Emulation Capacity between Idempotent Channels

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Abstract

We study the optimal rates of emulation (also called interconversion) between quantum channels. When the source and the target channels are idempotent, we give a single-letter expression for the zero-error emulation capacity in terms of structural properties of the range of the two channels. This expression shows that channel emulation is not reversible for general idempotent channels. Furthermore, we establish a strong converse rate that matches with the zero-error emulation capacity when the source or the target channel is either an identity or a completely dephasing channel.

1 Introduction

Given target and source quantum channels $\mathcal{F}$ and $\mathcal{G}$ (respectively), a central problem in quantum information theory is to understand the optimal interconversion rates k/n such that one can use n copies of $\mathcal{G}$, combined with an encoder and a decoder channel, to emulate the action of k copies of $\mathcal{F}$. In the special case where $\mathcal{F}$ is a perfect (noiseless) channel, this corresponds to the problem of channel coding for $\mathcal{G}$, which is central in Shannon theory [14]. Symmetrically, in the special case where $\mathcal{G}$ is the identity channel, this corresponds to the problem of channel simulation for $\mathcal{F}$ which is also well-studied within both classical and quantum Shannon theory [2–5].

In this article, we study the interconversion problem for more general channels. We start by considering the zero-error regime. We say that a channel $\mathcal{G}$ can emulate another channel $\mathcal{F}$ with rate r, if there exists an encoding and a decoding channel $\mathcal{E}$ and $\mathcal{D}$ (respectively) as well as a pair of integers (k, n) such that

$$\mathcal{F}^{\otimes k} = \mathcal{D}\mathcal{G}^{\otimes n}\mathcal{E},$$

and $r \leq k/n$. Such a rate r is called *achievable*. The supremum of achievable rates is called the zero-error emulation capacity of $\mathcal{F}$ by $\mathcal{G}$ and is denoted $C(\mathcal{G} \mapsto \mathcal{F})$. We note that, unlike most works in the literature on reverse Shannon theory [2], the model we consider here does not allow shared randomness or entanglement between the encoder and decoder. The first contribution of this work is a single-letter expression of $C(\mathcal{G} \mapsto \mathcal{F})$ when both $\mathcal{F}$ and $\mathcal{G}$ are assumed to be idempotent, i.e. they satisfy $\mathcal{F} \circ \mathcal{F} = \mathcal{F}$ and $\mathcal{G} \circ \mathcal{G} = \mathcal{G}$. Note that for an arbitrary channel Φ, the sequential composition $\Phi^n = \Phi \circ \cdots \circ \Phi$ has a subsequence that converges as $n \to \infty$ to an idempotent channel [19]. The capacities of sequential composition of channels have been studied in [8, 15, 16].

Theorem 1. *Let $\mathcal{F} = \mathcal{F} \circ \mathcal{F}$ and $\mathcal{G} = \mathcal{G} \circ \mathcal{G}$ be two idempotent channels with $\lambda(\mathcal{F}) \neq (1)$[1], then*

$$C(\mathcal{G} \mapsto \mathcal{F}) = \inf_{p \in [1, +\infty]} \frac{\log(\|\lambda(\mathcal{G})\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)}. \tag{1}$$

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[1]If $\lambda(\mathcal{F}) = (1)$, then $\mathcal{F}$ is basically a replacer channel (see Proposition 1) and it is simple to see that we can emulate it with any other idempotent channel $\mathcal{G}$ with an arbitrarily large rate.

In particular, when $\mathcal{F}$ or $\mathcal{G}$ is either the identity channel or the completely dephasing channel, it suffices to restrict the infimum to $p \in \{1, +\infty\}$:

$$C(\mathcal{G} \mapsto \mathcal{F}) = \min_{p \in \{1, +\infty\}} \frac{\log(\|\lambda(\mathcal{G})\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)}. \tag{2}$$

In the right-hand side of eq. (1), the two terms $\lambda(\mathcal{F})$ and $\lambda(\mathcal{G})$ are integer vectors which contain information on decomposition properties of $\mathcal{F}$ and $\mathcal{G}$. They are called the *shape vector* of their respective channel and their precise definition will be given in Section 3, Proposition 1. For now, it suffices to say that the definition of shape vector is specific to idempotent channel and that given an idempotent channel $\mathcal{F} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, $\lambda(\mathcal{F})$ can be computed in polynomial time in the dimension of $\mathcal{H}$ (see [8] for the algorithm). Shape vectors were first introduced by Kuperberg in [12] to study the problem of emulating *quantum memories* one with another. Also, for $u \in \mathbb{C}^n$ and $p \in [1, +\infty]$, $\|u\|_p$ denotes the ℓ_p-norm of u.

A remarkable aspect about this result is that it is a single-letter formula, i.e. the expression does not involve a limit. This comes from the fact that the norm of the shape vector is multiplicative under tensor product. Another implication is that the emulation capacity of idempotent channels is additive under tensor products of source channels, i.e. $C(\mathcal{G}_1 \otimes \mathcal{G}_2 \mapsto \mathcal{F}) = C(\mathcal{G}_1 \mapsto \mathcal{F}) + C(\mathcal{G}_2 \mapsto \mathcal{F})$.

One interesting consequence of this formula is that emulating $\mathcal{F}$ with $\mathcal{G}$ is in general not reversible. In fact, as shown in Example 1, there exists idempotent channels for which $C(\mathcal{G} \mapsto \mathcal{F})C(\mathcal{F} \mapsto \mathcal{G})^{-1} < 1$. This follows from the fact that the infimum in the formulas for $C(\mathcal{G} \mapsto \mathcal{F})$ and $C(\mathcal{F} \mapsto \mathcal{G})$ can be reached for different values of p.

Next, we consider the setting beyond the zero-error regime, where the emulation does not need to be exact: the rate r is achievable with error $\delta \in [0, 1]$ if there exists channels $\mathcal{E}, \mathcal{D}$ and k, n with $r \leq k/n$ such that

$$\frac{1}{2}\|\mathcal{F}^{\otimes k} - \mathcal{D}\mathcal{G}^{\otimes n}\mathcal{E}\|_\diamond \leq \delta,$$

where $\|\cdot\|_\diamond$ is the diamond norm (the completely bounded norm between trace classes). We believe that for idempotent channels, the capacity (with vanishing error) should be the same as the zero-error case. Here, we only proved a weaker statement: the capacity cannot be larger than the same formula except that we only take the minimum over $p \in \{1, +\infty\}$. More specifically, we establish a strong converse rate for the emulation capacity.

Theorem 2 (Strong converse rate). *Let $\mathcal{F} = \mathcal{F} \circ \mathcal{F}$ and $\mathcal{G} = \mathcal{G} \circ \mathcal{G}$ be two idempotent channels with $\lambda(\mathcal{F}) \neq (1)$. Let $(k_\nu)_{\nu \in \mathbb{N}}$ and $(n_\nu)_{\nu \in \mathbb{N}}$ be two integer sequences with $\lim_{\nu \to \infty} n_\nu = +\infty$ such that there exists $\varepsilon > 0$ satisfying:*

$$\lim_{\nu \to \infty} \frac{k_\nu}{n_\nu} \geq \min_{p \in \{1, +\infty\}} \frac{\log(\|\lambda(\mathcal{G})\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)} + \varepsilon. \tag{3}$$

Then, for all sequences of encoding and decoding channels $(\mathcal{E}_\nu)_{\nu \in \mathbb{N}}$, $(\mathcal{D}_\nu)_{\nu \in \mathbb{N}}$,

$$\frac{1}{2}\lim_{\nu \to \infty} \|\mathcal{F}^{\otimes k_\nu} - \mathcal{D}_\nu \mathcal{G}^{\otimes n_\nu} \mathcal{E}_\nu\|_\diamond = 1. \tag{4}$$

Note that for any two channels Φ and Ψ, we have that $\|\Phi - \Psi\|_\diamond \leq 2$. Therefore, Theorem 2 states that if one tries to emulate $\mathcal{F}$ with $\mathcal{G}$ with a rate greater than $\log(\|\lambda(\mathcal{G})\|_p)/\log(\|\lambda(\mathcal{F})\|_p)$ for $p \in \{1, +\infty\}$, the error in the limit of large block-length is maximal.

If $\mathcal{F}$ or $\mathcal{G}$ are identity channels or dephasing channels, then the achievability result of Theorem 1 and the strong converse rate in Theorem 2 match. In particular, for the case of $\mathcal{F} = \mathrm{Id}$, we obtain an expression for the quantum capacity of arbitrary idempotent channels together with a strong converse. In the case of self-adjoint idempotent channels (also known as tracial conditional expectation), this was proved in [9] (see also [13]).

We conjecture that Theorem 2 can be improved to take an infimum over all $p \in [1, +\infty]$, which would imply that the zero-error achievability in Theorem 1 is tight for all idempotent channels $\mathcal{F}, \mathcal{G}$ even if vanishing error is allowed.

The techniques used in our proofs stem from the theory of operator algebras. Nevertheless, familiarity with this theory is not required to understand the present article. We will introduce in Section 2 the basic theory of operator algebras that we use throughout our work.

This article is structured as follows. In Section 2, we introduce our notations and basic preliminaries on quantum information and operator algebra theories. In Section 3, we discuss the decomposition properties of idempotent channels. In Section 4, we prove Theorem 1. We prove Theorem 2 in Section 5. Finally, we apply the theory of approximate C^*-algebras recently developed by Kitaev [10] to strengthen the converse of Theorem 1 in the appendix.

2 Preliminaries and notations

All Hilbert spaces in this work are assumed to be finite-dimensional. Given a Hilbert space $\mathcal{H}$, we write $\mathcal{L}(\mathcal{H})$ the set of linear operators on $\mathcal{H}$. This set is endowed with the Hilbert-Schmidt inner product, denoted $\langle x, y \rangle := \mathrm{tr}(x^* y)$ for $x, y \in \mathcal{L}(\mathcal{H})$, with x^* the conjugate transpose of x and $\mathrm{tr}(\cdot)$ the trace on $\mathcal{L}(\mathcal{H})$. Note that x^* is the adjoint of x for the Hermitian product of $\mathcal{H}$. We denote $\mathbb{1} \in \mathcal{L}(\mathcal{H})$ the identity operator on $\mathcal{H}$. Given $x \in \mathcal{L}(\mathcal{H})$, we call the *(right) support* of x, denoted $\mathrm{supp}(x) \subseteq \mathcal{H}$, the orthogonal complement of its (right) kernel. We write x^0 the orthogonal projector on the support of x, parallel to its kernel. Furthermore, for $x \in \mathcal{L}(\mathcal{H})$, we write $\|x\|_1$ for its Schatten 1-norm and $\|x\|$ its Schatten ∞-norm, which coincide with its operator norm as a linear operator on $\mathcal{H}$. Recall that the set of Hermitian operators in $\mathcal{L}(\mathcal{H})$ is a partially ordered set: for $x = x^*, y = y^* \in \mathcal{L}(\mathcal{H})$, we write $x \leq y$ whenever $y - x$ is positive semidefinite.

A state is represented by a positive semidefinite operator $\rho \in \mathcal{L}(\mathcal{H})$ with trace one, which is called a *density operator*. We will take $\mathcal{D}(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})$ to be the set of density operators belonging to $\mathcal{L}(\mathcal{H})$. A measurement on a physical system $\mathcal{L}(\mathcal{H})$ is represented by a Positive Operator Valued Measure, POVM for short, which is a set of positive operators $\mu_1, \ldots, \mu_K \in \mathcal{L}(\mathcal{H})$, called *effects*, which sum to the identity, i.e. $\sum_{k=1}^{K} \mu_k = \mathbb{1}$.

A quantum channel $\mathcal{F} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$ is a completely positive trace preserving linear map. Each linear map $\mathcal{F} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$ admits an adjoint $\mathcal{F}^* : \mathcal{L}(\mathcal{H}') \to \mathcal{L}(\mathcal{H})$ for the Hilbert-Schmidt inner product such that for all $x \in \mathcal{L}(\mathcal{H})$, $y \in \mathcal{L}(\mathcal{H}')$, $\langle \mathcal{F}(x), y \rangle = \langle x, \mathcal{F}^*(y) \rangle$. When $\mathcal{F}$ is a channel, its adjoint $\mathcal{F}^*$ is a unital completely positive map. In Section 5, part of the discussion will be focused on the identity channel which we write $\mathrm{Id}_d : \mathcal{L}(\mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^d)$ and on the completely dephasing channel $\Delta_d : \mathcal{L}(\mathbb{C}^d) \to \mathcal{L}(\mathbb{C}^d)$, which maps every $x \in \mathcal{L}(\mathbb{C}^d)$ onto its diagonal $\mathrm{Diag}(x_{11}, \ldots, x_{dd})$ (w.r.t. the standard orthonormal basis). This channel is a 'classical identity' as it preserves perfectly the classical information while mapping its input state on a completely decorrelated state, hence a classical one.

We use, in Theorem 2, the diamond norm as a figure of merit for the quality of the emulation. We recall that for $\mathcal{F} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$, the diamond norm is defined as

$$\|\mathcal{F}\|_\diamond = \sup_{n \in \mathbb{N}^*} \|\mathcal{F} \otimes \mathrm{Id}_n\|_{1 \to 1},$$

with $\| \cdot \|_{1 \to 1}$ the 'one-to-one' norm, i.e. the map norm when the input and output spaces are endowed with the Schatten 1-norm, denoted $\| \cdot \|_1$. That is, for all linear maps $\mathcal{F} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$,

$$\|\mathcal{F}\|_{1 \to 1} = \sup_{\|x\|_1 \leq 1} \|\mathcal{F}(x)\|_1.$$

Then, we recall basic properties of the ℓ_p-norms. For $u \in \mathbb{C}^n$ and $p \in [1, +\infty)$, the ℓ_p-norm of u is defined as

$$\|u\|_p = \left(\sum_{k=1}^{n} |u_k|^p \right)^{\frac{1}{p}}, \tag{5}$$

and $\|u\|_\infty = \sup_k |u_k|$. In the discussions of the next sections, we will frequently make use of the fact that the ℓ_p-norm are *multiplicative* under the tensor product of vectors. Given $u \in \mathbb{C}^n$, $v \in \mathbb{C}^m$, the tensor product $u \otimes v \in \mathbb{C}^{nm}$ is the vector whose coordinates are the products of the coordinates of u and v, i.e. $(u_k v_l)_{k,l}$ for $k \in [n]$, $l \in [m]$. Then, for all $p \in [1, +\infty]$, $\|u \otimes v\|_p = \|u\|_p \|v\|_p$.

Finally, we end this section by introducing the notions of finite-dimensional $*$-algebras and their morphisms.

In this article, we actually consider only $*$-subalgebras of $\mathcal{L}(\mathcal{H})$. Such a $*$-subalgebra $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ is a linear subspace of $\mathcal{L}(\mathcal{H})$ which is stable under the multiplication and the $*$ operation of $\mathcal{L}(\mathcal{H})$, i.e. taking the transpose conjugate. That is, for all $x, y \in \mathcal{A}$, $xy \in \mathcal{A}$ and for all $x \in \mathcal{A}$, $x^* \in \mathcal{A}$. Furthermore, all

∗-algebras in this work are assumed to be *unital*, that is there is an identity $\mathbb{1} \in \mathcal{A}$ satisfying $\mathbb{1}^* = \mathbb{1}$ and, for all $x \in \mathcal{A}$, $\mathbb{1}x = x\mathbb{1} = x$. Such an identity operator is necessarily unique in each ∗-algebra.

Given two ∗-algebras $\mathcal{A}$ and $\mathcal{B}$, a ∗-homomorphism $\iota : \mathcal{A} \to \mathcal{B}$ is a linear map that preserves the product and the ∗ operation in $\mathcal{A}$. That is, for all $x, y \in \mathcal{A}$, $\iota(xy) = \iota(x)\iota(y)$ and $\iota(x^*) = \iota(x)^*$. In many texts, ∗-homomorphisms between unital ∗-algebras are assumed to be unital as well, that is they are assumed to map the identity of $\mathcal{A}$ on the identity of $\mathcal{B}$. Nevertheless, in this article, we will work with more general **subunital** ∗-homomorphisms. Such a map $\iota : \mathcal{A} \to \mathcal{B}$ maps the identity of $\mathcal{A}$ on $\iota(\mathbb{1}_\mathcal{A})$, which satisfies $\iota(\mathbb{1}_\mathcal{A}) \leq \mathbb{1}_\mathcal{B}$. Note that ∗-homomorphisms are completely positive [6].

We summarized all the notations we use throughout this article in Table 1.

$\mathcal{H}$	A finite-dimensional Hilbert space	
$\mathcal{L}(\mathcal{H})$	Set of linear operators on $\mathcal{H}$	
$\mathcal{D}(\mathcal{H})$	Set of density operators in $\mathcal{L}(\mathcal{H})$	
$\mathcal{A}, \mathcal{B}$	∗-subalgebras of $\mathcal{L}(\mathcal{H})$	
ρ, σ	Density operators	
$\mathrm{supp}(x)$	Support of x	
x^0	Orthogonal projector on the support of x	
x^*	Hermitian adjoint of x	
$\| \cdot \|_p$	ℓ_p-norm	Eq. (5)
$\mathcal{F}^*$	Adjoint of $\mathcal{F}$ for the Hilbert-Schmidt product	
$\widehat{\mathcal{F}}$	Reduced channel associated to $\mathcal{F}$	Prop. 1, eq. (9)
$\mathrm{Rg}(\mathcal{F})$	Range of $\mathcal{F}$	
$\lambda(\mathcal{F})$	Shape vector of $\mathcal{F}$	Prop. 1
$[n]$	The set of integers $\{1, \ldots, n\}$	

Table 1: Notations

3 Decomposition properties of idempotent channels

In this section, we discuss the decomposition properties of the range of idempotent channels, which are summarized in the following propositions. The range of a map $f : \Omega \to \Omega'$, denoted $\mathrm{Rg}(f)$, is the set of all possible images over its domain Ω, i.e. $\mathrm{Rg}(f) = \{ f(x) \mid x \in \Omega \}$. The range of an idempotent channel coincides with its set of fixed points and the structure of this space is well understood, see for example the lecture notes [19]. The following proposition summarizes the properties that we use.

Proposition 1 (Structure of the range of idempotent channels, see [19]). *Let $\mathcal{F} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be an idempotent channel, i.e. satisfying $\mathcal{F} = \mathcal{F} \circ \mathcal{F}$. We can write $\mathcal{H}$ as the direct sum*

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1, \tag{6}$$

with $\mathcal{H}_0 = \mathrm{supp}(\mathcal{F}(\mathbb{1}))$ and $\mathcal{H}_1 = \ker(\mathcal{F}(\mathbb{1}))$, and $\mathcal{H}_0$ can further be decomposed as the orthogonal direct sum

$$\mathcal{H}_0 = \bigoplus_{k=1}^{K} \mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}, \tag{7}$$

so that there is a set of density operators $\{\rho_k \in \mathcal{D}(\mathcal{H}_{k,2}) \mid k \in [K]\}$, such that

$$\mathrm{Rg}(\mathcal{F}) = \left(\bigoplus_{k=1}^{K} \mathcal{L}(\mathcal{H}_{k,1}) \otimes \rho_k \right) \oplus 0. \tag{8}$$

Let $e = \mathcal{F}(\mathbb{1})^0 \in \mathcal{L}(\mathcal{H})$ be the orthogonal projector on $\mathcal{H}_0$, and $V : \mathcal{H}_0 \to \mathcal{H}$ be the isometry embedding $\mathcal{H}_0$ into $\mathcal{H}$ such that $VV^* = e$ and $V^*V = \mathbb{1}_0$ the identity operator in $\mathcal{L}(\mathcal{H}_0)$. The reduced quantum channel *associated to* $\mathcal{F}$, denoted $\widehat{\mathcal{F}} : \mathcal{L}(\mathcal{H}_0) \to \mathcal{L}(\mathcal{H}_0)$ is then defined as

$$\widehat{\mathcal{F}} : x \mapsto V^*\mathcal{F}(VxV^*)V, \tag{9}$$

which is idempotent and whose range can be written as

$$\mathrm{Rg}(\widehat{\mathcal{F}}) = \bigoplus_{k=1}^{K} \mathcal{L}(\mathcal{H}_{k,1}) \otimes \rho_k. \tag{10}$$

Furthermore, we can express $\widehat{\mathcal{F}}$ and its adjoint $\widehat{\mathcal{F}}^*$ in the following way:

$$\widehat{\mathcal{F}}(x) = \sum_{k=1}^{K} \mathrm{tr}_{k,2}(P_k x P_k) \otimes \rho_k, \tag{11}$$

and

$$\widehat{\mathcal{F}}^*(x) = \sum_{k=1}^{K} \mathrm{tr}_{k,2}(P_k x P_k(\mathbb{1}_{d_k} \otimes \rho_k)) \otimes \mathbb{1}_{m_k}, \tag{12}$$

where, for all $k \in [K]$, P_k is the orthogonal projector onto $\mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}$, $d_k = \dim(\mathcal{H}_{k,1}) \geq 1$, $m_k = \dim(\mathcal{H}_{k,2})$ and $\mathrm{tr}_{k,2} : \mathcal{L}(\mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}) \to \mathcal{L}(\mathcal{H}_{k,1})$ is the partial trace over $\mathcal{H}_{k,2}$.

