Quadratic Equality Constrained Least Squares: Low-complexity ADMM for Global Optimality

Citation for the original published paper (version of record):
Wei, T., Huang, H., Wu, L. et al (2026). Quadratic Equality Constrained Least Squares:
Low-complexity ADMM for Global Optimality. IEEE Signal Processing Letters, 33: 361-365.
http://dx.doi.org/10.1109/LSP.2025.3646132

(article starts on next page)

Quadratic Equality Constrained Least Squares: Low-Complexity ADMM for Global Optimality

Tong Wei, *Member, IEEE*, Huiping Huang, *Member, IEEE*, Linlong Wu, *Senior Member, IEEE*, Chong-Yung Chi, *Life Fellow, IEEE*, Bhavani Shankar M. R., *Senior Member, IEEE*, and Björn Ottersten, *Fellow, IEEE*

***Abstract*—This letter addresses the quadratic equality constrained least squares (QEC-LS) problem, a class of non-convex optimization problems that arise in various signal processing and communication applications. We revisit the alternating direction method of multipliers (ADMM) approach to QEC-LS problem and investigate its convergence and efficiency. Despite the inherent non-convexity, the proposed ADMM algorithm is proved to converge globally only requiring the quadratic term equal to a positive constant. Numerical results demonstrate that our method achieves global optimality with significantly reduced complexity compared to existing approaches such as semidefinite relaxation and primal-dual methods.**

***Index Terms*—ADMM, global optimality, non-convex quadratic equality constraint.**

I. Introduction

QUADRATICALLY constrained quadratic programming (QCQP) problems with inequality constraints have been extensively studied in the literature [1], [2], [3]. In this paper, we focus on a special case of QCQP known as the quadratic equality constrained least squares (QEC-LS):

$$\underset{\mathbf{x}\in\mathbb{C}^N}{\text{minimize}}\ \|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2, \quad \text{subject to}\ \mathbf{x}^H\mathbf{x}=P, \tag{1}$$

where $\mathbf{y}\in\mathbb{C}^M$, $\mathbf{A}\in\mathbb{C}^{M\times N}$ is of full column rank, $\mathbb{C}$ denotes complex numbers, M and N are positive integers, $(\cdot)^H$ is the Hermitian transpose of a matrix or vector argument, and $P>0$ is a given parameter. Problem (1) aims to solve a least square problem on an ℓ_2-norm sphere, which has been investigated in [4], [5], [6] among others. This QEC-LS problem arises in a wide range of applications including radar systems [7], [8], wireless communications [9], integrated sensing and communication (ISAC) [10], array signal processing [11], and image processing [12], [13], where finding efficient and accurate solutions is critical.

Despite its broad applicability, solving the QEC-LS problem poses significant challenges due to the non-convex quadratic equality constraint. As an early investigation, [5] showed that strong duality can still hold for the non-convex problem (1) under mild conditions. Following this statement, the projection method proposed in [6], [14] is introduced as a numerical approach to solving problem (1). However, this method does not always guarantee global optimality unless the problem is well-posed. To address these challenges, semidefinite relaxation (SDR), which has been widely adopted as a convex reformulation technique, can recover globally optimal solutions to the QEC-LS problem [15], [16]. Despite its theoretical appeal, SDR requires lifting the original problem into a higher-dimensional semidefinite form, leading to a significant increase in computational burden. This becomes especially prohibitive for large-scale problems and real-time applications. Alternatively, the primal-dual method (PDM) offers iterative schemes that directly handle constraints through dual variable updates [10]. Although the PDM (based on the Karush–Kuhn–Tucker (KKT) conditions) offers a theoretically sound approach to the QEC-LS problem, it still requires determining the Lagrange multiplier via iterative methods, such as bisection or Newton's method. This process involves solving a nonlinear scalar equation to satisfy the power constraint, which becomes increasingly time-consuming in large-scale settings; see [2]. Moreover, the convergence speed of bisection depends on selecting a valid initial range for the dual variable, which is not always straightforward and may even require the eigenvalue decomposition. These practical limitations motivate the development of a more computationally efficient approach, especially for large-scale systems.

