Unveiling structure in linear equations: discerning and disregarding critical aspects

Educational Studies in Mathematics
https://doi.org/10.1007/s10649-025-10467-0

Unveiling structure in linear equations: discerning and disregarding critical aspects

Anna Holmlund[1]

Received: 30 May 2025 / Accepted: 19 November 2025

Abstract
Recognizing what is algebraically relevant in equations could potentially help students solve them more efficiently. In a research project involving 16-year-old students, a lesson was planned, analyzed, and modified in three cycles, using variation theory to explore which aspects of linear equations are decisive in making students aware of the structure of the equations. The results highlight the necessity to help students direct their attention to specific aspects of linear equations (e.g., the combination of operations), but also to aspects that are irrelevant for the algebraic structure (e.g., the numbers, the unknown's notation, and their positions relative to the equality sign and to commutative operations). The results contribute to research by highlighting that awareness of algebraic structure is not merely a set of operational skills, but an ability that relates to a specific mathematical content and develops through students experiencing critical aspects of that content.

Keywords Linear equations · Algebraic structure · Structure sense · Variation theory · Algebra education

1 Introduction

The difficulties for novices in algebra in identifying important features in equations have been known for a long time (Kieran, 1989). For example, Xie and Cai (2022) found that many students (aged 11–12) solved $9-(\cdot)=5$ correctly, but fewer succeeded with $9-x=5$, indicating they may interpret $(\cdot)$ and x differently. In another study, students (aged 16) were more often able to solve $819=39\cdot x$ using a calculator, compared to equations with decimal numbers such as $0.657=0.045\cdot x$ and $0.12=0.4\cdot x$ (Holmlund, 2024). These results suggest that students need support in identifying structural features within equations.

Although prior research highlights the importance of recognizing structure (Hoch & Dreyfus, 2004), studies on algebra structure sense rarely emphasize the algebraic content—the mathematical elements in an equation, such as operations, variables and equality— in

✉ Anna Holmlund
anna.holmlund@gu.se

[1] Department of Mathematical Sciences, University of Gothenburg, Chalmers University of Technology, Gothenburg, Sweden

other words, *what* students must discern to identify algebraic structures. Moreover, there is limited knowledge on how to foster this ability. The aim of this paper is therefore to explore the relationship between students' recognition of structure and the algebraic content. This is done using the setting of a learning study (Kullberg et al., 2024), a design research process using variation theory. The findings provide insights regarding algebra structure sense and guide the development of teaching practices in algebra.

2 Literature review

Before reviewing literature on how students become aware of algebraic structures—especially *what* they need to discern and the role of algebraic content—it is important to consider the different meanings of the term *structure* in relation to equations. If equations are equivalent, they share a *systemic structure* (Kieran, 1989). Thus x + 5 = 11 has the same structure as $x = 6$, but not as $x + 5 = 12$, and there is an infinite set of equivalent equations belonging to a systemic structure. The numbers play a central role in deciding which equations share systemic structure. By contrast, the medieval Arabic mathematicians generalized equations beyond the numbers and only discussed *one* structure of first-degree: $bx = c$ (Oaks & Alkhateeb, 2007), as all linear equations can be simplified to this structure. Likewise, Hoch and Dreyfus (2004) define *algebraic structure* as an internal order in an algebraic expression, referring to the relationship between the parts of the structure, that is, either apparent or can be revealed by transformations. In this article, the term *structure* refers to this latter definition. Also, the mathematical properties of linear equations can be considered on different levels in a parsing tree: seeing them all as one structure $ax = b$ or distinguishing between equations with different operations (e.g., $ax = b$ and $a + x = b$).

What is seen in an algebraic expression can depend both on the context, and on what the observer is able and prepared to perceive (Sfard & Linchevski, 1994). Recognizing structure presupposes knowledge about concepts, syntax, and procedures. For example, inability to handle equations as similar can be due to experiencing the symbol x as prompting a certain formal method (Xie & Cai, 2022), or perceiving decimal and negative coefficients in linear equations as complying with separate rules or difficult to align with concrete representations of the equation (Holmlund, 2024; Vlassis, 2002). Moreover, Sfard and Linchevski (1994) describe how a novice was "baffled" by $112 = 12x + 47$ after solving $7x + 157 = 248$, attributing the confusion to the student interpreting the equal sign as a prompt for action.

The fact that recognizing structure involves more than conceptual understanding of the components is hinted at when some researchers mention terms like intuition (e.g., Hosh & Dreyfus, 2004). Otte (2006) emphasizes that a relational expression must be evaluated by "visualizing" how the parts interact; if all details of the involved concepts were considered individually, it would be impossible to identify similarities "at a glance" since the difficulty lies in how they relate, rather than in their meanings. Thus, the *interactions* between objects seem more important for recognizing algebraic structure than focusing on the objects in isolation. What it entails to recognize structure is sometimes discussed under the term *structure sense* (Linchevski & Livneh, 1999) or *algebra structure sense* (Rojano, 2022). The definitions of these terms usually involve the following operational abilities: seeing an expression as a compound term, dividing an entity into substructures, recognizing structures as familiar and seeing connections between these, and recognizing possible and useful transformations (Hoch & Dreyfus, 2004, 2006). However, these definitions do not clarify *what* content students should recognize.

A review of articles on structure sense, focusing on *what* students should look for in equations and algebraic expressions, shows that the mathematical content is usually limited to specific kinds of structures, for example linear and quadratic equations (English & Sharry, 1996). However, *what* students should look for in the mathematical content to identify structure and solve equations is not always explicitly stated. A look at the tasks used in different studies shows that algebraic expressions are often formulated so that students should identify recurring visual features, for example similar expressions (sometimes in parentheses), numbers that share a factor, or other common characteristics, such as being squares (Hoch & Dreyfus, 2004, 2007; Maffia et al., 2025; Schüler-Meyer, 2017). Equations where similar parentheses can be added or subtracted are sometimes also used in items when measuring students' flexibility (Star & Rittle-Johnson, 2008). Occasionally, equations and expressions employed to assess students' structure sense include fractional expressions (Rüede, 2013), where the focus is on identifying or transforming them to reveal recurring visual features in numerator and denominator. Kirshner (1989) found that recognizing the spacing between symbols (e.g., the greater distance between a and b in $a + b$ than in ab, and a^b) is one aspect of the mathematical content that students use to identify operations. Studies on initial algebra learning differ partly in *what* content students should focus on to see structure. For example, understanding the minus sign as part of a term is emphasized (Linchevski & Livneh, 1999; Vlassis & Demonty, 2022), as is recognizing common factors in products and applying the order of operations by distinguishing between constant terms, product terms, and bracketed terms (Banerjee & Subramaniam, 2012). A few studies also discuss the relevance of *what not* to see to discern structure (e.g., English & Sharry, 1996).