Therefore, the range of $\widehat{\mathcal{F}}^*$ is a unital $*$-subalgebra of $\mathcal{L}(\mathcal{H}_0)$ and the following decomposition holds:

$$\mathrm{Rg}(\widehat{\mathcal{F}}^*) = \bigoplus_{k=1}^{K} \mathcal{L}(\mathcal{H}_{k,1}) \otimes \mathbb{1}_{m_k}. \tag{13}$$

We call the shape *of $\mathcal{F}$ the K-dimensional vector whose coordinates are the dimensions $(d_k)_{k \in [K]}$ sorted in non-increasing order, which we write $\lambda(\mathcal{F})$.*

Note that $\mathcal{F}$ and its reduced channel $\widehat{\mathcal{F}}$ both have the same shape vector, i.e. $\lambda(\mathcal{F}) = \lambda(\widehat{\mathcal{F}})$.

As an example of a shape vector, consider the qubit channel $\mathcal{F} : \mathcal{L}(\mathbb{C}^2) \to \mathcal{L}(\mathbb{C}^2)$ defined as

$$\mathcal{F} : x \mapsto \mathrm{tr}(x)|0\rangle\langle 0| = \begin{pmatrix} \mathrm{tr}(x) & 0 \\ 0 & 0 \end{pmatrix},$$

where the matrix expression is written in the basis $(|0\rangle, |1\rangle)$. We have

$$\mathcal{F}(\mathbb{1}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, in the notations of eq. (7), we get $\mathcal{H}_0 = \mathrm{supp}(\mathcal{F}(\mathbb{1})) = \mathbb{C}|0\rangle$, $\mathcal{H}_1 = \ker(\mathcal{F}(\mathbb{1})) = \mathbb{C}|1\rangle$, we can write $\mathrm{Rg}(\mathcal{F}) = \mathbb{C} \otimes |0\rangle\langle 0| \oplus 0$ and thus $\lambda(\mathcal{F}) = (1)$. To construct the reduced version of $\mathcal{F}$ we can take the isometry $V = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \mathcal{H}_0 \to \mathcal{H}$, where the expression of V is written in the basis $(|0\rangle, |1\rangle)$. Note that we clearly have

$$VV^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \mathcal{F}(\mathbb{1})^0.$$

Then, $\widehat{\mathcal{F}} : \mathcal{L}(\mathbb{C}) \to \mathcal{L}(\mathbb{C})$ is defined by the equation

$$\widehat{\mathcal{F}}(x) = V^* \mathcal{F}(V x V^*) V = x,$$

so that $\widehat{\mathcal{F}}$ is the identity on $\mathcal{L}(\mathbb{C})$.

Furthermore, we remark that we have $\lambda(\mathcal{F}) = (1)$ if and only if its reduced channel is a replacer channel, i.e. $\widehat{\mathcal{F}} : x \mapsto \mathrm{tr}(x)\rho$ with ρ a fixed state. This fact follows easily from eq. (11) taking $K = 1$.

Before we present the proof of Proposition 1, we need to state the two following lemmas which will be used in the proof.

Lemma 3. *Let $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ be a positive linear map[2], let $\mathcal{H}_0 = \mathrm{supp}(\Phi(\mathbb{1}))$ and let $V : \mathcal{H}_0 \to \mathcal{H}$ be an isometry embedding $\mathcal{H}_0$ into $\mathcal{H}$ so that $VV^* \in \mathcal{L}(\mathcal{H})$ is the orthogonal projection on $\mathcal{H}_0$. Then for all $x \in \mathcal{L}(\mathcal{H})$,*

$$VV^* \Phi(x) VV^* = \Phi(x).$$

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[2]Here 'positive' means that $x \geq 0$ implies $\Phi(x) \geq 0$.

Proof. Let $x \in \mathcal{L}(\mathcal{H})$ be positive semidefinite, then

$$0 \le x \le \|x\|\mathbb{1},$$

so that by positivity of Φ, we have

$$0 \le \Phi(x) \le \|x\|\Phi(\mathbb{1}).$$

Hence, $\Phi(x)$ is supported on the support of $\Phi(\mathbb{1})$ which is, by definition, $\mathcal{H}_0$. Furthermore, as x and $\mathbb{1}$ are positive semidefinite, $\Phi(x)$ and $\Phi(\mathbb{1})$ are positive semidefinite as well so that their respective range (as linear maps on $\mathcal{H}$) coincides with their support. Hence, we have

$$VV^*\Phi(x)VV^* = \Phi(x).$$

Now, if $x \in \mathcal{L}(\mathcal{H})$ is arbitrary, we can use the polarisation identity (see eq. (1.45) in [19]) to write

$$x = \frac{1}{4}\sum_{k=0}^{3} i^k(x + i^k\mathbb{1})^*(x + i^k\mathbb{1}),$$

so that x is a linear combination of four positive semidefinite elements of $\mathcal{L}(\mathcal{H})$. Now, by linearity, we get

$$VV^*\Phi(x)VV^* = \Phi(x).$$

$\square$

Furthermore, we remark that the set of fixed points and the range of an idempotent function coincide.

Lemma 4. *Let $f : \Omega \to \Omega$ be an arbitrary idempotent function, with Ω an arbitrary set, then*

$$\mathrm{Rg}(f) = \{x \in \Omega \mid f(x) = x\}. \tag{14}$$

Proof. If x is a fixed point, $x = f(x) \in \mathrm{Rg}(f)$. For the other inclusion, if $y = f(x)$, we also have $f(y) = f(f(x)) = f(x) = y$, so y is a fixed point. $\square$

Proposition 1 then basically follows from statements in Section 6.4 [19] and we detail it here only for the convenience of the reader. Similar derivation can be found in [1] in the Heisenberg picture.

Proof of Proposition 1. We will first prove that the reduced channel $\widehat{\mathcal{F}}$, defined in eq. (9), is actually a channel. It is clearly completely positive as it is a composition of completely positive maps and it is trace preserving because for all $x \in \mathcal{L}(\mathcal{H}_0)$,

$$\mathrm{tr}(\widehat{\mathcal{F}}(x)) = \mathrm{tr}(V^*\mathcal{F}(VxV^*)V) = \mathrm{tr}(VV^*\mathcal{F}(VxV^*)VV^*) = \mathrm{tr}(\mathcal{F}(VxV^*)) = \mathrm{tr}(VxV^*) = \mathrm{tr}(x),$$

where the third equality follows from Lemma 3. Furthermore, $\widehat{\mathcal{F}}$ is idempotent as, for $x \in \mathcal{L}(\mathcal{H}_0)$,

$$\begin{aligned}
\widehat{\mathcal{F}}(\widehat{\mathcal{F}}(x)) &= V^*\mathcal{F}(VV^*\mathcal{F}(VxV^*)VV^*)V \\
&= V^*\mathcal{F}(\mathcal{F}(VxV^*))V \\
&= V^*\mathcal{F}(VxV^*)V \\
&= \widehat{\mathcal{F}}(x).
\end{aligned}$$

The second equation follows from Lemma 3, and the third from the idempotence of $\mathcal{F}$.

Furthermore, $\widehat{\mathcal{F}}$ admits a full-rank fixed point. Indeed, $\widehat{\mathcal{F}}(V^*\mathcal{F}(\mathbb{1})V)$ is a fixed point of $\widehat{\mathcal{F}}$, since $\widehat{\mathcal{F}}$ is idempotent, and we have

$$\widehat{\mathcal{F}}(V^*\mathcal{F}(\mathbb{1})V) = V^*\mathcal{F}(VV^*\mathcal{F}(\mathbb{1})VV^*)V = V^*\mathcal{F}\mathcal{F}(\mathbb{1})V = V^*\mathcal{F}(\mathbb{1})V,$$

which is full-rank in $\mathcal{H}_0 = \mathrm{supp}(\mathcal{F}(\mathbb{1}))$, again using Lemma 3 for the second equality.

Therefore, we can apply Theorem 6.12 in [19] to the adjoint $\widehat{\mathcal{F}}^*$ of $\widehat{\mathcal{F}}$. Hence, the set of fixed points of $\widehat{\mathcal{F}}^*$ is a $*$-subalgebra of $\mathcal{L}(\mathcal{H}_0)$. By Lemma 4, since $\widehat{\mathcal{F}}^*$ is idempotent, its set of fixed points coincides with its range, so that its range is a $*$-subalgebra of $\mathcal{L}(\mathcal{H}_0)$. It is well known (see e.g. [17]), that if $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H}_0)$

is a $*$-subalgebra of $\mathcal{L}(\mathcal{H}_0)$, we can find a basis of $\mathcal{H}_0$ in which it can be expressed as an orthogonal direct sum as in eq. (13), which proves this equation. Furthermore, we get by the Proposition 1.5 of [19] that $\widehat{\mathcal{F}}^*$ takes the form of eq. (12). Then, taking the adjoint of this equation, we obtain that $\widehat{\mathcal{F}} = \widehat{\mathcal{F}}^{**}$ can be written as in eq. (11). Then, for $x \in \mathcal{L}(\mathcal{H}_0)$, we have that

$$\widehat{\mathcal{F}}(x) = \sum_{k=1}^{K} \mathrm{tr}_{k,2}(P_k x P_k) \otimes \rho_k \in \bigoplus_{k=1}^{K} \mathcal{L}(\mathcal{H}_{k,1}) \otimes \rho_k.$$

Conversely, if

$$x = \sum_{k=1}^{K} x_k \otimes \rho_k \in \bigoplus_{k=1}^{K} \mathcal{L}(\mathcal{H}_{k,1}) \otimes \rho_k,$$

where, for each $k \in [K]$, $x_k \in \mathcal{L}(\mathcal{H}_{k,1})$, we have that $P_l x_k \otimes \rho_k P_l = \delta_{k,l} x_k \otimes \rho_k$, since the spaces $\mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}$ are two by two orthogonal, so that $x = \widehat{\mathcal{F}}(x)$, and finally $x \in \mathrm{Rg}(\widehat{\mathcal{F}})$. This proves eq. (10).

Finally, to prove eq. (8), let $x \in \mathrm{Rg}(\mathcal{F})$. We then have

$$x = \mathcal{F}(x) = VV^*\mathcal{F}(x)VV^* = VV^*xVV^*, \tag{15}$$

so that $x \in \mathcal{L}(\mathcal{H}_0) \oplus 0$ where the 0 operator acts on $\mathcal{H}_1$. Then, $V^*xV \in \mathcal{L}(\mathcal{H}_0)$ and

$$\widehat{\mathcal{F}}(V^*xV) = V^*\mathcal{F}(VV^*xVV^*)V = V^*\mathcal{F}(x)V = V^*xV,$$

by Lemma 3 and by eq. (15). Hence,

$$V^*xV \in \bigoplus_{k=1}^{K} \mathcal{L}(\mathcal{H}_{k,1}) \otimes \rho_k,$$

by eq. (10) and

$$x = VV^*xVV^* \in \bigoplus_{k=1}^{K} \mathcal{L}(\mathcal{H}_{k,1}) \otimes \rho_k \oplus 0.$$

Conversely, if

$$x = \begin{pmatrix} x_0 & 0 \\ 0 & 0 \end{pmatrix},$$

with

$$x_0 \in \bigoplus_{k=1}^{K} \mathcal{L}(\mathcal{H}_{k,1}) \otimes \rho_k,$$

then

$$x = Vx_0V^* = V\widehat{\mathcal{F}}(x_0)V^* = VV^*\mathcal{F}(Vx_0V^*)VV^* = \mathcal{F}(Vx_0V^*) = \mathcal{F}(x),$$

so that $x \in \mathrm{Rg}(\mathcal{F})$. $\qquad \square$

We will now show that if $\mathcal{F}$ and $\mathcal{G}$ are two idempotent channels, it is equivalent to study the emulation of $\mathcal{F}$ by $\mathcal{G}$ and the emulation of their reduced version, i.e. $\widehat{\mathcal{F}}$ by $\widehat{\mathcal{G}}$. First, we can explicitly find an encoder and a decoder channel to convert any idempotent channel $\mathcal{F}$ into its reduced version $\widehat{\mathcal{F}}$ and vice versa as shown in the following proposition.

Proposition 2. *Let* $\mathcal{F} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ *be an idempotent channel, let* $\mathcal{H}_0 = \mathrm{supp}(\mathcal{F}(\mathbb{1}))$ *and let* $\widehat{\mathcal{F}} : \mathcal{L}(\mathcal{H}_0) \to \mathcal{L}(\mathcal{H}_0)$ *be its reduced quantum channel. Then, there is* $\widehat{\mathcal{E}} : \mathcal{L}(\mathcal{H}_0) \to \mathcal{L}(\mathcal{H})$, $\widehat{\mathcal{D}} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_0)$, $\mathcal{E} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_0)$ *and* $\mathcal{D} : \mathcal{L}(\mathcal{H}_0) \to \mathcal{L}(\mathcal{H})$ *such that*

$$\mathcal{F} = \widehat{\mathcal{D}}\widehat{\mathcal{F}}\widehat{\mathcal{E}}, \ and \ \widehat{\mathcal{F}} = \mathcal{D}\mathcal{F}\mathcal{E}. \tag{16}$$

Proof. Let $V : \mathcal{H}_0 \to \mathcal{H}$ be the isometry embedding $\mathcal{H}_0$ into $\mathcal{H}$ which defines $\widehat{\mathcal{F}}$ as in Proposition 1. For the first equality, let $\widehat{\mathcal{E}} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_0)$ be defined by

$$\widehat{\mathcal{E}} : x \mapsto V^*\mathcal{F}(x)V,$$

and $\widehat{\mathcal{D}} : \mathcal{L}(\mathcal{H}_0) \to \mathcal{L}(\mathcal{H})$ be defined by

$$\widehat{\mathcal{D}} : x \mapsto VxV^*.$$

Both $\widehat{\mathcal{E}}$ and $\widehat{\mathcal{D}}$ are completely positive trace preserving map. The complete positivity is easy. $\widehat{\mathcal{D}}$ is trace preserving because $\operatorname{tr}(VxV^*) = \operatorname{tr}(V^*Vx) = \operatorname{tr}(x)$ and $\widehat{\mathcal{E}}$ is trace preserving as well because for all $x \in \mathcal{L}(\mathcal{H})$,

$$\operatorname{tr}(V^*\mathcal{F}(x)V) = \operatorname{tr}(VV^*\mathcal{F}(x)VV^*) = \operatorname{tr}(\mathcal{F}(x)) = \operatorname{tr}(x),$$

where the second equality follows from Lemma 3. Then we have the factorisation

$$\mathcal{F} = \widehat{\mathcal{D}}\widehat{\mathcal{F}}\widehat{\mathcal{E}}.$$

Indeed, let $x \in \mathcal{L}(\mathcal{H})$, we have

$$\begin{aligned}
\widehat{\mathcal{D}}\widehat{\mathcal{F}}\widehat{\mathcal{E}}(x) &= VV^*\mathcal{F}(VV^*\mathcal{F}(x)VV^*)VV^* \\
&= \mathcal{F}(\mathcal{F}(x)) \\
&= \mathcal{F}(x),
\end{aligned}$$

where the second equality follows from Lemma 3. Now, for the factorisation of $\widehat{\mathcal{F}}$, let $\mathcal{E} = \widehat{\mathcal{D}}$ and $\mathcal{D} = \widehat{\mathcal{E}}$, for $x \in \mathcal{L}(\mathcal{H}_0)$, we have:

$$\begin{aligned}
\mathcal{D}\mathcal{F}\mathcal{E}(x) &= \widehat{\mathcal{E}}\mathcal{F}\widehat{\mathcal{D}}(x) \\
&= V^*\mathcal{F}(\mathcal{F}(VxV^*))V \\
&= V^*\mathcal{F}(VxV^*)V \\
&= \widehat{\mathcal{F}}(x).
\end{aligned}$$

$\square$

As a consequence of this proposition, we get that studying the interconversion of two idempotent channels $\mathcal{F}$ and $\mathcal{G}$, is equivalent to study the interconversion of their reduced versions $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{G}}$.

Proposition 3. *Let $\mathcal{F} : \mathcal{L}(\mathcal{H}_{\mathcal{F}}) \to \mathcal{L}(\mathcal{H}_{\mathcal{F}})$ and $\mathcal{G} : \mathcal{L}(\mathcal{H}_{\mathcal{G}}) \to \mathcal{L}(\mathcal{H}_{\mathcal{G}})$ be two idempotent channels and $\widehat{\mathcal{F}}$, $\widehat{\mathcal{G}}$ be their associated reduced channels, as defined in Proposition 1, then*

$$\inf_{\mathcal{D},\mathcal{E}} \|\mathcal{F} - \mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond = \inf_{\mathcal{D},\mathcal{E}} \|\widehat{\mathcal{F}} - \mathcal{D}\widehat{\mathcal{G}}\mathcal{E}\|_\diamond, \tag{17}$$

where the infimum are taken over encoders and decoders channels $\mathcal{E}$ and $\mathcal{D}$ between appropriate spaces.

Proof. Let $\mathcal{E} : \mathcal{L}(\mathcal{H}_{\mathcal{F}}) \to \mathcal{L}(\mathcal{H}_{\mathcal{G}})$ and $\mathcal{D} : \mathcal{L}(\mathcal{H}_{\mathcal{G}}) \to \mathcal{L}(\mathcal{H}_{\mathcal{F}})$ be a pair of channels. Let $\mathcal{E}_1, \mathcal{D}_1, \mathcal{E}_2, \mathcal{D}_2$ be channels such that

$$\widehat{\mathcal{F}} = \mathcal{D}_1\mathcal{F}\mathcal{E}_1, \text{ and } \mathcal{G} = \mathcal{D}_2\widehat{\mathcal{G}}\mathcal{E}_2, \tag{18}$$

which exist by Proposition 2. Then,

$$\begin{aligned}
\|\mathcal{F} - \mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond &= \|\mathcal{F} - \mathcal{D}\mathcal{D}_2\widehat{\mathcal{G}}\mathcal{E}_2\mathcal{E}\|_\diamond \\
&= \|\mathcal{D}_1\|_\diamond\|\mathcal{F} - \mathcal{D}\mathcal{D}_2\widehat{\mathcal{G}}\mathcal{E}_2\mathcal{E}\|_\diamond\|\mathcal{E}_1\|_\diamond \\
&\geq \|\mathcal{D}_1\mathcal{F}\mathcal{E}_1 - \mathcal{D}_1\mathcal{D}\mathcal{D}_2\widehat{\mathcal{G}}\mathcal{E}_2\mathcal{E}\mathcal{E}_1\|_\diamond \\
&= \|\widehat{\mathcal{F}} - \mathcal{D}_1\mathcal{D}\mathcal{D}_2\widehat{\mathcal{G}}\mathcal{E}_2\mathcal{E}\mathcal{E}_1\|_\diamond,
\end{aligned}$$

where we used in the second equality that as $\mathcal{E}_1, \mathcal{D}_1$ are channels, $\|\mathcal{E}_1\|_\diamond = \|\mathcal{D}_1\|_\diamond = 1$.