Recently, alternating direction method of multipliers (ADMM) has emerged as a powerful tool for constrained optimization, owing to its decomposition capability and empirical success in various non-convex settings [2], [17]. Although originally designed for convex problems, ADMM and its variants have been successfully extended to certain classes of non-convex problems with provable convergence guarantees under mild assumptions [18]. However, the analysis is used for non-convex

Received 18 September 2025; revised 3 December 2025; accepted 10 December 2025. Date of publication 19 December 2025; date of current version 2 January 2026. This work was supported in part by Luxembourg National Research Fund (FNR) through the CORE Project S3 under Grant C22/IS/17412681/S3, in part by HORIZON-MSCA through Project NEAT-6 G under Grant 101152670, in part by the National Science Technology Council (NSTC) under Grant NSTC 114-2221-E-007-008, and in part by FNR under Grant INTER/MOBILITY/2023/IS/18014377/MCR. The associate editor coordinating the review of this article and approving it for publication was Dr. Wenqiang Pu. *(Corresponding author: Huiping Huang.)*

Tong Wei, Bhavani Shankar M. R., and Björn Ottersten are with the Interdisciplinary Centre for Security, Reliability and Trust (SnT), University of Luxembourg, L-1855 Luxembourg City, Luxembourg (e-mail: tong.wei@uni.lu; bhavani.shankar@uni.lu; bjorn.ottersten@uni.lu).

Huiping Huang is with the Department of Electrical Engineering, Chalmers University of Technology, 41296 Gothenburg, Sweden (e-mail: huiping@chalmers.se).

Linlong Wu is with the School of Information and Communication Engineering, University of Electronic Science and Technology of China (UESTC), Chengdu 611731, China (e-mail: linlong.wu@uestc.edu.cn).

Chong-Yung Chi is with the Institute of Communications Engineering and Department of Electrical Engineering, National Tsing Hua University, Hsinchu 300044, Taiwan (e-mail: cychi@ee.nthu.edu.tw).

This article has supplementary downloadable material available at https://doi.org/10.1109/LSP.2025.3646132, provided by the authors.

Digital Object Identifier 10.1109/LSP.2025.3646132

objectives over convex feasible sets which can not be applied to problem (1).

In this letter, we propose an ADMM-based algorithm to solve the QEC-LS problem. Our method reformulates the original problem into an amenable form for the efficient ADMM updates. Despite the non-convexity, we show that the proposed algorithm converges to a globally optimal solution under specific, but practical, conditions. Global convergence of the ADMM algorithm is rigorously proven. Compared to the existing approaches, the proposed method achieves competitive performance with significantly reduced computational burden, i.e., orders of magnitude faster than the SDR and three times faster than the primal-dual method.

II. Global ADMM Solution

We can first reformulate problem (1) equivalently as

$$p^\star:\ \underset{\mathbf{x},\mathbf{z}}{\text{minimize}}\quad \|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2$$
$$\text{subject to}\quad \mathbf{z}-\mathbf{x}=\mathbf{0},\ \mathbf{z}^H\mathbf{z}=P, \tag{2}$$

where $\mathbf{z}$ is the additional variable. We introduce the following theorem[1] to reveal the duality property of (2).

Theorem 1: The strong duality holds for non-convex problem (2) under the mild condition $P>0$.

Proof: The Lagrangian of problem (2) is

$$\mathcal{L}_0(\mathbf{x},\mathbf{z},\mathbf{u},\nu)=\mathbf{y}^H\mathbf{y}-\mathbf{y}^H\mathbf{A}\mathbf{x}-\mathbf{x}^H\mathbf{A}^H\mathbf{y}+\mathbf{x}^H\mathbf{A}^H\mathbf{A}\mathbf{x}$$
$$+\nu(\mathbf{z}^H\mathbf{z}-P)+\Re(\mathbf{u}^H(\mathbf{z}-\mathbf{x})), \tag{3}$$

where ν and $\mathbf{u}$ denote the dual variables, and $\Re(\cdot)$ denotes the real part of the complex argument. If $\nu=0$ and $\mathbf{u}=\mathbf{0}$, the minimization of Lagrangian (3) reduces to the unconstrained least-squares problem, whose solution generally does not satisfy the constraint $\|\mathbf{x}\|^2=P$. Therefore, we focus on the non-trivial case of $\nu\neq 0$ and $\mathbf{u}\neq\mathbf{0}$. Let the complex gradient of (3) w.r.t. $\mathbf{x}^*$, and $\mathbf{z}^*$, respectively, equal to zero:

$$\nabla_{\mathbf{x}}\mathcal{L}_0(\mathbf{x},\mathbf{z},\mathbf{u},\nu)=-2\mathbf{A}^H\mathbf{y}+2\mathbf{A}^H\mathbf{A}\mathbf{x}-\mathbf{u}=\mathbf{0}, \tag{4a}$$
$$\nabla_{\mathbf{z}}\mathcal{L}_0(\mathbf{x},\mathbf{z},\mathbf{u},\nu)=2\nu\mathbf{z}+\mathbf{u}=\mathbf{0}. \tag{4b}$$

