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The pure spinor superparticle and 10D super-Yang-Mills amplitudes

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ABSTRACT: We present a prescription for computing tree-level scattering amplitudes in 10D super-Yang-Mills (SYM) theory using the pure spinor worldline formalism. The pure spinor formalism has proven to be a powerful framework for studying supersymmetric field theories, providing manifestly covariant and BRST-invariant formulations of amplitudes. By incorporating the worldline approach, we construct a first-quantized representation of SYM amplitudes in 10D, where interactions are encoded through the insertion of vertex operators along the particle's trajectory. We explicitly compute the N-point function, demonstrating an agreement with the $\alpha' \rightarrow 0$ limit of open superstring amplitudes and confirming that the kinematic numerators satisfy the expected BRST relations. Our results establish the pure spinor worldline formalism as a tool for studying scattering amplitudes and suggest further applications to 11D supergravity.

KEYWORDS: BRST Quantization, Scattering Amplitudes, Superstrings and Heterotic Strings

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1 Introduction

The pure spinor formalism for superstrings [1] has proven to be the most powerful and efficient framework for calculating string scattering amplitudes, surpassing other standard formalisms such as the Ramond-Neveu-Schwarz (RNS) and Green-Schwarz (GS) [2–6]. Similarly, the 10D pure spinor superparticle, constructed in [7] as the field theory limit of the pure spinor superstring, as well as its ambitwistorial counterpart [8–10], have been demonstrated to be highly effective for computing 10D super-Yang-Mills (SYM) amplitudes [11–13]. Furthermore, non-minimal pure spinor variables have been used to derive manifestly supersymmetric actions for maximally supersymmetric theories, including 10D SYM and 11D Supergravity [14, 15]. Together, these developments highlight the pure spinor superparticle as a promising framework for computing field-theory scattering amplitudes.

In string theory, amplitudes are typically formulated within a first-quantized framework, where we consider the quantum theory of a single string propagating through spacetime, and interactions are represented by insertions on the worldsheet of this string. In contrast, quantum field theory (QFT) is usually developed within a second-quantized approach, where particles are treated as excitations of a quantum field, and the individual trajectories of particles are not considered. However, the amplitudes of QFT can also be reformulated in a first-quantized manner, as illustrated by Strassler through the worldline formalism

in [16], firstly introduced for one-loop computations and partially extended to treat tree-level amplitudes in [17, 18]. This approach utilizes the path integral of a single particle, with interaction insertions representing the effects of background fields. The worldline formalism is particularly advantageous in scenarios where standard perturbative QFT techniques are unavailable or cumbersome, such as in the case of gravity. In particular, the worldline formalism is an effective tool for studying scattering amplitudes in 10D SYM theory and 11D Supergravity, as these theories can be expressed in a first-quantized framework that manifests supersymmetry.

In this paper, we will use the worldline formalism to compute tree-level scattering amplitudes of 10D SYM using the pure spinor formulation of the superparticle, which exhibits manifest super-Poincaré covariance [7]. The use of the worldline formalism for such calculations is challenging, mainly due to the structure of the contractions and operators associated with the worldline [17]. In this work, we aim to explore these structures within the framework of a BRST-quantized theory. A key aspect of our investigation is the insertion of pinching operators — interpreted as arising from higher-order corrections in the external fields — which are crucial for the BRST invariance of the amplitude. We demonstrate that these operators correspond to the higher-order poles in the OPE between two integrated vertices in string theory, $U(z)U(0)$, and have the role of contact terms in the amplitude, enabling us to express the 10D SYM amplitude in terms of cubic kinematic numerators that explicitly satisfy the generalized Jacobi identities.

The computation of scattering amplitudes within the pure spinor superparticle framework was addressed for the first time in [11], using canonical quantization. In this work, we will perform a similar construction, but using the path integral over the worldline instead of canonical quantization. In particular, we will explicitly produce all the necessary insertions on the worldline, including subtree diagrams, and interpret these as coming from BRST invariance direction on the action.

In [2], the tree-level N -point amplitude of 10D SYM was computed by taking the infinite tension limit of the disk open superstring amplitude. A key contribution of this paper was the identification of the BRST cohomology structure of the kinematical numerators, known as the BRST building blocks, which allow the amplitude to be bootstrapped using BRST invariance arguments. These building blocks naturally emerge from the superstring computation, and in our approach, we show that the same structure appears within the worldline formalism. Remarkably, the ideas presented in this work can be appropriately generalized to 11D for computing tree-level supergravity scattering amplitudes. We discuss these issues further in the accompanying papers [19, 20].

This paper is organized as follows. In section 2, we will review the 10D pure spinor superparticle and its BRST cohomology. The non-minimal pure spinors are introduced to construct the b -ghost, satisfying $\{Q, b\} = \frac{1}{2}P^2$. In section 2.3, we will couple the pure spinor superparticle to a SYM background, as was done in [7]. The insertions needed for the worldline tree amplitude will arise as the multiparticle expansion of the on-shell background fields, and their defining properties will be seen as deriving from the BRST consistency of the interacting action. In this section, the insertions needed for the worldline will be seen to satisfy the same master equations as the Berends-Giele currents of SYM. In section 3,

we will provide a prescription for the 10D SYM tree amplitude in the pure spinor worldline formalism and prove the BRST invariance of the prescription. In sections 3.3, 3.4, and 3.5 we will explicitly compute the 4-point, 5-point, and N-point amplitudes of the 10D SYM, showing that our result agrees with the amplitudes obtained as the low-energy limit of the open superstring in [2, 21, 22].

2 Pure spinor superparticle in 10D

The 10D pure spinor superparticle action in an Euclidean flat background is given by [7]

$$S = \int d\tau \left(\dot{x}^m p_m - \frac{1}{2} p^2 + p_\alpha \dot{\theta}^\alpha + \omega_\alpha \dot{\lambda}^\alpha \right), \quad (2.1)$$

where x^m is SO(10) vector with $m = 1, \dots, 10$, θ^α is a fermionic Majorana-Weyl spinor and λ^α is a bosonic Weyl spinor, with $\alpha = 1, \dots, 16$. The fields p_m , p_α , w_α are the conjugate momenta corresponding to x^m , θ^α , λ^α respectively. The spinor fields λ^α satisfy the pure spinor constraint

$$\lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta = 0, \quad (2.2)$$

where the $\gamma_{\alpha\beta}^m$ and $\gamma_m^{\alpha\beta}$ are the 16×16 symmetric Pauli matrices in 10D, which satisfy $(\gamma^{(m})^{\alpha\delta} \gamma_{\delta\beta}^{(n)}) = 2\eta^{mn} \delta_\beta^\alpha$. The constraints (2.2) generate the pure spinor gauge transformation

$$\delta\omega_\alpha = \Lambda_m (\gamma^m \lambda)_\alpha, \quad (2.3)$$

which leaves the action (2.1) invariant for arbitrary Λ_m . The constraints (2.2) and gauge symmetry (2.3) reduce the 16 complex components of each bosonic ghost to 11. The linear combinations of the ω_α field that are invariant under the pure spinor gauge transformations are the ghost charge current J and Lorentz current N^{mn} , given by

$$J = -\omega_\alpha \lambda^\alpha, \quad N_{mn} = \frac{1}{2} (\omega \gamma_{mn} \lambda). \quad (2.4)$$

Under the action of J , λ^α has ghost number +1 and ω_α has ghost number -1. The superparticle action (2.1) implies the following Euclidean canonical commutators

$$[p_m, x^n] = -\delta_m^n, \quad \{p_\alpha, \theta^\beta\} = \delta_\alpha^\beta, \quad [\omega_\alpha, \lambda^\beta] = -\delta_\alpha^\beta. \quad (2.5)$$

To be more precise, the Poisson brackets and commutators associated with the bosonic ghosts have to be defined in a way that preserves the pure spinor constraint, $[\omega_\alpha, (\lambda \gamma^m \lambda)] = 0$. This implies that $\{\omega_\alpha, \lambda^\beta\}$ has to be corrected by a projection operator K_α^β . For the computations carried over in this work, the existence of K_α^β has no consequence.

We will be interested in computing path integral correlation functions for the superparticle action. It is convenient to write the operator algebra obtained by quantization of (2.1) in terms of the path integral fields. We obtain the following free field contractions

$$\begin{aligned} p_m(\tau_1) x^n(\tau_2) &\sim -\delta_m^n \sigma_{12}, \\ p_\alpha(\tau_1) \theta^\beta(\tau_2) &\sim \delta_\alpha^\beta \sigma_{12}, \\ \omega_\alpha(\tau_1) \lambda^\beta(\tau_2) &\sim -\delta_\alpha^\beta \sigma_{12}, \end{aligned} \quad (2.6)$$

where we introduced the convenient notation $\frac{1}{2}\text{sign}(\tau_i - \tau_j) = \sigma_{ij}$. These contractions are equivalent to the canonical commutation relations. This can be checked by using the map between graded commutators and operator insertions on the path integral, given by

$$[A(\tau), B(\tau)] \sim A(\tau + \epsilon)B(\tau) \mp B(\tau)A(\tau - \epsilon), \quad (2.7)$$

where $\epsilon \rightarrow 0$ and the sign is $-$ for commutators, and $+$ for anticommutators.

The superparticle action is complemented by the BRST operator

$$Q = \lambda^\alpha d_\alpha, \quad (2.8)$$

where $d_\alpha = p_\alpha - \frac{1}{2}(\gamma^m \theta)_\alpha p_m$ are the fermionic constraints of the Brink-Schwarz superparticle [23]. The physical spectrum is defined as the cohomology of the BRST operator (2.8). To find the physical spectrum of the theory, it is convenient to split the BRST cohomology into different ghost number sectors. It can be proven that the cohomology is trivial for ghost numbers larger than 3, so an arbitrary wave function on the cohomology can be expanded as

$$\Psi(x, \theta, \lambda) = \Psi^{(0)} + \Psi^{(1)} + \Psi^{(2)} + \Psi^{(3)}. \quad (2.9)$$

where $\Psi^{(k)}$ has ghost number k . The wave functions on the ghost number sectors 0, 1, 2, and 3 describe, respectively, the ghosts, fields, antifields, and antighosts of the BV formulation of 10D SYM [7], so the pure spinor superparticle furnishes a description of 10 SYM. The ghost number 1 part is the most relevant for our purposes since it corresponds to the physical degrees of freedom of 10D SYM. To see this, one can write $\Psi^{(1)} = \lambda^\alpha A_\alpha$ where A^α is an arbitrary wave function of the matter variables, $A^\alpha = A^\alpha(x, \theta)$. The BRST cohomology implies equations for this wave function

$$Q\Psi^{(1)} = 0, \quad \delta\Psi^{(1)} = Q\Lambda, \quad (2.10)$$

that we recognize respectively as the linearized equations of motion and gauge symmetry of 10D SYM [24]

$$(\gamma^{mnpqr})^{\alpha\beta} D_\alpha A_\beta = 0, \quad \delta A_\alpha = D_\alpha \Lambda, \quad (2.11)$$

where $D_\alpha = \partial_\alpha + \frac{1}{2}(\gamma^m \theta)_\alpha \partial_m$ is the covariant superderivative.

In the usual BRST quantization of theories carrying diffeomorphism invariance, a b -ghost operator exists satisfying $\{Q, b\} = \frac{1}{2}p^2$. This is useful because it allows one to trivialize the BRST cohomology for off-shell fields (fields carrying a non-zero eigenvalue of p^2). Even though the pure spinor superparticle (2.1) is a BRST gauge-fixed action, this theory has no obvious analogous of the b -ghost operator because such an object must carry ghost number -1 . This motivates the enlargement of the action (2.1) through the introduction of a new set of variables, called the non-minimal pure spinor variables, which allows one to define a composite b -ghost operator.

2.1 Non-minimal variables

It was shown in [25] that one can define a composite b -ghost operator for the pure spinor string by adding non-minimal variables to the pure spinor formalism. The same thing can

be done for the pure spinor particle. The set of new variables consists of a bosonic pure spinor $\bar{\lambda}_\alpha$ and a fermionic spinor r_α satisfying the constraints

$$\bar{\lambda}\gamma^m\bar{\lambda} = 0, \quad \bar{\lambda}\gamma^m r = 0. \quad (2.12)$$

These non-minimal variables $\bar{\lambda}_\alpha$ and r^α enter into the enlarged action together with their respective conjugate momenta $\bar{\omega}^\alpha$ and s^α

$$S = \int d\tau \left(\dot{x}^m p_m - \frac{1}{2} p^2 + p_\alpha \dot{\theta}^\alpha + \omega_\alpha \dot{\lambda}^\alpha + \bar{\omega}^\alpha \dot{\bar{\lambda}}_\alpha + s^\alpha \dot{r}_\alpha \right). \quad (2.13)$$

To preserve the BRST cohomology, in such a way that this new set of variables describes the same physics as the action studied in the last subsection, the BRST charge must be modified according to the quartet argument [25]

$$Q = \lambda^\alpha d_\alpha + r_\alpha \bar{\omega}^\alpha. \quad (2.14)$$

Using the non-minimal variables, one can construct the composite b -ghost satisfying

$$\{Q, b\} = \frac{p^2}{2}. \quad (2.15)$$

The explicit form of the b -ghost was initially proposed in [25] and is a complicated expression. The expression for the b -ghost has been disentangled in [26] and a further simplification was proposed in [12] using the so-called physical operators, introduced in [27]. Throughout this paper, we will use this version for the b -ghost, which reads

$$b = -\mathbf{\Delta}_m \mathbf{A}^m, \quad (2.16)$$

where the bold letters denote the so-called physical operators defined in [12] and [27], given by

$$\mathbf{A}_m = \frac{(\bar{\lambda}\gamma_m d)}{2(\lambda\bar{\lambda})} + \frac{(\bar{\lambda}\gamma_{mnp} r)}{8(\lambda\bar{\lambda})^2} N^{np}, \quad (2.17)$$

$$\mathbf{\Delta}_m = -p_m - \frac{(\lambda\gamma^{mn} r)}{4(\lambda\bar{\lambda})} \mathbf{A}_n. \quad (2.18)$$

2.2 Uniqueness in the BRST cohomology

It proves useful to define another operator from the b -ghost field. This is obtained by promoting the canonical momenta in b to first-order differential operators, that is

$$p_m \rightarrow -\partial_m, \quad d_\alpha \rightarrow D_\alpha, \quad \omega_\alpha \rightarrow -\partial_{\lambda^\alpha}, \quad (2.19)$$

as suggested by equation (2.5). Doing this to the canonical momenta appearing in equation (2.16) defines the operator b_0 . It can be thought of as the zero-mode of the Laurent expansion of the string b -ghost, ($b(z) = b_0/z^2 + \dots$), but for the particle. In particular, equation (2.15) implies that the b_0 operator satisfies the BRST cohomology

$$\{Q, b_0\} = \square, \quad (2.20)$$

where \square is the Laplacian $\partial^m \partial_m$. This equation trivializes the BRST cohomology in the space orthogonal to the kernel of the Laplacian. Indeed, suppose that Φ is a BRST closed wave function with non-zero energy, i.e. $\{Q, \Phi\} = 0$ and $\square\Phi = h\Phi$ for $h \neq 0$. These conditions imply that

$$\Phi = \left\{ Q, \frac{b_0(\Phi)}{h} \right\}, \quad (2.21)$$

which establishes that every BRST closed wave function with non-zero energy is also BRST exact. As we will see, this is enough to reformulate the N -point tree-level amplitude as a question on BRST cohomology.

