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Citation for the original published paper (version of record):
Mårdby, G., Rowlett, J. (2025). Spectral Invariants of Integrable Polygons. Journal of Fourier Analysis and Applications, 31(6). http://dx.doi.org/10.1007/s00041-025-10202-6

Journal of Fourier Analysis and Applications (2025) 31:69
https://doi.org/10.1007/s00041-025-10202-6

Spectral Invariants of Integrable Polygons

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Received: 30 October 2024 / Revised: 22 September 2025 / Accepted: 9 October 2025

Abstract

An integrable polygon is one whose interior angles are fractions of π; that is to say of the form $\frac{\pi}{n}$ for positive integers n. We consider the Laplace spectrum on these polygons with the Dirichlet and Neumann boundary conditions, and we obtain new spectral invariants for these polygons. This includes new expressions for the spectral zeta function and zeta-regularized determinant as well as a new spectral invariant contained in the short-time asymptotic expansion of the heat trace. Moreover, we demonstrate relationships between the short-time heat trace invariants of general polygonal domains (not necessarily integrable) and smoothly bounded domains and pose conjectures and further related directions of investigation.

Keywords Laplace spectrum · Helmholtz equation · Polygonal domain · Laplace eigenvalues · Heat trace · Spectral zeta function · Zeta-regularized determinant · Polygonal billiard · Closed geodesic

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain in the Euclidean plane. We consider the Laplace eigenvalue problem, also known as Helmholtz's equation, that is to find all eigenfunctions $u : \Omega \to \mathbb{C}$ and eigenvalues $\lambda \in \mathbb{C}$ such that

$$\Delta u + \lambda u = 0 \text{ in } \Omega, \quad \Delta = \partial_x^2 + \partial_y^2. \tag{1}$$

The Helmholtz equation is perhaps the most fundamental partial differential equation of mathematics and physics, with numerous real-world applications including acoustic

Communicated by Hans G. Feichtinger.

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Published online: 05 November 2025

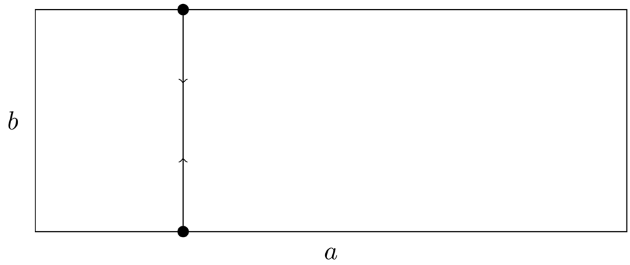

Fig. 1 Rectangles are integrable polygons, as their interior angles all measure $\frac{\pi}{2}$. If the rectangle has sides of lengths a and b, then the length of the shortest closed geodesic is twice the length of the shortest side, corresponding to the orbit running perpendicularly between the two longer sides

design, structural engineering, diffusion processes, quantum mechanics, and wave propagation. The function u is assumed to be in the Sobolev space $H^2(\Omega)$, and is further required to satisfy a boundary condition. Here, we consider the Dirichlet or Neumann boundary conditions, which respectively require the function or its normal derivative to vanish on the boundary. The set of Laplace eigenvalues is the *spectrum,* and any quantity that is defined in terms of the spectrum is a *spectral invariant.*

Although there exist fast and accurate methods for numerically calculating Laplace eigenvalues [12], the collection of domains for which the eigenvalues can be computed analytically in closed form is quite limited. Restricting to polygonal domains in the plane, this collection includes rectangles, equilateral triangles, isosceles right triangles, and hemi-equilateral triangles, also known as 30-60-90 triangles. By [20] these are precisely the polygons which are integrable, meaning they have all interior angles of the form π/n where $n \in \mathbb{N}$. It is a straightforward exercise in planar geometry to prove that all integrable polygons must be one of these four types as shown in Figures 1 – 4. By [34] (see also [44, Thm. 1]), the polygonal domains which strictly tessellate the plane are precisely rectangles, equilateral triangles, isosceles right triangles, and the 30-60-90 triangle. Therefore, polygons being integrable is equivalent to strictly tessellating the plane.

The Laplace eigenfunctions and eigenvalues of rectangles can be obtained using separation of variables by solving the one-dimensional Helmholtz equation. This was known to mathematicians and physicists in the 18th century; however proving that *all* eigenfunctions and eigenvalues are obtained by this method could not be demonstrated until functional analysis was developed in the 19th century [18]. The eigenfunctions of a square that are odd along a diagonal produce Dirichlet eigenvalues of isosceles right triangles [23]. For equilateral triangles, Lamé was the first to obtain expressions for eigenvalues and eigenfunctions [24–26]. At the end of the 19th century these eigenvalues and eigenfunctions were studied further by Pockels, who also noticed that the eigenfunctions of a regular rhombus and a regular hexagon are not trigonometric, i.e. cannot be expressed in terms of sines and cosines [42]. It was not until 2008 that McCartin proved that in fact, the only polygonal domains that have a complete set of trigonometric eigenfunctions are the integrable polygonal domains [34]. Rowlett et. al. generalized this result to higher dimensions, where polygons are replaced by polytopes [44] (Fig. 2).

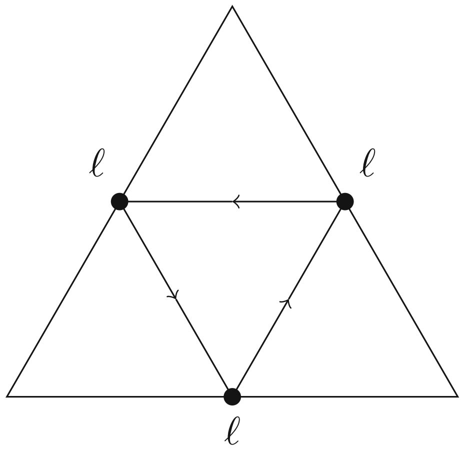

Fig. 2 Equilateral triangles are integrable polygons, as their interior angles all measure $\frac{\pi}{3}$. Their shortest closed geodesic is formed by connecting the midpoints of the three sides, creating an equilateral triangle with sides of length $\frac{\ell}{2}$ and therewith total length $3\frac{\ell}{2}$

Although Lamé obtained closed formulas for eigenvalues and eigenfunctions of equilateral triangles, similar to the case of rectangles, it is another matter to prove that these are *all* eigenvalues. Indeed, this was first rigorously demonstrated in the 1980s by Pinsky who gave a new way of deriving the eigenvalues and eigenfunctions and established completeness [40, 41]. In 1998, Prager gave a different method for deriving the eigenvalues and eigenfunctions of the equilateral triangle [43], and in 2002 McCartin gave another elementary method [31–33]. McCartin also showed that the eigenfunctions of an equilateral triangle with either two Dirichlet and one Neumann boundary condition or one Dirichlet and two Neumann boundary conditions are not trigonometric [32]. Similar to rectangles and isosceles right triangles, one obtains the eigenvalues of hemi-equilateral (30-60-90) triangles by considering the eigenfunctions of equilateral triangles that are odd along a nodal line [32].

In 1957 Brownell studied arbitrary polygonal domains with the Dirichlet boundary condition and conjectured that their heat trace expansion only has three terms [8]. A complicated and somewhat incomplete proof of this was given in [5, 6]. In 1988 van den Berg and Srisatkunarajah improved this result by obtaining a bound on the error term after the first three terms [46]. Moreover, they gave a detailed calculation of the third heat trace term for polygonal domains that had previously appeared in the literature without a complete proof [22], [36], attributed to unpublished work by Ray. Using the explicit expressions of the eigenvalues, Verhoeven calculated the heat trace of rectangles, isosceles right triangles, and equilateral triangles in his Bachelor thesis [47]. Although he did not explicitly compute the sharp remainder term, Verhoeven's techniques help us to obtain the sharp remainder term in the short time asymptotic expansion of the heat trace for integrable polygons. In the case of equilateral triangles, we compute the heat trace via an independent method and show that our expression is in fact equal to Verhoeven's. In doing so, we are able to prove directly that two different formulas (13), (24) for the eigenvalues of the equilateral triangle are in fact equivalent. This equivalence can be deduced by combining results from McCartin [32, 35], but to the best of our knowledge a direct proof has not previously appeared in the literature. Moreover, we obtain expressions for the spectral zeta functions and zeta-regularized determinants of integrable polygons, some of which appear to be new. We also prove that our expressions for the spectral zeta functions of integrable polygons turn out to be equal to those obtained by Aurell and Salomonson in [4, (108)], who were the first to obtain such closed-form expressions. We further show that differentiating these expressions one obtains the same results for the zeta-regularized determinant. By presenting these spectral invariants in their most explicit form, we aim to facilitate both practical computations and theoretical insights. Notably, our investigation of the heat trace leads to a conjecture for a new spectral invariant for convex polygonal domains, namely the length of their shortest closed geodesic. Durso proved that the length of the shortest closed geodesic is a spectral invariant in triangular domains by studying the singularities of the wave trace [16]. More generally she proved that the Poisson relation holds in polygonal domains, meaning that the times at which the wave trace is singular is contained in the set of lengths of closed geodesics in the domain. It is unknown if this containment is proper or an equality, so in order to show that the length of a certain closed geodesic is a spectral invariant, one must prove that the wave trace is singular when time is equal to that length. Our approach is via the heat trace, which may be slightly more accessible (Fig. 3).

Theorem 1.1 *Let Ω be an integrable polygonal domain. Let $\{\lambda_k\}_{k\geq 1}$ denote the Dirichlet eigenvalues of Ω. Then the heat trace has the short time asymptotic expansion*

$$\sum_{k\geq 1} e^{-\lambda_k t} = \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \sum_{i=1}^{n} \frac{\pi^2 - \gamma_i^2}{24\pi\gamma_i} + \mathcal{O}(e^{-(L^2-\epsilon)/(4t)}),\ \ t \to 0,\ \ \forall \epsilon > 0.$$

Above, $|\Omega|$ is the area of Ω, $|\partial\Omega|$ is its perimeter, γ_i are the measures of its interior angles, n is the number of sides, and L is the length of the shortest closed geodesic in Ω. If instead $\{\mu_k\}_{k\geq 0}$ denote the Neumann eigenvalues of Ω, then the heat trace

$$\sum_{k\geq 0} e^{-\mu_k t} = \frac{|\Omega|}{4\pi t} + \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \sum_{i=1}^{n} \frac{\pi^2 - \gamma_i^2}{24\pi\gamma_i} + \mathcal{O}(e^{-(L^2-\epsilon)/(4t)}),\ \ t \to 0,\ \ \forall \epsilon > 0.$$

The remainder estimate in both cases is sharp in the sense that $\frac{L^2}{4}$ is the supremum over all $c > 0$ such that the remainder is $\mathcal{O}(e^{-c/t})$ as $t \to 0$.

Theorem 1.1 is proved separately for each class of integrable polygons: in Theorem 2.4 for rectangles, Theorem 3.7 for equilateral triangles, Theorem 4.5 for isosceles right triangles, and Theorem 5.4 for hemi-equilateral triangles. This result is not surprising if one considers flat tori, as their heat trace consists of a leading term involving the volume, and an exponentially decaying remainder term with exponent of the same form as the remainder here for integrable polygons. In the following sections, §2–§5 we calculate the spectral zeta function, zeta-regularized determinant, and heat trace of rectangles, equilateral triangles, isosceles right triangles, and hemi-equilateral triangles. In the last section of the article §6 we present a brief comparison of the heat traces of flat tori, Euclidean space forms, convex polytopes, convex polygonal domains, and smoothly bounded domains. We conjecture that for Euclidean space forms as well

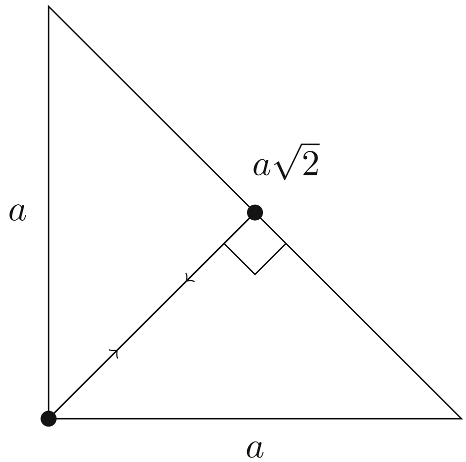

Fig. 3 Isosceles right triangles are integrable polygons, as their interior angles measure $\frac{\pi}{2}$ and $\frac{\pi}{4}$. Their shortest closed geodesic is the altitude that joins the right angle to the hypotenuse

as convex polytopes, the heat trace has a short time asymptotic expansion with a rapidly decaying remainder term of the same type as that of flat tori and integrable polygons. We then consider convex polygonal domains converging in the Hausdorff sense to a smoothly bounded domain and prove that the first three heat trace invariants converge to those of the smoothly bounded domain. In contrast, if smoothly bounded domains converge in the Hausdorff sense to a convex polygonal domain, then only the first two heat trace invariants converge; we show that the third does not. We hope to provide an inclusive introduction to the Laplace eigenvalue problem suitable for a broad readership and at the same time, inspire those readers well-versed in the field to investigate the many remaining open problems.

Notation and Abbreviations

For the reader's convenience, we include a summary of our notation and abbreviations in Table 1.

2 Spectral Invariants of Rectangles

Consider a rectangular domain $[0, a] \times [0, b]$ in the plane. One can separate variables and solve the Laplace eigenvalue equation on the two segments $[0, a]$ and $[0, b]$,

$$u_{xx} + u_{yy} + \lambda u(x, y) = 0, \quad 0 < x < a, \quad 0 < y < b, \tag{2}$$

imposing either the Dirichlet or Neumann boundary condition, respectively,

$$\begin{aligned} &(\text{DBC}) : u(0, y) = u(a, y) = u(x, 0) = u(x, b) = 0, \\ &(\text{NBC}) : u_x(0, y) = u_x(a, y) = u_y(x, 0) = u_y(x, b) = 0. \end{aligned}$$

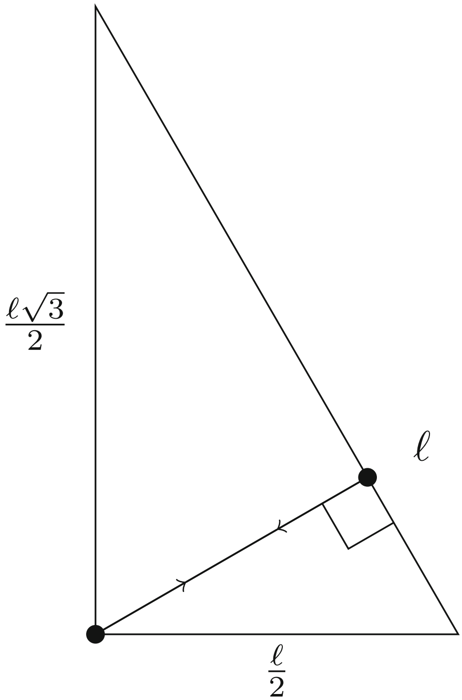

Fig. 4 Hemi-equilateral triangles are integrable polygons, as their interior angles measure $\frac{\pi}{2}$ and $\frac{\pi}{3}$ and $\frac{\pi}{6}$. Their shortest closed geodesic is the altitude that joins the right angle to the hypotenuse

In the Dirichlet case, the results of this calculation yields the eigenvalues and eigenfunctions

$$\lambda_{m,n} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right), \quad m, n \geq 1, \quad u_{m,n}(x, y) = \sin(m\pi x/a) \sin(n\pi y/b). \quad (3)$$

Having obtained these eigenvalues and eigenfunctions using separation of variables, one may follow [27, p. 83] to prove completeness, in other words that these are indeed *all* eigenfunctions and eigenvalues.

