Weingarten surfaces, the Codazzi equations and the membrane theory for the formfinding of tension structures, shells and vaults

Citation for the original published paper (version of record):
Adiels, E., Williams, C. (2025). Weingarten surfaces, the Codazzi equations and the membrane theory for the formfinding of
tension structures, shells and vaults. International Journal of Space Structures, 40(4): 242-269.
http://dx.doi.org/10.1177/09560599251388389

International Journal of SPACE STRUCTURES

Article

Weingarten surfaces, the Codazzi equations and the membrane theory for the formfinding of tension structures, shells and vaults

International Journal of Space Structures
2025, Vol. 40(4) 242–269

DOI: 10.1177/09560599251388389
journals.sagepub.com/home/sps
Sage

Emil Adiels[1] and Chris JK Williams[2]

Abstract

The formfinding of shell, masonry vault and fabric or cable net structures involves obtaining a geometry and a stress state in equilibrium under a dominant load case. In so doing we can give the formfinding model very different properties to those of the finished structure, for example modelling a fabric structure by a soap film. In order to obtain a formfound geometry we need to make a number of decisions about the geometrical and structural properties we aim to achieve. These decisions can be couched in a number of ways, and in this paper we concentrate on statements about the principal curvatures of the surface and the principal membrane stresses. A Weingarten surface has a functional relationship between the mean of the two principal curvatures and their product – the Gaussian curvature. Examples include minimal surfaces, surfaces of constant mean curvature and surfaces of constant Gaussian curvature. It is well known that the Codazzi equations enable us to obtain a spacing of the principal curvature lines on a Weingarten surface that is not arbitrary, but determined by the principal curvatures. The Codazzi equations are purely geometric but they are identical with the in plane components of the membrane equilibrium equations for the case when there is no tangential load. Zero-length or close-coiled springs are used in the Anglepoise lamp and have a length that is proportional to the tension, once the tension is sufficient for the coils to separate. We demonstrate that surfaces constructed from a fine grid of zero-length springs have a membrane stress such that the product of the principal stress is constant. If we add an isotropic stress we arrive at a condition similar to that for the curvature of a linear Weingarten surface. We provide a number of examples of the application of these ideas.

Keywords

Shells, formfinding, Weingarten surfaces, hyperelastic strain energy, masonry vaults, tension structures

Introduction

The formfinding of shell, masonry vault and fabric or cable net structures involves obtaining a geometry and a state of stress in equilibrium under a dominant load case. In so doing we can give the formfinding model – either a physical or a numerical model – very different properties to those of the finished structure. Modelling a fabric structure by a soap film is the best known example.[1,2] Other possibilities include modelling a masonry structure by a hanging stretchy latex rubber or Lycra model loaded with weights, which we then invert to form a masonry structure in compression.[3,4] Similarly a hanging chain model can be inverted to give the shape of a timber gridshell.[5] Slender timber structures may be subject to creep deformation, and so even though timber gridshells are lightweight structures, the permanent own weight load can be more important than short term wind and snow loads. Slender timber structures are subject to the possibility of creep buckling,[6]

[1]Block Research Group, Institute of Technology in Architecture, ETH Zürich, Zurich, Switzerland
[2]Department of Architecture & Civil Engineering, Chalmers University, Gothenburg, Sweden

Corresponding author:
Chris JK Williams, Department of Architecture & Civil Engineering, Chalmers University, 412 96 Gothenburg, Sweden.
Email: christopher.williams@chalmers.se

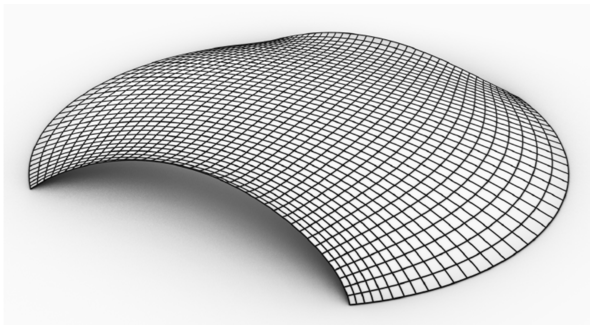

Figure 1. Inverted hanging shell. When hanging the zero-length springs shown in black are in tension balancing the own weight of the shell and an isotropic surface compression varying with height reducing away from the support, balancing the tangential component of the hanging weight. The net stress in the hanging model is always tensile and the isotropic compressive stress balances the tension in the zero-length springs at the boundary in the direction normal to the boundary. This model has tension coefficients that are independent of time, but do vary with position on the surface.

