(t) = \alpha(z_1(t))\,\mathrm{d}t,\ t \in (t_m, t_{m+1}], \\ z_1(t_m) = Y_m, \end{cases} \tag{3.3}$$

and then integrate the SDE

$$\begin{cases} \mathrm{d}Z_2(t) = \mathrm{d}B(t),\ t \in (t_m, t_{m+1}], \\ Z_2(t_m) = z_1(t_{m+1}). \end{cases} \tag{3.4}$$

We define the next numerical approximation as $Y_{m+1} = Z_2(t_{m+1}) \approx Y(t_{m+1})$.

The third and final step is to compute the approximation of the solution $X(t)$ of (3.1) at time $t = t_m$ as $X_m = \Phi^{-1}(Y_m)$. The proposed numerical schemes exhibits p-strong convergence of order 1 (for every $p \geq 1$) while preserving the invariant domain of the SDE (3.1).

Correction to proof of Theorems 3.5 **and** 4.4

The proof of the main theorems contain an error that I would like to highlight and correct here. We refer to the paper for the notation. On page 11, the following estimate is used:

$$\begin{aligned}|Y_m^{LS} - Y(t_m)| \leq{}& (L_H + |\mu|)\, L_H T \Delta t + (L_H + |\mu|)\, L_H T \Delta t \\ &+ \frac{1}{2} L_H T \Delta t + \left| \int_0^T \int_{\ell(s)}^s H'(Y(r))\, \mathrm{d}B(r)\, \mathrm{d}s \right| \\ &+ L_H \Delta t \sum_{k=0}^{m-1} |Y_k^{LS} - Y(t_k)|.\end{aligned}$$

This estimate should be replaced with

$$\begin{aligned}|Y_m^{LS} - Y(t_m)| \leq{}& (L_H + |\mu|)\, L_H T \Delta t + (L_H + |\mu|)\, L_H T \Delta t \\ &+ \frac{1}{2} L_H T \Delta t + \sup_{t\in[0,T]} \left| \int_0^t \int_{\ell(s)}^s H'(Y(r))\, \mathrm{d}B(r)\, \mathrm{d}s \right| \\ &+ L_H \Delta t \sum_{k=0}^{m-1} |Y_k^{LS} - Y(t_k)|.\end{aligned}$$

To complete the proofs of Theorems 3.5 and 4.4, the statement of Lemma A.1 should be replaced with

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left| \int_0^t \int_{\ell(s)}^s H'(Y(r))\, \mathrm{d}B(r)\, \mathrm{d}s \right|^p \right] \leq C(p) L_H^p T^{p/2} \Delta t^p,$$

where $C(p)$ is the BDG constant. The above estimate is proved in the same way as Lemma A.1, with the BDG inequality accounting for the additional supremum.

3.2 Paper II: Artificial Barriers for stochastic differential equations and for construction of boundary-preserving schemes

Short summary

We introduce the framework of *artificial barriers* to construct boundary-preserving numerical schemes to approximate the solutions to scalar Itô SDEs with an open invariant domain. The idea is to introduce auxiliary barrier processes to convert the considered SDE into an associated reflected SDE (RSDE) to be discretised using a modified boundary-preserving numerical scheme for RSDEs.

This approach achieves the same order of p-strong convergence as the used modified numerical scheme for RSDEs (for every $p \geq 2$), while preserving the invariant domain of the considered SDE.

Motivation

The search for generally applicable boundary-preserving numerical schemes for SDEs is an active research area. The ultimate goal is to develop a boundary-preserving numerical scheme with the same level of generality and convergence order as general numerical schemes for SDEs ignoring the domain.

Let us examine the applicability of the four approaches to construct boundary-preserving numerical schemes surveyed in Section 2.2.4. The transformation-, time-splitting- and gBm-based approaches all rely on regularity assumptions that limit the applicability. In contrast, truncation-based schemes are more widely applicable. Truncation-based schemes can, in fact, be regarded as special cases of artificial barrier methods. More precisely, combining artificial barriers with projection schemes for RSDEs yields numerical schemes that coincide with truncation-based ones.