Therefore,

$$\inf_{\mathcal{D},\mathcal{E}} \|\widehat{\mathcal{F}} - \mathcal{D}\widehat{\mathcal{G}}\mathcal{E}\|_\diamond \leq \inf_{\mathcal{D},\mathcal{E}} \|\mathcal{F} - \mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond.$$

We can apply the same line of argument but converting $\widehat{\mathcal{F}}$ to $\mathcal{F}$ and $\mathcal{G}$ to $\widehat{\mathcal{G}}$ to prove the converse inequality. $\square$

As mentioned in the introduction, the shape vector $\lambda(\mathcal{F})$ of an idempotent channel $\mathcal{F} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ can be computed efficiently in terms of the dimension of $\mathcal{H}$.

Proposition 4 (Theorem 2.6 and Algorithm 1 in [8])**.** *Given $\mathcal{F} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ an idempotent channel, there is an algorithm that computes $\lambda(\mathcal{F})$ in time $\mathcal{O}(d^6 \log(d))$ where d is the dimension of $\mathcal{H}$.*

We now show that the range and the shape vector of idempotent channels preserve the tensor product, a property that will be used in the proofs of our Theorems 1 and 2. Indeed, it will allow us to reduce the study of the problem of emulating k copies of $\mathcal{F}$ using n copies of $\mathcal{G}$ for general pairs of integers (k, n) to the case where $k = n$.

Proposition 5. *Let $\mathcal{F}_1 : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_1)$ and $\mathcal{F}_2 : \mathcal{L}(\mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_2)$ be two idempotent channels, we have*

$$\mathrm{Rg}(\mathcal{F}_1 \otimes \mathcal{F}_2) = \mathrm{Rg}(\mathcal{F}_1) \otimes \mathrm{Rg}(\mathcal{F}_2), \tag{19}$$

$$\mathrm{Rg}(\mathcal{F}_1^* \otimes \mathcal{F}_2^*) = \mathrm{Rg}(\mathcal{F}_1^*) \otimes \mathrm{Rg}(\mathcal{F}_2^*), \tag{20}$$

$$\widehat{\mathcal{F}_1 \otimes \mathcal{F}_2} = \widehat{\mathcal{F}_1} \otimes \widehat{\mathcal{F}_2}, \tag{21}$$

$$\lambda(\mathcal{F}_1 \otimes \mathcal{F}_2) = \lambda(\mathcal{F}_1) \otimes \lambda(\mathcal{F}_2), \tag{22}$$

where $\widehat{\mathcal{F}_1 \otimes \mathcal{F}_2}$, $\widehat{\mathcal{F}_2}$ and $\widehat{\mathcal{F}_2}$ are respectively the reduced channels associated to $\mathcal{F}_1 \otimes \mathcal{F}_2$, $\mathcal{F}_1$ and $\mathcal{F}_2$.

Proof. The proof of eq. (19) and eq. (20) is elementary and these equations actually hold for every linear maps $\Phi : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_1')$ and $\Psi : \mathcal{L}(\mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_2')$. Indeed, since $\mathrm{Rg}(\Phi) \otimes \mathrm{Rg}(\Psi) = \mathrm{Span}\{y_1 \otimes y_2 \mid y_1 \in \mathrm{Rg}(\Phi), y_2 \in \mathrm{Rg}(\Psi)\}$, we argue on product elements $y_1 \otimes y_2 \in \mathrm{Rg}(\Phi) \otimes \mathrm{Rg}(\Psi)$ with $y_1 \in \mathrm{Rg}(\Phi)$ and $y_2 \in \mathrm{Rg}(\Psi)$. We have

$$y_1 \otimes y_2 = \Phi(x_1) \otimes \Psi(x_2) = \Phi \otimes \Psi(x_1 \otimes x_2) \in \mathrm{Rg}(\Phi \otimes \Psi),$$

where $x_1 \in \mathcal{L}(\mathcal{H}_1)$ and $x_2 \in \mathcal{L}(\mathcal{H}_2)$. Conversely, for $y \in \mathrm{Rg}(\Phi \otimes \Psi)$ we have $y = \Phi \otimes \Psi(x)$, with $x \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Thus, we can write x as

$$x = \sum_{j=1}^{J} x_1^{(j)} \otimes x_2^{(j)},$$

with $x_i^{(j)} \in \mathcal{L}(\mathcal{H}_i)$ for $(i, j) \in [2] \times [J]$. Therefore,

$$y = \Phi \otimes \Psi(\sum_{j=1}^{J} x_1^{(j)} \otimes x_2^{(j)}) = \sum_{j=1}^{J} \Phi(x_1^{(j)}) \otimes \Psi(x_2^{(j)}) \in \mathrm{Rg}(\Phi) \otimes \mathrm{Rg}(\Psi).$$

Then, we prove the compatibility of taking the reduced version of idempotent channels with the tensor product. We write $\mathbb{1}_1$ and $\mathbb{1}_2$ respectively the identity operators in $\mathcal{L}(\mathcal{H}_1)$ and $\mathcal{L}(\mathcal{H}_2)$ and $\mathbb{1}_{1,2} = \mathbb{1}_1 \otimes \mathbb{1}_2$. We also write $\mathcal{H}_i^{(0)} = \mathrm{supp}(\mathcal{F}_i(\mathbb{1}_i))$ and $\mathcal{H}_i^{(1)} = \ker(\mathcal{F}_i(\mathbb{1}_i))$ for $i \in [2]$ and the corresponding isometries $V_i : \mathcal{H}_i^{(0)} \to \mathcal{H}_i$ defining $\widehat{\mathcal{F}_i}$. Then, $\mathcal{F}_1 \otimes \mathcal{F}_2(\mathbb{1}_{1,2}) = \mathcal{F}_1(\mathbb{1}_1) \otimes \mathcal{F}_2(\mathbb{1}_2)$, hence $\mathrm{supp}(\mathcal{F}_1 \otimes \mathcal{F}_2(\mathbb{1}_{1,2})) = \mathrm{supp}(\mathcal{F}_1(\mathbb{1}_1)) \otimes \mathrm{supp}(\mathcal{F}_2(\mathbb{1}_2))$ and $\widehat{\mathcal{F}_1 \otimes \mathcal{F}_2}$ maps $\mathcal{L}(\mathcal{H}_1^{(0)} \otimes \mathcal{H}_2^{(0)})$ into itself. If we write $e_{1,2} = \mathcal{F}_1 \otimes \mathcal{F}_2(\mathbb{1}_{1,2})^0$ the orthogonal projector onto $\mathcal{H}_1^{(0)} \otimes \mathcal{H}_2^{(0)}$ the support of $\mathcal{F}_1 \otimes \mathcal{F}_2(\mathbb{1}_{1,2})$, we also have that $e_{1,2} = e_1 \otimes e_2 = (V_1 \otimes V_2)(V_1 \otimes V_2)^*$ with $e_1 = V_1 V_1^* = \mathcal{F}_1(\mathbb{1}_1)^0$ and $e_2 = V_2 V_2^* = \mathcal{F}_2(\mathbb{1}_2)^0$ the orthogonal projectors onto the respective supports of $\mathcal{F}_i(\mathbb{1}_i)$, $i \in [2]$. Then for $x_1 \in \mathcal{L}(\mathcal{H}_1)$ and $x_2 \in \mathcal{L}(\mathcal{H}_2)$, we have

$$\begin{aligned}
\widehat{\mathcal{F}_1 \otimes \mathcal{F}_2}(x_1 \otimes x_2) &= (V_1 \otimes V_2)^* \mathcal{F}_1 \otimes \mathcal{F}_2((V_1 \otimes V_2)(x_1 \otimes x_2)(V_1 \otimes V_2)^*)(V_1 \otimes V_2) \\
&= V_1^* \mathcal{F}_1(V_1 x_1 V_1^*) V_1 \otimes V_2^* \mathcal{F}_2(V_2 x_2 V_2^*) V_2 \\
&= \widehat{\mathcal{F}_1}(x_1) \otimes \widehat{\mathcal{F}_2}(x_2).
\end{aligned}$$

Eq. (21) then follows on the whole space $\mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2) = \mathcal{L}(\mathcal{H}_1) \otimes \mathcal{L}(\mathcal{H}_2)$ by linearity.

The last equality, eq. (22), is then seen by injecting eq. (8) into eq. (19) and by the definition of the shape vector of an idempotent channel. $\square$

We end this section by the following remark on the origin of the concept of shape vectors.

Remark 1. *Note that the shape of an idempotent channel $\mathcal{F}$ coincides with the shape of the $*$-algebra $\mathrm{Rg}(\widehat{\mathcal{F}}^*)$, with $\widehat{\mathcal{F}}$ the reduced channel associated to $\mathcal{F}$. The notion of shape vector for $*$-algebras was introduced by Kuperberg in [12]. If $\mathcal{A}$ is a $*$-algebra, we can decompose*

$$\mathcal{A} \cong \bigoplus_{k=1}^{K} \mathcal{L}(\mathbb{C}^{d_k}),$$

for a set of integers $\{d_k \mid k \in [K]\}$ and we write $\lambda(\mathcal{A})$ the shape vector of $\mathcal{A}$, which coordinates are the integers $(d_k)_{k \in [K]}$ sorted in non-increasing order.[3]

4   Emulation of idempotent channels with zero error

Now that we have presented the decomposition properties of idempotent channels, we move on to proving Theorem 1 in this section, which is divided into two subsections. In the first one, we prove that every rate lesser than

$$\inf_{p \in [1, +\infty]} \frac{\log(\|\lambda(\mathcal{G})\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)} \tag{23}$$

is achievable. In the second one, we prove that any rate strictly greater than eq. (23) is not achievable.

Before presenting these proofs, we provide some examples for the expression eq. (23). In particular, we gather in Table 2 the expression of the emulation capacity $C(\mathcal{G} \mapsto \mathcal{F})$ when $\mathcal{F}$ or $\mathcal{G}$ is an identity quantum channel or a completely dephasing channel. In fact, when $\mathcal{F} = \mathrm{Id}_d$, then $\lambda(\mathcal{F}) = (d)$ and thus $\|\lambda(\mathcal{F})\|_p = d$ for all $p \geq 1$. As $\|\lambda(\mathcal{G})\|_p$ is non-increasing in p, this shows that $C(\mathcal{G} \mapsto \mathcal{F}) = \frac{\log(\|\lambda(\mathcal{G})\|_\infty)}{\log(d)}$. The other cases are similar. Note that, when $\mathcal{G} = \Delta_d$, $\log(\|\lambda(\Delta_d)\|_\infty) = \log(1) = 0$. Therefore, in Table 2, we divided the case $\mathcal{G} = \Delta_d$ in two more cases: the one where $\mathcal{F}$ is any idempotent channel such that at least one coordinate of $\lambda(\mathcal{F})$ is greater or equal to 2, and the one where $\lambda(\mathcal{F}) = (1, \ldots, 1)$, in which case, for all $p \in [1, +\infty[$,

$$\frac{\log(\|\lambda(\Delta_d)\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)} = \frac{p \log(\|\lambda(\Delta_d)\|_p)}{p \log(\|\lambda(\mathcal{F})\|_p)} = \frac{\log(d)}{\log(\|\lambda(\mathcal{F})\|_1)}.$$

Note that in the first case, i.e. $\lambda(\mathcal{F})$ has a coefficient greater or equal to two, the emulation capacity of $\mathcal{F}$ by the completely dephasing channel is zero, that is it is impossible to emulate a channel preserving some quantum information with a classical one, without error.

$\mathcal{F}$	$\mathcal{G}$	$C(\mathcal{G} \mapsto \mathcal{F})$
Id_d	$\mathcal{G}$	$\frac{\log(\|\lambda(\mathcal{G})\|_\infty)}{\log(d)}$
$\mathcal{F}$	Id_d	$\frac{\log(d)}{\log(\|\lambda(\mathcal{F})\|_1)}$
Δ_d	$\mathcal{G}$	$\frac{\log(\|\lambda(\mathcal{G})\|_1)}{\log(d)}$
$\mathcal{F}$ s.t. $\lambda(\mathcal{F}) \neq (1, \ldots, 1)$	Δ_d	0
$\mathcal{F}$ s.t. $\lambda(\mathcal{F}) = (1, \ldots, 1)$	Δ_d	$\frac{\log(d)}{\log(\|\lambda(\mathcal{F})\|_1)}$

Table 2: Emulation capacities when either $\mathcal{F}$ or $\mathcal{G}$ is the identity or the completely dephasing channel.

Then, we also note that, as a corollary to our Theorem 1, the emulation capacity of idempotent channels is additive under tensor products of source channels, which yields that the zero-error quantum and classical capacities of idempotent channels are additive.

Corollary 1. *Let $\mathcal{F}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ be three idempotent channels, then*

$$C(\mathcal{G}_1 \otimes \mathcal{G}_2 \mapsto \mathcal{F}) = C(\mathcal{G}_1 \mapsto \mathcal{F}) + C(\mathcal{G}_2 \mapsto \mathcal{F}). \tag{24}$$

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[3]In fact, in all this article and in Kuperberg's work [12], the order with respect to which the coordinates of shape vectors are sorted is actually irrelevant.

Proof. By Theorem 1, we have

$$C(\mathcal{G}_1 \otimes \mathcal{G}_2 \mapsto \mathcal{F}) = \inf_{p \in [1, +\infty]} \frac{\log(\|\lambda(\mathcal{G}_1 \otimes \mathcal{G}_2)\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)}$$

$$= \inf_{p \in [1, +\infty]} \frac{\log(\|\lambda(\mathcal{G}_1)\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)} + \inf_{p \in [1, +\infty]} \frac{\log(\|\lambda(\mathcal{G}_2)\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)}$$

$$= C(\mathcal{G}_1 \mapsto \mathcal{F}) + C(\mathcal{G}_2 \mapsto \mathcal{F}),$$

where we used Proposition 5 and the multiplicativity of the ℓ_p-norms over the tensor product in the second equation. □

The following example shows that the emulation capacity is not *reversible*, i.e. that $C(\mathcal{G} \mapsto \mathcal{F}) \neq C(\mathcal{F} \mapsto \mathcal{G})^{-1}$ in general.

Example 1. *Let $\mathcal{F} = \mathrm{Id}_4$ be the identity channel on $\mathcal{L}(\mathbb{C}^4)$ and $\mathcal{G} : \mathcal{L}(\mathbb{C}^4) \to \mathcal{L}(\mathbb{C}^4)$ be such that $\lambda(\mathcal{G}) = (2, 2)$. Then, by the first row of Table 2, we have*

$$C(\mathcal{G} \mapsto \mathcal{F}) = \frac{\log(\|\lambda(\mathcal{G})\|_\infty)}{\log(4)} = \frac{\log(2)}{\log(4)} = \frac{1}{2},$$

whereas by the second row,

$$C(\mathcal{F} \mapsto \mathcal{G}) = \frac{\log(4)}{\log(\|\lambda(\mathcal{G})\|_1)} = \frac{\log(4)}{\log(4)} = 1.$$

Therefore,

$$C(\mathcal{G} \mapsto \mathcal{F})^{-1} = 2 \neq 1 = C(\mathcal{F} \mapsto \mathcal{G}).$$

This example shows that neither the emulation capacity nor the strong converse rate given in Theorem 2 are reversible, since, as $\mathcal{F}$ is taken to be an identity channel in this example, both quantity match.

Although, when either $\mathcal{F}$ or $\mathcal{G}$ is an identity channel or a completely dephasing one, the infimum in eq. (1) is reached for $p \in \{1, +\infty\}$, we show in the following example, that this is not the case in general.

In particular, the fact that the infimum in eq. (1) can be reached for $p \notin \{1, +\infty\}$ implies that, for general idempotent channels $\mathcal{F}$ and $\mathcal{G}$, the chain of emulation $\mathcal{G} \mapsto \mathrm{Id}_d \mapsto \mathcal{F}$ (where each arrow represents an emulation) is less efficient than directly emulating $\mathcal{F}$ by $\mathcal{G}$.

Example 2. *Let $\mathcal{F} : \mathcal{L}(\mathbb{C}^8) \to \mathcal{L}(\mathbb{C}^8)$ and $\mathcal{G} : \mathcal{L}(\mathbb{C}^{15}) \to \mathcal{L}(\mathbb{C}^{15})$ be two idempotent channels such that $\lambda(\mathcal{F}) = (5, 3)$ and $\lambda(\mathcal{G}) = (10, 3, 1, 1)$, then, we numerically find that the infimum*

$$C(\mathcal{G} \mapsto \mathcal{F}) = \inf_{p \in [1, +\infty]} \frac{\log(\|\lambda(\mathcal{G})\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)} \approx 1.29916,$$

is reached for $p \approx 1.15401$. In contrast, the infimum

$$C(\mathcal{F} \mapsto \mathcal{G}) = \inf_{p \in [1, +\infty]} \frac{\log(\|\lambda(\mathcal{F})\|_p)}{\log(\|\lambda(\mathcal{G})\|_p)} = \frac{\log(5)}{\log(10)},$$

is reached for $p = +\infty$. We can verify that

$$C(\mathcal{F} \mapsto \mathcal{G})^{-1} = \frac{\log(10)}{\log(5)} \approx 1.43068 \neq C(\mathcal{G} \mapsto \mathcal{F}).$$

4.1 Achievability in Theorem 1

To prove the achievability direction of Theorem 1, we first show the following proposition in the case where only one copy of $\mathcal{F}$ is emulated for each copy of $\mathcal{G}$ used, we will then use Proposition 5 on the compatibility of the shape vector with the tensor product to conclude.

Proposition 6. *Let $\mathcal{F}$ and $\mathcal{G}$ be two idempotent channels such that, for all $p \in [1, +\infty]$,*

$$\|\lambda(\mathcal{F})\|_p < \|\lambda(\mathcal{G})\|_p.$$

Then, there exists $n \in \mathbb{N}^*$, $\mathcal{E}$ and $\mathcal{D}$ two channels such that

$$\mathcal{F}^{\otimes n} = \mathcal{D}\mathcal{G}^{\otimes n}\mathcal{E}.$$

To prove this proposition, we use the following theorem on embeddability of $*$-algebras.

Theorem 5 (Proven in [12]). *If $\mathcal{A}$ and $\mathcal{B}$ are two finite-dimensional $*$-algebras such that, for all $p \in [1, +\infty]$,*

$$\|\lambda(\mathcal{A})\|_p < \|\lambda(\mathcal{B})\|_p, \tag{25}$$

then, there is $n \in \mathbb{N}^*$ such that there exists a subunital injective $*$-homomorphism $\iota : \mathcal{A}^{\otimes n} \to \mathcal{B}^{\otimes n}$.

Proof. This theorem, despite not being stated explicitly in [12], is proven in the proof of Theorem 1.1 of [12]. $\qquad\square$

For our achievability proof, it will be useful to lift ι into a unital completely positive map.

In the statement of the following lemma, we will use the fact that it is always possible, given $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H}_\mathcal{A})$ a $*$-subalgebra of $\mathcal{L}(\mathcal{H}_\mathcal{A})$ with the same identity operator as $\mathcal{L}(\mathcal{H}_\mathcal{A})$, to construct a unital completely positive map $E_\mathcal{A} : \mathcal{L}(\mathcal{H}_\mathcal{A}) \to \mathcal{A}$ such that for all $x \in \mathcal{A}$, $E_\mathcal{A}(x) = x$. This fact is easily seen as if $\mathcal{A}$ is a $*$-subalgebra of $\mathcal{L}(\mathcal{H}_\mathcal{A})$ with the same identity operator, we can write the orthogonal direct sum decompositions

$$\mathcal{H}_\mathcal{A} = \bigoplus_{k=1}^{K} \mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}, \quad \mathcal{A} = \bigoplus_{k=1}^{K} \mathcal{L}(\mathcal{H}_{k,1}) \otimes \mathbb{1}_{m_k},$$

with $m_k = \dim(\mathcal{H}_{k,2})$ for $k \in [K]$. Then, we can take $E_\mathcal{A} : \mathcal{L}(\mathcal{H}_\mathcal{A}) \to \mathcal{A}$ defined by

$$E_\mathcal{A} : x \to \sum_{k=1}^{K} \mathrm{tr}_{k,2}(P_k x P_k) \otimes \frac{1}{m_k}\mathbb{1}_{m_k}, \tag{26}$$

which is called the *tracial conditional expectation on $\mathcal{A}$* (see e.g. Proposition 1.5 in [19]).

