CANTOR CORRELATIONS I. OPERATOR SYSTEMS AND CANTOR GAMES

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Abstract. We study no-signalling correlations over Cantor spaces, placing the product of infinitely many copies of a finite non-local game in a unified general setup. We define the subclasses of local, quantum spatial, approximately quantum and quantum commuting Cantor correlations and describe them in terms of states on tensor products of inductive limits of operator systems. We provide a correspondence between no-signalling (resp. approximately quantum, quantum commuting) Cantor correlations and sequences of correlations of the same type over the projections onto increasing number of finitely many coordinates. We introduce Cantor games, and associate canonically such a game to a sequence of finite input/output games, showing that the numerical sequence of the values of the games in the sequence converges to the corresponding value of the compound Cantor game.

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1. Introduction

Non-local games have been at the centre of the fruitful interactions between operator algebras and quantum information theory witnessed in the past decade (see e.g. [9, 22, 23, 26, 28, 29, 31]). These are games, played cooperatively by players Alice and Bob against a Verifier; in a single round of the game, the Verifier draws a pair (s,t) of inputs form the cartesian product $S \times T$ of two finite sets according to a certain probability distribution, and sends s (resp. t) to Alice (resp. Bob). The players respond with a pair (u,v) of outputs from the cartesian product $U \times V$ of two

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(perhaps different) finite sets; the tandem Alice-Bob wins the round if the quadruple (s, t, u, v) satisfies a given predicate, known to the players, and interpreted as the rules of the game. The players are not allowed to communicate during the course of the game, but they may agree beforehand on using a specific strategy.

Several types of strategies thus appear, depending on the physical model the players avail of (that is, the way of forming the joint physical system of their individual systems), leading to a type hierarchy of *no-signalling correlations* between them, expressed through a proper inclusion chain

$$\mathcal{C}_{\rm loc} \subseteq \mathcal{C}_{\rm qs} \subseteq \mathcal{C}_{\rm qa} \subseteq \mathcal{C}_{\rm qc} \subseteq \mathcal{C}_{\rm ns}, \tag{1}$$

where each of $\mathcal{C}_{\rm t}$ is the set of correlations between Alice and Bob observed during a repetition of game rounds. In particular, the class $\mathcal{C}_{\rm qc}$, arising by utilising the commuting operator model, strictly contains the the class $\mathcal{C}_{\rm qa}$ obtained by using liminal finite dimensional entanglement [17]. The inequality $\mathcal{C}_{\rm qa} \neq \mathcal{C}_{\rm qc}$ answers in the negative the Tsirelson problem in theoretical physics [36] and, simultaneously, thanks to [15, 18, 27], the Connes Embedding Problem in operator algebra theory [8]. At the heart of the equivalence between the aforementioned problems lie characterisations of the strategies from the classes $\mathcal{C}_{\rm qa}$ and $\mathcal{C}_{\rm qc}$ via states on, respectively, the minimal and maximal tensor products of two universal C*-algebras, each associated to one of the players.

The correspondence between strategies and states on the tensor product of these universal C*-algebras is not bijective; in fact, a given strategy may arise from multiple such states. This phenomenon lies at the core of quantum self-testing [11, 28], and leads to the necessity to use more economical, bijective, correspondences between strategies and states. This was achieved in [22, 32], where characterisations of strategies were obtained via states on different types of tensor products of universal *operator systems*, as opposed to C*-algebras, associated with the players of the game.

Parallel repetition [7, 34], that is, the formation of the product [24] of a sequence of copies of the game, played successively and independently, is at the base of defining and studying the asymptotic value of the game which, among others, is instrumental for demonstrating separation between classical and quantum models in device-independent cryptography. The product of countably many copies of a game can naturally be viewed as a single game, played over the Cantor spaces arising from the underlying finite sets of inputs/outputs. In this *Cantor setup*, the inputs/outputs are elements of those Cantor spaces; from an operational point of view, such *Cantor games* can be thought of as non-local games, in which the players receive a string of inputs of arbitrary (but equal) length, and are required to respond with strings of outputs of the same length. In fact, the Cantor setup is substantially more general, as the induced rules on the successively larger (but finite) input/output sets do not need to be products of a given common underlying rule, that is, the successive rounds captured within the compound Cantor game are not necessarily independent or identical.

The aim of the present paper is the study of strategies of Cantor games and the one-shot values (that is, the optimal winning probabilities in a single game round, according to the strategy type used) thereof. While a general definition and basic properties of no-signalling correlations over standard measure spaces was given in [6], the Cantor setup contains the crucial distinctive element of *inductivity*. More precisely, a no-signalling strategy Γ of a given Cantor game $\mathcal{G}$ corresponds in a unique way to a sequence $(\Gamma_n)_{n\in\mathbb{N}}$, where Γ_n is a no-signalling strategy of the restriction of $\mathcal{G}$ to its first n coordinates. We show that the passage from Γ to $(\Gamma_n)_{n\in\mathbb{N}}$ preserves the quantum commuting and approximately quantum correlation types, but not necessarily the quantum spatial correlation type. We define a universal operator system for each of the players in the Cantor setup, and show that the Cantor correlations of different types arise from states on different kinds of operator system tensor product [21] of these universal operator systems. As a consequence, the class of quantum commuting Cantor correlations is closed in the (natural to employ in our setting [5, 6]) Arveson BW topology [2]. Our main tools are drawn from operator algebra theory, including ultrapowers (see e.g. [1] and [33, Section 11.5]), operator system theory, including their tensor products [21], co-products [20] and inductive limits [25], and operator-valued information theory over abstract alphabets (see [5, 6, 14, 19]).

We apply the Cantor correlation setup to examine the behaviour of the values of Cantor games. We focus on the case where the question set is endowed with the uniform probability distribution, which, in the Cantor setting, corresponds to the product of uniform probability measures, and obtain a continuity result for the values of inductive sequences of games, inscribing the latter fact in a series of results about tensor norm expressions of game values (see [9, 10, 29]).

We have postponed some topics, naturally arising from our results, for future work, such as descriptions of synchronicity (which is examined in the upcoming article [4]), and consideration of inputs that are not independent and identically distributed.

In the sequel, we describe the content of the paper in more detail. Section 2 is preliminary and contains the necessary background in operator system theory and the operator-algebraic approach to no-signalling correlations in the finite case. After reviewing operator-valued information channels in Section 3, given Cantor spaces arising from sequences $X = (X_n)_{n\in\mathbb{N}}$ and $A = (A_n)_{n\in\mathbb{N}}$ of increasing sets of underlying finite coordinates, we introduce Alice's universal operator system $\mathcal{S}_{X,A}$ as a direct limit of an appropriately defined inductive sequence, where the embeddings take into account the uniform distributions over the question sets X_n, $n \in \mathbb{N}$. We show that the unital completely positive maps from $\mathcal{S}_{X,A}$ into the C*-algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H are in a canonical correspondence with the $\mathcal{B}(H)$-valued information channels indexed by elements of the Cantor space associated with X.

In Section 4, we define the main Cantor correlation types, namely those of local, quantum spatial, approximately quantum, quantum commuting and no-signalling ones, establish the inclusion chain (1) in the Cantor setup, and show that the Cantor no-signalling correlations correspond to states on the maximal tensor product $\mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B}$ (here $\mathcal{S}_{X,A}$ and $\mathcal{S}_{Y,B}$ are the Alice and Bob universal operator systems, respectively). En route, we provide a general result about diagonals of successive inductive limits in the operator system category. We establish the no-signalling type preservation for the operation $\Gamma \to (\Gamma_n)_{n\in\mathbb{N}}$.

The same property for the quantum commuting correlation type is provided in Section 5 and relies on an ultraproduct construction. It leads to a characterisation of quantum commuting Cantor correlations via states on the commuting tensor product $\mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B}$. An analogous characterisation is shown to hold for the approximately quantum correlations, this time via states on the minimal tensor product $\mathcal{S}_{X,A} \otimes_{\min} \mathcal{S}_{Y,B}$. While the classes of quantum commuting and, by definition, approximately quantum, Cantor correlations are closed in Arveson's BW topology, we show that this is not the case for the quantum spatial correlation type.

Section 6 is devoted to Cantor games and their values. We show that a finite game has the same value (of any type t among the local, quantum spatial, quantum commuting or no-signalling ones) as the canonical Cantor game arising from it after embedding the rules in the first coordinate of the corresponding Cantor spaces. By projecting on the first n coordinates, every Cantor game $\mathcal{G}$ gives rise to a decreasing sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of finite games. We show that the value $\omega_{\rm t}(\mathcal{G})$ of $\mathcal{G}$ is the limit of the (decreasing) sequence $(\omega_{\rm t}(\mathcal{G}_n))_{n\in\mathbb{N}}$ of finite game values. As a consequence, $\omega_{\rm t}(\mathcal{G})$ can be obtained as a limit of a (decreasing) sequence of norms of increasingly larger game tensors, canonically associated with $\mathcal{G}$, in the maximal, commuting and minimal tensor product of the corresponding universal operator systems. Finally, we discuss a class of examples to which our results apply.

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2. Finite no-signalling correlations

In this section, we provide the necessary background on operator systems, completely positive maps, no-signalling correlations over finite sets, and their operator systems characterisation, for later reference. We refer the reader to [30] for further background and details.

Given a vector *-space V over the complex field, let $M_n(V)$ be the vector *-space of all n by n matrices with entries in V, and $M_n(V)_h$ be the real vector space of all self-adjoint elements of $M_n(V)$. An *operator system* is a tuple $(V, (C_n)_{n\in\mathbb{N}}, e)$, where

V is a vector *-space, C_n is a proper cone in $M_n(V)_h$, the family $(C_n)_{n\in\mathbb{N}}$ is consistent in that $\alpha^* C_n \alpha \subseteq C_m$ for all scalar n by m matrices α, and the element $e \in C_1$ is an Archimedean matrix order unit for $(C_n)_{n\in\mathbb{N}}$. We usually write $M_n(V)^+ = C_n$. Given operator systems $(V, (C_n)_{n\in\mathbb{N}}, e)$ and $(V', (C'_n)_{n\in\mathbb{N}}, e')$ and a linear map ϕ, we let $\phi^{(n)} : M_n(V) \to M_n(V')$ be the (linear) map, given by $\phi^{(n)}((x_{i,j})_{i,j}) = (\phi(x_{i,j}))_{i,j}$. The map ϕ is called *positive* if $\phi(C_1) \subseteq C'_1$, and *completely positive* if $\phi^{(n)}$ is positive for every $n \in \mathbb{N}$; it is called a complete order embedding if it is injective and $\phi^{(n)}(M_n(V)) \cap M_n(V')^+ = \phi^{(n)}(M_n(V)^+)$, and a complete order isomorphism if it is a surjective complete order embedding.

If H is a Hilbert space, we denote by $\mathcal{B}(H)$ the space of all bounded linear operators acting on H and by I_H the identity operator on H. All Hilbert spaces we use will be assumed separable. After letting $\mathcal{B}(H)^+$ denote the cone of all positive operators in $\mathcal{B}(H)$ and making the identification $M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$, we have that every unital selfadjoint subspace $\mathcal{S} \subseteq \mathcal{B}(H)$ is an operator system with cones $M_n(\mathcal{S})^+ := M_n(\mathcal{S}) \cap M_n(\mathcal{B}(H))^+$; we call operator systems of the latter type *concrete*. By virtue of the Choi-Effros Theorem (see e.g. [30, Theorem 13.1]), every operator system is completely order isomorphic to a concrete operator system. We note that every operator system is an operator space in a canonical way, and write $\mathrm{CB}(\mathcal{S}, \mathcal{T})$ (resp. $\mathrm{UCP}(\mathcal{S}, \mathcal{T})$) for the operator space (resp. convex set) of completely bounded (resp. unital completely positive) maps from $\mathcal{S}$ into $\mathcal{T}$.

We recall the definitions of the operator system tensor products that will be used subsequently, and refer to [21] for further details. Given operator systems $\mathcal{S} \subseteq \mathcal{B}(H)$ and $\mathcal{T} \subseteq \mathcal{B}(K)$ (where H and K are Hilbert spaces), we write $\mathcal{S} \otimes \mathcal{T}$ for their algebraic tensor product; all three tensor products that we define have the latter as their underlying vector *-space. The *minimal tensor product* $\mathcal{S} \otimes_{\min} \mathcal{T}$ is equipped with the matricial cones that make the inclusion $\mathcal{S} \otimes \mathcal{T} \subseteq \mathcal{B}(H \otimes K)$ a complete order embedding (we denote by $H \otimes K$ the Hilbertian tensor product). The *commuting tensor product* $\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T}$ has matricial cones defined by letting $w \in M_n(\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T})^+$ if $(\phi \cdot \psi)^{(n)}(w) \in \mathcal{B}(L)^+$ whenever $\phi : \mathcal{S} \to \mathcal{B}(L)$ and $\psi : \mathcal{T} \to \mathcal{B}(L)$ are completely positive maps with commuting ranges, L is a Hilbert space and $\phi \cdot \psi : \mathcal{S} \otimes \mathcal{T} \to \mathcal{B}(L)$ is the linear map, given by $(\phi \cdot \psi)(u \otimes v) = \phi(u)\psi(v)$ (we say that ϕ and ψ form a *commuting pair*). Finally, the *maximal tensor product* $\mathcal{S} \otimes_{\max} \mathcal{T}$ has matricial cone structure, generated by the elementary tensors of the form $S \otimes T$, where $S \in M_n(\mathcal{S})^+$ and $T \in M_m(\mathcal{T})^+$.

For $d \in \mathbb{N}$, let $[d] = \{1, \dots, d\}$. Given operator systems $\mathcal{S}_i$, $i \in [d]$, their *coproduct* [16, 20] is a pair of the form $\left(\mathcal{S}_1 \oplus_1 \cdots \oplus_1 \mathcal{S}_d, (\iota_i)_{i=1}^d\right)$, where $\mathcal{S}_1 \oplus_1 \cdots \oplus_1 \mathcal{S}_d$ is an operator system, and $\iota_i : \mathcal{S}_i \to \mathcal{S}_1 \oplus_1 \cdots \oplus_1 \mathcal{S}_d$ is a unital complete order embedding, such that if $\mathcal{R}$ is an operator system and $\phi_i : \mathcal{S}_i \to \mathcal{R}$ is a unital completely positive map, $i \in [d]$, then there exists a unique unital completely positive map $\phi : \mathcal{S}_1 \oplus_1 \cdots \oplus_1 \mathcal{S}_d \to \mathcal{R}$, such that $\phi \circ \iota_i = \phi_i$, $i \in [d]$. We will write $\dot{\oplus}_{i=1}^d \mathcal{S}_i = \mathcal{S}_1 \oplus_1 \cdots \oplus_1 \mathcal{S}_d$. Given, in addition, operator systems $\mathcal{T}_i$, $i \in [d]$, and unital completely positive maps $\psi_i : \mathcal{S}_i \to \mathcal{T}_i$, $i \in [d]$, there exists a unique unital completely positive map $\dot{\oplus}_{i=1}^d \psi_i : \dot{\oplus}_{i=1}^d \mathcal{S}_i \to \dot{\oplus}_{i=1}^d \mathcal{T}_i$, such that $\left(\dot{\oplus}_{i=1}^d \psi_i\right)(\iota_i(u)) = (\iota_i \circ \psi_i)(u)$, $u \in \mathcal{S}_i$, $i \in [d]$. Indeed, the map $\iota_i \circ \psi_i : \mathcal{S}_i \to \dot{\oplus}_{j=1}^d \mathcal{T}_j$ is unital and completely positive, $i \in [d]$, and the existence of $\dot{\oplus}_{i=1}^d \psi_i$ follows from the universal property of the coproduct of the family $\{\mathcal{S}_i\}_{i=1}^d$.

We will require some preliminaries on inductive limits of operator systems, which we now include; we refer the reader to [25] for further details. Let

$$\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \cdots \tag{2}$$

be an inductive system in the operator system category; this means that $\mathcal{S}_k$ is an operator system and ϕ_k is a unital completely positive map for every $k \in \mathbb{N}$. The inductive limit of (2) is a pair $(\mathcal{S}, (\phi_{k,\infty})_{k\in\mathbb{N}})$, where $\mathcal{S}$ is an operator system and $\phi_{k,\infty} : \mathcal{S}_k \to \mathcal{S}$ is a unital completely positive map, $k \in \mathbb{N}$, with the property that if $\mathcal{R}$ is an operator system and $\rho_k : \mathcal{S}_k \to \mathcal{R}$, $k \in \mathbb{N}$, are unital completely positive maps, such that $\rho_{k+1} \circ \phi_k = \rho_k$, $k \in \mathbb{N}$, then there exists a unique unital completely positive map $\rho : \mathcal{S} \to \mathcal{R}$ such that $\rho \circ \phi_{k,\infty} = \rho_k$, $k \in \mathbb{N}$. Such an operator system $\mathcal{S}$ is unique up to a complete order isomorphism; we write $\mathcal{S} = \varinjlim \mathcal{S}_k$.

Given a finite set A, we let M_A be the algebra of all $|A| \times |A|$ matrices with complex entries, and $\mathcal{D}_A$ be its subalgebra of all diagonal matrices. Given another finite set X, we let $\mathcal{A}_{X,A} = \underbrace{\mathcal{D}_A * \cdots * \mathcal{D}_A}_{|X| \text{ times}}$ be the free product, amalgamated over the units, and

$$\mathcal{S}_{X,A} = \underbrace{\mathcal{D}_A \oplus_1 \cdots \oplus_1 \mathcal{D}_A}_{|X| \text{ times}}. \tag{3}$$

We note the unital completely order isomorphic inclusion $\mathcal{S}_{X,A} \subseteq \mathcal{A}_{X,A}$ [22, 32]. We write $e_{x,a}$, $x \in X$, $a \in A$, for the canonical generators of $\mathcal{S}_{X,A}$, that is, $e_{x,a} = \iota_x(\delta_a)$, where $\iota_x : \mathcal{D}_A \to \mathcal{S}_{X,A}$ is the inclusion map of the x-th copy of $\mathcal{D}_A$, and $(\delta_a)_{a\in A}$ is the canonical basis of $\mathcal{D}_A$.

Given finite sets X, Y, A and B, a *no-signalling correlation* over the quadruple (X, Y, A, B) is a family $\{(p(a, b|x, y))_{a,b} : x \in X, y \in Y\}$, where $p(\cdot, \cdot|x, y)$ is a probability distribution over $A \times B$ for every $(x, y) \in X \times Y$,

$$\sum_{b\in B} p(a, b|x, y) = \sum_{b\in B} p(a, b|x, y'), \quad x \in X, y, y' \in Y, a \in A,$$

and

$$\sum_{a\in A} p(a, b|x, y) = \sum_{a\in A} p(a, b|x', y), \quad x, x' \in X, y \in Y, b \in B.$$

Recall that a *positive operator-valued measure (POVM)* is a (finite) family $(E_i)_{i=1}^d$ of positive operators acting on a Hilbert space, such that $\sum_{i=1}^d E_i = I$. A no-signalling correlation p over (X, Y, A, B) is called *quantum commuting* if it has the form

$$p\,(a, b|x, y) = \langle P_{x,a} Q_{y,b} \xi, \xi\rangle, \tag{4}$$

where ξ is a unit vector in a Hilbert space H, and $(P_{x,a})_{a\in A}$ and $(Q_{y,b})_{b\in B}$, $x \in X$, $y \in Y$, are POVM's on H such that $P_{x,a}Q_{y,b} = Q_{y,b}P_{x,a}$ for all $x \in X$, $y \in Y$, $a \in A$ and $b \in B$. The correlation p is called *quantum spatial* if the Hilbert space in the representation (4) can be chosen of the form $H = H_A \otimes H_B$ for some Hilbert spaces H_A and H_B, and $P_{x,a}$ (resp. $Q_{y,b}$) has the form $P_{x,a} = P'_{x,a} \otimes I_{H_B}$ (resp. $Q_{y,b} = I_{H_A} \otimes Q'_{y,b}$). We further say that p is an *approximately quantum* correlation if p is the limit (in the vector space $\mathbb{R}^X \times \mathbb{R}^Y \times \mathbb{R}^A \times \mathbb{R}^B$) of quantum spatial correlations. We denote by $\mathcal{C}_{\rm ns}(X,Y,A,B)$ (resp. $\mathcal{C}_{\rm qc}(X,Y,A,B)$, $\mathcal{C}_{\rm qa}(X,Y,A,B)$, $\mathcal{C}_{\rm qs}(X,Y,A,B)$) the set of all no-signalling (resp. quantum commuting, approximately quantum, quantum spatial) correlations over the quadruple (X,Y,A,B). We refer the reader to [22] for further details regarding no-signalling correlations, and record here characterisations of correlation types in terms of operator system tensor products that will be needed in the sequel ([22, Corollary 3.2] and [22, Corollary 3.3]).

Theorem 2.1. *Let X, Y, A and B be finite sets. The no-signalling (resp. quantum commuting, approximately quantum) correlations $p = \{(p(a,b|x,y))_{a,b} : x \in X, y \in Y\}$ are in bijective correspondence to states $s : \mathcal{S}_{X,A} \otimes_{\rm max} \mathcal{S}_{Y,B} \to \mathbb{C}$ (resp. $s : \mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B} \to \mathbb{C}$, $s : \mathcal{S}_{X,A} \otimes_{\rm min} \mathcal{S}_{Y,B} \to \mathbb{C}$) via the assignment*

$$p(a,b|x,y) = s(e_{x,a} \otimes e_{y,b}), \quad x \in X, y \in Y, a \in A, b \in B.$$

3. The Cantor operator system

In this section, we review the definitions of operator-valued information channels [6], specialising to the context of Cantor spaces, introduce a class of operator systems that will be used in later sections, and describe their universal property.

3.1. **Operator-valued channels.** Let $\mathfrak{S}$ be a second countable compact Hausdorff space. We let $\mathfrak{B}_{\mathfrak{S}}$ be the Borel σ-algebra of $\mathfrak{S}$, $C(\mathfrak{S})$ be the C*-algebra of all continuous complex-valued functions on $\mathfrak{S}$, and $M(\mathfrak{S})$ be the space of all Radon measures on $\mathfrak{S}$. For a (separable) Hilbert space H, a *quantum probability measure (QPM)* over $\mathfrak{S}$ with values in $\mathcal{B}(H)$ is a map $E : \mathfrak{B}_{\mathfrak{S}} \to \mathcal{B}(H)^+$ such that $E(\emptyset) = 0$, $E(\mathfrak{S}) = I$, and $E\left(\cup_{i=1}^{\infty} \alpha_i\right) = \sum_{i=1}^{\infty} E(\alpha_i)$ in the strong operator topology whenever $(\alpha_i)_{i\in\mathbb{N}}$ is a sequence of mutually disjoint elements of $\mathfrak{B}_{\mathfrak{S}}$.