2.1 The Spectral Zeta Function and Zeta-Regularized Determinant of Rectangles

By our calculation of the eigenvalues under the Dirichlet boundary condition in (3) the spectral zeta function is

$$\zeta_{\square}(s) = \frac{1}{\pi^{2s}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(\frac{m^2}{a^2} + \frac{n^2}{b^2})^s}. \quad (4)$$

We begin by computing two equivalent expressions for the spectral zeta function. These expressions are relatively simple to obtain using known results, but it seems that the first expression is less widespread. Note that while the sum in (4) only converges for $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, it is well known that the spectral zeta function has an

Table 1 Notation and abbreviations

Δ	Laplace operator or Laplacian defined in (1)
DBC	Dirichlet boundary condition
NBC	Neumann boundary condition
$\sum_{m\in\mathbb{Z}}\sum'_{k\in\mathbb{Z}}$	sum over all integer pairs except (0, 0)
Γ	Gamma function
$\zeta_{\square}(s)$	Dirichlet spectral zeta function for a rectangle $[0, a] \times [0, b]$
$G_{\square}(s)$	function defined in (6)
$H^D_{\square}(t)$	Dirichlet heat trace for a rectangle $[0, a] \times [0, b]$
$H^N_{\square}(t)$	Neumann heat trace for a rectangle $[0, a] \times [0, b]$
$\zeta_{\blacksquare}(s)$	Dirichlet spectral zeta function for a square with sides of length a
$\zeta_{\triangledown}(s)$	Dirichlet spectral zeta function for an equilateral triangle with sides of length ℓ
$G_{\triangledown}(s)$	function defined in (20)
$H^D_{\triangledown}(t)$	Dirichlet heat trace for an equilateral triangle with sides of length ℓ
$H^N_{\triangledown}(t)$	Neumann heat trace for an equilateral triangle with sides of length ℓ
$\zeta_{\diamond}(s)$	Dirichlet spectral zeta function for an isosceles right triangle with legs of length a
$G_{\diamond}(s)$	function defined in (29)
$H^D_{\diamond}(t)$	Dirichlet heat trace for an isosceles right triangle with legs of length a
$H^N_{\diamond}(t)$	Neumann heat trace for an isosceles right triangle with legs of length a
$\zeta_{\heartsuit}(s)$	Dirichlet spectral zeta function for a hemi-equilateral (30-60-90) triangle with hypotenuse ℓ
$G_{\heartsuit}(s)$	function defined in (32)
$H^D_{\heartsuit}(t)$	Dirichlet heat trace for a hemi-equilateral (30-60-90) triangle with hypotenuse ℓ
$H^N_{\heartsuit}(t)$	Neumann heat trace for a hemi-equilateral (30-60-90) triangle with hypotenuse ℓ
$\zeta_R(s)$	Riemann zeta function
$\eta(z)$	Dedekind eta function
$L_3(s)$	Dirichlet L-function $1 - 2^{-s} + 4^{-s} - \cdots$
$L_4(s)$	Dirichlet L-function $1 - 3^{-s} + 5^{-s} + \cdots$

analytic continuation to $\mathbb{C}\backslash\{1\}$ with a simple pole at $s = 1$. Moreover, the formulas in Proposition 2.1 are well defined and analytic in $\mathbb{C}\backslash\{1\}$. Therefore, in the analytic continuation of the spectral zeta function, the formulas hold for all such s.

Proposition 2.1 *The spectral zeta function of a rectangle* $[0, a]\times[0, b]$ *with the Dirichlet boundary condition is equivalently given by the expressions*

$$\begin{aligned}\zeta_{\square}(s) &= \frac{1}{2}\left(\frac{b}{\pi}\right)^{2s}\left[-\zeta_R(2s) + \frac{a\sqrt{\pi}}{b}\frac{\zeta_R(2s-1)\Gamma(s-1/2)}{\Gamma(s)}\right] \\ &\quad + \left(\frac{ab}{\pi}\right)^{s}\frac{1}{\Gamma(s)}\sqrt{\frac{a}{b}}\sum_{n=1}^{\infty} n^{s-1/2}\sum_{d\mid n} d^{1-2s}\int_0^\infty x^{s-3/2}e^{-\pi a n(x+x^{-1})/b}\,dx\end{aligned} \tag{5}$$

and

$$\zeta_{\square}(s) = \frac{a^{2s}}{4}\left[G_{\square}(s) - 2\left(\frac{b}{a\pi}\right)^{2s}\zeta_R(2s) - \frac{2}{\pi^{2s}}\zeta_R(2s)\right],\ s \in \mathbb{C}\backslash\{1\}.$$

Here ζ_R denotes the Riemann zeta function, and

$$G_{\square}(s) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}{}' \frac{1}{\pi^{2s}|m+nz|^{2s}},\ z = ai/b,\ \mathrm{Re}(s) > 1. \tag{6}$$

Proof We write

$$\begin{aligned}\zeta_{\square}(s) &= \frac{1}{\pi^{2s}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(\frac{m^2}{a^2}+\frac{n^2}{b^2})^s}\\ &= \frac{1}{4\pi^{2s}}\left[\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}{}'\frac{1}{(\frac{m^2}{a^2}+\frac{n^2}{b^2})^s} - 2\zeta_R(2s)(a^{2s}+b^{2s})\right],\ \mathrm{Re}(s) > 1.\end{aligned}$$

By [10, p. 87], we have

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}{}'\frac{1}{(\frac{m^2}{a^2}+\frac{n^2}{b^2})^s} = 2a^{2s}\zeta_R(2s) + \frac{2ab^{2s-1}\sqrt{\pi}\zeta_R(2s-1)\Gamma(s-1/2)}{\Gamma(s)} + Q(s),$$
$$Q(s) = \frac{4(\pi ab)^s}{\Gamma(s)}\sqrt{\frac{a}{b}}\sum_{n=1}^{\infty}n^{s-1/2}\sum_{d|n}d^{1-2s}\int_0^{\infty}x^{s-3/2}e^{-\pi an(x+x^{-1})/b}dx,\ \mathrm{Re}(s) > 1.$$

Therefore, the first formula in the proposition follows for $\mathrm{Re}(s) > 1$, and hence for all $s \in \mathbb{C}\backslash\{1\}$ in the analytic continuation of $\zeta_{\square}(s)$. To obtain the second formula for $\zeta_{\square}(s)$ we write for $\mathrm{Re}(s) > 1$

$$\zeta_{\square}(s) = a^{2s}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{\pi^{2s}|m+nz|^{2s}},$$

where $z = ai/b$, which implies the second formula in the proposition. □

Corollary 2.2 *The zeta-regularized determinant of a rectangle* $[0, a] \times [0, b]$ *with the Dirichlet boundary condition is* $e^{-\zeta_{\square}'(0)}$ *with* $\zeta_{\square}'(0)$ *equivalently given by*

$$\zeta_{\square}'(0) = \frac{1}{2}\log(2b) + \frac{\pi a}{12b} + \sum_{n=1}^{\infty}\frac{1}{ne^{2\pi na/b}}\sum_{d|n}d,$$

$$\zeta_{\square}'(0) = \frac{1}{2}\log\left(\frac{2b}{|\eta(z)|^2}\right) = \frac{1}{2}\log(2b) + \frac{\pi a}{12b} - \sum_{n=1}^{\infty}\log(1-e^{-2\pi na/b}),\ z = ai/b.$$

Proof We differentiate our first expression from Proposition 2.1 and obtain

$$
\begin{aligned}
\zeta'_{\square}(s) &= \log\left(\frac{b}{\pi}\right)\left(\frac{b}{\pi}\right)^{2s}\left[-\zeta_R(2s)+\frac{a\sqrt{\pi}}{b}\frac{\zeta_R(2s-1)\Gamma(s-1/2)}{\Gamma(s)}\right] \\
&+\frac{1}{2}\left(\frac{b}{\pi}\right)^{2s}\left[-2\zeta'_R(2s)+\frac{a\sqrt{\pi}}{b}\frac{1}{\Gamma(s)^2}\Big[2\Gamma(s)\zeta'_R(2s-1)\Gamma(s-1/2)\right. \\
&\left.+\Gamma(s)\zeta_R(2s-1)\Gamma'(s-1/2)-\Gamma'(s)\zeta_R(2s-1)\Gamma(s-1/2)\Big]\right] \\
&-\left(\frac{ab}{\pi}\right)^{s}\frac{\Gamma'(s)}{\Gamma(s)^2}\sqrt{\frac{a}{b}}\sum_{n=1}^{\infty}n^{s-1/2}\sum_{d|n}d^{1-2s}\int_0^{\infty}x^{s-3/2}e^{-\pi an(x+x^{-1})/b}dx \\
&+\frac{1}{\Gamma(s)}\frac{d}{ds}\left[\left(\frac{ab}{\pi}\right)^{s}\sqrt{\frac{a}{b}}\sum_{n=1}^{\infty}n^{s-1/2}\sum_{d|n}d^{1-2s}\int_0^{\infty}x^{s-3/2}e^{-\pi an(x+x^{-1})/b}dx\right], \\
& s\in\mathbb{C}\backslash\{1\}.
\end{aligned}
$$

In Lemma A.1 we prove that we may differentiate termwise and under the integral in the last term above, and thereby obtain that this term vanishes at $s = 0$ due to the presence of $\frac{1}{\Gamma}$. If we write

$$
\phi = \frac{1}{\Gamma}, \text{ then } \phi'(0) = 1, \tag{7}
$$

due to the fact that $\lim_{s\to 0} s\Gamma(s) = 1$. Thus we obtain

$$
\begin{aligned}
\zeta'_{\square}(0) &= \log\left(\frac{b}{\pi}\right)\Big[-\zeta_R(0)\Big]+\frac{1}{2}\left[-2\zeta'_R(0)-\frac{a\sqrt{\pi}\Gamma'(0)\zeta_R(-1)\Gamma(-1/2)}{b\Gamma(0)^2}\right] \\
&+\phi'(0)\sqrt{\frac{a}{b}}\sum_{n=1}^{\infty}n^{-1/2}\sum_{d|n}d\int_0^{\infty}x^{-3/2}e^{-\pi an(x+x^{-1})/b}dx.
\end{aligned}
$$

We make the change of variable $x = e^t$, use the identity $e^{-t/2} = \cosh(t/2) - \sinh(t/2)$, together with the identities [15, Eq. 10.32.9, Eq. 10.39.2] to compute

$$
\int_0^{\infty}x^{-3/2}e^{-\pi an(x+x^{-1})/b}dx = \sqrt{\frac{b}{an}}e^{-2\pi an/b}. \tag{8}
$$

Moreover, we have

$$
\zeta_R(0) = -\frac{1}{2},\ \zeta'_R(0) = -\frac{\log(2\pi)}{2},\ \zeta_R(-1) = -\frac{1}{12},\ \Gamma(-1/2) = -2\sqrt{\pi}. \tag{9}
$$

Thus we obtain the first formula in the corollary.

Next we differentiate the second expression of Proposition 2.1:

$$\begin{aligned}\zeta_{\square}'(s) &= a^{2s}\log(a)\left[\frac{G_{\square}(s)}{2} - \left(\frac{b}{a\pi}\right)^{2s}\zeta_R(2s) - \frac{1}{\pi^{2s}}\zeta_R(2s)\right]\\ &\quad + a^{2s}\left[\frac{G_{\square}'(s)}{4} - \left(\frac{b}{a\pi}\right)^{2s}\log\left(\frac{b}{a\pi}\right)\zeta_R(2s) - \left(\frac{b}{a\pi}\right)^{2s}\zeta_R'(2s)\right.\\ &\quad\left. + \frac{1}{\pi^{2s}}\log(\pi)\zeta_R(2s) - \frac{1}{\pi^{2s}}\zeta_R'(2s)\right],\ \ s\in\mathbb{C}\backslash\{1\}.\end{aligned}$$

After inserting $s=0$, this becomes

$$\zeta_{\square}'(0) = \log(a)\left[\frac{G_{\square}(0)}{2}+1\right] + \frac{G_{\square}'(0)}{4} + \frac{1}{2}\log\left(\frac{4b}{a}\right).$$

We have by [39, p. 204-205] (see also [1, p. 1830-1831]),

$$G_{\square}(0) = -1,\ \ G_{\square}'(0) = -\frac{1}{12}\log\left((2\pi)^{24}\frac{(\eta(z)\bar{\eta}(z))^{24}}{\pi^{24}}\right),$$

where

$$\eta(\tau) = q^{1/12}\prod_{n=1}^{\infty}(1-q^{2n}),\ \ q=e^{\pi i\tau},\ \ \mathrm{Im}(\tau)>0, \tag{10}$$

is the Dedekind eta function. Since

$$\log(\eta(z)) = -\frac{\pi a}{12b} + \sum_{n=1}^{\infty}\log(1-e^{-2\pi na/b}),$$

we obtain

$$\zeta_{\square}'(0) = \frac{1}{2}\log\left(\frac{2b}{|\eta(z)|^2}\right) = \frac{1}{2}\log(2b) + \frac{\pi a}{12b} - \sum_{n=1}^{\infty}\log(1-e^{-2\pi na/b}).$$

It may be of some interest to verify explicitly that our two expressions of $\zeta_{\square}'(0)$ are equal.

To this end, it is enough to show that

$$-\sum_{n=1}^{\infty}\log(1-q^n) = \sum_{n=1}^{\infty}\frac{q^n}{n}\sum_{d|n}d$$

for $|q|<1$. We Taylor expand $\log(1-q^n)$ around 0 and obtain

$$-\sum_{n=1}^{\infty}\log(1-q^n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q^{nm}}{m} = \sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=1}^{\infty}q^{nm}.$$

If we write this as a power series in q, we see that q^n gets a contribution of $1/m$ if m divides n, and zero otherwise. Thus,

$$-\sum_{n=1}^{\infty} \log(1-q^n) = \sum_{n=1}^{\infty} q^n \sum_{d|n} \frac{1}{d} = \sum_{n=1}^{\infty} \frac{q^n}{n} \sum_{d|n} d,$$

where the last equality follows from the fact that $n/d \mapsto d$ is a bijection between the divisors of n. This completes the proof that the two expressions for $\zeta'_{\square}(0)$ are equal. □

Remark 2.3 We note that our expression for $\zeta'_{\square}(0)$ is equivalent to that given in [4, (105)]. In the case of a square, our expression for the spectral zeta function is equivalent to the first line of [4, (108)], and the derivative at $s = 0$ agrees with the expression given in [4, (113)].

2.2 The Heat Trace of a Rectangle and its Short Time Asymptotic Expansion

For the rectangle $[0, a] \times [0, b]$, the Dirichlet heat trace

$$H_{\square}^{D}(t) = \sum_{\lambda_{m,n}} e^{-\lambda_{m,n}t} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-\pi^2(m^2/a^2+n^2/b^2)t} = \sum_{m=1}^{\infty} e^{-\pi^2 m^2 t/a^2} \sum_{n=1}^{\infty} e^{-\pi^2 n^2 t/b^2}.$$

Since

$$\sum_{n=1}^{\infty} q^{n^2} = \frac{\Theta_3(q)-1}{2}, \quad \Theta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2},$$

the heat trace satisfies

$$H_{\square}^{D}(t) = \frac{\left(\Theta_3(e^{-\pi^2 t/a^2})-1\right)\left(\Theta_3(e^{-\pi^2 t/b^2})-1\right)}{4}.$$

Although we will not use this well known expression, we present it for the sake of completeness. Our focus here is on the asymptotic expansion of $H_{\square}^{D}(t)$ as $t \to 0$.

For any convex polygon Ω in the Euclidean plane with interior angles $\gamma_1, \ldots, \gamma_n$, its heat trace with the Dirichlet boundary condition admits the asymptotic expansion

$$\frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \sum_{i=1}^{n} \frac{\pi^2 - \gamma_i^2}{24\pi\gamma_i} + \mathcal{O}(e^{-c/t}), \ t \to 0,$$

for some $c > 0$ that has been estimated in [46]. For the Neumann boundary condition, an analogous estimate with such a remainder term remains an open problem. Verhoeven [47] used the Poisson summation formula to obtain the expression (11) that can be used not only to obtain further terms in the asymptotic expression, but also to determine the supremum of all such c for the remainder estimate above. We provide the sharp remainder here as well as the corresponding result for the Neumann boundary condition. It is interesting to note that certain terms appear with different signs according to the different boundary conditions.