Based on (4), the optimal solutions w.r.t. $\mathbf{x}$, and $\mathbf{z}$ are

$$\mathbf{x}^\star=(\mathbf{A}^H\mathbf{A})^{-1}\left(\mathbf{A}^H\mathbf{y}+\frac{1}{2}\mathbf{u}\right), \tag{5a}$$
$$\mathbf{z}^\star=-\frac{1}{2\nu}\mathbf{u}. \tag{5b}$$

Then, we can define the dual function of (3) by (6) shown at the bottom of this page. Further, the dual problem of (2) is given by

$$d^\star:\ \underset{\nu,\mathbf{u}}{\text{maximize}}\quad g(\nu,\mathbf{u}). \tag{7}$$

[1]Even though the strong duality holds for problem (1), see [5], the duality property of problem (2) is still unexplored.

Let the complex gradient of objective function in (7) w.r.t. ν and $\mathbf{u}^*$ to zeros, i.e.,

$$\nabla_\nu g(\nu,\mathbf{u})=\frac{\mathbf{u}^H\mathbf{u}}{4\nu^2}-P=0,$$
$$\nabla_{\mathbf{u}}g(\nu,\mathbf{u})=-\frac{1}{2\nu}\mathbf{u}-\frac{1}{2}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{u}-(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{y}=\mathbf{0}.$$

Then, we conclude that the optimal solutions w.r.t. ν and $\mathbf{u}$ should follow

$$\nu^\star=\pm\sqrt{\frac{\mathbf{u}^H\mathbf{u}}{4P}}, \tag{8a}$$
$$-\frac{1}{2\nu}\mathbf{u}^\star-\frac{1}{2}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{u}^\star=(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{y}. \tag{8b}$$

Substituting (8a) into (5b), we have $\mathbf{z}^{\star H}\mathbf{z}^\star=P$ which causes the term $\nu^\star(\mathbf{z}^{\star H}\mathbf{z}^\star-P)$ in Lagrangian function (3) to vanish, where $\mathbf{z}^\star$ denotes the optimal solution of problem (3). Combining (5) and (8), we can derive

$$\mathbf{z}^\star-\mathbf{x}^\star=-\frac{1}{2\nu^\star}\mathbf{u}^\star-(\mathbf{A}^H\mathbf{A})^{-1}\left(\mathbf{A}^H\mathbf{y}+\frac{1}{2}\mathbf{u}^\star\right)=\mathbf{0}. \tag{9}$$

Note that (9) makes the term $\Re(\mathbf{u}^{\star H}(\mathbf{z}^\star-\mathbf{x}^\star))$ in Lagrangian function (3) vanish. Then, we have

$$d^\star=g(\nu^\star,\mathbf{u}^\star)$$
$$=\mathbf{y}^H\mathbf{y}-\mathbf{y}^H\mathbf{A}\mathbf{x}^\star-\mathbf{x}^{\star H}\mathbf{A}^H\mathbf{y}+\mathbf{x}^{\star H}\mathbf{A}^H\mathbf{A}\mathbf{x}^\star=p^\star. \tag{10}$$

Based on (10), the strong duality holds for the problem (2). This completes the proof. □

Remark 1: For non-convex optimization problems, KKT conditions are generally only necessary for global optimality. However, for quadratic optimization problems with spherical constraints (e.g. the trust-region problem, a hard case), the dual problem is convex with a concave objective, and strong duality holds, implying that the KKT conditions are necessary and sufficient for global optimality, which is not necessarily unique [19].

Remark 2: If the strong duality holds, i.e., the duality gap is zero, and the dual problem is convex, then the saddle point for the Lagrangian is the global optimal solution of the problem (2).

Remark 3: Theorem 1 differs from [1] in two aspects: (i) In [1], the authors considered a **single inequality** constraint over the **real** field, while we address least squares with **two equality** constraints over the **complex** field [20], [21]. (ii) Their proof uses the S-procedure, whereas ours relies directly on the KKT conditions.