Let $\mathfrak{X}$ be a(nother) second countable compact Hausdorff space, equipped with a probability measure $\mu \in M(\mathfrak{X})$. An *operator-valued information channel* from $\mathfrak{X}$ to $\mathfrak{S}$ with values in $\mathcal{B}(H)$ is a family $E = E(\cdot|x)_{x\in\mathfrak{X}}$ of QPM's over $\mathfrak{S}$ such that, for every $\alpha \in \mathfrak{B}_{\mathfrak{S}}$, the function $x \mapsto E(\alpha|x)$ is weakly μ-measurable, that is, the functions $E_{\xi,\eta}(\alpha|\cdot) := \langle E(\alpha|\cdot)\xi, \eta\rangle$ are measurable for all $\xi, \eta \in H$ [6]. We write $\mathfrak{C}(\mathfrak{S}, \mathfrak{X}; H)$ for the set of all operator-valued information channels from $\mathfrak{X}$ to $\mathfrak{S}$ with values in $\mathcal{B}(H)$, and view its elements as measurable versions of families of POVM's in that the latter are operator-valued information channels over a pair of finite sets. We say that the channels $E, E' \in \mathfrak{C}(\mathfrak{S}, \mathfrak{X}; H)$ are μ-equivalent (and write $E \sim_\mu E'$) if

$$E(\alpha|x) = E'(\alpha|x) \ \ \mu\text{-almost everywhere},$$

for every $\alpha \in \mathfrak{B}_{\mathfrak{S}}$. We write $\mathfrak{C}_\mu(\mathfrak{S},\mathfrak{X};H)$ for the set of all $\sim_\mu$-equivalent classes of $\mathfrak{C}(\mathfrak{S},\mathfrak{X};H)$. Elements of $\mathfrak{C}_\mu(\mathfrak{S},\mathfrak{X};H)$ will be called *operator-valued μ-information channels* from $\mathfrak{X}$ to $\mathfrak{S}$ with values in $\mathcal{B}(H)$ (see [5] and note that a slightly different terminology was used therein); without risk of confusion, we will use the same symbol for an equivalence class and for a representative thereof. Let $L^\infty_\sigma(\mathfrak{X},\mu,\mathcal{B}(H))$ be the von Neumman algebra of all equivalent classes of weak* measurable essentially bounded functions $F : \mathfrak{X} \to \mathcal{B}(H)$, and note the canonical identification $L^\infty(\mathfrak{X},\mu)\bar{\otimes}\mathcal{B}(H) = L^\infty_\sigma(\mathfrak{X},\mu,\mathcal{B}(H))$. Here, and below, we use $\bar{\otimes}$ to denote the von Neumann algebra tensor product. For future reference, we recall the correspondence between μ-information channels with values in $\mathcal{B}(H)$ and unital completely positive maps from $C(\mathfrak{S})$ into $L^\infty_\sigma(\mathfrak{X},\mu,\mathcal{B}(H))$ established in [5, Theorem 3.11].

Theorem 3.1. *If $E \in \mathfrak{C}_\mu(\mathfrak{S},\mathfrak{X};H)$ then there exists a unital completely positive map $\Phi_E : C(\mathfrak{S}) \to L^\infty_\sigma(\mathfrak{X},\mu,\mathcal{B}(H))$ such that*

$$\langle \Phi_E(f)(x)\xi,\eta\rangle = \int_{\mathfrak{S}} f(a)dE_{\xi,\eta}(a|x) \quad \mu\text{-a.e.}, \quad f \in C(\mathfrak{S}), \xi,\eta \in H. \tag{5}$$

Conversely, if $\Phi : C(\mathfrak{S}) \to L^\infty_\sigma(\mathfrak{X},\mu,\mathcal{B}(H))$ is a unital completely positive map then there exists a (unique up to $\sim_\mu$-equivalence) channel $E \in \mathfrak{C}_\mu(\mathfrak{S},\mathfrak{X};H)$ such that $\Phi = \Phi_E$.

Recall that, if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, the *BW topology* [2] on the bounded subsets of the space $\mathcal{B}(\mathcal{X},\mathcal{Y}^*)$ of all bounded liner maps from $\mathcal{X}$ into $\mathcal{Y}^*$ is defined as the restriction of the point-weak* topology. The set $\mathfrak{C}_\mu(\mathfrak{S},\mathfrak{X};H)$ will be hereafter equipped with the topology (which we continue to refer to as the BW topology) according to which a net $(E^\lambda)_{\lambda\in\Lambda} \subseteq \mathfrak{C}_\mu(\mathfrak{S},\mathfrak{X};H)$ converges to an element $E \in \mathfrak{C}_\mu(\mathfrak{S},\mathfrak{X};H)$ if Φ_{E^λ} converges to Φ_E in the BW topology (see [5]). We note that, by [5, Theorem 3.14], the space $(\mathfrak{C}_\mu(\mathfrak{S},\mathfrak{X};H), \mathrm{BW})$ is compact. Since the operator projective tensor product $C(\mathfrak{S})\hat{\otimes}L^1(\mathfrak{X},\mu)\hat{\otimes}\mathcal{T}(H)$ is separable, the space $(\mathfrak{C}_\mu(\mathfrak{S},\mathfrak{X};H), \mathrm{BW})$ is metrisable (see e.g. [12, Theorem V.5.1]).

3.2. **Inductive channel families.** If X_1 and X_2 are finite sets, we write $X_1|X_2$ if there exists $d \in \mathbb{N}$ such that $X_2 = X_1 \times [d]$. Assuming that $X_2 = X_1 \times [d]$, let $\iota_{X_1,X_2} : \mathcal{D}_{X_1} \to \mathcal{D}_{X_2}$ be the unital *-monomorphism, given by $\iota_{X_1,X_2}(T) = T \otimes I_d$, after the canonical identification $\mathcal{D}_{X_2} = \mathcal{D}_{X_1} \otimes \mathcal{D}_{[d]}$.

A family $X = (X_n)_{n\in\mathbb{N}}$ of finite sets will be called *inductive* if $X_n|X_{n+1}$ for every $n \in \mathbb{N}$. The inductive limit of the sequence

$$\mathcal{D}_{X_1} \overset{\iota_{X_1,X_2}}{\longrightarrow} \mathcal{D}_{X_2} \overset{\iota_{X_2,X_3}}{\longrightarrow} \cdots \overset{\iota_{X_{n-1},X_n}}{\longrightarrow} \mathcal{D}_{X_n} \overset{\iota_{X_n,X_{n+1}}}{\longrightarrow} \cdots$$

in the category of C*-algebras will be denoted by $\mathcal{D}_X$. Assuming that $X_{n+1} = X_n \times [d_n^X]$, where $d_n^X \in \mathbb{N}$, $n \in \mathbb{N}$, we note that $\mathcal{D}_X$ is *-isomorphic to the C*-algebra $C(\Omega_X)$ of all continuous functions on the Cantor space $\Omega_X = \prod_{n=0}^\infty \left[d_n^X\right]$ (where we have set $d_0^X = |X_1|$); equivalently, $\mathcal{D}_X = \otimes_{n=0}^\infty \mathcal{D}_{[d_n^X]}$ as an infinite C*-algebraic tensor product.

All tracial algebras will be equipped with normalised traces, and dualities will always be with respect to the latter. For a finite set X_0, the (normalised) trace on $\mathcal{D}_{X_0}$ will be denoted by tr_{X_0}. We note that, if $X_1|X_2$ then the embedding ι_{X_1,X_2} is trace-preserving. For an inductive family $X = (X_n)_{n\in\mathbb{N}}$, we set $\tau_X = \otimes_{n=0}^{\infty}\mathrm{tr}_{[d_n^X]}$. We note that $\tau_X|_{\mathcal{D}_{X_n}} = \mathrm{tr}_{X_n}$, $n \in \mathbb{N}$. In the sequel, we write $L^1(\Omega_X)$ for $L^1(\Omega_X, \mu_X)$ and μ_{X_n} for the uniform probability measure on X_n, $n \in \mathbb{N}$. We note that $L^1(\Omega_X)$ coincides with the L^1-space $L^1(\mathcal{D}_X)$ of the C*-algebra $\mathcal{D}_X$ with respect to (the trace) τ_X.

The unital (completely) positive trace-preserving maps $\iota_{X_n,X_{n+1}} : \mathcal{D}_{X_n} \to \mathcal{D}_{X_{n+1}}$ give rise to the canonical conditional expectations $\mathcal{E}_{X_{n+1},X_n} : \mathcal{D}_{X_{n+1}} \to \mathcal{D}_{X_n}$; we note that $\mathcal{E}_{X_{n+1},X_n} = \mathrm{id}_{X_n} \otimes \mathrm{tr}_{d_n}$. As $\mathcal{D}_X \subseteq L^\infty(\Omega_X) \subseteq L^1(\Omega_X)$, the canonical map $\iota_X^{(n)} : \mathcal{D}_{X_n} \to \mathcal{D}_X$ induces a weak* continuous, unital *-monomorphism $\iota_{X,\infty}^{(n)} : \mathcal{D}_{X_n} \to L^\infty(\Omega_X)$, as well as a unital isometry $\iota_{X,1}^{(n)} : \mathcal{D}_{X_n} \to L^1(\Omega_X)$ with respect to the trace norm. The map $\iota_{X,\infty}^{(n)}$ admits a predual map $\mathcal{E}_{X_n} : L^1(\Omega_X) \to \mathcal{D}_{X_n}$, which is faithful, trace-preserving, and satisfies the identity $\mathcal{E}_{X_n} \circ \iota_{X,1}^{(n)} = \mathrm{id}_{\mathcal{D}_{X_n}}$. Thus, when identifying $\mathcal{D}_{X_n}$ as a subspace of $L^1(\Omega_X)$ via $\iota_{X,1}^{(n)}$, the map $\mathcal{E}_{X_n}$ is the canonical conditional expectation. At the von Neumann algebra level, there exists a faithful, trace-preserving, weak* continuous conditional expectation $\mathcal{E}_{X_n}^\infty : L^\infty(\Omega_X) \to \mathcal{D}_{X_n}$, which is the dual of the isometry $\iota_{X,1}^{(n)}$. If H is a Hilbert space, we further write $\tilde{\mathcal{E}}_{X_n}^\infty := \mathcal{E}_{X_n}^\infty \otimes \mathrm{id}_{\mathcal{B}(H)}$ for the conditional expectation

$$\tilde{\mathcal{E}}_{X_n}^\infty : L^\infty(\Omega_X)\bar{\otimes}\mathcal{B}(H) \to \mathcal{D}_{X_n} \otimes \mathcal{B}(H),$$

$\tilde{\mathcal{E}}_{X_{n+1},X_n}^\infty := \mathcal{E}_{X_{n+1},X_n}^\infty \otimes \mathrm{id}_{\mathcal{B}(H)}$ for the conditional expectation

$$\tilde{\mathcal{E}}_{X_{n+1},X_n}^\infty : \mathcal{D}_{X_{n+1}} \otimes \mathcal{B}(H) \to \mathcal{D}_{X_n} \otimes \mathcal{B}(H), \quad n \in \mathbb{N},$$

and $\tilde{\iota}_{X,\infty}^{(n)} : \mathcal{D}_{X_n} \otimes \mathcal{B}(H) \to L^\infty(\Omega_X)\bar{\otimes}\mathcal{B}(H)$ and $\tilde{\iota}_{X_n,X_{n+1}} : \mathcal{D}_{X_n} \otimes \mathcal{B}(H) \to \mathcal{D}_{X_{n+1}} \otimes \mathcal{B}(H)$ for the maps, given by $\tilde{\iota}_{X,\infty}^{(n)} = \iota_{X,\infty}^{(n)} \otimes \mathrm{id}_{\mathcal{B}(H)}$, and $\tilde{\iota}_{X_n,X_{n+1}} = \iota_{X_n,X_{n+1}} \otimes \mathrm{id}_{\mathcal{B}(H)}$, $n \in \mathbb{N}$. We note that the superscript/subscript ∞ is used to indicate that the domain or the range of the corresponding map is an L^∞-space.

We say that a family $(\Phi_n)_{n\in\mathbb{N}}$, where $\Phi_n : \mathcal{D}_{A_n} \to \mathcal{D}_{X_n} \otimes \mathcal{B}(H)$ is a unital completely positive map, $n \in \mathbb{N}$, is *inductive* if

$$\Phi_n = \tilde{\mathcal{E}}_{X_{n+1},X_n} \circ \Phi_{n+1} \circ \iota_{A_n,A_{n+1}}, \quad n \in \mathbb{N}, \tag{6}$$

that is, if the diagram

$$\begin{array}{ccc} \mathcal{D}_{A_n} & \xrightarrow{\iota_{A_n,A_{n+1}}} & \mathcal{D}_{A_{n+1}} \\ \Big\downarrow{\scriptstyle \Phi_n} & & \Big\downarrow{\scriptstyle \Phi_{n+1}} \\ \mathcal{D}_{X_n} \otimes \mathcal{B}(H) & \xleftarrow[\tilde{\mathcal{E}}_{X_{n+1},X_n}]{} & \mathcal{D}_{X_{n+1}} \otimes \mathcal{B}(H) \end{array}$$

is commutative for each $n \in \mathbb{N}$.

Theorem 3.2. *Let H be a Hilbert space, and $X = (X_n)_{n\in\mathbb{N}}$ and $A = (A_n)_{n\in\mathbb{N}}$ be inductive families of sets.*

(i) *If $\Phi : C(\Omega_A) \to L^\infty(\Omega_X)\bar{\otimes}\mathcal{B}(H)$ is a unital completely positive map and*

$$\Phi_n := \tilde{\mathcal{E}}^\infty_{X_n} \circ \Phi \circ \iota^{(n)}_A, \quad n \in \mathbb{N}, \tag{7}$$

then the family $(\Phi_n)_{n\in\mathbb{N}}$ is inductive.

(ii) *If $\Phi_n : \mathcal{D}_{A_n} \to \mathcal{D}_{X_n} \otimes \mathcal{B}(H)$ is a unital completely positive map, $n \in \mathbb{N}$, such that the family $(\Phi_n)_{n\in\mathbb{N}}$ is inductive then there exists a unique unital completely positive map $\Phi : C(\Omega_A) \to L^\infty(\Omega_X)\bar{\otimes}\mathcal{B}(H)$ satisfying (7).*

Proof. (i) Since

$$\tilde{\mathcal{E}}^\infty_{X_n} = \tilde{\mathcal{E}}_{X_{n+1},X_n} \circ \tilde{\mathcal{E}}^\infty_{X_{n+1}} \quad \text{and} \quad \iota^{(n)}_A = \iota^{(n+1)}_A \circ \iota_{A_n,A_{n+1}},$$

condition (6) is implied by (7).

(ii) For each $n \in \mathbb{N}$, let $\Psi_n : C(\Omega_A) \to L^\infty(\Omega_X)\bar{\otimes}\mathcal{B}(H)$ be defined by setting $\Psi_n := \tilde{\iota}^{(n)}_{X,\infty} \circ \Phi_n \circ \mathcal{E}^\infty_{A_n}|_{C(\Omega_A)}$; note that the maps Ψ_n are unital and completely positive. The sequence $(\Psi_n)_{n\in\mathbb{N}}$ has a BW cluster point $\Phi : C(\Omega_A) \to L^\infty(\Omega_X)\bar{\otimes}\mathcal{B}(H)$, whose existence follows from the compactness of $\mathrm{UCP}(C(\Omega_A), L^\infty(\Omega_X)\bar{\otimes}\mathcal{B}(H))$ in the BW topology ([5, Theorem 3.14]). Next we show that (7) is satisfied for the map Φ. Consider $k > n$ and note that

$$\iota^{(n)}_A = \iota^{(k)}_A \circ \iota_{A_n,A_k}, \quad \text{and} \quad \tilde{\mathcal{E}}^\infty_{X_n} = \tilde{\mathcal{E}}_{X_k,X_n} \circ \tilde{\mathcal{E}}^\infty_{X_k}$$

so that

$$\tilde{\mathcal{E}}^\infty_{X_n} \circ \tilde{\iota}^{(k)}_{X,\infty} = \tilde{\mathcal{E}}_{X_k,X_n} \quad \text{and} \quad \mathcal{E}^\infty_{A_k} \circ \iota^{(n)}_{A,\infty} = \iota_{A_n,A_k}$$

(here we have set $\tilde{\mathcal{E}}_{X_k,X_n} = \mathcal{E}_{X_k,X_n} \otimes \mathrm{id}_{\mathcal{B}(H)}$). Therefore, for $k > n$, we have

$$\tilde{\mathcal{E}}^\infty_{X_n} \circ \Psi_k \circ \iota^{(n)}_{A,\infty} = \tilde{\mathcal{E}}^\infty_{X_n} \circ \tilde{\iota}^{(k)}_{X,\infty} \circ \Phi_k \circ \mathcal{E}^\infty_{A_k} \circ \iota^{(n)}_{A,\infty} = \tilde{\mathcal{E}}_{X_k,X_n} \circ \Phi_k \circ \iota_{A_n,A_k} = \Phi_n,$$

where the last equation follows from the inductivity relations (6). Letting $k \to \infty$, we obtain (7).

We claim that the unital completely positive map Φ satisfying (7) is unique. Indeed, if Φ' is another such map satisfying (7), then, as $\tilde{\mathcal{E}}^\infty_{X_m} \circ \Phi(S) = \tilde{\mathcal{E}}^\infty_{X_m} \circ \Phi'(S)$, for every $S \in \iota^{(n)}_A(\mathcal{D}_{A_n})$, $m > n$, we get, using density arguments, that $\Phi(S) = \Phi'(S)$, giving the uniqueness. $\square$

3.3. **Definition and universal property.** We next identify a canonical operator system, associated with an inductive family of sets, which will serve as universal encoding object for each of the players of a non-local game over Cantor spaces. For $d \in \mathbb{N}$, recall the operator system $\mathcal{S}_{[d],A_n}$ defined, in (3), as a coproduct of d copies of $\mathcal{D}_{A_n}$. Denoting by ι_k the embedding of $\mathcal{D}_{A_n}$ in the k-th term of $\mathcal{S}_{[d],A_n}$, let $\beta_{d,A_n} : \mathcal{D}_{A_n} \to \mathcal{S}_{[d],A_n}$ be the unital completely positive map, given by

$$\beta_{d,A_n}(u) = \frac{1}{d}\left(\iota_1(u) + \cdots + \iota_d(u)\right), \quad u \in \mathcal{D}_{A_n}.$$

We have that the maps

$$\dot{\oplus}_{i=1}^{|X_n|}\iota_{A_n,A_{n+1}} : \mathcal{S}_{X_n,A_n} \to \mathcal{S}_{X_n,A_{n+1}}$$

and

$$\dot{\oplus}_{i=1}^{|X_n|}\beta_{d_n^X,A_{n+1}} : \mathcal{S}_{X_n,A_{n+1}} \to \mathcal{S}_{X_{n+1},A_{n+1}}$$

are unital and completely positive; thus, the composition

$$\gamma_{X_n,A_n} = \left(\dot{\oplus}_{i=1}^{|X_n|}\beta_{d_n^X,A_{n+1}}\right) \circ \left(\dot{\oplus}_{i=1}^{|X_n|}\iota_{A_n,A_{n+1}}\right)$$

is a unital completely positive map from $\mathcal{S}_{X_n,A_n}$ to $\mathcal{S}_{X_{n+1},A_{n+1}}$. We thus obtain an inductive sequence

$$\mathcal{S}_{X_1,A_1} \overset{\gamma_{X_1,A_1}}{\longrightarrow} \mathcal{S}_{X_2,A_2} \overset{\gamma_{X_2,A_2}}{\longrightarrow} \mathcal{S}_{X_3,A_3} \overset{\gamma_{X_3,A_3}}{\longrightarrow} \cdots \tag{8}$$

in the operator system category. We let $\mathcal{S}_{X,A} = \varinjlim \mathcal{S}_{X_n,A_n}$ be the corresponding inductive limit. We let $\gamma_{X,A}^{(n)} : \mathcal{S}_{X_n,A_n} \to \mathcal{S}_{X,A}$ be the canonical unital completely positive map, arising from the inductive sequence (8), $n \in \mathbb{N}$.

Remark 3.3. Let $(X_n)_{n\in\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}}$ be inductive families of sets. Using known results about inductive limits and coproducts in the operator system category, namely [25, Proposition 4.13] and [20, Section 8], one can show that the maps γ_{X_n,A_n} and $\gamma_{X,A}^{(n)}$ are unital complete order isomorphisms; thus, $\mathcal{S}_{X_n,A_n}$ can be canonically identified with an operator subsystem of $\mathcal{S}_{X,A}$, $n \in \mathbb{N}$. Since this fact will not be needed in the sequel, we do not include its proof.

Theorem 3.4. *Let H be a Hilbert space, and $(X_n)_{n\in\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}}$ be inductive families of sets. The unital completely positive maps $\Gamma : \mathcal{S}_{X,A} \to \mathcal{B}(H)$ are in a canonical bijective correspondence with the unital completely positive maps $\Phi : C(\Omega_A) \to L^\infty(\Omega_X)\bar{\otimes}\mathcal{B}(H)$.*

Proof. For $k \in \mathbb{N}$ and $\omega \in \mathcal{D}_{X_k}$, let $L_\omega : \mathcal{D}_{X_k} \otimes \mathcal{B}(H) \to \mathcal{B}(H)$ be the slice map, given by

$$L_\omega(S \otimes T) = \langle \omega, S\rangle T, \quad S \in \mathcal{D}_{X_k}, T \in \mathcal{B}(H)$$

(we recall that the duality is with respect to normalised traces).

By the universal property of the inductive limit, the unital completely positive maps $\Gamma : \mathcal{S}_{X,A} \to \mathcal{B}(H)$ are in a canonical correspondence with the sequences $(\Gamma_n)_{n\in\mathbb{N}}$ of unital completely positive maps, where $\Gamma_n : \mathcal{S}_{X_n,A_n} \to \mathcal{B}(H)$, $n \in \mathbb{N}$, satisfy the conditions

$$\Gamma_n = \Gamma_{n+1} \circ \gamma_{X_n,A_n}, \quad n \in \mathbb{N}. \tag{9}$$

On the other hand, the unital completely positive maps $\Gamma_n : \mathcal{S}_{X_n,A_n} \to \mathcal{B}(H)$ are in a canonical correspondence with unital completely positive maps $\Phi_n : \mathcal{D}_{A_n} \to \mathcal{D}_{X_n} \otimes \mathcal{B}(H)$ via the assignment $\Phi_n(\delta_a) = \sum_{x\in X_n} \delta_x \otimes \Gamma_n(e_{x,a})$. We note that the latter equality is equivalent to the identities

$$|X_n| L_{\delta_x}(\Phi_n(\delta_a)) = \Gamma_n(e_{x,a}), \quad x \in X_n, a \in A_n. \tag{10}$$

Finally, observe that condition (9) is equivalent to condition (6) being satisfied for the family $(\Phi_n)_{n\in\mathbb{N}}$, as follows from the fact that, if $x \in X_n$ and $a \in A_n$, then

$$\begin{aligned}
\Gamma_{n+1}(\gamma_{X_n,A_n}(e_{x,a})) &= \frac{1}{d_n^X}\sum_{\mu_n=1}^{d_n^X}\sum_{\lambda_n=1}^{d_n^A}\Gamma_{n+1}(e_{(x,\mu_n),(a,\lambda_n)}) \\
&= |X_n|\sum_{\mu_n=1}^{d_n^X}\sum_{\lambda_n=1}^{d_n^A} L_{\delta_{(x,\mu_n)}}(\Phi_{n+1}(\delta_{(a,\lambda_n)})) \\
&= |X_n|\sum_{\mu_n=1}^{d_n^X} L_{\delta_{(x,\mu_n)}}((\Phi_{n+1}\circ\iota_{A_n,A_{n+1}})(\delta_a)) \\
&= |X_n| L_{\iota_{X_n,X_{n+1}}(\delta_x)}((\Phi_{n+1}\circ\iota_{A_n,A_{n+1}})(\delta_a)) \\
&= |X_n| L_{\delta_x}((\tilde{\mathcal{E}}_{X_{n+1},X_n}\circ\Phi_{n+1}\circ\iota_{A_n,A_{n+1}})(\delta_a)).
\end{aligned}$$

The statement now follows from Theorem 3.2. □

Remark 3.5. We point out for further use that the statement of Theorem 3.4 is true and, up to our knowledge, part of folklore in the case where X and A are finite sets instead of inductive families; a proof readily follows from that of Theorem 3.4 with the straightforward modifications.