2.3 Superparticle in a SYM background

The worldline prescription depends on the coupling of the particle to an external background. We will perform this construction in this subsection, and we shall see that BRST symmetry implies that the external fields satisfy the equations of motion of multiparticle SYM [24].

The pure spinor superparticle action (2.1) can be coupled to a SYM background by introducing external fields. To describe the color degrees of freedom of SYM, one must also introduce fermions on the worldline. For $\text{SO}(N)$ (or $\text{U}(N)$) gauge group, one needs N real (or complex) fermions. For instance, for $\text{SO}(N)$, the Euclidean action is [7]

$$S^{\text{coupled}} = \int d\tau \left\{ \dot{x}^m \tilde{p}_m - \frac{1}{2} \tilde{p}^2 + \dot{\theta}^\alpha \tilde{p}_\alpha + \dot{\lambda}^\alpha \omega_\alpha - \frac{1}{2} \eta_I \dot{\eta}^I - g \eta_I \eta_J \left(\dot{\theta}^\alpha \mathbb{A}_\alpha^{IJ} + \Pi^m \mathbb{A}_m^{IJ} + \tilde{d}_\alpha \mathbb{W}^{IJ\alpha} + \frac{1}{2} N^{mn} \mathbb{F}_{mn}^{IJ} \right) \right\}, \quad (2.22)$$

where $\Pi^m = \dot{x}^m + \frac{1}{2} (\dot{\theta} \gamma^m \theta)$, and η_I are worldline fermions that encode the color degrees of freedom. For $\text{SO}(N)$, the indices I and J are in the fundamental representation, and the fields \mathbb{A}_α^{IJ} , \mathbb{A}_m^{IJ} , $\mathbb{W}^{IJ\alpha}$, \mathbb{F}_{mn}^{IJ} depend on x^m and θ^α and should be understood as valued in the Lie-algebra. For $\text{SU}(N)$, the indices I and J belong to the fundamental and antifundamental, as appropriate.

The external fields describe multiparticle SYM [28], as we will see shortly. It is convenient to use a basis T_a^{IJ} for the Lie-algebra and redefine the external fields in this basis, e.g. $\eta_I \eta_J \mathbb{A}_\alpha^{IJ} = \eta_I \eta_J T_a^{IJ} \mathbb{A}_\alpha^a$. Defining further $T_a = \eta_I \eta_J T_a^{IJ}$, it is clear that $[T_a, T_b] = i f_{ab}^c T_c$ if we normalize the internal fermions η_I appropriately, where f_{ab}^c are the structure constants of the Lie-algebra. We will omit the Lie-algebra generators and write simply $\mathbb{A}_\alpha \equiv T_a \mathbb{A}_\alpha^a$. We will see that the superfields \mathbb{A}_α , \mathbb{A}_m , \mathbb{W}^α and \mathbb{F}_{mn} satisfy a perturbation expansion where each coefficient is a Berends-Giele current. Performing this change of basis, the action reads

$$S^{\text{coupled}} = \int d\tau \left\{ \dot{x}^m \tilde{p}_m - \frac{1}{2} \tilde{p}^2 + \dot{\theta}^\alpha \tilde{p}_\alpha + \dot{\lambda}^\alpha \omega_\alpha - \frac{1}{2} \eta_I \dot{\eta}^I - g \left(\dot{\theta}^\alpha \mathbb{A}_\alpha + \Pi^m \mathbb{A}_m + \tilde{d}_\alpha \mathbb{W}^\alpha + \frac{1}{2} N^{mn} \mathbb{F}_{mn} \right) \right\}. \quad (2.23)$$

As happens for the free particle, this pure spinor action is complemented by a BRST charge, which is given by

$$Q^{\text{coupled}} = \lambda^\alpha \tilde{d}_\alpha. \quad (2.24)$$

Moreover, the space of physical states is once again identified with the cohomology of the BRST charge.

The fields \tilde{p}_m and \tilde{p}_α appearing in the action (2.23) and BRST charge (2.24) are not equal to the canonical momenta of the corresponding variables. Taking derivatives with respect to the velocities, we can compute the canonical momenta to be

$$\begin{aligned} p_m &= \frac{\partial L}{\partial \dot{x}^m} = \tilde{p}_m - g\mathbb{A}_m, \\ p_\alpha &= \frac{\partial L}{\partial \dot{\theta}^\alpha} = \tilde{p}_\alpha - g \left(\mathbb{A}_\alpha + \frac{1}{2}(\gamma^m \theta)_\alpha \mathbb{A}_m \right). \end{aligned}$$

Further insight is obtained if we transform the action (2.23) to express it in Hamiltonian form by performing a Legendre transformation on the velocities. The Hamiltonian is given by

$$\begin{aligned} H^{\text{coupled}}(x^m, p_m, \theta^\alpha, p_\alpha, \lambda^\alpha, \omega_\alpha) &= (\dot{x}^m p_m + \dot{\theta}^\alpha p_\alpha + \dot{\lambda}^\alpha \omega_\alpha - L) \Big|_{\dot{q}=\dot{q}(p)} \\ &= \frac{1}{2}p^2 + g \left(p^m \mathbb{A}_m + d_\alpha \mathbb{W}^\alpha + \frac{1}{2}N^{mn} \mathbb{F}_{mn} \right) + g^2 \left(\frac{1}{2}\mathbb{A}^2 + \mathbb{A}_\alpha \mathbb{W}^\alpha \right), \end{aligned} \quad (2.25)$$

where $d_\alpha = p_\alpha - \frac{1}{2}(\gamma^m \theta)_\alpha p_m$. The first term of the Hamiltonian is the energy of the free superparticle, $H = \frac{1}{2}p^2$, and the terms proportional to g are the perturbation due to the SYM background. We write $H^{\text{coupled}} = H + \mathbb{U}$, splitting the free and interacting pieces of the Hamiltonian. The object \mathbb{U} will turn out to contain the correct insertions for the worldline amplitude, and plays a similar role to the vertex operators of usual worldsheet string theory. However, as we will see, to describe the full tree-level amplitude, we must allow \mathbb{U} to be non-local. Using equation (2.25), we see that \mathbb{U} can be expanded as

$$\mathbb{U} = g \left(p^m \mathbb{A}_m + d_\alpha \mathbb{W}^\alpha + \frac{1}{2}N^{mn} \mathbb{F}_{mn} \right) + g^2 \left(\frac{1}{2}\mathbb{A}^2 + \mathbb{A}_\alpha \mathbb{W}^\alpha \right). \quad (2.26)$$

The Hamiltonian form of the action is given by

$$S^{\text{coupled}} = S - \int d\tau \mathbb{U}, \quad (2.27)$$

where $S = \int d\tau \left(\dot{x}^m p_m + \dot{\theta}^\alpha p_\alpha + \dot{\lambda}^\alpha \omega_\alpha - \frac{1}{2}p^2 \right)$ is the free superparticle action. It is also convenient to express the BRST charge Q^{coupled} in terms of the canonical momenta. It splits into the free BRST charge $Q = \lambda^\alpha d_\alpha$ and a perturbation term

$$Q^{\text{coupled}} = \lambda^\alpha \tilde{d}_\alpha = \lambda^\alpha d_\alpha + g\lambda^\alpha \mathbb{A}_\alpha = Q + \mathbb{V}, \quad (2.28)$$

where \mathbb{V} is defined as

$$\mathbb{V} = g\lambda^\alpha \mathbb{A}_\alpha. \quad (2.29)$$

The objects \mathbb{U} and \mathbb{V} will turn out to be the generating functions for the Berends-Giele currents of the theory.

The theory is only consistent when the background fields obey the BRST consistency conditions, as follows. The BRST charge must be nilpotent, $\{Q^{\text{coupled}}, Q^{\text{coupled}}\} = 0$. This implies

$$\{Q, \mathbb{V}\} = -\frac{1}{2}\{\mathbb{V}, \mathbb{V}\}. \quad (2.30)$$

Further, the BRST charge must be conserved in time, $[H^{\text{coupled}}, Q^{\text{coupled}}] = 0$. Expanding in terms of the background vertices, we find

$$[Q, \mathbb{U}] = [H, \mathbb{V}] + [\mathbb{U}, \mathbb{V}]. \quad (2.31)$$

These equations are equivalent to the SYM equations of motion for the external SYM fields. Indeed, expanding \mathbb{U} as in (2.26) and \mathbb{V} as in (2.29), the equation that follows from (2.30) is

$$(\gamma^{mnpqr})^{\alpha\beta} (D_\alpha \mathbb{A}_\beta + \{\mathbb{A}_\alpha, \mathbb{A}_\beta\}) = 0, \quad (2.32)$$

while (2.31) implies the equations

$$[\nabla_\alpha, \mathbb{A}_m] = [\partial_m, \mathbb{A}_\alpha] + (\gamma_m \mathbb{W})_\alpha, \quad (2.33)$$

$$\{\nabla_\alpha, \mathbb{W}^\beta\} = \frac{1}{4} (\gamma^{mn})_\alpha^\beta \mathbb{F}_{mn}, \quad (2.34)$$

$$[\nabla_\alpha, \mathbb{F}_{mn}] = \nabla_{[m} (\gamma_n] \mathbb{W})_\alpha, \quad (2.35)$$

where we defined the covariant derivatives $\nabla_\alpha = D_\alpha + \mathbb{A}_\alpha$, with $D_\alpha = \partial_\alpha + \frac{1}{2}(\theta\gamma^m)_\alpha \partial_m$, and $\nabla_m = \partial_m + \mathbb{A}_m$. These imply the equations of motion of 10D non-Abelian SYM [24, 28–30].

These equations can be solved order by order on the SYM coupling g using the perturbative method of Selivanov [24, 31, 32]. In this approach, one expands the fields in terms of multiparticle asymptotic states. Each asymptotic particle i is proportional to a plane wave $e^{ik_i \cdot x}$ and a color factor T^i .

To organize the expansion, we define words P containing letters from the natural numbers (eg. $P = 1253$), that might also stand for bracketing of words (eg. $P = [1, 4] = 14 - 41$) [24]. If P is a word containing the letters $P_1 \dots P_m$, then we define the corresponding momentum $k_P = k_{P_1} + \dots + k_{P_m}$. The Lie-algebra generator with a word as superscript, T^P , is defined as $T^P = T^{P_1} \dots T^{P_m}$. Each natural number is used as the index of an external state. We expand the superfield \mathbb{A}^α in terms of plane wave states

$$\mathbb{A}_\alpha = \sum_P \mathcal{A}_{\alpha P} T^P e^{k_P x} = \mathcal{A}_{\alpha i} T^i e^{k_i x} + \mathcal{A}_{\alpha ij} T^{ij} e^{k_{ij} x_{ij}} + \dots, \quad (2.36)$$

and similarly for \mathbb{A}_m , \mathbb{W}^α and \mathbb{F}^{mn} . The coefficients in the expansion are the Berends-Giele currents of SYM. We define the rank of an object with subscript P as $|P|$, where $|P|$ is the number of letters in the word P . The rank is also the number of particles in the respective state (e.g. $\mathcal{A}_{\alpha 12}$ has rank 2 and represents two-particle asymptotic states of SYM).

Collecting the non-Abelian SYM equations (2.32)–(2.35) order by order in the coupling g and the Lie-algebra generators T^P , one finds the equations that each of the rank- $|P|$ coefficients must satisfy. From now onwards, we also fix $g = 1$ after performing the expansion, to simplify the notation. The solution depends on the gauge choice. A convenient choice is the Lorenz gauge ($[\partial_m, \mathbb{A}^m] = 0$), where the coefficients can be determined recursively. For \mathcal{A}_α^P , for instance, one finds

$$\mathcal{A}_\alpha^P = \frac{1}{s_P} \sum_{RQ=P} \mathcal{A}_\alpha^{[RQ]}, \quad (2.37)$$

where

$$\mathcal{A}_\alpha^{[PQ]} = -\frac{1}{2}[\mathcal{A}_\alpha^P(k^P \cdot \mathcal{A}^Q) + \mathcal{A}_m^P(\gamma^m \mathcal{W}^P)_\alpha - (P \leftrightarrow Q)] \quad (2.38)$$

and $s_P = \frac{1}{2}k_P^2$. The notation $RQ = P$ refers to the deconcatenation sum of the word P — we sum over all *non-empty* words R and Q such that the union of the letters of R and Q , ordered such that R is to the left, gives P . For instance, if $P = 123$, then the tuple (R, Q) takes the two values $(1, 23)$ and $(12, 3)$. The full solution for the Berends-Giele currents in this gauge can be found in [33]. To achieve our goal, all we need to know is that the Berends-Giele constructed using the recursion (2.38) (and similarly for the remaining fields) are such that the action (2.27) is BRST consistent in the sense of equations (2.31) and (2.30).