In several of the studies mentioned above, students' abilities to look for structure *within* equations—such as finding similar expressions within parentheses or in a fractional expression—are assessed, thus not comparing the algebraic structure *between* equations. Alternatively, a study by English and Sharry (1996) examined what features students look at when sorting first- and second-degree equations. Besides identifying which features the students looked at (e.g., the use of different letters for the unknowns, one or more unknowns, one side with only a number, denominators with or without variables), they also observed that the high-achieving student looked more for features related to the equations' structure.

Most researchers agree that a student can be trained to discern structure (Hoch & Dreyfus, 2004; Rüede, 2013; Schüler-Meyer, 2017). Rüede (2013) found that novices tend to focus on visual similarities in algebraic expressions, whereas more experienced mathematicians access a broader range of interpretations. To help students develop this ability, teaching can involve comparing symbolic expressions with numerical equations, supporting them in seeing and assessing structural similarities (Muñoz-Porras & Xolocotzin, 2022; Schüler-Meyer, 2017; Tuominen et al., 2020). Giving students clues on how to achieve a goal (Janßen & Bikner-Ahsbahs, 2013) or making a structure explicit by giving it a name (Hoch & Dreyfus, 2010) can also help them to see structure and recognize it in new situations. Another activity that promotes recognizing structure is the practice of decomposing and recomposing expressions using operations (Kieran, 2018; Linchevski & Livneh, 1999; Venkat et al., 2019).

Researchers highlight the importance of integrating awareness of structure into teaching (Hoch & Dreyfus, 2004; Rüede, 2013), which could potentially improve equation solving. However, how students develop it and how it should be taught remain unclear.

To address this gap in research on algebra structure sense—*what* students must discern and how it can be taught—a learning study (Kullberg et al., 2024) has been employed. This is a form of collaboration between teachers and researchers where teaching is designed, analyzed, and improved in successive cycles. It aligns with design-based research (as described by Plomp, 2013, p. 20), but what makes learning study unique is its exclusive focus on *what* is to be learnt, with variation theory commonly used to study it. The question investigated in this paper is: In what ways is perceiving algebraic content relevant for discerning algebraic structure?

3 Theoretical background

What in some traditions is called *knowledge* is in variation theory understood as available ways of perceiving *something*. Thus, knowing negative numbers is not a mental state, but an ability to discern their properties, for example that adding negative numbers yields a sum smaller than each term. Variation theory describes learning as a relation between the learner and the world, where the world is richer than our perception of it. People do not perceive the world as it *is*, but as they experience it—shaped by what they are able to discern. Thus, learning occurs as students differentiate new aspects of a concept—wherefore teaching should provide opportunities to recognize variation in those aspects (Marton, 2015).

An act can never be detached from *content*. We cannot learn to "explain" without explaining *something*. Variation theory provides tools for investigating an act on a specific content called *the object of learning*. To master the object of learning, a student must discern some critical aspects simultaneously (Marton, 2015). Critical aspects are relational in nature; they could not be solely derived from disciplinary knowledge but are found in the ways students experience an object of learning, in the aspects that need to be discerned to see it in a more powerful way (Pang & Ki, 2016). Marton (2015) describes how critical aspects may not only concern aspects of the object of learning but can also involve non-necessary aspects, also called irrelevant aspects. He gives the example that while the rotation of a triangle is irrelevant for defining what a triangle is, learners still need to recognize that aspect as irrelevant. Differentiation is thus also essential for distinguishing an object from what it is not. According to variation theory, all learning is about being aware of something. In this study, the act on the object of learning is also to *be aware*, specifically, of the structure of linear equations. So, this investigation—analyzing how students make shifts in their awareness of the structure of equations depending on some critical aspects—revolves around the visual attributes of the linear equations and how students identify their structure. Using a theory that puts focus on the algebraic content will give insights into how the content is relevant when students discern structure.

To discern a critical aspect in an object (e.g., an algebraic expression), the learner first needs to perceive a *contrast* within that specific aspect while keeping the object's other aspects constant, thereby separating that critical aspect from everything else (Marton, 2015). Further, for a critical aspect to be differentiated from its context, it needs to be displayed in different settings while keeping the value of the critical aspect constant. This pattern of variation is called *generalization*. For students, these patterns of variation (contrast and generalization) can be shown deliberately in teaching, or through recalling previously met examples that can be compared with the current content.

4 Method

Three teachers from different upper secondary schools, in two municipalities in western Sweden, participated in the project, meeting digitally with the researcher nine times. They were selected because they all taught the same course in the electricity program for 16-year-old students. As solving linear equations is mandatory in lower secondary school, all participating students ($n = 45$) were expected to have some prior knowledge. Written informed consent was obtained from students and teachers. The project followed the standard process of a learning study (Kullberg et al., 2024). After investigating students' prior knowledge (pretests and five interviews), each cycle involved: (a) planning a research lesson together, that one teacher then taught in her class while video recording the classroom and one or two groups, (b) assessing learning, and (c) refining the lesson based on the analysis before starting a new cycle in another class (see Fig. 1). In this way, the learning study provided a useful methodology for combining teacher's professional knowledge with research to create opportunities for students to identfy structures in linear equations.

Both pretest ($n = 40$) and posttest ($n = 37$) were conducted on paper. Sample sizes differ due to student absences. The first task, identical in both tests, was used as an indicator of how well the students identify structure. It asked students to "sort the equations you think 'belong together' and resemble each other. Encircle the ones you think belong together and mark the first group as A, the second as B and so on" (see Table 1). This was followed by a question asking why those equations belong together. Thus, information was collected both as data on what equations students combine as similar or not (quantitative data further described in the analysis), and as descriptions in their justifications (used in the results to support the qualitative analysis). The choice of linear