A. ADMM Solution

To improve the convergence properties, ADMM generally updates the variables based on the augmented Lagrangian. The

$$g(\nu,\mathbf{u})=\inf_{\mathbf{x},\mathbf{z}}\mathcal{L}_0(\mathbf{x},\mathbf{z},\mathbf{u},\nu)=\mathcal{L}_0(\mathbf{x}^\star,\mathbf{z}^\star,\mathbf{u},\nu)$$
$$=\mathbf{y}^H\mathbf{y}-\mathbf{y}^H\mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{y}-\frac{1}{4\nu}\mathbf{u}^H\mathbf{u}-\nu P-\frac{1}{4}\mathbf{u}^H(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{u}-\Re(\mathbf{u}^H(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H\mathbf{y}). \tag{6}$$

augmented Lagrangian of problem (2) is

$$\mathcal{L}_1(\mathbf{x},\mathbf{z},\mathbf{u})=\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2+\Re(\mathbf{u}^H(\mathbf{z}-\mathbf{x}))+\frac{\rho}{2}\|\mathbf{z}-\mathbf{x}\|_2^2, \quad (11)$$

where $\mathbf{z}^H\mathbf{z}=P$ denotes the constraint for variable $\mathbf{z}$, $\mathbf{u}$ denotes the dual variable, and $\rho>0$ is the penalty parameter. Define $\mathbf{v}=\frac{1}{\rho}\mathbf{u}$, we have $\Re(\mathbf{u}^H(\mathbf{z}-\mathbf{x}))+\frac{\rho}{2}\|\mathbf{z}-\mathbf{x}\|_2^2=\frac{\rho}{2}\|\mathbf{z}-\mathbf{x}+\mathbf{v}\|_2^2-\frac{\rho}{2}\|\mathbf{v}\|_2^2$. Then, we can rewrite the scaled augmented Lagrangian (11) as

$$\mathcal{L}_2(\mathbf{x},\mathbf{z},\mathbf{v})=\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2+\frac{\rho}{2}\|\mathbf{z}-\mathbf{x}+\mathbf{v}\|_2^2-\frac{\rho}{2}\|\mathbf{v}\|_2^2. \quad (12)$$

Within the ADMM framework, each iteration involves minimizing the augmented Lagrangian $\mathcal{L}_2(\mathbf{x},\mathbf{z},\mathbf{v})$ with respect to the primal variables, followed by an update of the dual variable using the dual ascent method [22]. Specifically, ADMM updates the primal and dual variables sequentially as follows:

$$\mathbf{x}^{k+1}=\operatorname*{argmin}_{\mathbf{x}}\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2+\frac{\rho}{2}\|\mathbf{z}^k-\mathbf{x}+\mathbf{v}^k\|_2^2; \quad (13a)$$

$$\mathbf{z}^{k+1}=\operatorname*{argmin}_{\mathbf{z}}\frac{\rho}{2}\|\mathbf{z}-\mathbf{x}^{k+1}+\mathbf{v}^k\|_2^2,\ \text{s.t.}\ \mathbf{z}^H\mathbf{z}=P; \quad (13b)$$

$$\mathbf{v}^{k+1}=\operatorname*{argmax}_{\mathbf{v}}\frac{\rho}{2}\|\mathbf{z}^{k+1}-\mathbf{x}^{k+1}+\mathbf{v}\|_2^2-\frac{\rho}{2}\|\mathbf{v}\|_2^2. \quad (13c)$$

Subsequently, the solutions for the subproblems in (13) are, respectively, given by

$$\mathbf{x}^{k+1}=\left(\mathbf{A}^H\mathbf{A}+\frac{\rho}{2}\mathbf{I}\right)^{-1}\left(\mathbf{A}^H\mathbf{y}+\frac{\rho}{2}(\mathbf{z}^k+\mathbf{v}^k)\right); \quad (14a)$$

$$\mathbf{z}^{k+1}=\frac{\sqrt{P}}{\|\mathbf{x}^{k+1}-\mathbf{v}^k\|_2}(\mathbf{x}^{k+1}-\mathbf{v}^k); \quad (14b)$$

$$\mathbf{v}^{k+1}=\mathbf{v}^k+\mathbf{z}^{k+1}-\mathbf{x}^{k+1}. \quad (14c)$$

Note that a set of closed-form solutions are achieved for the subproblems in (14a) and (14b) which highlights the efficiency of proposed method.