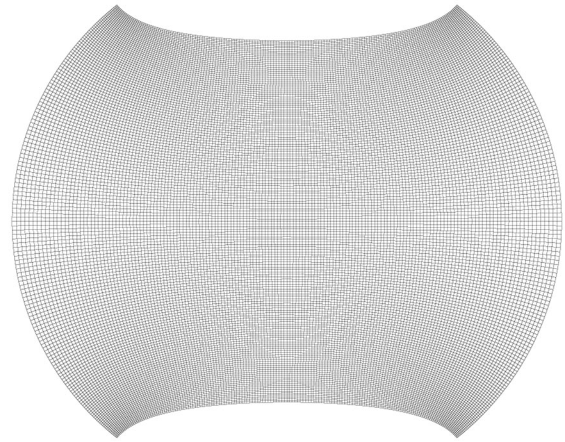

Figure 2. Structural grid of the shell in Figure 1.The grid in Figure 1 is drawn coarser in order to show the zero-length springs clearly. The analysis took approximately 15 s to converge on a 2017 MacBook Air using a program written using the Processing https://processing.org programming environment.

which is particularly difficult to predict because of the lack of data on the rate of creep and how it is influenced by stress, moisture content and so on. Essentially creep deformation increases the imperfections in the geometry of the structure, which may in turn reduce the buckling load.

Our aim in pursuing this work was to produce designs for a shell such as that in Figure 1, which has two large openings. It would be usual in the design of such a shell to introduce edge arches or stiffeners, and so concentrate forces into the boundary. However, in certain situations it might be preferable to have a more even distribution of stress and hence avoid unsightly edge members. In aiming for this we were very much influenced by the physical models of Heinz Isler,[4] in particular obtaining the upturned lip at the edge of the shell.

The distribution of forces in a shell, tension structure or masonry vault is influenced by the structural grid or pattern of masonry blocks and vice versa. And so we are very interested in the arrangement of the grid and its relationship with the curvature of the structure and the stress distribution. This means that we are equally concerned with structural and geometric equations, and have to ensure that we use the same mathematical notation for them both.

Fundamentals of the formfinding of shells and the contributions in this paper

Even though shell structures almost invariably experience internal bending moments under applied loads, such as wind or snow, the aim in the formfinding process is to make the structure work by membrane action under some idealised dominant load case. Membrane action means tensions and compressions in the local tangent plane to the surface with no bending moments or shear forces normal to the surface (Figures 1–5).

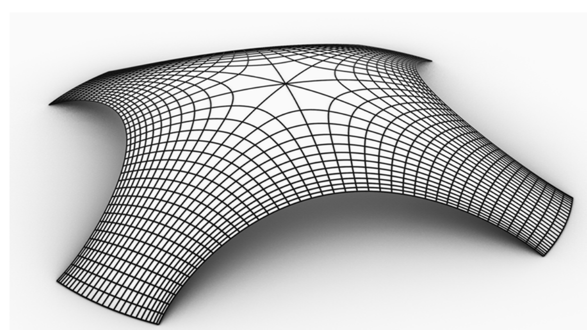

Figure 3. Inverted hanging shell of the same type as that in Figure 1, but with four openings. This model does have the same tension coefficient at all points on the surface. The longer springs have a higher tension, but they also have a higher spacing, so that membrane stress is approximately constant away from the boundaries.

