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A priori and a posteriori error estimates for discontinuous Galerkin time-discrete methods via maximal regularity

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Abstract

The maximal regularity property of discontinuous Galerkin methods for linear parabolic equations is used together with variational techniques to establish a priori and a posteriori error estimates of optimal order under optimal regularity assumptions. The analysis is set in the maximal regularity framework of UMD Banach spaces. Similar results were proved in an earlier work, based on the consistency analysis of Radau IIA methods. The present error analysis, which is based on variational techniques, is of independent interest, but the main motivation is that it extends to nonlinear parabolic equations; in contrast to the earlier work. Both autonomous and nonautonomous linear equations are considered.

Keywords Discontinuous Galerkin methods · Maximal regularity · Error estimates · Parabolic equations · UMD Banach space

Mathematics Subject Classification 65M12 · 65M15

1 Introduction

We consider the discretization of differential equations satisfying the maximal parabolic L^p-regularity property in unconditional martingale differences (UMD) Banach spaces by discontinuous Galerkin (dG) methods. We combine the maximal regular-

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ity property of the methods with variational techniques and establish optimal order, optimal regularity, a priori and a posteriori error estimates.

1.1 Maximal parabolic regularity

We consider an initial value problem for a linear parabolic equation,

$$\begin{cases} u'(t) + Au(t) = f(t), \quad 0 < t < T, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

in a UMD Banach space X with initial value $u_0 \in \mathscr{D}(A)$. Our structural assumption is that the closed operator $-A$ is the generator of an analytic semigroup on X having *maximal L^p-regularity*. This means that for vanishing initial value $u_0 = 0$, for any $T \in (0, \infty]$, for some, or, as it turns out, for all $p \in (1, \infty)$, and for any $f \in L^p((0, T); X)$ there exists a unique solution u of (1.1) such that $u' \in L^p((0, T); X)$; then we also have $Au \in L^p((0, T); X)$. As a consequence of the closed graph theorem, the solution u of (1.1) with $u_0 = 0$ satisfies the stability estimate

$$\|u'\|_{L^p((0,T);X)} + \|Au\|_{L^p((0,T);X)} \leqslant c_{p,X}\|f\|_{L^p((0,T);X)} \quad \forall f \in L^p((0, T); X) \tag{1.2}$$

with a constant $c_{p,X}$ independent of T, depending only on p and X; see, e.g., [6] and [10].

Since also $\|f\|_{L^p((0,T);X)} \leqslant \|u'\|_{L^p((0,T);X)} + \|Au\|_{L^p((0,T);X)}$ by the triangle inequality, we conclude that the norm of the sum $\|u' + Au\|_{L^p((0,T);X)}$ and the sum of the norms $\|u'\|_{L^p((0,T);X)} + \|Au\|_{L^p((0,T);X)}$ are equivalent on the Banach space

$$\left\{v \in W^{1,p}((0, T); X) \cap L^p((0, T); \mathscr{D}(A)) : v(0) = 0\right\}$$

with constants independent of T, for $T < \infty$. Here $\mathscr{D}(A) := \{v \in X : Av \in X\}$ is the domain of the operator A.

1.2 The numerical methods

We consider the discretization of the initial value problem (1.1) by dG methods.

Let $N \in \mathbb{N}, k = T/N$ be the constant time step, $t_n := nk, n = 0, \dots, N$, be a uniform partition of the time interval $[0, T]$, and $J_n := (t_n, t_{n+1}]$.

For $s \in \mathbb{N}_0$, we denote by $\mathbb{P}(s)$ and $\mathbb{P}_{X'}(s)$ the spaces of polynomials of degree at most s with coefficients in $\mathscr{D}(A)$ and in the dual X' of X, respectively, i.e., the elements g of $\mathbb{P}(s)$ and of $\mathbb{P}_{X'}(s)$, respectively, are of the form

$$g(t) = \sum_{j=0}^{s} t^j w_j, \quad w_j \in \mathscr{D}(A) \quad \text{and} \quad w_j \in X', \quad j = 0, \dots, s.$$

With this notation, let $\mathcal{V}_k^{\mathrm{c}}(s)$ and $\mathcal{V}_k^{\mathrm{d}}(s)$ be the spaces of continuous and possibly discontinuous piecewise elements of $\mathbb{P}(s)$, respectively,

$$\mathcal{V}_k^{\mathrm{c}}(s) := \{v \in C\big([0,T];\mathscr{D}(A)\big) : v|_{J_n} \in \mathbb{P}(s),\ n = 0,\dots,N-1\},$$
$$\mathcal{V}_k^{\mathrm{d}}(s) := \{v\colon [0,T] \to \mathscr{D}(A) \text{ with } v|_{J_n} \in \mathbb{P}(s),\ n = 0,\dots,N-1\}.$$

Thus, the functions v in $\mathcal{V}_k^{\mathrm{d}}(s)$ are $\mathscr{D}(A)$-valued, possibly discontinuous piecewise polynomials with respect to the intervals J_n, complemented by an initial value $v(0)$. The X-valued function spaces $\mathcal{X}_k^{\mathrm{c}}(s)$ and $\mathcal{X}_k^{\mathrm{d}}(s)$ are defined analogously, with coefficients $w_j \in X$.

We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X and X'.

For $q \in \mathbb{N}$, with starting value $U(0) = U_0 := u_0$ and $f \in L^p((0,T);X)$, we consider the discretization of the initial value problem (1.1) by the *discontinuous Galerkin method* dG$(q-1)$, i.e., we seek $U \in \mathcal{V}_k^{\mathrm{d}}(q-1)$ such that

$$\int_{J_n} \big(\langle U', v\rangle + \langle AU, v\rangle\big)\,\mathrm{d}t + \langle U_n^+ - U_n, v_n^+\rangle = \int_{J_n} \langle f, v\rangle\,\mathrm{d}t \quad \forall v \in \mathbb{P}_{X'}(q-1) \tag{1.3}$$

for $n = 0,\dots,N-1$. As usual, we use the notation $v_n := v(t_n)$, $v_n^+ := \lim_{s\searrow 0} v(t_n + s)$.

1.2.1 A reconstruction operator

With $0 < c_1 < \cdots < c_q = 1$ the Radau nodes in the interval $[0,1]$, let $t_{ni} := t_n + c_i k$, $i = 1,\dots,q$, be the intermediate nodes; we also use the notation $t_{n0} := t_n$. The *reconstruction operator* $\mathcal{V}_k^{\mathrm{d}}(q-1) \to \mathcal{V}_k^{\mathrm{c}}(q)$, $w \mapsto \widehat{w}$, defined via extended interpolation at the Radau nodes, cf. [13],

$$\widehat{w}(t_{nj}) = w(t_{nj}), \quad j = 0,\dots,q \quad (w(t_{n0}) = w_n), \tag{1.4}$$

plays a crucial role in our analysis. Note that the definition requires the initial value $w(0)$ and that the reconstruction operator also maps $\mathcal{X}_k^{\mathrm{d}}(q-1) \to \mathcal{X}_k^{\mathrm{c}}(q)$.