There is a point we must stress here. Since the currents are non-local (due to the factors of $1/s_P$), this multiparticle action will also carry non-local information, which is ultimately what allows one to describe subtree diagrams in this worldline formalism.

We now turn to study general properties of the operators defined in (2.26) and (2.29), without direct reference to their SYM components. The perturbative expansion of the SYM fields implies perturbative expansions for \mathbb{U} and \mathbb{V} . In the latter case, we find

$$\mathbb{V} = \sum_P V_P T^P = V_i T^i + V_{ij} T^{ij} + \dots \quad (2.39)$$

The V_P are Berends-Giele currents, which we will also refer to as unintegrated vertex currents. For \mathbb{U} , the second term in equation (2.26) is of order four in the internal fermions of the worldline, which means that we must expand (e.g. for $SU(N)$ group)

$$\mathbb{U} = \sum_P U_P T^P e^{k_P x} + \sum_{PQ} D_{PQ} (T^P \odot T^Q) e^{k_{PQ} x}, \quad (2.40)$$

where \odot is the symmetric tensor product. U_P are the integrated vertex currents, and D_P will be referred to as contact terms or *pinching operators*. We say that an object with P as a subscript is of rank- $|P|$, where $|P|$ is the number of letters in the word. The operators with a single index (U_i and V_i) represent single-particle states. The operators with two anti-symmetric indices (U_{ij} and V_{ij}) are two-particle states. Note that the coefficients V_P and U_P are non-local, since they can be written in terms of the SYM Berends-Giele currents defined previously. D_P generally has less propagators than the other objects, which justifies its interpretation as a contact term.

The second-order vertices represent quadratic effects equivalent to interaction with states containing two particles. In the string, we only need the rank-1 operators and can drop all of the higher-rank terms, due to conformal symmetry on the worldsheet. This is not possible for the particle. The amplitude is only BRST invariant after adding all of the higher-rank vertices, as we will see.

The equations obeyed by the vertices at each rank are obtained by matching powers of the SYM coupling and the Lie-algebra generators in (2.30) and (2.31), as we did for the SYM fields. Plugging in equation (2.39) in equation (2.30), one finds the following equations for the unintegrated vertex up to quadratic order (the exponential factors are implicit)

$$\{Q, V_i\} = 0, \quad (2.41)$$

$$\{Q, V_{ij}\} = -V_i V_j. \quad (2.42)$$

Plugging (2.40) into (2.31), one finds that up to quadratic order, the integrated vertices satisfy

$$[Q, U_i] = [H, V_i], \tag{2.43}$$

$$[Q, U_{ij}] = [H, V_{ij}] + U_i V_j - U_j V_i, \tag{2.44}$$

$$[Q, D_{ij}] = \frac{1}{2} ([U_i, V_j] + [U_j, V_i]). \tag{2.45}$$

The linear equations (2.41) and (2.43) are the usual equations of motion of the vertices of pure spinor SYM, while the equations for V_{ij} and U_{ij} give the two-particle SYM equations of motion [24, 28]. More generally, the equations obtained by the multiparticle vertices are

$$\{Q, V_P\} = - \sum_{R+Q=P} V_R V_Q, \tag{2.46}$$

$$[Q, U_P] = [H, V_P] + \sum_{RQ=P} U_R V_Q - U_Q V_R, \tag{2.47}$$

while the pinching operator satisfies

$$[Q, D_P] = \frac{1}{2} \sum_{RQ=P} [U_R, V_Q] + [U_Q, V_R]. \tag{2.48}$$

The deconcatenation sum $RQ = P$ was defined below equation (2.38). Note that (2.46) is exactly the equation satisfied by Berends-Giele currents.

2.4 Matching to string theory OPEs

There is a match between these operators and the objects that one encounters in the string amplitude, which we will now review. Remember that the basic objects of pure spinor string theory are the vertex operators $U^s(z)$ (conformal weight one and ghost number zero) and $V^s(z)$ (conformal weight zero and ghost number one). Both carry plane wave factors $e^{k \cdot X(z)}$ (in our conventions k is purely imaginary), which produce the Koba-Nielsen factor. In this subsection we use the superscript s to differentiate the string vertex operators from the objects we introduced before. We employ the OPE conventions of [2]. The string vertex operators are constrained to satisfy the BRST cohomology

$$\{Q, V^s\} = 0, \quad \{Q, U^s\} = \partial V^s. \tag{2.49}$$

One should notice the similarity between (2.49) and the equations obeyed by the single-particle vertices of the particle, (2.41) and (2.43). Indeed, there is a map between the vertex operators of the string (and the OPEs constructed from these objects) and the particle vertex currents introduced above, that we now describe. We will focus on the rank-two objects, which are simple to study.

The OPE between two integrated vertices in string theory is given by (we suppose that $z_2 > z_1$ and factor out the contribution to the Koba-Nielsen factor)

$$U_1^s(z_1) U_2^s(z_2) \sim (z_2 - z_1)^{-k_1 \cdot k_2} \left(\frac{2(1 + s_{12})}{z_{12}^2} D_{12}^s(z_2) + \frac{1}{z_{12}} U_{12}^s(z_2) \right). \tag{2.50}$$

where D_{12}^s is the double pole and $U_{12}^s(z_2)$ is the single pole. We have factorized $2(1 + s_{12})$ from the double pole for later convenience. The U_{12}^s operator represents a two-particle state coupling to the worldsheet, so we imagine it should be related to the vertex current U_{12} defined above. This is the right intuition.

To derive the equations for $[Q, D_{12}^s]$ and $[Q, U_{12}^s]$, we will compute the action of the BRST charge around an insertion containing $U_1^s(z_1)$ and $U_2^s(z_2)$. There are two ways we can perform the computation — by either acting with Q first and then taking the OPE, or first taking the OPE and then acting with Q .

In the first case, after acting with Q on $U_1^s U_2^s$ we obtain $\partial_1 V_1^s(z_1) U_2^s(z_2) + U_1^s(z_1) \partial_2 V_2^s(z_2)$. The OPE of this last expression might be computed with ease if we first introduce the multiparticle field L_{12} . This is the singular piece of the $U^s V^s$ OPE (in this subsection, singular means more singular than the Koba-Nielsen factor), defined such that

$$U_1^s(z_1) V_2^s(z_2) = (z_2 - z_1)^{-k_1 \cdot k_2} \left(\frac{1}{z_{12}} L_{12}(z_1, z_2) + : U_1^s(z_1) V_2^s(z_2) : \right) \quad (2.51)$$

$$\sim (z_2 - z_1)^{-k_1 \cdot k_2} \left(\frac{1}{z_{12}} L_{12}(z_2, z_2) + \partial_1 L_{12}(z_1, z_2)|_{z_1=z_2} + : U_1^s(z_2) V_2^s(z_2) : \right) \quad (2.52)$$

Explicitly, L_{12} can be written in terms of the SYM fields as

$$L_{12}(z_1, z_2) =: \left[- (k_2 \cdot A_1(z_1)) V_2(z_2) + W_1^\alpha(z_1) D_\alpha V_2(z_2) - \frac{1}{4} (\lambda(z_2) \gamma_{mn} A_2(z_2)) \mathcal{F}_1^{mn}(z_1) \right] e^{k_1 \cdot X(z_1) + k_2 \cdot X(z_2)} :. \quad (2.53)$$

Now, if we take the derivative with respect to z_2 and keep only divergent pieces, we find that

$$\begin{aligned} U_1^s(z_1) \partial_2 V_2^s(z_2) &\sim \frac{k_1 \cdot k_2}{z_{12}} (z_2 - z_1)^{-k_1 \cdot k_2} \left(\frac{1}{z_{12}} L_{12}(z_2, z_2) + \partial_1 L_{12}(z_1, z_2)|_{z_1=z_2} \right. \\ &+ : U_1^s(z_2) V_2^s(z_2) : \left. \right) + (z_2 - z_1)^{-k_1 \cdot k_2} \left(\frac{1}{z_{12}^2} L_{12}(z_2, z_2) + \frac{1}{z_{12}} \partial_2 L_{12}(z_2, z_2) \right) \\ &= (z_2 - z_1)^{-k_1 \cdot k_2} \left(\frac{1 + s_{12}}{z_{12}^2} L_{12}(z_2, z_2) + \frac{1}{z_{12}} \partial_2 L_{12}(z_2, z_2) \right. \\ &\left. + \frac{s_{12}}{z_{12}} [\partial_1 L_{12}(z_1, z_2)|_{z_1=z_2} + : U_1^s(z_2) V_2^s(z_2) :] \right) \end{aligned}$$

We recall that $s_{12} = k_1 \cdot k_2$. We can perform the same computation for $\partial_1 V_1^s(z_1) U_2^s(z_2)$. If we then match with the double pole obtained after applying Q in (2.50), we see that

$$Q D_{12}^s = \frac{1}{2} (L_{12} + L_{21}), \quad (2.54)$$

which is precisely analogous to (2.45). This leads us to split L_{12} into its symmetric and antisymmetric pieces. Collecting now the single pole, one finds that

$$Q \frac{U_{12}^s}{s_{12}} = \partial \frac{1}{s_{12}} (L_{12} - L_{21}) + : U_1^s V_2^s : - : V_1^s U_2^s : + Q(\dots). \quad (2.55)$$

This is analogous to (2.44) up to the BRST exact piece of the right-hand side, which is proportional to QD_{12}^s after using (2.54). In other words, we can map V_{12} to $\frac{(L_{12}-L_{21})}{s_{12}}$ and U_{12} to $\frac{U_{12}^s}{s_{12}}$ up to contact terms. In particular, we see that it is possible to construct the numerators of the particle vertex currents from the string theory OPEs. Note that it is natural that the equation for U_{12}^s does not exactly reproduce (2.44), since the definition of the string OPE depends on what point we choose to perform the Laurent expansion, and the string OPE also contains information about massive state excitations. Furthermore, we note that if one produces a new set of gauge-equivalent vertices $U^{s'}$ and $V^{s'}$ such that the corresponding L'_{ij} is antisymmetric, then the BRST cohomology of the single pole of the $U^{s'}U^{s'}$ OPE exactly reproduces (2.44).

In the string amplitude computation, it can be shown that the BRST trivial piece $L_{(12)}$ cancels — roughly, the double pole piece D_{12}^s can be incorporated into the L_{ij} to update $L_{ij} \rightarrow L_{[ij]}$ [2]. In this way, the string amplitude can be written solely in terms of $L_{[12]}$. This generalizes to more than two particles. The single poles of the string are (up to contact terms and BRST transformations) equal to the numerators of the Berends-Giele currents, while the higher-order poles are the contact terms. The interpretation of the worldline computation will be similar. Namely, the particle pinching operator D_{ij} will act as a contact term in the amplitude that allows one to enhance the BRST properties of the kinematical numerators. As in string theory, the cancellation of these contact terms can be understood as coming from integration by parts identities.

2.5 Multiparticle worldline insertions

In this subsection, we will furnish expressions different expressions for the worldline insertions. We start by giving the expression in terms of the SYM fields. First, we quote the results of [24] for the SYM Berends-Giele currents in Lorenz gauge. These are given in terms of the multiparticle SYM fields [24], which are the operators written in usual roman letters. At rank one, $\mathcal{A}^\alpha = A^\alpha$, and similarly for other fields. At rank two,

$$\mathcal{A}_\alpha^{12} = -\frac{1}{2s_{12}} [A_\alpha^1(k^1 \cdot A^2) + A_m^1(\gamma^m W^2)_\alpha - (12)], \tag{2.56}$$

$$\mathcal{A}_m^{12} = -\frac{1}{2s_{12}} [A_m^1(k^1 \cdot A^2) + A_n^1 F_{mn}^2 - (W^1 \gamma_m W^2) - (12)],$$

$$\mathcal{W}_{12}^\alpha = -\frac{1}{2s_{12}} [W_1^\alpha(k_1 \cdot A_2) + W_1^{m\alpha} A_m^2 + \frac{1}{2}(\gamma^{rs} W^1)^\alpha F_{rs}^2 - (12)],$$

$$\mathcal{F}_{12}^{mn} = -\frac{1}{2s_{12}} [F_1^{mn}(k_1 \cdot A_2) + F_1^{p[mn} A_p^2 + 2F_1^{mp} F_{2p}^n + 4\gamma_{\alpha\beta}^{[m} W_1^{n]\alpha} W_2^\beta - (12)].$$

Using these fields, we can construct the multiparticle worldline insertions,

$$V_1 = \lambda^\alpha \mathcal{A}_{1\alpha}, \tag{2.57}$$

$$V_{12} = \lambda^\alpha \mathcal{A}_{12\alpha}. \tag{2.58}$$

We can also construct the worldline insertions and pinching operators. Comparing equations (2.26) and (2.40), we find

$$U_1 = p^m \mathcal{A}_{1m} + d_\alpha \mathcal{W}_1^\alpha + \frac{1}{2} N^{mn} \mathcal{F}_{1mn}, \quad (2.59)$$

$$U_{12} = p^m \mathcal{A}_{12m} + d_\alpha \mathcal{W}_{12}^\alpha + \frac{1}{2} N^{mn} \mathcal{F}_{12mn}, \quad (2.60)$$

$$D_{12} = \frac{1}{2} (\mathcal{A}_1 \cdot \mathcal{A}_2 + \mathcal{W}_1 \mathcal{A}_2 + \mathcal{W}_2 \mathcal{A}_1). \quad (2.61)$$

Note that the worldline insertions are proportional to inverse powers of the Mandelstam variables and thus non-local. This procedure can be carried over to a higher number of external particles, and the BRST consistency conditions can be checked explicitly using the SYM equations of motion. We also note that D_{12} has less propagators than the other objects at rank two — this again generalizes to more particles.