Theorem 2.4 *Let $\Omega = [0, a] \times [0, b]$ be a rectangle that is not a square. Then the heat trace with the Dirichlet boundary condition admits the asymptotic expansion*

$$\begin{aligned} H_{\square}^{D}(t) = \frac{ab}{4\pi t} - \frac{a+b}{4\sqrt{\pi t}} + \frac{1}{4} + \frac{ab}{2\pi t} e^{-\min(a,b)^2/t} \\ - \frac{\min(a,b)}{2\sqrt{\pi t}} e^{-\min(a,b)^2/t} + \mathcal{O}\left(t^{-1} e^{-c/t}\right), \ t \to 0, \\ c = \min(\max(a,b)^2, 4\min(a,b)^2). \end{aligned}$$

The heat trace with the Neumann boundary condition admits the asymptotic expansion

$$\begin{aligned} H_{\square}^{N}(t) = \frac{ab}{4\pi t} + \frac{a+b}{4\sqrt{\pi t}} + \frac{1}{4} + \frac{ab}{2\pi t} e^{-\min(a,b)^2/t} \\ + \frac{\min(a,b)}{2\sqrt{\pi t}} e^{-\min(a,b)^2/t} + \mathcal{O}\left(t^{-1} e^{-c/t}\right), \quad t \to 0. \end{aligned}$$

If the rectangle is a square, $a = b$, then the respective expansions are

$$H_{\square}^{D}(t) = \frac{a^2}{4\pi t} - \frac{a}{2\sqrt{\pi t}} + \frac{1}{4} + \frac{a^2}{\pi t} e^{-a^2/t} - \frac{a}{\sqrt{\pi t}} e^{-a^2/t} + \mathcal{O}(t^{-1} e^{-2a^2/t}), \quad t \to 0,$$
$$H_{\square}^{N}(t) = \frac{a^2}{4\pi t} + \frac{a}{2\sqrt{\pi t}} + \frac{1}{4} + \frac{a^2}{\pi t} e^{-a^2/t} + \frac{a}{\sqrt{\pi t}} e^{-a^2/t} + \mathcal{O}(t^{-1} e^{-2a^2/t}), \quad t \to 0.$$

In all cases the remainders are sharp.

Remark 2.5 We note that the heat trace formulas for squares are not obtained by letting $a = b$ in the corresponding formulas for rectangles. In particular, the coefficients after the first three terms are inconsistent. The reason for this is that in the case $a = b$, certain sums in the heat trace can be combined, which effectively doubles some of the coefficients compared to the case $a \neq b$ where such grouping is not possible. In effect, then, one cannot interchange the limits $b \to a$ and $t \to 0$.

Proof By the Poisson summation formula,

$$\sum_{m \in \mathbb{Z}} e^{-\pi^2 m^2 t / a^2} = \frac{a}{\sqrt{\pi t}} \sum_{m \in \mathbb{Z}} e^{-m^2 a^2 / t}, \ a, t > 0,$$

which implies that

$$\sum_{m=1}^{\infty} e^{-\pi^2 m^2 t / a^2} = \frac{1}{2} \left(\frac{a}{\sqrt{\pi t}} - 1 \right) + \frac{a}{\sqrt{\pi t}} \sum_{m=1}^{\infty} e^{-m^2 a^2 / t}.$$

Consequently, the Dirichlet heat trace becomes

$$
\begin{aligned}
H_{\square}^{D}(t) &= \left(\frac{1}{2}\left(\frac{a}{\sqrt{\pi t}}-1\right)+\frac{a}{\sqrt{\pi t}}\sum_{m=1}^{\infty}e^{-m^2a^2/t}\right) \\
&\quad \left(\frac{1}{2}\left(\frac{b}{\sqrt{\pi t}}-1\right)+\frac{b}{\sqrt{\pi t}}\sum_{n=1}^{\infty}e^{-n^2b^2/t}\right) \\
&= \frac{ab}{4\pi t}-\frac{a+b}{4\sqrt{\pi t}}+\frac{1}{4}+\frac{ab}{2\pi t}\sum_{m=1}^{\infty}e^{-m^2a^2/t}+\frac{ab}{2\pi t}\sum_{n=1}^{\infty}e^{-n^2b^2/t} \\
&\quad -\frac{a}{2\sqrt{\pi t}}\sum_{m=1}^{\infty}e^{-m^2a^2/t}-\frac{b}{2\sqrt{\pi t}}\sum_{n=1}^{\infty}e^{-n^2b^2/t}+\frac{ab}{\pi t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}e^{-(m^2a^2+n^2b^2)/t}.
\end{aligned}
\tag{11}
$$

The proof in the Dirichlet case is then completed by calculating the leading order terms and determining the remainder by analyzing each of the three series. We note that Verhoeven did not compute the sharp remainder estimate in [47]. Instead, he stops with the formula above.

The eigenvalues of the rectangle $[0,a]\times[0,b]$ with the Neumann boundary condition are given by

$$
\lambda_{m,n}=\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right),\quad m,n\geq 0,
$$

so the heat trace becomes

$$
\begin{aligned}
H_{\square}^{N}(t) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}e^{-\pi^2(m^2/a^2+n^2/b^2)t}=1+\sum_{n=1}^{\infty}e^{-\pi^2n^2t/b^2}+\sum_{m=1}^{\infty}e^{-\pi^2m^2t/a^2}+H_{\square}^{D}(t) \\
&= 1+\frac{1}{2}\left(\frac{b}{\sqrt{\pi t}}-1\right)+\frac{b}{\sqrt{\pi t}}\sum_{n=1}^{\infty}e^{-n^2b^2/t}+\frac{1}{2}\left(\frac{a}{\sqrt{\pi t}}-1\right) \\
&\quad +\frac{a}{\sqrt{\pi t}}\sum_{m=1}^{\infty}e^{-m^2a^2/t}+H_{\square}^{D}(t) \\
&= \frac{ab}{4\pi t}+\frac{a+b}{4\sqrt{\pi t}}+\frac{1}{4}+\frac{ab}{2\pi t}\sum_{m=1}^{\infty}e^{-m^2a^2/t}+\frac{ab}{2\pi t}\sum_{n=1}^{\infty}e^{-n^2b^2/t} \\
&\quad +\frac{a}{2\sqrt{\pi t}}\sum_{m=1}^{\infty}e^{-m^2a^2/t}+\frac{b}{2\sqrt{\pi t}}\sum_{n=1}^{\infty}e^{-n^2b^2/t}+\frac{ab}{\pi t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}e^{-(m^2a^2+n^2b^2)/t}.
\end{aligned}
$$

Here we have used the Poisson summation formula and our calculation of $H_{\square}^{D}(t)$. The proof in the Neumann case is similarly completed by reading off the leading order terms and collecting the remainder.

When $a=b$, the heat traces simplify to

$$
H_{\square}^{D}(t)=\frac{a^2}{4\pi t}-\frac{a}{2\sqrt{\pi t}}+\frac{1}{4}+\frac{a^2}{\pi t}\sum_{m=1}^{\infty}e^{-m^2a^2/t}-\frac{a}{\sqrt{\pi t}}\sum_{m=1}^{\infty}e^{-m^2a^2/t}
$$

$$+\frac{a^2}{\pi t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}e^{-(m^2+n^2)a^2/t},$$

$$H_{\square}^{N}(t)=\frac{a^2}{4\pi t}+\frac{a}{2\sqrt{\pi t}}+\frac{1}{4}+\frac{a^2}{\pi t}\sum_{m=1}^{\infty}e^{-m^2a^2/t}+\frac{a}{\sqrt{\pi t}}\sum_{m=1}^{\infty}e^{-m^2a^2/t}$$

$$+\frac{a^2}{\pi t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}e^{-(m^2+n^2)a^2/t},$$

from which the claimed heat trace expansions follow.

□

Theorem 2.4 immediately implies that as $t \to 0$,

$$H_{\square}^{D}(t)=\frac{ab}{4\pi t}-\frac{a+b}{4\sqrt{\pi t}}+\frac{1}{4}+\mathcal{O}(e^{-((\min(a,b)^2-\epsilon)/t}),$$

$$H_{\square}^{N}(t)=\frac{ab}{4\pi t}+\frac{a+b}{4\sqrt{\pi t}}+\frac{1}{4}+\mathcal{O}(e^{-((\min(a,b)^2-\epsilon)/t}),$$

for any $\epsilon > 0$, but not $\epsilon = 0$. In particular, there is no maximal c such that the error term is $\mathcal{O}(e^{-c/t})$, but the supremum of all such values is $\min(a, b)^2$. To compare with [46], we note that their estimate for the exponent in the remainder term in case of rectangles is $\frac{\min(a,b)^2}{512}$. By [21, Prop. 8], $\min(a, b)^2$ is the square of half the length of the shortest closed geodesic in the rectangle. This proves Theorem 1.1 in the case of rectangles. This is not surprising considering the analogous result one can obtain for flat tori as discussed in §6.

We can easily generalize Theorem 2.4 to n-dimensional Euclidean boxes. For a Euclidean box, $\prod_{j=1}^{n}[0, a_j]$, a calculation similar to the one above yields the following heat trace in the Dirichlet and Neumann case, respectively:

$$\frac{1}{2^n}\prod_{j=1}^{n}\left(\frac{a_j}{\sqrt{\pi t}}-1\right)+\mathcal{O}(e^{-(\min(a_1,\ldots,a_n)^2-\epsilon)/t}),\ \forall\epsilon>0,\ t\to 0,$$

$$\frac{1}{2^n}\prod_{j=1}^{n}\left(\frac{a_j}{\sqrt{\pi t}}+1\right)+\mathcal{O}(e^{-(\min(a_1,\ldots,a_n)^2-\epsilon)/t}),\ \forall\epsilon>0,\ t\to 0.$$

We again observe that $\min(a_1, \ldots, a_n)^2$ is the square of half the length of the shortest closed geodesic in this Euclidean box.

3 Spectral Invariants of Equilateral Triangles

Let $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < y < x\sqrt{3},\ y < \sqrt{3}(1 - x)\}$ be an equilateral triangular domain. We note that the sides each have length one. We consider the Laplace eigenvalue problem with the Dirichlet boundary condition

$$\begin{cases} \Delta f(x, y) + \lambda f(x, y) = 0 \text{ in } \Omega, \\ f(x, y) = 0 \text{ on } \partial\Omega. \end{cases} \tag{12}$$

By [40, Thm. 1], the eigenvalues of (12) are

$$\lambda_{m,n} = \frac{16\pi^2}{27}(m^2 - mn + n^2), \text{ with } m, n \in \mathbb{Z} \text{ satisfying} \tag{13}$$

(A) $m + n \equiv 0 \pmod 3$,
(B) $m \neq 2n$,
(C) $n \neq 2m$,
(D) $m \neq -n$.

Although these conditions are stated in [40] and [41], there is no proof given that they are necessary and sufficient to guarantee that the associated $\lambda_{m,n}$ is an eigenvalue. A formula is given for the associated eigenfunction, but it could happen that the function vanishes identically, or that it does not satisfy the boundary condition. McCartin filled this gap by providing a beautiful pedagogical derivation of these expressions and showed how the conditions are necessary and sufficient [35]. We have a slightly different proof that some readers may find more accessible, and since it takes a mere page, we include it for the benefit of readers.

If $\lambda_{m,n}$ is an eigenvalue of Ω, a corresponding eigenfunction is given by

$$f_{m,n}(x, y) = \sum_{(m,n)} \pm e^{2\pi i/3(nx+(2m-n)y/\sqrt{3})}. \tag{14}$$

Here, the sum goes through the six pairs in

$$(-n, m-n), (-n, -m), (n-m, -m), (n-m, n), (m, n), (m, m-n) \tag{15}$$

from left to right and the sign alternates for each term. It is straightforward to verify that (m, n) satisfies (A) – (D) if and only if every pair above also satisfies (A) – (D). Explicitly, the eigenfunction corresponding to these six pairs is

$$\begin{aligned} f_{m,n}(x, y) = {} & e^{2\pi i/3((m-n)x-(m+n)y/\sqrt{3})} - e^{2\pi i/3(-mx+(m-2n)y/\sqrt{3})} \\ & + e^{2\pi i/3(-mx+(2n-m)y/\sqrt{3})} - e^{2\pi i/3(nx+(n-2m)y/\sqrt{3})} \\ & + e^{2\pi i/3(nx+(2m-n)y/\sqrt{3})} - e^{2\pi i/3((m-n)x+(m+n)y/\sqrt{3})}, \end{aligned} \tag{16}$$

or, equivalently,

$$\begin{aligned} f_{m,n}(x,y) = -2ie^{2\pi i/3(m-n)x}\sin\left(\frac{2\pi(m+n)y}{3\sqrt{3}}\right) - 2ie^{-2\pi i/3mx}\sin \\ \times\left(\frac{2\pi(m-2n)y}{3\sqrt{3}}\right) - 2ie^{2\pi i/3nx}\sin\left(\frac{2\pi(n-2m)y}{3\sqrt{3}}\right). \end{aligned} \tag{17}$$

We will now prove that $f_{m,n}$ is an eigenfunction of Ω with corresponding eigenvalue $\lambda_{m,n}$ if and only if (A), (B), (C), and (D) are satisfied. First, we note that

$$(m-n)^2 + \frac{(m+n)^2}{3} = m^2 + \frac{(m-2n)^2}{3} = n^2 + \frac{(n-2m)^2}{3} = \frac{4m^2-4mn+4n^2}{3},$$

from which it immediately follows that

$$\Delta f_{m,n} + \lambda_{m,n} f_{m,n} = 0.$$

Next, we examine when $f_{m,n}$ satisfies the boundary condition. At the boundary $y=0$, it is clear from (17) that $f_{m,n}$ vanishes. At $y = x\sqrt{3}$ we obtain

$$\begin{aligned} f_{m,n}(x,x\sqrt{3}) &= e^{2\pi i/3(-2nx)} - e^{\frac{2\pi i}{3}(-2nx)} + e^{2\pi i/3((2n-2m)x)} \\ &\quad - e^{2\pi i/3((2n-2m)x)} + e^{2\pi i/3(2mx)} - e^{2\pi i/3(2mx)} = 0. \end{aligned}$$

For $y = \sqrt{3}(1-x)$, we have

$$\begin{aligned} f_{m,n}(x,\sqrt{3}(1-x)) &= e^{2\pi i/3(-(m+n)+2mx)} - e^{2\pi i/3((m-2n)+(2n-2m)x)} \\ &\quad + e^{2\pi i/3((2n-m)-2nx)} \\ &\quad - e^{2\pi i/3((n-2m)+2mx)} + e^{2\pi i/3((2m-n)+(2n-2m)x)} \\ &\quad - e^{2\pi i/3((m+n)-2nx)} \\ &= e^{4\pi i/3mx}(e^{2\pi i/3(-m-n)} - e^{2\pi i/3(n-2m)}) \\ &\quad + e^{4\pi i/3(n-m)x}(e^{2\pi i/3(2m-n)} - e^{2\pi i/3(m-2n)}) \\ &\quad + e^{4\pi i/3(-nx)}(e^{2\pi i/3(2n-m)} - e^{2\pi i/3(m+n)}). \end{aligned}$$

This vanishes if and only if condition (A) holds. Indeed, if $m+n \equiv 0 \pmod 3$, then $m+n = 3k$ for some $k \in \mathbb{Z}$, and

$$\begin{aligned} f_{m,n}(x,\sqrt{3}(1-x)) &= e^{4\pi i/3mx}(e^{2\pi i/3(-3k)} - e^{2\pi i/3(3k-3m)}) \\ &\quad + e^{4\pi i/3(n-m)x}(e^{2\pi i/3(3m-3k)} - e^{2\pi i/3(3k-3n)}) \\ &\quad + e^{4\pi i/3(-nx)}(e^{2\pi i/3(3n-3k)} - e^{2\pi i/3(3k)}) = 0. \end{aligned}$$

If instead $m+n \equiv 1 \pmod 3$, so that $m+n = 3k+1$, then it simplifies to

$$f_{m,n}(x, \sqrt{3}(1-x)) = -i\sqrt{3}(e^{4\pi i/3mx} + e^{4\pi i/3(n-m)x} + e^{-4\pi i/3nx}),$$

which is not identically zero. For example, at $x = 3/4$ this equals $-i\sqrt{3}((-1)^m + (-1)^{n-m} + (-1)^n) \neq 0$. Similarly, if $m+n \equiv 2 \pmod 3$, then $m+n = 3k-1$ and

$$f_{m,n}(x, \sqrt{3}(1-x)) = i\sqrt{3}(e^{4\pi i/3mx} + e^{4\pi i/3(n-m)x} + e^{-4\pi i/3nx}) \not\equiv 0.$$

Next, we show that $f_{m,n} \equiv 0$ when either (B), (C), or (D) isn't satisfied. For this it suffices to check that $f_{m,2m}$, $f_{2n,n}$, and $f_{-n,n}$ are all identically zero. To show this, we compute that the six pairs in each case respectively are

$$\begin{aligned}
&(-2m, -m), (-2m, -m), (m, -m), (m, 2m), (m, 2m), (m, -m);\\
&(-n, n), (-n, -2n), (-n, -2n), (-n, n), (2n, n), (2n, n);\\
&(-n, -2n), (-n, n), (2n, n), (2n, n), (-n, n), (-n, -2n).
\end{aligned}$$

Due to the alternating signs in the definitions of $f_{m,2m}$, $f_{2n,n}$, and $f_{-n,n}$ it follows that they each vanish identically.