B. Convergence Analysis of ADMM

In this subsection, we present essential analyses regarding both the performance and convergence of the proposed ADMM algorithm. KKT conditions of problem (2) is equivalent to [22]

$$2\mathbf{A}^H\mathbf{A}\mathbf{x}^\star-2\mathbf{A}^H\mathbf{y}+\rho\mathbf{x}^\star-\rho(\mathbf{z}^\star+\mathbf{v}^\star)=\mathbf{0}; \quad (15a)$$

$$\rho\mathbf{z}^\star-\rho(\mathbf{x}^\star-\mathbf{v}^\star)-\nu\mathbf{z}^\star=\mathbf{0}; \quad (15b)$$

$$\mathbf{z}^{\star H}\mathbf{z}^\star=P; \quad (15c)$$

$$\mathbf{x}^\star-\mathbf{z}^\star=\mathbf{0}. \quad (15d)$$

Note that the ADMM updates in (14) are exactly solving the KKT conditions in (15) sequentially. Importantly, the $\mathbf{z}$-update in (14b) is jointly solving (15b) and (15c), and $\mathbf{v}$-update is gradually leading to the satisfaction of (15d). Then, we conclude that the proposed ADMM algorithm can converge to a stationary point at least, even if it is non-convex [23]. To further discuss the convergence performance, problem (2) can be reformulated as the general case

$$\operatorname*{minimize}_{\mathbf{x},\mathbf{z}}\ f(\mathbf{x})+g(\mathbf{z}),\quad \text{subject to}\ \mathbf{x}-\mathbf{z}=\mathbf{0}, \quad (16)$$

TABLE I
COMPLEXITY COMPARISON

Method	Complexity
SDR	$\mathcal{O}(N^6)$
PDM	$\mathcal{O}(2N^3+N\log(1/\epsilon)+MN^2)$
ADMM	$\mathcal{O}(N^3+K(MN+N^2+2N))$

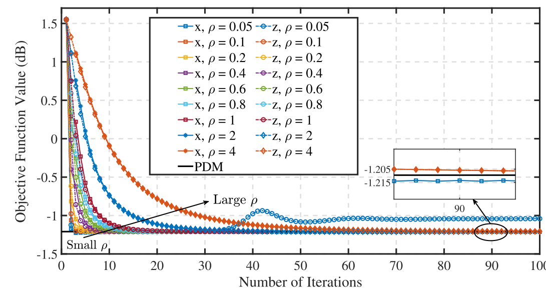

Fig. 1. Objective versus the number of iterations, $M=4000$, $N=1000$, and $\epsilon=10^{-6}$.

where $f(\mathbf{x})=\|\mathbf{y}-\mathbf{A}\mathbf{x}\|_2^2$ is closed, smooth, bounded, and convex, and the function

$$g(\mathbf{z})=\begin{cases}0, & \text{if}\ \mathbf{z}^H\mathbf{z}=P,\\ +\infty, & \text{otherwise}.\end{cases}$$

is the indicator function of the non-convex set $\{\mathbf{z}:\|\mathbf{z}\|_2^2=P\}$. It is proper and closed, but non-convex, which is different from [22]. Similarly, the augmented Lagrangian of (16) is given by

$$\tilde{\mathcal{L}}_\rho(\mathbf{x},\mathbf{z},\mathbf{u})=f(\mathbf{x})+g(\mathbf{z})+\Re(\mathbf{u}^H(\mathbf{z}-\mathbf{x}))+\frac{\rho}{2}\|\mathbf{z}-\mathbf{x}\|_2^2. \quad (17)$$

From Theorem 1, it is seen that strong duality holds for problem (2). Hence, there exists at least one saddle point for (17), and $\tilde{\mathcal{L}}_\rho(\mathbf{x}^*,\mathbf{z}^*,\mathbf{u})\le\tilde{\mathcal{L}}_\rho(\mathbf{x}^*,\mathbf{z}^*,\mathbf{u}^*)\le\tilde{\mathcal{L}}_\rho(\mathbf{x},\mathbf{z},\mathbf{u}^*)$

Proposition 2: For solving problem (2), ADMM in (14) iteratively satisfy: (1) Residual convergence; (2) Objective convergence; (3) Dual variable convergence; (4) Stationary point convergence.

Proof: Please see the supplemental materials. □

Remark 4: Theorem 1 establishes strong duality and Proposition 2 shows that the ADMM iterates converge to a stationary point of problem (2). Thus, we conclude that ADMM for solving QEC-LS can converge globally under the mild condition, see Theorem 1 and Corollary 2 in [23].