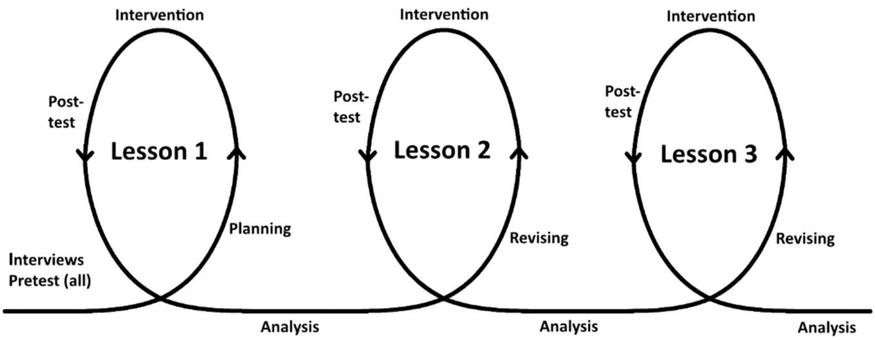

Fig. 1 Overview of the cycles in the learning study

Table 1 Equations used in the pre- and posttest

$352 = 63 + x$	$a = b \cdot x$	$0 = 0.8 \cdot x + 2.1$
$39 \cdot x = 819 + 3 \cdot x$	$0.4 \cdot x = 0.12$	$2.01 = 0.434 + x$
$0.657 = 0.045 \cdot y$	$-24 = 6 \cdot x$	$73 + y = 0$
$1.7 + 5 \cdot x = 2 \cdot x + 0.8$	$37 = 7 \cdot x + 2$	$3 \cdot y = 4$

equations in the task on the test was inspired by earlier studies, for example, varying the letter and including a zero (English & Sharry, 1996), decimal numbers with many digits (Holmlund, 2024), and including a negative number (Vlassis, 2002). The equations were also adjusted to vary in other aspects (for a detailed description of the design of the task and its potential, see Holmlund and Pejlare (2025)).

In the lesson, the word *structure* was not used, as we deemed the term as too formal and rigid to present it to the students. Instead, we used the term “type of equation” to indicate that some equations share the characteristics of similar internal order (i.e., algebraic structure). To direct students’ attention to different ways of combining operations in linear equation, we decided to introduce three “types of equations”: additive only, multiplicative only, and combined (addition and multiplication).

A lesson plan outlining teacher instruction was created and improved through three cycles, initially based on potentially critical aspects found in previous research, student interviews, and the pretest. The lesson mainly consisted of two tasks created by the team: *Sticky notes equation* and *Discussing claims*. In the Sticky notes equation task, using the pattern of *contrast* from variation theory, students were presented with pre-prepared equations differing only in the operations ($72 = x + 6$, $72 = 6 \cdot x$ and $72 = 6 \cdot x + 6$) to highlight that various structures of equations differ in the combination of operations. These were first discussed in groups of three, before the teacher made the difference in operations explicit. After this, the teacher pointed to the multiplicative equation, asking the student groups to create an equation of the same type. To do this, each group received a white sheet of paper with colorful sticky notes, marked with numbers, operations, equality signs, and unknowns in respectively green, orange, yellow, and purple. The equations created by the students were posted on the whiteboard, and the class discussed whether they matched the requested type (providing a *generalization* of equations, varying in many aspects but sharing the same combination of operations). Students were then asked to create an equation of another type, as the teacher pointed to one with both addition *and* multiplication. After that, all groups used calculators to solve one such equation, highlighting that equations of the same type are solved similarly. Following this, in the task Discussing claims, the student groups discussed and took a stance on five claims (distributed on a paper with one green and one red side) concerning whether two equations were of the same type (green) or not (red), see Fig. 3. The tasks were altered a little through the three cycles as shown in Table 2.

Since the learning object is to “identify the structure of linear equations”—not primarily to solve or understand the equation but to recognize its appearance—the analysis focused on examining instances in the data when students’ way of recognizing structure changed. Through the process, some aspects were formulated as potentially critical, at first based on previous research. These were designed to vary in the lesson, and the potentially critical aspects were then revised. The critical aspects were thus not predetermined but emerged as critical when the teachers and researcher found examples of how that aspect influenced students’ awareness of structure. The first part of the analysis was done by the team before, in between, and after the cycles of the learning study. In these analyses, the team reviewed students’ answers on the tests and video recordings of the teaching, searching for instances of when students changed in their awareness of structure. When such a change was found, it was noted which properties of the linear equation and its parts caused the registered change. After all data was collected, the researcher returned to the material to find support for the critical aspects found by the team and refine how they were formulated. Sequences from the data were then discussed with another researcher, and the most descriptive cases were chosen for display in the result section. The translations were

Table 2 Changes in the tasks between the three cycles

Task	Lesson 1	Lesson 2	Lesson 3
Sticky notes equation task, part 1	Students created *any* equations they wanted using sticky notes	The teacher asked for a specific "type of equation"	Same as in lesson two
Sticky notes equation task, part 2	All equations were compared with each other on the board	Equations of *one* type at a time were compared	First, two equations were compared, then all of that type were included
Sticky notes equation task, part 3	Students never received answers about "the right way of sorting"	The types of equations were named and a list of what is irrelevant for how an equation is solved was made on the board	Same as in lesson two
Discussing claims	Not used	The claims were discussed both in groups and in the whole class	Same as in lesson two
Additional tasks	Finishing the lesson, the teacher solved two equations on the board	–	In the beginning, a seemingly challenging equation with tricky numbers was shown to engage the students

made in the final editing and were made verbatim as long as the intention of the student was maintained.

To assess students' ability to recognize algebraically similar equations, two pairs of equations that were of the same structure but different in several irrelevant aspects were selected from the pre- and posttests: $3 \cdot y = 4$ with $-24 = 6 \cdot x$, and $3 \cdot y = 4$ with $0.4 \cdot x = 0.12$. The number of students identifying these as similar before and after the lesson were then compared. Some students were excluded due to missing the pretest, posttest, or research lesson, resulting in a relatively small sample size ($n = 33$).

5 Results

First, the five critical aspects influencing whether students identify structure are presented, followed by the analysis of the pre- and posttest.

5.1 Critical aspects for discerning structure

Here, each critical aspect (CA) is described, and examples from the data are given to support the claim that the aspect can be essential when students identify structure.

5.1.1 CA1: The combination of operations in an equation shows what structure the equation has

This entails that, when students look at the operations in equations, particularly their combination, they will identify equations that can be solved in a similar way (i.e., generalizing over numerical aspects).

This was first formulated as a tentative CA based on previous research. The pretest confirmed that some students do not naturally focus on operations. When asked to encircle equations that resemble each other and justify why, some students did not look for the combination of operations. For example, in Fig. 2, one student explained that equations labelled (a) include numbers less than one, (b) have x as variable, (c) have y as variable, and (d) consist of only variables.

Further support for the combination of operations being a CA can be seen in the following transcript from lesson 2, where students are asked to consider the statements displayed in Fig. 3.

Transcript 1
Bo: It doesn't look that way…
Bo takes the calculator to solve one of the equations.
Bo: 10000*x* … [preparing to calculate].
Carl: It's the same formula, they just don't have the same answer.
Bo pauses to examine the equations.
Bo: Yes, it is.
Carl: Yes.
Bo: It's the same type of equation.
Carl: Then it's green.