Remark 3.6. The statement of Theorem 3.4 remains true when the map Γ is a (not necessarily unital) completely positive map; this follows by inspection of the proof, together with the fact that the operator system inductive limit satisfies a universal property for inductive families of completely positive maps that are not necessarily unital. To show the latter fact, suppose that (2) is an inductive sequence in the operator system category, $\mathcal{R}$ is an operator system, and $\rho_k : \mathcal{S}_k \to \mathcal{R}$ are completely positive maps, such that $\rho_{k+1}\circ\phi_k = \rho_k$, $k \in \mathbb{N}$. Since the map ϕ_k in (2) is unital, we have that there exists $w \in \mathcal{R}$ such that $\rho_k(1_{\mathcal{S}_k}) = w$ for every $k \in \mathbb{N}$; clearly, $0 \le w \le 1_{\mathcal{R}}$. Let $(s_k)_{k\in\mathbb{N}}$ be an inductive sequence of states, where $s_k : \mathcal{S}_k \to \mathbb{C}$, and $s : \varinjlim \mathcal{S}_k \to \mathbb{C}$ be the associated state on the inductive limit operator system. Let $\alpha_k : \mathcal{S}_k \to \mathcal{R}$ be the map, given by $\alpha_k(u) = s_k(u)(1-w)$, and $\tilde{\rho}_k = \rho_k + \alpha_k$; thus, $\tilde{\rho}_k : \mathcal{S}_k \to \mathcal{R}$ is a unital completely positive map, $k \in \mathbb{N}$. Moreover,

$$(\tilde{\rho}_{k+1}\circ\phi_k)(u) = \rho_{k+1}(\phi_k(u)) + s(u)(1-w) = \tilde{\rho}_k(u), \quad u \in \mathcal{S}_k.$$

By the universal property of the inductive limit (for unital completely positive maps), there exists a unital completely positive map $\tilde{\rho} : \varinjlim \mathcal{S}_k \to \mathcal{R}$ such that $\tilde{\rho}\circ\phi_{k,\infty} = \tilde{\rho}_k$ for every $k \in \mathbb{N}$.

Let $\alpha : \varinjlim \mathcal{S}_k \to \mathcal{R}$ be the map, given by $\alpha(u) = s(u)(1-w)$. We have that

$$\tilde{\rho}^{(n)}(\phi_{k,\infty}^{(n)}(u)) - \alpha^{(n)}(\phi_{k,\infty}^{(n)}(u)) \in M_n(\mathcal{R})^+, \quad u \in M_n(\mathcal{S}_k)^+.$$

By density, $\rho := \tilde{\rho} - \alpha$ is completely positive. Finally, if $k \in \mathbb{N}$ then

$$(\rho \circ \phi_{k,\infty})(u) = \tilde{\rho}(\phi_{k,\infty}(u)) - \alpha(\phi_{k,\infty}(u)) = \tilde{\rho}_k(u) - \alpha_k(u) = \rho_k(u).$$

Remark 3.7. Let $(X_n)_{n\in\mathbb{N}}$ and $(A_n)_{n\in\mathbb{N}}$ be inductive families of sets. Suppose that $x \in X_n$ and $a \in A_n$ for some $n \in \mathbb{N}$. Equation (10) can be rewritten as

$$|X_n|\langle \delta_x \otimes \omega, \Phi_n(\delta_a)\rangle = \langle \Gamma_n(e_{x,a}), \omega\rangle, \quad \omega \in \mathcal{B}(H)_*,$$

that is,

$$\left\langle \frac{1}{\tau_X(\iota^{(n)}_{X,1}(\delta_x))}\iota^{(n)}_{X,1}(\delta_x) \otimes \omega, \Phi(\iota^{(n)}_{A,\infty}(\delta_a)) \right\rangle = \langle \Gamma(\gamma^{(n)}_{X,A}(e_{x,a})), \omega\rangle \tag{11}$$

for every $\omega \in \mathcal{B}(H)_*$. By identity (11) and uniform boundedness, the map $\Phi \to \Gamma$ is BW continuous. Since the linear span of the elements of the form $\frac{1}{\tau_X(\iota^{(n)}_{X,1}(\delta_x))}\iota^{(n)}_{X,1}(\delta_x)\otimes \omega$ is dense in $L^1(\Omega_X)\hat{\otimes}\mathcal{B}(H)_*$, we have, in fact, that the correspondence $\Phi \leftrightarrow \Gamma$ is a BW-BW homeomorphism.

We note that, as the predual of $L^\infty(\Omega_X)$, the space $L^1(\Omega_X)$ admits a canonical operator space structure, and that, if $C(\Omega_A)\hat{\otimes}L^1(\Omega_X)$ denotes the operator space projective tensor product, up to a canonical complete isometry we have that

$$\left(C(\Omega_A)\hat{\otimes}L^1(\Omega_X)\right)^* = \mathrm{CB}(C(\Omega_A), L^\infty(\Omega_X))$$

(see [13, Proposition 7.1.2]). By the previous paragraph, there exists a canonical weak*-homeomorphic order isomorphism between the positive cones of $\mathcal{S}^*_{X,A}$ and $\mathrm{UCP}(C(\Omega_A), L^\infty(\Omega_X))$. Passing to preduals, we obtain a order isomorphism between $\mathcal{S}^+_{X,A}$ and a dense subspace of the predual cone $\left(C(\Omega_A)\hat{\otimes}L^1(\Omega_X)\right)^+$ of the cone $\mathrm{CP}(C(\Omega_A), L^\infty(\Omega_X))$. A straightforward argument shows that the latter correspondence can be extended to the whole of $\mathcal{S}_{X,A}$. Through the latter identification, for $x \in X_n$ and $a \in A_n$, the element $\gamma^{(n)}_{X,A}(e_{x,a})$ corresponds to the elementary tensor $\chi_{\tilde{x}\times\tilde{a}} := \chi_{\tilde{x}} \otimes \chi_{\tilde{a}}$, where

$$\tilde{x} = \left\{xx' : x' \in \prod_{i=n+1}^{\infty}[d_i^X]\right\} \text{ and } \tilde{a} = \left\{aa' : a' \in \prod_{i=n+1}^{\infty}[d_i^A]\right\}. \tag{12}$$

4. Cantor no-signalling correlations

In this section we define no-signalling correlations over Cantor spaces and provide characterisations thereof in terms of states on the maximal operator system tensor product of operator systems from the class introduced in Section 3. In view of the nuclearity of abelian C*-algebras, in the sequel we will use the symbol $\otimes$ for their C*-algebraic tensor product.

Assume that S, T, U and V are finite sets. A no-signalling correlation $p = \{(p(u,v|s,t))_{u,v} : s \in S, t \in T\}$ (see Section 2) gives rise to the unital (completely) positive map $\Gamma_p : \mathcal{D}_U \otimes \mathcal{D}_V \to \mathcal{D}_S \otimes \mathcal{D}_T$, given by

$$\Gamma_p(\delta_u \otimes \delta_v) = \sum_{s\in S}\sum_{t\in T} p(u,v|s,t)\delta_s \otimes \delta_t, \quad u \in U, v \in V;$$

it is straightforward to verify that, moreover,

$$\Gamma_p(\mathcal{D}_U \otimes 1_{\mathcal{D}_V}) \subseteq \mathcal{D}_S \otimes 1_{\mathcal{D}_T} \text{ and } \Gamma_p(1_{\mathcal{D}_U} \otimes \mathcal{D}_V) \subseteq 1_{\mathcal{D}_S} \otimes \mathcal{D}_T. \tag{13}$$

Conversely, every unital (completely) positive map $\Gamma : \mathcal{D}_U \otimes \mathcal{D}_V \to \mathcal{D}_S \otimes \mathcal{D}_T$ satisfying the conditions (13) is easily seen to have the form $\Gamma = \Gamma_p$ for some no-signalling correlation p. Therefore, by abuse of terminology, we use the term "no-signalling correlation" in reference to unital (completely) positive maps satisfying (13). Fixing inductive families of finite sets $X = (X_n)_{n\in\mathbb{N}}$, $Y = (Y_n)_{n\in\mathbb{N}}$, $A = (A_n)_{n\in\mathbb{N}}$ and $B = (B_n)_{n\in\mathbb{N}}$, these observations justify the following definition.

Definition 4.1. A unital completely positive map

$$\Gamma : C(\Omega_A) \otimes C(\Omega_B) \to L^\infty(\Omega_X)\bar{\otimes}L^\infty(\Omega_Y)$$

will be called a *no-signalling correlation* over the quadruple (X, Y, A, B) if

$$\Gamma\left(C(\Omega_A) \otimes 1_B\right) \subseteq L^\infty(\Omega_X) \otimes 1_Y$$

and

$$\Gamma\left(1_A \otimes C(\Omega_B)\right) \subseteq 1_X \otimes L^\infty(\Omega_Y).$$

We denote by $\mathcal{C}_{\rm ns}(X, Y, A, B)$ the set of all no-signalling correlations over the quadruple (X, Y, A, B), and write $\mathcal{C}_{\rm ns}$ in case no confusion may arise.

Given a unital completely positive map

$$\Gamma : C(\Omega_A) \otimes C(\Omega_B) \to L^\infty(\Omega_X)\bar{\otimes}L^\infty(\Omega_Y),$$

define (unital completely positive) maps

$$\Gamma_n : \mathcal{D}_{A_n} \otimes \mathcal{D}_{B_n} \to \mathcal{D}_{X_n} \otimes \mathcal{D}_{Y_n}, \quad n \in \mathbb{N},$$

by setting

$$\Gamma_n := (\mathcal{E}^\infty_{X_n} \otimes \mathcal{E}^\infty_{Y_n}) \circ \Gamma \circ (\iota^{(n)}_A \otimes \iota^{(n)}_B); \tag{14}$$

by Theorem 3.2, the family $(\Gamma_n)_{n\in\mathbb{N}}$ is inductive, that is,

$$\Gamma_n = (\mathcal{E}_{X_{n+1},X_n} \otimes \mathcal{E}_{Y_{n+1},Y_n}) \circ \Gamma_{n+1} \circ (\iota_{A_n,A_{n+1}} \otimes \iota_{B_n,B_{n+1}}), \quad n \in \mathbb{N}; \tag{15}$$

we say that the family $(\Gamma_n)_{n\in\mathbb{N}}$ is *associated with* the map Γ.

Proposition 4.2. *Let $\Gamma : C(\Omega_A) \otimes C(\Omega_B) \to L^\infty(\Omega_X)\bar{\otimes}L^\infty(\Omega_Y)$ be a unital completely positive map and $(\Gamma_n)_{n\in\mathbb{N}}$ be the inductive family associated with Γ. The following are equivalent:*

(i) $\Gamma \in \mathcal{C}_{\rm ns}(X, Y, A, B)$;
(ii) $\Gamma_n \in \mathcal{C}_{\rm ns}(X_n, Y_n, A_n, B_n)$ *for every* $n \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii) Let $n \in \mathbb{N}$ and $f \in \mathcal{D}_{A_n}$. By symmetry, it suffices to show that $\Gamma_n(f \otimes 1_{B_n}) \in \mathcal{D}_{X_n} \otimes 1_{B_n}$. Since Γ is no-signalling, $\Gamma(\iota_A^{(n)}(f) \otimes 1_B) \in L^\infty(\Omega_X) \otimes 1_B$. The claim follows from the fact that $(\mathcal{E}_{X_n} \otimes \mathcal{E}_{Y_n})(L^\infty(\Omega_X) \otimes 1_B) = \mathcal{D}_{X_n} \otimes 1_{B_n}$.

(ii) $\Rightarrow$ (i) Assume that Γ_n is no-signalling for every $n \in \mathbb{N}$. It suffices to show that $\Gamma(\iota^{(n)}(f_n) \otimes 1_B) \in L^\infty(\Omega_X) \otimes 1_Y$ for every $f_n \in D_{A_n}$ and $n \in \mathbb{N}$. Indeed, if we then pick $f \in C(\Omega_A)$ and set $f_n := \mathcal{E}_{A_n}(f) \in \mathcal{D}_{A_n}$, we have that $f = \lim_{n\to\infty} \iota_A^{(n)}(f_n)$ in norm and, as Γ is continuous,

$$\Gamma(f \otimes 1_B) = \lim_{n\to\infty} \Gamma(\iota^{(n)}(f_n) \otimes 1_B) \in L^\infty(\Omega_X) \otimes 1_Y.$$

Let $f \in \mathcal{D}_{A_n}$ for some $n \in \mathbb{N}$. Let $k \geq n$, and note that

$$\begin{aligned}&\left((\mathcal{E}^\infty_{X_k} \otimes \mathcal{E}^\infty_{Y_k}) \circ \Gamma\right)(\iota_A^{(n)}(f) \otimes 1_B)\\ &= (\mathcal{E}^\infty_{X_k} \otimes \mathcal{E}^\infty_{Y_k}) \circ \Gamma \circ (\iota_A^{(k)} \otimes \iota_B^{(k)})(\iota_{A_n,A_k}(f) \otimes 1_{B_k}) = \Gamma_k(\iota_{A_n,A_k}(f) \otimes 1_{B_k}).\end{aligned}$$

Thus, $(\mathcal{E}^\infty_{X_k} \otimes \mathcal{E}^\infty_{Y_k}) \circ \Gamma(\iota_A^{(n)}(f) \otimes 1_B) \in \mathcal{D}_{X_k} \otimes 1_{Y_k}$ for all $k \geq n$ since Γ_k is no-signalling for all $k \in \mathbb{N}$. Hence,

$$(\iota_X^{(k)} \otimes \iota_Y^{(k)}) \circ (\mathcal{E}^\infty_{X_k} \otimes \mathcal{E}^\infty_{Y_k}) \circ \Gamma(\iota_A^{(n)}(f) \otimes 1_B) \in L^\infty(\Omega_X) \otimes 1_Y$$

for all $k \geq n$ and if we take the limits in norm as $k \to \infty$, we conclude that

$$\Gamma(\iota_A^{(n)}(f) \otimes 1_B) \in L^\infty(\Omega_X) \otimes 1_Y,$$

as desired. The result follows by symmetry. □

Given a no-signalling correlation Γ over the quadruple (X, Y, A, B), we write $p_{\Gamma,n}$ for the family of conditional probability distributions on $A_n \times B_n$, indexed by $X_n \times Y_n$, corresponding to Γ_n by (14), that is,

$$p_{\Gamma,n}(a, b|x, y) = |X_n||Y_n| \langle \delta_x \otimes \delta_y, \Gamma_n(\delta_a \otimes \delta_b)\rangle,$$

where $x \in X_n$, $y \in Y_n$, $a \in A_n$, $b \in B_n$ and, as before, the pairing is given by the normalised traces (that is, $\langle \delta_1 \otimes \delta_2, \omega_1 \otimes \omega_2\rangle = \operatorname{tr}_{X_n}(\delta_1\omega_1) \operatorname{tr}_{Y_n}(\delta_2\omega_2)$). By Proposition 4.2, the correlations $p_{\Gamma,n}$ are no-signalling. By Theorem 2.1, there exist states $s_{\Gamma_n} : \mathcal{S}_{X_n,A_n} \otimes_{\max} \mathcal{S}_{Y_n,B_n} \to \mathbb{C}$, such that

$$s_{\Gamma_n}(e_{x,a} \otimes e_{y,b}) = |X_n||Y_n|\langle \delta_x \otimes \delta_y, \Gamma_n(\delta_a \otimes \delta_b)\rangle$$

for $x \in X_n$, $y \in Y_n$, $a \in A_n$ and $b \in B_n$. The proof of the next lemma is similar to that of Theorem 3.4 and is omitted.

Lemma 4.3. *Let $(X_n)_{n\in\mathbb{N}}$, $(Y_n)_{n\in\mathbb{N}}$, $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be inductive families of sets, and $\Gamma_n : \mathcal{D}_{A_n} \otimes \mathcal{D}_{B_n} \to \mathcal{D}_{X_n} \otimes \mathcal{D}_{Y_n}$ be a no-signalling correlation, $n \in \mathbb{N}$. The family $(\Gamma_n)_{n\in\mathbb{N}}$ is inductive if and only if*

$$s_{\Gamma_{n+1}} \circ (\gamma_{X_n,A_n} \otimes \gamma_{Y_n,B_n}) = s_{\Gamma_n}, \quad n \in \mathbb{N}.$$

In the proof of the next theorem, we will need an auxiliary fact about operator system inductive limits. Let τ be an operator system tensor product (see [21]). We will say that τ *commutes with inductive limits* if, for every inductive sequence

$$\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \cdots$$

in the operator system category, and every operator system $\mathcal{T}$, we have that

$$\varinjlim(\mathcal{S}_k \otimes_\tau \mathcal{T}) \cong (\varinjlim \mathcal{S}_k) \otimes_\tau \mathcal{T},$$

up to a canonical complete order isomorphism.

Lemma 4.4. *Let τ be an operator system tensor product that commutes with inductive limits, and*

$$\mathcal{S}_1 \xrightarrow{\phi_1} \mathcal{S}_2 \xrightarrow{\phi_2} \mathcal{S}_3 \xrightarrow{\phi_3} \cdots$$

and

$$\mathcal{T}_1 \xrightarrow{\psi_1} \mathcal{T}_2 \xrightarrow{\psi_2} \mathcal{T}_3 \xrightarrow{\psi_3} \cdots$$

be inductive sequences in the operator system category with inductive limits $\mathcal{S}$ and $\mathcal{T}$, respectively. Then

$$\varinjlim(\mathcal{S}_k \otimes_\tau \mathcal{T}_k) \cong \mathcal{S} \otimes_\tau \mathcal{T},$$

up to a canonical complete order isomorphism.

Proof. For brevity, we will use the symbol id_k to denote the identity map on either $\mathcal{S}_k$ or $\mathcal{T}_k$, depending on the context. Write $\mathcal{R} := \varinjlim(\mathcal{S}_k \otimes_\tau \mathcal{T}_k)$ and set $\theta_n = \phi_n \otimes \mathrm{id}_{\mathcal{T}}$; thus, $\theta_n : \mathcal{S}_n \otimes_\tau \mathcal{T} \to \mathcal{S}_{n+1} \otimes_\tau \mathcal{T}$ is a unital completely positive map, $n \in \mathbb{N}$. Trivially,

$$\theta_n \circ (\mathrm{id}_n \otimes \psi_{k,\infty}) = (\mathrm{id}_{n+1} \otimes \psi_{k,\infty}) \circ (\phi_n \otimes \mathrm{id}_k), \quad k \in \mathbb{N}.$$

On the other hand, using [25, Remark 2.15], we have that the diagram

$$\begin{array}{ccccccc}
\mathcal{S}_n \otimes_\tau \mathcal{T}_n & \xrightarrow{\mathrm{id}_n \otimes \psi_n} & \mathcal{S}_n \otimes_\tau \mathcal{T}_{n+1} & \xrightarrow{\mathrm{id}_n \otimes \psi_{n+1}} & \mathcal{S}_n \otimes_\tau \mathcal{T}_{n+2} & \longrightarrow & \cdots \\
\big\downarrow {\scriptstyle \phi_{n,n} \otimes \mathrm{id}_n} & & \big\downarrow {\scriptstyle \phi_{n,n+1} \otimes \mathrm{id}_{n+1}} & & \big\downarrow {\scriptstyle \phi_{n,n+2} \otimes \mathrm{id}_{n+2}} & & \\
\mathcal{S}_n \otimes_\tau \mathcal{T}_n & \xrightarrow{\phi_n \otimes \psi_n} & \mathcal{S}_{n+1} \otimes_\tau \mathcal{T}_{n+1} & \xrightarrow{\phi_{n+1} \otimes \psi_{n+1}} & \mathcal{S}_{n+2} \otimes_\tau \mathcal{T}_{n+2} & \longrightarrow & \cdots
\end{array}$$

yields a canonical unital completely positive map

$$\gamma_n : \mathcal{S}_n \otimes_\tau \mathcal{T} \to \mathcal{R},$$

such that

$$\gamma_n \circ (\mathrm{id}_n \otimes \psi_{k,\infty}) = (\phi_k \otimes \psi_k)_\infty \circ (\phi_{n,k} \otimes \mathrm{id}_k), \quad k \geq n.$$

Thus, the pair $(\mathcal{R}, (\gamma_n)_{n \in \mathbb{N}})$ satisfies

$$\begin{aligned}
(\gamma_{n+1} \circ \theta_n) \circ (\mathrm{id}_n \otimes \psi_{k,\infty}) &= \gamma_{n+1} \circ (\mathrm{id}_{n+1} \otimes \psi_{k,\infty}) \circ (\phi_n \otimes \mathrm{id}_k) \\
&= (\phi_k \otimes \psi_k)_\infty \circ (\phi_{n+1,k} \otimes \mathrm{id}_k) \circ (\phi_n \otimes \mathrm{id}_k) \\
&= (\phi_k \otimes \psi_k)_\infty \circ (\phi_{n,k} \otimes \mathrm{id}_k) \\
&= \gamma_n \circ (\mathrm{id}_n \otimes \psi_{k,\infty})
\end{aligned}$$

for each $k \geq n+1$, hence $\gamma_{n+1} \circ \theta_n = \gamma_n$ and by the universal property of the inductive limit $\varinjlim(\mathcal{S}_k \otimes_\tau \mathcal{T})$ (see [25, Definition 2.13]) there exists a canonical unital completely positive map $\alpha : \mathcal{S} \otimes_\tau \mathcal{T} \to \mathcal{R}$, such that

$$\alpha \circ (\phi_{n,\infty} \otimes \mathrm{id}_{\mathcal{T}}) = \gamma_n, \quad n \in \mathbb{N}.$$

Similarly, the diagram

$$\begin{array}{ccccccc}
\mathcal{S}_1 \otimes_\tau \mathcal{T}_1 & \xrightarrow{\phi_1 \otimes \psi_1} & \mathcal{S}_2 \otimes_\tau \mathcal{T}_2 & \xrightarrow{\phi_2 \otimes \psi_2} & \mathcal{S}_3 \otimes_\tau \mathcal{T}_3 & \longrightarrow & \cdots \\
\big\downarrow {\scriptstyle \mathrm{id}_1 \otimes \psi_{1,\infty}} & & \big\downarrow {\scriptstyle \mathrm{id}_2 \otimes \psi_{2,\infty}} & & \big\downarrow {\scriptstyle \mathrm{id}_3 \otimes \psi_{3,\infty}} & & \\
\mathcal{S}_1 \otimes_\tau \mathcal{T} & \xrightarrow{\phi_1 \otimes \mathrm{id}_{\mathcal{T}}} & \mathcal{S}_2 \otimes_\tau \mathcal{T} & \xrightarrow{\phi_2 \otimes \mathrm{id}_{\mathcal{T}}} & \mathcal{S}_3 \otimes_\tau \mathcal{T} & \longrightarrow & \cdots,
\end{array}$$

yields a canonical unital completely positive map

$$\beta : \mathcal{R} \to \mathcal{S} \otimes_\tau \mathcal{T},$$

such that

$$\beta \circ (\phi_n \otimes \psi_n)_\infty = (\phi_{n,\infty} \otimes \mathrm{id}_{\mathcal{T}}) \circ (\mathrm{id}_n \otimes \psi_{n,\infty}), \quad n \in \mathbb{N}.$$

We show that the maps α and β are inverse to each other; indeed,

$$\begin{aligned}
\alpha \circ \beta \circ (\phi_n \otimes \psi_n)_\infty &= \alpha \circ (\phi_{n,\infty} \otimes \mathrm{id}_{\mathcal{T}}) \circ (\mathrm{id}_n \otimes \psi_{n,\infty}) \\
&= \gamma_n \circ (\mathrm{id}_n \otimes \psi_{n,\infty}) = (\phi_n \otimes \psi_n)_\infty \circ (\phi_{n,n} \otimes \mathrm{id}_n) \\
&= (\phi_n \otimes \psi_n)_\infty
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, $\alpha \circ \beta = \mathrm{id}$. On the other hand,

$$\begin{aligned}
\beta \circ \alpha \circ (\phi_{n,\infty} \otimes \psi_{k,\infty}) &= \beta \circ \alpha \circ (\phi_{n,\infty} \otimes \mathrm{id}_{\mathcal{T}}) \circ (\mathrm{id}_n \otimes \psi_{k,\infty}) \\
&= \beta \circ \gamma_n \circ (\mathrm{id}_n \otimes \psi_{k,\infty}) \\
&= \beta \circ (\phi_k \otimes \psi_k)_\infty \circ (\phi_{n,k} \otimes \mathrm{id}_k) \\
&= (\phi_{k,\infty} \otimes \mathrm{id}_{\mathcal{T}}) \circ (\mathrm{id}_k \otimes \psi_{k,\infty}) \circ (\phi_{n,k} \otimes \mathrm{id}_k) \\
&= \phi_{n,\infty} \otimes \psi_{k,\infty}
\end{aligned}$$

for all $n, k \in \mathbb{N}$ with $k \geq n$, showing that $\beta \circ \alpha = \mathrm{id}$. □

For the formulation of the next theorem, we recall the notation (12) for the cylinders associated with elements $x \in X_n$ and $a \in A_n$; we employ similar notation for cylinders based on $y \in Y_n$ and $b \in B_n$.