1.3 Main results

We establish the following optimal order, optimal regularity, a priori and a posteriori error estimates.

Theorem 1.1 *(A priori error estimates) Let $p \in (1,\infty)$ and assume that the solution of (1.1) is sufficiently regular, that is, $u_0 \in \mathscr{D}(A)$ and $u \in W^{q,p}\big((0,T);\mathscr{D}(A)\big)$. Then, the dG approximation $U \in \mathcal{V}_k^{\mathrm{d}}(q-1)$ satisfies the estimate*

$$\|A(u-U)\|_{L^p((0,T);X)} \leqslant Ck^q \|Au^{(q)}\|_{L^p((0,T);X)}. \tag{1.5}$$

Furthermore, if in addition $u \in W^{q+1,p}\big((0,T);X\big)$, then for the reconstruction $\widehat{U} \in \mathcal{V}_k^{\mathrm{c}}(q)$, we have

$$\begin{aligned}\|(u-\widehat{U})'\|_{L^p((0,T);X)} + \|A(u-\widehat{U})\|_{L^p((0,T);X)} \\ \leqslant Ck^q\big(\|u^{(q+1)}\|_{L^p((0,T);X)} + \|Au^{(q)}\|_{L^p((0,T);X)}\big).\end{aligned} \tag{1.6}$$

The constant C depends on q, p, and X, but it is independent of the solution u, of T, and of the time step k.

We remark that, since $W^{q,p}((0,T);\mathscr{D}(A)) \subset C([0,T];\mathscr{D}(A))$, the regularity assumption $u \in W^{q,p}\big((0,T);\mathscr{D}(A)\big)$ actually implies $u(0) \in \mathscr{D}(A)$.

Theorem 1.2 *(A posteriori error estimate) For initial value $u_0 \in \mathscr{D}(A)$, let $R(t) := \widehat{U}'(t) + A\widehat{U}(t) - f(t)$ be the residual of the reconstruction $\widehat{U} \in \mathcal{V}_k^{\mathrm{c}}(q)$ of the dG approximation $U \in \mathcal{V}_k^{\mathrm{d}}(q-1)$. Then, the following maximal regularity a posteriori error estimate holds:*

$$\|R\|_{L^p((0,t);X)} \leqslant \|(u-\widehat{U})'\|_{L^p((0,t);X)} + \|A(u-\widehat{U})\|_{L^p((0,t);X)} \leqslant c_{p,X}\|R\|_{L^p((0,t);X)} \tag{1.7}$$

for all $0 < t \leqslant T$ for any $p \in (1,\infty)$ with the constant $c_{p,X}$ from (1.2). Furthermore, the estimator is of optimal asymptotic order of accuracy,

$$\|R\|_{L^p((0,T);X)} \leqslant Ck^q\big(\|u^{(q+1)}\|_{L^p((0,T);X)} + \|Au^{(q)}\|_{L^p((0,T);X)}\big), \tag{1.8}$$

provided that $u \in W^{q,p}\big((0,T);\mathscr{D}(A)\big) \cap W^{q+1,p}\big((0,T);X\big)$.

Our proofs of Theorems 1.1 and 1.2 rely on the maximal regularity property of the dG methods from [3] and on variational techniques. The variational technique is applicable also in the case of nonlinear parabolic equations; see [2]; this is our main motivation.

Similar error estimates were recently established in [3]. The proofs in [3] rely on properties of the Radau IIA methods; in particular, the proof of the optimality of $\|R\|_{L^p((0,T);X)}$ in [3] is significantly lengthier and more involved.

As the proof of Theorem 1.2 is very short, we give it here.

Proof Our assumption $u_0 \in \mathscr{D}(A)$ ensures that the reconstruction $\widehat{U}$ of the dG approximation U belongs to $\mathcal{V}_k^{\mathrm{c}}(q)$. The residual R, the amount by which $\widehat{U}$ misses being an exact solution of the differential equation in (1.1), is a computable quantity, depending only on the numerical solution $\widehat{U}$ and the given forcing term f. Replacing f in the residual by $u' + Au$, we see that the error $\hat{e} := u - \widehat{U}$ satisfies the *error equation*

$$\hat{e}'(t) + A\hat{e}(t) = -R(t), \quad t \in (t_n, t_{n+1}],\ n = 0,\dots,N-1; \quad \hat{e}(0) = 0. \tag{1.9}$$

Now, the triangle inequality and the maximal L^p-regularity (1.2) of the operator A applied to the error equation (1.9) yield (1.7), i.e., the asserted lower and upper a posteriori error estimators.

In view of the representation (1.9) of R, the optimality (1.8) of the residual is an immediate consequence of the a priori error estimate (1.6). □

We present the a priori error analysis in Section 2. In Section 3 we extend these results to the case of nonautonomous linear parabolic equations. The Appendix contains proofs of relevant interpolation error estimates.

2 A priori error estimates

In this section we prove Theorem 1.1.

2.1 A discrete $\ell^p(X)$-norm and its equivalence to the continuous $L^p(X)$-norm

We introduce the discrete $\ell^p(X)$-norm $\|\cdot\|_{\ell^p((0,T);X)}$ on the space $\mathcal{X}_{k,0}^{\mathrm{d}}(q-1) := \{v \in \mathcal{X}_k^{\mathrm{d}}(q-1) : v(0)=0\}$ and on $\mathcal{X}_{k,0}^{\mathrm{c}}(q) := \{v \in \mathcal{X}_k^{\mathrm{c}}(q) : v(0)=0\}$ by

$$\|v\|_{\ell^p((0,T);X)} := \Big(\sum_{\ell=0}^{N-1}\Big(k\sum_{i=1}^{q}\|v(t_{\ell i})\|_X^p\Big)\Big)^{1/p} \tag{2.1}$$

and show that it is equivalent to the continuous $\|\cdot\|_{L^p((0,T);X)}$-norm; see also [1, Lemma 3.9].