The approach developed in this paper is inspired by the discretisation of RSDEs, for which boundary-preservation is both crucial and well-developed. By establishing a link between SDEs with invariant domains and RSDEs, we can exploit existing boundary-preserving numerical schemes for RSDEs to construct corresponding schemes for our setting.

Contributions

The contributions of Paper II are:

- A general problem-independent framework to construct boundary-preserving numerical schemes for SDEs with an open invariant domain, leveraging modified numerical schemes for RSDEs.
- Proof that the order of p-strong convergence is equal to the order of p-strong convergence of the used modified numerical scheme for RSDEs (for every $p \geq 2$). For coefficient functions that are globally Lipschitz continuous on the invariant domain, we combine artificial barrier with a modified projected EM scheme to achieve p-strong convergence of order $1/2-$ (for every $p \geq 2$).

- Numerical verification of boundary-preservation and of 2-strong convergence of order $1/2-$.

Detailed summary

Paper II studies scalar SDEs of the form

$$\begin{cases} \mathrm{d}X(t) = f(X(t))\,\mathrm{d}t + g(X(t))\,\mathrm{d}B(t),\ t \in (0,T], \\ X(0) = x_0 \in \mathcal{D}, \end{cases} \tag{3.5}$$

where $f, g : \mathcal{D} \to \mathbb{R}$ are globally Lipschitz continuous functions satisfying the Feller condition for unattainable boundary points on the open invariant domain $\mathcal{D} \subsetneq \mathbb{R}$. Thus, $\mathcal{D}$ is such that

$$\mathbb{P}\left(X(t) \in \mathcal{D},\ \text{for all } t \in [0,T]\right) = 1.$$

As in Section 2.1, we refer the interested reader to, for example, Chapter 15 in Karlin and Taylor (1981) for more details on the Feller condition. We do not impose any assumptions on f and g outside the domain $\mathcal{D}$. In particular, f and g may, for instance, be of superlinear growth outside $\mathcal{D}$. In contrast to Paper I, $\mathcal{D}$ can either be bounded or half-bounded in Paper II.

The construction of the framework that we call artificial barriers consists of three steps. In step 1, with $R \in \mathbb{N}$, we define a sequence of open domains $(\mathcal{D}_R)_{R\in\mathbb{N}}$ approaching $\mathcal{D}$ from the inside and we modify the coefficient functions of (3.5) to obtain the auxiliary Itô SDEs

$$\begin{cases} \mathrm{d}X_R(t) = f_R(X_R(t))\,\mathrm{d}t + g_R(X_R(t))\,\mathrm{d}B(t),\ t \in (0,T], \\ X_R(0) = x_0 \in \mathcal{D}_R. \end{cases} \tag{3.6}$$

Here $f_R, g_R : \mathcal{D}_R \to \mathbb{R}$ are constructed such that $X_R(t) \in \mathcal{D}_R \subsetneq \mathcal{D}$ for all $t \in [0,T]$, almost surely, and such that $X_R \to X$, as $R \to \infty$, in the p-strong sense (for every $p \geq 2$). The reason for this modification of the invariant domain is that most numerical schemes for RSDEs can reach the boundary points of the invariant domain (to then be reflected into the invariant domain again), we avoid this issue by moving the boundary points slightly inside $\mathcal{D}$.

Step 2 introduces an auxiliary process ξ_R that solves the RSDE obtained by augmenting (3.6) with a barrier process

$$\begin{cases} \mathrm{d}\xi_R(t) = f_R(\xi_R(t))\,\mathrm{d}t + g_R(\xi_R(t))\,\mathrm{d}B(t) + \mathrm{d}L(t),\ t \in (0,T], \\ \xi_R(0) = x_0 \in \mathcal{D}_R, \end{cases} \tag{3.7}$$

where $L(t)$ is a stochastic process that, intuitively, forces $\xi_R \in \mathrm{cl}\,\mathcal{D}_R = \mathcal{D}_R \cup \partial\mathcal{D}_R$. As $X_R(t) \in \mathcal{D}_R$ for all $t \in [0,T]$, almost surely, it follows that $X_R(t) = \xi_R(t)$ for all $t \in [0,T]$, almost surely, by uniqueness of solutions of RSDEs (see, for instance, Pilipenko (2014)).