Finally, we compute that

$$f_{m,n}(0, y) = -2i\left[\sin\left(\frac{2\pi y(m+n)}{3\sqrt{3}}\right) + \sin\left(\frac{2\pi y(m-2n)}{3\sqrt{3}}\right) + \sin\left(\frac{2\pi y(n-2m)}{3\sqrt{3}}\right)\right].$$

Sines with different frequencies are linearly independent. Consequently as long as $|m+n|$, $|m-2n|$ and $|n-2m|$ are not all equal, then $f_{m,n}(0, y)$ is not the zero function. Since $f_{m,n}$ is a real analytic function on $\mathbb{R}^2$ it then follows that $f_{m,n}$ is also not the zero function. We therefore compute

$$\begin{aligned}
&m^2 + n^2 + 2mn = m^2 + 4n^2 - 4mn = n^2 + 4m^2 - 4mn\\
&\iff 6mn = 3n^2 \iff n = 0 \text{ or } 2m = n.
\end{aligned}$$

Since $2m \neq n$ by (C), this would require $n = 0$, but then $m = 0$ which violates (D). We have therewith shown that each orbit in (15) satisfying (A)–(D) gives rise to a (nontrivial) Laplace eigenfunction that satisfies the Dirichlet boundary condition. The fact that each orbit gives rise to a *distinct* eigenfunction, and that the collection of all of these functions constitutes an orthogonal basis for L^2 on the triangle follows from [41].

3.1 The Spectral Zeta Function and Zeta-Regularized Determinant of an Equilateral Triangle

For the equilateral triangle with side length ℓ, the spectral zeta function corresponding to the Dirichlet boundary condition is

$$\zeta_\nabla(s) = \sum_{\lambda_{m,n}} \frac{1}{\lambda_{m,n}^s} = \frac{1}{6}\left(\frac{27\ell^2}{16\pi^2}\right)^s \sum_{\lambda_{m,n}} \frac{1}{(m^2-mn+n^2)^s}, \ \operatorname{Re}(s) > 1.$$

We sum according to (13). Each eigenvalue occurs six times its actual multiplicity, hence we divide by 6 to correctly account for multiplicities. Our first result is an expression for this spectral zeta function that we have not encountered in the literature.

Proposition 3.1 *The spectral zeta function for an equilateral triangle with side lengths equal to ℓ and the Dirichlet boundary condition is equivalently*

$$\zeta_\nabla(s) = \frac{1}{6}\left(\frac{3\ell}{4\pi}\right)^{2s}\Bigg[-4\zeta_R(2s) + \frac{2^{2s}\sqrt{\pi}\zeta_R(2s-1)\Gamma(s-1/2)}{\Gamma(s)3^{s-1/2}} + \frac{4\pi^s 2^{s-1/2}}{\Gamma(s)3^{s/2-1/4}}\sum_{n=1}^\infty n^{s-1/2}\sum_{d|n} d^{1-2s}(-1)^n \int_0^\infty x^{s-3/2}e^{-\pi n\sqrt{3}(x+x^{-1})/2}dx\Bigg], \tag{18}$$

$$\zeta_\nabla(s) = \frac{1}{6}\left(\frac{9\ell^2}{16\pi^2}\right)^s G_\nabla(s) - \left(\frac{9\ell^2}{16\pi^2}\right)^s \zeta_R(2s), \ s \in \mathbb{C}\setminus\{1\}, \tag{19}$$

$$G_\nabla(s) = \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}{}' \frac{1}{|m+kz|^{2s}}, \ z = \frac{-3+i\sqrt{3}}{2}, \ \operatorname{Re}(s) > 1. \tag{20}$$

Proof To calculate the sum defining the spectral zeta function we begin by restricting the sum to nonzero pairs $(m,n) = (m, 3k-m)$ satisfying (A):

$$\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}{}' \frac{1}{(m^2 - m(3k-m) + (3k-m)^2)^s} = \frac{1}{3^s}\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}{}' \frac{1}{(m^2-3km+3k^2)^s}, \ \operatorname{Re}(s) > 1.$$

By [10, p. 87], we have

$$\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}{}' \frac{1}{(m^2-3km+3k^2)^s} = 2\zeta_R(2s) + \frac{2^{2s}\sqrt{\pi}\zeta_R(2s-1)\Gamma(s-1/2)}{\Gamma(s)3^{s-1/2}} + Q(s),$$

$$Q(s) = \frac{4\pi^s 2^{s-1/2}}{\Gamma(s)3^{s/2-1/4}}\sum_{n=1}^\infty n^{s-1/2}\sum_{d|n} d^{1-2s}(-1)^n \int_0^\infty x^{s-3/2}e^{-\pi n\sqrt{3}(x+x^{-1})/2}dx, \ \operatorname{Re}(s) > 1.$$

When we sum over the pairs (m, n) with $m = 2n$, we get

$$\sum_{n\in\mathbb{Z},n\neq 0} \frac{1}{((2n)^2 - 2n\cdot n + n^2)^s} = \frac{2}{3^s}\zeta_R(2s), \ \ \mathrm{Re}(s) > 1.$$

We get the same result when we sum over the pairs (m, n) with $n = 2m$ and $m = -n$. Thus, recalling the factor of $\frac{1}{6}$, we obtain the first expression for $\zeta_\nabla(s)$ in the proposition.

The second expression follows from noting that

$$\begin{aligned}&\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}{}' \frac{1}{(m^2 - 3km + 3k^2)^s}\\ &= \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}{}' \frac{1}{|m + kz|^{2s}} = G_\nabla(s), \ \ z = \frac{-3 + i\sqrt{3}}{2}, \ \ \mathrm{Re}(s) > 1.\end{aligned}$$

□

Remark 3.2 Aurell & Salomonson [4] gave an expression for the spectral zeta function of the equilateral triangle in the third line of their equation (108) as

$$\left(\frac{4\pi}{3\ell}\right)^{-2s} [L_3(s)\zeta_R(s) - \zeta_R(2s)]. \tag{21}$$

Here L_3 is the Dirichlet L-series $1 - 2^{-s} + 4^{-s} - 5^{-s} + \cdots$. One can show using [50, Example 1, p. 280] that this expression is equivalent to our (19). We thank Anders Södergren for suggesting this reference to demonstrate the equivalence.

Corollary 3.3 *The derivative of the spectral zeta function of an equilateral triangle with sides of length ℓ and the Dirichlet boundary condition is equivalently given by the expressions*

$$\begin{aligned}\zeta_\nabla'(0) &= \frac{2}{3}\log\left(\frac{3\ell}{2}\right) + \frac{\pi\sqrt{3}}{36} + \frac{2}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{ne^{\pi n\sqrt{3}}}\sum_{d|n} d,\\ \zeta_\nabla'(0) &= \frac{2}{3}\log\left(\frac{3\ell}{2|\eta(z)|}\right),\\ \zeta_\nabla'(0) &= \frac{2}{3}\log(\pi\ell) + \frac{7}{12}\log(3) - \log\Gamma(1/3).\end{aligned}$$

Above, η is the Dedekind eta function.

Proof We have by Proposition 3.1

$$\zeta_\nabla'(s) = \frac{1}{3}\log\left(\frac{3\ell}{4\pi}\right)\left(\frac{3\ell}{4\pi}\right)^{2s}\Bigg[-4\zeta_R(2s) + \frac{2^{2s}\sqrt{\pi}\zeta_R(2s-1)\Gamma(s-1/2)}{\Gamma(s)3^{2s-1/2}}$$

$$
\begin{aligned}
&+ \frac{4\pi^s 2^{s-1/2}}{\Gamma(s) 3^{s/2-1/4}} \sum_{n=1}^{\infty} n^{s-1/2} \sum_{d|n} d^{1-2s} (-1)^n \int_0^\infty x^{s-3/2} e^{-\pi n \sqrt{3}(x+x^{-1})/2} dx \Bigg] \\
&- \frac{1}{6} \left(\frac{3\ell}{4\pi}\right)^{2s} 8\zeta_R'(2s) + \frac{\sqrt{3\pi}}{\Gamma(s)^2} \left(\frac{3\ell^2}{4\pi^2}\right)^s \left[\log\left(\frac{4}{3}\right) \zeta_R(2s-1)\Gamma(s)\Gamma(s-1/2)\right. \\
&+ 2\zeta_R'(2s-1)\Gamma(s)\Gamma(s-1/2) - \zeta_R(2s-1)\Gamma'(s)\Gamma(s-1/2) \\
&\quad \left. + \zeta_R(2s-1)\Gamma(s)\Gamma'(s-1/2)\right] \\
&+ \frac{1}{6} \frac{d}{ds} \left[\frac{4\pi^s 2^{s-1/2}}{\Gamma(s) 3^{s/2-1/4}} \sum_{n=1}^{\infty} n^{s-1/2} \sum_{d|n} d^{1-2s} (-1)^n \int_0^\infty x^{s-3/2} e^{-\pi n \sqrt{3}(x+x^{-1})/2} dx \right], \\
&\quad s \in \mathbb{C}\backslash\{1\},
\end{aligned}
$$

and we compute

$$
\begin{aligned}
&\frac{d}{ds} \left[\frac{4\pi^s 2^{s-1/2}}{\Gamma(s) 3^{s/2-1/4}} \sum_{n=1}^{\infty} n^{s-1/2} \sum_{d|n} d^{1-2s} (-1)^n \int_0^\infty x^{s-3/2} e^{-\pi n \sqrt{3}(x+x^{-1})/2} dx \right] \\
&= -\frac{\Gamma'(s)}{\Gamma(s)^2} \frac{4\pi^s 2^{s-1/2}}{3^{s/2-1/4}} \sum_{n=1}^{\infty} n^{s-1/2} \sum_{d|n} d^{1-2s} (-1)^n \int_0^\infty x^{s-3/2} e^{-\pi n \sqrt{3}(x+x^{-1})/2} dx \\
&\quad + \frac{1}{\Gamma(s)} \frac{d}{ds} \left[\frac{4\pi^s 2^{s-1/2}}{3^{s/2-1/4}} \sum_{n=1}^{\infty} n^{s-1/2} \sum_{d|n} d^{1-2s} (-1)^n \int_0^\infty x^{s-3/2} e^{-\pi n \sqrt{3}(x+x^{-1})/2} dx \right].
\end{aligned}
$$

By Lemma A.1, the last term vanishes as $s \to 0$, and we thus obtain

$$
\begin{aligned}
\zeta_\nabla'(0) &= \frac{1}{3} \log\left(\frac{3\ell}{4\pi}\right) (-4\zeta_R(0)) - \frac{4}{3}\zeta_R'(0) - \frac{\sqrt{3\pi}\,\zeta_R(-1)\Gamma(-1/2)\Gamma'(0)}{6\Gamma(0)^2} \\
&\quad + \frac{2\sqrt[4]{3}}{3\sqrt{2}} \phi'(0) \sum_{n=1}^{\infty} n^{-1/2} \sum_{d|n} d(-1)^n \int_0^\infty x^{-3/2} e^{-\pi n \sqrt{3}(x+x^{-1})/2} dx.
\end{aligned}
$$

Here, ϕ is again given by (7). By (7), (8), and (9), this simplifies to the first expression in the corollary.

Next we differentiate (19)

$$
\zeta_\nabla'(s) = \left(\frac{9\ell^2}{16\pi^2}\right)^s \left[\log\left(\frac{9\ell^2}{16\pi^2}\right) \left(\frac{1}{6} G_\nabla(s) - \zeta_R(2s)\right) + \frac{1}{6} G_\nabla'(s) - 2\zeta_R'(2s)\right].
$$

At $s = 0$ this becomes

$$
\zeta_\nabla'(0) = \log\left(\frac{9\ell^2}{16\pi^2}\right) \left(\frac{1}{6} G_\nabla(0) - \zeta_R(0)\right) + \frac{1}{6} G_\nabla'(0) - 2\zeta_R'(0).
$$

Our $G_\nabla(s) = L^{2s}G(s)$ with G defined in [1, p. 1830-1831], and the parameter $L = \pi$. With the calculations of [1, p. 1830-1831] (see also [39, p. 204-205]) we have

$$G(0) = -1, \quad G'(0) = -\frac{1}{12}\log((2\pi)^{24}L^{-24}(\eta(z)\overline{\eta}(z))^{24}) \overset{L=\pi}{=} -2\log(2|\eta(z)|^2),$$

hence

$$G_\nabla(0) = -1, \quad G'_\nabla(0) = -\log(\pi^2) - 2\log(2|\eta(z)|^2), \quad z = \frac{-3+i\sqrt{3}}{2}.$$

Thus, we obtain the second expression for $\zeta'_\nabla(0)$ in the corollary. Since

$$|\eta(z)| = \frac{3^{1/8}\Gamma(1/3)^{3/2}}{2\pi},$$

the third expression for $\zeta'_\nabla(0)$ also follows. □

Remark 3.4 Since our expression is equal to that in (21) for $\zeta_\nabla(s)$, we have calculated the derivative of that expression at $s = 0$ and obtained

$$\zeta'_\nabla(0) = \frac{2}{3}\log\ell + \frac{5}{6}\log 3 - \frac{1}{2}\log 2 + \frac{1}{6}\log\pi - \frac{1}{2}\log\frac{\Gamma(1/3)}{\Gamma(2/3)}.$$

Using identities for the Dedekind eta function and the Gamma function, it is straightforward to show that this is equal to the second expression in the corollary as well as [4, (110)].

3.2 The Heat Trace of Equilateral Triangles and an Alternative Expression for the Eigenvalues

There is another common expression for the eigenvalues of an equilateral triangle in the literature (see e.g. [32]), namely

$$\lambda_{m,n} = \frac{4\pi^2}{27r^2}(m^2 + mn + n^2), \;\; m, n \geq 1.$$

Here, r is the radius of the inscribed circle of the triangle. If the triangle has side lengths each equal to ℓ, then $r = \ell/\sqrt{12}$. Therefore this expression for the eigenvalues in case $\ell = 1$ is simply

$$\lambda_{m,n} = \frac{16\pi^2}{9}(m^2 + mn + n^2), \;\; m, n \geq 1. \tag{22}$$

This is different from the expression (13) given by Pinsky [40, 41] and Lamé [24–26]. It may appear simpler for computations because it no longer involves the conditions (A)–(D) on the integers, and their range is $\mathbb{N}$ rather than $\mathbb{Z}$. At the same time, the connection to eigenfunctions is obfuscated, as is the multiplicity of the eigenvalues.

By [40, 41], expressing the eigenvalues as (13), we know that there are six pairs that correspond to one linearly independent eigenfunction, given by the six pairs in (15). Each distinct orbit gives rise to a distinct, linearly independent eigenfunction. Hence to calculate spectral invariants like the spectral zeta function or the heat trace, it suffices to sum over all integers (m, n) satisfying (A)–(D) and then divide by six. It is not at all clear how to account for multiplicities using the expression (22). Here we use the heat trace to show how to account for the multiplicities correctly if one wishes to use (22) to compute eigenvalues and spectral invariants of equilateral triangles.