In the formfinding of structures the geometry is treated as an unknown. However, before discussing that it is worth considering the simpler case of when the geometry is known. The membrane theory of shells has equilibrium equations in three directions and three

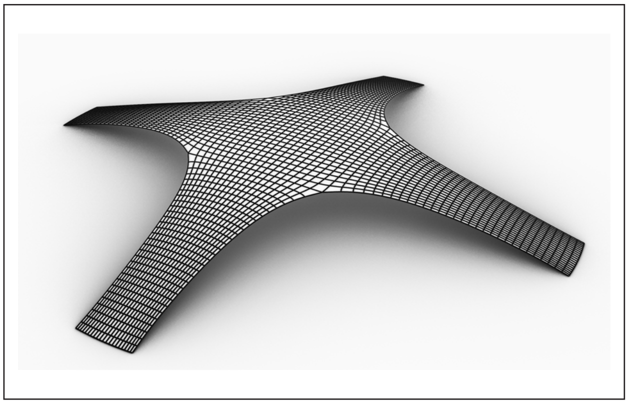

Figure 4. Inverted hanging shell of the same type as that in Figure 3, but with a different grid. This model does have the same tension coefficient at all points on the surface. The area of longer springs is moved from the centre of the shell in Figure 3 to the mid-points of the boundaries, as shown in detail in Figure 5.

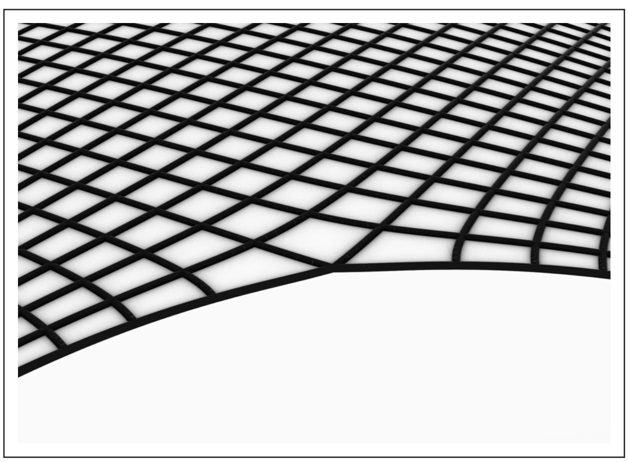

Figure 5. Detail of shell in Figure 4.

unknown components of membrane stress. We therefore have the same number of equations as unknowns, if we know the shape of the shell and the loading upon it. The three equations can be reduced to one partial differential equation which is elliptic if the Gaussian curvature is positive and hyperbolic if the Gaussian curvature is negative.[7] This means that shells with positive Gaussian curvature and negative Gaussian curvature behave very differently, and require different boundary support to be statically determinate, and avoid being mechanisms. There are a number of ways to reduce the three equations of equilibrium to one equation,[8] and the simplest way is to derive Puchers equation which is described in §113 of Timoshenko and Woinowsky-Krieger.[7]

In formfinding we generally reverse the problem and specify certain properties of the state of stress and the geometry is treated as an unknown. The problem will be

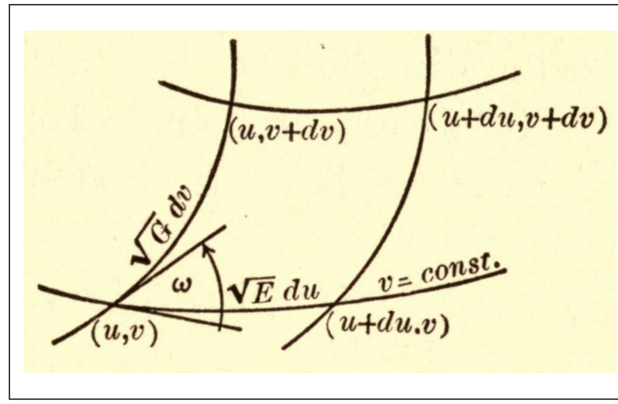

Figure 6. Figure 11 from Eisenhart[10] showing u-v coordinate curves on a surface. We employ θ^1 and θ^2 instead of u and v enabling us to use the tensor notation.[11–13]