Theorem 4.5. *Let $X = (X_n)_{n \in \mathbb{N}}$, $Y = (Y_n)_{n \in \mathbb{N}}$, $A = (A_n)_{n \in \mathbb{N}}$ and $B = (B_n)_{n \in \mathbb{N}}$ be inductive families of finite sets.*

(i) *If $\Gamma \in \mathcal{C}_{\mathrm{ns}}(X, Y, A, B)$ then there exists a state $s_\Gamma : \mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B} \to \mathbb{C}$ such that*

$$s_\Gamma(\chi_{\tilde{x} \times \tilde{a}} \otimes \chi_{\tilde{y} \times \tilde{b}}) = |X_n||Y_n| \langle \chi_{\tilde{x}} \otimes \chi_{\tilde{y}}, \Gamma(\chi_{\tilde{a}} \otimes \chi_{\tilde{b}}) \rangle,$$

for all $x \in X_n$, $y \in Y_n$, $a \in A_n$, $b \in B_n$, and all $n \in \mathbb{N}$.

(ii) *If s is a state on $\mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B}$ then there exists $\Gamma \in \mathcal{C}_{\rm ns}(X,Y,A,B)$ such that $s = s_\Gamma$.*

Proof. (i) Let Γ be no-signalling and, for each $n \in \mathbb{N}$, set

$$\Gamma_n := (\mathcal{E}^\infty_{X_n} \otimes \mathcal{E}^\infty_{Y_n}) \circ \Gamma \circ (\iota_A^{(n)} \otimes \iota_B^{(n)}).$$

By Proposition 4.2, Γ_n is no-signalling and hence, by Theorem 2.1, there exist states $s_n : \mathcal{S}_{X_n,A_n} \otimes_{\max} \mathcal{S}_{Y_n,B_n} \to \mathbb{C}$, $n \in \mathbb{N}$, such that

$$s_n(e_{x,a} \otimes e_{y,b}) = |X_n||Y_n|\langle \delta_x \otimes \delta_y, \Gamma_n(\delta_a \otimes \delta_b)\rangle, \quad n \in \mathbb{N}, \tag{16}$$

for every $(x,y,a,b) \in X_n \times Y_n \times A_n \times B_n$. By Theorem 3.2 and Lemma 4.3,

$$s_{n+1} \circ (\gamma_{X_n,A_n} \otimes \gamma_{Y_n,B_n}) = s_n,$$

and therefore by the universal property of inductive limits, Lemma 4.4 and [25, Theorem 4.34] there exists a state $s : \mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B} \to \mathbb{C}$ such that $s \circ (\gamma^{(n)}_{X,A} \otimes \gamma^{(n)}_{Y,B}) = s_n$, $n \in \mathbb{N}$. Using (16), it follows that, if $x \in X_n$, $y \in Y_n$, $a \in A_n$ and $b \in B_n$ then

$$\begin{aligned} s(\chi_{\tilde{x}\times\tilde{a}} \otimes \chi_{\tilde{y}\times\tilde{b}}) &= s_n(e_{x,a} \otimes e_{y,b}) \\ &= |X_n||Y_n|\langle \delta_x \otimes \delta_y, (\mathcal{E}^\infty_{X_n} \otimes \mathcal{E}^\infty_{Y_n}) \circ \Gamma \circ (\iota_A^{(n)} \otimes \iota_B^{(n)})(\delta_a \otimes \delta_b)\rangle \\ &= |X_n||Y_n|\langle (\iota_X^{(n)} \otimes \iota_Y^{(n)})(\delta_x \otimes \delta_y), \Gamma(\chi_{\tilde{a}} \otimes \chi_{\tilde{b}})\rangle \\ &= |X_n||Y_n|\langle \chi_{\tilde{x}} \otimes \chi_{\tilde{y}}, \Gamma(\chi_{\tilde{a}} \otimes \chi_{\tilde{b}})\rangle. \end{aligned}$$

(ii) Setting $s_n = s \circ (\gamma^{(n)}_{X,A} \otimes \gamma^{(n)}_{Y,B})$, $n \in \mathbb{N}$, we have that s_n is a state on the tensor product $\mathcal{S}_{X_n,A_n} \otimes_{\max} \mathcal{S}_{Y_n,B_n}$ and consequently it gives rise, via Theorem 2.1, to a no-signalling correlation Γ_n over (X_n, Y_n, A_n, B_n), $n \in \mathbb{N}$. Note that $s_{n+1} \circ (\gamma_{X_n,A_n} \otimes \gamma_{Y_n,B_n}) = s_n$, $n \in \mathbb{N}$, and thus, by Lemma 4.3, the family $(\Gamma_n)_{n\in\mathbb{N}}$ is inductive. By Theorem 3.2, there exists a (unique) unital completely positive map $\Gamma : C(\Omega_A) \otimes C(\Omega_B) \to L^\infty(\Omega_X)\bar{\otimes}L^\infty(\Omega_Y)$ that satisfies the relations

$$(\mathcal{E}^\infty_{X_n} \otimes \mathcal{E}^\infty_{Y_n}) \circ \Gamma \circ (\iota_A^{(n)} \otimes \iota_B^{(n)}) = \Gamma_n, \quad n \in \mathbb{N};$$

by Proposition 4.2, Γ is no-signalling. If $x \in X_n$, $y \in Y_n$, $a \in A_n$ and $b \in B_n$, then

$$\begin{aligned} s_\Gamma(\chi_{\tilde{x}\times\tilde{a}} \otimes \chi_{\tilde{y}\times\tilde{b}}) &= |X_n||Y_n|\langle \chi_{\tilde{x}} \otimes \chi_{\tilde{y}}, \Gamma(\chi_{\tilde{a}} \otimes \chi_{\tilde{b}})\rangle \\ &= |X_n||Y_n|\langle \delta_x \otimes \delta_y, \Gamma_n(\delta_a \otimes \delta_b)\rangle = s_n(e_{x,a} \otimes e_{y,b}) \\ &= (s \circ (\gamma^{(n)}_{X,A} \otimes \gamma^{(n)}_{Y,B}))(e_{x,a} \otimes e_{y,b}) = s(\chi_{\tilde{x}\times\tilde{a}} \otimes \chi_{\tilde{y}\times\tilde{b}}) \end{aligned}$$

and, since the elements $\chi_{\tilde{x}\times\tilde{a}} \otimes \chi_{\tilde{y}\times\tilde{b}}$, when n varies, form a generating set for $\mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B}$, we have that $s_\Gamma = s$. □

5. The type hierarchy

In this section, we consider other types of correlations over Cantor spaces, that lie within the class of all no-signalling correlations defined in Section 4, and obtain corresponding operator algebraic descriptions. We require some preparations; in the next subsection, we develop the bipartite versions of operator-valued channels from Subsection 3.1 that will be needed in the sequel.

5.1. **Bipartite operator-valued channels.** Let $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{S}$ and $\mathfrak{T}$ be second countable compact Hausdorff spaces, $\mu \in M(\mathfrak{X})$ and $\nu \in M(\mathfrak{Y})$ be probability measures, and H be a Hilbert space. Given $E \in \mathfrak{C}_\mu(\mathfrak{S}, \mathfrak{X}; H)$ and $F \in \mathfrak{C}_\nu(\mathfrak{T}, \mathfrak{Y}; H)$, and denoting by $\mathfrak{f}$ the flip between the first and the second tensor terms in the three-leg expressions below, we let

$$\phi_E : C(\mathfrak{S}) \to L^\infty(\mathfrak{X}, \mu)\bar{\otimes} L^\infty(\mathfrak{Y}, \nu)\bar{\otimes}\mathcal{B}(H)$$

and

$$\phi_F : C(\mathfrak{T}) \to L^\infty(\mathfrak{X}, \mu)\bar{\otimes} L^\infty(\mathfrak{Y}, \nu)\bar{\otimes}\mathcal{B}(H)$$

be the maps, defined by setting

$$\phi_E(f) = \mathfrak{f}(1_{\mathfrak{Y}} \otimes \Phi_E(f)) \ \text{ and } \ \phi_F(g) = 1_{\mathfrak{X}} \otimes \Phi_F(g).$$

We say that E and F form a commuting pair if ϕ_E and ϕ_F have commuting ranges.

Theorem 5.1. *Let $E \in \mathfrak{C}_\mu(\mathfrak{S}, \mathfrak{X}; H)$ and $F \in \mathfrak{C}_\nu(\mathfrak{T}, \mathfrak{Y}; H)$ be operator-valued channels that form a commuting pair. Then there exists a unique, up to $\sim_{\mu\times\nu}$-equivalence, channel $E \cdot F \in \mathfrak{C}_{\mu\times\nu}(\mathfrak{S} \times \mathfrak{T}, \mathfrak{X} \times \mathfrak{Y}; H)$ such that*

$$(E \cdot F)(\alpha \times \beta|x, y) = E(\alpha|x)F(\beta|y) \quad \mu \times \nu\text{-a.e.}, \quad \alpha \in \mathfrak{B}_{\mathfrak{S}}, \beta \in \mathfrak{B}_{\mathfrak{T}}. \tag{17}$$

Proof. Since ϕ_E and ϕ_F are unital completely positive maps with commuting ranges, by [30, Theorem 12.8], there exists a unique unital completely positive map

$$\phi_E \cdot \phi_F : C(\mathfrak{S}) \otimes C(\mathfrak{T}) \to L^\infty(\mathfrak{X} \times \mathfrak{Y}, \mu \times \nu)\bar{\otimes}\mathcal{B}(H),$$

such that

$$(\phi_E \cdot \phi_F)(f \otimes g) = \phi_E(f)\phi_F(g), \quad f \in C(\mathfrak{S}), g \in C(\mathfrak{T}).$$

Noting the canonical identification $C(\mathfrak{S} \times \mathfrak{T}) \cong C(\mathfrak{S}) \otimes C(\mathfrak{T})$, we consider $\phi_E \cdot \phi_F$ as a map from $C(\mathfrak{S} \times \mathfrak{T})$ into $L^\infty(\mathfrak{X} \times \mathfrak{Y}, \mu \times \nu)\bar{\otimes}\mathcal{B}(H)$. By Theorem 3.1, there exists a unique $E \cdot F \in \mathfrak{C}_{\mu\times\nu}(\mathfrak{S} \times \mathfrak{T}, \mathfrak{X} \times \mathfrak{Y}; H)$ such that, for any $h \in C(\mathfrak{S} \times \mathfrak{T})$ and $\xi, \eta \in H$, we have

$$\langle(\phi_E \cdot \phi_F)(h)(x, y)\xi, \eta\rangle = \int_{\mathfrak{S}\times\mathfrak{T}} h(a, b)d(E \cdot F)_{\xi,\eta}(a, b|x, y) \quad \mu \times \nu\text{-a.e.}$$

Applying approximation arguments similar to those in the proof of [6, Lemma 3.1], we obtain (17). □

Theorem 5.1 easily yields the following corollary; the detailed proof is omitted.

Corollary 5.2. *Let H and K be Hilbert spaces, $E \in \mathfrak{C}_\mu(\mathfrak{S}, \mathfrak{X}; H)$ and $F \in \mathfrak{C}_\nu(\mathfrak{T}, \mathfrak{Y}; K)$. Then there exists $E \otimes F \in \mathfrak{C}_{\mu\times\nu}(\mathfrak{S}\times\mathfrak{T}, \mathfrak{X}\times\mathfrak{Y}; H\otimes K)$ such that, for all $\alpha \in \mathfrak{B}_\mathfrak{S}$ and $\beta \in \mathfrak{B}_\mathfrak{T}$, we have that*

$$(E \otimes F)(\alpha \times \beta|x, y) = E(\alpha|x) \otimes F(\beta|y) \ \ \mu \times \nu\text{-almost everywhere.}$$

Remark 5.3. We fix inductive families of sets $X = (X_n)_{n\in\mathbb{N}}$, $Y = (Y_n)_{n\in\mathbb{N}}$, $A = (A_n)_{n\in\mathbb{N}}$ and $B = (B_n)_{n\in\mathbb{N}}$ and let Ω_X, Ω_Y, Ω_A and Ω_B be their respective Cantor spaces. Given $E \in \mathfrak{C}_{\mu_X}(\Omega_A, \Omega_X; H)$ and $n \in \mathbb{N}$, we denote by $E_n \in \mathfrak{C}(A_n, X_n; H)$ the ($\mathcal{B}(H)$-valued) information channel (from X_n to A_n) for which the equality

$$\Phi_{E_n} = \tilde{\mathcal{E}}^{\infty}_{X_n} \circ \Phi_E \circ \iota^{(n)}_{A,\infty} \tag{18}$$

is satisfied, $n \in \mathbb{N}$. Set

$$\Psi_n = \tilde{\iota}^{(n)}_{X,\infty} \circ \Phi_{E_n} \circ \mathcal{E}^{\infty}_{A_n}|_{C(\Omega_A)}, \quad n \in \mathbb{N};$$

thus, $(\Psi_n)_{n\in\mathbb{N}} \subseteq \mathrm{UCP}(C(\Omega_A), L^\infty(\Omega_X)\bar{\otimes}\mathcal{B}(H))$. We have that $\lim_{n\to\infty} \Psi_n = \Phi_E$ in the BW topology. Indeed, note that

$$\lim_{n\to\infty} \tilde{\iota}^{(n)}_{X,\infty} \circ \tilde{\mathcal{E}}^{\infty}_{X_n} = \mathrm{id}_{L^\infty(\Omega_X)\bar{\otimes}\mathcal{B}(H)}$$

in the BW topology, and

$$\lim_{n\to\infty} \iota^{(n)}_{A,\infty} \circ \mathcal{E}^{\infty}_{A_n}|_{C(\Omega_A)} = \mathrm{id}_{C(\Omega_A)}$$

in the point-norm topology. Fix $\omega \in \mathcal{B}(H)_*$ with $\|\omega\|_1 \leq 1$, $\epsilon > 0$ and $f \in C(\Omega_A)$, and let $N \in \mathbb{N}$ be such that

$$\left\|(\iota^{(n)}_{A,\infty} \circ \mathcal{E}^{\infty}_{A_n})(f) - f\right\| < \frac{\epsilon}{2}$$

and

$$\left|\langle\omega, (\tilde{\iota}^{(n)}_{X,\infty} \circ \tilde{\mathcal{E}}^{\infty}_{X_n} \circ \Phi_E)(f)\rangle - \langle\omega, \Phi_E(f)\rangle\right| < \frac{\epsilon}{2}$$

whenever $n \geq N$. Then

$$\begin{aligned}
&|\langle\omega, \Psi_n(f)\rangle - \langle\omega, \Phi_E(f)\rangle| \\
&\leq \left|\langle\omega, \Psi_n(f)\rangle - \langle\omega, (\tilde{\iota}^{(n)}_{X,\infty} \circ \tilde{\mathcal{E}}^{\infty}_{X_n} \circ \Phi_E)(f)\rangle\right| \\
&\qquad + \left|\langle\omega, (\tilde{\iota}^{(n)}_{X,\infty} \circ \tilde{\mathcal{E}}^{\infty}_{X_n} \circ \Phi_E)(f)\rangle - \langle\omega, \Phi_E(f)\rangle\right| \\
&= \left|\langle\omega, (\tilde{\iota}^{(n)}_{X,\infty} \circ \tilde{\mathcal{E}}^{\infty}_{X_n} \circ \Phi_E \circ \iota^{(n)}_{A,\infty} \circ \mathcal{E}^{\infty}_{A_n})(f)\rangle - \langle\omega, (\tilde{\iota}^{(n)}_{X,\infty} \circ \tilde{\mathcal{E}}^{\infty}_{X_n} \circ \Phi_E)(f)\rangle\right| \\
&\qquad + \left|\langle\omega, (\tilde{\iota}^{(n)}_{X,\infty} \circ \tilde{\mathcal{E}}^{\infty}_{X_n} \circ \Phi_E)(f)\rangle - \langle\omega, \Phi_E(f)\rangle\right| \leq \epsilon
\end{aligned}$$

for every $n \geq N$.

Remark 5.4. The channels $E \in \mathfrak{C}_{\mu_X}(\Omega_A, \Omega_X; H)$ and $F \in \mathfrak{C}_{\mu_Y}(\Omega_B, \Omega_Y; H)$ form a commuting pair if and only if the (finite) channels E_n and F_m obtained via equation (18) from E and F, respectively, also do so, for all n and m. Indeed, assume that

(E, F) is a commuting pair, then the channels $\Phi_E \circ \iota_A^{(n)}$, $\Phi_F \circ \iota_B^{(m)}$ form a commuting pair and we can write

$$\phi_{E_n}(f) = (\mathcal{E}_{X_n}^{\infty} \otimes 1_{Y_m} \otimes \mathrm{id}_{\mathcal{B}(H)})(\Phi_E(\iota_A^{(n)}(f))_{1,3})$$

and

$$\phi_{F_m}(g) = (1_{X_n} \otimes \mathcal{E}_{Y_m}^{\infty} \otimes \mathrm{id}_{\mathcal{B}(H)})(\Phi_F(\iota_B^{(m)}(g))_{2,3}),$$

from which the statement follows immediately (in the displayed equations, we have used standard leg notation).

On the other hand, assume that Φ_{E_n} and Φ_{F_m} form a commuting pair for every $n, m \in \mathbb{N}$. Then the maps $\tilde{\iota}_{X,\infty}^{(n)} \circ \Phi_{E_n} \circ \mathcal{E}_{A_n}|_{C(\Omega_A)}$ and $\tilde{\iota}_{Y,\infty}^{(m)} \circ \Phi_{F_m} \circ \mathcal{E}_{B_n}|_{C(\Omega_B)}$ also form a commuting pair. By Remark 5.3 the latter unital completely positive maps converge to Φ_E and Φ_F respectively in the BW topology. By taking iterated limits we conclude that Φ_E and Φ_F have commuting ranges.

Lemma 5.5. *Let $\Gamma^1 : \mathcal{S}_{X,A} \to \mathcal{B}(H)$ and $\Gamma^2 : \mathcal{S}_{Y,B} \to \mathcal{B}(H)$ be unital completely positive maps, and let*

$$\Phi^1 : C(\Omega_A) \to L^{\infty}(\Omega_X) \bar{\otimes} \mathcal{B}(H), \qquad \Phi^2 : C(\Omega_B) \to L^{\infty}(\Omega_Y) \bar{\otimes} \mathcal{B}(H)$$

be their corresponding channels ariding via Theorem 3.4. Then Γ^1 and Γ^2 form a commuting pair if and only if Φ^1 and Φ^2 do so.

Proof. We work at finite levels and pass to the limit. For n, m, let

$$\Gamma_n^1 : \mathcal{S}_{X_n, A_n} \to \mathcal{B}(H), \quad \Gamma_m^2 : \mathcal{S}_{Y_m, B_m} \to \mathcal{B}(H),$$

and the corresponding

$$\Phi_n^1 : \mathcal{D}_{A_n} \to \mathcal{D}_{X_n} \otimes \mathcal{B}(H), \qquad \Phi_m^2 : \mathcal{D}_{B_m} \to \mathcal{D}_{Y_m} \otimes \mathcal{B}(H),$$

with

$$\Phi_n^1(\delta_a) = \sum_{x \in X_n} \delta_x \otimes \Gamma_n^1(e_{x,a}), \qquad \Phi_m^2(\delta_b) = \sum_{y \in Y_m} \delta_y \otimes \Gamma_m^2(e_{y,b}).$$

We compute

$$\big[(\Phi_n^1(\delta_a))_{1,3},\, (\Phi_m^2(\delta_b))_{2,3}\big] = \sum_{x \in X_n} \sum_{y \in Y_m} \delta_x \otimes \delta_y \otimes \big[\Gamma_n^1(e_{x,a}),\, \Gamma_m^2(e_{y,b})\big]. \tag{19}$$

If Γ^1 and Γ^2 have commuting ranges then $[\Gamma_n^1(e_{x,a}), \Gamma_m^2(e_{y,b})] = 0$ for all x, y, a, b, and (19) vanishes on generators; by linearity Φ_n^1 and Φ_m^2 have commuting ranges for all $n, m \in \mathbb{N}$. By Remark 5.4 Φ^1 and Φ^2 have commuting ranges.