Proposition 3.5 *The heat trace for the equilateral triangle with side length ℓ and the Dirichlet boundary condition is equivalently given by the expressions*

$$\begin{aligned} H_{\nabla}^{D}(t) &= \frac{\Theta_3(q^3)\Theta_3(q^9) + \Theta_2(q^3)\Theta_2(q^9) - 3\Theta_3(q^3) + 2}{6}, \\ \Theta_2(q) &= \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2}, \ \ \Theta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2}, \ \ q = e^{-16\pi^2 t/(27\ell^2)}, \\ H_{\nabla}^{D}(t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-16\pi^2 t/(9\ell^2)(m^2+mn+n^2)}. \end{aligned} \tag{23}$$

As a consequence, the eigenvalues are the values

$$\frac{16\pi^2}{9\ell^2}(m^2 + mn + n^2), \quad m, n \geq 1. \tag{24}$$

For each pair (m, n) with $m, n \geq 1$ there is exactly one orbit of the form (15) where each of the six pairs in the orbit satisfies conditions (A), (B), (C), and (D).

Proof For the equilateral triangle with side length ℓ, the heat trace with the Dirichlet boundary condition is

$$H_{\nabla}^{D}(t) = \frac{1}{6}\sum_{\lambda_{m,n}} e^{-16\pi^2 t/(27\ell^2)(m^2-mn+n^2)}.$$

The sum goes through all pairs $(m, n) \in \mathbb{Z}^2$ satisfying (A), (B), (C), and (D). To compensate for the fact that the six pairs in (15) all correspond to the same eigenvalue, we have divided by 6.

Then, for $q = e^{-16\pi^2 t/(27\ell^2)}$, we get when summing over *all* integer pairs

$$\begin{aligned} \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} q^{m^2-mn+n^2} &= \sum_{n\in\mathbb{Z}} q^{3n^2/4} \sum_{m\in\mathbb{Z}} q^{(m-n/2)^2} \\ &= \sum_{n\in 2\mathbb{Z}} q^{3n^2/4} \sum_{m\in\mathbb{Z}} q^{m^2} + \sum_{n\in 2\mathbb{Z}+1} q^{3n^2/4} \sum_{m\in\mathbb{Z}} q^{(m+1/2)^2} \\ &= \Theta_3(q)\Theta_3(q^3) + \Theta_2(q)\Theta_2(q^3). \end{aligned} \tag{25}$$

Now let us only sum over the pairs (m, n) which satisfy (A). We write $n = 3k - m$ and obtain

$$h_1(t) = \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} q^{m^2 - m(3k-m) + (3k-m)^2}$$
$$= \sum_{k\in\mathbb{Z}} q^{9k^2/4} \sum_{m\in\mathbb{Z}} q^{3(m-3k/2)^2} = \Theta_3(q^3)\Theta_3(q^9) + \Theta_2(q^3)\Theta_2(q^9).$$

To obtain $H^D_\nabla(t)$, we must subtract the contribution from the pairs (m, n) with $m = 2n$ or $n = 2m$ or $m = -n$:

$$h_2(t) = \sum_{n\in\mathbb{Z}} q^{(2n)^2 - 2n\cdot n + n^2} = \Theta_3(q^3),$$
$$h_3(t) = \sum_{m\in\mathbb{Z}} q^{m^2 - m\cdot 2m + (2m)^2} = \Theta_3(q^3),$$
$$h_4(t) = \sum_{n\in\mathbb{Z}} q^{(-n)^2 - (-n)n + n^2} = \Theta_3(q^3).$$

Thus, we get

$$H^D_\nabla(t) = \frac{h_1(t) - h_2(t) - h_3(t) - h_4(t) + 2}{6}$$
$$= \frac{\Theta_3(q^3)\Theta_3(q^9) + \Theta_2(q^3)\Theta_2(q^9) - 3\Theta_3(q^3) + 2}{6}. \tag{26}$$

The plus 2 appears because when we subtract the contribution from pairs (m, n) with $m = 2n$ or $n = 2m$ or $m = -n$, we subtract the contribution from (0, 0) three times.

Now, [47] starts with (23) and computes

$$H^D_\nabla(t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q^{3(m^2+mn+n^2)} = \frac{1}{6}\left[\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} q^{3(m^2+mn+n^2)} - 3\sum_{m\in\mathbb{Z}} q^{3m^2} + 2\right], \tag{27}$$

which by (25) agrees with (26). In particular, the two expressions for the heat trace are identical. Since the heat trace uniquely determines the spectrum, this proves the equivalence between the two expressions for the eigenvalues of equilateral triangles. □

Remark 3.6 Observe that in the expression (24), it is not at all obvious that each eigenvalue occurs precisely *six times*, but this is indeed the case. While this equivalence can be deduced by combining McCartin's works [32, 35], the argument presented here provides, to the best of our knowledge, the first direct proof. It is also interesting to recall the observation made in [47, Corollary 2.2.6]: there is no bounded domain in

$\mathbb{R}^2$ whose spectrum has the form

$$c(m^2 - mn + n^2), \quad m, n \in \mathbb{Z}.$$

In particular, the restrictions (A) – (D) on the pairs $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ are essential to the correct expression for the eigenvalues of the equilateral triangle.

3.3 Short Time Asymptotic Expansion of the Heat Trace

Here we obtain further terms in the short time asymptotic expansion of the heat trace. We note that [47] obtained a related formula but rather than extracting further terms explicitly, collected all terms that vanish as $t \to 0$ into one big-O term using the asymptotic behavior of Jacobi theta functions.

Theorem 3.7 *The Dirichlet heat trace for an equilateral triangle with sides of length ℓ has the asymptotic expansion*

$$H_\nabla^D(t) = \frac{\ell^2\sqrt{3}}{16\pi t} - \frac{3\ell}{8\sqrt{\pi t}} + \frac{1}{3} - \frac{3\ell}{4\sqrt{\pi t}} e^{-9\ell^2/(16t)} + \frac{\ell^2\sqrt{3}}{8\pi t} e^{-3\ell^2/(4t)} + \mathcal{O}(t^{-1}e^{-9\ell^2/(4t)}), \ t \to 0.$$

The Neumann heat trace has the asymptotic expansion

$$H_\nabla^N(t) = \frac{\ell^2\sqrt{3}}{16\pi t} + \frac{3\ell}{8\sqrt{\pi t}} + \frac{1}{3} + \frac{3\ell}{4\sqrt{\pi t}} e^{-9\ell^2/(16t)} + \frac{\ell^2\sqrt{3}}{8\pi t} e^{-3\ell^2/(4t)} + \mathcal{O}(t^{-1}e^{-9\ell^2/(4t)}), \ t \to 0.$$

The remainders are sharp.

Proof By Proposition 3.5 the Dirichlet heat trace is

$$\begin{aligned} H_\nabla^D(t) &= \frac{\Theta_3(q^3)\Theta_3(q^9) + \Theta_2(q^3)\Theta_2(q^9) - 3\Theta_3(q^3) + 2}{6} \\ &= \frac{1}{6}\Bigg[\sum_{m\in\mathbb{Z}} e^{-16\pi^2 t/(9\ell^2)m^2} \sum_{n\in\mathbb{Z}} e^{-16\pi^2 t/(3\ell^2)n^2} \\ &+ \sum_{m\in\mathbb{Z}} e^{-16\pi^2 t/(9\ell^2)(m+\frac{1}{2})^2} \sum_{n\in\mathbb{Z}} e^{-16\pi^2 t/(3\ell^2)(n+\frac{1}{2})^2} - 3\sum_{m\in\mathbb{Z}} e^{-16\pi^2 t/(9\ell^2)m^2} + 2\Bigg]. \end{aligned}$$

By the Poisson summation formula,

$$\sum_{m\in\mathbb{Z}} e^{-16\pi^2 t/(9\ell^2)m^2} = \frac{3\ell}{4\sqrt{\pi t}} \sum_{m\in\mathbb{Z}} e^{-9\ell^2 m^2/(16t)},$$

$$\sum_{n\in\mathbb{Z}} e^{-16\pi^2 t/(3\ell^2)n^2} = \frac{\ell}{4}\sqrt{\frac{3}{\pi t}} \sum_{n\in\mathbb{Z}} e^{-3\ell^2 n^2/(16t)},$$

$$\sum_{m\in\mathbb{Z}} e^{-16\pi^2 t/(9\ell^2)(m+\frac{1}{2})^2} = \frac{3\ell}{4\sqrt{\pi t}} \sum_{m\in\mathbb{Z}} (-1)^m e^{-9\ell^2 m^2/(16t)},$$
$$\sum_{n\in\mathbb{Z}} e^{-16\pi^2 t/(3\ell^2)(n+\frac{1}{2})^2} = \frac{\ell}{4}\sqrt{\frac{3}{\pi t}} \sum_{n\in\mathbb{Z}} (-1)^n e^{-3\ell^2 n^2/(16t)}.$$

This gives

$$\begin{aligned}
H_\nabla^D(t) = \frac{1}{6}\Bigg[& \frac{3\ell^2\sqrt{3}}{16\pi t}\left(1+2\sum_{m=1}^{\infty} e^{-9\ell^2 m^2(16t)}\right)\left(1+2\sum_{n=1}^{\infty} e^{-3\ell^2 n^2/(16t)}\right) \\
& + \frac{3\ell^2\sqrt{3}}{16\pi t}\left(1+2\sum_{m=1}^{\infty}(-1)^m e^{-9\ell^2 m^2/(16t)}\right)\left(1+2\sum_{n=1}^{\infty}(-1)^n e^{-3\ell^2 n^2/(16t)}\right) \\
& - \frac{9\ell}{4\sqrt{\pi t}}\left(1+2\sum_{m=1}^{\infty} e^{-9\ell^2 m^2/(16t)}\right)+2\Bigg] \\
= & \frac{\ell^2\sqrt{3}}{16\pi t} - \frac{3\ell}{8\sqrt{\pi t}} + \frac{1}{3} - \frac{3\ell}{4\sqrt{\pi t}}\sum_{m=1}^{\infty} e^{-9\ell^2 m^2/(16t)} + \frac{\ell^2\sqrt{3}}{8\pi t}\sum_{n=1}^{\infty} e^{-3\ell^2 n^2/(4t)} \\
& + \frac{\ell^2\sqrt{3}}{8\pi t}\sum_{m=1}^{\infty} e^{-9\ell^2 m^2/(4t)} + \frac{\ell^2\sqrt{3}}{4\pi t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-3\ell^2(3m^2+n^2)/(4t)} \\
& + \frac{\ell^2\sqrt{3}}{4\pi t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-3\ell^2(3(2m-1)^2+(2n-1)^2)/(16t)},
\end{aligned}$$

which proves the Dirichlet case.

The eigenvalues of the same equilateral triangle with the Neumann boundary condition are

$$\lambda_{m,n} = \frac{16\pi^2}{9\ell^2}(m^2+mn+n^2),\ \ m,n \geq 0,$$

so its heat trace becomes

$$H_\nabla^N(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} e^{-16\pi^2 t/(9\ell^2)(m^2+mn+n^2)} = 1 + 2\sum_{m=1}^{\infty} e^{-16\pi^2 t/(9\ell^2)m^2} + H_\nabla^D(t).$$

By the Poisson summation formula,

$$\sum_{m=1}^{\infty} e^{-16\pi^2 t/(9\ell^2)m^2} = \frac{1}{2}\left(\frac{3\ell}{4\sqrt{\pi t}} - 1\right) + \frac{3\ell}{4\sqrt{\pi t}}\sum_{m=1}^{\infty} e^{-9\ell^2 m^2/(16t)},$$

so, we have

$$\begin{aligned}
H_\nabla^N(t) = & \frac{\ell^2\sqrt{3}}{16\pi t} + \frac{3\ell}{8\sqrt{\pi t}} + \frac{1}{3} + \frac{3\ell}{4\sqrt{\pi t}}\sum_{m=1}^{\infty} e^{-9\ell^2 m^2/(16t)} + \frac{\ell^2\sqrt{3}}{8\pi t}\sum_{n=1}^{\infty} e^{-3\ell^2 n^2/(4t)} \\
& + \frac{\ell^2\sqrt{3}}{8\pi t}\sum_{m=1}^{\infty} e^{-9\ell^2 m^2/(4t)}
\end{aligned}$$

$$+\frac{\ell^2\sqrt{3}}{4\pi t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}e^{-3\ell^2(3m^2+n^2)/(4t)}+\frac{\ell^2\sqrt{3}}{4\pi t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}e^{-3\ell^2(3(2m-1)^2+(2n-1)^2)/(16t)}.$$

This completes the proof. □

It follows from Theorem 3.7 that

$$H_{\nabla}^{D}(t)=\frac{\ell^2\sqrt{3}}{16\pi t}-\frac{3\ell}{8\sqrt{\pi t}}+\frac{1}{3}+\mathcal{O}(e^{-(9\ell^2-\epsilon)/(16t)}),\ \ t\to 0,$$
$$H_{\nabla}^{N}(t)=\frac{\ell^2\sqrt{3}}{16\pi t}+\frac{3\ell}{8\sqrt{\pi t}}+\frac{1}{3}+\mathcal{O}(e^{-(9\ell^2-\epsilon)/(16t)}),\ \ t\to 0,$$

for any $\epsilon>0$. Consequently, the supremum over all $c>0$ such that the remainder is $\mathcal{O}(e^{-c/t})$ is $\frac{9\ell^2}{16}$. By [16, p. 43], this is the square of half the length of the shortest closed geodesic in the equilateral triangle of side length ℓ. Thus, Theorem 1.1 follows in the case of equilateral triangles.

4 Spectral Invariants of Isosceles Right Triangles

By [3], the eigenvalues of an isosceles right triangle with area $\frac{a^2}{2}$, and therewith legs of length a are

$$\lambda_{m,n}=\frac{\pi^2(m^2+n^2)}{a^2},\ \ m>n\geq 1. \tag{28}$$

The fact that these are *all* eigenvalues including multiplicities follows from [23, p. 168]; see also [19] and [37, p. 756].

4.1 The Spectral Zeta Function and Zeta-Regularized Determinant of Isosceles Right Triangles

Our first result for the isosceles right triangle concerns two equivalent expressions for the spectral zeta function. To the best of our knowledge these expressions are new.