The previous equations give one set of solution to the consistency conditions, but this solution is not unique. Indeed, the worldline insertions are gauge-dependent objects, and it is useful to exploit this gauge freedom to introduce other representations of V_P and U_P . These will be constructed with the b -ghost operator introduced in section 2.1. First, we notice that the equations (2.46) mean that the unintegrated vertices V_P are Berends-Giele currents [32, 34]. The natural ansatz for V_P is

$$V_P = \frac{1}{s_P} \sum_{RQ=P} [U_R, V_Q], \quad (2.62)$$

which determines the higher-rank V_P in terms of lower point operators. We can check that this satisfies the correct consistency equation (2.46). Acting with the BRST charge on (2.62) and using equation (2.47), we find

$$\begin{aligned} \{Q, V_P\} &= \frac{1}{s_P} \sum_{RQ=P} \left\{ [[H, V_R], V_Q] + \sum_{R_1 R_2=R} [U_{R_1} V_{R_2}, V_Q] - [V_{R_1} U_{R_2}, V_Q] - \sum_{Q_1 Q_2=Q} [U_R, V_{Q_1} V_{Q_2}] \right\} \\ &= \frac{1}{s_P} \sum_{RQ=P} \left\{ -k_R \cdot k_Q V_R V_Q - \sum_{R_1 R_2=R} V_{R_1} [U_{R_2}, V_Q] - \sum_{Q_1 Q_2=Q} [U_R, V_{Q_1}] V_{Q_2} \right\} \\ &= -\frac{1}{s_P} \left\{ \sum_{RQ=P} k_R \cdot k_Q V_R V_Q + \sum_{RQ_1 Q_2=P} V_R [U_{Q_1}, V_{Q_2}] + \sum_{R_1 R_2 Q=P} [U_{R_1}, V_{R_2}] V_Q \right\}. \quad (2.63) \end{aligned}$$

To obtain the second equality, we used $[H, V_R] = \partial_\tau V_R = (k_R \cdot P) V_R$, so that $[[H, V_R], V_Q] = -k_R \cdot k_Q V_R V_Q$, and also applied the Jacobi identity for the remaining terms in the first and second line. If we now plug in the iterative definition (2.62) into (2.63), we see that

$$\begin{aligned} \{Q, V_P\} &= -\frac{1}{s_P} \sum_{RQ=P} \left\{ k_R \cdot k_Q + s_R + s_Q \right\} V_R V_Q \\ &= -\sum_{RQ=P} V_R V_Q, \quad (2.64) \end{aligned}$$

where we remember that $s_P = k_P^2/2$. We have also used the identity $k_P^2 = (k_R + k_Q)^2 \Rightarrow s_P = k_R \cdot k_Q + s_R + s_Q$. This establishes that the recursive relation (2.62) satisfies the consistency condition (2.46).

Using the b -ghost, it is also possible to construct a recursion for the integrated vertices. For single particles, we define

$$\tilde{U}_i = [b, V_i]. \tag{2.65}$$

We can use equation (2.16) to furnish an expression for \tilde{U}_i in terms of the physical operators

$$\tilde{U}_i = -\hat{\mathbf{A}}_m(V_i)\Delta^m + \hat{\Delta}^m(V_i)\mathbf{A}_m. \tag{2.66}$$

It is possible to establish that \tilde{U}_i is related to U_i of equation (2.60) by a BRST-exact piece [12, 35]. Further, one can easily check that such an object satisfies the BRST consistency (2.43) by using the equation $\{Q, b\} = H$. Likewise, we define recursively

$$\tilde{V}_P = \frac{1}{s_P} \sum_{RQ=P} [\tilde{U}_R, \tilde{V}_Q], \tag{2.67}$$

$$\tilde{U}_P = [b, \tilde{V}_P] = -\hat{\mathbf{A}}_m(\tilde{V}_P)\Delta^m + \hat{\Delta}^m(\tilde{V}_P)\mathbf{A}_m. \tag{2.68}$$

All of these fields satisfy the BRST consistency conditions necessary for the consistency of the coupled action. Using the tilded insertions, no pinching operators are necessary. Indeed, the equation for the pinching operator say that its BRST variation is the symmetric piece of the multiparticle field,

$$[Q, \tilde{D}_P] = \frac{1}{2} \sum_{RQ=P} ([\tilde{U}_R, \tilde{V}_Q] + [\tilde{U}_Q, \tilde{V}_R]). \tag{2.69}$$

But for the tilded insertions, one can easily see that $[\tilde{U}_R, \tilde{V}_Q] = -\hat{\mathbf{A}}_m(\tilde{V}_R)\hat{\Delta}^m(\tilde{V}_Q) + \hat{\Delta}^m(\tilde{V}_R)\hat{\mathbf{A}}_m(\tilde{V}_Q)$ is antisymmetric in $R \leftrightarrow Q$ (\mathbf{A}_m and V_P are fermionic, while Δ_m is bosonic). Thus the r.h.s. of the last equation vanishes, and $[Q, \tilde{D}_P] = 0$. But this is defining equation on \tilde{D}_P — it is the only constraint we need to impose on the pinching operators to define a BRST invariant amplitude. So we can simply take $\tilde{D}_P = 0$ without losing any information.

We can produce another yet set of insertions that are given in terms of the b_0 operator, which is the differential operator obtained by performing the substitution (2.19) to the b -ghost field (2.16). We first go to Siegel gauge by imposing $b_0(V'_i) = 0$. If we now define $U'_i = [b, V'_i]$, we can check that

$$[U'_i, V'_j] = b_0(V'_i V'_j). \tag{2.70}$$

The same thing holds true if we recursively construct the higher rank vertices as

$$V'_P = \frac{1}{s_P} \sum_{RQ=P} b_0(V'_R V'_Q), \tag{2.71}$$

$$U'_P = [b, V'_P]. \tag{2.72}$$

Since $b_0(V'_R V'_Q) = [U'_R, V'_Q]$ in the Siegel gauge, it is clear that this set of multiparticle vertices satisfy the BRST consistency conditions. Moreover, since the b_0 operator is nilpotent, the unintegrated vertices satisfy

$$b_0(V'_P) = 0. \tag{2.73}$$

for all P .

We can also study the pinching operators in the Siegel gauge. One finds

$$[Q, D'_P] = \frac{1}{2} \sum_{RQ=P} b_0(V'_R V'_Q) + b_0(V'_Q V'_R) = 0, \tag{2.74}$$

which means that we can take $D'_P = 0$. Here, we have used the anti-commutativity of the V_P currents.

Using uniqueness in the BRST cohomology as in subsection (2.2), one can show that any two sets of worldline insertions satisfying the consistency conditions are related rank by rank by BRST-exact pieces and total derivatives. We will use this to show that the worldline amplitude prescription does not depend on which set of insertions one uses. The operators constructed with the b -ghost will prove particularly useful.

3 Tree-level amplitude in 10D

In this section, we will formulate a prescription to construct the tree-level scattering amplitudes of 10D $\mathcal{N} = 1$ SYM using the pure spinor superparticle, and then check that we obtain results that agree with the $\alpha' \rightarrow 0$ limit of the open superstring.

In SYM, tree-level amplitudes can be decomposed into color-ordered amplitudes. The N -point amplitude A_N can be written as

$$A_N(1, 2, \dots, N) = g^{N-2} \sum_{\sigma(1, \dots, N)/\mathbb{Z}_N} \text{Tr}(T^{\sigma_1} \dots T^{\sigma_N}) \mathcal{A}_N(\sigma_1, \sigma_2, \dots, \sigma_N),$$

where g is the SYM coupling constant, the T^a are a basis of the Lie-algebra, and $\sigma(1, \dots, N)$ are the non-cyclic permutations of the external indices $\{1, \dots, N\}$. The factor $\mathcal{A}_N(\sigma_1, \dots, \sigma_N)$ is the color-ordered amplitude, and only receives contributions from sub-diagrams in which the external legs are arranged in the order $\sigma_1, \dots, \sigma_N$.

In the worldline formulation, the correlator $\mathcal{A}_N(\sigma_1, \dots, \sigma_N)$ is written as a path integral over the fields defined on the particle worldline. The construction begins with the choice of the worldline, which can be the line connecting any pair of external states. These are interpreted as the initial and final states of the superparticle. The remaining legs are then represented by vertices (V_P and U_P) coupling to the worldline, and can be seen as interactions of the superparticle with the SYM background. The amplitude is completely determined from the vertices and the worldline Green function, without the need to sew lower-point tree graphs to the worldline [17]. We note however that the inserted vertices, being generally non-local, naturally contain the information of lower-point tree graphs.

The two particles that define the worldline are represented by unintegrated vertex operators located at the endpoints of the worldline, $\tau = \pm\infty$. Due to the invariance of the worldline under global translations of τ , one of the remaining insertions must also be represented by an unintegrated vertex. The amplitude does not depend on the position of this third unintegrated insertion, and we fix it to be at $\tau = 0$. The remaining particles are represented by integrated vertices, whose proper time is integrated over the worldline. The amplitude \mathcal{A}_N is given by the sum of all amplitudes with N external states in the correct ordering. In particular, the interval of integration of the integrated vertices is such that this ordering is respected.

We will suppose that the external states are ordered clockwise as $1, 2, \dots, N$. When using this ordering, it is convenient to choose the worldline to be the line connecting 1 and $N - 1$, and also fix the position of the insertion V_N (this is convenient because it ensures that we will not need to insert any multiparticle unintegrated vertices on the worldline). The propagation of the superparticle in the SYM background is described by the path integral

$$\begin{aligned} \mathcal{A} &= \left\langle V_1(\infty) \left\{ \int d\tau \mathbb{U} \right\} V_{N-1}(-\infty) \right\rangle \\ &= \sum \frac{1}{(N-2)!} \left\langle V_1(\infty) \left(\int d\tau \mathbb{U} \right)^{N-3} V_N(0) V_{N-1}(-\infty) \right\rangle, \end{aligned} \quad (3.1)$$

where \mathbb{U} is the multiparticle superfield (2.26), described in the last section, and we have used the residual translation symmetry of the worldline to fix the N th insertion as an unintegrated vertex. Expanding \mathbb{U} into multiparticle states and restricting the sum to obey the correct number of external states and ordering, we find the color-ordered amplitude \mathcal{A}_N as

$$\begin{aligned} \mathcal{A}_N &= \sum_{|R_1|+\dots+|R_k|=N-3} \int d\tau_1 \dots d\tau_k \\ &\times \langle V_1(\infty)(U_{R_1} + D_{R_1})(\tau_1) \dots (U_{R_k} + D_{R_k})(\tau_k) V_N(0) V_{N-1}(-\infty) \rangle, \end{aligned} \quad (3.2)$$

where U_P is the multiparticle integrated vertices and D_R are the pinching operators, as defined in section 2.

This correlator is evaluated by first contracting all momenta, using the worldline contractions (2.6). This has the effect of promoting the canonical momenta to differential operators of the zero mode variables, up to the factors of $\sigma_{ij} = \frac{1}{2} \text{sign}(\tau_i - \tau_j)$

$$p_m \rightarrow -\partial_m, \quad d_\alpha \rightarrow D_\alpha, \quad \omega_\alpha \rightarrow -\partial_{\lambda^\alpha}. \quad (3.3)$$

After that, one must perform the integration over the zero modes. One advantage of using the Hamiltonian action is that since there are no insertions of \dot{x}^m on the worldline, one does not have to worry about possible ill-defined products of Dirac deltas.

3.1 Integration over zero modes

After getting rid of the canonical momenta, one still needs to integrate over x^m , θ^α and λ^α to obtain a number out of equation (3.2). The integration over x^m furnishes a Koba-Nielsen-like measure on the space of τ_i , as follows.

After contracting p_m , the amplitude (3.2) is proportional to the exponential factor

$$\mathcal{A}_N \propto \int \mathcal{D}x \left\{ - \int d\tau p_m \dot{x}^m \right\} \prod_P \{ k_P \cdot x(\tau_P) \}. \quad (3.4)$$

This integral can be easily performed if we collect the plane wave factors under the integral, by writing $k_j \cdot x(\tau_j) = \int d\tau \delta(\tau - \tau_j) k_j \cdot x(\tau)$. The integral over the x^m zero mode $x^m = \text{const.}$ furnish the momentum conservation delta function, which we omit. The integral over the non-zero modes furnishes, under integration by parts, a Dirac delta, as we can easily see above

$$\int \mathcal{D}x \left\{ \int d\tau \left(\dot{p}_m + \sum_j \delta(\tau - \tau_j) k_{jm} \right) x^m \right\} = \delta \left(\dot{p}_m + \sum_j \delta(\tau - \tau_j) k_{jm} \right). \quad (3.5)$$

We can use this Dirac delta to integrate over the p_m field next. The equation

$$\dot{p}_m(\tau) = - \sum_j \delta(\tau - \tau_j) k_{jm} \quad (3.6)$$

uniquely determines the canonical momenta p_m once we impose that p_m must be equal to the momentum of the boundary states at $\pm\infty$. We find

$$p^m(\tau) = - \sum_P \sigma_{\tau P} k_P^m. \quad (3.7)$$

It can also be useful to employ the alternative expression

$$p^m(\tau) = -\frac{1}{2} \left(\sum_{\tau_P < \tau} k_P^m - \sum_{\tau_P > \tau} k_P^m \right), \quad (3.8)$$

which has the advantage of also being well-defined for $\tau = \tau_P$. We can use equation (3.8) to evaluate the factor of the path integral that comes from the kinetic energy of the particle. We obtain the following measure for the τ_P integration

$$\left\{ \frac{1}{2} \int d\tau p^2 \right\} = \prod_i \left\{ \frac{1}{2} (\tau_{i+1} - \tau_i) \left(\sum_{j \leq i} k_{P_j} \right)^2 \right\}, \quad (3.9)$$

which can also be written as

$$\left\{ \frac{1}{2} \int d\tau p^2 \right\} = \prod_i \left\{ -\frac{1}{2} \tau_i k_{P_i} \cdot \left(\sum_{j < i} k_{P_j} - \sum_{j > i} k_{P_j} \right) \right\}. \quad (3.10)$$

After integration against $\prod_i d\tau_i$, the factor (3.9) will furnish the channels of the amplitude corresponding to the worldline. Note that there are also channels corresponding to lower-point tree graphs sewed to the worldline; the corresponding Mandelstam variables come from the vertex operator insertions, as in equation (3.4).