Let us focus on the space $\mathcal{X}_{k,0}^{\mathrm{d}}(q-1)$; the proof for the space $\mathcal{X}_{k,0}^{\mathrm{c}}(q)$ is completely analogous. First, it is easily seen that the continuous norm is dominated by the discrete norm,

$$\|v\|_{L^p((0,T);X)} \leqslant \tilde{c}_{q,p}\|v\|_{\ell^p((0,T);X)} \quad \forall v \in \mathcal{X}_{k,0}^{\mathrm{d}}(q-1). \tag{2.2}$$

Indeed, with ℓ_{mi} the Lagrange polynomials $\ell_i \in \mathbb{P}_{q-1}$ for the Radau points $c_1,\dots,c_q$, shifted to the subinterval J_m, we have

$$\begin{aligned}\int_{J_m}\|v(t)\|_X^p\,\mathrm{d}t &= \int_{J_m}\Big\|\sum_{i=1}^{q}\ell_{mi}(t)v(t_{mi})\Big\|_X^p\,\mathrm{d}t \leqslant k\Big(\sum_{i=1}^{q}\|\ell_{mi}\|_{L^\infty(J_m)}\|v(t_{mi})\|_X\Big)^p\\ &\leqslant \Big(\sum_{i=1}^{q}\|\ell_{mi}\|_{L^\infty(J_m)}^{p'}\Big)^{p/p'}\Big(\sum_{i=1}^{q}k\|v(t_{mi})\|_X^p\Big) = c_{q,p}k\sum_{i=1}^{q}\|v(t_{mi})\|_X^p\end{aligned}$$

with p' the dual exponent of p, and by summing over m, we obtain (2.2); cf. [3].

Next, we prove that the discrete norm is dominated by the continuous norm in the reference element $[0,1]$ for polynomials v of degree at most $q-1$ with coefficients in X; a scaling argument then shows that this is the case in arbitrary intervals $(0,T)$. We use the Lagrange form of v,

$$v(t) = \sum_{i=1}^{q}\ell_i(t)v(c_i), \quad t \in [0,1].$$

Let i be such that

$$\|v(c_i)\|_X = \max_{1\leqslant j\leqslant q}\|v(c_j)\|_X. \tag{2.3}$$

We want to show that

$$\|v(c_i)\|_X \leqslant c\|v\|_{L^p((0,1);X)}.$$

We have, for arbitrary but small $\delta > 0$,

$$\|v\|^p_{L^p((0,1);X)} \geqslant \int_{c_i-\delta}^{c_i} \|v(t)\|^p_X \,\mathrm{d}t = \int_{c_i-\delta}^{c_i} \Big\|\ell_i(t)v(c_i) + \sum_{\substack{j=1\\ j\neq i}}^{q} \ell_j(t)v(c_j)\Big\|^p_X \,\mathrm{d}t,$$

whence, in view of (2.3),

$$\|v\|^p_{L^p((0,1);X)} \geqslant \|v(c_i)\|^p_X \int_{c_i-\delta}^{c_i} \Big(|\ell_i(t)| - \sum_{\substack{j=1\\ j\neq i}}^{q} |\ell_j(t)|\Big)^p \,\mathrm{d}t. \tag{2.4}$$

Now, for any $\varepsilon_1, \varepsilon_2 \in (0, 1)$, for sufficiently small δ, we have

$$|\ell_i(t)| \geqslant 1 - \varepsilon_1, \quad |\ell_j(t)| \leqslant \varepsilon_2, \quad j \neq i, \quad \forall t \in [c_i - \delta, c_i],$$

and (2.4) yields

$$\|v\|^p_{L^p((0,1);X)} \geqslant \delta\big(1 - \varepsilon_1 - (q-1)\varepsilon_2\big)^p \|v(c_i)\|^p_X$$

and the desired property follows easily.

The proof for the space $\mathcal{X}^{\mathrm{c}}_{k,0}(q)$ is completely analogous. In this case we shift the Lagrange polynomials $\hat{\ell}_i \in \mathbb{P}_q$ for the points $c_0 = 0, c_1, \ldots, c_q$, to a subinterval J_m, and use the fact that $t_{m0} = t_m = t_{m-1,q}$.

Notice that it is obvious from (1.4) that the discrete $\ell^p(X)$-norms of an element $v \in \mathcal{X}^{\mathrm{d}}_{k,0}(q-1)$ and of its reconstruction $\hat{v} \in \mathcal{X}^{\mathrm{c}}_{k,0}(q)$ coincide,

$$\|\hat{v}\|_{\ell^p((0,T);X)} = \|v\|_{\ell^p((0,T);X)} \quad \forall v \in \mathcal{X}^{\mathrm{d}}_{k,0}(q-1).$$

Since the equivalence constants of the discrete $\|\cdot\|_{\ell^p((0,T);X)}$ and the continuous $\|\cdot\|_{L^p((0,T);X)}$-norms are independent of T, the corresponding discrete $\|\cdot\|_{\ell^p((0,t_n);X)}$ and continuous $\|\cdot\|_{L^p((0,t_n);X)}$-seminorms, $n = 1, \ldots, N$, are also equivalent with constants independent of n. Of course, the discrete $\|\cdot\|_{\ell^p((0,t_n);X)}$-seminorm is the term on the right-hand side of (2.1) with N replaced by n. (It is only a seminorm on $\mathcal{X}^{\mathrm{d}}_{k,0}(q-1)$ if $n < N$.)

2.2 Maximal parabolic regularity of the dG method

We shall use the notation ∂_k for the backward difference operator,

$$\partial_k v := \frac{v(\cdot) - v(\cdot - k)}{k}, \quad v \in \mathcal{X}^{\mathrm{d}}_{k,0}(q-1) \cup \mathcal{X}^{\mathrm{c}}_{k,0}(q)$$

with $v = 0$ in the interval $[-k, 0)$. Obviously, ∂_k commutes with the reconstruction operator, $\partial_k \hat{v} = \widehat{\partial_k v}$, $v \in \mathcal{X}^{\mathrm{d}}_{k,0}(q-1)$.

In the case of vanishing initial value $u_0 = 0$, for the reconstruction $\widehat{U}$ and the dG approximation U we have the following maximal parabolic regularity result

$$\begin{aligned} \|\partial_k \widehat{U}\|_{\ell^p((0,T);X)} + \|\widehat{U}'\|_{L^p((0,T);X)} + \|A\widehat{U}\|_{L^p((0,T);X)} \\ + \|AU\|_{L^p((0,T);X)} \leqslant C_{p,X}\|f\|_{L^p((0,T);X)}, \end{aligned} \tag{2.5}$$

with $C_{p,X}$ a method-dependent constant, independent of N and T. The estimate for the last three terms on the left-hand side of (2.5) is given in [3, (1.9) and the last line on p. 186]; the proof relies on the interpretation of dG methods in [3] as modified Radau IIA methods and on the maximal regularity property of Radau IIA methods from [8].