Proposition 4.1 *The spectral zeta function of an isosceles right triangle with area $a^2/2$ and with the Dirichlet boundary condition is equivalently given by the expressions*

$$\zeta_{\diamond}(s)=-\frac{1}{4}\left(\frac{a}{\pi}\right)^{2s}\zeta_R(2s)-\frac{1}{2^{s+1}}\left(\frac{a}{\pi}\right)^{2s}\zeta_R(2s)+\frac{\sqrt{\pi}}{4}\left(\frac{a}{\pi}\right)^{2s}\frac{\zeta_R(2s-1)\Gamma(s-1/2)}{\Gamma(s)}$$
$$+\frac{1}{2}\left(\frac{a^2}{\pi}\right)^{s}\frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}n^{s-1/2}\sum_{d|n}d^{1-2s}\int_0^{\infty}x^{s-3/2}e^{-\pi n(x+x^{-1})}dx$$

and

$$\zeta_{\diamond}(s)=\frac{a^{2s}}{2}\left[\frac{1}{4}G_{\diamond}(s)-\frac{1}{\pi^{2s}}\zeta_R(2s)-\frac{1}{(2\pi^2)^s}\zeta_R(2s)\right],\ \ s\in\mathbb{C}\backslash\{1\}.$$

Here,

$$G_{\diamond}(s) = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}}{}' \frac{1}{\pi^{2s}|m+ni|^{2s}}, \quad \mathrm{Re}(s) > 1. \tag{29}$$

Proof For $\mathrm{Re}(s) > 1$, we have

$$\zeta_{\diamond}(s) = \sum_{\lambda_{m,n}} \frac{1}{\lambda_{m,n}^s} = \left(\frac{a}{\pi}\right)^{2s} \sum_{m>n\geq 1} \frac{1}{(m^2+n^2)^s} = \frac{1}{2}\zeta_{\blacksquare}(s) - \frac{1}{2}\left(\frac{a^2}{2\pi^2}\right)^s \zeta_R(2s), \tag{30}$$

where $\zeta_{\blacksquare}$ is the spectral zeta function of a square with sides of length a. The result is thus an immediate consequence of Proposition 2.1. □

Remark 4.2 Similar to the case of equilateral triangles, one can show that our expression for the isosceles right triangle is equal to that given in the second equation of [4, (108)],

$$\zeta_{\diamond}(s) = \frac{1}{2}\left(\frac{\pi}{a}\right)^{-2s} \left[L_4(s)\zeta_R(s) - (1+2^{-s})\zeta_R(2s)\right].$$

Corollary 4.3 *The zeta-regularized determinant of an isosceles right triangle with area $a^2/2$ and with the Dirichlet boundary condition is $e^{-\zeta_{\diamond}'(0)}$ with $\zeta_{\diamond}'(0)$ equivalently given by*

$$\zeta_{\diamond}'(0) = \frac{\log(4a^3)}{4} + \frac{\pi}{24} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{ne^{2\pi n}} \sum_{d|n} d,$$

$$\zeta_{\diamond}'(0) = \frac{1}{4}\log\left(\frac{4a^3}{|\eta(i)|^2}\right) = \frac{\log(4a^3)}{4} + \frac{\pi}{24} - \frac{1}{2}\sum_{n=1}^{\infty} \log(1-e^{-2\pi n}).$$

Proof From (30), we get

$$\zeta_{\diamond}'(s) = \frac{1}{2}\zeta_{\blacksquare}'(s) - \frac{1}{2}\left(\frac{a^2}{2\pi^2}\right)^s \log\left(\frac{a^2}{2\pi^2}\right)\zeta_R(2s) - \left(\frac{a^2}{2\pi^2}\right)^s \zeta_R'(2s), \quad s \in \mathbb{C}\backslash\{1\}.$$

Corollary 2.2 then gives the following equivalent expressions

$$\zeta_{\diamond}'(0) = \frac{1}{2}\left[\frac{1}{2}\log(2a) + \frac{\pi}{12} + \sum_{n=1}^{\infty} \frac{1}{ne^{2\pi n}} \sum_{d|n} d\right] - \frac{1}{2}\log\left(\frac{a^2}{2\pi^2}\right)\zeta_R(0) - \zeta_R'(0),$$

$$\zeta_{\diamond}'(0) = \frac{1}{4}\log\left(\frac{2a}{|\eta(i)|^2}\right) + \frac{1}{4}\log\left(\frac{a^2}{2\pi^2}\right) + \frac{1}{2}\log(2\pi),$$

which respectively simplify to those in the corollary. □

Remark 4.4 Differentiating the expression in [4, (108)] for the spectral zeta function of the isosceles right triangle and setting $s = 0$ it becomes

$$\zeta_{\diamond}'(0) = \frac{3}{4}\log a + \log 2 + \frac{3}{8}\log\pi - \frac{1}{2}\log\Gamma(1/4). \tag{31}$$

Using identities for the Dedekind eta function and the Gamma function, one can show that this is equivalent to the expressions in our corollary as well as [4, (111)].

4.2 The Heat Trace of Isosceles Right Triangles

For the isosceles right triangle with area $a^2/2$ and with the Dirichlet boundary condition, the heat trace is

$$H_{\diamond}^{D}(t) = \sum_{\lambda_{n,m}} e^{-\lambda_{n,m}t} = \sum_{m>n\geq 1} e^{-\pi^2(m^2+n^2)t/a^2}.$$

Let $q = e^{-\pi^2 t/a^2}$ and note that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{m^2+n^2} = 2\sum_{m>n\geq 1} q^{m^2+n^2} + \sum_{m=1}^{\infty} q^{2m^2}.$$

Since

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{m^2+n^2} &= \left(\frac{\Theta_3(q)-1}{2}\right)^2,\\ \sum_{m=1}^{\infty} q^{2m^2} &= \frac{\Theta_3(q^2)-1}{2},\end{aligned}$$

it follows that

$$\begin{aligned}H_{\diamond}^{D}(t) &= \frac{1}{2}\left[\left(\frac{\Theta_3(q)-1}{2}\right)^2 - \frac{\Theta_3(q^2)-1}{2}\right]\\ &= \frac{\Theta_3(q)^2 - 2\Theta_3(q) - 2\Theta_3(q^2) + 3}{8}.\end{aligned}$$

Now we use our expressions for the heat trace for the isosceles right triangle to obtain further terms in the short time asymptotic expansion of the heat trace as well as a sharp remainder term.

Theorem 4.5 *The heat trace with the Dirichlet boundary condition for an isosceles right triangle of area $\frac{a^2}{2}$ admits the asymptotic expansion*

$$H_\diamond^D(t) = \frac{a^2}{8\pi t} - \frac{a(2+\sqrt{2})}{8\sqrt{\pi t}} + \frac{3}{8} - \frac{a}{2\sqrt{2\pi t}} e^{-a^2/(2t)} + \frac{a^2}{2\pi t} e^{-a^2/t} + \mathcal{O}(t^{-1/2} e^{-a^2/t}), \ \ t \to 0.$$

The heat trace with the Neumann boundary condition admits the asymptotic expansion

$$H_\diamond^N(t) = \frac{a^2}{8\pi t} + \frac{a(2+\sqrt{2})}{8\sqrt{\pi t}} + \frac{3}{8} + \frac{a}{2\sqrt{2\pi t}} e^{-a^2/(2t)} + \frac{a^2}{2\pi t} e^{-a^2/t} + \mathcal{O}(t^{-1/2} e^{-a^2/t}), \ \ t \to 0.$$

The remainders are sharp.

Proof We have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty e^{-\pi^2(m^2+n^2)t/a^2} = 2H_\diamond^D(t) + \sum_{m=1}^\infty e^{-\frac{2\pi^2 m^2}{a^2} t}$$

and

$$\sum_{m=1}^\infty e^{-\pi^2 m^2 t/a^2} = \frac{1}{2}\left(\frac{a}{\sqrt{\pi t}} - 1\right) + \frac{a}{\sqrt{\pi t}} \sum_{m=1}^\infty e^{-m^2 a^2/t},$$
$$\sum_{m=1}^\infty e^{-2\pi^2 m^2 t/a^2} = \frac{1}{2}\left(\frac{a}{\sqrt{2\pi t}} - 1\right) + \frac{a}{\sqrt{2\pi t}} \sum_{m=1}^\infty e^{-m^2 a^2/(2t)}$$

by the Poisson summation formula, hence

$$\begin{aligned} H_\diamond^D(t) &= \frac{1}{2}\left[\left(\frac{1}{2}\left(\frac{a}{\sqrt{\pi t}} - 1\right) + \frac{a}{\sqrt{\pi t}} \sum_{m=1}^\infty e^{-m^2a^2/t}\right)^2 - \frac{1}{2}\left(\frac{a}{\sqrt{2\pi t}} - 1\right) \right. \\ &\quad \left. - \frac{a}{\sqrt{2\pi t}} \sum_{m=1}^\infty e^{-m^2 a^2/(2t)}\right] \\ &= \frac{a^2}{8\pi t} - \frac{a(2+\sqrt{2})}{8\sqrt{\pi t}} + \frac{3}{8} - \frac{a}{2\sqrt{2\pi t}} \sum_{m=1}^\infty e^{-m^2a^2/(2t)} + \frac{a^2}{2\pi t} \sum_{m=1}^\infty e^{-m^2a^2/t} \\ &\quad - \frac{a}{2\sqrt{\pi t}} \sum_{m=1}^\infty e^{-m^2a^2/t} + \frac{a^2}{2\pi t}\left(\sum_{m=1}^\infty e^{-m^2a^2/t}\right)^2. \end{aligned}$$

The eigenvalues of an isosceles right triangle with area $a^2/2$ with the Neumann boundary condition are

$$\lambda_{m,n} = \frac{\pi^2(m^2+n^2)}{a^2}, \ \ m \geq n \geq 0,$$

so the corresponding heat trace becomes

$$\begin{aligned} H_{\diamond}^N(t) &= \sum_{m\geq n\geq 0} e^{-\pi^2(m^2+n^2)t/a^2} = 1 + \sum_{m=1}^{\infty} e^{-\pi^2 m^2 t/a^2} + \sum_{n=1}^{\infty} e^{-2\pi^2 n^2 t/a^2} + H_{\diamond}^D(t) \\ &= \frac{a^2}{8\pi t} + \frac{a(2+\sqrt{2})}{8\sqrt{\pi t}} + \frac{3}{8} + \frac{a}{2\sqrt{2\pi t}} \sum_{m=1}^{\infty} e^{-m^2 a^2/(2t)} + \frac{a^2}{2\pi t} \sum_{m=1}^{\infty} e^{-m^2 a^2/t} \\ &\quad + \frac{a}{2\sqrt{\pi t}} \sum_{m=1}^{\infty} e^{-m^2 a^2/t} + \frac{a^2}{2\pi t}\left(\sum_{m=1}^{\infty} e^{-m^2 a^2/t}\right)^2 . \end{aligned}$$

The proof is now completed by collecting leading order terms. □

Theorem 4.5 shows that

$$\begin{aligned} H_{\diamond}^D(t) &= \frac{a^2}{8\pi t} - \frac{a(2+\sqrt{2})}{8\sqrt{\pi t}} + \frac{3}{8} + \mathcal{O}(e^{-(a^2-\epsilon)/(2t)}), \ \ t \to 0, \\ H_{\diamond}^N(t) &= \frac{a^2}{8\pi t} + \frac{a(2+\sqrt{2})}{8\sqrt{\pi t}} + \frac{3}{8} + \mathcal{O}(e^{-(a^2-\epsilon)/(2t)}), \ \ t \to 0, \end{aligned}$$

for any $\epsilon > 0$. Again, $a^2/2$ is the square of half the length of the shortest closed geodesic in the isosceles right triangle (see [16, p. 43]), hence Theorem 1.1 follows in this case.

5 Spectral Invariants of Hemi-Equilateral (30-60-90) Triangles

By [32], the eigenvalues of the 30-60-90 triangle with hypotenuse of length ℓ are given by

$$\lambda_{m,n} = \frac{4\pi^2}{27r^2}(m^2+mn+n^2) = \frac{16\pi^2}{9\ell^2}(m^2+mn+n^2), \ \ m > n \geq 1.$$

Here, r is the radius of the inscribed circle of the equilateral triangle obtained by doubling the hemi-equilateral triangle. McCartin shows in [32] how antisymmetric eigenfunctions of equilateral triangles form a complete set of eigenfunctions for 30-60-90 triangles [13]; see also [23, p. 168].

5.1 The Spectral Zeta Function and Zeta-Regularized Determinant of Hemi-Equilateral Triangles

Our first result for these triangles contains two equivalent expressions for the spectral zeta function that to the best of our knowledge are new.

Proposition 5.1 *The spectral zeta function of the hemi-equilateral triangle with hypotenuse of length ℓ and with the Dirichlet boundary condition is equivalently given by the expressions*

$$\zeta_{\heartsuit}(s) = \frac{1}{12}\left(\frac{3\ell}{4\pi}\right)^{2s}\Bigg[-4\zeta_R(2s) - \frac{6}{3^s}\zeta_R(2s) + \frac{2^{2s}\sqrt{\pi}\zeta_R(2s-1)\Gamma(s-1/2)}{\Gamma(s)3^{s-1/2}} + \frac{4\pi^s 2^{s-1/2}}{\Gamma(s)3^{s/2-1/4}}\sum_{n=1}^{\infty} n^{s-1/2}\sum_{d|n} d^{1-2s}(-1)^n \int_0^\infty x^{s-3/2} e^{-\pi n\sqrt{3}(x+x^{-1})/2} dx\Bigg],$$

$$\zeta_{\heartsuit}(s) = \frac{1}{12}\left(\frac{3\ell}{4}\right)^{2s}\left[G_{\heartsuit}(s) - \frac{6}{\pi^{2s}}\zeta_R(2s) - \frac{6}{(3\pi^2)^s}\zeta_R(2s)\right], \ s \in \mathbb{C}\backslash\{1\}.$$

Here,

$$G_{\heartsuit}(s) = \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}{}' \frac{1}{\pi^{2s}|m+kz|^{2s}}, \ z = \frac{-3+i\sqrt{3}}{2}, \ \mathrm{Re}(s) > 1. \tag{32}$$

Proof The spectral zeta function is for $\mathrm{Re}(s) > 1$

$$\zeta_{\heartsuit}(s) = \left(\frac{3\ell}{4\pi}\right)^{2s} \sum_{m>n\geq 1} \frac{1}{(m^2+mn+n^2)^s}.$$

Since

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(m^2+mn+n^2)^s} = 2\sum_{m>n\geq 1}\frac{1}{(m^2+mn+n^2)^s} + \frac{1}{3^s}\zeta_R(2s),$$

we can rewrite $\zeta_{\heartsuit}$ as

$$\zeta_{\heartsuit}(s) = \frac{1}{2}\zeta_{\nabla}(s) - \frac{1}{2}\left(\frac{3\ell^2}{16\pi^2}\right)^s \zeta_R(2s), \ s \in \mathbb{C}\backslash\{1\}, \tag{33}$$

where ζ_{∇} is the spectral zeta function of the corresponding equilateral triangle with sides of length ℓ. The result now follows from Proposition 3.1. □

Corollary 5.2 *The zeta-regularized determinant of the hemi-equilateral triangle with hypotenuse of length ℓ and with the Dirichlet boundary condition is $e^{-\zeta'_{\heartsuit}(0)}$ with $\zeta'_{\heartsuit}(0)$ equivalently given by*

$$\zeta'_{\heartsuit}(0) = \frac{5}{6}\log(\ell) + \frac{7}{12}\log(3) - \frac{5}{6}\log(2) + \frac{\pi\sqrt{3}}{72} + \frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{ne^{\pi n\sqrt{3}}}\sum_{d|n} d,$$

$$\zeta'_{\heartsuit}(0) = \frac{1}{2}\zeta'_{\nabla}(0) + \frac{1}{4}\log\left(\frac{3\ell^2}{4}\right).$$

Proof The proof follows by differentiating (33) and setting $s = 0$. □

Remark 5.3 We have verified that our expression for the spectral zeta function of the hemi-equilateral triangle in (33) agrees with that given in [4, (108)],

$$\zeta_{\heartsuit}(s) = \frac{1}{2}\left(\frac{9\ell^2}{16\pi^2}\right)^s \left[L_3(s)\zeta_R(s) - (1+3^{-s})\zeta_R(2s)\right].$$

Differentiating the equivalent expression and evaluating at $s = 0$ one can show that it agrees with our expressions above as well as that given in [4, (112)].

5.2 The Heat Trace of Hemi-Equilateral Triangles

For the hemi-equilateral triangle with hypotenuse of length ℓ, we let

$$q = e^{-16\pi^2 t/(9\ell^2)}.$$

By (27),

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{m^2+mn+n^2} = \frac{\Theta_3(q)\Theta_3(q^3) + \Theta_2(q)\Theta_2(q^3) - 3\Theta_3(q) + 2}{6}.$$

On the other hand, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{m^2+mn+n^2} = 2\sum_{m>n\geq 1} q^{m^2+mn+n^2} + \sum_{m=1}^{\infty} q^{3m} = 2H^D_{\heartsuit}(t) + \frac{\Theta_3(q^3)-1}{2}.$$

By comparing, we obtain the heat trace for the Dirichlet boundary condition

$$\begin{aligned} H^D_{\heartsuit}(t) = \sum_{m>n\geq 1} q^{m^2+mn+n^2} &= \frac{1}{2}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q^{m^2+mn+n^2} - \sum_{m=1}^{\infty} q^{3m}\right] \\ &= \frac{\Theta_3(q)\Theta_3(q^3) + \Theta_2(q)\Theta_2(q^3) - 3\Theta_3(q) - 3\Theta_3(q^2) + 5}{12}. \end{aligned}$$

We calculate the short time asymptotic expansion of the heat trace, obtaining further terms and a sharp remainder.