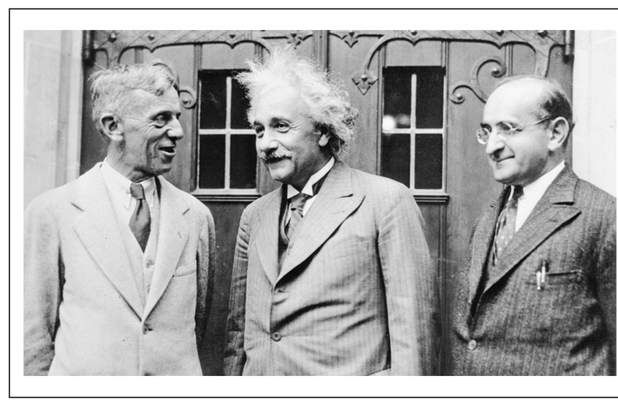

Figure 7. Albert Einstein with two of his colleagues at Princeton University in 1933, Luther Eisenhart on the left and Walther Mayer on the right.

elliptic if the shell is all in tension or all in compression and hyperbolic if the product of the principal membrane stress is negative. However, in some cases we might impose a mixture of geometric and stress constraints and we have to be careful to ensure that we have sufficient constraints for the problem to be determinate, but not too many constraints so that there is no solution.

In specifying the geometry we need the shape of the surface representing the structure, but also the geometry of the structural grid which might represent members of a gridshell or the stone blocks of a vault.[9]

We define the position of a point on a surface using surface coordinates or parameters. Traditionally these coordinates are chosen to be u and v as shown in Figure 6, taken from Eisenhart,[10] but we will instead use θ^1 and θ^2, or θ^α where Greek letters may have the values 1 or 2, to be consistent with the tensor notation.[11,12] Eisenhart[13] later wrote a book on differential geometry using the tensor notation, perhaps influenced by his work with Einstein (Figure 7).

In general it is convenient to align the structural grid with the coordinate grid and we shall do this. However, there might be circumstances when they are not aligned, in which case we have essentially two coordinate systems, and the tensor notation is particularly suited to a change of coordinate system.

> The tensor notation that we shall be using for differential geometry is described in detail in Appendix A and the structural equations that we need are derived in Appendix B, which includes bending moments. The reason for this is that a proper understanding of the membrane theory is best achieved by treating it as a special case of the bending theory.

The unknowns that we have to deal with are the geometric quantities $E = a_{11}$, $F = a_{12} = a_{21}$ and $G = a_{22}$, which are the coefficients of the first fundamental form. Struik[14] and Eisenhart[10] use E, F and G, while Green and Zerna[11] use $a_{\alpha\beta}$ from the tensor notation.

Next we have the $e = L = D = b_{11}$, $f = M = D' = b_{12} = b_{21}$ and $g = N = D'' = b_{22}$ which are the coefficients of the second fundamental form. Struik uses e, f and g and Eisenhart rather confusingly uses D, D' and D'', other authors commonly use L, M and N while Green & Zerna use $b_{\alpha\beta}$.

Finally we have the components of the symmetric membrane stress tensor. Green & Zerna use n^{11}, $n^{12} = n^{21}$ and n^{22}, but we shall use $\sigma^{\alpha\beta}$ for the components of the membrane stress tensor $\boldsymbol{\sigma}$, which has units force over length. Books on differential geometry do not include stresses, still less bending moments.

The components of the first and second fundamental forms must satisfy the Gauss-Codazzi equations,[14] that is Gauss's Theorema Egregium and the Peterson-Mainardi-Codazzi equations. The Gauss-Codazzi equations are three partial differential equations. In addition we have the equilibrium equations, so that we have six equations and nine unknowns, $a_{\alpha\beta}$, $b_{\alpha\beta}$ and $\sigma^{\alpha\beta}$.

Thus we need three more equations to perform formfinding. For a prestressed or hanging equal mesh or Tchebychev[15] net we have the shear stresses component $\sigma^{12} = 0$ and the lengths $\sqrt{a_{11}} = \sqrt{a_{22}} = constant$. Thus we have a mixture of structural and geometric constraints.