It remains to prove the estimate for the first term on the left-hand side of (2.5). Now, the nodal values $\widehat{U}_{ni} = U_{ni} = U(t_{ni}), n = 0, \dots, N-1, i = 1, \dots, q$, satisfy the Radau IIA equations with the nodal values $f(t_{ni})$ of the forcing term replaced by the averages f_{ni},

$$f_{ni} := \frac{1}{\int_{J_n} \ell_{ni}(s)\,\mathrm{d}s} \int_{J_n} \ell_{ni}(s) f(s)\,\mathrm{d}s, \quad n = 0, \dots, N-1,\ i = 1, \dots, q;$$

see [3, Lemma 2.2]. Therefore, according to the maximal regularity result of the Radau IIA methods in [9, Lemma 3.6], we have

$$\sum_{i=1}^{q} \|(\partial \widehat{U}_{ni})_{n=0}^{N-1}\|^p_{\ell^p(X)} \leqslant C_{p,q,X} \sum_{i=1}^{q} \|(f_{ni})_{n=0}^{N-1}\|^p_{\ell^p(X)}; \tag{2.6}$$

here, $\widehat{U}_{-1,1} = \cdots = \widehat{U}_{-1,q} = 0$, and for a sequence $(v_n)_{n\in\mathbb{N}_0} \subset X$, we used the notation

$$\partial v_n := \frac{v_n - v_{n-1}}{k} \quad \text{and} \quad \|(v_n)_{n=0}^{N-1}\|_{\ell^p(X)} := \Big(k\sum_{n=0}^{N-1}\|v_n\|_X^p\Big)^{1/p}$$

for the backward difference quotient and for the discrete $\ell^p(X)$-norm $\|\cdot\|_{\ell^p(X)}$.

Notice that the term on the left-hand side of (2.6) coincides with the first term on the left-hand side of (2.5) raised to the power p. Furthermore,

$$\sum_{i=1}^{q} \|(f_{ni})_{n=0}^{N-1}\|^p_{\ell^p(X)} \leqslant \gamma \|f\|^p_{L^p((0,T);X)} \tag{2.7}$$

with a constant γ depending only on $c_1 \dots, c_q$, and p; see [[3], (2.12)]. Now, (2.6) and (2.7) yield

$$\|\partial_k \widehat{U}\|_{\ell^p((0,T);X)} \leqslant C_{p,X}\|f\|_{L^p((0,T);X)},$$

and the proof of (2.5) is complete.

Obviously, (2.5) is also valid with T replaced by $t_n, n = 1, \dots, N$.

Let us note that the estimate for the first term on the left-hand side of (2.5) is not needed in the error analysis for linear parabolic equations; however, it plays a key role in the error analysis for nonlinear parabolic equations; see [2]. The analogous estimate for Radau IIA methods from [9, Lemma 3.6] was used there in the error analysis for Radau IIA methods for nonlinear parabolic equations.

Notice also that due to the equivalence of the discrete $\ell^p(X)$- and the continuous $L^p(X)$-norms established in section 2.1, $\|\partial_k \widehat{U}\|_{\ell^p((0,T);X)}$ can be replaced by $\|\partial_k \widehat{U}\|_{L^p((0,T);X)}$ in the maximal regularity estimate (2.5).

Logarithmically quasi-maximal parabolic regularity results for dG methods were earlier established in [11] in general Banach spaces for autonomous equations and in [12] in Hilbert spaces for nonautonomous equations. These works are not based on the maximal regularity theory and therefore cover also the cases of variable time steps as well as the critical exponents $p = 1, \infty$.

More recently, another approach to the maximal regularity of dG time discretization was presented in [7]. This is based on studying the dG approximation of a temporally regularized Green's function. The result allows quasi-uniform meshes but is restricted to $q \geqslant 2$.

2.3 An interpolant and its approximation properties

We shall use a standard interpolant $\tilde{v} \in \mathcal{X}_k^{\mathrm{d}}(q-1)$ of a function $v \in C([0,T]; X)$ such that $\tilde{v}(t_n) = v(t_n), n = 0, \dots, N$, and $v - \tilde{v}$ is in each subinterval J_n orthogonal to polynomials of degree at most $q - 2$ (with the second condition being void for $q = 1$). Thus, $\tilde{v}$ is determined in J_n by the conditions

$$\begin{cases} \tilde{v}(t_{n+1}) = v(t_{n+1}), \\ \displaystyle\int_{J_n} \big(v(t) - \tilde{v}(t)\big) t^j \, \mathrm{d}t = 0, \quad j = 0, \dots, q-2; \end{cases} \tag{2.8}$$

cf., e.g., [15, (12.9)] and [14, §§3.1–3.3] for the case of Hilbert spaces. Note that $\tilde{v} \in \mathcal{V}_k^{\mathrm{d}}(q-1)$ if $v \in C([0,T]; \mathscr{D}(A))$.

The approximation property (valid for $1 \leqslant p \leqslant \infty, q \geqslant 1$)

$$\|v - \tilde{v}\|_{L^p((0,T);X)} \leqslant Ck^q \|v^{(q)}\|_{L^p((0,T);X)} \tag{2.9}$$

will play an important role in our analysis. Furthermore, we will use the following approximation properties of the reconstruction $\hat{\tilde{v}}$ of $\tilde{v}$,

$$\|v - \hat{\tilde{v}}\|_{L^p((0,T);X)} \leqslant Ck^q \|v^{(q)}\|_{L^p((0,T);X)}, \tag{2.10}$$

$$\|(v - \hat{\tilde{v}})'\|_{L^p((0,T);X)} \leqslant Ck^q \|v^{(q+1)}\|_{L^p((0,T);X)}. \tag{2.11}$$

We shall prove existence and uniqueness of $\tilde{v}$ as well as the approximation properties (2.9), (2.10), and (2.11) in the appendix.

Notice that the orthogonality condition in (2.8) can be equivalently written in the form

$$\int_{J_n} \langle v(t) - \tilde{v}(t), w(t)\rangle \,\mathrm{d}t = 0 \quad \forall w \in \mathbb{P}_{X'}(q-2). \tag{2.12}$$

2.4 A priori error estimates

Using the interpolant $\tilde{u} \in \mathcal{V}_k^{\mathrm{d}}(q-1)$ of u, we decompose the error $e = u - U$ as

$$e = \rho + \vartheta \quad \text{with} \quad \rho := u - \tilde{u} \quad \text{and} \quad \vartheta := \tilde{u} - U \in \mathcal{V}_k^{\mathrm{d}}(q-1).$$

By replacing v by Au in (2.9) and recalling our assumption $u \in W^{q,p}\big((0,T);\mathscr{D}(A)\big)$, we obtain the desired estimate for the interpolation error ρ,

$$\|A\rho\|_{L^p((0,T);X)} \leqslant Ck^q \|Au^{(q)}\|_{L^p((0,T);X)}, \tag{2.13}$$

Therefore, to prove (1.5), it remains to bound ϑ.

Since $\rho = u - \tilde{u}$ is orthogonal to v' in J_n for any $v \in \mathbb{P}_{X'}(q-1)$, cf. (2.12), and vanishes at the nodes t_n, integration by parts shows that it has the following crucial property,

$$\int_{J_n} \langle \rho', v\rangle \,\mathrm{d}t + \langle \rho_n^+ - \rho_n, v_n^+\rangle = 0 \quad \forall v \in \mathbb{P}_{X'}(q-1); \tag{2.14}$$

cf. [15, (12.13)].