If Φ^1 and Φ^2 form a commuting pair, by Remark 5.4, $[(\Phi_n^1(\delta_a))_{1,3},\, (\Phi_m^2(\delta_b))_{2,3}] = 0$, apply the slice $L_{\delta_x \otimes \delta_y}$ to (19) to obtain $[\Gamma_n^1(e_{x,a}), \Gamma_m^2(e_{y,b})] = 0$ for all x, y, a, b; by linearity this yields $[\Gamma_n^1(u), \Gamma_m^2(v)] = 0$ for all u, v and hence by density Γ^1 and Γ^2 have commuting ranges.

□

5.2. **An ultraproduct channel construction.** We collect some details about ultraproducts that we will need in the sequel, and refer the reader to [1] for further background. Fix a free ultrafilter ω on $\mathbb{N}$. For a sequence $(\mathcal{X}_n)$ of Banach spaces, set $\ell^\infty(\mathcal{X}_n) = \{(x_n) : \sup_n \|x_n\| < \infty\}$ and $\mathcal{N}_\omega = \{(x_n) : \lim_\omega \|x_n\| = 0\}$. Then the space $(\mathcal{X}_n)^\omega := \ell^\infty(\mathcal{X}_n)/\mathcal{N}_\omega$, endowed with the norm $\|[x_n]\| = \lim_\omega \|x_n\|$ (where $[x_n]$ denotes the coset containing the sequence $(x_n)_{n\in\mathbb{N}}$), is a Banach space, called the *Banach space ultraproduct* of $(\mathcal{X}_n)$. For a sequence $(H_n)_{n\in\mathbb{N}}$ of Hilbert spaces, the Banach space ultraproduct H^ω is a Hilbert space when endowed with the inner product $\langle [x_n], [y_n]\rangle = \lim_\omega \langle x_n, y_n\rangle$. For a sequence $(\mathcal{M}_n)_{n\in\mathbb{N}}$ of C*-algebras, the ultraproduct $(\mathcal{M}_n)^\omega$ is again a C*-algebra when equipped with the pointwise multiplication and involution of sequences. If $(T_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence, where $T_n \in \mathcal{B}(H_n)$, the formula $\pi_\omega([T_n])[x_n] = [T_n x_n]$ defines an isometric *-homomorphism $\pi_\omega : (\mathcal{B}(H_n))^\omega \to \mathcal{B}(H^\omega)$. For simplicity we will write $[T_n]_\omega$ for $\pi_\omega([T_n])$.

Let $\mathcal{M}_n \subseteq \mathcal{B}(H_n)$ be von Neumann algebras, $n \in \mathbb{N}$. Write $\pi_\omega : (\mathcal{M}_n)^\omega \to \mathcal{B}(H^\omega)$ for the canonical representation induced on the ultraproduct Hilbert space H^ω. The *abstract ultraproduct* [1, Definition 3.5]

$$\prod^{\omega}(\mathcal{M}_n, H_n) := \overline{\pi_\omega((\mathcal{M}_n)^\omega)}^{\mathrm{SOT}} \subseteq \mathcal{B}(H^\omega)$$

is the strong-operator closure of $\pi_\omega((\mathcal{M}_n)^\omega)$. In particular, when $\mathcal{M}_n = \mathcal{B}(H_n)$ one has $\prod^\omega(\mathcal{B}(H_n), H_n) = \mathcal{B}(H^\omega)$ (see [1, Lemma 3.4]).

Lemma 5.6. *Let $\mathcal{S}$ be an operator system, H_n be a Hilbert space, $n \in \mathbb{N}$, and $\Phi_n : \mathcal{S} \to \mathcal{B}(H_n)$ unital completely positive maps for every $n \in \mathbb{N}$. Then the map*

$$\Phi^\omega : \mathcal{S} \longrightarrow \mathcal{B}(H^\omega), \quad \textit{given by} \quad \Phi^\omega(s) := \pi_\omega\big([\Phi_n(s)]\big),$$

is unital and completely positive.

Moreover, if $\mathcal{T}$ is another operator system and $\Psi_n : \mathcal{T} \to \mathcal{B}(H_n)$ are unital completely positive maps such (Φ_n, Ψ_n) is a commuting pair for every n, then $(\Phi^\omega, \Psi^\omega)$ is a commuting pair.

Proof. The map from $\mathcal{S}$ into $\ell^\infty(\mathcal{B}(H_n))$, sending an element $s \in \mathcal{S}$ to the sequence $(\Phi_n(s))$, is unital and completely positive. The quotient map $\ell^\infty(\mathcal{B}(H_n)) \to (\mathcal{B}(H_n))^\omega$ is a unital $*$-homomorphism because $\mathcal{I}_\omega = \{(x_n) : \lim_\omega \|x_n\| = 0\}$ is a closed two-sided $*$-ideal. Finally, $\pi_\omega : (\mathcal{B}(H_n))^\omega \to \mathcal{B}(H^\omega)$ is a unital $*$-homomorphism. Thus the composition $s \mapsto (\Phi_n(s)) \mapsto [\Phi_n(s)] \mapsto \pi_\omega([\Phi_n(s)])$ is unital and completely positive, proving the first claim.

To prove commutation, fix $s \in \mathcal{S}$, $t \in \mathcal{T}$ and $\xi = [\xi_n] \in H^\omega$. Then

$$\Phi^\omega(s)\,\Psi^\omega(t)\,\xi = \big[\Phi_n(s)\Psi_n(t)\,\xi_n\big] = \big[\Psi_n(t)\Phi_n(s)\,\xi_n\big] = \Psi^\omega(t)\,\Phi^\omega(s)\,\xi,$$

since $\Phi_n(s)$ and $\Psi_n(t)$ commute for each n. Hence $[\Phi^\omega(s), \Psi^\omega(t)] = 0$ in $\mathcal{B}(H^\omega)$, as required. □

5.3. **Definitions and characterisations.** Motivated by the hierarchy of types in the case of correlations over finite input and output sets, we now adapt the definitions of no-signalling correlation types from [6] to the Cantor setup.

Definition 5.7. Let $X = (X_n)_{n\in\mathbb{N}}$, $Y = (Y_n)_{n\in\mathbb{N}}$, $A = (A_n)_{n\in\mathbb{N}}$ and $B = (B_n)_{n\in\mathbb{N}}$ be inductive families of sets. A unital completely positive map $\Gamma : C(\Omega_A)\otimes C(\Omega_B) \to L^\infty(\Omega_X)\bar{\otimes}L^\infty(\Omega_Y)$ is called a

(i) *local correlation* if it is a finite convex combination of maps of the form $\Phi \otimes \Psi$, where $\Phi : C(\Omega_A) \to L^\infty(\Omega_X)$ and $\Psi : C(\Omega_B) \to L^\infty(\Omega_Y)$ are unital completely positive maps;

(ii) *quantum spatial correlation* if there exist separable Hilbert spaces H and K, a unit vector $\xi \in H\otimes K$, and operator-valued channels $E \in \mathfrak{C}_{\mu_X}(\Omega_A, \Omega_X; H)$ and $F \in \mathfrak{C}_{\mu_Y}(\Omega_B, \Omega_Y; K)$, such that

$$\langle g, \Gamma(h)\rangle = \langle g \otimes \xi\xi^*, \Phi_{E\otimes F}(h)\rangle \tag{20}$$

whenever $h \in C(\Omega_A)\otimes C(\Omega_B)$ and $g \in L^1(\Omega_X)\hat{\otimes}L^1(\Omega_Y)$.

(iii) *quantum approximate correlation* if $\Gamma \in \overline{\mathcal{C}_{\rm qs}}^{\rm BW}$;

(iv) *quantum commuting correlation* if there exist a separable Hilbert space H, a unit vector $\xi \in H$, and operator-valued channels $E \in \mathfrak{C}_{\mu_X}(\Omega_A, \Omega_X; H)$ and $F \in \mathfrak{C}_{\mu_Y}(\Omega_B, \Omega_Y; H)$ that form a commuting pair, such that

$$\langle g, \Gamma(h)\rangle = \langle g \otimes \xi\xi^*, \Phi_{E\cdot F}(h)\rangle \tag{21}$$

whenever $h \in C(\Omega_A)\otimes C(\Omega_B)$ and $g \in L^1(\Omega_X)\hat{\otimes}L^1(\Omega_Y)$.

In the context of Definition 5.7 (iv), we will say that the triple (H, E, F, ξ) is a realisation of the correlation Γ. We denote by $\mathcal{C}_{\rm loc}(X, Y, A, B)$ (resp. $\mathcal{C}_{\rm qs}(X, Y, A, B)$, $\mathcal{C}_{\rm qa}(X, Y, A, B)$, $\mathcal{C}_{\rm qc}(X, Y, A, B)$) the set of all local, (resp. quantum spatial, quantum approximate, quantum commuting) no-signalling correlations over (X, Y, A, B), and simply use $\mathcal{C}_{\rm t}$ when the quadruple (X, Y, A, B) is clear from the context.

Theorem 5.8. *Let* $\Gamma : C(\Omega_A)\otimes C(\Omega_B) \to L^\infty(\Omega_X)\bar{\otimes}L^\infty(\Omega_Y)$ *be a unital completely positive map and* $(\Gamma_n)_{n\in\mathbb{N}}$ *be its associated inductive family of maps. The following are equivalent:*

(i) $\Gamma \in \mathcal{C}_{\rm qc}(X, Y, A, B)$;

(ii) $\Gamma_n \in \mathcal{C}_{\rm qc}(X_n, Y_n, A_n, B_n)$ *for every* $n \in \mathbb{N}$.

Proof. We recall that $\tilde{\mathcal{E}}^\infty_{X_n} : L^\infty(\Omega_X)\bar{\otimes}\mathcal{B}(H) \to \mathcal{D}_{X_n} \otimes \mathcal{B}(H)$ is the canonical expectation.

(i)⇒(ii) Let $\Gamma \in \mathcal{C}_{\rm qc}$, and let H be a Hilbert space, $\xi \in H$ be a unit vector, and $E \in \mathfrak{C}_{\mu_X}(\Omega_A, \Omega_X; H)$ and $F \in \mathfrak{C}_{\mu_Y}(\Omega_B, \Omega_Y; H)$ be channels forming a commuting pair, such that (21) is satisfied. Further, let $E_n \in \mathfrak{C}(A_n, X_n; H)$ and $F_n \in \mathfrak{C}(B_n, Y_n; H)$ be the channels such that

$$\Phi_{E_n} = \tilde{\mathcal{E}}^\infty_{X_n} \circ \Phi_E \circ \iota^{(n)}_A \quad \text{and} \quad \Phi_{F_n} = \tilde{\mathcal{E}}^\infty_{Y_n} \circ \Phi_F \circ \iota^{(n)}_B, \quad n \in \mathbb{N}.$$

By Remark 5.4, (E_n, F_n) is a commuting pair for every $n \in \mathbb{N}$.

We show that Γ_n is a quantum commuting correlation with realisation (H, E_n, F_n, ξ). Indeed, if $x \in X_n$, $y \in Y_n$, $a \in A_n$ and $b \in B_n$ then

$$\begin{aligned}
&\langle \delta_x \otimes \delta_y, \Gamma_n(\delta_a \otimes \delta_b)\rangle = \\
&= \langle \delta_x \otimes \delta_y, (\mathcal{E}^\infty_{X_n} \otimes \mathcal{E}^\infty_{Y_n}) \circ \Gamma \circ (\iota_A^{(n)} \otimes \iota_B^{(n)})(\delta_a \otimes \delta_b)\rangle \\
&= \langle (\iota_{X,1}^{(n)} \otimes \iota_{Y,1}^{(n)})(\delta_x \otimes \delta_y), \Gamma \circ (\iota_A^{(n)} \otimes \iota_B^{(n)})(\delta_a \otimes \delta_b)\rangle \\
&= \langle (\iota_{X,1}^{(n)} \otimes \iota_{Y,1}^{(n)})(\delta_x \otimes \delta_y) \otimes \xi\xi^*, \Phi_{E\cdot F} \circ (\iota_A^{(n)} \otimes \iota_B^{(n)})(\delta_a \otimes \delta_b)\rangle \\
&= \langle (\delta_x \otimes \delta_y) \otimes \xi\xi^*, (\widetilde{\mathcal{E}^\infty_{X_n} \otimes \mathcal{E}^\infty_{Y_n}}) \circ \Phi_{E\cdot F} \circ (\iota_A^{(n)} \otimes \iota_B^{(n)})(\delta_a \otimes \delta_b)\rangle \\
&= \langle (\delta_x \otimes \delta_y) \otimes \xi\xi^*, \Phi_{E_n\cdot F_n}(\delta_a \otimes \delta_b)\rangle,
\end{aligned}$$

where the last equality follows from the fact that

$$\begin{aligned}
&(\widetilde{\mathcal{E}^\infty_{X_n} \otimes \mathcal{E}^\infty_{Y_n}}) \circ \Phi_{E\cdot F} \circ (\iota_A^{(n)} \otimes \iota_B^{(n)})(\delta_a \otimes \delta_b) \\
&= (\widetilde{\mathcal{E}^\infty_{X_n} \otimes \mathcal{E}^\infty_{Y_n}})((\Phi_E(\iota_A^{(n)}(\delta_a)))_{1,3}(\Phi_F(\iota_B^{(n)}(\delta_b)))_{2,3}) \\
&= (\tilde{\mathcal{E}}^\infty_{X_n}(\Phi_E(\iota_A^{(n)}(\delta_a)))_{1,3}(\tilde{\mathcal{E}}^\infty_{Y_n}(\Phi_F(\iota_B^{(n)}(\delta_b)))_{2,3}.
\end{aligned}$$

(ii)⇒(i) By assumption, for each $n \in \mathbb{N}$ there exist a Hilbert space H_n, a unit vector $\xi_n \in H_n$, and channels

$$E_n \in \mathfrak{C}(A_n, X_n; H_n) \ \text{ and } \ F_n \in \mathfrak{C}(B_n, Y_n; H_n),$$

forming a commuting pair and realising Γ_n:

$$\langle g, \Gamma_n(h)\rangle = \langle g \otimes \xi_n\xi_n^*, \Phi_{E_n\cdot F_n}(h)\rangle, \quad g \in \mathcal{D}_{X_n} \otimes \mathcal{D}_{Y_n},\ h \in \mathcal{D}_{A_n} \otimes \mathcal{D}_{B_n}. \tag{22}$$

In addition, we have the inductivity relations

$$\Gamma_n = (\mathcal{E}_{X_{n+1},X_n} \otimes \mathcal{E}_{Y_{n+1},Y_n}) \circ \Gamma_{n+1} \circ (\iota_{A_n,A_{n+1}} \otimes \iota_{B_n,B_{n+1}}), \quad n \in \mathbb{N}. \tag{23}$$

Combining equations (22) and (23), for all $g \in \mathcal{D}_{X_n} \otimes \mathcal{D}_{Y_n}$, $h \in \mathcal{D}_{A_n} \otimes \mathcal{D}_{B_n}$ we have

$$\begin{aligned}
&\left\langle g \otimes \xi_n\xi_n^*, \Phi_{E_n\cdot F_n}(f)\right\rangle = \langle g, \Gamma_n(h)\rangle \\
&= \langle g, (\mathcal{E}_{X_{n+1},X_n} \otimes \mathcal{E}_{Y_{n+1},Y_n}) \circ \Gamma_{n+1} \circ (\iota_{A_n,A_{n+1}} \otimes \iota_{B_n,B_{n+1}})(h)\rangle \\
&= \langle (\iota_{X_{n+1},X_n} \otimes \iota_{Y_{n+1},Y_n})(g), \Gamma_{n+1} \circ (\iota_{A_n,A_{n+1}} \otimes \iota_{B_n,B_{n+1}})(h)\rangle \\
&= \langle (\iota_{X_{n+1},X_n} \otimes \iota_{Y_{n+1},Y_n})(g) \otimes \xi_{n+1}\xi_{n+1}^*, \Phi_{E_{n+1}\cdot F_{n+1}} \circ (\iota_{A_n,A_{n+1}} \otimes \iota_{B_n,B_{n+1}})(h)\rangle.
\end{aligned}$$

Thus, for all $g \in \mathcal{D}_{X_n} \otimes \mathcal{D}_{Y_n}$, $h \in \mathcal{D}_{A_n} \otimes \mathcal{D}_{B_n}$, we obtain

$$\begin{aligned}
\left\langle g \otimes \xi_n\xi_n^*, \Phi_{E_n\cdot F_n}(h)\right\rangle = \Big\langle g \otimes \xi_{n+1}\xi_{n+1}^*, &(\widetilde{\mathcal{E}_{X_{n+1},X_n} \otimes \mathcal{E}_{Y_{n+1},Y_n}}) \\
&\circ \Phi_{E_{n+1}\cdot F_{n+1}}\big((\iota_{A_n,A_{n+1}} \otimes \iota_{B_n,B_{n+1}})(h)\big)\Big\rangle.
\end{aligned} \tag{24}$$

Define the channels

$$\Phi'_{E_n} := \tilde{\iota}^{(n)}_{X,\infty} \circ \Phi_{E_n} \circ \mathcal{E}^\infty_{A_n}|_{C(\Omega_A)} : C(\Omega_A) \to L^\infty(\Omega_X)\bar{\otimes}\mathcal{B}(H_n),$$

$$\Phi'_{F_n} := \tilde{\iota}^{(n)}_{Y,\infty} \circ \Phi_{F_n} \circ \mathcal{E}^{\infty}_{B_n}|_{C(\Omega_B)} : C(\Omega_B) \to L^\infty(\Omega_Y)\bar{\otimes}\mathcal{B}(H_n),$$

which form a commuting pair at each level n. By Theorem 3.4 (with $H = H_n$), there exist unique unital completely positive maps

$$\Theta_n : \mathcal{S}_{X,A} \to \mathcal{B}(H_n) \text{ and } \Lambda_n : \mathcal{S}_{Y,B} \to \mathcal{B}(H_n),$$

such that

$$\begin{aligned} |X_m|\, L_{\iota^{(m)}_{X,1}(\delta_x)}\big(\Phi'_{E_n}(\iota^{(m)}_A(\delta_a))\big) &= \Theta_n\big(\gamma^{(m)}_{X,A}(e_{x,a})\big), \\ |Y_m|\, L_{\iota^{(m)}_{Y,1}(\delta_y)}\big(\Phi'_{F_n}(\iota^{(m)}_B(\delta_b))\big) &= \Lambda_n\big(\gamma^{(m)}_{Y,B}(e_{y,b})\big) \end{aligned} \tag{25}$$

for all $m \in \mathbb{N}$, $x \in X_m$, $a \in A_m$, $y \in Y_m$, $b \in B_m$ (see (10)). In particular, by Lemma 5.5 we obtain that (Θ_n, Λ_n) a commuting pair, $n \in \mathbb{N}$.

Fix a free ultrafilter ω on $\mathbb{N}$ and form the Hilbert ultrapower H^ω and the abstract ultraproduct $\prod^\omega \big(\mathcal{B}(H_n), H_n\big) = \mathcal{B}(H^\omega)$ (see Subsection 5.2). Let $\xi = [\xi_n]$; thus, ξ is a unit vector in H^ω. By Lemma 5.6, we have unital completely positive maps $\Theta^\omega : \mathcal{S}_{X,A} \to \mathcal{B}(H^\omega)$ and $\Lambda^\omega : \mathcal{S}_{Y,B} \to \mathcal{B}(H^\omega)$, given by

$$\Theta^\omega(s) := [\Theta_n(s)]_\omega \text{ and } \Lambda^\omega(t) := [\Lambda_n(t)]_\omega,$$

that form a commuting pair. Apply Theorem 3.4 again (with $H = H^\omega$) to obtain unital completely positive maps

$$\Phi : C(\Omega_A) \to L^\infty(\Omega_X)\bar{\otimes}\mathcal{B}(H^\omega) \text{ and } \Psi : C(\Omega_B) \to L^\infty(\Omega_Y)\bar{\otimes}\mathcal{B}(H^\omega),$$

satisfying

$$\begin{aligned} |X_k|\, L_{\iota^{(k)}_{X,1}(\delta_x)}\big(\Phi(\iota^{(k)}_A(\delta_a))\big) &= \Theta^\omega\big(\gamma^{(k)}_{X,A}(e_{x,a})\big), \\ |Y_k|\, L_{\iota^{(k)}_{Y,1}(\delta_y)}\big(\Psi(\iota^{(k)}_B(\delta_b))\big) &= \Lambda^\omega\big(\gamma^{(k)}_{Y,B}(e_{y,b})\big) \end{aligned} \tag{26}$$

for all $x \in X_k$, $a \in A_k$, $y \in Y_k$, $b \in B_k$, $k \in \mathbb{N}$. By Lemma 5.5 again, the pair (Φ, Ψ) is commuting.