Lemma 6. *Let $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H}_\mathcal{A})$, $\mathcal{B} \subseteq \mathcal{L}(\mathcal{H}_\mathcal{B})$ be two finite-dimensional unital $*$-subalgebras of $\mathcal{L}(\mathcal{H}_\mathcal{A})$ and $\mathcal{L}(\mathcal{H}_\mathcal{B})$ respectively, where we assume that the identity operator in $\mathcal{A}$ coincides with the one in $\mathcal{L}(\mathcal{H}_\mathcal{A})$, denoted $\mathbb{1}_\mathcal{A}$ and the one in $\mathcal{B}$ coincides with the one in $\mathcal{L}(\mathcal{H}_\mathcal{B})$, denoted $\mathbb{1}_\mathcal{B}$. Let $\iota : \mathcal{A} \to \mathcal{B}$ be an injective subunital $*$-homomorphism.*

Then, writing $\mathcal{H}_\mathcal{B}^{(0)} = \mathrm{supp}(\iota(\mathbb{1}_\mathcal{A})) \subseteq \mathcal{H}_\mathcal{B}$, we have that $P_\iota = \iota(\mathbb{1}_\mathcal{A}) \in \mathcal{L}(\mathcal{H}_\mathcal{B})$ is the orthogonal projector on $\mathcal{H}_\mathcal{B}^{(0)}$. We denote $V : \mathcal{H}_\mathcal{B}^{(0)} \to \mathcal{H}_\mathcal{B}$ the isometry embedding $\mathcal{H}_\mathcal{B}^{(0)}$ into $\mathcal{H}_\mathcal{B}$ such that $VV^* = P_\iota$ and we define the reduced $*$-homomorphism $\widehat{\iota} : \mathcal{A} \to \mathcal{L}(\mathcal{H}_\mathcal{B}^{(0)})$ by

$$\widehat{\iota} : x \mapsto V^*\iota(x)V. \tag{27}$$

Then $\widehat{\iota} : \mathcal{A} \to \mathcal{L}(\mathcal{H}_\mathcal{B}^{(0)})$ is a unital injective $*$-homomorphism and $\widehat{\iota}(\mathcal{A})$ is a unital $*$-subalgebra of $\mathcal{L}(\mathcal{H}_\mathcal{B}^{(0)})$ where both $\widehat{\iota}(\mathcal{A})$ and $\mathcal{L}(\mathcal{H}_\mathcal{B}^{(0)})$ share the same identity operator. We will write $\widehat{\iota}^{-1} : \widehat{\iota}(\mathcal{A}) \to \mathcal{A}$ the inverse of $\widehat{\iota}$, which is itself a unital injective $*$-homomorphism.

We write $E_\mathcal{A} : \mathcal{L}(\mathcal{H}_\mathcal{A}) \to \mathcal{A}$ and $E_{\widehat{\iota}(\mathcal{A})} : \mathcal{L}(\mathcal{H}_\mathcal{B}^{(0)}) \to \widehat{\iota}(\mathcal{A})$ the tracial conditional expectations on $\mathcal{A}$ and $\widehat{\iota}(\mathcal{A})$ respectively. Then, the maps

$$\widetilde{\iota} : \mathcal{L}(\mathcal{H}_\mathcal{A}) \to \mathcal{B} \subseteq \mathcal{L}(\mathcal{H}_\mathcal{B})$$
$$x \mapsto \iota(E_\mathcal{A}(x)) + \frac{\mathrm{tr}(x)}{\mathrm{tr}(\mathbb{1}_\mathcal{A})}(\mathbb{1}_\mathcal{B} - P_\iota), \tag{28}$$

and

$$\widetilde{\iota}^{-1} : \mathcal{L}(\mathcal{H}_\mathcal{B}) \to \mathcal{A} \subseteq \mathcal{L}(\mathcal{H}_\mathcal{A})$$
$$x \mapsto \widehat{\iota}^{-1}(E_{\widehat{\iota}(\mathcal{A})}(V^*xV)) \tag{29}$$

are unital completely positive and satisfy for all $x \in \mathcal{A}$,

$$\widetilde{\iota}^{-1} \circ \widetilde{\iota}(x) = x. \tag{30}$$

Proof. First, we prove that $P_\iota = \iota(\mathbb{1}_\mathcal{A})$ is an orthogonal projector on $\mathrm{supp}(\iota(\mathbb{1}_\mathcal{A}))$. As ι is a $*$-homomorphism, it preserves both the operation of taking the adjoint and the multiplication, so that $P_\iota^* = \iota(\mathbb{1}_\mathcal{A})^* = \iota(\mathbb{1}_\mathcal{A}^*) = P_\iota$ and $P_\iota^2 = \iota(\mathbb{1}_\mathcal{A})^2 = \iota(\mathbb{1}_\mathcal{A}^2) = P_\iota$.

We move on to prove that $\widehat{\iota} : \mathcal{A} \to \mathcal{L}(\mathcal{H}_\mathcal{B}^{(0)})$, with $\mathcal{H}_\mathcal{B}^{(0)} = \mathrm{supp}(\iota(\mathbb{1}_\mathcal{A}))$ is a $*$-homomorphism, which is also unital and injective. First $\widehat{\iota}$ is clearly linear. Let $x \in \mathcal{A}$,

$$\widehat{\iota}(x)^* = (V^* \iota(x) V)^* = V \iota(x)^* V^* = V \iota(x^*) V^* = \widehat{\iota}(x^*),$$

and for $y \in \mathcal{A}$,

$$\widehat{\iota}(xy) = V^* \iota(xy) V = V^* \iota(x) \iota(y) V \overset{(a)}{=} V^* V V^* \iota(x) V V^* \iota(y) V = \widehat{\iota}(x) \widehat{\iota}(y).$$

The equality (a) holds by applying Lemma 3 on the map $\Phi = \iota$ which is positive as it is a $*$-homomorphism[4]. To show the unitality, we have

$$\widehat{\iota}(\mathbb{1}_\mathcal{A}) = V^* \iota(\mathbb{1}_\mathcal{A}) V = V^* P_\iota V = V^* V V^* V = \mathbb{1}_{\mathcal{H}_\mathcal{B}^{(0)}},$$

with $\mathbb{1}_{\mathcal{H}_\mathcal{B}^{(0)}}$ the identity operator in $\mathcal{L}(\mathcal{H}_\mathcal{B}^{(0)})$. We finish with the injectivity property. Let $x \in \mathrm{ker}(\widehat{\iota})$,

$$0 = \widehat{\iota}(x) = V^* \iota(x) V.$$

Hence, by Lemma 3

$$\iota(x) = V V^* \iota(x) V V^* = 0,$$

so that $x \in \mathrm{ker}(\iota)$, thus $x = 0$ by injectivity of ι.

Since $\widehat{\iota}$ is a unital $*$-homomorphism, $\widehat{\iota}(\mathcal{A})$ is a $*$-subalgebra of $\mathcal{L}(\mathcal{H}_\mathcal{B}^{(0)})$ and has the same identity operator as $\mathcal{L}(\mathcal{H}_\mathcal{B}^{(0)})$, so that the tracial conditional expectation $E_{\widehat{\iota}(\mathcal{A})}$ is well-defined by eq. (26). Furthermore, as $\widehat{\iota}$ is injective $\widehat{\iota}^{-1}$ is well-defined and is a unital $*$-homomorphism as well.

Now, we show that $\widetilde{\iota}$ and $\widetilde{\iota}^{-1}$ are unital completely positive maps. By unitality of $E_\mathcal{A}$, $E_{\widehat{\iota}(\mathcal{A})}$ and $\widehat{\iota}^{-1}$, it is clear that $\widetilde{\iota}$ and $\widetilde{\iota}^{-1}$ are unital. Furthermore, $\widetilde{\iota}^{-1}$ is the composition of completely positive maps and is thus completely positive. Moreover, as ι was assumed to be subunital, $\iota(\mathbb{1}_\mathcal{A}) = P_\iota \le \mathbb{1}_\mathcal{B}$ so that $\widetilde{\iota}$ is completely positive.

We also have that $\widetilde{\iota}(\mathcal{L}(\mathcal{H}_\mathcal{A})) \subseteq \mathcal{B}$. Indeed, as $\mathcal{B}$ is unital of unit $\mathbb{1}_\mathcal{B}$ and as the range of ι is in $\mathcal{B}$, $\mathbb{1}_\mathcal{B} - P_\iota \in \mathcal{B}$ and $\iota(E_\mathcal{A}(x)) \in \mathcal{B}$ for all $x \in \mathcal{L}(\mathcal{H}_\mathcal{B})$. Hence $\widetilde{\iota}(\mathcal{A}) \subseteq \mathcal{B}$.

Finally, note that $\mathbb{1}_\mathcal{B} - P_\iota$ is the orthogonal projector on $\mathrm{ker}(P_\iota)$ which is orthogonal to $\mathrm{supp}(P_\iota) = \mathrm{Rg}(V)$, so that $(\mathbb{1}_\mathcal{B} - P_\iota) V = 0$. Hence, for $x \in \mathcal{A}$,

$$\widetilde{\iota}^{-1}(\widetilde{\iota}(x)) = \widehat{\iota}^{-1}(E_{\widehat{\iota}(\mathcal{A})}(V^* \iota(E_\mathcal{A}(x)) V)) = \widehat{\iota}^{-1}(E_{\widehat{\iota}(\mathcal{A})}(\widehat{\iota}(x))) = \widehat{\iota}^{-1}(\widehat{\iota}(x)) = x.$$

$\square$

We now prove Proposition 6.

Proof of Proposition 6. Let $\mathcal{F} : \mathcal{L}(\mathcal{H}_\mathcal{F}) \to \mathcal{L}(\mathcal{H}_\mathcal{F})$ and $\mathcal{G} : \mathcal{L}(\mathcal{H}_\mathcal{G}) \to \mathcal{L}(\mathcal{H}_\mathcal{G})$ be two idempotent channels. We can suppose without loss of generality that $\mathcal{F} = \widehat{\mathcal{F}}$ and $\mathcal{G} = \widehat{\mathcal{G}}$ as, by Proposition 3, we can convert $\mathcal{G}$ into $\mathcal{F}$ if and only if we can convert the reduced channel of $\mathcal{G}$ into the one of $\mathcal{F}$. Therefore, by Proposition 1, we can suppose that both $\mathrm{Rg}(\mathcal{G}^*)$ and $\mathrm{Rg}(\mathcal{F}^*)$ are unital $*$-subalgebras of $\mathcal{L}(\mathcal{H}_\mathcal{G})$ and $\mathcal{L}(\mathcal{H}_\mathcal{F})$ respectively and that the identity operators of $\mathcal{L}(\mathcal{H}_\mathcal{F})$ and $\mathrm{Rg}(\mathcal{F}^*)$ coincide as well as those of $\mathcal{L}(\mathcal{H}_\mathcal{G})$ and $\mathrm{Rg}(\mathcal{G}^*)$. Thus, by assumption, for all $p \in [1, +\infty]$,

$$\|\lambda(\mathrm{Rg}(\mathcal{F}^*))\|_p = \|\lambda(\mathcal{F})\|_p < \|\lambda(\mathcal{G})\|_p = \|\lambda(\mathrm{Rg}(\mathcal{G}^*))\|_p.$$

Therefore, by Theorem 5, there is an $n \in \mathbb{N}^*$ such that there exists an injective subunital $*$-homomorphism $\iota : \mathrm{Rg}(\mathcal{F}^*)^{\otimes n} \to \mathrm{Rg}(\mathcal{G}^*)^{\otimes n}$ between the two $*$-algebras $\mathrm{Rg}(\mathcal{F}^*)^{\otimes n}$ and $\mathrm{Rg}(\mathcal{G}^*)^{\otimes n}$. Then, by Proposition 5, $\mathrm{Rg}(\mathcal{F}^*)^{\otimes n} = \mathrm{Rg}(\mathcal{F}^{*\otimes n})$ and $\mathrm{Rg}(\mathcal{G}^*)^{\otimes n} = \mathrm{Rg}(\mathcal{G}^{*\otimes n})$, so that ι injects $\mathrm{Rg}(\mathcal{F}^{*\otimes n})$ into $\mathrm{Rg}(\mathcal{G}^{*\otimes n})$. We then show that we have the factorisation:

$$\mathcal{F}^{*\otimes n} = \widetilde{\iota}^{-1} \circ \mathcal{G}^{*\otimes n} \circ \widetilde{\iota} \circ \mathcal{F}^{*\otimes n}, \tag{31}$$

---

[4] Actually, $*$-homomorphisms are even completely positive [7].

with $\widetilde{\iota}$ and $\widetilde{\iota}^{-1}$ being defined as in Lemma 6. Indeed, for $x \in \mathcal{L}(\mathcal{H}_{\mathcal{F}}^{\otimes n})$, $\widetilde{\iota}(\mathcal{F}^{*\otimes n}(x)) \in \mathrm{Rg}(\mathcal{G}^{*\otimes n})$. Then, since $\mathcal{G}^{*\otimes n}$ is idempotent, every element of its range is one of its fixed points, so that $\mathcal{G}^{*\otimes n}(\widetilde{\iota}(\mathcal{F}^{*\otimes n}(x))) = \widetilde{\iota}(\mathcal{F}^{*\otimes n}(x))$. As $\mathcal{F}^{*\otimes n}(x) \in \mathrm{Rg}(\mathcal{F}^{*\otimes n})$, we can finally apply eq. (30) to get that $\widetilde{\iota}^{-1}\widetilde{\iota}(\mathcal{F}^{*\otimes n}(x)) = \mathcal{F}^{*\otimes n}(x)$.

We then write $\mathcal{E}^* = \widetilde{\iota}^{-1}$ and $\mathcal{D}^* = \widetilde{\iota} \circ \mathcal{F}^{*\otimes n}$. By Lemma 6, both $\mathcal{E}^*$ and $\mathcal{D}^*$ are unital completely positive, so that their adjoint $\mathcal{E}$ and $\mathcal{D}$ are channels. Taking the adjoint of eq. (31), we finally get

$$\mathcal{F}^{\otimes n} = \mathcal{D}\mathcal{G}^{\otimes n}\mathcal{E}.$$

$\square$

Then we can prove the achievability in Theorem 1.

Proof of achievability in Theorem 1. Let $\mathcal{F}, \mathcal{G}$ be idempotent channels and

$$r < \inf_{p \in [1,+\infty]} \frac{\log(\|\lambda(\mathcal{G})\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)}.$$

By density of the rational numbers into the reals, there exists $k, n \in \mathbb{N}^*$ such that

$$r \le \frac{k}{n} < \inf_{p \in [1,+\infty]} \frac{\log(\|\lambda(\mathcal{G})\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)}.$$

Then, for every $p \in [1,+\infty]$, we have

$$\|\lambda(\mathcal{F}^{\otimes k})\|_p < \|\lambda(\mathcal{G}^{\otimes n})\|_p,$$

where we used Proposition 5 on the shape vectors of the tensor product of channels.

Then, by Proposition 6, there is $m \in \mathbb{N}^*$, $\mathcal{E}$ and $\mathcal{D}$ channels such that

$$\mathcal{F}^{\otimes km} = \mathcal{D}\mathcal{G}^{\otimes nm}\mathcal{E},$$

and $r \le k/n = km/(nm)$ is thus an achievable rate. $\square$

4.2 Converse of Theorem 1

We now move on to the proof of the converse of Theorem 1. To prove this direction, we use a result of [12] which is almost a converse to Theorem 5.

Theorem 7 (Forward direction of Theorem 1.1 in [12])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two finite-dimensional $*$-algebras. If there is an injective $*$-homomorphism of $\mathcal{A}$ into $\mathcal{B}$, then for all $p \in [1,+\infty]$,*

$$\|\lambda(\mathcal{A})\|_p \le \|\lambda(\mathcal{B})\|_p. \tag{32}$$

Note that Theorems 5 and 7 are not converse one from another because the inequality between the ℓ_p-norms of the shape vectors is strict in one case but not in the other.

The converse of Theorem 1 then follows from Theorem 7 combined with the following proposition.

Proposition 7. *Let $\mathcal{F} : \mathcal{L}(\mathcal{H}_{\mathcal{F}}) \to \mathcal{L}(\mathcal{H}_{\mathcal{F}})$ and $\mathcal{G} : \mathcal{L}(\mathcal{H}_{\mathcal{G}}) \to \mathcal{L}(\mathcal{H}_{\mathcal{G}})$ be two idempotent channels equal to their reduced channels, i.e. $\mathcal{F} = \widehat{\mathcal{F}}$ and $\mathcal{G} = \widehat{\mathcal{G}}$. Suppose that there are two channels $\mathcal{E}, \mathcal{D}$ satisfying*

$$\mathcal{F} = \mathcal{D}\mathcal{G}\mathcal{E}. \tag{33}$$

We write $\mathbb{1}_{\mathcal{F}}$ and $\mathbb{1}_{\mathcal{G}}$ the identity operator in $\mathcal{L}(\mathcal{H}_{\mathcal{F}})$ and $\mathcal{L}(\mathcal{H}_{\mathcal{G}})$ respectively. To simplify our notations, we also denote $e_{\mathcal{G}\mathcal{E}} = \mathcal{G}\mathcal{E}(\mathbb{1}_{\mathcal{F}})^0$. Define

$$\widetilde{\mathcal{G}}^* : x \mapsto e_{\mathcal{G}\mathcal{E}}\mathcal{G}^*(x)e_{\mathcal{G}\mathcal{E}}. \tag{34}$$

Then, $e_{\mathcal{G}\mathcal{E}} \mathrm{Rg}(\mathcal{G}^*)e_{\mathcal{G}\mathcal{E}}$ is a unital $*$-subalgebra of $\mathcal{L}(\mathcal{H}_{\mathcal{G}})$ whose identity operator is $e_{\mathcal{G}\mathcal{E}}$ and $\widetilde{\mathcal{G}}^*\mathcal{D}^*$ restricted to the $*$-algebra $\mathrm{Rg}(\mathcal{F}^*)$ is an injective unital $*$-homomorphism which embeds $\mathrm{Rg}(\mathcal{F}^*)$ into $e_{\mathcal{G}\mathcal{E}} \mathrm{Rg}(\mathcal{G}^*)e_{\mathcal{G}\mathcal{E}}$ as a subalgebra.

Furthermore, for all $p \in [1,+\infty]$

$$\|\lambda(e_{\mathcal{G}\mathcal{E}} \mathrm{Rg}(\mathcal{G}^*)e_{\mathcal{G}\mathcal{E}})\|_p \le \|\lambda(\mathrm{Rg}(\mathcal{G}^*))\|_p.$$

The proof of this proposition uses the following lemma.

Lemma 8. *Let* $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$ *be a quantum channel and write* $e = \Phi(\mathbb{1})^0$. *Then,*

    i) for all $x \in \mathcal{L}(\mathcal{H}')$, $\Phi^*(x) = \Phi^*(exe)$,

    ii) for all positive semidefinite $x \in \mathcal{L}(\mathcal{H}')$, $\Phi^*(x) = 0$ *implies* $exe = 0$.

Proof. Let $\mathcal{H}'_0 = \mathrm{supp}(\Phi(\mathbb{1}))$ and $V : \mathcal{H}'_0 \to \mathcal{H}'$ be the isometry such that $e = VV^*$. Let $y \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{L}(\mathcal{H}')$, then

$$\mathrm{tr}(y\Phi^*(x)) = \mathrm{tr}(\Phi(y)x) = \mathrm{tr}(VV^*\Phi(y)VV^*x) = \mathrm{tr}(e\Phi(y)ex) = \mathrm{tr}(y\Phi^*(exe)),$$

where the second equality follows by Lemma 3. Therefore $\Phi^*(x) = \Phi(exe)$.