Subtracting the dG method (1.3) from the corresponding relation for the exact solution, and using the splitting $e = \rho + \vartheta$ as well as relation (2.14) for ρ, we obtain the following equation for $\vartheta \in \mathcal{V}_k^{\mathrm{d}}(q-1)$,

$$\int_{J_n} \big(\langle \vartheta', v\rangle + \langle A\vartheta, v\rangle\big) \,\mathrm{d}t + \langle \vartheta_n^+ - \vartheta_n, v_n^+\rangle = -\int_{J_n} \langle A\rho, v\rangle \,\mathrm{d}t \tag{2.15}$$

for all $v \in \mathbb{P}_{X'}(q-1)$ and $n = 0, \dots, N-1$. Below we also use the reconstruction $\hat{\vartheta} = \hat{\tilde{u}} - \widehat{U} \in \mathcal{V}_k^{\mathrm{c}}(q)$ of ϑ.

Since $\vartheta(0) = 0$, the maximal regularity property (2.5) of the dG method applied to the error equation (2.15) yields

$$\|\hat{\vartheta}'\|_{L^p((0,T);X)} + \|A\hat{\vartheta}\|_{L^p((0,T);X)} + \|A\vartheta\|_{L^p((0,T);X)} \leqslant C_{p,X}\|A\rho\|_{L^p((0,T);X)},$$

and thus, in view of (2.13),

$$\|\hat{\vartheta}'\|_{L^p((0,T);X)} + \|A\hat{\vartheta}\|_{L^p((0,T);X)} + \|A\vartheta\|_{L^p((0,T);X)} \leqslant Ck^q \|Au^{(q)}\|_{L^p((0,T);X)}. \tag{2.16}$$

Now, (1.5) follows immediately from (2.13) and (2.16). To prove (1.6), we write

$$\hat{e} = u - \widehat{U} = (u - \hat{\tilde{u}}) + (\hat{\tilde{u}} - \widehat{U}) = \hat{\rho} + \hat{\vartheta}$$

and combine (2.16) with the estimates for $\hat{\rho} := u - \hat{\hat{u}}$ and $\hat{\rho}'$ from (2.10) and (2.11), namely

$$\|A\hat{\rho}\|_{L^p((0,T);X)} \leqslant Ck^q \|Au^{(q)}\|_{L^p((0,T);X)},$$
$$\|\hat{\rho}'\|_{L^p((0,T);X)} \leqslant Ck^q \|u^{(q+1)}\|_{L^p((0,T);X)}.$$

3 Extension to nonautonomous equations

In this section, we extend the maximal parabolic regularity stability estimates for dG methods to nonautonomous parabolic equations by a perturbation argument. For similar ideas and results, we refer to [12] and [9, §3.6], [4] for the dG method with piecewise constant elements and for Radau IIA methods, respectively. Furthermore, we establish optimal order a priori and a posteriori error estimates.

We consider an initial value problem for a nonautonomous linear parabolic equation,

$$\begin{cases} u'(t) + A(t)u(t) = f(t), & 0 < t < T, \\ u(0) = u_0, \end{cases} \tag{3.1}$$

in a UMD Banach space X.

Our structural assumptions on $A(t)$ are that all operators $A(t), t \in [0, T]$, share the same domain $\mathscr{D}(A)$, $-A(t)$ is the generator of an analytic semigroup on X having maximal L^p-regularity, for every $t \in [0, T]$, $A(t)$ induce equivalent norms on $\mathscr{D}(A)$,

$$\|A(t)v\|_X \leqslant c\|A(\tilde{t})v\|_X \quad \forall t, \tilde{t} \in [0, T]\ \forall v \in \mathscr{D}(A), \tag{3.2}$$

and $A(t)\colon \mathscr{D}(A) \to X$ satisfies the Lipschitz condition with respect to t,

$$\|\big(A(t) - A(\tilde{t})\big)v\|_X \leqslant L|t - \tilde{t}|\|A(s)v\|_X \quad \forall t, \tilde{t} \in [0, T]\ \forall v \in \mathscr{D}(A), \tag{3.3}$$

for all $s \in [0, T]$.

With starting value $U(0) = U_0 := u_0$ and source term $f \in L^p((0, T); X)$, we consider the discretization of the initial value problem (3.1) by the dG$(q-1)$ method, i.e., we seek $U \in \mathcal{V}_k^{\mathrm{d}}(q - 1)$ such that

$$\int_{J_n} \big(\langle U', v\rangle + \langle A(t)U, v\rangle\big)\, \mathrm{d}t + \langle U_n^+ - U_n, v_n^+\rangle = \int_{J_n} \langle f, v\rangle\, \mathrm{d}t \quad \forall v \in \mathbb{P}_{X'}(q - 1) \tag{3.4}$$

for $n = 0, \dots, N - 1$; cf. (1.3).

3.1 Maximal parabolic regularity

Here, for vanishing starting value $U_0 = 0$, we establish maximal parabolic regularity of dG methods for the nonautonomous parabolic equation (3.1) via a perturbation argument.

For a fixed t_m, $m \in \{1, \dots, N\}$, we write (3.4) in the form

$$\int_{J_n} \langle U', v\rangle \, \mathrm{d}t + \int_{J_n} \langle A(t_m)U, v\rangle \, \mathrm{d}t + \langle U_n^+ - U_n, v_n^+\rangle \\ = \int_{J_n} \langle (A(t_m) - A(t))U, v\rangle \, \mathrm{d}t + \int_{J_n} \langle f, v\rangle \, \mathrm{d}t \quad \forall v \in \mathbb{P}_{X'}(q-1), \tag{3.5}$$

$n = 0, \dots, m-1$. Since the time t is frozen at t_m on the left-hand side of (3.5), we can apply the maximal parabolic regularity estimate (2.5) for dG methods for autonomous equations and obtain

$$\|\partial_k \widehat{U}\|_{\ell^p((0,t_\ell);X)} + \|\widehat{U}'\|_{L^p((0,t_\ell);X)} + \|A(t_m)\widehat{U}\|_{L^p((0,t_\ell);X)} + E_\ell \\ \leqslant C_{p,X}\big(G_m + \|f\|_{L^p((0,t_m);X)}\big), \quad \ell = 1, \dots, m, \tag{3.6}$$

with

$$G_m := \|(A(t_m) - A(\cdot))U\|_{L^p((0,t_m);X)}, \ E_\ell := \|A(0)U\|_{L^p((0,t_\ell);X)}, \ \ell = 1, \dots, m,$$

and $E_0 := 0$. Notice that to render E_ℓ independent of m, we took advantage of the equivalence of norms (3.2) and replaced the argument t_m of the operator A by 0. We also extended the interval of integration from $(0, t_\ell)$ to $(0, t_m)$ in the definition of G_m. Our goal is to bound E_m by a Gronwall argument.