For a fabric structure based on a soap film we have the mean curvature $H = 0$ to define the surface and $a_{12} = 0$ together with $a_{11} = constant$ if we want the cutting pattern to follow geodesic coordinates[14] on the surface. This follows from the fact that $\mathbf{a}_{2,1} = \mathbf{a}_{1,2}$ and so

$$\mathbf{a}_{1,1} \cdot \mathbf{a}_2 = \left(\mathbf{a}_1 \cdot \mathbf{a}_2\right)_{,1} - \mathbf{a}_1 \cdot \mathbf{a}_{2,1}$$

$$= a_{12,1} - \frac{a_{11,2}}{2} = 0$$

if $a_{12} = 0$ and $a_{11} = constant$. Hence the Christoffel symbol $\Gamma^2_{\cdot 11} = 0$ in (0), and this is the condition for the coordinate curves $\theta^2 = constant$ to be geodesics.

However the easiest way to ensure $H = 0$ is to specify an unloaded surface containing an isotropic membrane stress with components $\sigma^{\alpha\beta} = Ta^{\alpha\beta}$, where T is a constant surface tension, in which case the unloaded equilibrium equation in the normal direction is simply $\sigma^{\alpha\beta} b_{\alpha\beta} = Ta^{\alpha\beta} b_{\alpha\beta} = Tb^{\alpha}_{\alpha} = 2TH = 0$. The surface can be easily modelled using constant strain finite elements[16] together with constraints to form the geodesics. The resulting non-linear equations can be integrated using Verlet integration[17] or dynamic relaxation,[2,18] which are essentially the same thing.

Thus in both these simple cases formfinding involves making a mixture of three geometric and structural statements. We can, of course make just one statement to define a surface, perhaps in the Monge form $z = z(x, y)$, but we need two more statements to define the grid on the surface, which is equally, if not more important.

An elastic membrane is one in which there is a relationship between the membrane stress components $\sigma^{\alpha\beta}$ and the strain in the surface. Strain is the change in lengths on the surface between the current configuration, with lengths given by $a_{\alpha\beta}$, and some initial configuration in which the coefficients of the first fundamental form were $A_{\alpha\beta} = A_{\beta\alpha}$. This gives rise to the strain tensor with components $\gamma_{\alpha\beta} = \left(a_{\alpha\beta} - A_{\alpha\beta}\right)/2$.

In formfinding we are free to choose $A_{\alpha\beta}$ as we wish because $A_{\alpha\beta}$ do not have to satisfy the Gauss-Codazzi equations. In other words the initial configuration does not have to 'fit together' without stretching certain parts of it or allowing the Gaussian curvature to change freely.

A hyperelastic or Green elastic sheet is a sheet whose stress-strain relationship derives from a strain energy density function.[19,20] The Green strain energy function is named after George Green (1793–1841)[21] who should not be confused with Albert Edward Green (1912–1999).[11,20] A Cauchy elastic material[19] is a material in which the stress is a unique function of the strain, but which does not admit a strain energy density function, although quite how such a material could exist without breaking the first law of thermodynamics is unclear.

We shall assume that we have a Green elastic material, and purely geometric constraints, such as constant lengths or geodesics, which correspond to allowing some stiffnesses to tend to infinity. The constraints then become Lagrange multipliers which can be simply included as unknowns in the relaxation process, including a rate of change of the Lagrange multipliers. If a cable of an equal mesh net is too long we increase the tension, if it is too short we reduce the tension, and we can, if we are so minded, include the rate of change of tension as a damped variable.