To finish the computation of the amplitude, one must also perform the integration of the zero modes of θ^α and λ^α . This is carried out by projecting the remaining expression onto the pure spinor measure, which is the unique measure in the cohomology of the BRST charge with the correct ghost number of +3, given by

$$\langle (\theta \gamma^m \lambda) (\theta \gamma^n \lambda) (\theta \gamma^p \lambda) (\theta \gamma_{mnp} \theta) \rangle = 1. \quad (3.11)$$

3.2 BRST invariance of the amplitude

One can prove that the amplitude (3.2) is BRST closed and invariant. This means that the value of the amplitude does not depend on which cohomology class the worldline insertions U_P and V_P belong to. Let us prove BRST closedness. First of all, for legibility, let us omit the τ dependence of the operators in equation (3.2), and define the sum of the unintegrated vertex with the pinching operator as $\mathbb{U}_P = U_P + D_P$. Then (3.2) can be rewritten as

$$\mathcal{A}_N = \sum_{|R_1| + \dots + |R_k| = N-3} \int d\tau_1 \dots d\tau_k \langle V_1 \mathbb{U}_{R_1} \mathbb{U}_{R_2} \dots \mathbb{U}_{R_k} V_N V_{N-1} \rangle. \quad (3.12)$$

Now let the BRST charge act on (3.12). When Q hits a unintegrated vertex \mathbb{U}_P , using equations (2.47) and (2.47) one obtains

$$Q\mathbb{U}_P = \partial V_P + \sum_{RQ=P} \left(U_{[RV_Q]} + [U_{\{R, V_Q\}}] \right). \quad (3.13)$$

Thus, $Q\mathcal{A}_N$ is given by the following sum

$$\begin{aligned} Q\mathcal{A}_N &= - \sum_{k=1}^{N-3} \sum_{|R_1|+\dots+|R_k|=N-3} \int d\tau_1 \dots d\tau_k \\ &\times \sum_{i=1}^k \left\langle V_1 \mathbb{U}_{P_1} \dots \mathbb{U}_{P_{i-1}} \left(\partial V_{P_i} + \sum_{R_i Q_i=P_i} \left(U_{[R_i V_{Q_i}]} + [U_{\{R_i, V_{Q_i}\}}] \right) \right) \mathbb{U}_{P_{i+1}} \dots \mathbb{U}_{P_k} V_N V_{N-1} \right\rangle. \end{aligned} \quad (3.14)$$

The integral over τ_i might be performed right away for the total derivative terms ∂V_{P_i} . Integrating over τ_i , each of these terms in the last sum gives

$$\begin{aligned} &- \sum_k \sum_i \int d\tau_1 \dots d\tau_k \langle V_1 \mathbb{U}_{P_1} \dots \mathbb{U}_{P_{i-1}} \partial V_{P_i} \mathbb{U}_{P_{i+1}} \dots \mathbb{U}_{P_k} V_N V_{N-1} \rangle \\ &= - \sum_k \sum_i \int d\tau_1 \dots \hat{d}\tau_i \dots d\tau_k \langle V_1 \mathbb{U}_{P_1} \dots \mathbb{U}_{P_{i-1}} (\tau_{i-1}) V_{P_i} (\tau_{i-1} - \epsilon) \mathbb{U}_{P_{i+1}} \dots \mathbb{U}_{P_k} V_N V_{N-1} \rangle \\ &\quad + \int d\tau_1 \dots \hat{d}\tau_i \dots d\tau_k \langle V_1 \mathbb{U}_{P_1} \dots \mathbb{U}_{P_{i-1}} V_{P_i} (\tau_{i+1} + \epsilon) \mathbb{U}_{P_{i+1}} (\tau_{i+1}) \dots \mathbb{U}_{P_k} V_N V_{N-1} \rangle, \end{aligned} \quad (3.15)$$

where $\hat{d}\tau_i$ indicates that the $d\tau_i$ differential is omitted. After relabeling τ_i and τ_{i+1} , the last equation becomes

$$\begin{aligned} &- \sum_k \sum_i \int d\tau_1 \dots d\tau_{i-1} d\tau d\tau_{i+2} \dots d\tau_k \\ &\times \langle V_1 \mathbb{U}_{P_1} \dots (V_{P_i} (\tau + \epsilon) U_{P_{i+1}} (\tau) - U_{P_i} (\tau) V_{P_{i+1}} (\tau - \epsilon)) \dots \mathbb{U}_{P_k} V_N V_{N-1} \rangle, \end{aligned} \quad (3.16)$$

where we have dropped the terms in which V_{P_i} approaches $+\infty$ and V_{P_k} approaches $-\infty$. The subtraction $V_{P_i} (\tau + \epsilon) U_{P_{i+1}} (\tau) - U_{P_i} (\tau) V_{P_{i+1}} (\tau - \epsilon)$ can be written as an operator insertion on the worldline after appropriate normal ordering

$$\begin{aligned} &V_{P_i} (\tau + \epsilon) U_{P_{i+1}} (\tau) - U_{P_i} (\tau) V_{P_{i+1}} (\tau - \epsilon) \\ &= V_{P_i} U_{P_{i+1}} (\tau) - U_{P_i} V_{P_{i+1}} (\tau) + \frac{1}{2} \left([U_{P_i}, V_{P_{i+1}}] + [U_{P_{i+1}}, V_{P_i}] \right) (\tau) \\ &= V_{[P_i} U_{P_{i+1}]} + [U_{\{P_i, V_{P_{i+1}\}}}. \end{aligned} \quad (3.17)$$

Therefore, after summing equation (3.15) over all P and all i and normal ordering, we obtain

$$\begin{aligned} &\sum_{k=1}^{N-3} \sum_{|R_1|+\dots+|R_k|=N-3} \int d\tau_1 \dots d\tau_k \\ &\times \sum_i \sum_{R_i Q_i=P_i} \left\langle V_1 \mathbb{U}_{P_1} \dots \mathbb{U}_{P_{i-1}} \left(U_{[R_i V_{Q_i}]} + [U_{\{R_i, V_{Q_i}\}}] \right) (\tau_i) \mathbb{U}_{P_{i+1}} \dots \mathbb{U}_{P_k} V_N V_{N-1} \right\rangle, \end{aligned} \quad (3.18)$$

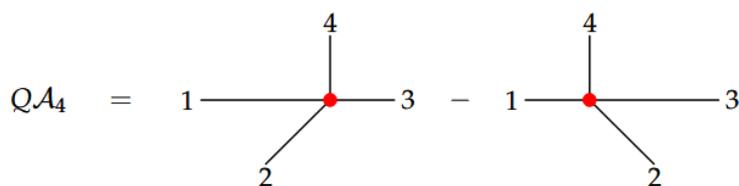


Figure 1. The action of Q kills off internal propagators in the amplitude. There is a relative sign from whether the insertion is being collapsed to a vertex in its future or in its past. In this example, 2 is being collapsed to its past in the first diagram and to its future in the second one.

so the two kinds of terms in equation (3.14) cancel, and the amplitude is indeed BRST closed. Pictorially, the BRST charge acts by cutting propagators on the amplitude, as in figure 1.

As mentioned before, the amplitude is also BRST invariant. This means that the value of the amplitude does not depend on what specific choice of vertex operators is made, as long as the choice is consistent in the sense of equations (2.46) and (2.47). The demonstration follows the same steps as the proof performed above.

3.3 4-point amplitude

In this subsection we will compute the 4-point amplitude to give a first example of the prescription detailed in the last subsections. The tree-level 4-pt amplitude contains only single-particle insertions and is given by

$$\mathcal{A}_4 = \int_{-\infty}^{\infty} d\tau_2 \langle V_1(\infty) U_2(\tau_2) V_4(0) V_3(-\infty) \rangle. \tag{3.19}$$

To evaluate this expression, we will first integrate over x^m , obtaining a kinematic measure such as in (3.9). Then we will use the worldline contractions of the non-zero modes to promote the canonical momenta to differential operators as in (3.3). Finally, we should project onto the pure spinor measure (3.11) to obtain the amplitude in terms of the external polarizations, but we leave this last step implicit. There exists software that automates this last step [2].

The x^m dependence of the correlator in equation (3.19) comes from the plane wave factors of the worldline insertions. Using the results of subsection 3.1, the path integral over the x^m and p_m fields furnishes

$$\int \mathcal{D}x \mathcal{D}p \left\{ - \int d\tau \left(p_m \dot{x}^m - \frac{1}{2} p^2 \right) \right\} \prod_{j=1}^4 \{ k_j \cdot x(\tau_j) \} = \{ (\sigma_{2i} s_{2i}) \tau_2 \}, \tag{3.20}$$

where we have also defined $s_{ij} = k_i \cdot k_j$ and used the massless condition $k_i^2 = 0$. We can check that this factor gives the correct Mandelstam poles for the amplitude. The regions of integration $\tau_2 > 0$ and $\tau_2 < 0$ correspond to the subdiagrams of figure 2.

When integrated against $d\tau$, the kinematic measure gives the correct propagators corresponding to each of the diagrams in figure 2. Indeed, we can notice that

$$\int_0^{\infty} d\tau_2 \text{ (3.4) } \{ (\sigma_{2i} s_{2i}) \tau_2 \} = \int_0^{\infty} d\tau_2 \text{ (3.4) } \{ -s_{21} \tau_2 \} = \frac{1}{s_{21}}, \tag{3.21}$$

$$\int_{-\infty}^0 d\tau_2 \text{ (3.4) } \{ (\sigma_{2i} s_{2i}) \tau_2 \} = \int_{-\infty}^0 d\tau_2 \text{ (3.4) } \{ +s_{23} \tau_2 \} = \frac{1}{s_{23}}, \tag{3.22}$$

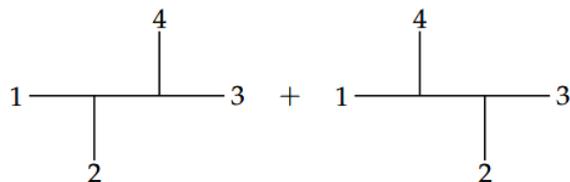


Figure 2. The two subdiagrams that contribute to the 4-pt function. Note that time flows from the right to the left in our figures.

which is the correct result. This behavior generalizes to tree amplitudes with more external legs, but some of the poles come from the multiparticle vertices as in equation (2.40). Now, we consider the contraction of the canonical momenta. Let us denote by \hat{U} the differential operator obtained by substituting the canonical momenta of the integrated vertex U using the rules of equations (3.3). One then obtains for the 4-pt function (3.19) the following expression,

$$\mathcal{A}_4 = \int d\tau_2 \{(\sigma_{2i} s_{2i}) \tau_2\} \langle \sigma_{21} \hat{U}_2(V_1) V_4 V_3 + \sigma_{23} V_1 \hat{U}_2(V_4) V_3 + \sigma_{24} V_1 V_4 \hat{U}_2(V_3) \rangle, \quad (3.23)$$

where each vertex operator also carries a factor of (3.4) $\{k \cdot x\}$ where x^m is the constant mode of the position variable. For concreteness, we also perform the τ_2 integration, obtaining

$$\begin{aligned} \mathcal{A}_4 = & \frac{1}{2s_{21}} \langle -\hat{U}_2(V_1) V_4 V_3 + V_1 \hat{U}_2(V_4) V_3 + V_1 V_4 \hat{U}_2(V_3) \rangle \\ & + \frac{1}{2s_{23}} \langle -\hat{U}_2(V_1) V_4 V_3 - V_1 \hat{U}_2(V_4) V_3 + V_1 V_4 \hat{U}_2(V_3) \rangle. \end{aligned} \quad (3.24)$$

Each of the terms in the last equation corresponds to one of the subdiagrams in figure 2. It would be natural to expect that the summand corresponding to the channel s_{21} would have a numerator where \hat{U}_2 acts only on V_1 (as figure 2 suggests), and similarly for the channel s_{23} . In our partial result, however, this is not true, unless it is possible to somehow integrate by parts the action of \hat{U}_2 in equation (3.24). That this is possible in the pure spinor formalism is not clear at first, because the integration over θ^α and λ^α is obtained by projection onto the pure spinor measure (3.11) instead of ordinary integration. In appendix A we will argue that this is indeed allowed.