For the second statement, let $x \in \mathcal{L}(\mathcal{H}')$ be positive semidefinite such that $\Phi^*(x) = 0$. Then

$$0 = \mathrm{tr}(\Phi^*(x)) = \mathrm{tr}(\Phi(\mathbb{1})x) = \mathrm{tr}(e\Phi(\mathbb{1})ex) \geq \lambda_{\min}(\Phi(\mathbb{1}))\,\mathrm{tr}(exe),$$

with $\lambda_{\min}(\Phi(\mathbb{1}))$ the minimal non-zero eigenvalue of $\Phi(\mathbb{1})$. Therefore, $\mathrm{tr}(exe) = 0$ and, by positivity of x, $exe = 0$. $\qquad\square$

Then, we move on to the proof of Proposition 7. This proof uses the Kadison-Schwarz inequality (eq. (5.2) in [19]), which is a generalisation of the well-known Cauchy-Schwarz inequality to maps on operators. For every channel $\mathcal{F} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$, for all $x \in \mathcal{L}(\mathcal{H}')$,

$$\mathcal{F}^*(x^*)\mathcal{F}^*(x) \leq \mathcal{F}^*(x^*x). \tag{35}$$

Furthermore, in this proof, we will use the notion of *multiplicative domain* of linear maps. Given $\mathcal{F}^* : \mathcal{L}(\mathcal{H}') \to \mathcal{L}(\mathcal{H})$ a linear map, the multiplicative domain of $\mathcal{F}^*$ is the set

$$\{x \in \mathcal{L}(\mathcal{H}') \mid \forall y \in \mathcal{L}(\mathcal{H}'),\ \mathcal{F}^*(xy) = \mathcal{F}^*(x)\mathcal{F}^*(y) \text{ and } \mathcal{F}^*(yx) = \mathcal{F}^*(y)\mathcal{F}^*(x)\}.$$

If $\mathcal{F}^*$ is furthermore Hermitian preserving then its multiplicative domain is closed under taking the adjoint and is therefore a $*$-algebra. Moreover, in this case, $\mathcal{F}^*$ acts as a (non necessarily unital) $*$-homomorphism on its multiplicative domain. Furthermore, when $\mathcal{F}^*$ satisfies the Kadison-Schwarz inequality, by Theorem 5.7 in [19], we can write its multiplicative domain as

$$\{x \in \mathcal{L}(\mathcal{H}') \mid \mathcal{F}^*(x^*x) = \mathcal{F}^*(x^*)\mathcal{F}^*(x) \text{ and } \mathcal{F}^*(xx^*) = \mathcal{F}^*(x)\mathcal{F}^*(x^*)\}.$$

In particular, if $\mathcal{F}^*$ is unital completely positive[5], it satisfies the Kadison-Schwarz inequality (eq. (35)) and we can use this second characterisation of its multiplicative domain.

Proof of Proposition 7. Let $\mathcal{F} : \mathcal{L}(\mathcal{H}_\mathcal{F}) \to \mathcal{L}(\mathcal{H}_\mathcal{F})$ and $\mathcal{G} : \mathcal{L}(\mathcal{H}_\mathcal{G}) \to \mathcal{L}(\mathcal{H}_\mathcal{G})$ be two idempotent channels equal to their reduced channels, i.e. $\mathcal{F} = \widehat{\mathcal{F}}$ and $\mathcal{G} = \widehat{\mathcal{G}}$, and let $\mathcal{D}, \mathcal{E}$ be two channels such that

$$\mathcal{F} = \mathcal{D}\mathcal{G}\mathcal{E}.$$

Taking the adjoint of this equation, we get

$$\mathcal{F}^* = \mathcal{E}^*\mathcal{G}^*\mathcal{D}^* = \mathcal{E}^*\mathcal{G}^*\mathcal{G}^*\mathcal{D}^*,$$

where the right-hand side follows from the idempotence of $\mathcal{G}^*$. Let $x \in \mathcal{L}(\mathcal{H}_\mathcal{F})$, applying Lemma 8 to the channel $\Phi = \mathcal{G}\mathcal{E}$, we get that

$$\mathcal{F}^*(x) = \mathcal{E}^*\mathcal{G}^*(e_{\mathcal{G}\mathcal{E}}\mathcal{G}^*\mathcal{D}^*(x)e_{\mathcal{G}\mathcal{E}}) = \mathcal{E}^*\mathcal{G}^*\widetilde{\mathcal{G}}^*\mathcal{D}^*(x),$$

---

[5]In fact, it suffices for $\mathcal{F}^*$ to be subunital 2-positive for it to satisfy the Kadison-Schwarz inequality, see Section 5.2 of [19] for more details.

with $\widetilde{\mathcal{G}}^*$ defined in the statement of Proposition 7. Let $x \in \mathrm{Rg}(\mathcal{F}^*)$. By the assumption that $\mathcal{F} = \widehat{\mathcal{F}}$, $\mathrm{Rg}(\mathcal{F}^*)$ is a $*$-subalgebra of $\mathcal{L}(\mathcal{H}_{\mathcal{F}})$, thus $x^*x \in \mathrm{Rg}(\mathcal{F}^*)$, and as $\mathcal{F}^*$ is idempotent, xx^* is a fixed point of $\mathcal{F}^*$, so that we have:

$$
\begin{aligned}
x^*x &= \mathcal{F}^*(x^*x) \\
&= \mathcal{E}^*\mathcal{G}^*\widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*x) \\
&\overset{(a)}{\geq} \mathcal{E}^*\mathcal{G}^*(\widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*)\widetilde{\mathcal{G}}^*\mathcal{D}^*(x)) \\
&\overset{(b)}{\geq} \mathcal{E}^*\mathcal{G}^*(\widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*))\mathcal{E}^*\mathcal{G}^*(\widetilde{\mathcal{G}}^*\mathcal{D}^*(x)) \\
&= x^*x,
\end{aligned}
$$

where we used the Kadison-Schwarz inequality to get both (a) and (b). Since the left-hand side and the right-hand side of this chain of inequalities coincide, we conclude that (a) and (b) are in fact equalities. Therefore,

$$
\mathcal{E}^*\mathcal{G}^*(\widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*x) - \widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*)\widetilde{\mathcal{G}}^*\mathcal{D}^*(x)) = 0,
$$

where, by the Kadison-Schwarz inequality,

$$
\widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*x) - \widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*)\widetilde{\mathcal{G}}^*\mathcal{D}^*(x) \geq 0.
$$

Therefore, by the second statement in Lemma 8 applied to the channel $\Phi = \mathcal{G}\mathcal{E}$, we obtain

$$
\widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*x) - \widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*)\widetilde{\mathcal{G}}^*\mathcal{D}^*(x) = 0.
$$

We can swap the roles of x and x^* in (a) and (b) to find that

$$
\widetilde{\mathcal{G}}^*\mathcal{D}^*(xx^*) - \widetilde{\mathcal{G}}^*\mathcal{D}^*(x)\widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*) = 0.
$$

Therefore, x is in the multiplicative domain of $\widetilde{\mathcal{G}}^*\mathcal{D}^*$. Thus, $\widetilde{\mathcal{G}}^*\mathcal{D}^*$ restricted to $\mathrm{Rg}(\mathcal{F}^*)$ is an injective unital $*$-homomorphism into the $*$-algebra $e_{\mathcal{G}\mathcal{E}}\,\mathrm{Rg}(\mathcal{G}^*)e_{\mathcal{G}\mathcal{E}}$, which is a $*$-subalgebra of $\mathcal{L}(\mathcal{H}_{\mathcal{G}})$ and whose identity operator is $e_{\mathcal{G}\mathcal{E}}$. It is injective as, for all $x \in \mathrm{Rg}(\mathcal{F}^*)$,

$$
x = \mathcal{F}^*(x) = \mathcal{E}^*\mathcal{G}^*\widetilde{\mathcal{G}}^*\mathcal{D}^*(x),
$$

hence $\widetilde{\mathcal{G}}^*\mathcal{D}^*(x) = 0$ implies $x = 0$. It is unital as it maps the identity of $\mathcal{L}(\mathcal{H}_{\mathcal{F}})$ to $e_{\mathcal{G}\mathcal{E}}$ which is the identity operator of the $*$-algebra $e_{\mathcal{G}\mathcal{E}}\,\mathrm{Rg}(\mathcal{G}^*)e_{\mathcal{G}\mathcal{E}}$. Indeed, as both $\mathcal{G}$ and $\mathcal{D}$ are channels, $\mathcal{G}^*$ and $\mathcal{D}^*$ are unital so that $\mathcal{G}^*\mathcal{D}^*(\mathbb{1}_{\mathcal{F}}) = \mathbb{1}_{\mathcal{G}}$ and $\widetilde{\mathcal{G}}^*\mathcal{D}^*(\mathbb{1}_{\mathcal{F}}) = e_{\mathcal{G}\mathcal{E}}$.

Now, to show that $e_{\mathcal{G}\mathcal{E}}\,\mathrm{Rg}(\mathcal{G}^*)e_{\mathcal{G}\mathcal{E}}$ is a $*$-subalgebra of $\mathcal{L}(\mathcal{H}_{\mathcal{G}})$ which satisfies for all $p \in [1, +\infty]$, $\|\lambda(e_{\mathcal{G}\mathcal{E}}\,\mathrm{Rg}(\mathcal{G}^*)e_{\mathcal{G}\mathcal{E}})\|_p \leq \|\lambda(\mathrm{Rg}(\mathcal{G}^*))\|_p$, it suffices to remark that, by Proposition 1, we can decompose the Hilbert space $\mathcal{H}_{\mathcal{G}}$ as

$$
\mathcal{H}_{\mathcal{G}} = \bigoplus_{k=1}^{K} \mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2},
$$

and the $*$-algebra $\mathrm{Rg}(\mathcal{G}^*)$ as

$$
\mathrm{Rg}(\mathcal{G}^*) = \bigoplus_{k=1}^{K} \mathcal{L}(\mathcal{H}_{k,1}) \otimes \mathbb{1}_{m_k},
$$

with $m_k = \dim(\mathcal{H}_{k,2})$ for all $k \in [K]$. Then $\mathcal{G}\mathcal{E}(\mathbb{1}_{\mathcal{F}}) \in \mathrm{Rg}(\mathcal{G})$ can itself be decomposed as

$$
\mathcal{G}\mathcal{E}(\mathbb{1}_{\mathcal{F}}) = \sum_{k=1}^{K} x_k \otimes \rho_k,
$$

with $\{x_k \in \mathcal{L}(\mathcal{H}_{k,1}) \mid k \in [K]\}$ and $\{\rho_k \in \mathcal{D}(\mathcal{H}_{k,2}) \mid k \in [K]\}$. Thus,

$$
e_{\mathcal{G}\mathcal{E}} = \mathcal{G}\mathcal{E}(\mathbb{1}_{\mathcal{F}})^0 = \sum_{k=1}^{K} x_k^0 \otimes \rho_k^0,
$$

16

and

$$e_{\mathcal{G}\mathcal{E}} \operatorname{Rg}(\mathcal{G}^*) e_{\mathcal{G}\mathcal{E}} = \bigoplus_{k=1}^{K} x_k^0 \mathcal{L}(\mathcal{H}_{k,1}) x_k^0 \otimes \rho_k^0.$$

Hence, $e_{\mathcal{G}\mathcal{E}} \operatorname{Rg}(\mathcal{G}^*) e_{\mathcal{G}\mathcal{E}}$ is a $*$-subalgebra for all $k \in [K]$, the k-th coordinate of the shape vector of $e_{\mathcal{G}\mathcal{E}} \operatorname{Rg}(\mathcal{G}^*) e_{\mathcal{G}\mathcal{E}}$ satisfies

$$\lambda(e_{\mathcal{G}\mathcal{E}} \operatorname{Rg}(\mathcal{G}^*) e_{\mathcal{G}\mathcal{E}})_k \leq \lambda(\operatorname{Rg}(\mathcal{G}^*))_k,$$

which allows us to conclude. $\qquad\square$

We can now prove the converse of Theorem 1.

Converse of Theorem 1. Let $\mathcal{F} : \mathcal{L}(\mathcal{H}_\mathcal{F}) \to \mathcal{L}(\mathcal{H}_\mathcal{F})$ and $\mathcal{G} : \mathcal{L}(\mathcal{H}_\mathcal{G}) \to \mathcal{L}(\mathcal{H}_\mathcal{G})$ be two idempotent channels, let $k, n \in \mathbb{N}^*$ such that there are two channels $\mathcal{E}$ and $\mathcal{D}$ satisfying

$$\mathcal{F}^{\otimes k} = \mathcal{D}\mathcal{G}^{\otimes n}\mathcal{E}.$$

By Proposition 3, then, if we denote $\widehat{\mathcal{F}^{\otimes k}}$ and $\widehat{\mathcal{G}^{\otimes n}}$ the reduced channels associated to $\mathcal{F}^{\otimes k}$ and $\mathcal{G}^{\otimes n}$, this condition is equivalent to

$$\widehat{\mathcal{F}^{\otimes n}} = \mathcal{D}'\widehat{\mathcal{G}^{\otimes n}}\mathcal{E}',$$

for some channels $\mathcal{D}'$, $\mathcal{E}'$. Thus, by Proposition 7, there is an injective unital $*$-homomorphism from $\operatorname{Rg}(\widehat{\mathcal{F}^{\otimes k}}^*)$ into the $*$-algebra $e_{\widehat{\mathcal{G}^{\otimes n}}\mathcal{E}'} \operatorname{Rg}(\widehat{\mathcal{G}^{\otimes n}}^*) e_{\widehat{\mathcal{G}^{\otimes n}}\mathcal{E}'}$, which is itself a $*$-subalgebra of $\mathcal{L}(\mathcal{H}_\mathcal{G})$. Then, by Theorem 7 and the second statement of Proposition 7, we have for all $p \in [1, +\infty]$:

$$\|\lambda(\widehat{\mathcal{F}^{\otimes k}})\|_p = \|\lambda(\operatorname{Rg}(\widehat{\mathcal{F}^{\otimes k}}^*))\|_p \leq \|\lambda(e_{\widehat{\mathcal{G}^{\otimes n}}\mathcal{E}'} \operatorname{Rg}(\widehat{\mathcal{G}^{\otimes n}}^*) e_{\widehat{\mathcal{G}^{\otimes n}}\mathcal{E}'})\|_p \leq \|\lambda(\operatorname{Rg}(\widehat{\mathcal{G}^{\otimes n}}^*))\|_p = \|\lambda(\widehat{\mathcal{G}^{\otimes n}}^*)\|.$$

By the multiplicativity property of the shape vectors and the ℓ_p-norms with the tensor product (Proposition 5), we get for all $p \in [1, +\infty]$:

$$\|\lambda(\mathcal{F})\|_p^k = \|\lambda(\widehat{\mathcal{F}})\|_p^k \leq \|\lambda(\widehat{\mathcal{G}})\|_p^k = \|\lambda(\mathcal{G})\|_p^n.$$

Thus,

$$\frac{k}{n} \leq \inf_{p \in [1, +\infty]} \frac{\log(\|\lambda(\mathcal{G})\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)},$$

which ends the proof. $\qquad\square$

Finally, we give a slightly stronger version of the converse of Theorem 1 when allowing for dimension dependent errors in the Appendix. This stronger result is an application of approximate C^*-algebras, which were recently introduced by Kitaev in [10].

5   Emulation of idempotent channels with errors

In Theorem 1, the converse states that it is impossible to emulate an idempotent channel $\mathcal{F}$ by another idempotent channel $\mathcal{G}$ at a rate higher than the infimum in eq. (1) without error. In this section, we prove Theorem 2 which shows that we can not achieve better rates than

$$\inf_{p \in \{1, +\infty\}} \frac{\log(\|\lambda(\mathcal{G})\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)},$$

even if one allows errors. This strong converse rate is probably not tight in general, as the emulation capacity is expressed in Theorem 1 as an infimum over all $p \in [1, +\infty]$, we therefore conjecture that our strong converse rate can be improved to the infimum over all $p \in [1, +\infty]$.

Theorem 2 is actually a corollary of the following theorem, which gives a lower bound on the minimal error achievable for the one-shot emulation of $\mathcal{F}$ by $\mathcal{G}$. Note that this theorem can be seen as a generalisation of eq. (18) in [11] about emulation of identity channels. In fact, even in the special case of identity channels, we strengthen the bound of [11] by a factor of 2.

Theorem 9. *Let $\mathcal{F}$ and $\mathcal{G}$ be two idempotent channels, then*

$$\frac{1}{2} \inf_{\mathcal{D},\mathcal{E}} \|\mathcal{F} - \mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond \geq 1 - \min_{p \in \{1,+\infty\}} \frac{\|\lambda(\mathcal{G})\|_p}{\|\lambda(\mathcal{F})\|_p}. \tag{36}$$

To prove this theorem, we use the Holevo-Helstrom bound applied to the problem of channel discrimination.

Lemma 10 (see e.g. Theorem 3.52 in [18]). *Let $\Phi_1,\ \Phi_2 : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$ be two quantum channels. For any choice of auxiliary system $\mathcal{L}(\mathcal{H}'')$ of dimension $d = \dim(\mathcal{H}'')$, for any choice of positive semidefinite operator $\mu \in \mathcal{L}(\mathcal{H}' \otimes \mathcal{H}'')$ satisfying $\mu \leq \mathbb{1}$ and of density operator $\sigma \in \mathcal{D}(\mathcal{H} \otimes \mathcal{H}'')$,*

$$\frac{1}{2}\|\Phi_1 - \Phi_2\|_\diamond \geq \operatorname{tr}(\mu \Phi_1 \otimes \operatorname{Id}_d(\sigma)) - \operatorname{tr}(\mu \Phi_2 \otimes \operatorname{Id}_d(\sigma)). \tag{37}$$

We can now prove Theorem 9.

Proof of Theorem 9. Let $\mathcal{F} : \mathcal{L}(\mathcal{H}_\mathcal{F}) \to \mathcal{L}(\mathcal{H}_\mathcal{F})$ and $\mathcal{G} : \mathcal{L}(\mathcal{H}_\mathcal{G}) \to \mathcal{L}(\mathcal{H}_\mathcal{G})$ be two idempotent channels. If we denote $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{G}}$ their respective reduced channels, we showed as Proposition 3 that

$$\inf_{\mathcal{D},\mathcal{E}} \|\mathcal{F} - \mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond = \inf_{\mathcal{D},\mathcal{E}} \|\widehat{\mathcal{F}} - \mathcal{D}\widehat{\mathcal{G}}\mathcal{E}\|_\diamond.$$

Therefore, to simplify our notations, we will suppose without loss of generality that $\mathcal{F}$ and $\mathcal{G}$ are equal to their reduced version, i.e. $\mathcal{F} = \widehat{\mathcal{F}}$ and $\mathcal{G} = \widehat{\mathcal{G}}$. We prove separately the two following equations:

$$\frac{1}{2} \inf_{\mathcal{D},\mathcal{E}} \|\mathcal{F} - \mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond \geq 1 - \frac{\|\lambda(\mathcal{G})\|_1}{\|\lambda(\mathcal{F})\|_1}, \tag{38}$$

and

$$\frac{1}{2} \inf_{\mathcal{D},\mathcal{E}} \|\mathcal{F} - \mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond \geq 1 - \frac{\|\lambda(\mathcal{G})\|_\infty}{\|\lambda(\mathcal{F})\|_\infty}, \tag{39}$$

which taken together amount to the statement of Theorem 9.

To prove each equation, we use Lemma 10 on the channels $\Phi_1 = \mathcal{F}$ and $\Phi_2 = \mathcal{D}\mathcal{G}\mathcal{E}$ and we find an Ansatz (σ, μ) in eq. (37) such that on the one hand

$$\operatorname{tr}(\mu \mathcal{F} \otimes \operatorname{Id}_d(\sigma)) = 1,$$

and, on the other hand

$$\operatorname{tr}(\mu \mathcal{D}\mathcal{G}\mathcal{E} \otimes \operatorname{Id}_d(\sigma)) \leq \frac{\|\lambda(\mathcal{G})\|_p}{\|\lambda(\mathcal{F})\|_p},$$

with $p = 1$ for one Ansatz and $p = +\infty$ for the other.