In step 3, we discretise the RSDE in (3.7) to approximate ξ_R using a modified version of a classical numerical scheme for RSDEs. These steps yields a p-strongly convergent (for every $p \geq 2$) approximation procedure for SDEs of the form (3.5) that is boundary-preserving. The proposed numerical schemes inherit the order of p-strong convergence from the modified discretisation procedure for the associated RSDE (for every $p \geq 2$). In particular, artificial barriers combined with a modified version of the classical projected EM scheme for RSDEs yields p-strong convergence of order $1/2-$ (for every $p \geq 2$).

3.3 Paper III: Positivity-preserving schemes for some nonlinear stochastic PDEs

Short summary

We propose and study a positivity-preserving temporal numerical scheme for strong approximations of the solutions to some multidimensional stochastic heat equations perturbed by temporal white noise (that is, (2.35) with $f = 0$ and in any spatial dimension $d \geq 1$) with positive solutions. The proposed numerical scheme combines a Lie–Trotter time splitting discussed in Section 2.2.3 with a gBm-based positivity-preserving numerical scheme discussed in Section 2.2.4. The proposed numerical scheme achieves temporal 2-strong convergence of order $1/2$ while preserving positivity of the considered SPDE.

Motivation

To the best of our knowledge, no previous temporal numerical schemes exist that preserve the positivity of solutions of SPDEs of the form (2.35) (with $f = 0$ and in any spatial dimension $d \geq 1$). We mention, however, the work Yang et al. (2022) that constructs and analyses a positivity-preserving numerical scheme for the special case $f = 0$ and $g(r) = r$ in the SPDE in (2.35).

Contributions

The contributions of Paper III are:

- Construction of a positivity-preserving temporal numerical scheme for the strong approximation of the solutions for some multidimensional SPDEs perturbed by temporal white noise with solutions that are positive.
- Numerical verification of positivity-preservation and of 2-strong convergence of order $1/2$ in time.

Detailed summary

Paper III considers $(1+d)$-dimensional SPDEs perturbed by temporal white noise of the form

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + g(u(t,x))\dfrac{\mathrm{d}B}{\mathrm{d}t}(t), \\ u(0,x) = u_0(x) \geq 0, \end{cases} \tag{3.8}$$

for $(t,x) \in (0,T] \times (0,1)^d$, subject to homogeneous Dirichlet boundary conditions. Here $g : [0,\infty) \to \mathbb{R}$ is $\mathcal{C}^1$ with bounded derivative and satisfies $g(0) = 0$, $B(t)$ is a Brownian motion, and we assume that $u_0 : [0,1]^d \to [0,\infty)$ is bounded, globally Lipschitz continuous and satisfies homogeneous Dirichlet boundary conditions. Note that the driving noise is not depending on the spatial variable $x \in (0,1)^d$. We study time-discrete explicit approximations in Paper III.

The assumptions on g imply that

$$\mathbb{P}\left(u(t,x) \in \mathcal{D} = [0,\infty), \text{ for all } (t,x) \in [0,T] \times [0,1]\right) = 1$$

and enable us to apply a gBm-based positivity-preserving numerical scheme for SDEs as discussed in Section 2.2.4.

The idea of the proposed scheme is to apply a Lie–Trotter time splitting introduced in Section 2.2.3 to decompose (3.8) into Itô SDEs

$$\frac{\partial u}{\partial t}(t,x) = g(u(t,x))\frac{\mathrm{d}B}{\mathrm{d}t}(t), \tag{3.9}$$

subject to homogeneous Dirichlet boundary conditions, and the deterministic heat equation

$$\frac{\partial v}{\partial t}(t,x) = \Delta v(t,x), \tag{3.10}$$

also subject to homogeneous Dirichlet boundary conditions. Note that (3.9) can be defined as an Itô SDE for each fixed $x \in (0,1)^d$.