After performing such integration by parts, we rewrite the 4-pt amplitude as

$$\mathcal{A}_4 = -\frac{1}{s_{21}} \langle \hat{U}_2(V_1) V_4 V_3 \rangle + \frac{1}{s_{23}} \langle V_1 V_4 \hat{U}_2(V_3) \rangle. \quad (3.25)$$

It is of interest to understand the BRST cohomology of the numerators appearing in tree amplitudes. The last equation can be written as

$$\mathcal{A}_4 = \frac{1}{s} n_s + \frac{1}{u} n_u, \quad (3.26)$$

where we introduced the kinematical numerators

$$\hat{U}_2(V_1) V_4 V_3 = -n_s, \quad V_1 \hat{U}_2(V_4) V_3 = n_t, \quad V_1 V_4 \hat{U}_2(V_3) = n_u. \quad (3.27)$$

The requirement that the amplitude be BRST closed fixes the BRST cohomology of the kinematical numerators n_i . Indeed, the 4-pt amplitude can only obey $Q\mathcal{A}_4 = 0$ for every value of the Mandelstam variables if there exists some operator \mathcal{O} such that

$$Qn_s = -s\mathcal{O}, \quad Qn_u = u\mathcal{O}. \quad (3.28)$$

The objects satisfying equation (3.28) were studied thoroughly in [2]. These are the BRST building blocks, that we review in appendix B. We use the BRST building blocks to organize our expressions.

We now return to equation (3.25) and compute the action of the integrated vertex on the unintegrated vertex, $\hat{U}_i(V_j)$. Using the equations of motion of SYM, one finds

$$\hat{U}_i(V_j) = -\left((A_i \cdot k_j)V_j + (W_i \gamma^m \lambda)A_{jm} + Q(W_i A_j)\right). \quad (3.29)$$

This can be further rewritten if we use the identity

$$Q(A_1 \cdot A_2) = k_1 \cdot A_2 V_1 + (\lambda \gamma^m W_1)A_2^m + (1 \leftrightarrow 2). \quad (3.30)$$

which can be checked explicitly from (2.60). We find

$$\frac{1}{s_{12}}\hat{U}_2(V_1) = \lambda^\alpha A_{\alpha 12} + \frac{1}{s_{12}}QD_{12} = V_{21} + \frac{1}{s_{12}}QD_{12}. \quad (3.31)$$

Plugging this into (3.25), the BRST-exact piece decouples trivially after integration by parts. So the four-point function can be written as

$$\mathcal{A}_4 = -\langle V_{12}V_3V_4 \rangle - \langle V_1V_{23}V_4 \rangle, \quad (3.32)$$

where we also used the fact that V_{ij} is antisymmetric in i and j (notice also that V is fermionic). We have thus seen that the BRST exact pieces decouple and the amplitude can be written in terms of the Berends-Giele currents V_P , which are constructed with the BRST building blocks (see appendix B). This is the correct result obtained by the $\alpha' \rightarrow 0$ limit of the superstring [2].

This simple argument for the decoupling of the BRST exact piece of $\hat{U}(V)$ is no longer true at higher points, because there the BRST charge can hit other integrated vertices after integration by parts. This will produce residual terms, but they cancel exactly with the contributions coming from the pinching operators in equation (3.2).

3.4 5-point amplitude

For the color ordering 12345, the tree level 5-point amplitude coming from (3.2) is given by

$$\begin{aligned} \mathcal{A}_5 = & \int_{-\infty}^{\infty} d\tau_2 \int_{-\infty}^{\tau_2} d\tau_3 \langle V_1(\infty)U_2(\tau_2)U_3(\tau_3)V_5(0)V_4(-\infty) \rangle \\ & + \int_{-\infty}^{\infty} d\tau \langle V_1(\infty)(U_{23}(\tau) + D_{23}(\tau))V_5(0)V_4(-\infty) \rangle, \end{aligned} \quad (3.33)$$

where in the second line we included the contribution from the two-particle integrated vertex U_{23} of equation (2.60) and also from the pinching operator D_{23} of equation (2.61).

As before, the x^m path integral gives a kinematical measure that is integrated in τ to give the propagators of the amplitude. After this is done, one integrates over the

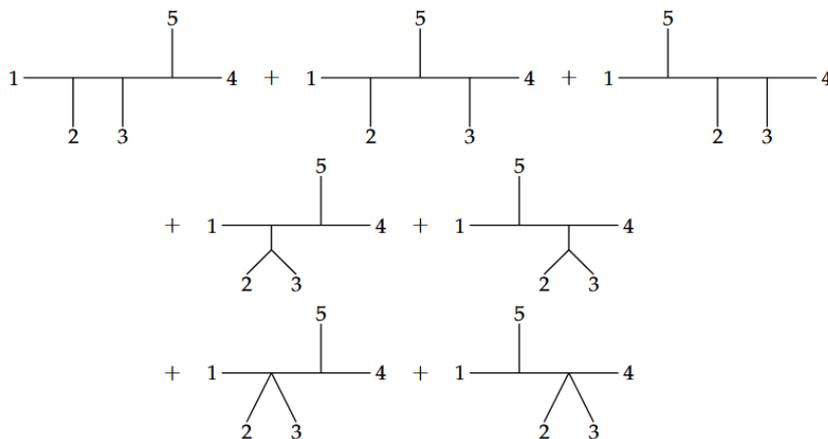


Figure 3. The diagrams that contribute to the 5-pt function. We have split the diagrams according to the region of integration. On the first line, all the diagrams have only single particle insertions, while in the second line the insertions of U_{23} are represented by a subtree. Finally, the third line includes contact terms, represented by the pinching operators D_{23} .

canonical momenta, which has the effect of promoting the integrated vertices to differential operators. Computing the correlator, we find a sum of sixteen contractions in the first line of equation (3.33) and three in the second line. We can simplify this expression by integrating the action of the integrated vertices by parts, as before. To do this, we must first perform a gauge transformation of the vertex operators that takes advantage of the properties of the b -ghost.

The gauge transformation that we will perform is

$$U_P \mapsto \tilde{U}_P, \tag{3.34}$$

where the tilded vertices, defined in subsection 2.5, are constructed from the b -ghost operator. In particular, the pinching operator turns out to be BRST closed and can be taken to be zero without loss of generality as explained below equation (2.69). After this gauge transformation is performed, the amplitude can be written as

$$\begin{aligned} \mathcal{A}_5 = & \int_{-\infty}^{\infty} d\tau_2 \int_{-\infty}^{\tau_2} d\tau_3 \langle V_1(\infty) \tilde{U}_2(\tau_2) \tilde{U}_3(\tau_3) V_5(0) V_4(-\infty) \rangle \\ & + \int_{-\infty}^{\infty} d\tau \langle V_1(\infty) \tilde{U}_{23}(\tau) V_5(0) V_4(-\infty) \rangle. \end{aligned} \tag{3.35}$$

The first term in the equation above corresponds to the three diagrams in the first line of figure 3, while the second term corresponds to the diagrams in the second line.

We shall now proceed to the computation of the correlator. The path integral over the canonical momenta is equivalent to replacing the canonical momenta in the integrated vertex insertions with the corresponding differential operators, up to σ_{ij} factors. For instance

$$\langle p_m(\tau_1) \mathcal{O}(x(\tau_2)) \rangle = \sigma_{12} \langle \partial_m \mathcal{O}(x(\tau_2)) \rangle. \tag{3.36}$$

Performing this substitution in equation (3.35), one defines the hatted operators \hat{U}_i and \hat{U}_{ij} . Using the properties of the b ghost, it is possible to integrate these operators by parts

(see appendix A), which greatly simplifies the computation of the correlator. To illustrate, let us take the first integral in equation (3.35) and consider the interval of integration $\infty > \tau_2 > \tau_3 > 0$, corresponding to the first diagram of figure 3. The corresponding piece of the amplitude is equal to

$$\begin{aligned} & \int_0^\infty d\tau_2 \int_0^{\tau_2} d\tau_3 e^{-\tau_2 s_{12}} e^{-\tau_3(s_{13}+s_{23})} \langle V_1(\infty) \tilde{U}_2(\tau_2) \tilde{U}_3(\tau_3) V_5(0) V_4(-\infty) \rangle \\ &= -\frac{1}{2} \int_0^\infty d\tau_2 \int_0^{\tau_2} d\tau_3 e^{-\tau_2 s_{12}} e^{-\tau_3(s_{13}+s_{23})} \langle (\hat{U}_2(V_1) \tilde{U}_3 V_5 V_4 - V_1 \hat{U}_2(\tilde{U}_3 V_5 V_4)) \rangle \\ &= -\int_0^\infty d\tau_2 \int_0^{\tau_2} d\tau_3 e^{-\tau_2 s_{12}} e^{-\tau_3(s_{13}+s_{23})} \langle \hat{U}_2(V_1) \tilde{U}_3 V_5 V_4 \rangle, \end{aligned} \tag{3.37}$$

where we integrated the action of \hat{U}_2 by parts in the last line. Using the same argument for the action of \tilde{U}_3 and performing the integral over the kinematic measure, one obtains the value of the correlator together with the corresponding kinematical poles of the amplitude

$$\frac{1}{s_{12}s_{45}} \langle \hat{U}_3(\hat{U}_2(V_1)) V_5 V_4 \rangle. \tag{3.38}$$

This computation is conceptually important because it demonstrates how the contractions coming from the worldline can be arranged as Berends–Giele currents. A useful pictorial interpretation of this result is that, in the corresponding diagram of this correlator — specifically the first diagram of figure 3 — each integrated vertex acts only on the leg where it sits.

If we repeat this procedure to all channels, we obtain the following expression for the amplitude

$$\begin{aligned} \mathcal{A}_5 &= \frac{1}{s_{12}s_{45}} \langle \hat{U}_3(\hat{U}_2(V_1)) V_5 V_4 \rangle + \frac{1}{s_{12}s_{34}} \langle \hat{U}_2(V_1) V_5 \hat{U}_3(V_4) \rangle + \frac{1}{s_{15}s_{34}} \langle V_1 V_5 \hat{U}_3(\hat{U}_2(V_4)) \rangle \\ &+ \frac{1}{s_{45}} \langle \hat{U}_{23}(V_1) V_5 V_4 \rangle + \frac{1}{s_{15}} \langle V_1 V_5 \hat{U}_{23}(\hat{U}_4) \rangle. \end{aligned} \tag{3.39}$$

To compare with the previous result for SYM amplitudes in the pure spinor formalism of [2], we will write this result in terms of the BRST building blocks T_{ij} and T_{ijk} . As it turns out, any set of composite operators satisfying the right BRST cohomology can be used to express the amplitude, and different composite operators are related by BRST-exact pieces and contact terms that cancel in the full color-ordered amplitude. The BRST cohomology and symmetries of T_{ij} and T_{ijk} are enough to obtain our result.

The BRST building blocks are composite fields that satisfy the following BRST cohomology [2]

$$\{Q, T_{12}\} = s_{12} V_1 V_2, \tag{3.40}$$

$$\{Q, T_{123}\} = s_{123} T_{12} V_3 + s_{12}(T_{13} V_2 - T_{12} V_3 + V_1 T_{23}), \tag{3.41}$$

and the symmetries $T_{(12)} = T_{(12)3} = T_{123} + \text{cyclic} = 0$. In order to express our result (3.39) in terms of the BRST building blocks, we note that the composite vertices $\hat{U}_i(V_j)$, $\hat{U}_i(\hat{U}_j(V_k))$ and $\hat{U}_{ij}(V_k)$ satisfy the same relation under BRST transformation, namely

$$\{Q, \hat{U}_2(V_1)\} = s_{12} V_2 V_1, \tag{3.42}$$

$$\{Q, \hat{U}_3(\hat{U}_2(V_1))\} = s_{123} \hat{U}_2(V_1) V_3 + s_{12}(\hat{U}_3(V_2) V_1 + V_2 \hat{U}_3(V_1) - V_3 \hat{U}_2(V_1)). \tag{3.43}$$

This set of equations means that the composite vertices can be mapped to the BRST building blocks up to contact terms. Let us see how this plays out. At two-particles, one can see that $\{Q, \hat{U}_2(V_1) - T_{21}\} = 0$. The uniqueness of the BRST cohomology then implies that the composite vertex is related to this building block at this rank by a BRST exact piece; i.e. there exists E_{21} such that

$$\hat{U}_2(V_1) = T_{21} + [Q, E_{21}]. \tag{3.44}$$

The anti-symmetry of both the composite vertex and the BRST building block implies that E_{ij} is also antisymmetric, $E_{(ij)} = 0$. For the three-particle composite fields, one can find from (3.41), (3.43) and (3.44),

$$\{Q, \hat{U}_3(\hat{U}_2(V_1))\} = \{Q, T_{123} - s_{123}V_3E_{21} + s_{21}(E_{32}V_1 - E_{31}V_2 + E_{21}V_3)\}. \tag{3.45}$$

The uniqueness of the BRST cohomology then implies that

$$\hat{U}_3(\hat{U}_2(V_1)) = T_{123} - s_{123}V_3E_{21} + s_{21}(E_{32}V_1 - E_{31}V_2 + E_{21}V_3) + [Q, \dots]. \tag{3.46}$$

The terms that come multiplied by Mandelstam variables are named contact terms. The result thus obtained is that the composite field is equal to the BRST building blocks plus contact terms and BRST-exact pieces. The $[Q_0, \dots]$ in the last equation does not contribute to the 5-pt function, but if we were interested in the higher point functions, we would need to include them, and they would give contact terms for the higher-order composite fields. Proceeding similarly, one finds that

$$s_{23}\hat{U}_{23}(V_1) = T_{321} - s_{123}V_1E_{23} + s_{23}(E_{12}V_3 - E_{13}V_2 + E_{23}V_1) + [Q, \dots]. \tag{3.47}$$

To write the result (3.39) in terms of the BRST building blocks, we plug the equations (3.44), (3.46) and (3.47). The contact terms E_{ij} cancel between diagrams and the final result for the 5-point function is

$$\mathcal{A}_5 = \frac{\langle T_{123}V_5V_4 \rangle}{s_{12}s_{45}} + \frac{\langle T_{12}T_{34}V_5 \rangle}{s_{12}s_{34}} + \frac{\langle V_1T_{432}V_5 \rangle}{s_{15}s_{34}} + \frac{\langle T_{321}V_4V_5 \rangle}{s_{23}s_{45}} + \frac{\langle V_1T_{234}V_5 \rangle}{s_{23}s_{15}}. \tag{3.48}$$

This result agrees with the low energy limit of the pure spinor superstring disk amplitude [2]. The same result can be obtained without the b -ghost by writing down all the contractions coming from (3.33), and using the BRST cohomology to obtain an expression similar to (3.39), but including pinching operators. Then one can again argue that the pinching operators cancel with the BRST exact piece of the multiparticle vertices, and the result is again that of eq. (3.48).