C. Complexity Analysis

We summarize the complexity of the proposed ADMM and some benchmarks in Table I, in which ϵ denotes the accuracy level of PDM and K denotes the total iteration number of ADMM.

III. SIMULATION RESULTS

Simulations are conducted on a Dell laptop equipped with an Intel Core i7-10610 U CPU (1.80 GHz, 2.30 GHz Turbo) and 16 GB RAM, running MATLAB R2023b. We compare the performance of proposed ADMM algorithm with SDR [15] and primal-dual method [10], in terms of solution accuracy, objective value, and running time. Throughout the simulations, the data

TABLE II
PERFORMANCE AND COMPUTATIONAL COMPLEXITY COMPARISON WITH STATE-OF-THE-ART METHODS

Method, $P = 0.1$, $\rho = 0.2$	$N = 200$	$N = 250$	$N = 300$	$N = 350$	$N = 400$	$N = 450$
SDR, time cons. (s)	4.1197	6.4038	10.2645	15.1859	28.8541	55.7395
Primal-dual method, time cons. (s)	0.0155	0.0199	0.0295	0.0410	0.0570	0.1158
ADMM, time cons. (s) $\epsilon = 10^{-6}$	0.0125	0.0139	0.0188	0.0251	0.0382	0.0451
ADMM, time cons. (s) $\epsilon = 10^{-10}$	0.0107	0.0119	0.0152	0.0223	0.0390	0.0384
ADMM, time cons. (s) $\epsilon = 10^{-14}$	0.0111	0.0126	0.0139	0.0203	0.0349	0.0377
SDR, solution err.	0	0	0	0	0	0
ADMM, solution err., $\epsilon = 10^{-6}$	$0.1195e^{-6}$	$0.1160e^{-6}$	$0.1080e^{-6}$	$0.1027e^{-6}$	$0.0969e^{-6}$	$0.0897e^{-6}$
ADMM, solution err., $\epsilon = 10^{-10}$	$0.1827e^{-10}$	$0.1593e^{-10}$	$0.1524e^{-10}$	$0.1651e^{-10}$	$0.1417e^{-10}$	$0.1477e^{-10}$
ADMM, solution err., $\epsilon = 10^{-14}$	$4.6610e^{-13}$	$0.3080e^{-12}$	$0.2158e^{-12}$	$0.5260e^{-12}$	$0.3312e^{-12}$	$0.6218e^{-12}$
Primal-dual method, solution err.	$4.1898e^{-13}$	$9.2742e^{-8}$	$0.0274e^{-4}$	$0.1349e^{-4}$	$0.2781e^{-4}$	$0.4323e^{-4}$

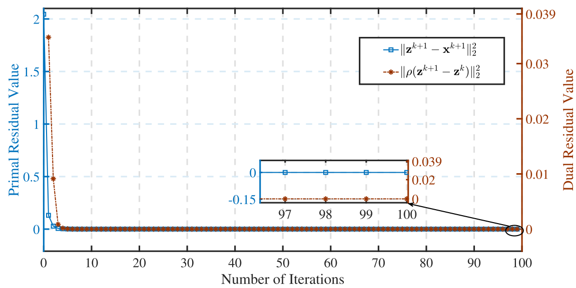

Fig. 2. Residual versus the number of iterations, $M = 4000$, $N = 1000$, and $\rho = 0.2$.

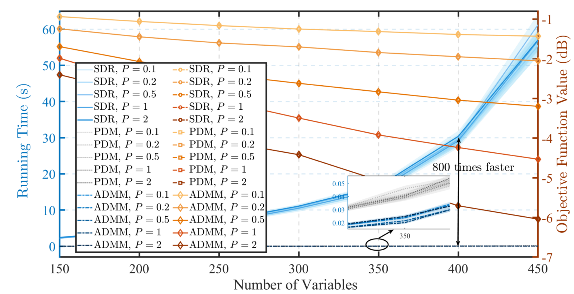

Fig. 3. Running time and objective function value versus the number of variables, $M = 400$, $\rho = 0.2$ and $\epsilon = 10^{-14}$.