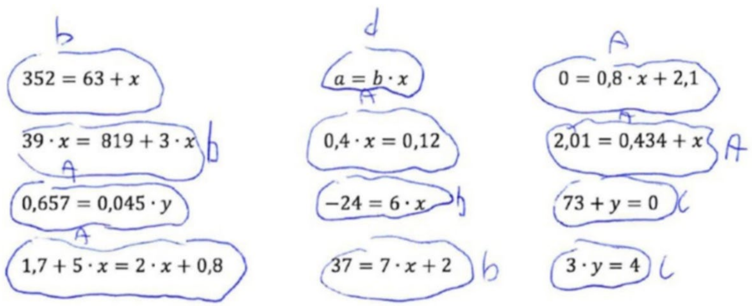

Fig. 2 A student's encircling of which equations are similar to him

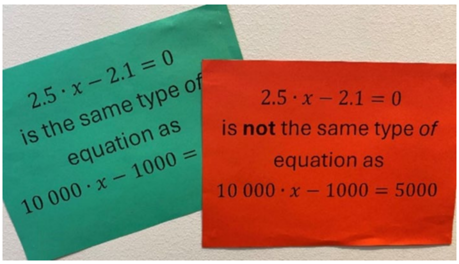

Fig. 3 Statements discussed

The transcript reveals Bo's changing awareness of structure. From first wanting to solve one equation, he perceives *something* that alters his action—from computing to agreeing that the equations are of the same type. This happens as Carl draws Bo's attention to the operations by using the word *formula*, a term that highlights similarities in the combination of operations. This supports the argument that awareness of structure concerns *what* we look at in the appearance of linear equations, and that directing students' attention to the CA of "the combination of operations" can change their way of seeing the equations. When the teacher afterwards asked the whole class why it is the same type of equation, Bo said "it's the same type of calculation."

In Transcript 1, a tension arises between comparing the systemic structure of an equation (i.e., whether they are equivalent) and their algebraic structure. This is illustrated when Bo shifts from wanting to compare the solutions to comparing the formulas (i.e., combination of operations).

5.1.2 CA2: The numbers have no bearing on the structure of the equation

When identifying different types of linear equations, aspects relating to the numbers in the equation (i.e., the specific number's attributes, like the number domain or the representation used) are not relevant to what structure the equation has. It is therefore important that students see numbers as a non-relevant aspect to generalize the structure beyond it.

In the pretest, numbers came to the fore of some students' awareness as an aspect that makes equations similar. As an example, one of the groups in Fig. 2 was formed based on the criterion that the equations include numbers less than one. Another student explained that since $-24 = 6 \bullet x$ has a negative number, it must be calculated and thought about in a different manner. Numbers were in various ways explicitly mentioned by nine of 40 in the pretest as a feature that makes equations similar, but implicitly also in the encircling by others. Thus, students' attention when looking at equations can be guided by the numbers, either the numbers' visual features or their properties.

Support of the argument that students can become aware of the algebraic structure of linear equations by attending to CA2 can be seen in transcript 2, from the third lesson. In groups of three, the students are about to create an equation of the same type as $72 = 6 \bullet x$, using sticky notes. The students have just identified multiplication as the operation of the equation and put down "0.53" as a start, while discussing how to proceed:

Transcript 2
David: Let's see all the numbers.
David puts all sticky notes on the table and looks at the numbers. They have −100, −12, 0.004, 0.1, and 0.1212.
David: We can do like this. [adds a sticky note so it says"0.53 ="].
Fredric: Equals to 0.004*x*, or?
Eric: Yes, sure.
David: Does it exist?
Fredric: Does it exist? I don't know. Because it's the decimal numbers that are confusing.
David: To be safe, we take this one. [taking 0.1, creating $0.53 = 0.1 \cdot x$].
Fredric: Oh my…
Eric: Just hope we don't have to solve this equation.
David: But it's an easy one. It is 500 something…
Eric: It is 530.
Fredric: Even if it might not be correct, at least we're doing an equation.

Overall, there are several indications that the students find the numbers to be an important feature when creating the equation. First, the transcript displays how they do not consider the negative numbers but choose the decimal numbers. Secondly, they express that decimal numbers—especially 0.004—are confusing, and they end up with choosing the number with the least number of digits.

The varying degree of attention to the numbers also contributes to whether the students focus on the algebraic structure of the equation. Initially, the students' focus is on comparing the new equation they are creating with the one on the board ($72 = 6 \cdot x$), identifying its structure as multiplicative and thereby considering the algebraic structure. Then, at the start of Transcript 2, their focus is no longer primarily on the algebraic structure, but on the numbers *within* the new equation. David asks if "it exists" with a concern for the validity of the new equation's equivalence. After deciding to go with a simpler number (0.1), they again switch focus to the algebraic structure as Fredric notes that their solution "might not be correct" but they have formulated an equation.

5.1.3 CA3 (a special case of CA1 and CA2): The minus sign contributes to the equation's structure when it is an operation, not as a part of a negative number

The minus sign in front of a coefficient for x indicates a negative coefficient (which is not critical for the structure CA2), but in front of a constant term, it indicates the operation subtraction (i.e., reversed addition) which is critical (CA1). It is therefore essential for students to discern the role of the minus sign when they examine the structure.

An example comes from the second lesson, where the teacher has just led a discussion on what does *not* matter when solving equations and "negative or positive numbers" is one of the points written on the whiteboard. In the following transcript, three students are discussing whether the statement "$0.15555 \cdot x = 0.6578533$ is the same type of equation as $-44 \cdot x = 30$" is true or not.

Transcript 3

Henric points with his pencil, first at 0.15555 and then at −44.

Henric: The same, doesn't matter – negative or positive doesn't matter [compares the numbers] and then it is times x [compares the operations] and there is 'equal to'. So, it should be the same, right?

Gabriella: Okay, so that's how you're thinking.

Henric: Or?

Gabriella: However, there is a negative number there [−44].

Henric: Does it matter if it is negative?

Henric points at the board where it says that negative/positive numbers do not matter.

Isac: You're thinking it's minus there.

Henric: I think that it doesn't have to [inaudible]. It's the same computation. Though, it does actually become…

Isac tries to understand Henric and points comparingly at 0.15555 and −44. Meanwhile, Henric changes focus:

Henric: Though, it's another computation for this [shifts to seeing −44 as subtraction]. There, we add 44 on both sides [points at −44 and 30] and there, we subtract 0.657 [points at $0.15555 \cdot x = 0.6578533$].

Gabriella: That is true.