Now, by Theorem 3.1, there exist operator-valued channels $E \in \mathfrak{C}_{\mu_X}(\Omega_A, \Omega_X; H)$ and $F \in \mathfrak{C}_{\mu_Y}(\Omega_B, \Omega_Y; H)$ such that $\Phi = \Phi_E$ and $\Psi = \Psi_F$, and since E and F form a commuting pair, by Theorem 5.1 we obtain the channel $E \cdot F$, giving rise to the unital completely positive map

$$\Phi_{E\cdot F} : C(\Omega_A) \otimes C(\Omega_B) \to L^\infty(\Omega_X)\bar{\otimes}L^\infty(\Omega_Y)\bar{\otimes}\mathcal{B}(H^\omega).$$

Fix $n \in \mathbb{N}$, $a \in A_n$, $b \in B_n$, and take any $k \geq n$, $x \in X_k$, $y \in Y_k$. By inductivity and (22),

$$\begin{aligned} &\Big\langle (\iota^{(k)}_{X,1} \otimes \iota^{(k)}_{Y,1})(\delta_x \otimes \delta_y),\ \Gamma\big((\iota^{(n)}_A \otimes \iota^{(n)}_B)(\delta_a \otimes \delta_b)\big)\Big\rangle \\ &\quad = \Big\langle \delta_x \otimes \delta_y,\ \Gamma_k\big((\iota_{A_n,A_k} \otimes \iota_{B_n,B_k})(\delta_a \otimes \delta_b)\big)\Big\rangle \\ &\quad = \Big\langle \delta_x \otimes \delta_y \otimes \xi_k\xi_k^*,\ \Phi_{E_k\cdot F_k}\big((\iota_{A_n,A_k} \otimes \iota_{B_n,B_k})(\delta_a \otimes \delta_b)\big)\Big\rangle. \end{aligned} \tag{27}$$

Using (26), we have

$$\begin{aligned}
& \Big\langle (\iota_{X,1}^{(k)} \otimes \iota_{Y,1}^{(k)})(\delta_x \otimes \delta_y) \otimes \xi\xi^*,\ \Phi_{E\cdot F}(\iota_A^{(n)}(\delta_a) \otimes \iota_B^{(n)}(\delta_b)) \Big\rangle \\
= \ & \Big\langle L_{\iota_{X,1}^{(k)}(\delta_x)}(\Phi(\iota_A^{(n)}(\delta_a)))\, L_{\iota_{Y,1}^{(k)}(\delta_y)}(\Psi(\iota_B^{(n)}(\delta_b)))\,\xi,\ \xi \Big\rangle \\
= \ & \Big\langle L_{\iota_{X,1}^{(k)}(\delta_x)}(\Phi(\iota_A^{(k)} \circ \iota_{A_n,A_k}(\delta_a)))\, L_{\iota_{Y,1}^{(k)}(\delta_y)}(\Psi(\iota_B^{(k)} \circ \iota_{B_n,B_k}(\delta_b)))\,\xi,\ \xi \Big\rangle \\
= \ & \frac{1}{|X_k|\,|Y_k|} \Big\langle \Theta^\omega\left(\gamma_{X,A}^{(k)}\left(\sum_{\vec\lambda} e_{x,(a,\vec\lambda)}\right)\right) \Lambda^\omega\left(\gamma_{Y,B}^{(k)}\left(\sum_{\vec\mu} e_{y,(b,\vec\mu)}\right)\right)\xi,\ \xi \Big\rangle,
\end{aligned}$$

where the summations are over $\vec\lambda = (\lambda_{n+1},\dots,\lambda_k) \in [d_{n+1}^A] \times \dots \times [d_k^A]$ and $\vec\mu = (\mu_{n+1},\dots,\mu_k) \in [d_{n+1}^B] \times \dots \times [d_k^B]$. Taking the limit along the ultrafilter and using (25), we obtain

$$\begin{aligned}
& \Big\langle (\iota_{X,1}^{(k)} \otimes \iota_{Y,1}^{(k)})(\delta_x \otimes \delta_y) \otimes \xi\xi^*,\ \Phi_{E\cdot F}(\iota_A^{(n)}(\delta_a) \otimes \iota_B^{(n)}(\delta_b)) \Big\rangle \\
= \ & \frac{1}{|X_k|\,|Y_k|} \Big\langle \Theta^\omega(\gamma_{X,A}^{(k)}(\sum_{\vec\lambda} e_{x,(a,\vec\lambda)}))\, \Lambda^\omega(\gamma_{Y,B}^{(k)}(\sum_{\vec\mu} e_{y,(b,\vec\mu)}))\,\xi,\ \xi \Big\rangle \\
= \ & \lim_{m\to\omega} \frac{1}{|X_k|\,|Y_k|} \Big\langle \Theta_m(\gamma_{X,A}^{(k)}(\sum_{\vec\lambda} e_{x,(a,\vec\lambda)}))\, \Lambda_m(\gamma_{Y,B}^{(k)}(\sum_{\vec\mu} e_{y,(b,\vec\mu)}))\,\xi_m,\ \xi_m \Big\rangle \\
= \ & \lim_{m\to\omega} \Big\langle L_{\iota_{X,1}^{(k)}(\delta_x)}(\Phi'_{E_m}(\iota_A^{(n)}(\delta_a)))\, L_{\iota_{Y,1}^{(k)}(\delta_y)}(\Phi'_{F_m}(\iota_B^{(n)}(\delta_b)))\,\xi_m,\ \xi_m \Big\rangle \\
= \ & \lim_{m\to\omega} \Big\langle \delta_x \otimes \delta_y \otimes \xi_m\xi_m^*,\ ((\tilde{\mathcal{E}}_{X_k}^\infty \circ \Phi'_{E_m})(\iota_A^{(n)}(\delta_a)))_{1,3}\, ((\tilde{\mathcal{E}}_{Y_k}^\infty \circ \Phi'_{F_m})(\iota_B^{(n)}(\delta_b)))_{2,3} \Big\rangle;
\end{aligned}$$

if $m \geq k$ then, by the inductivity relations (24) and equation (27), we have

$$\begin{aligned}
& \Big\langle \delta_x \otimes \delta_y \otimes \xi_m\xi_m^*,\ ((\tilde{\mathcal{E}}_{X_k}^\infty \circ \Phi'_{E_m})(\iota_A^{(n)}(\delta_a)))_{1,3}\, ((\tilde{\mathcal{E}}_{Y_k}^\infty \circ \Phi'_{F_m})(\iota_B^{(n)}(\delta_b)))_{2,3} \Big\rangle \\
= & \Big\langle \delta_x \otimes \delta_y \otimes \xi_m\xi_m^*, ((\tilde{\mathcal{E}}_{X_m,X_k} \circ \Phi_{E_m})(\iota_{A_n,A_m}(\delta_a)))_{1,3} ((\tilde{\mathcal{E}}_{Y_m,Y_k} \circ \Phi_{F_m})(\iota_{B_n,B_m}(\delta_b)))_{2,3} \Big\rangle \\
= & \Big\langle \delta_x \otimes \delta_y \otimes \xi_m\xi_m^*,\ \Big(\mathcal{E}_{X_m,X_k} \widetilde{\otimes} \mathcal{E}_{Y_m,Y_k} \circ \Phi_{E_m\cdot F_m} \circ (\iota_{A_n,A_m} \otimes \iota_{B_n,B_m})\Big)(\delta_a \otimes \delta_b) \Big\rangle \\
= & \Big\langle \delta_x \otimes \delta_y \otimes \xi_k\xi_k^*, (\Phi_{E_k\cdot F_k} \circ (\iota_{A_n,A_k} \otimes \iota_{B_n,B_k}))\,(\delta_a \otimes \delta_b) \Big\rangle \\
= & \Big\langle (\iota_{X,1}^{(k)} \otimes \iota_{Y,1}^{(k)})(\delta_x \otimes \delta_y),\ \Gamma\big((\iota_A^{(n)} \otimes \iota_B^{(n)})(\delta_a \otimes \delta_b)\big) \Big\rangle.
\end{aligned}$$

Combining the previous calculations, we conclude that

$$\Big\langle (\iota_{X,1}^{(k)} \otimes \iota_{Y,1}^{(k)})(\delta_x \otimes \delta_y),\ \Gamma\big((\iota_A^{(n)} \otimes \iota_B^{(n)})(\delta_a \otimes \delta_b)\big) \Big\rangle$$
$$= \Big\langle (\iota_{X,1}^{(k)} \otimes \iota_{Y,1}^{(k)})(\delta_x \otimes \delta_y) \otimes \xi\xi^*,\ \Phi_{E\cdot F}(\iota_A^{(n)}(\delta_a) \otimes \iota_B^{(n)}(\delta_b)) \Big\rangle;$$

therefore $\Gamma \in \mathcal{C}_{\rm qc}(X,Y,A,B)$. □

Corollary 5.9. *Let X, Y, A and B be inductive families of finite sets. Then the set $\mathcal{C}_{\rm qc}(X,Y,A,B)$ is closed in the BW topology.*

Proof. The claim follows from Theorem 5.8, the fact that the map $\Gamma \to \Gamma_n$ is continuous in the BW topology, and the fact that the class of all quantum commuting correlations over a quadruple of finite sets is closed (see Theorem 2.1). □

Proposition 5.10. *Let X, Y, A and B be inductive families of finite sets. Then, writing $\mathcal{C}_{\rm t} = \mathcal{C}_{\rm t}(X,Y,A,B)$, we have*

$$\mathcal{C}_{\rm loc} \subseteq \mathcal{C}_{\rm qs} \subseteq \mathcal{C}_{\rm qa} \subseteq \mathcal{C}_{\rm qc} \subseteq \mathcal{C}_{\rm ns}.$$

Proof. We show that quantum commuting correlations are no-signalling. Assume that $\Gamma \in \mathcal{C}_{\rm qc}$; we claim that $\Gamma(C(\Omega_A) \otimes 1_B) \subseteq L^\infty(\Omega_X) \otimes 1_Y$ and $\Gamma(1_A \otimes C(\Omega_B)) \subseteq 1_X \otimes L^\infty(\Omega_Y)$. Fix $h \in C(\Omega_A)$; it suffices to show that $L_\omega(\Gamma(h \otimes 1_B)) \in \mathbb{C} \cdot 1_Y$ for all $\omega \in L^1(\Omega_X)$. Let $g \in L^1(\Omega_Y)$ and note that

$$\begin{aligned}
&\langle g, L_\omega(\Gamma(h \otimes 1_B))\rangle = \langle g \otimes \omega, \Gamma(h \otimes 1_B)\rangle \\
&= \langle (\omega \otimes g) \otimes \xi\xi^*, \Phi_{E\cdot F}(h \otimes 1_B)\rangle = \langle (\omega \otimes g) \otimes \xi\xi^*, \Phi_E(h) \otimes 1_Y\rangle \\
&= \langle g, L_{\omega\otimes\xi\xi^*}(\Phi_E(h)) \cdot 1_Y\rangle
\end{aligned}$$

and, since g is arbitrary, we have that

$$L_\omega(\Gamma(h \otimes 1_B)) = L_{\omega\otimes\xi\xi^*}(\Phi_E(h)) \cdot 1_Y$$

for every $\omega \in L^1(\Omega_X)$, as desired. The fact that $\Gamma(1_A \otimes C(\Omega_B)) \subseteq 1_X \otimes L^\infty(\Omega_Y)$ follows by symmetry.

The first inclusion can be shown in a standard way following the finite case, the second one is trivial, while the third follows from Corollary 5.9 and the fact that $\mathcal{C}_{\rm qs} \subseteq \mathcal{C}_{\rm qc}$. □

Our next aim is to obtain an operator algebraic description of quantum commuting and approximately quantum correlations over Cantor spaces.

Remark 5.11. Given a correlation Λ of quantum commuting type over a quadruple (S,T,U,V) of finite sets, let s_Λ be the (unique) state of $\mathcal{S}_{S,U} \otimes_{\rm c} \mathcal{S}_{T,V}$ corresponding to Λ via Theorem 2.1. By Lemma 4.3, if Γ_n is a no-signalling correlation over (X_n, Y_n, A_n, B_n), $n \in \mathbb{N}$, then the family $(\Gamma_n)_{n\in\mathbb{N}}$ is inductive if and only if

$$s_{\Gamma_{n+1}} \circ (\gamma_{X_n,A_n} \otimes \gamma_{Y_n,B_n}) = s_{\Gamma_n}, \quad n \in \mathbb{N}. \tag{28}$$

Lemma 5.12. *Let* $\mathrm{t} \in \{\mathrm{loc}, \mathrm{qs}, \mathrm{qc}, \mathrm{ns}\}$, $n \in \mathbb{N}$ *and* $\Gamma_n \in \mathcal{C}_{\mathrm{t}}(X_n, Y_n, A_n, B_n)$. *Then the correlation* $\tilde{\Gamma}_n$ *defined by letting*

$$\tilde{\Gamma}_n = (\iota_{X,\infty}^{(n)} \otimes \iota_{Y,\infty}^{(n)}) \circ \Gamma_n \circ (\mathcal{E}_{A_n}^{\infty} \otimes \mathcal{E}_{B_n}^{\infty})|_{C(\Omega_A)\otimes C(\Omega_B)}$$

belongs to $\mathcal{C}_{\mathrm{t}}(X, Y, A, B)$.

Proof. We only include the proof for the case $\mathrm{t} = \mathrm{qc}$; the case $\mathrm{t} = \mathrm{qs}$ is similar and the cases $\mathrm{t} = \mathrm{loc}, \mathrm{ns}$ are immediate. Let (H, E, F, ξ) be a realisation of Γ_n; thus, $\Phi_E : \mathcal{D}_{A_n} \to \mathcal{D}_{X_n} \otimes \mathcal{B}(H)$ and $\Phi_F : \mathcal{D}_{B_n} \to \mathcal{D}_{Y_n} \otimes \mathcal{B}(H)$ are unital completely positive map and $\xi \in H$ is a unit vector. Let $\tilde{E} \in \mathfrak{C}_{\mu}(\Omega_A, \Omega_X; H)$ and $\tilde{F} \in \mathfrak{C}_{\mu}(\Omega_B, \Omega_Y; H)$ be the operator-valued channels, satisfying

$$\Phi_{\tilde{E}} = \iota_{X,\infty}^{(n)} \circ \Phi_E \circ \mathcal{E}_{A_n}^{\infty}|_{C(\Omega_A)} \ \text{ and } \ \Phi_{\tilde{F}} = \iota_{Y,\infty}^{(n)} \circ \Phi_F \circ \mathcal{E}_{B_n}^{\infty}|_{C(\Omega_B)}.$$

It is straightforward to check that $(H, \tilde{E}, \tilde{F}, \xi)$ is a realisation of $\tilde{\Gamma}_n$. □

For the formulation of the next theorem, recall once again the notation (12) for cylinders in Cantor spaces.

Theorem 5.13. *Let* $X = (X_n)_{n\in\mathbb{N}}$, $Y = (Y_n)_{n\in\mathbb{N}}$, $A = (A_n)_{n\in\mathbb{N}}$ *and* $B = (B_n)_{n\in\mathbb{N}}$ *be inductive families of finite sets, and let* $\Gamma : C(\Omega_A) \otimes C(\Omega_B) \to L^{\infty}(\Omega_X)\bar{\otimes}L^{\infty}(\Omega_Y)$ *be a no-signalling correlation. The following are equivalent:*

(i) $\Gamma \in \mathcal{C}_{\mathrm{qc}}(X, Y, A, B)$ *(resp.* $\Gamma \in \mathcal{C}_{\mathrm{qa}}(X, Y, A, B)$*);*

(ii) *there exists a state* $s : \mathcal{S}_{X,A} \otimes_{\mathrm{c}} \mathcal{S}_{Y,B} \to \mathbb{C}$ *(resp.* $s : \mathcal{S}_{X,A} \otimes_{\min} \mathcal{S}_{Y,B} \to \mathbb{C}$*), such that*

$$s(\chi_{\tilde{x}\times\tilde{a}} \otimes \chi_{\tilde{y}\times\tilde{b}}) = |X_n||Y_n|\langle \chi_{\tilde{x}} \otimes \chi_{\tilde{y}}, \Gamma(\chi_{\tilde{a}} \otimes \chi_{\tilde{b}})\rangle,$$

for all $x \in X_n$, $y \in Y_n$, $a \in A_n$, $b \in B_n$, *and all* $n \in \mathbb{N}$.

Proof. We first establish the equivalence in the quantum commuting case.

(i)⇒(ii) Assume that $\Gamma \in \mathcal{C}_{\mathrm{qc}}(X, Y, A, B)$, and let H be a separable Hilbert space, $\xi \in H$ be a unit vector, and $E \in \mathfrak{C}_{\mu_X}(\Omega_A, \Omega_X; H)$ and $F \in \mathfrak{C}_{\mu_Y}(\Omega_B, \Omega_Y; H)$ be operator-valued channels, satisfying (21). Let $\Phi_E : C(\Omega_A) \to L^{\infty}(\Omega_X)\bar{\otimes}\mathcal{B}(H)$ and $\Phi_F : C(\Omega_B) \to L^{\infty}(\Omega_Y)\bar{\otimes}\mathcal{B}(H)$ be the unital completely positive maps, associated with E and F, respectively, via Theorem 3.1. Let $(\Phi_n)_{n\in\mathbb{N}}$ and $(\Psi_n)_{n\in\mathbb{N}}$ be the inductive families, associated with Φ_E and Φ_F, respectively, via (7). Let $\tilde{\Phi}_n : \mathcal{S}_{X_n,A_n} \to \mathcal{B}(H)$ and and $\tilde{\Psi}_n : \mathcal{S}_{Y_n,B_n} \to \mathcal{B}(H)$ be the unital completely positive maps, arising from Φ_n and Ψ_n, respectively, through Remark 3.5. It follows from the proof of Theorem 3.4 that

$$\tilde{\Phi}_n = \tilde{\Phi}_{n+1} \circ \gamma_{X_n,A_n} \ \text{ and } \ \tilde{\Psi}_n = \tilde{\Psi}_{n+1} \circ \gamma_{Y_n,B_n}, \quad n \in \mathbb{N}.$$

By the universal property of the operator system inductive limit, there exist unital completely positive maps $\tilde{\Phi} : \mathcal{S}_{X,A} \to \mathcal{B}(H)$ and and $\tilde{\Psi} : \mathcal{S}_{Y,B} \to \mathcal{B}(H)$, such that

$$\tilde{\Phi} \circ \gamma_{X,A}^{(n)} = \tilde{\Phi}_n \ \text{ and } \ \tilde{\Psi} \circ \gamma_{Y,B}^{(n)} = \tilde{\Psi}_n, \quad n \in \mathbb{N}.$$

Since the pair (Φ_E, Φ_F) is commuting, so is $(\tilde{\Phi}, \tilde{\Psi})$. By the definition of the operator system commuting tensor product, the map $\tilde{\Phi} \cdot \tilde{\Psi} : \mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B} \to \mathcal{B}(H)$ is (unital and) completely positive. Let $s : \mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B} \to \mathbb{C}$ be the state, given by

$$s(u) = \langle (\tilde{\Phi} \cdot \tilde{\Psi})(u), \xi, \xi \rangle, \quad u \in \mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B}.$$

Set $s_n := s \circ (\gamma_X^{(n)} \otimes \gamma_A^{(n)})$; thus, s_n is a state on $\mathcal{S}_{X_n,A_n} \otimes_{\rm c} \mathcal{S}_{Y_n,B_n}$. Let Γ_n be given via (14), $n \in \mathbb{N}$, and observe that, if $x \in X_n$, $y \in Y_n$, $a \in A_n$ and $b \in B_n$, then

$$\begin{aligned} s(\chi_{\tilde{x} \times \tilde{a}} \otimes \chi_{\tilde{y} \times \tilde{b}}) &= s_n(e_{x,a} \otimes e_{y,b}) = \langle \tilde{\Phi}_n(e_{x,a}) \tilde{\Psi}_n(e_{y,b}) \xi, \xi \rangle \\ &= \langle L_{\delta_x}(\Phi_n(\delta_a)) L_{\delta_y}(\Psi_n(\delta_b)) \xi, \xi \rangle \\ &= |X_n||Y_n| \langle \delta_x \otimes \delta_y, \Gamma_n(\delta_a \otimes \delta_b) \rangle \\ &= |X_n||Y_n| \langle (\iota_{X,1}^{(n)} \otimes \iota_{Y,1}^{(n)})(\delta_x \otimes \delta_y), \Gamma(\chi_{\tilde{a}} \otimes \chi_{\tilde{b}}) \rangle \\ &= |X_n||Y_n| \langle \chi_{\tilde{x}} \otimes \chi_{\tilde{y}}, \Gamma(\chi_{\tilde{a}} \otimes \chi_{\tilde{b}}) \rangle. \end{aligned}$$

(ii)$\Rightarrow$(i) Setting

$$s_n = s \circ (\gamma_{X,A}^{(n)} \otimes \gamma_{Y,B}^{(n)}), \quad n \in \mathbb{N},$$

we obtain a family $(s_n)_{n \in \mathbb{N}}$ of states, where $s_n : \mathcal{S}_{X_n,A_n} \otimes_{\rm c} \mathcal{S}_{Y_n,B_n} \to \mathbb{C}$, $n \in \mathbb{N}$, satisfying (28) and therefore quantum commuting correlations Γ_n over (X_n, Y_n, A_n, B_n) which by Remark 5.11 form an inductive family. Theorem 3.2 gives rise to a unital completely positive map $\Gamma : C(\Omega_A) \otimes C(\Omega_B) \to L^\infty(\Omega_X) \bar{\otimes} L^\infty(\Omega_Y)$ that satisfies

$$(\mathcal{E}_{X_n}^\infty \otimes \mathcal{E}_{Y_n}^\infty) \circ \Gamma \circ (\iota_A^{(n)} \otimes \iota_B^{(n)}) = \Gamma_n, \quad n \in \mathbb{N}.$$

By Theorems 2.1 and 5.8, $\Gamma \in \mathcal{C}_{\rm qc}(X, Y, A, B)$.

We now consider the approximately quantum case. Assume that the implication (i)$\Rightarrow$(ii) holds in the case where $\Gamma \in \mathcal{C}_{\rm qs}(X, Y, A, B)$; to conclude it in the full generality, let $\Gamma \in \mathcal{C}_{\rm qa}(X, Y, A, B)$ and $(\Gamma^{(k)})_{k \in \mathbb{N}} \subseteq \mathcal{C}_{\rm qs}(X, Y, A, B)$ be a sequence with BW limit Γ. Let $s_k : \mathcal{S}_{X,A} \otimes_{\min} \mathcal{S}_{Y,B} \to \mathbb{C}$ be a state yielding $\Gamma^{(k)}$, $k \in \mathbb{N}$. Let $(s_{k_l})_{l \in \mathbb{N}}$ be a subsequence such that $s_{k_l} \to_{l \to \infty} s$ in the weak* topology; then $s(\gamma_{X,A}^{(k_l)}(e_{x,a}) \otimes \gamma_{Y,B}^{(k_l)}(e_{y,b}))$ agrees with $|X_{k_l}||Y_{k_l}| \langle \chi_{\tilde{x}} \otimes \chi_{\tilde{y}}, \Gamma(\chi_{\tilde{a}} \otimes \chi_{\tilde{b}}) \rangle$ for all $x \in X_{k_l}$, $y \in Y_{k_l}$, $a \in A_{k_l}$ and $b \in B_{k_l}$ and all $l \in \mathbb{N}$. By uniform boundedness, the density of the linear span of the elements $\chi_{\tilde{a}} \otimes \chi_{\tilde{b}}$ in $C(\Omega_A) \otimes C(\Omega_B)$, and the weak* density of the elements $\chi_{\tilde{x}} \otimes \chi_{\tilde{y}}$ in $L^\infty(\Omega_X) \bar{\otimes} L^\infty(\Omega_Y)$, we conclude that $s = s_\Gamma$.

To complete the proof of the implication (i)$\Rightarrow$(ii), we note that in the case where $\Gamma \in \mathcal{C}_{\rm qs}(X, Y, A, B)$ the statement follows readily by inspecting the proof of the same implication in the quantum commuting case, working with tensor, instead of operator products, and using the fact that unital completely positive maps on the individual terms tensor to a unital completely positive map on the minimal operator system tensor product.

For the implication (ii)$\Rightarrow$(i) in the approximately quantum case, let $s : \mathcal{S}_{X,A} \otimes_{\min} \mathcal{S}_{Y,B} \to \mathbb{C}$ be a state, and $s_n = s \circ (\gamma_{X,A}^{(n)} \otimes_{\min} \gamma_{Y,B}^{(n)})$; thus, s_n is a state on $\mathcal{S}_{X_n,A_n} \otimes_{\min} \mathcal{S}_{Y_n,B_n}$, $n \in \mathbb{N}$. By Theorem 2.1, s_n gives rise to a no-signalling correlation Γ_n :

$\mathcal{D}_{A_n} \otimes \mathcal{D}_{B_n} \to \mathcal{D}_{X_n} \otimes \mathcal{D}_{Y_n}$ of approximately quantum type. Let $\tilde{\Gamma}_n : C(\Omega_A) \otimes C(\Omega_B) \to L^\infty(\Omega_X)\bar{\otimes}L^\infty(\Omega_Y)$ be the map given by letting

$$\tilde{\Gamma}_n = (\iota^{(n)}_{X,\infty} \otimes \iota^{(n)}_{Y,\infty}) \circ \Gamma_n \circ (\mathcal{E}^\infty_{A_n} \otimes \mathcal{E}^\infty_{B_n})|_{C(\Omega_A)\otimes C(\Omega_B)}. \tag{29}$$

Using the argument from Remark 5.3, we see that $\tilde{\Gamma}_n \to_{n\to\infty} \Gamma$ in the BW topology.