Theorem 5.4 *The heat trace for the hemi-equilateral triangle with hypotenuse of length ℓ with the Dirichlet boundary condition admits the asymptotic expansion*

$$\begin{aligned} H^D_{\heartsuit}(t) = &\frac{\ell^2\sqrt{3}}{32\pi t} - \frac{\ell(3+\sqrt{3})}{16\sqrt{\pi t}} + \frac{5}{12} - \frac{\ell}{8}\sqrt{\frac{3}{\pi t}}e^{-3\ell^2/(16t)} \\ &- \frac{3\ell}{8\sqrt{\pi t}}e^{-9\ell^2/(16t)} + \mathcal{O}(t^{-1}e^{-3\ell^2/(4t)}),\ t \to 0. \end{aligned}$$

The heat trace with the Neumann boundary condition admits the asymptotic expansion

$$H_{\heartsuit}^{N}(t) = \frac{\ell^2\sqrt{3}}{32\pi t} + \frac{\ell(3+\sqrt{3})}{16\sqrt{\pi t}} + \frac{5}{12} + \frac{\ell}{8}\sqrt{\frac{3}{\pi t}}e^{-3\ell^2/(16t)} + \frac{3\ell}{8\sqrt{\pi t}}e^{-9\ell^2/(16t)} + \mathcal{O}(t^{-1}e^{-3\ell^2/(4t)}),\ \ t \to 0.$$

The remainders are sharp.

Proof Since the heat trace of the equilateral triangle and that of the hemi-equilateral triangle are related via

$$H_{\nabla}^{D}(t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-16\pi^2 t/(9\ell^2)(m^2+mn+n^2)} = 2H_{\heartsuit}^{D}(t) + \sum_{m=1}^{\infty} e^{-16\pi^2 t/(3\ell^2)m^2},$$

by the Poisson summation formula

$$\sum_{m=1}^{\infty} e^{-16\pi^2 t/(3\ell^2)m^2} = \frac{1}{2}\left(\frac{\ell}{4}\sqrt{\frac{3}{\pi t}} - 1\right) + \frac{\ell}{4}\sqrt{\frac{3}{\pi t}}\sum_{m=1}^{\infty} e^{-3\ell^2 m^2/(16t)}$$

it follows that

$$H_{\heartsuit}^{D}(t) = \frac{1}{2}\left[H_{\nabla}^{D}(t) - \frac{1}{2}\left(\frac{\ell}{4}\sqrt{\frac{3}{\pi t}} - 1\right) - \frac{\ell}{4}\sqrt{\frac{3}{\pi t}}\sum_{m=1}^{\infty} e^{-3\ell^2 m^2/(16t)}\right].$$

We apply Theorem 3.7 to obtain

$$\begin{aligned} H_{\heartsuit}^{D}(t) = &\frac{\ell^2\sqrt{3}}{32\pi t} - \frac{\ell(3+\sqrt{3})}{16\sqrt{\pi t}} + \frac{5}{12} - \frac{\ell}{8}\sqrt{\frac{3}{\pi t}}\sum_{m=1}^{\infty} e^{-3\ell^2 m^2/(16t)} \\ &- \frac{3\ell}{8\sqrt{\pi t}}\sum_{m=1}^{\infty} e^{-9\ell^2 m^2/(16t)} + \frac{\ell^2\sqrt{3}}{16\pi t}\sum_{n=1}^{\infty} e^{-3\ell^2 n^2/(4t)} \\ &+ \frac{\ell^2\sqrt{3}}{16\pi t}\sum_{m=1}^{\infty} e^{-9\ell^2 m^2/(4t)} + \frac{\ell^2\sqrt{3}}{8\pi t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-3\ell^2(3m^2+n^2)/(4t)} \\ &+ \frac{\ell^2\sqrt{3}}{8\pi t}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-3\ell^2(3(2m-1)^2+(2n-1)^2)/(16t)}. \end{aligned}$$

This proves the theorem in the Dirichlet case. The eigenvalues of the hemi-equilateral triangle with hypotenuse of length ℓ and the Neumann boundary condition are

$$\lambda_{m,n} = \frac{16\pi^2}{9\ell^2}(m^2 + mn + n^2),\ \ m \geq n \geq 0,$$

so the heat trace becomes

$$H_{\heartsuit}^{N}(t) = \sum_{m \geq n \geq 0} e^{-\frac{16\pi^2}{9\ell^2}(m^2+mn+n^2)t} = 1 + \sum_{m=1}^{\infty} e^{-\frac{16\pi^2}{9\ell^2}m^2 t} + \sum_{n=1}^{\infty} e^{-\frac{16\pi^2}{3\ell^2}n^2 t} + H_{\heartsuit}^{D}(t).$$

We have by the Poisson summation formula

$$\sum_{m=1}^{\infty} e^{-16\pi^2 t/(9\ell^2)m^2} = \frac{1}{2}\left(\frac{3\ell}{4\sqrt{\pi t}} - 1\right) + \frac{3\ell}{4\sqrt{\pi t}} \sum_{m=1}^{\infty} e^{-9\ell^2 m^2/(16t)},$$
$$\sum_{n=1}^{\infty} e^{-16\pi^2 t/(3\ell^2)n^2} = \frac{1}{2}\left(\frac{\ell}{4}\sqrt{\frac{3}{\pi t}} - 1\right) + \frac{\ell}{4}\sqrt{\frac{3}{\pi t}} \sum_{n=1}^{\infty} e^{-3\ell^2 n^2/(16t)}.$$

Then

$$\begin{aligned} H_{\heartsuit}^{N}(t) &= \frac{\ell^2\sqrt{3}}{32\pi t} - \frac{\ell(3+\sqrt{3})}{16\sqrt{\pi t}} + \frac{5}{12} - \frac{\ell}{8}\sqrt{\frac{3}{\pi t}} \sum_{m=1}^{\infty} e^{-3\ell^2 m^2/(16t)} \\ &\quad - \frac{3\ell}{8\sqrt{\pi t}} \sum_{m=1}^{\infty} e^{-9\ell^2 m^2/(16t)} + \frac{\ell^2\sqrt{3}}{16\pi t} \sum_{n=1}^{\infty} e^{-3\ell^2 n^2/(4t)} \\ &\quad + \frac{\ell^2\sqrt{3}}{16\pi t} \sum_{m=1}^{\infty} e^{-9\ell^2 m^2/(4t)} + \frac{\ell^2\sqrt{3}}{8\pi t} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-3\ell^2(3m^2+n^2)/(4t)} \\ &\quad + \frac{\ell^2\sqrt{3}}{8\pi t} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-3\ell^2(3(2m-1)^2+(2n-1)^2)/(16t)}. \end{aligned}$$

which proves the theorem in the Neumann case. □

It follows from Theorem 5.4 that

$$H_{\heartsuit}^{D}(t) = \frac{\ell^2\sqrt{3}}{32\pi t} - \frac{\ell(3+\sqrt{3})}{16\sqrt{\pi t}} + \frac{5}{12} + \mathcal{O}(e^{-(3\ell^2-\epsilon)/(16t)}), \quad t \to 0,$$
$$H_{\heartsuit}^{N}(t) = \frac{\ell^2\sqrt{3}}{32\pi t} + \frac{\ell(3+\sqrt{3})}{16\sqrt{\pi t}} + \frac{5}{12} + \mathcal{O}(e^{-(3\ell^2-\epsilon)/(16t)}), \quad t \to 0,$$

for any $\epsilon > 0$. Once again, $3\ell^2/16$ is the square of half the length of the shortest closed geodesic in the hemi-equilateral triangle with hypotenuse of length ℓ as proven in [16, p. 43]. This completes the proof of Theorem 1.1.

6 Heat Traces of Flat tori, Convex Polygonal Domains and a Comparison with Smoothly Bounded Domains

A full-rank lattice $\Gamma \subset \mathbb{R}^n$ is a discrete additive subgroup of the additive group $(\mathbb{R}^n, +)$. In fact, every discrete additive subgroup of $\mathbb{R}^n$ is a lattice, albeit not necessarily full-rank. A full-rank lattice $\Gamma \subset \mathbb{R}^n$ gives rise to a smooth, compact Riemannian manifold known as a flat torus, obtained as the quotient $\mathbb{R}^n/\Gamma$. Its Riemannian metric is inherited from the Euclidean (flat) metric on $\mathbb{R}^n$. The eigenvalues of the Laplacian on the flat torus $\mathbb{R}^n/\Gamma$ are the values $4\pi^2||y||^2$ for all y in the dual lattice Γ^*, with multiplicities counted according to how many distinct y have the same length; recall

$$\Gamma^* = \{y \in \mathbb{R}^n : y \cdot x \in \mathbb{Z},\ \forall x \in \Gamma\}.$$

The Poisson summation formula is the relation (see [11, p. 125])

$$\sum_{\gamma \in \Gamma^*} e^{-4\pi^2 t||\gamma^*||^2} = \frac{\mathrm{vol}(\mathbb{R}^n/\Gamma)}{(4\pi t)^{n/2}} \sum_{\gamma \in \Gamma} e^{-||\gamma||^2/(4t)}.$$

We recognize the left side as the heat trace of the flat torus. Thus, we have the asymptotic expansion

$$\sum_{k \geq 0} e^{-\lambda_k t} = \frac{\mathrm{vol}(\mathbb{R}^n/\Gamma)}{(4\pi t)^{n/2}} \left(1 + m(\gamma_1)e^{-||\gamma_1||^2/(4t)} + \mathcal{O}(e^{-||\gamma_2||^2/(4t)})\right), \quad t \to 0.$$

Above, $\{\lambda_k\}_{k\geq 0}$ are the eigenvalues of the flat torus, $m(\gamma_1)$ is the number of $\gamma \in \Gamma$ of minimal positive length given by $||\gamma_1||$, with the next shortest length given by $||\gamma_2||$. We then observe that the shortest closed geodesic in $\mathbb{R}^n/\Gamma$ has length $||\gamma_1||$. Consequently, the asymptotic expansion of the heat trace consists of the usual leading term, together with a remainder term that is of the form $\mathcal{O}(t^{-n/2}e^{-L^2/(4t)})$ with L the length of the shortest closed geodesic in the flat torus. This leads us to make a conjecture about the short time asymptotic expansion of the heat trace in similarly flat settings.

A compact Riemannian manifold with curvature identically equal to zero is known as a Euclidean space form. The fundamental groups of compact Euclidean space forms are examples of crystallographic groups. These are discrete groups of Euclidean isometries with compact quotients. It is interesting to note that in two dimensions, the fundamental domains of crystallographic groups are precisely the integrable polygonal domains of this study. In two dimensions, all space forms are diffeomorphic to either a flat torus or a Klein bottle. There are 10 diffeomorphism classes of compact 3-dimensional Euclidean space forms, and 75 classes in dimension 4. Every Euclidean space form is a quotient of a flat torus by a finite group of isometries, and in each dimension there are only finitely many diffeomorphism classes of Euclidean space forms, although the complete classification is known only in low dimensions. We refer to [49] and [28] for further details about Euclidean space forms. Due to the vanishing of their curvature, similar to the case of flat tori, we reasonably expect their heat traces to have a similar form.

Conjecture 6.1 *Assume that M is an n-dimensional Euclidean space form. Then its heat trace admits an asymptotic expansion of the form*

$$\sum_{k \geq 0} e^{-\lambda_k t} = t^{-n/2} \left(\frac{\mathrm{vol}(M)}{(4\pi t)^{n/2}} + \mathcal{O}(e^{-L^2/(4t)})\right), \quad t \to 0.$$

Here, L is the length of the shortest closed geodesic in M.

In higher dimensions, strictly tessellating polytopes as defined in [44, Definition 1] are analogous to integrable polygons in dimension two. Indeed, one could reasonably define an integrable polytope to be a strictly tessellating polytope in the sense of [44]. Heuristically, our definition of a polytope is a bounded domain in Euclidean space such that its boundary is piecewise smooth and consists of flat boundary faces. In two dimensions, for example, a polytope is a bounded, connected polygonal domain. We suggest that it is reasonable that all polytopes admit a heat trace expansion that behaves analogously to the two-dimensional case.

Conjecture 6.2 *Assume that M is a polytope in $\mathbb{R}^n$. Then its heat trace with either the Dirichlet or Neumann boundary condition admits an asymptotic expansion of the form*

$$\sum_{k\geq 0} e^{-\lambda_k t} = t^{-n/2}\left(\sum_{j=0}^{n} a_j t^{j/2} + \mathcal{O}(e^{-c/t})\right), \quad t \to 0.$$

The coefficient a_0 is given by $a_0 = (4\pi)^{-n/2}\operatorname{vol}(M)$ with $\operatorname{vol}(M)$ the n-dimensional (Lebesgue) volume of the polytope. The coefficient a_1 can be expressed with a universal constant together with the total $(n-1)$-dimensional volume of the boundary faces of the polytope. Analogously, the coefficients a_j for $2 \leq j \leq n-1$ can be expressed with a universal constant together with the total $(n-j)$-dimensional volume of the $(n-j)$-dimensional intersections of the boundary faces. The coefficient a_n can be expressed in terms of the angles in the polytope and its boundary faces as well as angles between the intersections of these. The supremum over all $c > 0$ such that this remainder estimate holds is $L^2/4$ with L the length of the shortest closed geodesic in M.

The coefficients a_j for $0 \leq j \leq n-1$ are motivated by locality principles [38] that generalize Kac's principle of not feeling the boundary [22, p. 9]. The idea is that on the interior of each $(n-j)$-dimensional subset of the boundary, away from its edges, the heat kernel in M can be modelled as the heat kernel in $\mathbb{R}^{n-j}$. The leading term in the heat trace then comes simply from the $(n-j)$-dimensional volumes of these subsets, together with certain universal constants.

6.1 A Comparison of Heat Trace Invariants of Smoothly Bounded Domains and Polygonal Domains

We conclude with a comparison of the heat trace expansion of smoothly bounded planar domains to that of polygonal domains that need not be integrable. We therefore recall the short time asymptotic expansion of the heat trace in these contexts.

Proposition 6.3 *Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain. For the Dirichlet boundary condition, the heat trace of Ω satisfies*

$$H(t) \sim \frac{a_{-1}}{t} + \frac{a_{-1/2}}{\sqrt{t}} + a_0 + a_{1/2}\sqrt{t}, \ t \to 0,$$

where

$$a_{-1} = \frac{|\Omega|}{4\pi}, \quad a_{-1/2} = -\frac{|\partial\Omega|}{8\sqrt{\pi}}, \quad a_0 = \frac{1}{12\pi}\int_{\partial\Omega} k(s)ds, \quad a_{1/2} = \frac{1}{256\sqrt{\pi}}\int_{\partial\Omega} k(s)^2 ds, \tag{34}$$

with $k(s)$ being the Gauss curvature of the boundary. If in addition Ω is convex, then $a_0 = 1/6$. If instead Ω is a convex n-sided polygon with interior angles $\gamma_1,\ldots,\gamma_n$, then

$$a_0 = \sum_{i=1}^{n} \frac{\pi^2 - \gamma_i^2}{24\pi\gamma_i}. \tag{35}$$

Proof The formulas given by (34) can be found in [48]. Moreover, by [38, Thm. 6.10, Remark 6.15] we have $a_0 = \frac{\chi(\Omega)}{6}$, which equals $1/6$ if Ω is convex. Finally, (35) follows from [38, Thm. 6.10]. □

As a consequence, we will see that the first two heat trace coefficients of a sequence of smoothly bounded convex domains that converge to a convex polygonal domain converge to that of the polygonal domain. However, the third heat trace coefficient does *not* converge to that of the polygonal domain.