First, according to (3.6),

$$E_m^p \leqslant C\big(G_m^p + \|f\|^p_{L^p((0,t_m);X)}\big). \tag{3.7}$$

Furthermore,

$$G_m^p = \|(A(t_m) - A(\cdot))U\|^p_{L^p((0,t_m);X)} = \sum_{\ell=0}^{m-1} \|(A(t_m) - A(\cdot))U\|^p_{L^p(J_\ell;X)},$$

and thus, in view of the Lipschitz condition (3.3) with $s = 0$,

$$G_m^p \leqslant L^p \sum_{\ell=0}^{m-1} (t_m - t_\ell)^p \|A(0)U\|^p_{L^p(J_\ell;X)}.$$

Now, $t_m - t_\ell \leqslant T - t_\ell$, $\ell = 0, \dots, m-2$, and $t_m - t_{m-1} = k$, and the previous estimate yields

$$G_m^p \leqslant L^p \sum_{\ell=0}^{m-2} (T - t_\ell)^p (E_{\ell+1}^p - E_\ell^p) + L^p k^p (E_m^p - E_{m-1}^p).$$

Neglecting the nonpositive quantity $-E_{m-1}^p$ in the last term on the right-hand side, summing by parts, and estimating the coefficient $(T - t_{m-2})^p$ of E_{m-1}^p from above by T^p, we have

$$G_m^p \leqslant L^p \sum_{\ell=1}^{m-1} a_\ell E_\ell^p + L^p k^p E_m^p, \tag{3.8}$$

with $a_\ell := (T - t_{\ell-1})^p - (T - t_\ell)^p$, $\ell = 1, \dots, m-2$, $a_{m-1} := T^p$. Thus, (3.7) yields

$$E_m^p \leqslant C\|f\|^p_{L^p((0,t_m);X)} + C \sum_{\ell=1}^{m-1} a_\ell E_\ell^p + Ck^p E_m^p.$$

Hence, for $Ck^p \leqslant 1/2$,

$$E_m^p \leqslant C\|f\|^p_{L^p((0,t_m);X)} + C \sum_{\ell=1}^{m-1} a_\ell E_\ell^p, \quad m = 2, \dots, N. \tag{3.9}$$

Since the sum $\sum_{\ell=1}^m a_\ell$ is uniformly bounded,

$$\sum_{\ell=1}^{m} a_\ell = (T - t_0)^p - (T - t_{m-1})^p + T^p \leqslant 2T^p,$$

a discrete Gronwall-type argument applied to (3.9) leads to

$$E_m^p \leqslant C\|f\|^p_{L^p((0,t_m);X)}. \tag{3.10}$$

By combining (3.10) with (3.8) and (3.6), we obtain, for sufficiently small k, the desired maximal parabolic regularity stability estimate

$$\begin{aligned} &\|\partial_k \widehat{U}\|_{\ell^p((0,t_m);X)} + \|\widehat{U}'\|_{L^p((0,t_m);X)} + \|A(t_m)\widehat{U}\|_{L^p((0,t_m);X)} \\ &\quad + \|A(t_m)U\|_{L^p((0,t_m);X)} \leqslant C_{p,X,T}\|f\|_{L^p((0,t_m);X)}, \quad m = 1, \dots, N, \end{aligned} \tag{3.11}$$

with a constant $C_{p,X,T}$ independent of k. Notice that, due to the equivalence of norms (3.2), $A(t_m)$ can be replaced by $A(s)$ on the left-hand side of (3.11), for arbitrary $s \in [0, T]$.

3.2 A priori error estimates

Here, we establish the analogue of Theorem 1.1 in the nonautonomous case.

Theorem 3.1 *(A priori error estimates) Assume that the solution of (3.1) is sufficiently regular, that is, $u_0 \in \mathscr{D}(A)$ and $u \in W^{q,p}\big((0, T); \mathscr{D}(A)\big)$. Then, for sufficiently small k, the dG approximation $U \in \mathcal{V}_k^{\mathrm{d}}(q-1)$ of (3.4) satisfies the estimate*

$$\|A(s)(u - U)\|_{L^p((0,T);X)} \leqslant Ck^q \|A(s)u^{(q)}\|_{L^p((0,T);X)} \tag{3.12}$$

for any $s \in [0, T]$. Furthermore, if in addition $u \in W^{q+1,p}\big((0, T); X\big)$, for the reconstruction $\widehat{U} \in \mathcal{V}_k^{\mathrm{c}}(q)$ of U, we have

$$\begin{aligned}\|(u-\widehat{U})'\|_{L^p((0,T);X)} + \|A(s)(u-\widehat{U})\|_{L^p((0,T);X)} \\ \leqslant Ck^q\big(\|u^{(q+1)}\|_{L^p((0,T);X)} + \|A(s)u^{(q)}\|_{L^p((0,T);X)}\big).\end{aligned} \tag{3.13}$$

The constant C depends on q, p, L, X, and T, but it is independent of the solution u and of the time step k.

Proof We proceed as in the proof of Theorem 1.1. In particular, we decompose the error $e = u - U$ in the form

$$e = \rho + \vartheta \quad \text{with} \quad \rho := u - \tilde{u} \quad \text{and} \quad \vartheta := \tilde{u} - U \in \mathcal{V}_k^{\mathrm{d}}(q-1).$$

The error ρ can be estimated as in Section 2.4; for instance, the analogue of (2.13) reads

$$\|A(s)(u-\tilde{u})\|_{L^p((0,T);X)} \leqslant ck^q\|A(s)u^{(q)}\|_{L^p((0,T);X)}, \tag{3.14}$$

for any $s \in [0, T]$.

Furthermore, subtracting the dG method (3.4) from the corresponding equation for the exact solution u of (3.1) and using the identity (2.14) for ρ, we obtain the following equation for ϑ,

$$\int_{J_n} \big(\langle \vartheta', v\rangle + \langle A(t)\vartheta, v\rangle\big)\,\mathrm{d}t + \langle \vartheta_n^+ - \vartheta_n, v_n^+\rangle = -\int_{J_n} \langle A(t)\rho, v\rangle\,\mathrm{d}t \tag{3.15}$$

for all $v \in \mathbb{P}_{X'}(q-1)$ and $n = 0, \dots, N-1$.