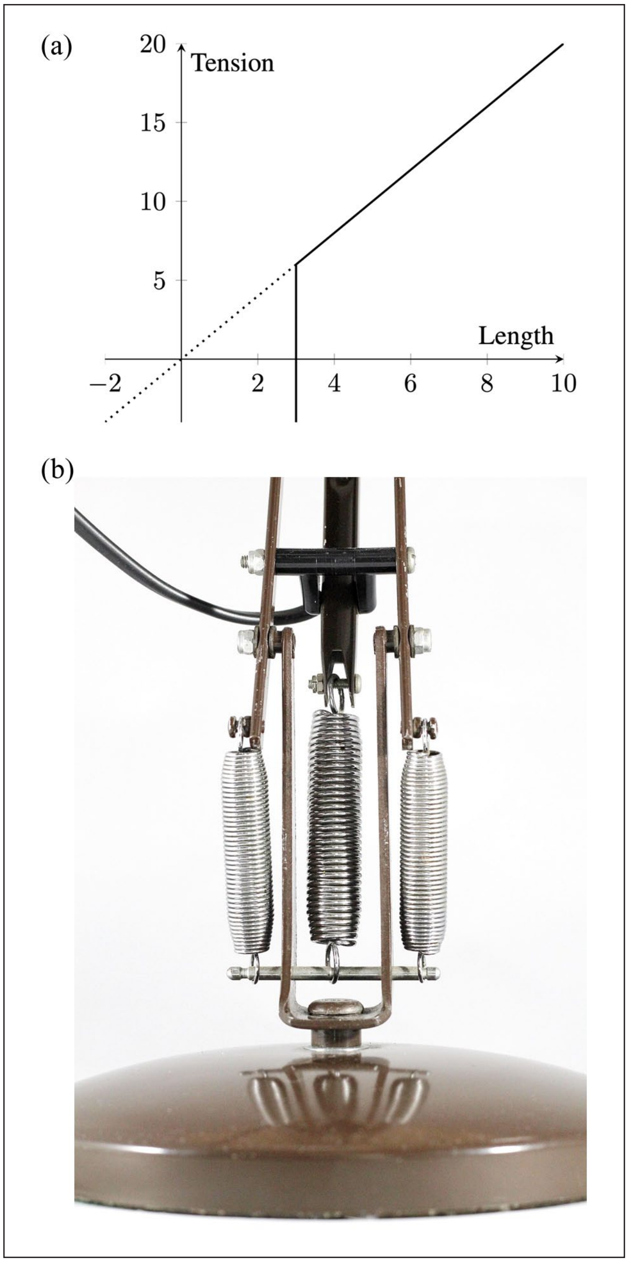

Figure 8. Zero-length springs: (a) length/tension relationship for zero-length spring and (b) springs of an Anglepoise lamp. Image: TheGoodEndedHappily, CC BY-SA 4.0.

A soap film is an elastic membrane in which the strain energy per unit area is a constant equal to the isotropic surface tension. Hence minimising the surface area and minimising the strain energy are identical problems.

A particularly useful hyperelastic sheet is a continuum made from a very large number of zero-length or close-coiled springs that have a length that is proportional to the tension, once the tension is sufficient for the coils to separate. Figure 8(a) shows the tension / length relationship for a zero-length spring. The coils touch when unloaded and only when a certain tension is reached do they begin to separate in such a way that thereafter the tension is proportional to the length. Thus, even though the springs do not have a zero length when unstressed, we use the term zero-length spring to mean a spring with a linear length/tension relationship which can be extrapolated back to the origin.

It would seem that the practical manufacture of zero-length springs was pioneered independently by George Carwardine[22] for use in the Anglepoise lamp (Figure 8(b)) and Lucien LaCoste[23] for use in gravimeters.

The terms tension coefficient and force density were coined by Southwell[24] and Linkwitz and Schek,[25–27] respectively, to mean the tension in a structural element divided by its length. Therefore we can describe a zero-length spring as a member with constant tension coefficient or force density.

> We shall demonstrate that the product of the two principal membrane stresses remains constant as a zero-length spring surface is deformed.

We can draw a principal curvature line on a surface by starting at an arbitrary point and moving backwards or forwards in one of the two principal curvature directions. This involves solving the eigenvalue-eigenvector problem[14]

$$\left(b_{\alpha\beta} - \lambda a_{\alpha\beta}\right) d\theta^{\beta} = 0$$

in which the eigenvalues, λ, are the principal curvatures and the eigenvectors, $d\theta^2 / d\theta^1$, are the principal curvature directions. This only fails at umbilic points at which the principal curvatures are equal so that all directions are principal curvature directions. All points on a sphere are umbilics.

We can draw a number of principal curvature lines by starting at a number of arbitrary points, and the spacing of the principal curvature lines is arbitrarily determined by the starting points.

A Weingarten surface is one in which there is a functional relationship between the mean and Gaussian curvatures, and it is well known[10] that the Codazzi equations enable us to obtain a spacing of the principal curvature lines on a Weingarten surface that is not arbitrary, but determined by the principal curvatures.

A linear Weingarten surface has a linear relationship between the mean and Gaussian curvatures and examples include minimal surfaces, surfaces of constant mean curvature and surfaces of constant Gaussian curvature.