We iteratively approximate (3.9) and solve (3.10) with the initial value for one being the output from the other. This can be formulated as follows. Given $u_m : [0,1] \to \mathbb{R}$, for some $m = 0, \dots, M-1$, approximating $u(t_m, \cdot) : [0,1] \to \mathbb{R}$, we define the approximation at the next time point t_{m+1} as

$$\begin{aligned} u_{m+1}(x) = \int_{(0,1)^d} & G(\Delta t, x, y) \\ & \times \left(\exp\left(h(u_m(y))\Delta B_m - \frac{h(u_m(y))^2 \Delta t}{2} \right) u_m(y) \right) \mathrm{d}y, \end{aligned} \quad (3.11)$$

for $x \in (0,1)^d$, where G is the heat kernel defined in (2.33), $h(r) = g(r)/r$ and where $\Delta B_m = B(t_{m+1}) - B(t_m)$ denotes the Brownian increment. The integral kernel without the heat kernel factor $G(\Delta t, x, y)$ of (3.11) comes from expressing $g(r) = rh(r)$ in (3.9) and approximation by a geometric Brownian motion on the small interval $[t_m, t_{m+1}]$ for each $x \in (0,1)^d$. The representation (3.11) is then obtained by integrating (3.10) with initial value at t_m being the geometric Brownian motion obtained from (3.9). It is clear from (3.11) that the sequence $u_0, \dots, u_M$ defines positive spatial functions. Note that the first value in the sequence u_0 is the initial value of (3.8).

The proposed numerical scheme defined by (3.11) exhibits temporal 2-strong convergence of order $1/2$ when applied to (3.8) and this is verified in one and two spatial dimensions by numerical experiments. This order of convergence is expected since the solution of (3.8) is Hölder continuous with exponent $1/2$ in time.

3.4 Paper IV: Analysis of a positivity-preserving splitting scheme for some semilinear stochastic heat equations

Short summary

We propose the first fully discrete positivity-preserving numerical scheme for strong approximation for some semilinear stochastic heat equations perturbed by space-time white noise (that is, (2.37) with $f = 0$) with positive solutions. Similarly to Paper III, the proposed numerical scheme combines a Lie–Trotter time splitting discussed in Section 2.2.3 with a gBm-based positivity-preserving numerical scheme for SDEs in the sense of Section 2.2.4 for the time discretisation. In contrast to Paper III, due to too low regularity of space-time white noise, we can not apply a Lie–Trotter time splitting directly to the SPDE. Instead, we first discretise the SPDE in space using an FD discretisation.

The proposed numerical scheme converges in the 2-strong sense of order $1/4$ in time and of order $1/2$ in space, under some assumptions. The 2-strong convergence of order $1/2$ in space follows from Gyöngy (1998).

Motivation

To the best of our knowledge, no previous numerical schemes exist that preserve the positivity of solutions of SPDE of the form (2.37) with $f = 0$. Paper IV is continuing the work in Paper III, where we apply a similar time discretisation to the SPDE perturbed by space-time white noise in (2.37) with $f = 0$ whose solutions are positive almost surely.

Classical space and time discretisations for this problem have previously been studied in, for example, Gyöngy (1998, 1999), but these fully discrete numerical schemes are not guaranteed to remain positive for all times $t \in (0, T]$.

Contributions

The contributions of Paper IV are:

- Construction of the first positivity-preserving numerical scheme for the strong approximation for some semilinear SPDEs perturbed by space-white noise with positive solutions.
- Proof of positivity-preservation and of 2-strong convergence of order $1/4$ in time.
- Numerical verification of positivity-preservation and of 2-strong convergence of order $1/4$ in time.

Detailed summary

Paper IV considers $(1+1)$-dimensional semilinear SPDE of the form

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + g(u(t,x))\dfrac{\partial^2 W}{\partial t \partial x}(t,x), \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = u_0(x) \in \mathcal{D} = [0,\infty), \end{cases} \tag{3.12}$$

for $(t,x) \in (0,T] \times (0,1)$. Here $g : [0,\infty) \to \mathbb{R}$ is continuously differentiable with bounded derivative and satisfies $g(0) = 0$, $W(t,x)$ is a Wiener sheet, and $u_0 : [0,1] \to [0,\infty)$ is $\mathcal{C}^3$ and satisfies $u_0(0) = u_0(1) = 0$. These assumptions imply that

$$\mathbb{P}\left(u(t,x) \in \mathcal{D} = [0,\infty), \text{ for all } (t,x) \in [0,T] \times [0,1]\right) = 1$$

and that we can apply a gBm-based positivity-preserving numerical scheme for SDEs as discussed in Section 2.2.4. If $g(r) = r$ for $r \in (0,\infty)$, then (3.12) is the well-known parabolic Anderson model. We refer to Carmona and Molchanov (1994); König (2016) for treatments and applications of the parabolic Anderson model. We study fully discrete explicit approximations in Paper IV.