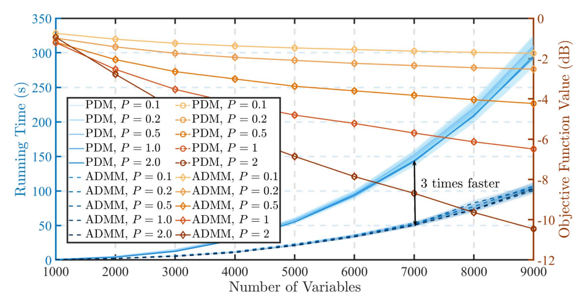

Fig. 4. Running time and objective function value versus the number of variables, $M = 4000$, $\rho = 0.2$ and $\epsilon = 10^{-14}$.

for $\mathbf{y} \in \mathbb{C}^M$ and $\mathbf{A} \in \mathbb{C}^{M\times N}$ are generated randomly, where the entries of $\mathbf{y}$ follow a zero-mean complex Gaussian distribution with covariance $\frac{1}{M}$, i.e., $\mathcal{CN}(0, \frac{1}{M})$, and the entries of $\mathbf{A}$ follow $\mathcal{CN}(0, \frac{1}{MN})$. Meanwhile, we generate the initial value $\mathbf{z} \in \mathcal{CN}(0, \frac{1}{N})$ and dual variable $\mathbf{u} = \mathbf{0}$ for ADMM. Finally, we terminate ADMM algorithm when the iteration number reaches the maximum iteration $K = 100$ or both the primal residual $\|\mathbf{z}^{k+1} - \mathbf{x}^{k+1}\|_2^2$ and the dual residual $\|\rho(\mathbf{z}^{k+1} - \mathbf{z}^k)\|_2^2$ less than the threshold ϵ.

Fig. 1 illustrates the convergence of the objective function over iterations. We observe that as the penalty parameter ρ decreases, the proposed ADMM method tends to converge faster. However, if ρ becomes too small, convergence performance can no longer be guaranteed, which is consistent with the previous discussions [3], [18]. Meanwhile, the proposed algorithm demonstrates fast and stable convergence across a wide range of ρ values. For instance, when $\rho = 0.2$, the algorithm typically converges within 11 iterations, highlighting its practical efficiency in solving large-scale QEC-LS problems.

Fig. 2 shows the evolution of the primal and dual residuals versus iteration number. The residual decreases monotonically to zero, indicating stable convergence of the ADMM updates. This aligns with the theoretical convergence properties discussed earlier. Thus, we chose the penalty value as $\rho = 0.2$ and the threshold $\epsilon = 10^{-14}$ to achieve a fast convergence and accurate solution, simultaneously, in the sequel. To evaluate the performance and the stability of the proposed ADMM algorithm, hereafter, we run the algorithm with 30 times Monte Carlo simulations and then calculate the average running time and objective function value.

Fig. 3 presents the running time and objective function value versus the number of variables, i.e., N ranging from 150 to 450. We observed that to achieve the same objective value, SDR requires the most time consumption compared to PDM and ADMM. Meanwhile, the ADMM method runs twice faster than PDM when $N = 350$.

Fig. 4 shows the running time and objective function value versus the number of variables, i.e., N ranging from 1000 to 9000. Again, to achieve the same objective value, ADMM can run fater than PDM, i.e., three times faster when $N = 7000$, which indicates the computational efficiency of ADMM. Table II shows the performance under different threshold values of ϵ. It is observed that as ϵ decreases, the solution error between ADMM and SDR decreases without a significant increase in running time.

IV. SUMMARY

We investigated the QEC-LS problem and presented a low-complexity ADMM-based algorithm that achieves global optimality under mild condition, i.e., $P > 0$. Unlike traditional approaches such as SDR and primal-dual methods, which either suffer from high computational cost or require careful tuning, the proposed method offers a simple yet effective iterative scheme with guaranteed convergence. Numerical results validate the efficiency and accuracy of the proposed approach, highlighting its potential for large-scale and real-time applications in signal processing and communications.

References

[1] S. Boyd and L. Vandenberghe, "Appendix B: Problems involving two quadratic functions," in *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004, pp. 653–659.

[2] K. Huang and N. D. Sidiropoulos, "Consensus-ADMM for general quadratically constrained quadratic programming," *IEEE Trans. Signal Process.*, vol. 64, no. 20, pp. 5297–5310, Oct. 2016.

[3] H. Huang, H. C. So, and A. M. Zoubir, "Convergence analysis of consensus-ADMM for general QCQP," *Signal Process.*, vol. 208, Jul. 2023, Art. no. 108991.