Since $\vartheta(0) = 0$, the maximal regularity property of the dG method for nonautonomous equations (see (3.11)) applied to the error equation (3.15) yields

$$\|\hat{\vartheta}'\|_{L^p((0,T);X)} + \|A(s)\hat{\vartheta}\|_{L^p((0,T);X)} + \|A(s)\vartheta\|_{L^p((0,T);X)} \leqslant C_{p,X}\|A(\cdot)\rho\|_{L^p((0,T);X)},$$

for any $s \in [0, T]$, and thus, in view of (3.14),

$$\begin{aligned}\|\hat{\vartheta}'\|_{L^p((0,T);X)} + \|A(s)\hat{\vartheta}\|_{L^p((0,T);X)} \\ + \|A(s)\vartheta\|_{L^p((0,T);X)} \leqslant Ck^q\|A(s)u^{(q)}\|_{L^p((0,T);X)}.\end{aligned}$$

The proof can now be completed as in the case of Theorem 1.1. □

3.3 A posteriori error estimates

Let R be the *residual* of the reconstruction $\widehat{U}$,

$$R(t) := \widehat{U}'(t) + A(t)\widehat{U}(t) - f(t), \quad t \in (t_n, t_{n+1}], \quad n = 0, \dots, N-1.$$

Then, the error $\hat{e} := u - \widehat{U}$ satisfies the *error equation*

$$\hat{e}'(t) + A(t)\hat{e}(t) = -R(t), \quad t \in (t_n, t_{n+1}],\ n = 0, \dots, N-1; \quad \hat{e}(0) = 0. \quad (3.16)$$

Let us now fix an $s \in (0, T]$. To apply the *maximal L^p-regularity* of the operator $A(s)$, frozen at time s, we rewrite (3.16) in the form

$$\hat{e}'(t) + A(s)\hat{e}(t) = [A(s) - A(t)]\hat{e}(t) - R(t), \quad t \in (0, s].$$

Proceeding as in the proof of the maximal regularity property for nonautonomous parabolic equations in the continuous case, cf., e.g., [4, §4.2] for the derivation of a posteriori estimates for Radau IIA methods, we obtain the desired a posteriori error estimate

$$\|\hat{e}'\|_{L^p((0,s);X)} + \|A(s)\hat{e}\|_{L^p((0,s);X)} \leqslant C\|R\|_{L^p((0,s);X)}, \quad 0 < s \leqslant T,$$

for any $p \in (1, \infty)$, with a constant C depending on p, X, L, and T, but independent of s.

Furthermore, the triangle inequality applied to (3.16) yields a lower a posteriori error estimator,

$$\|R\|_{L^p((0,s);X)} \leqslant \|\hat{e}'\|_{L^p((0,s);X)} + \|A(\cdot)\hat{e}\|_{L^p((0,s);X)}, \quad 0 < s \leqslant T.$$

As in the autonomous case, we see that the a posteriori error estimator is of optimal order as an immediate consequence of the a priori error estimate (3.13).

A Interpolation error estimates

We prove error estimates for $\tilde{v}$ and $\hat{\tilde{v}}$. We shall use a standard argument based on the Bramble–Hilbert lemma; cf. [5, (4.4.4)]. The key ingredients are reproduction of polynomials and boundedness with respect to a relevant Sobolev norm.

A.1 Existence and uniqueness of $\tilde{v}$

The interpolant $\tilde{v}$ of a function $v \in C([0, T]; X)$ can be expressed in terms of the value $v(t_{n+1})$ of v at t_{n+1} and the Legendre coefficients $v_0, \dots, v_{q-2} \in X$ of v,

$$v_i := \frac{1}{\|L_{ni}\|^2_{L^2(J_n)}} \int_{J_n} L_{ni}(t)v(t)\,\mathrm{d}t. \quad (A.1)$$

More precisely, for $t \in J_n$,

$$\tilde{v}(t) = (P_{q-2}v)(t) + L_{n,q-1}(t)\Big[v(t_{n+1}) - \sum_{i=0}^{q-2} v_i\Big], \quad P_{q-2}v = \sum_{i=0}^{q-2} L_{ni}v_i, \quad (A.2)$$

with P_{q-2} the piecewise L^2-projection onto $\mathcal{X}_k^{\mathrm{d}}(q-2)$; here, L_{ni} are the Legendre polynomials L_i of degree i shifted to the interval J_n,

$$L_{ni}\big(\tfrac{1}{2}(t_n + t_{n+1} + ks)\big) = L_i(s), \quad s \in [-1, 1];$$

see [14, (3.2)]. Indeed, the part $P_{q-2}v$ of $\tilde{v}$ on the right-hand side of (A.2) is due to the orthogonality of $v - \tilde{v}$ to the orthogonal polynomials $L_{ni}, i = 0, \ldots, q-2$. The coefficient of $L_{n,q-1}$ is determined by the interpolation condition $\tilde{v}(t_{n+1}) = v(t_{n+1})$ and the property $L_{ni}(t_{n+1}) = 1$ of the Legendre polynomials, which yields $(P_{q-2}v)(t_{n+1}) = v_0 + \cdots + v_{q-2}$. In particular, $\tilde{v} = v$ for $v \in \mathcal{X}_k^{\mathrm{d}}(q-1)$.

A.2 Proof of the approximation property (2.9)

We first consider interpolation of functions on the unit interval $(0, 1)$. Here we assume $1 \leqslant p \leqslant \infty$ and $q \geqslant 1$ and hence Sobolev's embedding $W^{q,p}((0,1); X) \subset C([0,1]; X)$ holds. From (A.1) and (A.2) it is clear that the interpolation operator $C([0,1]; X) \to W^{q,p}((0,1); X)$, $v \mapsto \tilde{v}$, is bounded, so that by Sobolev's inequality,

$$\|\tilde{v}\|_{W^{q,p}((0,1);X)} \leqslant C\|v\|_{C([0,1];X)} \leqslant C\|v\|_{W^{q,p}((0,1);X)}. \tag{A.3}$$

Moreover,

$$\tilde{v} = v \quad \forall v \in \mathbb{P}_X(q-1). \tag{A.4}$$

Hence, by a standard argument based on the Bramble–Hilbert lemma, we have

$$\|v - \tilde{v}\|_{W^{q,p}((0,1);X)} \leqslant C|v|_{W^{q,p}((0,1);X)}. \tag{A.5}$$

Here $|v|_{W^{q,p}((0,1);X)} = \|v^{(q)}\|_{L^p((0,1);X)}$ denotes the seminorm.

In fact, by the Bramble–Hilbert lemma there is a Taylor polynomial $\bar{v} \in \mathbb{P}_X(q-1)$ such that

$$\|v - \bar{v}\|_{W^{q,p}((0,1);X)} \leqslant C|v|_{W^{q,p}((0,1);X)}. \tag{A.6}$$

Hence, since by (A.4) $\tilde{\bar{v}} = \bar{v}$, and by (A.3), (A.6),

$$\begin{aligned}\|v - \tilde{v}\|_{W^{q,p}((0,1);X)} &\leqslant \|v - \bar{v}\|_{W^{q,p}((0,1);X)} + \|\bar{v} - \tilde{v}\|_{W^{q,p}((0,1);X)} \\ &= \|v - \bar{v}\|_{W^{q,p}((0,1);X)} + \|(\bar{v} - v)\tilde{}\,\|_{W^{q,p}((0,1);X)} \\ &\leqslant C\|v - \bar{v}\|_{W^{q,p}((0,1);X)} \leqslant C|v|_{W^{q,p}((0,1);X)},\end{aligned}$$

which is (A.5).