3.5 N-point amplitude

The N -point SYM tree-level amplitude was computed in [33] using the Berends-Giele currents constructed from the BRST building blocks. The result is given by

$$\mathcal{A}'_N = \sum_{j=1}^{N-2} \langle M_{1\dots j} M_{j+1\dots N-1} V_N \rangle, \tag{3.49}$$

where the Berends-Giele currents $M_{i\dots j}$ are defined iteratively through their BRST cohomology. They satisfy

$$QM_{1\dots p} = \sum_{j=1}^{p-1} M_{1\dots j} M_{j+1\dots p}, \quad (3.50)$$

with $M_i = V_i$. One should notice that this is the same recursion relation that defines the multiparticle unintegrated vertices as in equation (2.46). We will now show that the worldline amplitude of equation (3.2) can be written in the same way as in equation (3.49), using a potentially different set of Berends-Giele currents. We will then use the uniqueness in the BRST cohomology to establish that the value of the amplitude is independent of the representation of the Berends-Giele currents, proving that the worldline prescription gives the same result as equation (3.49). For convenience, we will write down the worldline amplitude prescription (equation (3.2)) again

$$\begin{aligned} \mathcal{A}_N = & \sum_{|R_1|+\dots+|R_k|=N-3} \int_{\tau_j > \tau_{j+1}} d\tau_1 \dots d\tau_k \\ & \times \langle V_1(\infty) V_N(0) V_{N-1}(-\infty) (U_{R_1} + D_{R_1})(\tau_1) \dots (U_{R_k} + D_{R_k})(\tau_k) \rangle. \end{aligned} \quad (3.51)$$

Gauge invariance allows us to rewrite \mathcal{A}_N in terms of the b -ghost vertex insertions, which are denoted by a tilde, as follows

$$\mathcal{A}_N = \sum_{k=1}^{N-3} \sum_{\sum_{i=1}^k |P_i|=N-3} \int_{\tau_j > \tau_{j+1}} d\tau_1 \dots d\tau_k \langle V_1 \tilde{U}_{P_1} \dots \tilde{U}_{P_k} V_N V_{N-1} \rangle. \quad (3.52)$$

The worldline wick contractions promote the vertex insertions to differential operators as explained in subsection 3.4. Due to the properties of the b -ghost, we can integrate the action of these operators by parts, obtaining a sum of strings of operators acting on V_1 or V_{N-1} according to where τ_i is greater or smaller than zero

$$\begin{aligned} \mathcal{A}_N = & \sum_{k=1}^{N-3} \sum_{\sum_{i=1}^k |P_i|=N-3} \sum_j (-1)^j \int_{\tau_j > 0 > \tau_{j+1}} d\tau_1 \dots d\tau_k \\ & \times \left\langle \left(\hat{U}_{P_1} \dots \hat{U}_{P_j} V_1 \right) V_N \left(\hat{U}_{P_{j+1}} \dots \hat{U}_{P_k} V_{N-1} \right) \right\rangle. \end{aligned} \quad (3.53)$$

Although this expression seems complicated, we can readily identify it as a sum of Berends-Giele currents as defined in (3.50). For instance, summing over all partitions of $j+1, \dots, N-2$, we find

$$\sum_{P_i=j+1, \dots, N-3} \int_{0 > \tau_{j+1}} d\tau_{j+1} \dots d\tau_k \left(\hat{U}_{P_{j+1}} \dots \hat{U}_{P_k} V_{N-1} \right) = \tilde{V}_{j+1\dots N-1}, \quad (3.54)$$

where the multiparticle field $\tilde{V}_{j+1\dots N-1}$ is defined as in (2.62) — the factors of $1/s_P$ come from the proper time integration. Thus equation (3.53) can be written as

$$\mathcal{A}_N = - \sum_{j=1}^{N-2} \langle \tilde{V}_{1\dots j} \tilde{V}_{j+1\dots N-1} V_N \rangle, \quad (3.55)$$

where we used the shuffle symmetry of the Berends-Giele currents [32]. A similar argument can be constructed without using the b -ghost, but one needs to subtract contact terms appropriately when defining the Berends-Giele currents to guarantee the correct BRST cohomology.

We will now prove that every set of Berends-Giele currents gives rise to the same scattering amplitude. Suppose that

$$\mathcal{A}_N = - \sum_{j=1}^{N-2} \langle V_{1\dots j} V_{j+1\dots N-1} V_N \rangle, \quad (3.56)$$

for some set of Berends-Giele currents satisfying

$$QV_P = - \sum_{RQ=P} V_R V_Q. \quad (3.57)$$

Suppose further that there exists another set of Berends-Giele currents and construct \mathcal{A}'_N as

$$\mathcal{A}'_N = - \sum_{j=1}^{N-2} \langle V'_{1\dots j} V'_{j+1\dots N-1} V_N \rangle, \quad (3.58)$$

Suppose further that $V'_i = V_i$. The two sets of Berends-Giele currents will be related to each other by contact terms and BRST-exact pieces. For instance

$$Q(V'_{12} - V_{12}) = 0 \Rightarrow V'_{12} = V_{12} + QE_{12}, \quad (3.59)$$

for some E_{12} , since the BRST cohomology is trivial for off-shell fields by what is argued in subsection 2.2. This implies that

$$\begin{aligned} QV'_{123} &= QV_{123} + Q(-E_{12}V_3 + V_1E_{23}) \\ \Rightarrow V'_{123} &= V_{123} - E_{12}V_3 + V_1E_{23} + Q(E_{123}), \end{aligned} \quad (3.60)$$

for some E_{123} . This process can be continued for all ranks of the currents. Schematically

$$V'_P = V_P + Q(E_P) + \text{contact terms}. \quad (3.61)$$

Now consider \mathcal{A}'_N , and expand the V'_P appearing in the expansion as V_P , plus contact terms and BRST exact pieces. We will fix $P = 1 \dots k$ and track the term proportional to E_P to see that it cancels. This term comes from V'_P and also from every higher rank current V'_{PQ} , where $Q = (k+1) \dots j$ for some j . Let us study the case $Q = k+1$,

$$\begin{aligned} QV'_{P(k+1)} &= -V'_P V'_{k+1} + \dots \\ &= -V_P V'_{k+1} - QE_P V'_{k+1} + \dots \\ &= -V_P V'_{k+1} - Q(E_P V'_{k+1}) + \dots, \\ \Rightarrow V'_{P(k+1)} &= V_{P(k+1)} - E_P V'_{k+1} + \dots, \end{aligned} \quad (3.62)$$

where the \dots include all terms that do not have E_P . Similarly

$$\begin{aligned} QV'_{P(k+1)(k+2)} &= -V_P V'_{(k+1)(k+2)} - Q(E_P V'_{(k+1)(k+2)}) + \dots \\ \Rightarrow V'_{P(k+1)(k+2)} &= V_{P(k+1)(k+2)} - E_P V'_{(k+1)(k+2)} + \dots \end{aligned} \quad (3.63)$$

In general, one finds (for $Q = (k + 1) \dots j$)

$$\begin{aligned} QV'_{PQ} &= -V_P V'_Q - Q(E_P V'_Q) + \dots \\ \Rightarrow V'_{PQ} &= V_{PQ} - E_P V'_Q + \dots \end{aligned} \quad (3.64)$$

Using this equation, the terms proportional to E_P are easily seen to cancel in \mathcal{A}'_N . Indeed, fix $P = 1 \dots k$. Then (here, Q is a shortcut for $((k + 1) \dots N - 1)$, and $Q_1 Q_2$ run over the deconcatenation sum of Q)

$$\begin{aligned} \mathcal{A}'_N &= - \sum_{j=1}^{N-2} \langle V'_{1\dots j} V'_{j+1\dots N-1} V_N \rangle \\ &= - \langle V'_P V'_Q V_N \rangle - \sum_{Q_1 Q_2 = Q} \langle V'_{PQ_1} V'_{Q_2} V_N \rangle + \dots \\ &= - \langle V_P V'_Q V_N \rangle - \langle Q E_P V'_Q V_N \rangle - \sum_{Q_1 Q_2 = Q} \langle V'_{PQ_1} V'_{Q_2} V_N \rangle + \dots \\ &= - \langle V_P V'_Q V_N \rangle + \langle E_P Q V'_Q V_N \rangle - \sum_{Q_1 Q_2 = Q} \langle V'_{PQ_1} V'_{Q_2} V_N \rangle + \dots \\ &= - \langle V_P V'_Q V_N \rangle - \sum_{Q_1 Q_2 = Q} \langle E_P V'_{Q_1} V'_{Q_2} V_N \rangle - \sum_{Q_1 Q_2 = Q} \langle V'_{PQ_1} V'_{Q_2} V_N \rangle + \dots \\ &= - \langle V_P V'_Q V_N \rangle - \sum_{Q_1 Q_2 = Q} \langle V_{PQ_1} V'_{Q_2} V_N \rangle + \dots, \end{aligned} \quad (3.65)$$

where the \dots contain no term proportional to E_P . A similar argument can be constructed for the cancellation of the E_P where $P \neq 1 \dots k$ (for instance by using shuffle symmetry). After all of the E_P terms are canceled, one finds that

$$\begin{aligned} \mathcal{A}'_N &= - \sum_{j=1}^{N-2} \langle V'_{1\dots j} V'_{j+1\dots N-1} V_N \rangle \\ &= - \sum_{j=1}^{N-2} \langle V_{1\dots j} V_{j+1\dots N-1} V_N \rangle = \mathcal{A}_N. \end{aligned} \quad (3.66)$$

Thus the amplitude has the same numerical value for all consistent sets of Berends-Giele currents. In particular, the worldline prescription gives the same result as the one obtained from the BRST building blocks of [2].

4 Discussions

We have developed a prescription for computing tree-level scattering amplitudes in the 10D SYM theory using the pure spinor worldline formalism for the superparticle. In particular, we have studied the necessary insertions on the worldline, and showed that the pinching operators act as contact terms that enhance the BRST properties of the kinematical numerators, elucidating how this phenomenon first observed in string theory also appears in the particle. In particular, one might interpret the multiparticle fields appearing in the computation of the amplitude as Berends-Giele currents. Once this is done, the computation of the amplitude boils down to the computation of these currents, which singles out the known result in terms

of the BRST building blocks. In this sense, the computation of the amplitude is formulated as a recursive construction in BRST cohomology, as in [21]. We notice that, in the worldline, this reformulation arises quite naturally from the integration by parts procedure that we carried over. The use of the Hamiltonian path integral (as opposed to the usual Lagrangian) in our computation of the worldline amplitude was established as a way to avoid ill-defined distributions (coming from products of Dirac deltas) in the formalism.

Our study provides a foundation for further investigation into the worldline formalism of other supersymmetric theories. A natural next step is the extension to 11D supergravity, using the 11D pure spinor superparticle formalism [36–44]. We start this research program in our companion papers [19, 20]. Furthermore, the ideas developed in the present manuscript might shed light on the construction of N-point amplitudes for ambitwistor models constructed from their particle counterparts [45–47]. Additionally, exploring loop-level amplitudes within this formalism may offer new insights into the interplay between worldline methods and traditional Feynman diagram techniques. By establishing the effectiveness of the worldline approach for 10D SYM, we pave the way for a broader application of first-quantized techniques in field theory, potentially simplifying amplitude computations in supersymmetric and gravitational theories.

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A Integration by parts

The last step in the evaluation of the scattering amplitude (3.2) is the integration over the worldline zero modes. While in the minimal pure spinor formalism this is described ad-hoc by the projection on the measure (3.11), the non-minimal pure spinor formalism allows one to justify this result from first principles. One defines the quantum mechanical correlator $\langle \mathcal{O} \rangle$ as the path integral over the non-minimal variables with insertion \mathcal{O}

$$\mathcal{A}_N = \int \mathcal{D}\Phi \mathcal{N} e^{-S} \mathcal{O}, \quad (\text{A.1})$$

where $\mathcal{D}\Phi$ represents the measure for integration over all non-minimal fields and \mathcal{N} is a regularization factor that ensures that the integral over the bosonic fields is finite. After integrating out x^m and the various non-zero modes using the contractions of equation (2.6), one is left with an integral over the zero modes of λ^α , $\bar{\lambda}_\alpha$, r_α and θ^α

$$\mathcal{A}_N = \int [d\lambda][d\bar{\lambda}][dr] d^{16}\theta \mathcal{N} f(\lambda, \bar{\lambda}, r, \theta), \quad (\text{A.2})$$

for some function f . The holomorphic top-form on the respective spaces gives the zero-mode integration measure

$$[d\lambda] \lambda^{\alpha_1} \lambda^{\alpha_2} \lambda^{\alpha_3} = (\epsilon T^{-1})_{\beta_1 \dots \beta_{11}}^{\alpha_1 \alpha_2 \alpha_3} d\lambda^{\beta_1} \dots d\lambda^{\beta_{11}}, \quad (\text{A.3})$$

$$[d\bar{\lambda}] \bar{\lambda}_{\alpha_1} \bar{\lambda}_{\alpha_2} \bar{\lambda}_{\alpha_3} = (\epsilon T)_{\alpha_1 \alpha_2 \alpha_3}^{\beta_1 \dots \beta_{11}} d\bar{\lambda}_{\beta_1} \dots d\bar{\lambda}_{\beta_{11}}, \quad (\text{A.4})$$

$$[dr] = (\epsilon T^{-1})_{\beta_1 \dots \beta_{11}}^{\alpha_1 \alpha_2 \alpha_3} \bar{\lambda}_{\alpha_1} \bar{\lambda}_{\alpha_2} \bar{\lambda}_{\alpha_3} \left(\frac{\partial}{\partial r_{\beta_1}} \right) \dots \left(\frac{\partial}{\partial r_{\beta_{11}}} \right). \quad (\text{A.5})$$

where the tensors $(\epsilon T^{-1})_{\beta_1 \dots \beta_{11}}^{\alpha_1 \alpha_2 \alpha_3}$ and $(\epsilon T)_{\alpha_1 \alpha_2 \alpha_3}^{\beta_1 \dots \beta_{11}}$ are symmetric and gamma matrix traceless in $(\alpha_1, \alpha_2, \alpha_3)$ and antisymmetric in $[\beta_1, \dots, \beta_{11}]$. Their explicit form can be found in [25], but this will not be needed here. The presence of the regulator \mathcal{N} is necessary because the integral over the unbounded bosonic ghosts diverges in its absence. This regulator \mathcal{N} is BRST trivial and thus cannot change the value of the amplitude. It is given by

$$\mathcal{N} = \{-\{Q, \bar{\lambda}_\alpha \theta^\alpha\}\} = \{-\bar{\lambda}_\alpha \lambda^\alpha - r_\alpha \theta^\alpha\}, \quad (\text{A.6})$$

where only the zero-modes of the fields appearing in the last equation contribute. Due to BRST invariance, the amplitude does not depend on the specific choice of regulator. Plugging equation (A.6) into (A.2), the integration furnishes exactly the rule of equation (3.11) [25, 48].