We begin with the case $p = 1$. As we saw in Proposition 1, we can decompose $\mathcal{H}_\mathcal{F}$ as an orthogonal direct sum

$$\mathcal{H}_\mathcal{F} = \bigoplus_{k=1}^{K} \mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2},$$

so that

$$\operatorname{Rg}(\mathcal{F}) = \bigoplus_{k=1}^{K} \mathcal{L}(\mathcal{H}_{k,1}) \otimes \rho_k,$$

with, for all $k \in [K]$, $\rho_k \in \mathcal{D}(\mathcal{H}_{k,2})$. To simplify our notations, we write $d_k = \dim(\mathcal{H}_{k,1})$ and $m_k = \dim(\mathcal{H}_{k,2})$ for all $k \in [K]$, so that $\|\lambda(\mathcal{F})\|_1 = \sum_{k=1}^{K} d_k$. Now, for each $k \in [K]$, let $\{|\nu\rangle_k \mid \nu \in [d_k]\}$ be an orthonormal basis of $\mathcal{H}_{k,1}$. We define our Ansatz σ as being, basically, the sum of the maximally correlated states on the $\mathcal{H}_{k,1}$ spaces for $k \in [K]$:

$$\sigma = \frac{1}{\|\lambda(\mathcal{F})\|_1} \sum_{k=1}^{K} \sum_{\nu=1}^{d_k} |\nu\rangle\langle\nu|_k \otimes \frac{\mathbb{1}_{m_k}}{m_k} \otimes |\nu\rangle\langle\nu|_k \otimes \frac{\mathbb{1}_{m_k}}{m_k}, \tag{40}$$

where $|\nu\rangle\langle\nu|_k = |\nu\rangle_k\langle\nu|_k$ for notational simplicity. Then, we take

$$\mu = \sum_{k=1}^{K}\sum_{\nu=1}^{d_k} |\nu\rangle\langle\nu|_k \otimes \mathbb{1}_{m_k} \otimes |\nu\rangle\langle\nu|_k \otimes \mathbb{1}_{m_k}. \tag{41}$$

We now have:

$$\mathrm{tr}(\mu\mathcal{F}\otimes\mathrm{Id}(\sigma))$$
$$= \mathrm{tr}(\mathcal{F}^*\otimes\mathrm{Id}(\mu)\sigma)$$
$$= \mathrm{tr}(\mu\sigma)$$
$$= \frac{1}{\|\lambda(\mathcal{F})\|_1} \mathrm{tr}\Big(\Big(\sum_{k=1}^{K}\sum_{\nu=1}^{d_k} |\nu\rangle\langle\nu|_k \otimes \mathbb{1}_{m_k} \otimes |\nu\rangle\langle\nu|_k \otimes \mathbb{1}_{m_k}\Big)\Big(\sum_{k'=1}^{K}\sum_{\nu'=1}^{d_{k'}} |\nu'\rangle\langle\nu'|_{k'} \otimes \frac{\mathbb{1}_{m_{k'}}}{m_{k'}} \otimes |\nu'\rangle\langle\nu'|_{k'} \otimes \frac{\mathbb{1}_{m_{k'}}}{m_{k'}}\Big)\Big)$$
$$= \frac{1}{\|\lambda(\mathcal{F})\|_1} \mathrm{tr}\Big(\sum_{k=1}^{K}\sum_{\nu=1}^{d_k} |\nu\rangle\langle\nu|_k \otimes |\nu\rangle\langle\nu|_k\Big)$$
$$= \frac{1}{\|\lambda(\mathcal{F})\|_1} \sum_{k=1}^{K} d_k$$
$$= 1,$$

where, for the second equality, we used the fact that as $|\nu\rangle\langle\nu|_k \in \mathcal{L}(\mathcal{H}_{k,1})$, $|\nu\rangle\langle\nu|_k \otimes \mathbb{1}_{m_k} \in \mathrm{Rg}(\mathcal{F}^*)$ for all $k \in [K]$ so that $\mathcal{F}^*\otimes\mathrm{Id}(\mu) = \mu$. For the upper bound on $\mathrm{tr}(\mu\mathcal{DGE}\otimes\mathrm{Id}(\sigma))$, we have

$$\mathrm{tr}(\mu\mathcal{DGE}\otimes\mathrm{Id}(\sigma))$$
$$= \frac{1}{\|\lambda(\mathcal{F})\|_1} \mathrm{tr}\Big(\Big(\sum_{\substack{k \in [K] \\ \nu \in [d_k]}} |\nu\rangle\langle\nu|_k \otimes \mathbb{1}_{m_k} \otimes |\nu\rangle\langle\nu|_k \otimes \mathbb{1}_{m_k}\Big)\Big(\sum_{\substack{k' \in [K] \\ \nu' \in [d_{k'}]}} \mathcal{DGE}\big(|\nu'\rangle\langle\nu'|_{k'} \otimes \frac{\mathbb{1}_{m_{k'}}}{m_{k'}}\big) \otimes |\nu'\rangle\langle\nu'|_{k'} \otimes \frac{\mathbb{1}_{m_{k'}}}{m_{k'}}\Big)\Big),$$

so that

$$\mathrm{tr}(\mu\mathcal{DGE}\otimes\mathrm{Id}(\sigma)) = \frac{1}{\|\lambda(\mathcal{F})\|_1} \mathrm{tr}\Big(\sum_{k=1}^{K}\sum_{\nu=1}^{d_k}(|\nu\rangle\langle\nu|_k \otimes \mathbb{1}_{m_k})(\mathcal{DGE}\big(|\nu\rangle\langle\nu|_k \otimes \frac{\mathbb{1}_{m_k}}{m_k}\big))\Big). \tag{42}$$

From Proposition 1, we can decompose $\mathcal{H}_\mathcal{G}$ and $\mathcal{G}$ as

$$\mathcal{H}_\mathcal{G} = \bigoplus_{k=1}^{K'} \mathcal{H}'_{k,1} \otimes \mathcal{H}'_{k,2},$$

and

$$\mathcal{G} : x \mapsto \sum_{k=1}^{K'} \mathrm{tr}_{k,2}(P'_k x P'_k) \otimes \rho'_k,$$

with P'_k the orthogonal projector on $\mathcal{H}'_{k,1}\otimes\mathcal{H}'_{k,2}$, $\mathrm{tr}_{k,2}(\cdot)$ the partial trace over $\mathcal{H}'_{k,2}$ and $\rho'_k \in \mathcal{D}(\mathcal{H}'_{k,2})$ for all $k \in [K']$. We can therefore factorise $\mathcal{G}$ into two channels, one being the sum of the partial traces and the other being the sum of tensor products with the states ρ'_k for $k \in [K']$. We write $\mathcal{H}_{\mathcal{G}_1} = \bigoplus_{k=1}^{K'} \mathcal{H}'_{k,1}$ and we define $\mathcal{G}_1 : \mathcal{L}(\mathcal{H}_\mathcal{G}) \to \mathcal{L}(\mathcal{H}_{\mathcal{G}_1})$ and $\mathcal{G}_2 : \mathcal{L}(\mathcal{H}_{\mathcal{G}_1}) \to \mathcal{L}(\mathcal{H}_\mathcal{G})$ as

$$\mathcal{G}_1 : x \mapsto \sum_{k=1}^{K'} \mathrm{tr}_{k,2}(P'_k x P'_k), \tag{43}$$

$$\mathcal{G}_2 : x \mapsto \sum_{k=1}^{K'} \widetilde{P}'_k x \widetilde{P}'_k \otimes \rho'_k, \tag{44}$$

where $\widetilde{P}'_k \in \mathcal{L}(\mathcal{H}_{\mathcal{G}_1})$ is the orthogonal projector on $\mathcal{H}'_{k,1}$ for all $k \in [K']$. Now, $\mathcal{G}_1$ and $\mathcal{G}_2$ are channels and we can write $\mathcal{G} = \mathcal{G}_2 \circ \mathcal{G}_1$. Furthermore, let $\mathbb{1}_{\mathcal{G}_1}$ be the identity operator in $\mathcal{L}(\mathcal{H}_{\mathcal{G}_1})$, we then have, similarly as in the proof of eq. (18) in [11],

$$
\begin{aligned}
\|\lambda(\mathcal{G})\|_1 &= \sum_{k=1}^{K'} \dim(\mathcal{H}'_{k,1}) \\
&= \dim(\mathcal{H}_{\mathcal{G}_1}) \\
&= \operatorname{tr}(\mathbb{1}_{\mathcal{G}_1}) \\
&\stackrel{(a)}{=} \operatorname{tr}(\mathcal{G}_2^* \mathcal{D}^*(\mathbb{1}_{\mathcal{F}})) \\
&= \sum_{k=1}^{K} \sum_{\nu=1}^{d_k} \operatorname{tr}(\mathcal{G}_2^* \mathcal{D}^*(|\nu\rangle\langle\nu|_k \otimes \mathbb{1}_{m_k})) \\
&\stackrel{(b)}{\geq} \sum_{k=1}^{K} \sum_{\nu=1}^{d_k} \operatorname{tr}(\mathcal{G}_2^* \mathcal{D}^*(|\nu\rangle\langle\nu|_k \otimes \mathbb{1}_{m_k}) \mathcal{G}_1 \mathcal{E}(|\nu\rangle\langle\nu|_k \otimes \frac{\mathbb{1}_{m_k}}{m_k})) \\
&= \sum_{k=1}^{K} \sum_{\nu=1}^{d_k} \operatorname{tr}(|\nu\rangle\langle\nu|_k \otimes \mathbb{1}_{m_k} \mathcal{D}\mathcal{G}_2 \mathcal{G}_1 \mathcal{E}(|\nu\rangle\langle\nu|_k \otimes \frac{\mathbb{1}_{m_k}}{m_k})) \\
&= \operatorname{tr}(\sum_{k=1}^{K} \sum_{\nu=1}^{d_k} (|\nu\rangle\langle\nu|_k \otimes \mathbb{1}_{m_k})(\mathcal{D}\mathcal{G}\mathcal{E}(|\nu\rangle\langle\nu|_k \otimes \frac{\mathbb{1}_{m_k}}{m_k}))),
\end{aligned}
$$

where (a) follows from the fact that $\mathcal{D}\mathcal{G}_2 : \mathcal{L}(\mathcal{H}_{\mathcal{G}_1}) \to \mathcal{L}(\mathcal{H}_{\mathcal{F}})$ is a channel, so that $\mathcal{G}_2^* \mathcal{D}^*$ is unital. Then (b) follows from the fact that for all $k \in [K]$ and $\nu \in [d_k]$, $|\nu\rangle\langle\nu|_k \otimes \frac{\mathbb{1}_{m_k}}{m_k}$ is a state so that $\operatorname{tr}(\mathcal{G}_1 \mathcal{E}_1(|\nu\rangle\langle\nu|_k \otimes \frac{\mathbb{1}_{m_k}}{m_k})) = 1$, hence $0 \leq \mathcal{G}_1 \mathcal{E}_1(|\nu\rangle\langle\nu|_k \otimes \frac{\mathbb{1}_{m_k}}{m_k}) \leq \mathbb{1}_{\mathcal{G}_1}$, and that as $\mathcal{G}_2^* \mathcal{D}^*$ is completely positive, $\mathcal{G}_2^* \mathcal{D}^*(|\nu\rangle\langle\nu|_k \otimes \mathbb{1}_{m_k}) \geq 0$. We then simply use that if $A \geq 0$ and $B \leq \mathbb{1}$, $\operatorname{tr}(A) \geq \operatorname{tr}(AB)$. When injecting the last equation into eq. (42), we find

$$
\operatorname{tr}(\mu \mathcal{D}\mathcal{G}\mathcal{E}(\sigma)) \leq \frac{\|\lambda(\mathcal{G})\|_1}{\|\lambda(\mathcal{F})\|_1}.
$$

We now move on to the proof of eq. (39), i.e. the case $p = +\infty$. To deal with the case where the blocks of the range of $\mathcal{F}^*$ have multiplicities, we factorise $\mathcal{F} = \mathcal{F}_2 \circ \mathcal{F}_1$ as we did for $\mathcal{G}$ in eq. (43), so that we have the decompositions

$$
\begin{aligned}
\mathcal{H} &= \bigoplus_{k=1}^{K} \mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}, \\
\mathcal{F} &: x \mapsto \sum_{k=1}^{K} \operatorname{tr}_{k,2}(P_k x P_k) \otimes \rho_k, \\
\mathcal{F}_1 &: x \mapsto \sum_{k=1}^{K} \operatorname{tr}_{k,2}(P_k x P_k), \\
\mathcal{F}_2 &: x \mapsto \sum_{k=1}^{K} \widetilde{P}_k x \widetilde{P}_k \otimes \rho_k,
\end{aligned}
$$

where, for $k \in [K]$, $\rho_k \in \mathcal{D}(\mathcal{H}_{k,2})$, P_k is the orthogonal projector on $\mathcal{H}_{k,1} \otimes \mathcal{H}_{k,2}$ and $\widetilde{P}_k \in \mathcal{L}(\mathcal{H}_{\mathcal{F}_1})$ is the orthogonal projector on $\mathcal{H}_{k,1}$ with $\mathcal{H}_{\mathcal{F}_1} = \bigoplus_{k=1}^{K} \mathcal{H}_{k,1}$. Then, although we have the factorisation $\mathcal{F} = \mathcal{F}_2 \circ \mathcal{F}_1$, we also have the identity $\mathcal{F}_1 = \mathcal{F}_1 \circ \mathcal{F}$. Therefore, we can write

$$
\|\mathcal{F}_1 - \mathcal{F}_1 \mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond = \|\mathcal{F}_1 \circ (\mathcal{F} - \mathcal{D}\mathcal{G}\mathcal{E})\|_\diamond \leq \|\mathcal{F}_1\|_\diamond \|\mathcal{F} - \mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond = \|\mathcal{F} - \mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond,
$$

where the last equality follows from the fact that $\mathcal{F}_1$ is a channel and thus, $\|\mathcal{F}_1\|_\diamond = 1$. Therefore, to prove eq. (39), we will prove that

$$
\|\mathcal{F}_1 - \mathcal{F}_1 \mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond \geq 1 - \frac{\|\lambda(\mathcal{G})\|_\infty}{\|\lambda(\mathcal{F})\|_\infty},
$$

using Lemma 10. Let $k_\infty \in [K]$ be such that $\mathcal{H}_{k_\infty,1}$ has maximal dimension. To simplify our notations, we write $d = \dim(\mathcal{H}_{k_\infty,1}) = \|\lambda(\mathcal{F})\|_\infty$ and we let $\{|\nu\rangle \mid \nu \in [d]\}$ be an orthonormal basis of $\mathcal{H}_{k_\infty,1}$. The Ansatz we choose for σ and μ is then the maximally entangled state on $\mathcal{H}_{k_\infty,1}$:

$$\mu = \sigma = \frac{1}{d} \sum_{\nu,\eta=1}^{d} |\nu\nu\rangle\langle\eta\eta|. \tag{45}$$

Then,

$$\begin{aligned}
\operatorname{tr}(\mu \mathcal{F}_1 \otimes \operatorname{Id}(\sigma)) &= \frac{1}{d^2} \operatorname{tr}\Big(\Big(\sum_{\nu,\eta=1}^{d} |\nu\nu\rangle\langle\eta\eta| \Big) \Big(\sum_{\nu',\mu'=1}^{d} \mathcal{F}_1(|\nu'\rangle\langle\eta'|) \otimes |\nu'\rangle\langle\eta'| \Big) \Big) \\
&= \frac{1}{d^2} \operatorname{tr}\Big(\sum_{\nu,\eta,\nu',\eta'=1}^{d} |\nu\nu\rangle\langle\eta\eta|\nu'\nu'\rangle\langle\eta'\eta'| \Big) \\
&= \frac{1}{d^2} \Big(\sum_{\eta,\nu'=1}^{d} \langle\eta\eta|\nu'\nu'\rangle \Big) \Big(\sum_{\nu,\eta'=1}^{d} \langle\eta'\eta'|\nu\nu\rangle \Big) \\
&= 1,
\end{aligned}$$

where we used the fact that as $|\nu'\rangle\langle\eta'| \in \mathcal{L}(\mathcal{H}_{k_\infty,1})$ for all $\nu', \eta' \in [d]$, we have $\mathcal{F}_1(|\nu'\rangle\langle\eta'|) = |\nu'\rangle\langle\eta'|$.

For the upper bound on $\operatorname{tr}(\mu \mathcal{F}_1 \mathcal{D} \mathcal{G} \mathcal{E} \otimes \operatorname{Id}(\sigma))$, we use techniques much similar to those of the proof of Theorem 2.3 in [8]. First, we remark that the quantity $\operatorname{tr}(\mu \mathcal{F}_1 \mathcal{D} \mathcal{G} \mathcal{E} \otimes \operatorname{Id}(\sigma))$ is actually very similar to the *entanglement fidelity* of $\mathcal{F}_1 \mathcal{D} \mathcal{G} \mathcal{E}$ and can be easily expressed in terms of its Kraus operators, which we write, for now, $\{K_i \mid i \in [R]\}$[6]. Then:

$$\begin{aligned}
\operatorname{tr}(\mu \mathcal{F}_1 \mathcal{D} \mathcal{G} \mathcal{E} \otimes \operatorname{Id}(\sigma)) &= \frac{1}{d^2} \operatorname{tr}\Big(\Big(\sum_{\nu,\eta=1}^{d} |\nu\nu\rangle\langle\eta\eta| \Big) \Big(\sum_{i=1}^{R} \sum_{\nu',\eta'=1}^{d} K_i |\nu'\rangle\langle\eta'| K_i^* \otimes |\nu'\rangle\langle\eta'| \Big) \Big) \\
&= \frac{1}{d^2} \sum_{i=1}^{R} \sum_{\nu,\eta=1}^{d} \operatorname{tr}(|\nu\rangle\langle\eta| K_i |\eta\rangle\langle\nu| K_i^*) \\
&= \frac{1}{d^2} \sum_{i=1}^{R} \sum_{\nu,\eta=1}^{d} \langle\eta| K_i |\eta\rangle\langle\nu| K_i^* |\nu\rangle \\
&= \frac{1}{d^2} \sum_{i=1}^{R} |\operatorname{tr}(\widetilde{P}_\infty K_i)|^2,
\end{aligned}$$

where we write $\widetilde{P}_\infty = \widetilde{P}_{k_\infty} = \sum_\nu |\nu\rangle\langle\nu| \in \mathcal{L}(\mathcal{H}_{\mathcal{F}_1})$ the orthogonal projector onto $\mathcal{H}_{k_\infty,1}$. Now, we use the fact that the channel is of the form $\mathcal{F}_1 \mathcal{D} \mathcal{G} \mathcal{E}$. First, we factorise $\mathcal{G} = \mathcal{G}_2 \circ \mathcal{G}_1$ using the notations of eq. (43) and we furthermore define the pinching map $\mathcal{P} : \mathcal{L}(\mathcal{H}_{\mathcal{G}_1}) \to \mathcal{L}(\mathcal{H}_{\mathcal{G}_1})$ as

$$\mathcal{P} : x \mapsto \sum_{k=1}^{K'} \widetilde{P}_k' x \widetilde{P}_k', \tag{46}$$

with $\widetilde{P}_k' \in \mathcal{L}(\mathcal{H}_{\mathcal{G}_1})$ the orthogonal projector on $\mathcal{H}_{k,1}'$ for all $k \in [K']$. We then have the full factorisation $\mathcal{G} = \mathcal{G}_2 \circ \mathcal{P} \circ \mathcal{G}_1$. For notational simplicity, we write respectively $\{D_a\}_a$, $\{E_c\}_c$ the Kraus operators of the channels $\mathcal{F}_1 \circ \mathcal{D} \circ \mathcal{G}_2$ and $\mathcal{G}_1 \circ \mathcal{E}$. Hence, we can write Kraus operators $\{K_i\}_i$ of $\mathcal{F}_1 \mathcal{D} \mathcal{G} \mathcal{E}$ as $\{D_a \widetilde{P}_b' E_c\}_{a,b,c}$. Note that we can express $\|\lambda(\mathcal{G})\|_\infty$ in terms of the Kraus operators of $\mathcal{P}$ as $\|\lambda(\mathcal{G})\|_\infty = \max_b \operatorname{tr}(\widetilde{P}_b')$. Therefore, we have

$$\operatorname{tr}(\mu \mathcal{F}_1 \mathcal{D} \mathcal{G} \mathcal{E} \otimes \operatorname{Id}(\sigma)) = \frac{1}{d^2} \sum_{a,b,c} |\operatorname{tr}(\widetilde{P}_\infty D_a \widetilde{P}_b' E_c)|^2.$$

---

[6]We do not suppose this Kraus decomposition to be minimal.