Since $|v|_{W^{j,p}((0,1);X)} \leqslant \|v\|_{W^{q,p}((0,1);X)}$ and $|v|_{L^\infty((0,1);X)} \leqslant C\|v\|_{W^{q,p}((0,1);X)}$, from (A.5) we now conclude

$$\begin{aligned}&|v - \tilde{v}|_{W^{j,p}((0,1);X)} \leqslant C|v|_{W^{q,p}((0,1);X)}, \quad j = 0, \ldots, q, \\ &\|v - \tilde{v}\|_{L^\infty((0,1);X)} \leqslant C|v|_{W^{q,p}((0,1);X)}.\end{aligned}$$

Finally, by a change of variables and with a slight abuse of notation, we have

$$|v|_{W^{j,p}((0,1);X)} = k^{j-1/p}|v|_{W^{j,p}(J_n;X)},$$
$$\|v\|_{L^\infty((0,1);X)} = \|v\|_{L^\infty(J_n;X)}.$$

This proves

$$|v-\tilde{v}|_{W^{j,p}(J_n;X)} \leqslant Ck^{q-j}|v|_{W^{q,p}(J_n;X)}, \quad j=0,\dots,q, \tag{A.7}$$
$$\|v-\tilde{v}\|_{L^\infty(J_n;X)} \leqslant Ck^{q-1/p}|v|_{W^{q,p}(J_n;X)}. \tag{A.8}$$

As a consequence of (A.7) with $j=0$ (or of (A.8)) we have (2.9). This is the only case that we need in the present work; the other cases are included for the sake of completeness.

The approximation property (A.8) for general p and for Banach spaces is an analogue of [15, (12.10)] for $p=2$ and for Hilbert spaces.

A.3 Proof of the approximation properties (2.10) and (2.11)

We begin by proving that the operator $v \mapsto \hat{\tilde{v}}$ reproduces polynomials of degree at most q. Again we assume $1 \leqslant p \leqslant \infty$.

Lemma A.1 *(Reproduction property) Assume that $v \in \mathcal{X}_k^{\mathrm{c}}(q)$. Then, the following reproduction property holds:*

$$\hat{\tilde{v}} = v. \tag{A.9}$$

In particular, if $v \in \mathcal{X}_k^{\mathrm{c}}(q-1)$, then $\hat{\tilde{v}} = \tilde{v} = v$.

Proof Let us consider the Lagrange interpolant $w \in \mathcal{X}_k^{\mathrm{d}}(q-1)$, with $w(0)=v(0)$, of v at the Radau nodes,

$$w(t_{nj}) = v(t_{nj}), \quad j=1,\dots,q, \tag{A.10}$$

for $n=0,\dots,N-1$. Then, it follows immediately from the definition (1.4) that $v \in \mathcal{X}_k^{\mathrm{c}}(q)$ is the reconstruction of $w \in \mathcal{X}_k^{\mathrm{d}}(q-1)$, that is, $v=\widehat{w}$.

It remains to show that $\tilde{v}=w$, that is, that w satisfies the orthogonality condition

$$\int_{t_n}^{t_{n+1}} \big(v(t)-w(t)\big)t^j\,\mathrm{d}t = 0, \quad j=0,\dots,q-2.$$

This can be easily seen by using the Lagrange form of the interpolation error $v(t)-w(t)=(v_n-w_n^+)\ell_{n0}(t)$, with ℓ_{n0} the polynomial of degree q vanishing at the nodes $t_{n1},\dots,t_{nq}$ and taking the value 1 at the node t_{n0}, and the orthogonality of ℓ_{n0} to polynomials of degree at most $q-2$ or, equivalently, by the exactness of the Radau quadrature rule for polynomials of degree $2q-2$.

For $v \in \mathcal{X}_k^{\mathrm{c}}(q-1)$, we obviously have $w=v$ by (A.10), i.e., $\tilde{v}=v$. □

We next consider the boundedness of $v \mapsto \hat{\tilde{v}}$. Let $v \in C([0,T];X)$. The hat operator interpolates $\tilde{v}$ at the node values $\hat{\tilde{v}}(t_{nj}) = \tilde{v}(t_{nj})$, $j = 0,\dots,q$, where $\tilde{v}(t_{n0}) = \tilde{v}(t_n) = v(t_n)$ and $\tilde{v}(t_{nq}) = \tilde{v}(t_{n+1}) = v(t_{n+1})$. Thus, in view of (A.1) and (A.2), $\hat{\tilde{v}}|_{J_n}$ depends only on the values of v in $[t_n, t_{n+1}]$. More precisely, after a transformation to the unit interval, we have

$$\|\hat{\tilde{v}}\|_{W^{q+1,p}((0,1);X)} \leqslant C\|v\|_{C([0,1];X)} \leqslant C\|v\|_{W^{q+1,p}((0,1);X)}.$$

Moreover, according to (A.9),

$$\hat{\tilde{v}} = v \quad \forall v \in \mathbb{P}_X(q).$$

By the same argument as in Subsection A.2, we conclude

$$\begin{aligned} |v - \hat{\tilde{v}}|_{W^{j,p}((0,1);X)} &\leqslant C|v|_{W^{q+1,p}((0,1);X)}, \quad j = 0,\dots,q+1,\\ \|v - \hat{\tilde{v}}\|_{L^\infty((0,1);X)} &\leqslant C|v|_{W^{q+1,p}((0,1);X)}, \end{aligned}$$

which proves

$$\begin{aligned} |v - \hat{\tilde{v}}|_{W^{j,p}(J_n;X)} &\leqslant Ck^{q+1-j}|v|_{W^{q+1,p}(J_n;X)}, \quad j = 0,\dots,q+1,\\ \|v - \hat{\tilde{v}}\|_{L^\infty(J_n;X)} &\leqslant Ck^{q+1-1/p}|v|_{W^{q+1,p}(J_n;X)}. \end{aligned}$$

In particular, for $j = 1$, we obtain (2.11).

Since $v \mapsto \hat{\tilde{v}}$ also reproduces polynomials of degree at most $q-1$, the same argument shows

$$|v - \hat{\tilde{v}}|_{W^{j,p}(J_n;X)} \leqslant Ck^{q-j}|v|_{W^{q,p}(J_n;X)}, \quad j = 0,\dots,q, \tag{A.11}$$

$$\|v - \hat{\tilde{v}}\|_{L^\infty(J_n;X)} \leqslant Ck^{q-1/p}|v|_{W^{q,p}(J_n;X)}. \tag{A.12}$$

From (A.11) for $j = 0$ (or from (A.12)), we obtain (2.10).

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