Henric: I don't know, it feels… It says there that negative numbers don't matter [points at the board].

The group thinks briefly while the teacher asks them to make their decision.

Henric: I say red.

Isac takes the paper with the statements and holds up the red side.

In this paragraph, we can see that the way the minus sign is discerned changes the way that Henric and Gabriella see the structure of the equation. Henric first has his attention on the different parts of the equations being similar. Both Henric and Gabriella can see that -44 represents a negative number. However, after subtraction is introduced by Isac, Henric's awareness of the minus sign shifts into seeing the minus sign as subtraction and makes him unable to see the multiplicative structure. Still, he tries to combine the two ways of experiencing -44 when he reads from the whiteboard that the sign of the numbers does not matter. To see the similar algebraic structure, he would need to see the negative number as multiplied by x while simultaneously disregarding the meaning of the minus sign as subtraction. Unexpectedly, Henric also states that $0.15555 \cdot x = 0.6578533$ would call for subtraction—in contrast to his previous reasoning, his focus is now on one of the positive coefficients in the equation (i.e., $+0.6578533$) overlooking the operation.

Initially in this transcript, Henric displays his comparison of algebraic structures not only in words—but also by his physical movements. He first points to one part of the equation, then to the corresponding part in the other equation, repeating this process for different components. A similar movement is done by Isac when comparing 0.15555 and -44 (see Fig. 4). Even though the students turn to considering the minus sign as subtraction, they keep their attention on comparing the two equations.

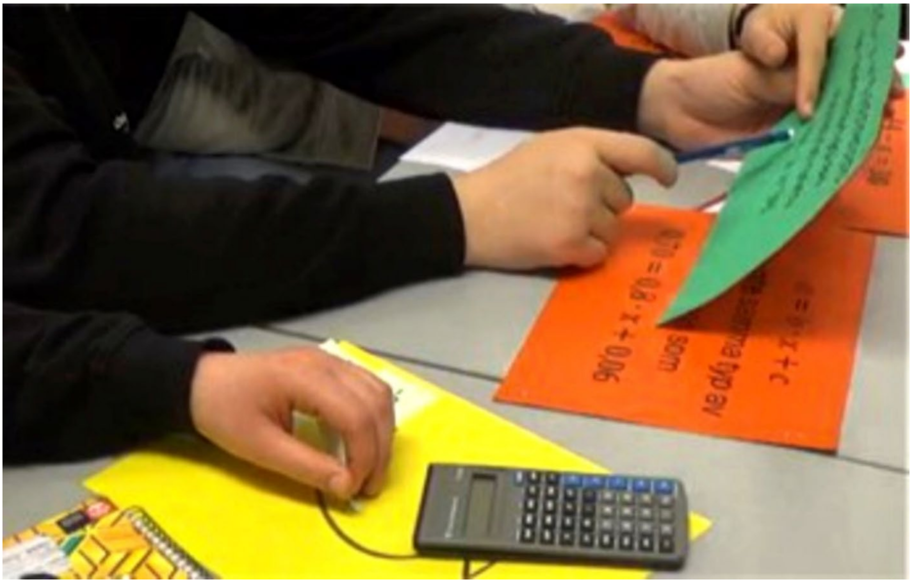

Fig. 4 Henric and Isac points at -44

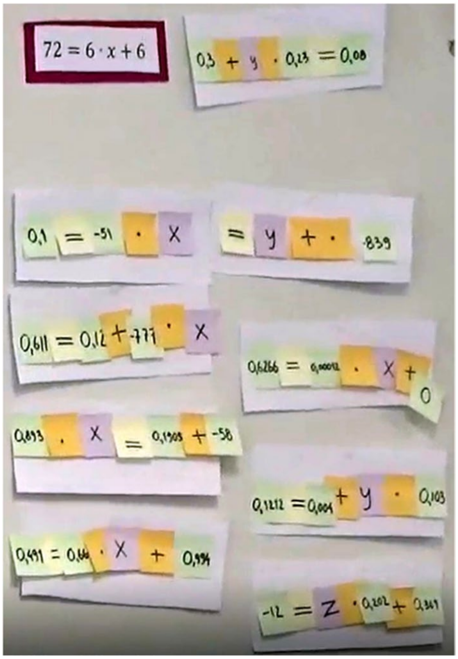

Fig. 5 Equations created by students in the third cycle of the lesson

Another sequence that displays this CA comes from the third lesson when the students have constructed equations of the same type as $72 = 6 \cdot x + 6$ and the teacher has put them on the board (see Fig. 5). The class agrees that the sticky notes forming "$= y + \cdot - 839$" do not follow the same pattern but disagrees on whether $0.1 = -51 \cdot x$ does. One student explains that it "has minus 51. You must add 51." Also here, how the minus sign is viewed impacts how the structure is perceived. The teacher then creates a pattern of contrast between $0.1 = x - 51$ and $0.1 = -51 \cdot x$, keeping the numbers the same, but changing the role of the minus sign.

5.1.4 CA4: The symbol for the unknown has no bearing on the structure of the equation

The algebraic structure is independent of the choice of symbol used for the unknown number. Therefore, it is essential that students see similar or different letters in two equations (e.g., x, y, or z) as an irrelevant feature when comparing them with each other.

This CA was tentatively formulated based on previous research. In the pretest, some students referred to the choice of letters, explicitly for five of the 40 students, and implicitly for others. For example, one student described $0.657 = 0.045 \cdot y$, $73 + y = 0$, and $3 \cdot y = 4$ as similar since the variable is y, not x.

CA4 was not discussed extensively in the classes, but in Fig. 6, we can see that for all the multiplicative equations the students created in the second lesson, they choose x rather than y or z. Admittedly, x was on top of the pile of the sticky notes they received. Still, using another letter seems to be something they paid attention to. The following transcript