[4] G. Golub and U. von Matt, "Quadratically constrained least squares and quadratic problems," *Numer. Math.*, vol. 59, pp. 561–580, 1991.

[5] T. P. Dinh and A. L. T. Hoai, "Lagrangian stability and global optimality in nonconvex quadratic minimization over Euclidean balls and spheres," *J. Convex Anal.*, vol. 2, no. 1, pp. 263–276, 1995.

[6] Z. Zhang and Y. Huang, "A projection method for least squares problems with a quadratic equality constraint," *SIAM J. Matrix Anal. Appl.*, vol. 25, no. 1, pp. 188–212, 2003.

[7] B. Tang, J. Liu, H. Wang, and Y. Hu, "Constrained radar waveform design for range profiling," *IEEE Trans. Signal Process.*, vol. 69, pp. 1924–1937, 2021.

[8] T. Wei, K. V. Mishra, L. Wu, and B. S. M. R. Rao, "Cramér-Rao bounds for wideband near-field sensing," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process.*, 2025, pp. 1–5.

[9] I. Nevat, T. Yang, K. Avnit, and J. Yuan, "MIMO detection with high-level modulations using power equality constraints," *IEEE Trans. Veh. Technol.*, vol. 59, no. 7, pp. 3383–3392, Sep. 2010.

[10] F. Liu, L. Zhou, C. Masouros, A. Li, W. Luo, and A. Petropulu, "Toward dual-functional radar-communication systems: Optimal waveform design," *IEEE Trans. Signal Process.*, vol. 66, no. 16, pp. 4264–4279, Aug. 2018.

[11] M. Wang, F. Gao, S. Jin, and H. Lin, "An overview of enhanced massive MIMO with array signal processing techniques," *IEEE J. Sel. Topics Signal Process.*, vol. 13, no. 5, pp. 886–901, Sep. 2019.

[12] E. E. Zelniker and I. V. L. Clarkson, "Maximum-likelihood estimation of circle parameters via convolution," *IEEE Trans. Image Process.*, vol. 15, no. 4, pp. 865–876, Apr. 2006.

[13] D. Ikami, T. Yamasaki, and K. Aizawa, "Fast and robust estimation for unit-norm constrained linear fitting problems," in *Proc. IEEE/CVF Conf. Comput. Vis. Pattern Recognit.*, 2018, pp. 8147–8155.

[14] T. Vu, R. Raich, and X. Fu, "On convergence of projected gradient descent for minimizing a large-scale quadratic over the unit sphere," in *Proc. IEEE Int. Workshop Mach. Learn. Signal Process.*, 2019, pp. 1–6.

[15] Z.-q. Luo, W.-k. Ma, A. M. c. So, Y. Ye, and S. Zhang, "Semidefinite relaxation of quadratic optimization problems," *IEEE Signal Process. Mag.*, vol. 27, no. 3, pp. 20–34, May 2010.

[16] Y. Huang and D. P. Palomar, "Rank-constrained separable semidefinite programming with applications to optimal beamforming," *IEEE Trans. Signal Process.*, vol. 58, no. 2, pp. 664–678, Feb. 2010.

[17] L. Zhao, R. Zhou, and W. Pu, "An efficient and globally optimal algorithm for nonconvex QCQP with one equality constraint," 2025. [Online]. Available: https://arxiv.org/abs/2506.03692

[18] M. Hong, Z.-Q. Luo, and M. Razaviyayn, "Convergence analysis of alternating direction method of multipliers for a family of nonconvex problems," *SIAM J. Optim.*, vol. 26, no. 1, pp. 337–364, 2016.

[19] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[20] A. Beck and Y. C. Eldar, "Strong duality in nonconvex quadratic optimization with two quadratic constraints," *SIAM J. Optim.*, vol. 17, no. 3, pp. 844–860, 2006.

[21] A. L. Fradkov and V. A. Jakubovic, "The S-procedure and a duality relations in nonconvex problems of quadratic programming," *Vestnik Leningrad Univ. Math.*, vol. 6, pp. 101–109, 1979.

[22] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multipliers," *Found. Trends Mach. Learn.*, vol. 3, no. 1, pp. 1–122, 2011.

[23] Y. Wang, W. Yin, and J. Zeng, "Global convergence of ADMM in nonconvex nonsmooth optimization," *J. Sci. Comput.*, vol. 78, no. 1, pp. 29–63, 2019.