Theorem 6.4 *Let $\{\Omega_k\}$ be a sequence of convex smoothly bounded domains in $\mathbb{R}^2$ and let Ω be a convex polygon such that $\Omega_k \to \Omega$ in the Hausdorff distance. For the Dirichlet boundary condition, the heat trace coefficients satisfy*

$$a_j(\Omega_k) \to a_j(\Omega), \quad j = -1, -1/2, \quad a_0(\Omega_k) \not\to a_0(\Omega).$$

Proof With the assumptions of convexity and Hausdorff convergence, it follows that the areas $|\Omega_k|$ and perimeters $|\partial\Omega_k|$ converge to $|\Omega|$ and $|\partial\Omega|$, respectively. So we now consider the third heat trace coefficient. By Proposition 6.3, $a_0(\Omega_k) = 1/6$ for every k. We will show that $a_0(\Omega) > 1/6$, from which the result follows. By Proposition 6.3,

$$\begin{aligned} a_0(\Omega) &= \sum_{i=1}^{n} \frac{\pi^2 - \gamma_i^2}{24\pi\gamma_i} = \frac{\pi}{24}\sum_{i=1}^{n}\frac{1}{\gamma_i} - \frac{1}{24\pi}\sum_{i=1}^{n}\gamma_i \\ &= \frac{\pi}{24}\sum_{i=1}^{n}\frac{1}{\gamma_i} - \frac{1}{24\pi}\pi(n-2) = \frac{\pi}{24}\sum_{i=1}^{n}\frac{1}{\gamma_i} - \frac{n-2}{24}. \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$n^2 \le \sum_{i=1}^{n}\frac{1}{\gamma_i}\sum_{i=1}^{n}\gamma_i = \sum_{i=1}^{n}\frac{1}{\gamma_i}\pi(n-2),$$

so that

$$\sum_{i=1}^{n} \frac{1}{\gamma_i} \geq \frac{n^2}{\pi(n-2)}.$$

Thus,

$$a_0(\Omega) \geq \frac{\pi}{24} \frac{n^2}{\pi(n-2)} - \frac{n-2}{24} = \frac{1}{6} + \frac{1}{6(n-2)} > \frac{1}{6}. \tag{36}$$

□

Using the notation of Theorem 6.4, it follows that

$$\lim_{k\to\infty} a_0(\Omega_k) \neq a_0\left(\lim_{k\to\infty} \Omega_k\right).$$

In other words, the map $\Omega \to a_0(\Omega)$ is not continuous in the Hausdorff topology. Intuitively, this failure arises because the third heat trace coefficient encodes different geometric information in the smooth and polygonal cases. For smooth domains, it depends on the integrated boundary curvature, while for polygons, it depends on the interior angles at the corners. Although a sequence of smooth curves can approximate a corner arbitrarily well in shape, we cannot expect the curves to capture the singular corner contributions appearing in the polygonal coefficient. It is interesting to note that if instead we approximate a smoothly bounded domain by polygonal domains, this third heat trace coefficient of the polygonal domains *converges* to that of the smoothly bounded domain.

Theorem 6.5 (See [29], Thm. 4.4.1) *Let $\{\Omega_k\}$ be a sequence of N_k-sided convex polygons with interior angles $\gamma_{k,j}$, for $k \geq 1$ and $1 \leq j \leq N_k$. Assume that $\overline{\Omega_k} \to \overline{\Omega}$ in Hausdorff, with Ω being a nonempty smoothly bounded convex domain. Then the first three heat trace coefficients of Ω_k converge to those of Ω.*

Proof The first two heat trace coefficients converge thanks to the assumptions of Hausdorff convergence and convexity. By [30, Lemma 4.7], the interior angles $\gamma_{k,j}$ all tend to π as the polygons tend to the smoothly bounded domain in Hausdorff convergence. Next, we show that $N_k \to \infty$ as $k \to \infty$. Suppose instead that there is an $M > 0$ such that $N_k \leq M$ for all k. Since the angles all tend to π, there is an $N \geq 1$ such that $\gamma_{k,j} > \pi - \frac{2\pi}{M}$ for all $k \geq N$ and $1 \leq j \leq N_k$. Then, for $k \geq N$,

$$\pi(N_k - 2) = \sum_{j=1}^{N_k} \gamma_{k,j} > N_k \left(\pi - \frac{2\pi}{M}\right),$$

which implies that $N_k > M$, a contradiction.

Now, the term a_0 for each k is

$$a_0(\Omega_k) = \frac{\pi}{24} \sum_{k=1}^{N_k} \frac{1}{\gamma_{k,j}} - \frac{N_k}{24} + \frac{1}{12}.$$

Following the proof of [29, Thm. 4.4.1], we can write $\gamma_{k,j} = \pi(1 - f(k, j))$, $k \geq 1$, $1 \leq j \leq N_k$, from which it follows that $\sum_{j=1}^{N_k} f(k, j) = 2$ for every k and

$$a_0(\Omega_k) = \frac{1}{6} + \frac{1}{24} \sum_{j=1}^{N_k} \frac{f(k, j)^2}{1 - f(k, j)}.$$

If we then write

$$\epsilon_k = \max_{1 \leq j \leq N_k} f(k, j),$$

then $\epsilon_k \to 0$ because the angles tend to π. We therefore obtain that

$$0 \leq \sum_{j=1}^{N_k} \frac{f(k, j)^2}{1 - f(k, j)} \leq \frac{\epsilon_k}{1 - \epsilon_k} \sum_{j=1}^{N_k} f(k, j) = \frac{2\epsilon_k}{1 - \epsilon_k} \to 0 \text{ as } k \to \infty.$$

Thus, $a_0(\Omega_k) \to \frac{1}{6} = a_0(\Omega)$ as $k \to \infty$. □

6.2 Concluding Remarks

There are numerous modes of geometric convergence for domains, Riemannian manifolds, and more general types of possibly singular spaces. Under different modes of convergence, one can study the behavior of spectral invariants of a sequence and compare to those of the limit space, as long as it is possible to define a Laplace spectrum on the elements of the sequence and also on the limit space. Interestingly, one can define a Laplace spectrum on very singular spaces, including but not limited to noncollapsed limits under Gromov-Hausdorff convergence [9], rough Riemannian manifolds [7], and RCD spaces [2]. In some cases, it is even possible to define notions of curvature, from which one could hope to obtain higher order heat trace invariants. As a first step, one could investigate the convergence of the most elementary spectral invariants: the individual eigenvalues. Convergence of individual eigenvalues under Gromov-Hausdorff convergence to noncollapsed limits of compact manifolds with Ricci curvature bounded below was shown by Cheeger and Colding [9]. In the same setting, the associated heat kernels also converge [14]. However, the convergence of other spectral invariants can be much more subtle, because in essence it could involve several limiting processes that need not commute. In the simple setting of Hausdorff convergence of planar domains, if we remove the assumption of convexity, a quantity as simple as the perimeters of the domains need not converge! There are many interesting problems one could study in the general field of spectral geometry, exploring relationships between the Laplace spectrum and the underlying geometry, and we welcome both newcomers and seasoned researchers to join us in exploring!

Appendix A Estimates

Here we show that certain quantities are bounded and therewith justify our calculations of the zeta-regularized determinants.

Lemma A.1 *For any* $a, b > 0$*, the quantity*

$$\frac{d}{ds}\left[\left(\frac{ab}{\pi}\right)^s \sqrt{\frac{a}{b}} \sum_{n=1}^{\infty} n^{s-1/2} \sum_{d|n} d^{1-2s} \int_0^{\infty} x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx\right].$$

is bounded in a neighborhood of $s = 0$*. In particular,*

$$\lim_{s\to 0} \frac{1}{\Gamma(s)} \frac{d}{ds}\left[\left(\frac{ab}{\pi}\right)^s \sqrt{\frac{a}{b}} \sum_{n=1}^{\infty} n^{s-1/2} \sum_{d|n} d^{1-2s} \int_0^{\infty} x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx\right] = 0.$$

Proof Since

$$\begin{aligned}
&\frac{d}{ds}\left[\left(\frac{ab}{\pi}\right)^s \sqrt{\frac{a}{b}} \sum_{n=1}^{\infty} n^{s-1/2} \sum_{d|n} d^{1-2s} \int_0^{\infty} x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx\right] \\
&= \left(\frac{ab}{\pi}\right)^s \log\left(\frac{ab}{\pi}\right) \sqrt{\frac{a}{b}} \sum_{n=1}^{\infty} n^{s-1/2} \sum_{d|n} d^{1-2s} \int_0^{\infty} x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx \\
&+ \left(\frac{ab}{\pi}\right)^s \sqrt{\frac{a}{b}} \frac{d}{ds}\left[\sum_{n=1}^{\infty} n^{s-1/2} \sum_{d|n} d^{1-2s} \int_0^{\infty} x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx\right],
\end{aligned}$$

it is enough to show that

$$\frac{d}{ds}\left[\sum_{n=1}^{\infty} n^{s-1/2} \sum_{d|n} d^{1-2s} \int_0^{\infty} x^{s-3/2} e^{-\pi na(x+x^{-1})/b} dx\right]$$

is bounded in a neighborhood of zero, say $s \in (-1, 1)$. We will in fact show that we may differentiate termwise and differentiate under the integral sign, from which the lemma will follow. Let

$$f_N(s) = \sum_{n=1}^{N} n^{s-1/2} \sum_{d|n} d^{1-2s} \int_0^{\infty} x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx, \ \ N \geq 1, \ \ s \in (-1, 1).$$

By definition of infinite sums, f_N converges pointwise to

$$f(s) = \sum_{n=1}^{\infty} n^{s-1/2} \sum_{d|n} d^{1-2s} \int_0^{\infty} x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx.$$

Now, we need to show that $f_N'(s)$ converges uniformly to some function. We have

$$f_N'(s) = \sum_{n=1}^{N} n^{s-1/2} \sum_{d|n} d^{1-2s} (\log(n) - 2\log(d)) \int_0^\infty x^{s-3/2} e^{-\pi a n (x+x^{-1})/b} dx$$
$$+ \sum_{n=1}^{N} n^{s-1/2} \sum_{d|n} d^{1-2s} \frac{d}{ds} \left[\int_0^\infty x^{s-3/2} e^{-\pi a n (x+x^{-1})/b} dx \right].$$

To proceed, we want to show that we may differentiate under the integral sign. Let

$$h(s,x) = x^{s-3/2} e^{-\pi a n (x+x^{-1})/b}, \;\; s \in (-1,1), \;\; x \in (0,\infty).$$

Fix s. Since $h(s,x) \to 0$ as $x \to 0$ and $h(s,x)$ decays exponentially as $x \to \infty$, it follows that $h(s,x)$ is Lebesgue-integrable over $x \in (0,\infty)$. Moreover,

$$\frac{\partial h}{\partial s} = \log(x) x^{s-3/2} e^{-\pi a n (x+x^{-1})/b}$$

exists for all $s \in (-1,1)$ and $x \in (0,\infty)$. Finally, let

$$\theta(x) = \begin{cases} \log(x) x^{-5/2} e^{-\pi a n (x+x^{-1})/b}, & 0 < x < 1, \\ \log(x) e^{-\pi a n (x+x^{-1})/b}, & x \geq 1. \end{cases}$$

By construction we have $|\frac{\partial h}{\partial s}| \leq \theta(x)$ for all $s \in (-1,1)$ and $x \in (0,\infty)$, and $\theta(x)$ is Lebesgue-integrable over $x \in (0,\infty)$ by the same arguments as for $h(s,x)$. Thus, it follows from [17, Thm. 2.27] that

$$\frac{d}{ds} \left[\int_0^\infty x^{s-3/2} e^{-\pi a n (x+x^{-1})/b} dx \right] = \int_0^\infty \log(x) x^{s-3/2} e^{-\pi a n (x+x^{-1})/b} dx.$$

Therefore,

$$f_N'(s) = \sum_{n=1}^{N} n^{s-1/2} \sum_{d|n} d^{1-2s} (\log(n) - 2\log(d)) \int_0^\infty x^{s-3/2} e^{-\pi a n (x+x^{-1})/b} dx$$
$$+ \sum_{n=1}^{N} n^{s-1/2} \sum_{d|n} d^{1-2s} \int_0^\infty \log(x) x^{s-3/2} e^{-\pi a n (x+x^{-1})/b} dx.$$

To show that $f_N'(s)$ converges uniformly, we use Weierstrass' M-test. Write $f_N'(s) = \sum_{n=1}^{N} g_n(s)$ where

$$g_n(s) = \sum_{d|n} n^{s-1/2} d^{1-2s} \Bigg[(\log(n) - 2\log(d)) \int_0^\infty x^{s-3/2} e^{-\pi a n (x+x^{-1})/b} dx$$

$$+ \int_0^\infty \log(x) x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx \Bigg].$$

We need to bound $|g_n(s)|$ by some sequence M_n such that $\sum_{n=1}^\infty M_n$ converges. We have

$$\begin{aligned}
|g_n(s)| &\leq \sum_{d|n} n^{s-1/2} d^{1-2s} \Bigg| (\log(n) - 2\log(d)) \int_0^\infty x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx \\
&\quad + \int_0^\infty \log(x) x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx \Bigg| \\
&\leq n^{9/2} \Bigg[3\log(n) \int_0^\infty x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx \\
&\quad + \int_0^\infty |\log(x)| x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx \Bigg].
\end{aligned}$$

To obtain a bound on the first integral, we compute

$$\begin{aligned}
&\int_0^\infty x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx \\
&= \int_0^1 x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx + \int_1^\infty x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx \\
&\leq \int_0^1 x^{-5/2} e^{-\pi an(x+x^{-1})/b} dx + \int_1^\infty e^{-\pi an(x+x^{-1})/b} dx \\
&\leq \int_0^1 x^{-3} e^{-\pi anx^{-1}/b} dx + \int_1^\infty e^{-\pi anx/b} dx \\
&= \frac{b}{\pi an} \left(2 + \frac{b}{\pi an} \right) e^{-\pi an/b}.
\end{aligned}$$

Similarly for the second integral,

$$\begin{aligned}
&\int_0^\infty |\log(x)| x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx \\
&\leq \int_0^1 |\log(x)| x^{-3} e^{-\pi anx^{-1}/b} dx + \int_1^\infty \log(x) e^{-\pi anx/b} dx \\
&\leq \int_0^1 x^{-4} e^{-\pi anx^{-1}/b} dx + \int_1^\infty x e^{-\pi anx/b} dx \\
&= \frac{b}{\pi an} \left(2 + \frac{3b}{\pi an} + \frac{2b^2}{(\pi an)^2} \right) e^{-\pi an/b}.
\end{aligned}$$

Thus,

$$|g_n(s)| \leq n^{9/2} \left[\frac{3b}{\pi a} \left(2 + \frac{b}{\pi an} \right) + \frac{b}{\pi an} \left(2 + \frac{3b}{\pi an} + \frac{2b^2}{(\pi an)^2} \right) \right] e^{-\pi an/b}. \tag{37}$$

In particular, there are constants $C > 0$ and $M \geq 1$ such that $|g_n(s)| \leq Cn^M e^{-\pi an/b}$ for all $n \geq 1$ and $s \in (-1, 1)$. Since

$$\sum_{n=1}^{\infty} Cn^M e^{-\pi an/b}$$

converges, it follows from Weierstrass' M-test that $f_N'(s)$ converges uniformly on $(-1, 1)$. This in turn implies that we can differentiate f termwise (see e.g. [45, Thm. 7.17]), i.e.

$$\begin{aligned}
&\frac{d}{ds}\left[\sum_{n=1}^{\infty} n^{s-1/2} \sum_{d|n} d^{1-2s} \int_0^{\infty} x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx\right] \\
&= \sum_{n=1}^{\infty} n^{s-1/2} \sum_{d|n} d^{1-2s} (\log(n) - 2\log(d)) \int_0^{\infty} x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx \\
&\quad + \sum_{n=1}^{\infty} n^{s-1/2} \sum_{d|n} d^{1-2s} \int_0^{\infty} \log(x) x^{s-3/2} e^{-\pi an(x+x^{-1})/b} dx.
\end{aligned}$$

In particular, we can by (37) conclude that the derivative is bounded for $s \in (-1, 1)$. □

Acknowledgements JR is grateful to Michiel van den Berg, Sebastian Götte, and Klaus Kröncke for insightful suggestions. Both authors thank Norm Hurt for correspondence that inspired this work. We thank Anders Södergren for showing how to use identities in [50] to relate our expressions for the spectral zeta functions to the expressions involving Dirichlet L-series in [4]. We kindly thank the anonymous referees for providing constructive critiques that improved the quality of our work.

Funding Open access funding provided by Chalmers University of Technology. No funding.

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