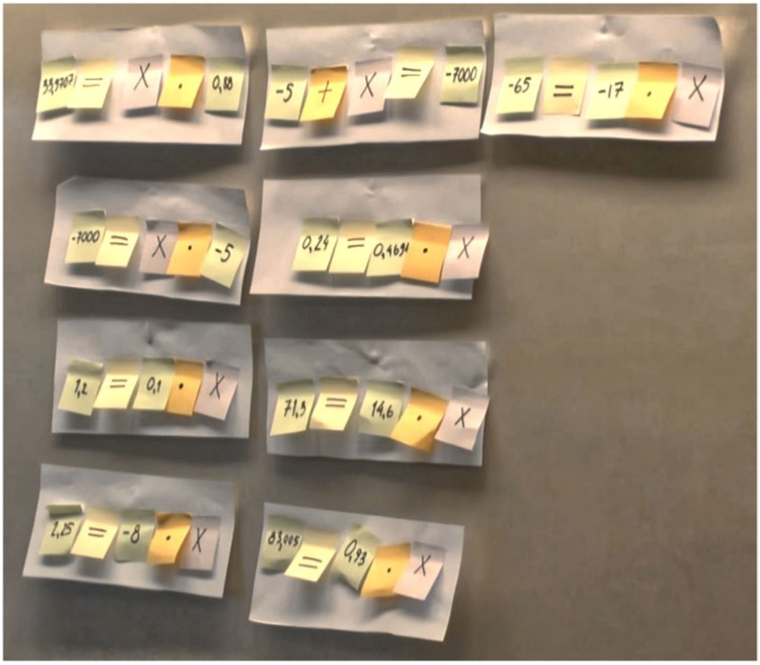

Fig. 6 Equations created to be of the same type as $72 = 6 \cdot x$

displays the discussion between three students a bit later in the lesson when they were asked to make an equation of the same type as $72 = 6 \cdot x + 6$. The group has just put two sticky notes on the white sheet, yielding “63.7 =”.

Transcript 4

Isac: Should we take *y*? It’s the same.

Henric: It’s the same.

Gabriella: Doesn’t matter.

Henric: There are two here. [He has discovered the note with *z*].

Henric: Should we go with *z*?

Henric puts *z* and when their equation $63.7 = 0.32z + 0.6583$ is put on the board, it is the first equation where x is not the unknown. Thus, some students comment:

S1: There’s a *z*!

S2: You took *z*?

S3: *z*?!

Isac: It’s the same as *x*.

The notation of the unknown number seems to be a feature that students look at (based on the answers in the pretest) but is also, as seen in transcript 4, an issue that raises students’ concern for the syntactic rules of algebraic expressions. However, when the students discern this CA, that the type of symbol for the unknown is irrelevant, they can proceed to look at the algebraic structure of the equations.

5.1.5 CA5: Equations with unknowns and coefficients in different positions can still have the same structure

Positions of coefficients and unknowns in the symbolic expressions came to the focal awareness of some students as an important feature on different occasions in the project and sometimes hindered students' ability to discern similar structures in equations. However, the mathematical properties allow reordering an expression while keeping the same meaning: multiplication and addition are commutative for real numbers and the equality is symmetric (e.g., $3 \cdot x = 4$ has the same structure as $36 = x \cdot 9$). Therefore, it is essential to help students to not primarily look at the positions of the unknown and the coefficients in the symbolic expressions, but at their relations.

In the pretest, position was explicitly mentioned when indicating similar equations by four of the 40 students. For example, having the expression with x or a single number on the same side of the equals sign (e.g., $352 = 63 + x$ and $0.657 = 0.045 \cdot y$ "has only one number left of the equality sign") or indicating specific positions, like the equations "ending" with a number or an unknown. For some other students, position was not explicitly mentioned but could be seen as a feature taken into account when grouping equations (see Holmlund & Pejlare, 2025).

This CA, with the commutativity of multiplication, appears in the first lesson during the Sticky notes equation task. The whiteboard shows three numbered groups that include (though it is not explicitly stated) equations with only multiplication (group one), with both addition and multiplication (group two), and with only addition (group three). The teacher now asks the class where on the board to put the student-created equation $36 = x \cdot 9$:

Transcript 5

John:	I want to say that that one is not right.
Teacher:	Where do you want it?
Kevin:	There, in three [equations with addition].
Teacher:	Why?
John:	But you've mixed up the digits and the numbers!
Leo:	It should be in three, three!
Teacher:	Do you want this in three?
John:	It should be in the first [equations with multiplication].
Max:	It should not be in the third, it should be in the first.
John:	Good.
Teacher:	Why?
Max:	Cause 5 times x, this is 9 times x [compares with the equation that is on top of the board in group 1: $37 = 5 \cdot x$].
Leo:	Aha, you just turned it [sees that $x \cdot 9$ can be written as $9 \cdot x$].

The equations in the group of additive equations present x to the left of the constant, as in $72 = x + 6$. Positioning the unknown preceding the number seems to be the reason why some students want to place the multiplicative equation in the group of additive equations. This can be seen in the comment from Leo, first claiming that the equation should be in the third category but then realizing the property of commutativity for multiplication.

Examples of students' need to discern the order of the sides in an equality as not relevant for the algebraic structure can be seen in the Sticky notes equation task. As shown in Fig. 6, most student-created multiplicative equations in the second lesson placed the x-term on the right—mirroring the example $72 = 6 \cdot x$ even though this is a less common way to write an equation. When later asked to create equations with both addition and multiplication (e.g., $72 = 6 \bullet x + 6$), students placed x on the right side for all equations; wherefore, it seems this dimension was not varied enough in the task to make students discern the symmetric property of equality. However, later in the same lesson, when discussing features that do not matter when solving equations, one student remarked that it does not matter which side of the equals sign an expression is on, acknowledging the symmetry of equality.

5.2 Change in awareness

Comparing the first task in the pre- and posttest in Tables 3 and 4, there are indications of improvement in students' ability to group multiplicative equations as similar. These tables suggest that students find it harder to see similarity between two multiplicative equations when one includes a negative number than when one includes decimal numbers.

By examining the change in proportions of students' abilities to classify multiplicative equations as similar (the fourth columns in Table 3 and 4), the calculation of approximate 95% confidence intervals for the difference in proportions based on paired samples from all students, $n = 33$, (see Table 5) was calculated. They suggest that the difference for the entire group, from pretest to posttest, lies between 11 and 55 percentage points (seeing $3 \cdot y = 4$ and $-24 = 6 \cdot x$ as similar), respectively, between 1 and 42 percentage points (seeing $3 \cdot y = 4$ and $0.4 \cdot x = 0.12$ as similar). The results suggest that it *is* possible to change the way students perceive linear equations as similar or not, and that systematic variation of the CAs displayed is a powerful way to do it.