It therefore suffices to show that $\tilde{\Gamma}_n \in \mathcal{C}_{\rm qa}(X, Y, A, B)$, $n \in \mathbb{N}$. To see that, fix $n \in \mathbb{N}$, and let $(\Theta_k)_{k\in\mathbb{N}} \subseteq \mathcal{C}_{\rm qs}(X_n, Y_n, A_n, B_n)$ be a sequence, such that $\Theta_k \to_{k\to\infty} \Gamma_n$. It is straightforward to see that, if $\tilde{\Theta}_k$ arises from Θ_k as in (29) then $\tilde{\Theta}_k \to_{k\to\infty} \tilde{\Gamma}_n$ in the BW topology. On the other hand, if H and K are separable Hilbert spaces, $\xi \in H \otimes K$ is a unit vector, and $E \in \mathfrak{C}_{\mu_{X_n}}(A_n, X_n; H)$ and $F \in \mathfrak{C}_{\mu_{Y_n}}(B_n, Y_n; K)$ are operator-valued channels such that

$$\langle \Theta_k(\delta_a \otimes \delta_b), \delta_x \otimes \delta_y\rangle = \langle \delta_x \otimes \delta_y \otimes \xi\xi^*, (\Phi_E \otimes \Phi_F)(\delta_a \otimes \delta_b)\rangle,$$

then $\tilde{\Theta}_k$ has the form (20) for the channels $\tilde{E}$ and $\tilde{F}$, satisfying

$$\Phi_{\tilde{E}} = \iota^{(n)}_{X,\infty} \circ \Phi_E \circ \mathcal{E}^\infty_{A_n}|_{C(\Omega_A)} \ \text{ and } \ \Phi_{\tilde{F}} = \iota^{(n)}_{Y,\infty} \circ \Phi_F \circ \mathcal{E}^\infty_{B_n}|_{C(\Omega_B)}. \tag{30}$$

The proof is complete. □

The next statement, which is a straightforward consequence of Theorem 5.13, complements Theorem 5.8 in the approximately quantum case.

Corollary 5.14. *Let $\Gamma : C(\Omega_A) \otimes C(\Omega_B) \to L^\infty(\Omega_X)\bar{\otimes}L^\infty(\Omega_Y)$ be a unital completely positive map and $(\Gamma_n)_{n\in\mathbb{N}}$ be its associated inductive family of maps. Then $\Gamma \in \mathcal{C}_{\rm qa}(X, Y, A, B)$ if and only if $\Gamma_n \in \mathcal{C}_{\rm qa}(X_n, Y_n, A_n, B_n)$ for every $n \in \mathbb{N}$.*

Remark 5.15. It is straightforward to see that, if $\Gamma \in \mathcal{C}_{\rm qs}(X, Y, A, B)$ (resp. $\Gamma \in \mathcal{C}_{\rm loc}(X, Y, A, B)$) then $\Gamma_n \in \mathcal{C}_{\rm qs}(X_n, Y_n, A_n, B_n)$ (resp. $\Gamma_n \in \mathcal{C}_{\rm loc}(X, Y, A, B)$) for every $n \in \mathbb{N}$. We finish this section by showing that Corollary 5.14 does not hold in the quantum spatial case and that the class $\mathcal{C}_{\rm qs}(X, Y, A, B)$ is not closed.

Theorem 5.16. *There exist inductive families of finite sets $X = (X_n)_{n\in\mathbb{N}}$, $Y = (Y_n)_{n\in\mathbb{N}}$, $A = (A_n)_{n\in\mathbb{N}}$ and $B = (B_n)_{n\in\mathbb{N}}$ and a no-signalling correlation $\Gamma : C(\Omega_A) \otimes C(\Omega_B) \to L^\infty(\Omega_X)\bar{\otimes}L^\infty(\Omega_Y)$ such that, if $(\Gamma_n)_{n\in\mathbb{N}}$ is its associated inductive family, then $\Gamma_n \in \mathcal{C}_{\rm qs}(X_n, Y_n, A_n, B_n)$ for every $n \in \mathbb{N}$, but $\Gamma \notin \mathcal{C}_{\rm qs}(X, Y, A, B)$.*

Proof. By [35], there exist finite sets $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{A}$, $\mathbb{B}$, such that $\mathcal{C}_{\rm qs}(\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B})$ is not closed, that is, there exist correlations $p_n \in \mathcal{C}_{\rm qs}(\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B})$, $n \in \mathbb{N}$, and a correlation $p \in \mathcal{C}_{\rm qa}(\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B}) \setminus \mathcal{C}_{\rm qs}(\mathbb{X}, \mathbb{Y}, \mathbb{A}, \mathbb{B})$ such that $p_n(a, b|x, y) \to p(a, b|x, y)$ for all a, b, x, y.

Let f and f_n be the states on $\mathcal{S}_{\mathbb{X},\mathbb{A}} \otimes_{\min} \mathcal{S}_{\mathbb{Y},\mathbb{B}}$ yielding p and p_n respectively via Theorem 2.1; thus,

$$p(a, b|x, y) = f(e_{x,a} \otimes e_{y,b}) \ \text{ and } p_n(a, b|x, y) = f_n(e_{x,a} \otimes e_{y,b})$$

for all $n \in \mathbb{N}$. Set $X_n = \prod_{i=0}^{n-1} \mathbb{X}$ and let $X = (X_n)_{n\in\mathbb{N}}$ be the corresponding inductive family of sets; define the families Y, A, B similarly. Let $s_n : \mathcal{S}_{X_n,A_n} \otimes_{\min} \mathcal{S}_{Y_n,B_n} \to \mathbb{C}$,

$n \in \mathbb{N}$, be the linear maps, defined inductively by letting $s_1 = f_1$ and

$$s_{n+1}\left(e_{(xx'),(aa')} \otimes e_{(yy'),(bb')}\right) = s_n(e_{x,a} \otimes e_{y,b}) f_{n+1}(e_{x',a'} \otimes e_{y',b'}),$$

where $x \in X_n$, $x' \in \mathbb{X}$, $y \in Y_n$, $y' \in \mathbb{Y}$, $a \in A_n$, $a' \in \mathbb{A}$ and $b \in B_n$, $b' \in \mathbb{B}$. To see that the maps s_n are well-defined, we refer to the commutativity of the minimal tensor product and the universal property of the C*-algebra $\mathcal{A}_{X_{n+1},A_{n+1}}$, according to which the map

$$\mathcal{A}_{X_{n+1},A_{n+1}} \mapsto \mathcal{A}_{X_n,A_n} \otimes_{\min} \mathcal{A}_{\mathbb{X},\mathbb{A}}; \quad e_{xx',aa'} \mapsto e_{x,a} \otimes e_{x',a'},$$

gives rise to a $*$-homomorphism that restricts to a unital completely positive map from $\mathcal{S}_{X_{n+1},A_{n+1}}$ to $\mathcal{S}_{X_n,A_n} \otimes_{\min} \mathcal{S}_{\mathbb{X},\mathbb{A}}$, $n \in \mathbb{N}$. Moreover,

$$\begin{aligned}
& s_{n+1}(\gamma_{X_n,A_n}(e_{x,a}) \otimes \gamma_{Y_n,B_n}(e_{y,b})) \\
&= \frac{1}{|\mathbb{X}||\mathbb{Y}|} \sum_{x' \in \mathbb{X}} \sum_{y' \in \mathbb{Y}} \sum_{a' \in \mathbb{A}} \sum_{b' \in \mathbb{B}} s_{n+1}(e_{(xx'),(aa')} \otimes e_{(yy'),(bb')}) \\
&= \frac{1}{|\mathbb{X}||\mathbb{Y}|} \sum_{x' \in \mathbb{X}} \sum_{y' \in \mathbb{Y}} \sum_{a' \in \mathbb{A}} \sum_{b' \in \mathbb{B}} s_n(e_{x,a} \otimes e_{y,b}) f_{n+1}(e_{x',a'} \otimes e_{y',b'}) = s_n(e_{x,a} \otimes e_{y,b}).
\end{aligned}$$

By Lemma 4.3, the family $(s_n)_{n\in\mathbb{N}}$ gives rise to the inductive family of correlations $(\Gamma_n)_{n\in\mathbb{N}}$. It is easy to see that, since p_n is of quantum spatial type, so is Γ_n, $n \in \mathbb{N}$. Let $\Gamma : C(\Omega_A) \otimes C(\Omega_B) \to L^\infty(\Omega_X)\bar{\otimes}L^\infty(\Omega_Y)$ be the unique unital completely positive map associated to the sequence $(\Gamma_n)_{n\in\mathbb{N}}$ and $s : \mathcal{S}_{X,A} \otimes_{\min} \mathcal{S}_{Y,B} \to \mathbb{C}$ be the corresponding state, arising via Theorem 5.13. Then $s \circ (\gamma_{X,A}^{(n)} \otimes \gamma_{Y,B}^{(n)}) = s_n$, $n \in \mathbb{N}$.

We now show that $\Gamma \notin \mathcal{C}_{\rm qs}(X, Y, A, B)$. Fix $x \in \mathbb{X}$, $y \in \mathbb{Y}$, $a \in \mathbb{A}$ and $b \in \mathbb{B}$, consider the sets $\Lambda_x^n = \{(x_i)_i \in \Omega_X : x_n = x\}$ and define Λ_a^n, Λ_y^n and Λ_b^n similarly. Assuming, towards a contradiction, that Γ is in $\mathcal{C}_{\rm qs}(X, Y, A, B)$ we can find Hilbert spaces H and K, unital completely positive maps $\Phi : C(\Omega_A) \to L^\infty(\Omega_X)\bar{\otimes}\mathcal{B}(H)$ and $\Psi : C(\Omega_B) \to L^\infty(\Omega_Y)\bar{\otimes}\mathcal{B}(K)$, and a unit vector $\xi \in H \otimes K$, such that

$$\begin{aligned}
& \langle \chi_{\Lambda_x^n} \otimes \chi_{\Lambda_y^n} \otimes \xi\xi^*, \Phi(\chi_{\Lambda_a^n}) \otimes \Psi(\chi_{\Lambda_b^n})\rangle \\
&= \langle \chi_{\Lambda_x^n} \otimes \chi_{\Lambda_y^n}, \Gamma(\chi_{\Lambda_a^n} \otimes \chi_{\Lambda_b^n})\rangle = s(\chi_{\Lambda_x^n \times \Lambda_a^n} \otimes \chi_{\Lambda_y^n \times \Lambda_b^n}) \\
&= \sum_{x',y',a',b'} s((\gamma_{X,A}^{(n)} \otimes \gamma_{Y,B}^{(n)})(e_{(x'x),(a'a)} \otimes e_{(y'y),(b'b)})) \\
&= \sum_{x',y',a',b'} s_n(e_{(x'x),(a'a)} \otimes e_{(y'y),(b'b)}) \\
&= \sum_{x',y',a',b'} s_{n-1}(e_{x',a'} \otimes e_{y',b'}) f_n(e_{x,a} \otimes e_{y,b}) = |X_{n-1}||Y_{n-1}| f_n(e_{x,a} \otimes e_{y,b}),
\end{aligned}$$

where the summation is over $(x', y', a', b') \in X_{n-1} \times Y_{n-1} \times A_{n-1} \times B_{n-1}$.

Let $E_{x,a}^n = \frac{1}{|X_{n-1}|} L_{\chi_{\Lambda_x^n}}(\Phi(\chi_{\Lambda_a^n})) \in \mathcal{B}(H)$ and $F_{y,b}^n = \frac{1}{|Y_{n-1}|} L_{\chi_{\Lambda_y^n}}(\Phi(\chi_{\Lambda_b^n})) \in \mathcal{B}(K)$. Then

$$\langle (E_{x,a}^n \otimes F_{y,b}^n)\xi, \xi\rangle = f_n(e_{x,a} \otimes e_{y,b}) = p_n(a, b|x, y),$$

for all $x \in \mathbb{X}$, $y \in \mathbb{Y}$, $a \in \mathbb{A}$ and $b \in \mathbb{B}$. Moreover, $(E^n_{x,a})_{a\in\mathbb{A}}$ and $(F^n_{y,b})_{b\in\mathbb{B}}$ are families of POVM's for each $x \in \mathbb{X}$ and $y \in \mathbb{Y}$. Choose subnets $(E^{n_\alpha}_{x,a})_\alpha$, $(F^{n_\alpha}_{y,b})_\alpha$ converging to $E_{x,a}$ and $F_{y,b}$, respectively, in the weak* topology. We have

$$\langle (E_{x,a} \otimes F_{y,b})\xi, \xi\rangle = \lim_\alpha \langle (E^{n_\alpha}_{x,a} \otimes F^{n_\alpha}_{y,b})\xi, \xi\rangle = \lim_{n\to\infty} p_n(a, b|x, y) = p(a, b|x, y). \tag{31}$$

As the families $(E_{x,a})_{a\in\mathbb{A}}$, $(F_{y,b})_{b\in\mathbb{B}}$, $x \in \mathbb{X}$, $y \in \mathbb{Y}$ are again POVM's, identity (31) contradicts the fact that p is not of quantum spatial type. □

Corollary 5.17. *There exist inductive families of finite sets X, Y, A and B for which the set $\mathcal{C}_{\rm qs}(X, Y, A, B)$ is not closed in the BW topology.*

Proof. Let $(\Gamma_n)_{n\in\mathbb{N}}$ be the inductive family from Theorem 5.16 and let $\tilde{\Gamma}_n$ be the associated no-signalling correlations via Lemma 5.12. By Lemma 5.12, $\tilde{\Gamma}_n \in \mathcal{C}_{\rm qs}$. As $\tilde{\Gamma}_n \to \Gamma$ in the BW topology, the statement follows from Theorem 5.16. □

Remark. Since the elements of the form $(\gamma^{(n)}_{X,A}\otimes\gamma^{(n)}_{Y,B})(e_{x,a}\otimes e_{y,b})$, where $x \in X_n$, $y \in Y_n$, $a \in A_n$ and $b \in B_n$, $n \in \mathbb{N}$, generate a dense subspace of the operator system $\mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B}$, the correspondence between $\mathcal{C}_{\rm qc}(X, Y, A, B)$ (resp. $\mathcal{C}_{\rm qa}(X, Y, A, B)$) and the state space of $\mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B}$ (resp. $\mathcal{S}_{X,A} \otimes_{\min} \mathcal{S}_{Y,B}$) from Theorem 5.13 is bijective.

6. Cantor games

In this section, we define values of non-local games over Cantor sets, based on the correlation types studied in the previous sections, and establish continuity results thereof. We recall that, if S, T, U and V are finite sets, a non-local game over the quadruple (S, T, U, V) is a pair $G = (\lambda, \mu)$, where $\lambda : S\times T\times U\times V \to \{0, 1\}$ and μ is a probability measure on $S \times T$; here, S (resp. U) is interpreted as the set of inputs (resp. outputs) for player Alice, and T (resp. V) – as the set of inputs (resp. outputs) for player Bob. Alice and Bob play collaboratively against a third party, Verifier. In each round, the Verifier choses a pair $(s, t) \in S \times T$ of questions according to the probability measure μ, and the players return a pair $(u, v) \in U \times V$; the tandem Alice-Bob wins (resp. loses) the round if $\lambda(s, t, u, v) = 1$ (resp. $\lambda(s, t, u, v) = 0$). Given a correlation type t over (S, T, U, V), the t-*value* of G is the parameter

$$\omega_{\rm t}(\lambda, \mu) = \sup_{p\in\mathcal{C}_{\rm t}} \sum_{s\in S}\sum_{t\in T} \mu(s, t) \sum_{u\in U}\sum_{v\in V} \lambda(s, t, u, v)p(u, v|s, t)$$

(we note that $\omega_{\rm qs}(\lambda, \mu) = \omega_{\rm qa}(\lambda, \mu)$).

Measurable games and their values were defined in [6]; here, we specialise those to the case of Cantor topological spaces. For an inductive family X of finite sets, we let for brevity $\mathfrak{B}_X = \mathfrak{B}_{\Omega_X}$. Let $X = (X_n)_{n\in\mathbb{N}}$, $Y = (Y_n)_{n\in\mathbb{N}}$, $A = (A_n)_{n\in\mathbb{N}}$ and $B = (B_n)_{n\in\mathbb{N}}$ be inductive families of finite sets. A *Cantor game* is a pair (κ, μ), where $\kappa \subseteq \Omega_X \times \Omega_Y \times \Omega_A \times \Omega_B$ is a closed subset and μ is a probability measure on $\Omega_X \times \Omega_Y$. In the sequel, we consider only the case where μ is the uniform probability measure, that is, $\mu = \mu_X \times \mu_Y$.

We equip $X_n \times Y_n$ with the uniform probability measure, and hence identify a non-local game $\mathcal{G}_n$ over (X_n, Y_n, A_n, B_n) with its rule function $\lambda_n : X_n \times Y_n \times A_n \times B_n \to \{0,1\}$, $n \in \mathbb{N}$. Let, further $\kappa_{\mathcal{G}_n}$ be the subset of $\Omega_X \times \Omega_Y \times \Omega_A \times \Omega_B$, given by

$$\kappa_{\mathcal{G}_n} = \{((x_k)_k), ((y_k)_k), ((a_k)_k), ((b_k)_k) : \\ \big((x_k)_{k=0}^{n-1}, (y_k)_{k=0}^{n-1}, (a_k)_{k=0}^{n-1}, (b_k)_{k=0}^{n-1}\big) \in \operatorname{supp} \lambda_n\}, \tag{32}$$

and note that $\kappa_{\mathcal{G}_n}$ is (open and) closed. We say that the family $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is *nested* if $\kappa_{\mathcal{G}_{n+1}} \subseteq \kappa_{\mathcal{G}_n}$ for every n. For a nested family $\mathcal{G} = (\mathcal{G}_n)_{n\in\mathbb{N}}$ of games, we set $\kappa_{\mathcal{G}} := \cap_{n\in\mathbb{N}} \kappa_{\mathcal{G}_n}$, and note that $\kappa_{\mathcal{G}}$ is a closed subset of $\Omega_X \times \Omega_Y \times \Omega_A \times \Omega_B$. We call the Cantor games of the latter form *nested*.

Lemma 6.1. *Every Cantor game $\kappa \subseteq \Omega_X \times \Omega_Y \times \Omega_A \times \Omega_B$ is nested.*

Proof. Let $\pi_n : \Omega_X \times \Omega_Y \times \Omega_A \times \Omega_B \to X_n \times Y_n \times A_n \times B_n$ be the projection, and set $\kappa_n = \pi_n^{-1}(\pi_n(\kappa))$, $n \in \mathbb{N}$. Clearly, $\kappa_{n+1} \subseteq \kappa_n$ for every n, and $\kappa \subseteq \cap_{n=1}^{\infty} \kappa_n$. Assuming that $\omega \in \cap_{n=1}^{\infty} \kappa_n$, let $\omega_n \in X_n \times Y_n \times A_n \times B_n$ and $\omega_n' \in \prod_{i\geq n} [d_i^X] \times [d_i^Y] \times [d_i^A] \times [d_i^B]$ be such that $\omega_n \in \pi_n(\kappa)$ and $\omega = \omega_n \omega_n'$, $n \in \mathbb{N}$. Since $\omega_n \in \pi_n(\kappa)$, there exists $\omega_n'' \in \prod_{i\geq n} [d_i^X] \times [d_i^Y] \times [d_i^A] \times [d_i^B]$, such that $\omega^{(n)} := \omega_n \omega_n'' \in \kappa$. Let ω' be a cluster point of the sequence $(\omega^{(n)})_{n\in\mathbb{N}}$; since κ is closed, $\omega' \in \kappa$. Since $\pi_n(\omega) = \pi_n(\omega')$ for infinitely many $n \in \mathbb{N}$, we have that $\omega = \omega'$, implying that $\omega \in \kappa$. □

In the sequel, we write for brevity $\mu_{XY} = \mu_X \times \mu_Y$. By [5, Theorem 3.11], every correlation $\Gamma : C(\Omega_A) \otimes C(\Omega_B) \to L^\infty(\Omega_X) \bar{\otimes} L^\infty(\Omega_Y)$ gives rise to a (unique) classical information channel $p_\Gamma \in \mathfrak{C}_{\mu_{XY}}(\Omega_A \times \Omega_B, \Omega_X \times \Omega_Y; \mathbb{C})$, viewed as a $\Omega_X \times \Omega_Y$-measurable family $p_\Gamma = (p_\Gamma(\cdot,\cdot|x,y)_{x,y})_{(x,y)\in\Omega_X\times\Omega_Y}$ of Borel probability measures over $\Omega_A \times \Omega_B$, such that

$$\Gamma(f)(x,y) = \int_{\Omega_A\times\Omega_B} f(a,b) dp_\Gamma(a,b|x,y), \quad \mu_{XY}\text{-almost everywhere}, \tag{33}$$

for every $f \in C(\Omega_A) \otimes C(\Omega_B)$. We have that the map Γ extends uniquely to the space of all bounded Borel functions on $\Omega_A \times \Omega_B$ and satisfies

$$\Gamma(\chi_\delta)(x,y) = p_\Gamma(\delta|x,y), \ \delta \in \mathfrak{B}_A \otimes \mathfrak{B}_B, \ \ \mu_X \times \mu_Y\text{-almost everywhere}. \tag{34}$$

Indeed, using Stinespring's Theorem and [3, Theorem 2.6.3], the map Γ can be extended uniquely to the space of bounded measurable functions on $\Omega_A \times \Omega_B$ in such a way that it has the following property: for every uniformly bounded sequence of measurable functions $(f_n)_{n\in\mathbb{N}}$ which converges pointwise to zero, the sequence $(\Gamma(f_n))_{n\in\mathbb{N}}$ converges strongly to zero. In particular, taking a uniformly bounded sequence of continuous functions $(f_n)_{n\in\mathbb{N}}$ converging pointwise to χ_δ, $\delta \in \mathfrak{B}_A \otimes \mathfrak{B}_B$, for $\xi, \eta \in L^2(X \times Y)$ we obtain

$$\begin{aligned}\langle \Gamma(\chi_\delta)\xi, \eta\rangle &= \lim_{n\to\infty} \langle \Gamma(f_n)\xi, \eta\rangle \\ &= \lim_{n\to\infty} \iint f_n(a,b) dp_\Gamma(a,b|x,y)\xi(x,y)\overline{\eta(x,y)} d\mu_{XY}(x,y) \\ &= \langle p_\Gamma(\delta|\cdot,\cdot)\xi, \eta\rangle,\end{aligned}$$

giving $\Gamma(\chi_\delta)(x,y) = p_\Gamma(\delta|x,y)$ μ_{XY}-almost everywhere.