By the Cauchy-Schwarz inequality on operators, for x and y of adequate dimensions, we have $|\operatorname{tr}(x^*y)|^2 \le \operatorname{tr}(x^*x)\operatorname{tr}(y^*y)$. Hence,

$$
\begin{aligned}
\operatorname{tr}(\mu \mathcal{F}_1 \mathcal{D}\mathcal{G}\mathcal{E}(\sigma)) &= \frac{1}{d^2} \sum_{a,b,c} |\operatorname{tr}(\widetilde{P}_\infty D_a \widetilde{P}'_b E_c)|^2 \\
&= \frac{1}{d^2} \sum_{a,b,c} |\operatorname{tr}(D_a \widetilde{P}'_b \widetilde{P}'_b E_c \widetilde{P}_\infty)|^2 \\
&\le \frac{1}{d^2} \sum_{a,b,c} \operatorname{tr}(D_a \widetilde{P}'_b D_a^*) \operatorname{tr}(\widetilde{P}_\infty E_c^* \widetilde{P}'_b E_c) \\
&\overset{(a)}{=} \frac{1}{d^2} \sum_b \Big(\operatorname{tr}(\widetilde{P}'_b) \sum_c \operatorname{tr}(\widetilde{P}_\infty E_c^* \widetilde{P}'_b E_c) \Big) \\
&\le \frac{\max_b \operatorname{tr}(\widetilde{P}'_b)}{d^2} \sum_{b,c} \operatorname{tr}(\widetilde{P}_\infty E_c^* \widetilde{P}'_b E_c) \\
&\overset{(b)}{=} \frac{\max_b \operatorname{tr}(\widetilde{P}'_b)}{d^2} \sum_c \operatorname{tr}(\widetilde{P}_\infty E_c^* E_c) \\
&\overset{(c)}{=} \frac{\max_b \operatorname{tr}(\widetilde{P}'_b)}{d^2} \operatorname{tr}(\widetilde{P}_\infty) \\
&= \frac{\|\lambda(\mathcal{G})\|_\infty}{\|\lambda(\mathcal{F})\|_\infty}.
\end{aligned}
$$

For (a), we used the fact that the operators $\{D_a\}_a$ are the Kraus operators of $\mathcal{F}_1 \mathcal{D}\mathcal{G}_2$, which is a channel and hence trace preserving. For (b), we used that $\sum_b \widetilde{P}'_b = \mathbb{1}_{\mathcal{G}_1}$. Finally, for (c), we used the Kraus relation $\sum_c E_c^* E_c = \mathbb{1}_\mathcal{F}$ for the channel $\mathcal{G}_1 \circ \mathcal{E}$. This ends the proof of eq. (39). $\qquad \square$

We can now present the proof of Theorem 2.

Proof of Theorem 2. Let $\mathcal{F}$, $\mathcal{G}$ be two idempotent channels, $(k_\nu)_{\nu\in\mathbb{N}}$, $(n_\nu)_{\nu\in\mathbb{N}}$ be two integer sequences and $\varepsilon > 0$ satisfying the assumptions of Theorem 2. Let $p_{\min} \in \{1, +\infty\}$ be such that

$$
\min_{p\in\{1,+\infty\}} \frac{\log(\|\lambda(\mathcal{G})\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)} = \frac{\log(\|\lambda(\mathcal{G})\|_{p_{\min}})}{\log(\|\lambda(\mathcal{F})\|_{p_{\min}})}.
$$

Then, we have

$$
\lim_{\nu\to\infty} \frac{k_\nu}{n_\nu} \ge \frac{\log(\|\lambda(\mathcal{G})\|_{p_{\min}})}{\log(\|\lambda(\mathcal{F})\|_{p_{\min}})} + \varepsilon.
$$

Then, for all ν big enough,

$$
\frac{k_\nu}{n_\nu} \ge \frac{\log(\|\lambda(\mathcal{G})\|_{p_{\min}})}{\log(\|\lambda(\mathcal{F})\|_{p_{\min}})} + \frac{\varepsilon}{2},
$$

so that

$$
\frac{1}{\|\lambda(\mathcal{F})\|_{p_{\min}}^{\frac{\varepsilon}{2}n_\nu}} \ge \frac{\|\lambda(\mathcal{G})\|_{p_{\min}}^{n_\nu}}{\|\lambda(\mathcal{F})\|_{p_{\min}}^{k_\nu}}.
$$

As, by assumption, $n_\nu \underset{\nu\to\infty}{\to} \infty$ and $\|\lambda(\mathcal{F})\|_{p_{\min}} > 1$, we get that

$$
\lim_{\nu\to\infty} \frac{\|\lambda(\mathcal{G})\|_{p_{\min}}^{n_\nu}}{\|\lambda(\mathcal{F})\|_{p_{\min}}^{k_\nu}} = 0. \tag{47}
$$

However, by Theorem 9, for all $\nu \in \mathbb{N}$, all encoding and decoding channels $\mathcal{E}_\nu$, $\mathcal{D}_\nu$,

$$
\frac{1}{2}\|\mathcal{F}^{\otimes k_\nu} - \mathcal{D}_\nu \mathcal{G}^{\otimes n_\nu} \mathcal{E}_\nu\|_\diamond \ge 1 - \frac{\|\lambda(\mathcal{G}^{\otimes n_\nu})\|_{p_{\min}}}{\|\lambda(\mathcal{F}^{\otimes k_\nu})\|_{p_{\min}}} = 1 - \frac{\|\lambda(\mathcal{G})\|_{p_{\min}}^{n_\nu}}{\|\lambda(\mathcal{F})\|_{p_{\min}}^{k_\nu}},
$$

where we used for the right-hand side equality the fact that both the shape vector and the ℓ_p-norms are compatible with the tensor product, i.e. Proposition 5. Then, Theorem 2 follows by injecting eq. (47) in this last inequality. $\qquad \square$

Acknowledgements

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A An application of approximate C^*-algebras

In a recent article, Kitaev introduced and studied approximate C^*-algebras and almost idempotent quantum channels, in order to find an algebraic framework which allows to naturally express the effects of the noise on quantum information [10]. As most of our proofs in Section 4 exploit the decomposition properties of finite-dimensional $*$-algebras, we use approximate C^*-algebras to strengthen the converse of Theorem 1 and show that it holds even if a small error is allowed.

Theorem 11. *Let $\mathcal{F} : \mathcal{L}(\mathcal{H}_\mathcal{F}) \to \mathcal{L}(\mathcal{H}_\mathcal{F})$ and $\mathcal{G} : \mathcal{L}(\mathcal{H}_\mathcal{G}) \to \mathcal{L}(\mathcal{H}_\mathcal{G})$ be two idempotent channels and $\mathcal{E}$, $\mathcal{D}$ be two channels. Let $d = \dim(\mathcal{H}_\mathcal{F})$, $\delta_{\max}$ be the absolute constant[7] of Theorem 12 and $\lambda_{\min,n}$ be the minimal non-zero eigenvalue of $\mathcal{G}^{\otimes n}\mathcal{E}(\mathbb{1}_{\mathcal{F}^{\otimes k}})$. If*

$$\|\mathcal{F}^{\otimes k} - \mathcal{D}\mathcal{G}^{\otimes n}\mathcal{E}\|_\diamond \leq \frac{\delta_{\max}\lambda_{\min,n}}{6d^k},$$

then

$$\frac{k}{n} \leq \inf_{p \in [1,+\infty]} \frac{\log(\|\lambda(\mathcal{G})\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)}.$$

Note that in Theorem 11, the error allowed depends on the dimension of the input space of $\mathcal{F}$, on the number of copies of $\mathcal{F}$ obtained and the number of copies of $\mathcal{G}$ used as well as on the encoder used for the emulation. This theorem is therefore less operationally motivated than Theorems 1 and 2, but it still strengthens the converse of Theorem 1 and its proof leads to nice mathematical developments, especially on multiplicative domains (see Theorem 13).

Note that an interesting direction would be to extend our Theorem 1 to the problem of emulating almost idempotent channels one with another, but we leave this for future work.

The Appendix is divided as follows. In Subsection A.1, we recall, for completeness, the *Error Reduction Theorem* for approximate C^*-algebras. We then move on to proving Theorem 13 on approximate multiplicative domain of maps satisfying the Kadison-Schwarz inequality in Subsection A.2, which will be used in the proof of Theorem 11 in Subsection A.3.

Note that this Appendix assumes familiarity with basic concepts from operator algebras.

A.1 The Error Reduction Theorem

We begin by recalling the definitions of ε-C^*-algebras and δ-homomorphisms.

Definition 1 (ε-C^*-algebra, Definition 2.1 in [10]). *An ε-Banach algebra is a Banach space $\mathcal{A}$, endowed with a bilinear multiplication map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that*

$$\forall x, y \in \mathcal{A}, \quad \|xy\| \leq (1+\varepsilon)\|x\|\|y\|,$$
$$\forall x, y, z \in \mathcal{A}, \quad \|(xy)z - x(yz)\| \leq \varepsilon\|x\|\|y\|\|z\|.$$

A $*\varepsilon$-Banach algebra is a complex ε-Banach algebra with a conjugate linear involution $x \mapsto x^*$ satisfying the equations:

$$\forall x, y \in \mathcal{A}, \quad \|x^*\| = \|x\|, \quad (xy)^* = y^*x^*. \tag{48}$$

---

[7]Here, 'absolute' means that $\delta_{\max}$ does neither depend on the dimension of $\mathcal{H}_\mathcal{F}$ nor of $\mathcal{H}_\mathcal{G}$.

An ε-C^*-algebra is one satisfying the following property:

$$\forall x \in \mathcal{A}, \ \|x^*x\| \geq (1 - \varepsilon)\|x\|^2. \tag{49}$$

The unit element $\mathbb{1} \in \mathcal{A}$ should satisfy the following conditions:

$$\|x\mathbb{1} - x\| \leq \varepsilon \|x\|, \ \|\mathbb{1}x - x\| \leq \varepsilon \|x\|, \ \|\|\mathbb{1}\| - 1| \leq \varepsilon. \tag{50}$$

If $\mathcal{A}$ is involutive, we also have

$$\mathbb{1}^* = \mathbb{1}. \tag{51}$$

A δ-homomorphism is then naturally defined in the following way.

Definition 2 (δ-homomorphism, Definition 2.2 in [10])**.** *A δ-homomorphism from an ε'-Banach algebra $\mathcal{A}'$ to an ε''-Banach algebra $\mathcal{A}''$ is a bounded linear map $v : \mathcal{A}' \to \mathcal{A}''$ that almost preserves the unit and the multiplication:*

$$\begin{aligned} \forall x, y \in \mathcal{A}', \ &\|v(\mathbb{1}) - \mathbb{1}\| \leq \delta, \\ &\|v(xy) - v(x)v(y)\| \leq \delta\|x\|\|y\|. \end{aligned} \tag{52}$$

A non-unital δ-homomorphism is defined by imposing only the second condition. In the $*$-algebra setting, it is also required that $v(x^*) = v(x)^*$.

A δ-inclusion is a δ-homomorphism such that

$$\forall x \in \mathcal{A}', \ (1 - \delta)\|x\| \leq \|v(x)\| \leq (1 + \delta)\|x\|.$$

The Error Reduction Theorem then states that if there is a δ-inclusion of an exact C^*-algebra $\mathcal{A}$ into an ε-C^*-algebra $\mathcal{B}$ for $\delta \leq \delta_{\max}$ and $\varepsilon \leq \varepsilon_{\max}$, which are constants that do not depend on the dimensions of $\mathcal{A}$ and $\mathcal{B}$, one can 'correct' this δ-inclusion into a less noisy approximate inclusion.

Theorem 12 (Error Reduction Theorem, Corollary 8.3 in [10])**.** *There exists some positive constants ε_{max}, δ_{max} and c_0 such that for all $\varepsilon < \varepsilon_{max}$, if a finite-dimensional C^*-algebra $\mathcal{A}$ is δ_{max}-included into an ε-C^*-algebra $\mathcal{B}$, then there is also a $(c_0 \, \varepsilon)$-inclusion. If the original inclusion is bijective, then so is the new inclusion.*

Note that as every C^*-algebra is a 0-C^*-algebra, Theorem 12 states that when we have a $\delta_{\max}$-inclusion between two C^*-algebras, it is possible to lift it to an *exact* inclusion. This can be interpreted as a case of *complete correction* in the sense that the reduction of the error parameter δ is total.

A.2 Approximate Multiplicative Domains

We used in the proof of the converse of Theorem 1 the fact that, for a map $\mathcal{F}^* : \mathcal{L}(\mathcal{H}') \to \mathcal{L}(\mathcal{H})$, which satisfies the Kadison-Schwarz inequality, its multiplicative domain can be expressed as

$$\{x \in \mathcal{L}(\mathcal{H}') \mid \mathcal{F}^*(x^*x) = \mathcal{F}^*(x^*)\mathcal{F}^*(x) \text{ and } \mathcal{F}^*(xx^*) = \mathcal{F}^*(x)\mathcal{F}^*(x^*)\}.$$

In order to prove Theorem 11, we have to extend this result to the approximate case. For $\mathcal{F}^* : \mathcal{L}(\mathcal{H}') \to \mathcal{L}(\mathcal{H})$ a linear map satisfying the Kadison-Schwarz inequality, we define, for $\delta \geq 0$ the following sets:

$$\begin{aligned} \mathcal{C}_\delta^R &= \{x \in \mathcal{L}(\mathcal{H}')|\|\mathcal{F}^*(x^*x) - \mathcal{F}^*(x^*)\mathcal{F}^*(x)\| \leq \delta\|x\|^2\}, \\ \mathcal{C}_\delta^L &= \{x \in \mathcal{L}(\mathcal{H}')|\|\mathcal{F}^*(xx^*) - \mathcal{F}^*(x)\mathcal{F}^*(x^*)\| \leq \delta\|x\|^2\}, \\ \mathcal{C}_\delta &= \mathcal{C}_\delta^R \cap \mathcal{C}_\delta^L. \end{aligned}$$

We can show the following theorem.

Theorem 13. *Let $\mathcal{F}^* : \mathcal{L}(\mathcal{H}') \to \mathcal{L}(\mathcal{H})$ be a linear map satisfying the Kadison-Schwarz inequality. Then, for all $\delta > 0$, $\mathcal{F}^*$ is 2δ-multiplicative on $\mathcal{C}_\delta$, i.e.*

$$\forall x, y \in \mathcal{C}_\delta, \|\mathcal{F}^*(xy) - \mathcal{F}^*(x)\mathcal{F}^*(y)\| \leq 2\delta\|x\|\|y\|. \tag{53}$$

More precisely, we have that

$$\forall x \in \mathcal{C}_\delta^L, \forall y \in \mathcal{C}_\delta^R, \|\mathcal{F}^*(xy) - \mathcal{F}^*(x)\mathcal{F}^*(y)\| \leq 2\delta\|x\|\|y\|. \tag{54}$$

The proof of this theorem uses the following lemma.

Lemma 14. *Let $\{x_k \in \mathcal{L}(\mathcal{H}) \mid k \in [K]\}$ and $\{\widetilde{x}_k \in \mathcal{L}(\mathcal{H}) \mid k \in [K]\}$ be two sets of operators such that for all k even,*

$$x_k \leq \widetilde{x}_k,$$

and for k odd,

$$x_k = \widetilde{x}_k.$$

Then, for all $t \in \mathbb{R}$,

$$\sum_{k=1}^{K} t^k x_k \leq \sum_{k=1}^{K} t^k \widetilde{x}_k. \tag{55}$$

Proof of Lemma 14. Let $t \in \mathbb{R}$:

$$\sum_{k=1}^{K} t^k \widetilde{x}_k - \sum_{k=1}^{K} t^k x_k = \sum_{k \in [K], \ k \text{ even}} t^k (\widetilde{x}_k - x_k) \geq 0,$$

where the right-hand side inequality follows from the assumption that $\widetilde{x}_k \geq x_k$ for all k even and also that, for all $t \in \mathbb{R}$, $t^k \geq 0$ for k even. $\qquad\square$

Proof of Theorem 13. This proof generalises the ones of Theorems 5.4 and 5.7 in [19]. In this proof, we let $\mathbb{1}$ denote the identity operator of $\mathcal{L}(\mathcal{H}')$.

We have that $x \in \mathcal{C}_\delta^L$ if and only if $x^* \in \mathcal{C}_\delta^R$ and vice versa. Therefore, it suffices to show that for all $x, y \in \mathcal{C}_\delta^R$,

$$\|\mathcal{F}^*(x^*y) - \mathcal{F}^*(x^*)\mathcal{F}^*(y)\| \leq 2\delta \|x\| \|y\|. \tag{56}$$

Let $x, y \in \mathcal{C}_\delta^R$, $t \in \mathbb{R}$ and $z = tx + y$. As $\mathcal{F}^*$ satisfies the Kadison-Schwarz inequality,

$$0 \leq \mathcal{F}^*(z^*z) - \mathcal{F}^*(z^*)\mathcal{F}^*(z).$$

If we develop the right-hand side, we get:

$$0 \leq t^2 Q + tV + W,$$

with

$$\begin{aligned} Q &= \mathcal{F}^*(x^*x) - \mathcal{F}^*(x^*)\mathcal{F}^*(x), \\ W &= \mathcal{F}^*(y^*y) - \mathcal{F}^*(y^*)\mathcal{F}^*(y), \\ v &= \mathcal{F}^*(x^*y) - \mathcal{F}^*(x^*)\mathcal{F}^*(y), \\ V &= v + v^*. \end{aligned}$$

Then, as $\mathcal{F}^*$ satisfies the Kadison-Schwarz inequality, $0 \leq Q$ and $0 \leq W$. Furthermore, $Q \leq \|Q\|\mathbb{1} \leq \delta \|x\|^2 \mathbb{1}$ and $W \leq \|W\|\mathbb{1} \leq \delta \|y\|^2 \mathbb{1}$ by assumption. Therefore, by Lemma 14, for all $t \in \mathbb{R}$:

$$0 \leq t^2 Q + tV + W \leq t^2 \delta \|x\|^2 \mathbb{1} + tV + \delta \|y\|^2 \mathbb{1}.$$

As $V = v + v^*$ is Hermitian, we have $V = UDU^*$ with U a unitary matrix and D a diagonal matrix with real coefficients. Conjugating the last inequality by U, we get

$$0 \leq t^2 \delta \|x\|^2 \mathbb{1} + tD + \delta \|y\|^2 \mathbb{1}.$$

The right-hand side of this equation is simply a diagonal operator, whose diagonal coefficients are quadratic polynomials in t. As this equation gives us that this operator is positive semidefinite for all $t \in \mathbb{R}$, each of these quadratic polynomials is always non-negative, and can thus have at most a single root. Therefore, if we write $\{D_i\}_i$ the diagonal entries of D, the discriminant of the polynomial

$$\delta \|x\|^2 t^2 + D_i t + \delta \|y\|^2,$$

has to be non-positive for all indices i, that is

$$D_i^2 \leq 4\delta^2 \|x\|^2 \|y\|^2.$$

So that:
$$|D_i| \leq 2\delta \|x\| \|y\|.$$

As the Schatten ∞-norm of $V = UDU^*$ is $\|V\| = \max_i |D_i|$, we have:
$$\|v + v^*\| = \|V\| \leq 2\delta \|x\| \|y\|.$$

We can apply the same line of argument replacing y with iy to find
$$\|iv - iv^*\| \leq 2\delta \|x\| \|y\|. \tag{57}$$

We can decompose $v = h_1 + ih_2$ with h_1 and h_2 Hermitian operators[8], to find:
$$\|v + v^*\| = 2\|h_1\| \leq 2\delta \|x\| \|y\|,$$
$$\|iv - iv^*\| = 2\|h_2\| \leq 2\delta \|x\| \|y\|.$$

Therefore,
$$\|v\| \leq \|h_1\| + \|h_2\| \leq 2\delta \|x\| \|y\|, \tag{58}$$

which is what we claimed. $\qquad\square$

A.3 Proof of Theorem 11

Finally, we prove in this subsection Theorem 11 in the same way as we proved the converse of Theorem 1, that is by first proving the one-shot case and then using the compatibility of the shape vector and the ℓ_p-norms with the tensor product to conclude.