Table 3 Proportion of students that encircle $3 \cdot y = 4$ and $-24 = 6 \bullet x$ as "similar"

Cycle	Pretest	Posttest	Difference
Lesson one (n=6)	33%	33%	0
Lesson two (n=17)	41%	82%	41%-points
Lesson three (n=10)	30%	70%	40%-points

Table 4 Proportion of students that encircle $3 \cdot y = 4$ and $0.4 \cdot x = 0.12$ as "similar"

Cycle	Pretest	Posttest	Difference
Lesson one (n=6)	50%	33%	– 17%-points
Lesson two (n=17)	59%	82%	24%-points
Lesson three (n=10)	40%	80%	40%-points

Table 5 Discordant pairs table for circling $3 \cdot y = 4$ as "similar" to $-24 = 6 \cdot x$, respectively, to $0.4 \cdot x = 0.12$

Pretest	Posttest		Pretest	Posttest	
	Not similar	*Similar*		*Not similar*	*Similar*
Not similar	7	14	*Not similar*	6	10
Similar	3	9	*Similar*	3	14

6 Discussion

This study set out to explore in what ways perceiving algebraic content is relevant for discerning structure—an issue that has been rather implicitly treated in previous research on algebra structure sense. Through a learning study, five critical aspects of linear equations were identified. These aspects—influencing whether students perceive structure or not—encompass *several* components of the equation and involve how the parts *relate* to the structure. They include both content to discern and content to disregard.

Algebra structure sense has previously been characterized as a set of abilities described in operational terms (Hoch & Dreyfus, 2004), with limited attention given to the content of algebraic expressions. The findings of this study add to this body of research by demonstrating that recognition of structure depends on *how* the components of algebraic content are perceived as contributing to the overall structure. Students' various ways of perceiving the components of an equation have been documented before, such as considering the notation of the unknown as relevant (Xie & Cai, 2022), detaching the minus sign from the intended terms (Linchevski & Livneh, 1999) or seeing the equality sign as a prompt to do something (Sfard & Linchevski, 1994). In this study, concepts like negative numbers and variables become topical again, but now in the context of whether they have bearing on the structure or not.

The results of this study call for a broader view of what visual features to attend to in algebraic expressions—such as similar combination of operations—and thus complements prior research that has largely focused on other features, like similar numbers or parentheses. Only one of the critical aspects found in this study is inherent to linear equations: the combination of operations (CA1), whereas the other four can be called irrelevant aspects (CA2–5), as they do not have bearing on the structure: the choice of numbers (including negative numbers), the symbol for the unknown, and the position of the unknowns and coefficients. Irrelevant or non-necessary aspects are described by Marton in a general context (2015, p.33) as aspects that need to be separated from the learning object and thus generalized across. This broadens the question of *what* aspects of linear equations students need to attend to when developing awareness of algebraic structure, by raising the question of what they should *not* attend to. Discussing that awareness of algebraic structures involves both attending to and intentionally *not* attending to (i.e., generalizing) certain aspects can be compared to Kieran's (2018) argument that there is a dual face of activity that promotes algebraic thinking: seeing structure (e.g., interpreting $0.4 \cdot x = 0.12$ as a multiplicative relation) and generalizing (seeing it as an instance of $a \cdot x = b$ where $a, b \in \mathbb{R}$). In this sense, the result of this study shows that awareness of algebraic structure involves learning to discern certain aspects of the content, while keeping others out of focus in one's awareness.

This study's focus on a context where numbers should be ignored to make algebraic structure visible (CA2) diverges from other studies on structure sense that promote analyzing numbers, for example by identifying factors (e.g., in $4x - 12$) or squared numbers (e.g., in $x^2 - 81$). Thus, the result adds tension between procedural and structural perspectives on teaching linear equations, as it suggests that to discern algebraic structure, students are required to overlook features that are critical for solving (e.g., the numbers). To avoid confusing students, the roles of numbers in algebra need to be clarified, and teachers need to consider which structures students should practice seeing, since systemic structures emphasize numbers (are the equations equivalent?), whereas algebraic structures put less emphasis on the numbers (do the equations have the same internal order?). In this intervention, an insecurity emerges among students as to whether "same" kind of equations refer to the equations as sharing algebraic structure or sharing systemic structure. In transcript 1, Bo wants to calculate the answer, but Carl replies that they indeed have “the same formula, they just don’t have the same answer.” There are reasons to consider an increased emphasis on the algebraic way of considering expressions as “similar.” The equation $0.4 \cdot x = 12$ has little in common with $-6 \cdot x = 24$ regarding the properties and manipulations of the numbers—topics usually considered in arithmetic—but are algebraically the same equations when generalized as $a \cdot x = b$. Using the concept of computational discourses, Sfard (2008, p.123) notes that “what appears to be different at one discursive level may conflate into ‘the same thing’ at the higher discursive level.” Thus “the same” in algebra may be considered “different” in arithmetic.

To support the discernment of algebraic structure, students should be encouraged to compare equations. One example is given in transcript 3 when Henric compares $0.15555 \cdot x = 0.6578533$ with $-44 \cdot x = 30$ by moving his pen from one part of the equation to the corresponding part in the other equation. This comparison would not have been possible if he had only been looking at one equation. Additionally, this study gives an example of teaching that specifically aims at developing students’ awareness of algebraic structure by looking at—not solving—the equations. It is a concrete example of the ideas expressed by several researchers on decomposing and recomposing equations (Kieran, 2018; Linchevski & Livneh, 1999; Venkat et al., 2019), but instead of decomposing numbers and expressions by their operations, the focus is now on the visual symbolic expressions. The decomposition of the equations, emphasized by the different colors of sticky notes, offers the students the opportunity to discern the different parts of the equation. The decomposition also enables a mapping between the equations (see also Schüler-Meyer, 2017), and the different colors of the notes assists the teachers in directing students’ attention to specific symbols, for example the operations in their orange color.

A question for further research is the relation between identifying structure and solving the equations. From the tests in this study, it was not obvious that identifying the structure led to solving the equation correctly. For example, on the posttest, one student wrote that $0.657 = 0.045 \cdot y$ belongs to the group of equations that has “multiplication in the equations and then you just calculate with division,” but on the next page tried to solve $0.657 = 0.045 \cdot x$ by subtracting 0.045 from both sides. Exploring how the identification of algebraic structure can become a natural part of students’ equation solving is important for learning how to best support them in solving equations efficiently.

This study is limited by the small sample size and the short duration of the intervention. Moreover, when students are asked to sort equations, the data collected here might not have captured all the underlying reasons for how they grouped the equations. To address this, other types of tasks or additional interviews could provide further insights. The critical aspects identified in this study are limited to linear equations, though other critical aspects would likely emerge with other types of equations. Nevertheless, the idea that awareness of algebraic structure involves attending to aspects of the content still applies—considering *what* students need to discern and *what* to disregard.

Funding Open access funding provided by University of Gothenburg. This project was supported by the ULF governmental funding (Education–Learning–Research) through Chalmers University of Technology and is part of a PhD-project at CUL (Centre for Educational Science and Teacher Research) at The University of Gothenburg.

Data availability The data that support the findings of this study are not publicly available in order to maintain participant privacy, but they are available from the corresponding author upon reasonable request.