If, further, π is a Borel probability measure on $\Omega_X \times \Omega_Y$, let $\pi \otimes p_\Gamma$ be the *compound measure* of π and p_Γ, that is, the Borel probability measure on $\Omega_X \times \Omega_Y \times \Omega_A \times \Omega_B$, given by

$$(\pi \otimes p_\Gamma)(M) = \int_X p_\Gamma(M_{x,y}|x,y) d\pi(x,y), \quad M \in \mathfrak{B}_X \otimes \mathfrak{B}_Y \otimes \mathfrak{B}_A \otimes \mathfrak{B}_B, \tag{35}$$

where

$$M_{x,y} := \{(a,b) \in \Omega_A \times \Omega_B : (x,y,a,b) \in M\}$$

is the (x,y)-section of M (see [19] and [6]). For $\mathrm{t} \in \{\mathrm{loc}, \mathrm{qs}, \mathrm{qc}, \mathrm{ns}\}$, let the t-*value* $\omega_{\mathrm{t}}(\kappa, \mu_{XY})$ of κ with respect to the measure μ_{XY} be given by

$$\omega_{\mathrm{t}}(\kappa, \mu_{XY}) = \sup_{\Gamma \in \mathcal{C}_{\mathrm{t}}} (\mu_{XY} \otimes p_\Gamma)(\kappa).$$

We note that if κ_1 and κ_2 are Cantor games with $\kappa_1 \subseteq \kappa_2$ then

$$\omega_{\mathrm{t}}(\kappa_1, \mu_{XY}) \le \omega_{\mathrm{t}}(\kappa_2, \mu_{XY}), \quad \mathrm{t} \in \{\mathrm{loc}, \mathrm{qs}, \mathrm{qc}, \mathrm{ns}\}. \tag{36}$$

Remark 6.2. The setup of no-signalling correlations in this paper is established using almost everywhere defined operator-valued information channels. Since we are about to exploit the framework developed in [6], we note that the compound measure $\mu_{XY} \otimes p_\Gamma$ is independent of the use of μ_{XY}-information channels as opposed to everywhere defined information channels. We refer the reader to [6, Remark 6.1] for a detailed argument.

Lemma 6.3. *Let X, Y, A and B be inductive families of finite sets, κ be a Cantor game over (X,Y,A,B) and $\Gamma, \Gamma_n \in \mathcal{C}_{\mathrm{ns}}(X,Y,A,B)$, $n \in \mathbb{N}$, be such that $\Gamma_n \to_{n\to\infty} \Gamma$ in the BW topology. If a subset $\kappa \subseteq \Omega_X \times \Omega_Y \times \Omega_A \times \Omega_B$ is closed and open then*

$$(\mu_{XY} \otimes p_{\Gamma_n})(\kappa) \to_{n\to\infty} (\mu_{XY} \otimes p_\Gamma)(\kappa).$$

Proof. If $f \in C(\Omega_X \times \Omega_Y)$ and $g \in C(\Omega_A \times \Omega_B)$ then

$$\begin{aligned}\langle \mu_{XY} \otimes p_\Gamma, f \otimes g\rangle &= \int_{\Omega_X\times\Omega_Y\times\Omega_A\times\Omega_B} f(x,y)g(a,b)d(\mu_{XY}\otimes p_\Gamma)(x,y,a,b) \\ &= \int_{\Omega_X\times\Omega_Y} \left(\int_{\Omega_A\times\Omega_B} g(a,b)dp_\Gamma(a,b|x,y)\right) f(x,y)d\mu_{XY}(x,y) \\ &= \int_{\Omega_X\times\Omega_Y} \Gamma(g)(x,y)f(x,y)d\mu_{XY}(x,y).\end{aligned}$$

If κ is closed and open then χ_κ is the finite sum of functions of the form $\chi_\alpha \otimes \chi_\beta$, where $\alpha \subseteq \Omega_{X\times Y}$ and $\beta \subseteq \Omega_{A\times B}$. The claim is now immediate. □

For a closed and open subset $\kappa \subseteq \Omega_X \times \Omega_Y \times \Omega_A \times \Omega_B$, we write $\Gamma(\kappa) : \Omega_X \times \Omega_Y \to \mathbb{C}$ for the (measurable) function, given by $\Gamma(\kappa)(x,y) = \Gamma(\chi_{\kappa_{x,y}})(x,y)$, and note that the proof of Lemma 6.3 shows that

$$\langle \mu_{XY} \otimes p_\Gamma, \chi_\kappa \rangle = \int_{\Omega_X\times\Omega_Y} \Gamma(\kappa)(x,y) d\mu_{XY}(x,y).$$

Lemma 6.4. *Fix $n \in \mathbb{N}$ and let $\mathcal{G}_n = (X_n, Y_n, A_n, B_n, \lambda_n)$ be a non-local game. If $\Gamma_n \in \mathcal{C}_{\rm t}$ and $\tilde{\Gamma}_n = (\iota_{X,\infty}^{(n)} \otimes \iota_{Y,\infty}^{(n)}) \circ \Gamma_n \circ (\mathcal{E}_{A_n}^\infty \otimes \mathcal{E}_{B_n}^\infty)|_{C(\Omega_A)\otimes C(\Omega_B)}$ then*

$$(\mu_{X_nY_n} \otimes p_{\Gamma_n})(\operatorname{supp}\lambda_n) = (\mu_{XY} \otimes p_{\tilde{\Gamma}_n})(\kappa_n);$$

consequently, $\omega_{\rm t}(\mathcal{G}_n, \mu_{X_n,Y_n}) = \omega_{\rm t}(\kappa_n, \mu_{XY})$, ${\rm t} \in \{\rm loc, qs, qc, ns\}$.

Proof. Letting $G_n = \operatorname{supp}\lambda_n$ and κ_n be defined as in (32), using the definition of the compound measure and (34) we have that

$$\begin{aligned}(\mu_{XY} \otimes p_{\tilde{\Gamma}_n})(\kappa_n) &= \langle 1_{XY}, \tilde{\Gamma}_n(\kappa_n)\rangle \\ &= \langle 1_{XY}, (\iota_{X,\infty}^{(n)} \otimes \iota_{Y,\infty}^{(n)}) \circ \Gamma_n \circ (\mathcal{E}_{A_n}^\infty \otimes \mathcal{E}_{B_n}^\infty)(\kappa_n)\rangle \\ &= \langle (\mathcal{E}_{X_n} \otimes \mathcal{E}_{Y_n})(1_{XY}), \Gamma_n \circ (\mathcal{E}_{A_n}^\infty \otimes \mathcal{E}_{B_n}^\infty)(\kappa_n)\rangle \\ &= \langle 1_{X_nY_n}, \Gamma_n(G_n)\rangle = (\mu_{X_nY_n} \otimes p_{\Gamma_n})(G_n).\end{aligned}$$

Now, since $\Gamma_n \in \mathcal{C}_{\rm t}(X_n, Y_n, A_n, B_n) \implies \tilde{\Gamma}_n \in \mathcal{C}_{\rm t}(X,Y,A,B)$, by Lemma 5.12 we obtain

$$\omega_{\rm t}(\mathcal{G}_n, \mu_{X_n,Y_n}) \leq \omega_{\rm t}(\kappa_n, \mu_{XY}).$$

Next let $\Gamma \in \mathcal{C}_{\rm t}(X,Y,A,B)$, set

$$\Gamma_n = \left(\mathcal{E}_{X_n}^\infty \otimes \mathcal{E}_{Y_n}^\infty\right) \circ \Gamma \circ \left(\iota_A^{(n)} \otimes \iota_B^{(n)}\right), \quad n \in \mathbb{N},$$

and note that $(\mu_{X_nY_n} \otimes p_{\Gamma_n})(G_n) = (\mu_{XY} \otimes p_\Gamma)(\kappa_n)$. Indeed,

$$\begin{aligned}(\mu_{X_nY_n} \otimes p_{\Gamma_n})(G_n) &= \langle 1_{X_nY_n}, \Gamma_n(G_n)\rangle \\ &= \langle 1_{X_nY_n}, (\mathcal{E}_{X_n}^\infty \otimes \mathcal{E}_{Y_n}^\infty) \circ \Gamma \circ (\iota_A^{(n)} \otimes \iota_B^{(n)})(G_n) \\ &= \langle \mathcal{E}_{X_n} \otimes \mathcal{E}_{Y_n})(1_{XY}), (\mathcal{E}_{X_n}^\infty \otimes \mathcal{E}_{Y_n}^\infty) \circ \Gamma(\kappa_n)\rangle \\ &= \langle 1_{XY}, \Gamma(\kappa_n)\rangle = (\mu_{XY} \otimes p_\Gamma)(\kappa_n).\end{aligned}$$

Now since $\Gamma \in \mathcal{C}_{\rm t}(X,Y,A,B) \implies \Gamma_n \in \mathcal{C}_{\rm t}(X_n,Y_n,A_n,B_n)$ (see Theorems 4.5 and 5.8, and Remark 5.15), one obtains $\omega_{\rm t}(\mathcal{G}_n, \mu_{X_nY_n}) = \omega_{\rm t}(\kappa_n, \mu_{XY})$. □

Theorem 6.5. *Let $\mathcal{G} = (\mathcal{G}_n)_{n\in\mathbb{N}}$ be a nested family, where $\mathcal{G}_n$ is a non-local game over (X_n, Y_n, A_n, B_n), $n \in \mathbb{N}$. If ${\rm t} \in \{\rm loc, qs, qc, ns\}$ then*

$$\lim_{n\to\infty} \omega_{\rm t}(\mathcal{G}_n, \mu_{X_nY_n}) = \inf_{n\in\mathbb{N}} \omega_{\rm t}(\mathcal{G}_n, \mu_{X_nY_n}) = \omega_{\rm t}(\mathcal{G}, \mu_{XY}).$$

Proof. Let $\kappa = \kappa_{\mathcal{G}}$ and $\kappa_n = \pi_n^{-1}(\operatorname{supp}\lambda_n)$. We have $\kappa = \cap_{n\in\mathbb{N}}\kappa_n$. Fix $\Gamma \in \mathcal{C}_{\rm t}$ and note that for any $m \geq n$, by monotonicity of the measures as $(\kappa_n)_{n\in\mathbb{N}}$ is a decreasing sequence of sets, one has

$$(\mu_{XY} \otimes p_\Gamma)(\kappa_m) \leq (\mu_{XY} \otimes p_\Gamma)(\kappa_n).$$

By taking supremum over $\Gamma \in \mathcal{C}_{\rm t}$ we obtain that $\omega_{\rm t}(\kappa_m, \mu_{XY}) \leq \omega_{\rm t}(\kappa_n, \mu_{XY})$. By Lemma 6.4,

$$\omega_{\rm t}(\mathcal{G}_m, \mu_{X_mY_m}) \leq \omega_{\rm t}(\mathcal{G}_n, \mu_{X_nY_n}).$$

Again by Lemma 6.4,

$$\omega_{\rm t}(\mathcal{G}, \mu_{XY}) \leq \inf_{n\in\mathbb{N}} \omega_{\rm t}(\mathcal{G}_n, \mu_{X_nY_n}).$$

For the converse, fix $\epsilon > 0$, and let $\Gamma_m \in \mathcal{C}_{\rm t}$ such that

$$\omega_{\rm t}(\mathcal{G}_m, \mu_{X_mY_m}) - (\mu_{XY} \otimes p_{\Gamma_m})(\kappa_m) < \frac{\epsilon}{m}, \tag{37}$$

and assume, without loss of generality, that $(\tilde{\Gamma}_m)_{m\in\mathbb{N}}$ converges to Γ in the BW topology. By monotonicity of the measures again,

$$(\mu_{XY} \otimes p_{\tilde{\Gamma}_m})(\kappa_m) \leq (\mu_{XY} \otimes p_{\tilde{\Gamma}_m})(\kappa_n) \quad \text{for all } m \geq n,$$

and thus, using the convergence of $\tilde{\Gamma}_m$ to Γ, Lemma 6.3, Lemma 6.4 and (37), we get

$$\inf_{m\in\mathbb{N}} \omega_{\rm t}(\kappa_m, \mu_{XY}) \leq (\mu_{XY} \otimes p_\Gamma)(\kappa_n). \tag{38}$$

By the monotonicity of measure and the fact that $\kappa = \cap_{n\in\mathbb{N}}\kappa_n$, we obtain

$$(\mu_{XY} \otimes p_\Gamma)(\kappa) = \lim_{n\in\mathbb{N}} (\mu_{XY} \otimes p_\Gamma)(\kappa_n). \tag{39}$$

Finally, combining relations (38) and (39) one obtains the desired result. □

Our next aim is to provide tensor norm descriptions of the quantum spatial and the quantum commuting value of a Cantor game. Suppose first that $G = (\lambda, \mu)$ is a non-local game over a quadruple (S, T, U, V) of finite sets. The element

$$t_G := \sum_{(x,y)\in S\times T} \sum_{(a,b)\in U\times V} \mu(x,y)\lambda(x,y,a,b) e_{x,a} \otimes e_{y,b}$$

of the (algebraic) tensor product $\mathcal{S}_{S,U} \otimes \mathcal{S}_{T,V}$ is usually referred to as the *full game tensor* of G. It can easily be seen from Theorem 2.1 that

$$\omega_{\rm qs}(\lambda,\mu) = \|t_G\|_{\min}, \quad \omega_{\rm qc}(\lambda,\mu) = \|t_G\|_{\rm c} \quad \text{and} \quad \omega_{\rm ns}(\lambda,\mu) = \|t_G\|_{\max}. \tag{40}$$

Assume that $\mathcal{G} = (\mathcal{G}_n)_{n\in\mathbb{N}}$ is a nested family, where $\mathcal{G}_n$ is a non-local game over (X_n, Y_n, A_n, B_n) with rule function λ_n, $n \in \mathbb{N}$. Letting $\kappa_n = \kappa_{\mathcal{G}_n}$, write

$$t_{\mathcal{G}}^{(n)} = \frac{1}{|X_n||Y_n|} \sum_{(x,y)\in X_n\times Y_n} \sum_{(a,b)\in A_n\times B_n} \lambda_n(x,y,a,b) \chi_{\tilde{x}\times\tilde{a}} \otimes \chi_{\tilde{y}\times\tilde{b}},$$

considered as an element of $\mathcal{S}_{X,A} \otimes \mathcal{S}_{Y,B}$.

Lemma 6.6. *Let $\mathcal{G} = (\mathcal{G}_n)_{n\in\mathbb{N}}$ be a nested family of games.*

(i) *If $\tau \in \{\min, \mathrm{c}, \max\}$ then $t_{\mathcal{G}}^{(n+1)} \le t_{\mathcal{G}}^{(n)}$ in $\mathcal{S}_{X,A} \otimes_\tau \mathcal{S}_{Y,B}$, $n \in \mathbb{N}$.*

(ii) *The limit $\lim_{n\to\infty} t_{\mathcal{G}}^{(n)}$ exists in the weak* topology of $\mathcal{S}_{X,A}^{**} \bar{\otimes} \mathcal{S}_{Y,B}^{**}$.*

Proof. Write $G_n = \mathrm{supp}(\lambda_n)$ for brevity.

(i) Fix $n \in \mathbb{N}$. Since the family $\mathcal{G}$ is nested,

$$\mathrm{supp}(\lambda_{n+1}) \subseteq \mathrm{supp}(\lambda_n) \times \left([d_n^X] \times [d_n^Y] \times [d_n^A] \times [d_n^B]\right).$$

Letting

$$G'_{n+1} = \{(xx', yy', aa', bb') : (x,y,a,b) \in G_n,\\ (x',y',a',b') \in [d_n^X] \times [d_n^Y] \times [d_n^A] \times [d_n^B]\},$$

we have

$$\begin{aligned} & \frac{1}{|X_n||Y_n|} \sum_{(x,y,a,b)\in G_n} \chi_{\tilde{x}\times\tilde{a}} \otimes \chi_{\tilde{y}\times\tilde{b}} \\ & \qquad - \frac{1}{|X_{n+1}||Y_{n+1}|} \sum_{(x,y,a,b)\in G_{n+1}} \chi_{\tilde{x}\times\tilde{a}} \otimes \chi_{\tilde{y}\times\tilde{b}} \\ = \;& \frac{1}{|X_n||Y_n|} \cdot \frac{1}{d_n^X d_n^Y} \sum_{(x,y,a,b)\in G'_{n+1}} \chi_{\tilde{x}\times\tilde{a}} \otimes \chi_{\tilde{y}\times\tilde{b}} \\ & \qquad - \frac{1}{|X_{n+1}||Y_{n+1}|} \sum_{(x,y,a,b)\in G_{n+1}} \chi_{\tilde{x}\times\tilde{a}} \otimes \chi_{\tilde{y}\times\tilde{b}} \\ = \;& \frac{1}{|X_{n+1}||Y_{n+1}|} \sum_{(x,y,a,b)\in G'_{n+1}\setminus G_{n+1}} \chi_{\tilde{x}\times\tilde{a}} \otimes \chi_{\tilde{y}\times\tilde{b}}. \end{aligned}$$

Since $\chi_{\tilde{x}\times\tilde{a}} \in \mathcal{S}_{X,A}^+$ and $\chi_{\tilde{y}\times\tilde{b}} \in \mathcal{S}_{Y,B}^+$, we have that

$$\sum_{(x,y,a,b)\in G'_{n+1}\setminus G_{n+1}} \chi_{\tilde{x}\times\tilde{a}} \otimes \chi_{\tilde{y}\times\tilde{b}} \in (\mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B})^+,$$

and hence

$$t_{\mathcal{G}}^{(n)} - t_{\mathcal{G}}^{(n+1)} \ge \frac{1}{|X_{n+1}||Y_{n+1}|} \sum_{(x,y,a,b)\in G'_{n+1}\setminus G_{n+1}} \chi_{\tilde{x}\times\tilde{a}} \otimes \chi_{\tilde{y}\times\tilde{b}} \ge 0$$

in $\mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B}$. The rest of the conclusions follow from the fact that

$$(\mathcal{S}_{X,A} \otimes_{\max} \mathcal{S}_{Y,B})^+ \subseteq (\mathcal{S}_{X,A} \otimes_{\mathrm{c}} \mathcal{S}_{Y,B})^+ \subseteq (\mathcal{S}_{X,A} \otimes_{\min} \mathcal{S}_{Y,B})^+.$$

(ii) Consider $t_{\mathcal{G}}^{(n)}$ as an element of $\mathcal{S}_{X,A}^{**} \bar{\otimes} \mathcal{S}_{Y,B}^{**}$, $n \in \mathbb{N}$. By (i), the sequence $(t_{\mathcal{G}}^{(n)})_{n\in\mathbb{N}}$ is monotone decreasing and bounded from below (by the zero element). The conclusion is now immediate. □

In view of Lemma 6.6, set $t_{\mathcal{G}} := \text{w}^*\text{-}\lim_{n\to\infty} t_{\mathcal{G}}^{(n)}$, considered as an element of $\mathcal{S}_{X,A}^{**}\bar{\otimes}\mathcal{S}_{Y,B}^{**}$.

Theorem 6.7. *Let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a nested family, where $\mathcal{G}_n$ is a non-local game over (X_n, Y_n, A_n, B_n), $n \in \mathbb{N}$. Then*

(i) $\omega_{\rm ns}(\mathcal{G}, \mu_{XY}) = \lim_{n\to\infty} \|t_{\mathcal{G}}^{(n)}\|_{\max}$;

(ii) $\omega_{\rm qc}(\mathcal{G}, \mu_{XY}) = \lim_{n\to\infty} \|t_{\mathcal{G}}^{(n)}\|_{\rm c}$, *and*

(iii) $\omega_{\rm qs}(\mathcal{G}, \mu_{XY}) = \lim_{n\to\infty} \|t_{\mathcal{G}}^{(n)}\|_{\min} \geq \|t_{\mathcal{G}}\|_{\min}$.

Proof. The statements follow from (40) and Theorem 6.5. □

Example 6.8 (The IID case)**.** Let X_0, Y_0, A_0 and B_0 be finite sets, and $\lambda : X_0\times Y_0\times A_0 \times B_0 \to \{0,1\}$ be a rule function of a game over the quadruple (X_0, Y_0, A_0, B_0). Write $E = \text{supp}(\lambda)$ and, letting $X_n = X_0^n$, $Y_n = Y_0^n$, $A_n = A_0^n$ and $B_n = B_0^n$, $n \in \mathbb{N}$, consider the game over (X_n, Y_n, A_n, B_n), with rule function whose support is the set

$$E_n = \{((x_i)_{i=1}^n, (y_i)_{i=1}^n, (a_i)_{i=1}^n, (b_i)_{i=1}^n) : (x_i, y_i, a_i, b_i) \in E \text{ for every } i \in [n]\}.$$

Let $X = (X_n)_{n=1}^\infty$, $Y = (Y_n)_{n=1}^\infty$, $A = (A_n)_{n=1}^\infty$ and $B = (B_n)_{n=1}^\infty$ be the corresponding inductive sequences, and embed E_n in the first n coordinates, yielding a set $\kappa_n \subseteq \Omega_X \times \Omega_Y \times \Omega_A \times \Omega_B$, $n \in \mathbb{N}$. The sequence $(\kappa_n)_{n=1}^\infty$ is nested and, if $\kappa = \cap_{n\in\mathbb{N}}\kappa_n$, then

$$\kappa = \{((x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty, (a_i)_{i=1}^\infty, (b_i)_{i=1}^\infty) : (x_i, y_i, a_i, b_i) \in E \text{ or } \text{ for every } i \in \mathbb{N}\}.$$

The Cantor game κ encodes the infinite parallel repetition of the game E (that is, the product of infinitely many copies of E).

Example 6.9 (Markov type)**.** We describe a class of examples of non-IID Cantor games, and therefore of games, to which Theorems 6.5 and 6.7 apply. As in Example 6.8, let X_0, Y_0, A_0 and B_0 be finite sets, and $\lambda : X_0 \times Y_0 \times A_0 \times B_0 \to \{0,1\}$ be a rule function of a game over the quadruple (X_0, Y_0, A_0, B_0). Write $E = \text{supp}(\lambda)$ and consider the game over (X_n, Y_n, A_n, B_n), with rule function whose support is the set

$$\begin{aligned} E_n = \{((x_i)_{i=1}^n, (y_i)_{i=1}^n, (a_i)_{i=1}^n, (b_i)_{i=1}^n) &: (x_i, y_i, a_i, b_i) \in E \text{ or} \\ &(x_{i+1}, y_{i+1}, a_{i+1}, b_{i+1}) \in E, \text{ for every } i \in [n-1]\}. \end{aligned}$$

Let $X = (X_n)_{n=1}^\infty$, $Y = (Y_n)_{n=1}^\infty$, $A = (A_n)_{n=1}^\infty$ and $B = (B_n)_{n=1}^\infty$ be the corresponding inductive sequences, and embed E_n in the first n coordinates, yielding a set $\kappa_n \subseteq \Omega_X \times \Omega_Y \times \Omega_A \times \Omega_B$, $n \in \mathbb{N}$. A straightforward inspection shows that the sequence $(\kappa_n)_{n=1}^\infty$ is nested. The corresponding Cantor game has a set κ of admissible quadruples, given by

$$\begin{aligned} \kappa = \{((x_i)_{i=1}^\infty, (y_i)_{i=1}^\infty, (a_i)_{i=1}^\infty, (b_i)_{i=1}^\infty) &: (x_i, y_i, a_i, b_i) \in E \text{ or} \\ &(x_{i+1}, y_{i+1}, a_{i+1}, b_{i+1}) \in E, \text{ for every } i \in \mathbb{N}\}; \end{aligned}$$

heuristically, this means that the rules of the Cantor game require that, in any two individual rounds of the underlying finite game, the players win at least once.

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