The construction of the proposed positivity-preserving numerical scheme for (3.12) consists of three steps. In step 1, we spatially discretise (3.12) using the standard FD method as introduced in Section 2.4 with $N \in \mathbb{N}$ as the discretisation parameter. The $\mathbb{R}^{N-1}$-valued solution process $u^N : [0,T] \to \mathbb{R}^{N-1}$ of the system of Itô SDEs

$$\begin{cases} \mathrm{d}u^N(t) = N^2 D^N u^N(t)\,\mathrm{d}t + \sqrt{N} g(u^N(t))\,\mathrm{d}W^N(t),\ t \in (0,T], \\ u_i^N(0) = u_0(x_i),\ i \in \{1,\ldots,N-1\}, \end{cases} \tag{3.13}$$

approximates the mild solution $u(t,x)$ of (3.12) on the spatial grid points $x_1,\ldots,x_{N-1}$. As in Section 2.4, $N^2 D^N$ is the discrete $(N-1)\times(N-1)$ Laplacian matrix defined in (2.42) and $W^N(t)$ is a vector of size $N-1$ consisting of independent Brownian motions generated from the Wiener sheet process $W(t,x)$.

In step 2, we apply a Lie–Trotter time splitting as introduced in Section 2.2 to split (3.13) into a diagonal $(N-1)$-dimensional system of Itô SDEs

$$\mathrm{d}v^N(t) = \sqrt{N} g(v^N(t))\,\mathrm{d}W^N(t) \tag{3.14}$$

and a non-diagonal $(N-1)$-dimensional system of ODEs

$$\mathrm{d}v^N(t) = N^2 D^N v^N(t)\,\mathrm{d}t. \tag{3.15}$$

The time integration of (3.13) by a Lie–Trotter splitting is given by iteratively approximating (3.14) and integrating (3.15) with starting value for one being the output value from the other one.

We initialise the approximating sequence with $u_{0,n} = u_0(x_n)$ for $n = 1, \ldots, N-1$. Suppose that $(u_{m,n})_{n=1}^{N-1}$, for some $m = 0, \ldots, M-1$, approximating the mild solution $u(t_m, \cdot)$ of (3.12) at time $t = t_m$ on the spatial grid points $x_1, \ldots, x_{N-1}$ is given. We define the approximation $(u_{m+1,n})_{n=1}^{N-1}$ of the mild solution $u(t_{m+1}, \cdot)$ at the next time point t_{m+1} as

$$\begin{aligned} u_{m+1,n} = \sum_{k=1}^{N-1} & G_{nk}^N(\Delta t) \\ & \times \exp\left(\sqrt{N} h(u_{m,k}) \Delta_{m,n} W - \frac{N h(u_{m,k})^2 \Delta t}{2}\right) u_{m,k}, \end{aligned} \tag{3.16}$$

for $n = 1, \ldots, N-1$, where $\left(G_{ij}^N(t)\right)_{i,j=1}^{N-1} = \exp\left(t N^2 D^N\right)$ is the discrete heat kernel associated with (3.15), $h(r) = g(r)/r$ and where $\Delta_{m,n} W = W_n^N(t_{m+1}) - W_n^N(t_m)$. The random variables $(\Delta_{m,n} W)_{0 \leq m \leq M-1, 1 \leq n \leq N-1}$ are independent and normally distributed with mean 0 and variance Δt.

The proposed positivity-preserving numerical scheme exhibits 2-strong temporal convergence of order $1/4$ and spatial convergence of order $1/2$. This is expected since the mild solution of (3.12) is Hölder continuous with exponent $1/4$ in time and with exponent $1/2$ in space.