The measure (A.6) commutes with the b field defined in (2.16). To show this, note first that

$$[b, \{Q, \bar{\lambda}_\alpha \theta^\alpha\}] = -[Q, \{b, \bar{\lambda}_\alpha \theta^\alpha\}]. \quad (\text{A.7})$$

since $[H, \bar{\lambda}_\alpha \theta^\alpha] = 0$. It is simple to check that the (anti)commutator of the physical operators with $\bar{\lambda}_\alpha \theta^\alpha$ vanishes

$$[\mathbf{\Delta}_m, \bar{\lambda}_\alpha \theta^\alpha] = \frac{(\bar{\lambda} \gamma^m r)}{2(\bar{\lambda} \lambda)} = 0, \quad \{\mathbf{A}^m, \bar{\lambda}_\alpha \theta^\alpha\} = \frac{(\bar{\lambda} \gamma^m \bar{\lambda})}{2(\bar{\lambda} \lambda)} = 0, \quad (\text{A.8})$$

where we used the non-minimal pure spinor constraints. This establishes that $[b, \mathcal{N}] = 0$. It can actually be shown that $[\mathbf{A}_m, \mathcal{N}] = [\mathbf{\Delta}^m, \mathcal{N}] = 0$. This is an important result because it allows one to integrate the action of b by parts in a correlator. This can be demonstrated with ease in the Siegel gauge. A generic kinematical numerator will be given by

$$\langle \mathcal{O}_1[U'_P, \mathcal{O}_2] \rangle = \int [d\lambda][d\bar{\lambda}][dr] d^{16}\theta \mathcal{N} (\mathcal{O}_1[U'_P, \mathcal{O}_2]), \quad (\text{A.9})$$

for unspecified operators $\mathcal{O}_{1,2}$. Writing $U'_P = -\hat{\mathbf{A}}_m(V'_P)\Delta^m + \hat{\Delta}^m(V'_P)\mathbf{A}_m$, we find

$$\begin{aligned} \langle \mathcal{O}_1[U'_P, \mathcal{O}_2] \rangle &= \langle \mathcal{O}_1 \left(-\hat{\mathbf{A}}_m(V'_P)\hat{\Delta}^m(\mathcal{O}_2) + \hat{\Delta}^m(V'_P)\mathbf{A}_m(\mathcal{O}_2) \right) \rangle \\ &= \langle \left(\hat{\mathbf{A}}_m(V'_P)\hat{\Delta}^m(\mathcal{O}_1) - \hat{\Delta}^m(V'_P)\mathbf{A}_m(\mathcal{O}_1) \right) \mathcal{O}_2 \rangle \\ &= -\langle [U'_P, \mathcal{O}_1] \mathcal{O}_2 \rangle, \end{aligned} \quad (\text{A.10})$$

where we integrated by parts in the second line, and used the fact that \mathbf{A}_m is fermionic while Δ^m is bosonic. We have also used the Siegel gauge condition $b_0(V'_P) = 0$, and the fact that the physical operators commute with the measure \mathcal{N} . Finally, we have also used the fact that N^{mn} acts as a total derivative on pure spinor space,

$$\int [d\lambda] \lambda^\alpha (\gamma_{mn})_\alpha^\beta \partial_{\lambda^\beta} [f(\lambda)] = 0, \tag{A.11}$$

provided that the function f is sufficiently well-behaved at the boundaries.

In Siegel gauge, this implies that the kinematical numerators explicitly satisfy generalized Jacobi identities, as show in [12]. Indeed, the kinematical numerators appearing in the tree amplitude in Siegel gauge are generally given by

$$\langle b_0(V'_A V'_B) V'_C V'_D \rangle. \tag{A.12}$$

Using the fact that b_0 is a quadratic differential operator, we find

$$b_0(V'_A V'_B V'_C) = b_0(V'_A V'_B) V'_C + b_0(V'_B V'_C) V'_A + b_0(V'_C V'_A) V'_B. \tag{A.13}$$

Integrating by parts, we obtain the following generalized Jacobi identities

$$\langle b_0(V'_A V'_B) V'_C V'_D \rangle + \langle b_0(V'_B V'_C) V'_A V'_D \rangle + \langle b_0(V'_C V'_A) V'_B V'_D \rangle = 0. \tag{A.14}$$

B BRST building blocks

In [2], the contractions of the N-point function are organized using the BRST building blocks. They behave like the composite operators $\hat{U}_{i_1} \cdots \hat{U}_{i_k}(V)$ in the sense that they describe multiparticle states and obey the BRST cohomology of equation (2.46). But they also satisfy symmetries that are reminiscent of the generalized Jacobi identities and can be used to construct the Berends-Giele currents of SYM. In this appendix, we will briefly explain their construction. The interested reader should consult the original reference [2].

The composite fields that appear in the worldline prescription (3.2) are given by strings of integrated vertices U_{P_i} acting on the unintegrated vertices $V_i, \hat{U}_{P_1} \cdots \hat{U}_{P_k}(V_i)$. At the first level, we obtain the two-particle composite field

$$\hat{U}_i(V_j) = -\left((A_i \cdot k_j) V_j + (W_i \gamma^m \lambda) A_{jm} + Q(W_i A_j) \right). \tag{B.1}$$

The worldline prescription also forces us to consider the composite fields representing a larger number of particles, such as

$$\begin{aligned} & \hat{U}_3 \hat{U}_2 V_1, \\ & \hat{U}_{[23]} V_1, \\ & \hat{U}_4 \hat{U}_3 \hat{U}_2 V_1, \\ & \vdots \end{aligned} \tag{B.2}$$

The string theory computation contains equivalent objects. These are the composite fields $L_{ijkn\dots}$, defined as the single pole contraction of integrated and unintegrated vertices.

At first order, we define the composite field L_{21} is computed by taking the single pole in the OPE of U_2 colliding with V_1 ,

$$L_{21}(z_1) = \lim_{z_2 \rightarrow z_1} (z_2 - z_1)U_2(z_2)V_1(z_1). \tag{B.3}$$

At higher orders, the composite field $L_{ijkn\dots}$ is defined by taking $z_i \rightarrow z_j$, $z_k \rightarrow z_n$, etc in the OPE. These can be obtained recursively. For instance,

$$L_{2131\dots n1} = \lim_{z_n \rightarrow z_1} (z_n - z_1)U_n(z_n)L_{2131\dots}(z_1). \tag{B.4}$$

The equivalence between the composite vertices of the particle and the $L_{ijkn\dots}$ can be established with ease since the extraction of the single pole in $U(z)\mathcal{O}(0)$ where \mathcal{O} has conformal weight zero is equivalent to the particle contraction $[U, \mathcal{O}]$. The composite fields satisfy the BRST cohomology

$$QL_{21} = s_{12}V_1V_2, \tag{B.5}$$

$$QL_{2131} = s_{123}L_{21}V_3 + s_{12}[L_{31}V_2 + V_1L_{32} - L_{21}V_3], \tag{B.6}$$

and similarly for higher-order vertices. This can be obtained by using the descent equation $[Q, U] = \partial V$. This BRST cohomology allows one to assign a diagrammatic interpretation to the composite fields, as done in [2]. This form of the BRST cohomology is fixed by the requirement of BRST invariance of the tree-amplitude. Let us see how this works out for the color-ordered 4-pt function. If we write generally

$$\mathcal{A}_4 = \frac{n_s}{s} + \frac{n_t}{t}, \tag{B.7}$$

then this can only satisfy $Q\mathcal{A}_4 = 0$ for arbitrary values of the external momenta if $Qn_s = s\mathcal{O}$ and $Qn_t = -t\mathcal{O}$ for some \mathcal{O} . This is exactly the general structure of BRST cohomology of the composite fields given by equation (B.5), so it is intuitive that the composite fields correspond to the kinematical numerators of the tree-amplitude.

Different kinematical numerators have different properties. It is of interest to enhance the properties of the kinematical numerators by performing generalized gauge transformations, which is equivalent to the cancellation of contact terms between diagrams. The contact terms correspond to the BRST exact pieces of the composite fields.

The BRST exact piece of L_{21} can be found by direct inspection to be the symmetric part,

$$L_{12} + L_{21} = -QD_{12}. \tag{B.8}$$

where $D_{12} = (A_1 \cdot A_2 + W_1A_2 + W_2A_1)$. In the string, this might be established with ease by considering the BRST charge acting on the OPE of two integrated vertices $U_1(z)U_2(0)$. By first taking the double pole and then acting with Q , we obtain $-[Q, D_{12}](0)/z$ where $-D_{12}$ is the (symmetric) double pole of the OPE. By first acting with the BRST charge and then taking the double pole, we obtain $(L_{21} + L_{12})(0)/z^2$, and since the order of operations cannot alter the result, this establishes that $L_{\{12\}}$ is BRST exact on the string. For the superparticle, one notes that the consistency condition (2.45) implies that $\hat{U}_1(V_2) + \hat{U}_2(V_1)$ must be BRST trivial.

When we subtract the BRST exact pieces, we define the BRST building blocks $T_{ij\dots}$. For instance,

$$T_{12} = L_{[12]} = L_{21} - L_{\{21\}} = L_{21} + \frac{1}{2}QD_{12}. \quad (\text{B.9})$$

This is the rank-2 BRST building block. It satisfies the same BRST cohomology as L_{21} , but has the advantage of also being anti-symmetric,

$$T_{\{12\}} = 0. \quad (\text{B.10})$$

The higher-order BRST building blocks are constructed similarly. From the BRST cohomology of the rank-2 composite field, we obtain

$$\begin{aligned} QL_{2131} &= s_{123}T_{12}V_3 + s_{12}\left[T_{13}V_2 + V_1T_{23} - T_{12}V_3\right] \\ &\quad - \frac{1}{2}Q\left(s_{123}D_{12}V_3 + s_{12}\left[D_{13}V_2 - V_1D_{23} - D_{12}V_3\right]\right). \end{aligned} \quad (\text{B.11})$$

This implies that

$$\begin{aligned} Q\left(L_{2131} + \frac{1}{2}s_{123}D_{12}V_3 + \frac{1}{2}s_{12}\left[D_{13}V_2 - V_1D_{23} - D_{12}V_3\right]\right) \\ = s_{123}T_{12}V_3 + s_{12}\left[T_{13}V_2 + V_1T_{23} - T_{12}V_3\right]. \end{aligned} \quad (\text{B.12})$$

We define T_{123} as the object inside parenthesis, but stripped of its BRST exact pieces,

$$T_{123} = L_{2131} + \frac{1}{2}s_{123}D_{12}V_3 + \frac{1}{2}s_{12}\left[D_{13}V_2 - V_1D_{23} - D_{12}V_3\right] + Q(\dots). \quad (\text{B.13})$$

The construction of the BRST building blocks goes on in this iterative manner. The BRST building blocks satisfy the BRST cohomology appropriate for kinematical numerators, e.g.

$$QT_{12} = s_{12}V_1V_2, \quad (\text{B.14})$$

$$QT_{123} = s_{123}T_{12}V_3 + s_{12}\left[T_{13}V_2 + V_1T_{23} - T_{12}V_3\right], \quad (\text{B.15})$$

and similarly for higher rank BRST building blocks. The subtraction of the BRST exact pieces also enhances the symmetry properties of these objects, e.g.

$$T_{\{12\}} = 0, \quad (\text{B.16})$$

$$T_{\{12\}3} = T_{\{123\}} = 0. \quad (\text{B.17})$$

The composite fields obtained in the worldline formalism are equivalent to the $L_{ijkn\dots}$ composite fields, as argued. In particular, they satisfy the same BRST cohomology as the $L_{ijkn\dots}$ and the BRST building blocks $T_{ijk\dots}$. By arguments of uniqueness in the BRST cohomology, we might relate any such set of operators by the subtraction of contact terms and BRST exact pieces. As we have seen, the worldline amplitude is naturally written in terms of the $\hat{U}_i \cdots \hat{U}_j V_k$ composite fields. The pinching operators conspire to kill off the BRST exact pieces of these fields, enhancing the symmetry properties of the numerators. Crucially, this procedure allows one to bootstrap the tree-level amplitude from BRST invariance once one computes the composite fields $\hat{U}_i \cdots \hat{U}_j V_k$, as we did.

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