Declarations

Competing interests The author declares no competing interests.

References

Banerjee, R., & Subramaniam, K. (2012). Evolution of a teaching approach for beginning algebra. *Educational Studies in Mathematics, 80*(3), 351–367. https://doi.org/10.1007/s10649-011-9353-y

English, L. D., & Sharry, P. V. (1996). Analogical reasoning and the development of algebraic abstraction. *Educational Studies in Mathematics, 30*(2), 135–157. https://doi.org/10.1007/BF00302627

Hoch, M., & Dreyfus, T. (2004). Structure sense in high school algebra: The effect of brackets. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 49–56). PME.

Hoch, M., & Dreyfus, T. (2006). Structure sense versus manipulation skills: An unexpected result. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 305–312). PME.

Hoch, M., & Dreyfus, T. (2007). Recognising an algebraic structure. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education,* CERME 5 (pp. 436–445). University of Cyprus and ERME.

Hoch, M., & Dreyfus, T. (2010). Developing Katy's algebraic structure sense. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education,* CERME 6. (pp. 529–538). Institut National de Recherche Pédagogique and ERME.

Holmlund, A. (2024). How numbers influence students when solving linear equations. *Mathematical Thinking and Learning, 27*(3), 305–322. https://doi.org/10.1080/10986065.2024.2314067

Holmlund, A., & Pejlare, J. (2025). Assessing students' emerging awareness of algebraic structures. In M. Bosch, G. Bolondi, S. Carreira, C. Spagnolo, & M. Gaidoschik (Eds.),*Proceedings of the Fourteenth Congress of the European Society for Research in Mathematics Education,* CERME 14 (pp. 518–525). Free University of Bozen-Bolzano and ERME.

Janßen, T., & Bikner-Ahsbahs, A. (2013). Networking theories in a design study on the development of algebraic structure sense. In B. Ubuz, C. Haser, & M. A. Mariotti (Eds.), *Proceedings of the 8th Congress of the European Society for Research in Mathematics Education,* CERME 8 (pp. 2830–2839). Middle East Technical University and ERME.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 33–56). Reston: Lawrence Erlbaum, Hillsdale.

Kieran, C. (2018). Seeking, using, and expressing structure in numbers and numerical operations: A fundamental path to developing early algebraic thinking. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5-to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 79–105). ICME-13 Monographs. Springer. https://doi.org/10.1007/978-3-319-68351-5_4

Kirshner, D. (1989). The visual syntax of algebra. *Journal for Research in Mathematics Education, 20*(3), 274–287. https://doi.org/10.5951/jresematheduc.20.3.0274

Kullberg, A., Ingerman, Å., & Marton, F. (2024). *Planning and analyzing teaching: Using the variation theory of learning*. Taylor & Francis.

Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics, 40*(2), 173–196. https://doi.org/10.1023/A:1003606308064

Maffia, A., Manolino, C., & Miragliotta, E. (2025). There is more to algebra than meets the eye: The case of blindness. *Educational Studies in Mathematics, 119*(1), 63–77. https://doi.org/10.1007/s10649-025-10394-0

Marton, F. (2015). *Necessary conditions of learning*. Routledge. https://doi.org/10.4324/9781315816876

Muñoz-Porras, V., & Xolocotzin, U. (2022). Developing structure sense with digital technologies: Introducing the MEx platform. In T. Rojano (Ed.), *Algebra structure sense development amongst diverse learners* (pp. 67–87). Routledge.

Oaks, J. A., & Alkhateeb, H. M. (2007). Simplifying equations in arabic algebra. *Historia Mathematica, 34*(1), 45–61. https://doi.org/10.1016/j.hm.2006.02.006

Otte, M. (2006). Mathematical epistemology from a Peircean semiotic point of view. *Educational Studies in Mathematics, 61*, 11–38. https://doi.org/10.1007/s10649-006-0082-6

Pang, M. F., & Ki, W. W. (2016). Revisiting the idea of "critical aspects." *Scandinavian Journal of Educational Research, 60*(3), 323–336. https://doi.org/10.1080/00313831.2015.1119724

Plomp, T. (2013). Educational design research: An introduction. In T. Plomp & N. Nieveen (Eds.), *Educational design research part A: An introduction* (pp. 9–51). SLO.

Rojano, T. (2022). Algebra Structure Sense: Conceptual Approaches and Elements for Its Development. In *Algebra Structure Sense Development amongst Diverse Learners.* (pp. 1-19). Routledge.

Rüede, C. (2013). How secondary level teachers and students impose personal structure on fractional expressions and equations—an expert-novice study. *Educational Studies in Mathematics, 83*, 387–408. https://doi.org/10.1007/s10649-012-9462-2

Schüler-Meyer, A. (2017). Students' development of structure sense for the distributive law. *Educational Studies in Mathematics, 96*(1), 17–32. https://doi.org/10.1007/s10649-017-9765-4

Sfard, A. (2008). Thinking as communicating: Human development, the growth of discourses, and mathematizing. *Cambridge University Press*. https://doi.org/10.1017/CBO9780511499944

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification—The case of algebra. *Educational Studies in Mathematics, 26*, 191–228. https://doi.org/10.1007/BF01273663

Star, J. R., & Rittle-Johnson, B. (2008). Flexibility in problem solving: The case of equation solving. *Learning and Instruction, 18*(6), 565–579. https://doi.org/10.1016/j.learninstruc.2007.09.018

Tuominen, J., Andersson, C., & Björklund Boistrup, L. (2020). Critical aspects of equations when explored as a part-whole structure. In Y. Liljekvist, L. Björklund Boistrup, J. Häggström, L. Mattsson, O. Olande, & H. Palmér (Eds.), *Sustainable mathematics education in a digitalized world: Proceedings of MADIF12. The twelfth research seminar of the Swedish Society for Research in Mathematics Education, Växjö, 2020* (pp. 223–233). SMDF.

Venkat, H., Askew, M., Watson, A., & Mason, J. (2019). Architecture of mathematical structure. *For the Learning of Mathematics, 39*(1), 13–17.

Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics, 49*(3), 341–359. https://doi.org/10.1023/A:1020229023965

Vlassis, J., & Demonty, I. (2022). The role of algebraic thinking in dealing with negative numbers. *ZDM – Mathematics Education, 54*(6), 1243–1255. https://doi.org/10.1007/s11858-022-01402-1

Xie, S., & Cai, J. (2022). Fifth graders' learning to solve equations: The impact of early arithmetic strategies. *ZDM – Mathematics Education, 54*, 1169–1179. https://doi.org/10.1007/s11858-022-01417-8