3.5 Paper V: Boundary-preserving weak approximation for some semilinear stochastic partial differential equations

Short summary

We propose the first boundary-preserving numerical scheme for weak approximation for some semilinear SPDEs with bounded invariant domains and with coefficient functions that are only assumed to satisfy regularity assumptions on the invariant domain. The behaviour of the coefficient functions outside the invariant domain is of no importance and can, for instance, exhibit superlinear growth. The proposed fully discrete boundary-preserving numerical scheme consists of a FD discretisation in space and a Lie–Trotter time splitting combined with exact integration and exact sampling in time. This approach enables us to guarantee that the numerical approximations are confined to the invariant domain of the SPDE and it enables us to prove weak convergence of order $1/4$ in time and of order $1/2$ in space, under some assumptions, for globally Lipschitz continuous test functions.

Motivation

The first positivity-preserving numerical scheme for the SPDE in (2.37) with $f = 0$ was proposed and studied in Paper IV. Many interesting SPDEs are not of this form and there is therefore a need to construct boundary-preserving numerical schemes for the general case (2.37).

As mentioned in the summary of Paper IV, classical numerical schemes are known to converge in the 2-strong sense to the mild solution (see Gyöngy (1999)) of the considered SPDE with bounded invariant domain (possibly with truncated coefficient functions). However, these classical numerical schemes do not preserve the invariant domain of the SPDE.

Contributions

The contributions of Paper V are:

- Construction of the first boundary-preserving numerical scheme for the weak approximation for some semilinear SPDEs perturbed by space-time white noise with a bounded invariant domain.
- Proof of boundary-preservation and of weak convergence of order $1/4$ in time for globally Lipschitz continuous test functions.
- Numerical verification of boundary-preservation and of weak convergence of order $1/4$ in time for globally Lipschitz continuous test functions.

Detailed summary

Paper V considers $(1+1)$-dimensional semilinear SPDEs of the form

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + f(u(t,x)) + g(u(t,x))\dfrac{\partial^2 W}{\partial t \partial x}(t,x), \\ u(t,0) = u(t,1) = 0, \\ u(0,x) = u_0(x) \in \mathcal{D} = [-1,1], \end{cases} \tag{3.17}$$

for $(t, x) \in (0, T] \times (0, 1)$. Here $f, g : \mathcal{D} \to \mathbb{R}$ are $\mathcal{C}^2$ and $\mathcal{C}^3$ functions, respectively, that satisfy some decay estimates near the boundary points of the invariant domain $\mathcal{D} = [-1, 1]$, $W(t, x)$ is a Wiener sheet, and $u_0 : [0, 1] \to \mathcal{D}$ is of class $\mathcal{C}^3$ and satisfies $u_0(0) = u_0(1) = 0$. The specific choice of $\mathcal{D} = [-1, 1]$ in (3.17) is for ease of presentation. The generalisation of (3.17) to $\mathcal{D} = [a, b]$, for $-\infty < a < b < \infty$, is straightforward. We study fully discrete approximations in Paper V.

In Paper V, we propose to combine the ideas of Paper IV with exact sampling for SDEs to construct a boundary-preserving numerical scheme that converges weakly to the mild solution of (3.17). This approach enables us to treat a large family of SPDEs of the form (3.17) with bounded invariant domains and to obtain weak convergence with respect to globally Lipschitz continuous test functions. The first part of the numerical scheme is the same as in Paper IV (up to (3.15)). In contrast to Paper IV, both the diagonal system of SDEs in (3.14) and the non-diagonal system of ODEs in (3.15) are solved exactly. The former is solved in the weak stochastic sense using componentwise exact sampling and the latter is solved deterministically using the matrix exponential $\exp\left(\Delta t N^2 D^N\right)$, as in Paper IV. The assumptions on f and g enable us to apply exact sampling as introduced by Beskos and Roberts (2005); Beskos et al. (2006).

The proposed boundary-preserving numerical scheme exhibits temporal weak convergence of order $1/4$ and spatial weak convergence of order $1/2$, under some assumptions, for globally Lipschitz continuous test functions.

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