Proposition 8. *Let $\mathcal{F} : \mathcal{L}(\mathcal{H}_\mathcal{F}) \to \mathcal{L}(\mathcal{H}_\mathcal{F})$ and $\mathcal{G} : \mathcal{L}(\mathcal{H}_\mathcal{G}) \to \mathcal{L}(\mathcal{H}_\mathcal{G})$ be two idempotent channels equal to their reduced channels, i.e. $\mathcal{F} = \widehat{\mathcal{F}}$ and $\mathcal{G} = \widehat{\mathcal{G}}$, such that there exists two channels $\mathcal{E}$ and $\mathcal{D}$ satisfying*
$$\|\mathcal{F} - \mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond \leq \frac{\delta_{\max}\lambda_{\min,1}}{6d}, \tag{59}$$

with $d = \dim(\mathcal{H}_\mathcal{F})$, $\delta_{\max}$ is the constant of Theorem 12 and $\lambda_{\min,1}$ is the minimal non-zero eigenvalue of $\mathcal{G}\mathcal{E}(\mathbb{1}_\mathcal{F})$. We write $e_{\mathcal{G}\mathcal{E}} = \mathcal{G}\mathcal{E}(\mathbb{1}_\mathcal{F})^0$ the orthogonal projector on the support $\mathrm{supp}(\mathcal{G}\mathcal{E}(\mathbb{1}_\mathcal{F}))$ and we define
$$\widetilde{\mathcal{G}}^* : x \mapsto e_{\mathcal{G}\mathcal{E}}\mathcal{G}^*(x)e_{\mathcal{G}\mathcal{E}}.$$

Then, $\widetilde{\mathcal{G}}^*\mathcal{D}^*$ is a $\delta_{\max}$-inclusion from the $*$-algebra $\mathrm{Rg}(\mathcal{F}^*)$ into the $*$-algebra $e_{\mathcal{G}\mathcal{E}} \mathrm{Rg}(\mathcal{G}^*)e_{\mathcal{G}\mathcal{E}}$. By the Error Reduction Theorem 12, there is therefore an injective unital $*$-homomorphism embedding $\mathrm{Rg}(\mathcal{F}^*)$ into $e_{\mathcal{G}\mathcal{E}} \mathrm{Rg}(\mathcal{G}^*)e_{\mathcal{G}\mathcal{E}}$, thus, for all $p \in [1, +\infty]$,
$$\|\lambda(\mathcal{F})\|_p \leq \|\lambda(\mathcal{G})\|_p.$$

We divide the proof that the map $\widetilde{\mathcal{G}}^*\mathcal{D}^*$ restricted to $\mathrm{Rg}(\mathcal{F}^*)$ satisfies all the properties of a $\delta_{\max}$-inclusion into $e_{\mathcal{G}\mathcal{E}} \mathrm{Rg}(\mathcal{G}^*)e_{\mathcal{G}\mathcal{E}}$ in multiple lemmas, each of which being devoted to prove one of these properties.

We begin with the one related to the almost preservation of the Schatten ∞-norm. We use the *completely bounded operator norm* in its proof, which is defined for all linear maps $\mathcal{F} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$ as
$$\|\mathcal{F}\|_{\mathrm{cb}} = \sup_{n \in \mathbb{N}^*} \|\mathcal{F} \otimes \mathrm{Id}_n\|_{\infty \to \infty},$$

with $\|\cdot\|_{\infty \to \infty}$ the 'infinity-to-infinity' norm, i.e. the map norm of linear maps from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H}')$ when the input and output spaces are endowed with the Schatten ∞-norms. That is:
$$\|\mathcal{F}\|_{\infty \to \infty} = \sup_{\|x\| \leq 1} \|\mathcal{F}(x)\|.$$

Note that for any linear map $\mathcal{F} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$, we have
$$\|\mathcal{F}\|_\diamond = \|\mathcal{F}^*\|_{\mathrm{cb}}.$$

---

[8] Taking $h_1 = \frac{1}{2}(v + v^*)$, $h_2 = \frac{1}{2i}(v - v^*)$.

Lemma 15 (Almost norm preservation). *Let $\mathcal{F}$, $\mathcal{G}$, $\widetilde{\mathcal{G}}$, $\mathcal{E}$ and $\mathcal{D}$ be defined as in the statement of Proposition 8 and let*

$$\delta \geq \|\mathcal{F} - \mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond = \|\mathcal{F}^* - \mathcal{E}^*\mathcal{G}^*\mathcal{D}^*\|_{\mathrm{cb}}.$$

Then, for all $x \in \mathrm{Rg}(\mathcal{F}^*)$,

$$(1 - \delta)\|x\| \leq \|\widetilde{\mathcal{G}}^*\mathcal{D}^*(x)\| \leq \|x\|.$$

Note that this lemma implies that as long as $\|\mathcal{F} - \mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond \leq \delta < 1$, $\widetilde{\mathcal{G}}^*\mathcal{D}^*$ is injective as every element of its kernel $x \in \ker(\widetilde{\mathcal{G}}^*\mathcal{D}^*)$ satisfies $(1 - \delta)\|x\| \leq \|\widetilde{\mathcal{G}}^*\mathcal{D}^*(x)\| = 0$, hence $\|x\| = 0$ so that $x = 0$.

Proof. The right-hand side inequality follows from the fact that $\widetilde{\mathcal{G}}^*\mathcal{D}^* : \mathcal{L}(\mathcal{H}_\mathcal{F}) \to \mathcal{L}(\mathcal{H}_\mathcal{G})$ is a subunital completely positive map (when the identity considered in the output space of $\widetilde{\mathcal{G}}^*\mathcal{D}^*$ is the identity of $\mathcal{L}(\mathcal{H}_\mathcal{G})$). For the left-hand side inequality, recall that, by Lemma 8, for all $x \in \mathcal{L}(\mathcal{H}_\mathcal{F})$,

$$\mathcal{E}^*\mathcal{G}^*\mathcal{D}^*(x) = \mathcal{E}^*\mathcal{G}^*\widetilde{\mathcal{G}}^*\mathcal{D}^*(x).$$

Then, we have for all $x \in \mathrm{Rg}(\mathcal{F}^*)$:

$$
\begin{aligned}
\|x\| - \|\widetilde{\mathcal{G}}^*\mathcal{D}^*(x)\| &\overset{(a)}{\leq} \|x\| - \|\mathcal{E}^*\mathcal{G}^*\widetilde{\mathcal{G}}^*\mathcal{D}^*(x)\| \\
&\overset{(b)}{\leq} \|x - \mathcal{E}^*\mathcal{G}^*\mathcal{D}^*(x)\| \\
&\overset{(c)}{=} \|\mathcal{F}^*(x) - \mathcal{E}^*\mathcal{G}^*\mathcal{D}^*(x)\| \\
&\leq \|\mathcal{F}^* - \mathcal{E}^*\mathcal{G}^*\mathcal{D}^*\|_{\mathrm{cb}}\|x\| \\
&\leq \delta\|x\|,
\end{aligned}
$$

where (a) follows from the fact that $\|\mathcal{E}^*\mathcal{G}^*\widetilde{\mathcal{G}}^*\mathcal{D}^*(x)\| \leq \|\widetilde{\mathcal{G}}^*\mathcal{D}^*(x)\|$, since $\mathcal{E}^*\mathcal{G}^*$ is a unital completely positive map. Then, (b) follows from the triangular inequality. Finally, (c) follows from the fact that $\mathcal{F}^*(x) = x$ for all $x \in \mathrm{Rg}(\mathcal{F}^*)$. The last inequality can be rewritten as

$$(1 - \delta)\|x\| \leq \|\widetilde{\mathcal{G}}^*\mathcal{D}^*(x)\|.$$

$\square$

We then move on to the proof that $\widetilde{\mathcal{G}}^*\mathcal{D}^* : \mathrm{Rg}(\mathcal{F}^*) \to e_{\mathcal{G}\mathcal{E}}\mathrm{Rg}(\mathcal{G}^*)e_{\mathcal{G}\mathcal{E}}$ is unital, where we recall that the unit of the $*$-algebra $e_{\mathcal{G}\mathcal{E}}\mathrm{Rg}(\mathcal{G}^*)e_{\mathcal{G}\mathcal{E}}$ is $e_{\mathcal{G}\mathcal{E}}$, as, for every element $x = e_{\mathcal{G}\mathcal{E}}x'e_{\mathcal{G}\mathcal{E}} \in e_{\mathcal{G}\mathcal{E}}\mathrm{Rg}(\mathcal{G}^*)e_{\mathcal{G}\mathcal{E}}$, $e_{\mathcal{G}\mathcal{E}}x = xe_{\mathcal{G}\mathcal{E}} = x$. Also note that for a general δ-inclusion, one only requires almost unitality.

Lemma 16 (Exact unitality). *Let $\mathcal{F}$, $\mathcal{G}$, $\widetilde{\mathcal{G}}$, $\mathcal{E}$ and $\mathcal{D}$ be defined as in Proposition 8. Then:*

$$\widetilde{\mathcal{G}}^*\mathcal{D}^*(\mathbb{1}_\mathcal{F}) = e_{\mathcal{G}\mathcal{E}}.$$

Proof. As $\mathcal{G}$ and $\mathcal{D}$ are assumed to be channels, their adjoints are unital maps on $\mathcal{L}(\mathcal{H}_\mathcal{G})$, so that

$$\mathcal{G}^*\mathcal{D}^*(\mathbb{1}_\mathcal{F}) = \mathbb{1}_\mathcal{G}.$$

Therefore

$$\widetilde{\mathcal{G}}^*\mathcal{D}^*(\mathbb{1}_\mathcal{F}) = e_{\mathcal{G}\mathcal{E}}\mathcal{G}^*(\mathcal{D}^*(\mathbb{1}_\mathcal{F}))e_{\mathcal{G}\mathcal{E}} = e_{\mathcal{G}\mathcal{E}}\mathbb{1}_\mathcal{G}e_{\mathcal{G}\mathcal{E}} = e_{\mathcal{G}\mathcal{E}}.$$

$\square$

Finally, to prove the almost preservation of the multiplication on $\mathrm{Rg}(\mathcal{F}^*)$ by $\widetilde{\mathcal{G}}^*\mathcal{D}^*$, we will need the following additional lemma.

Lemma 17. *Let $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}')$ be a channel, write $d = \dim(\mathcal{H})$, $e = \Phi(\mathbb{1})^0$ and $\lambda_{\min,1}$ the minimal non-zero eigenvalue of $\Phi(\mathbb{1})$. Then, for all positive semidefinite $x \in \mathcal{L}(\mathcal{H}')$,*

$$\frac{\lambda_{\min,1}}{d}\|exe\| \leq \|\Phi^*(x)\|. \tag{60}$$

Proof. Let $x \in \mathcal{L}(\mathcal{H}')$ be positive semidefinite, then

$$
\begin{aligned}
\|\Phi^*(x)\| &\geq \frac{1}{d} \operatorname{tr}(\Phi^*(x)) \\
&= \frac{1}{d} \operatorname{tr}(x\Phi(\mathbb{1})) \\
&\geq \frac{\lambda_{\min,1}}{d} \operatorname{tr}(exe) \\
&\geq \frac{\lambda_{\min,1}}{d} \|exe\|.
\end{aligned}
$$

$\square$

Note that this lemma implies the second statement of Lemma 8. We can now move on to the proof of the almost preservation of the multiplication, which uses our Theorem 13 on the approximate multiplicative domain of maps satisfying the Kadison-Schwarz inequality.

Lemma 18 (Almost preservation of the multiplication). *Let $\mathcal{F}$, $\mathcal{G}$, $\widetilde{\mathcal{G}}$, $\mathcal{E}$ and $\mathcal{D}$ be defined as in the statement of Proposition 8, let*

$$
\delta \geq \|\mathcal{F} - \mathcal{D}\mathcal{G}\mathcal{E}\|_\diamond,
$$

let $\lambda_{\min,1}$ be the minimal non-zero eigenvalue of $\widetilde{\mathcal{G}}^*\mathcal{E}^*(\mathbb{1}_\mathcal{F})$, $d = \dim(\mathcal{H}_\mathcal{F})$ and let $x, y \in \operatorname{Rg}(\mathcal{F}^*)$. We then have:

$$
\|\widetilde{\mathcal{G}}^*\mathcal{D}^*(xy) - \widetilde{\mathcal{G}}^*\mathcal{D}^*(x)\widetilde{\mathcal{G}}^*\mathcal{D}^*(y)\| \leq \frac{6\delta d}{\lambda_{\min,1}}\|x\|\|y\|.
$$

Proof. We prove that, for all $x \in \operatorname{Rg}(\mathcal{F}^*)$,

$$
\|\widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*x) - \widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*)\widetilde{\mathcal{G}}^*\mathcal{D}^*(x)\| \leq \frac{3\delta d}{\lambda_{\min,1}}\|x\|^2,
$$

$$
\|\widetilde{\mathcal{G}}^*\mathcal{D}^*(xx^*) - \widetilde{\mathcal{G}}^*\mathcal{D}^*(x)\widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*)\| \leq \frac{3\delta d}{\lambda_{\min,1}}\|x\|^2,
$$

and then use Theorem 13 to conclude. First, to simplify our notations, we let $\Phi^* = \mathcal{E}^*\mathcal{G}^*\mathcal{D}^* = \mathcal{E}^*\mathcal{G}^*\widetilde{\mathcal{G}}^*\mathcal{D}^*$. We have:

$$
\begin{aligned}
\|\Phi^*(x^*x) - \Phi^*(x^*)\Phi^*(x)\| &= \|\Phi^*(x^*x) - \Phi^*(x^*)\Phi^*(x) + x^*x - x^*x\| \\
&\leq \|\Phi^*(x^*x) - x^*x\| + \|x^*x - \Phi^*(x^*)\Phi^*(x) + x^*\Phi^*(x) - x^*\Phi^*(x)\| \\
&\overset{(a)}{\leq} \delta\|x\|^2 + \|x\|\|x - \Phi^*(x)\| + \|x^* - \Phi^*(x^*)\|\|\Phi^*(x)\| \\
&\overset{(b)}{\leq} 3\delta\|x\|^2,
\end{aligned}
$$

where (a) follows from the fact that as $x \in \operatorname{Rg}(\mathcal{F}^*)$, we have

$$
\|\Phi(x^*x) - x^*x\| = \|\Phi^*(x^*x) - \mathcal{F}^*(x^*x)\| \leq \|\Phi^* - \mathcal{F}^*\|_{\mathrm{cb}}\|x^*x\| \leq \delta\|x\|^2.
$$

By the same argument, we also have that $\|x - \Phi^*(x)\| \leq \delta\|x\|$ and $\|x^* - \Phi^*(x^*)\| \leq \delta\|x\|$, hence (b) follows.

Then, applying the Kadison-Schwarz inequality, we get:

$$
\mathcal{E}^*\mathcal{G}^*\widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*x) \geq \mathcal{E}^*\mathcal{G}^*(\widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*)\widetilde{\mathcal{G}}^*\mathcal{D}^*(x)) \geq \mathcal{E}^*\mathcal{G}^*\widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*)\mathcal{E}^*\mathcal{G}^*\widetilde{\mathcal{G}}^*\mathcal{D}^*(x).
$$

Therefore:

$$
\|\mathcal{E}^*\mathcal{G}^*(\widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*x) - \widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*)\widetilde{\mathcal{G}}^*\mathcal{D}^*(x))\| \leq \|\Phi^*(x^*x) - \Phi^*(x^*)\Phi^*(x)\| \leq 3\delta\|x\|^2.
$$

Thus, by Lemma 17, we obtain

$$
\|\widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*x) - \widetilde{\mathcal{G}}^*\mathcal{D}^*(x^*)\widetilde{\mathcal{G}}^*\mathcal{D}^*(x)\| \leq \frac{3d\delta}{\lambda_{\min,1}}\|x\|^2.
$$

We can apply the exact same line of reasoning to prove the same inequality but with the roles of x and x^* swapped. $\square$

Gathering all these lemmas, the proof of Proposition 8 is then as follows.

Proof of Proposition 8. Let $\delta \geq \|\mathcal{F} - \mathcal{DGE}\|_\diamond$. Since we assumed that $\mathcal{F}$ and $\mathcal{G}$ are equal to their reduced channels, i.e. $\mathcal{F} = \widehat{\mathcal{F}}$ and $\mathcal{G} = \widehat{\mathcal{G}}$, $\mathrm{Rg}(\mathcal{F}^*)$ is a $*$-subalgebra of $\mathcal{L}(\mathcal{H}_\mathcal{F})$ and $e_{\mathcal{GE}} \mathrm{Rg}(\mathcal{G}^*) e_{\mathcal{GE}}$ is a $*$-subalgebra of $\mathcal{L}(\mathcal{H}_\mathcal{G})$. By Lemmas 15, 16 and 18, $\widetilde{\mathcal{G}}^* \mathcal{D}^*$ is a $\max\{\delta, \frac{6d\delta}{\lambda_{\min,1}}\}$-inclusion (we trivially have that $\widetilde{\mathcal{G}}^* \mathcal{D}^*$ is Hermitian preserving). However, as

$$d = \mathrm{tr}(\mathbb{1}_\mathcal{F}) = \mathrm{tr}(\mathcal{GE}(\mathbb{1}_\mathcal{F})) \geq \lambda_{\min,1},$$

$\frac{d}{\lambda_{\min,1}} \geq 1$ so that $\delta \leq \frac{6d}{\lambda_{\min,1}}\delta$. Finally, $\widetilde{\mathcal{G}}^* \mathcal{D}^*$ is a $\frac{6d\delta}{\lambda_{\min,1}}$-inclusion of $\mathrm{Rg}(\mathcal{F}^*)$ into $e_{\mathcal{GE}} \mathrm{Rg}(\mathcal{G}^*) e_{\mathcal{GE}}$.

Then, if we suppose that

$$\frac{6d\delta}{\lambda_{\min,1}} \leq \delta_{\max},$$

i.e.

$$\|\mathcal{F} - \mathcal{DGE}\|_\diamond \leq \delta \leq \frac{\delta_{\max} \lambda_{\min,1}}{6d},$$

by the Error Reduction Theorem (Theorem 12), there is an injective $*$-homomorphism from $\mathrm{Rg}(\mathcal{F}^*)$ into $e_{\mathcal{GE}} \mathrm{Rg}(\mathcal{G}^*) e_{\mathcal{GE}}$, thus by Kuperberg's theorem (Theorem 7) and the second part of Proposition 7, for all $p \in [1, +\infty]$,

$$\|\lambda(\mathcal{F})\|_p \leq \|\lambda(e_{\mathcal{GE}} \mathrm{Rg}(\mathcal{G}^*) e_{\mathcal{GE}})\|_p \leq \|\lambda(\mathcal{G})\|_p.$$

$\square$

We can finally show Theorem 11.

Proof of Theorem 11. Basically, we simply apply Proposition 8 to $\widehat{\mathcal{F}}^{\otimes k} : \mathcal{L}(\mathcal{H}_\mathcal{F}^{\otimes k}) \to \mathcal{L}(\mathcal{H}_\mathcal{F}^{\otimes k})$ and $\widehat{\mathcal{G}}^{\otimes n} : \mathcal{L}(\mathcal{H}_\mathcal{G}^{\otimes n}) \to \mathcal{L}(\mathcal{H}_\mathcal{G}^{\otimes n})$, with $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{G}}$ the reduced channels respectively associated to $\mathcal{F}$ and $\mathcal{G}$. Just note that the dimension of $\mathcal{H}_\mathcal{F}^{\otimes k}$ is d^k, with $d = \dim(\mathcal{H}_\mathcal{F})$. Thus, by Proposition 8, if the hypotheses of Theorem 11 are met, for all $p \in [1, +\infty]$, we have:

$$\|\lambda(\mathcal{F}^{\otimes k})\|_p = \|\lambda(\widehat{\mathcal{F}}^{\otimes k})\|_p \leq \|\lambda(\widehat{\mathcal{G}}^{\otimes n})\|_p = \|\lambda(\mathcal{G}^{\otimes n})\|_p.$$

This implies, by the compatibility of the ℓ_p-norms of the shape vector with the tensor product, that

$$\frac{k}{n} \leq \inf_{p \in [1, +\infty]} \frac{\log(\|\lambda(\mathcal{G})\|_p)}{\log(\|\lambda(\mathcal{F})\|_p)} = C(\mathcal{G} \mapsto \mathcal{F}),$$

where the right-hand side equality is the first statement in Theorem 1.

$\square$