It is less well known that in the absence of tangential loads on a shell working by membrane action, the equilibrium equations in the tangential directions are identical to the Codazzi equations, and hence the spacing of the principal membrane stress lines is uniquely determined, provided that we have a functional relationship between the mean and product of the two principal membrane stresses.

We demonstrate that under certain conditions an isotropic hyperelastic sheet will have such a functional relationship between the mean and product of the two principal membrane stresses.

If we add an isotropic tension or compression to the zero-length spring membrane stress we arrive at the same condition for membrane stress as we do for curvature of a linear Weingarten surface. Thus the results for linear Weingarten surface can be applied directly to the spacing of the principal membrane stress trajectories, which are important for structural design. In general the principal curvature directions and the principal membrane stress directions do not coincide, unless this is imposed as a constraint.
It is 'conventional' in the design of shell structures to have a relatively 'flimsy' shell supported on 'substantial' edge beams, arches and cables. One of the aims of this work is to formfind shapes in which the forces that would have been concentrated in the edge beams or arches are distributed into the shell itself, more like the shells that we see in nature.[28]

We use a large number of 'standard results' which are relegated to appendices, as described above. Many of these results are found in Green and Zerna,[11] in Green and Adkins[20] or in Eisenhart,[10] but not always in a form suitable for our use, or derived in a manner consistent with our narrative. The book Pinkall and Gross[29] *Differential Geometry: From Elastic Curves to Willmore Surfaces* is interesting because it covers essentially what we would call the bending theory of shells, but without mentioning concepts such as 'force', 'moment', 'stress' or 'tensor'.

Comparison with existing methods of formfinding

In order to discuss existing methods of formfinding it is helpful to divide the process into three parts:

1. Define a conceptual model involving geometrical and structural quantities. If the model is a continuum model this will involve differential equations or the minimisation of surface integrals. If the model is discrete the equations will be simultaneous equations which are often non-linear.

Examples of such conceptual models include the Force Density Method,[25] Thrust Network Analysis,[30] Membrane Equilibrium Analysis[31] and the use of machine learning.[32] The force density method allows one to treat the elements of a cable net as inextensional in which case the tensions in an element can no longer be calculated from the strain and instead the force densities or tension coefficients are treated as additional unknowns.

2. For a numerical solution a continuum model has to be converted to a discrete model. This almost invariably involves use of the finite element method, and other methods, such as the finite difference method can be considered as special cases of the finite element method with appropriate shape functions. Particle methods, such as smoothed particle hydrodynamics and peridynamics can be considered as the use of finite elements which overlap and merge into each other. Isogeometric analysis is the use of finite element shape functions borrowed from computer aided design, including NURBS[33] and Bézier triangles.[34]
3. Numerical solution of the equations. Here the techniques can be broadly described as implicit matrix methods or explicit relaxation methods. Since the equations are often non-linear, matrix methods may involve techniques such as Newton-Raphson. Relaxation techniques were introduced by Southwell[35] include Verlet integration[17] and dynamic relaxation,[18] which is effectively Verlet integration applied to a static problem, leading to a damped oscillation about the equilibrium state.
   From an engineering point of view it does not matter whether matrix or relaxation techniques are used to solve the equations, and the equations produced by the finite element method can equally well be solved using either technique. The Force Density Method[25] uses matrices, but the equations could equally well be solved using relaxation.

Those who are so minded can write their own software using, for example C++ and OpenGL, or Processing https://processing.org. Alternatively they or they can use an environment such as Daniel Piker's Kangaroo3d http://kangaroo3d.com in Rhino Grasshopper.

Principal stress trajectories

The stress in a membrane is a symmetric second order tensor, in exactly the same way that curvature of a surface can be described by a symmetric second order tensor. This means that we have orthogonal principal membrane stresses just as we have orthogonal principal curvatures. In the from finding process we are interested in both the principal stress directions and the principal curvature directions, and we may align structural members and cladding panels either with the stresses or the curvatures. Ideally we would align the principal curvatures and the principal stresses, but this may be difficult.

In the same way that we can plot principal curvature directions we can plot the principal stress trajectories by

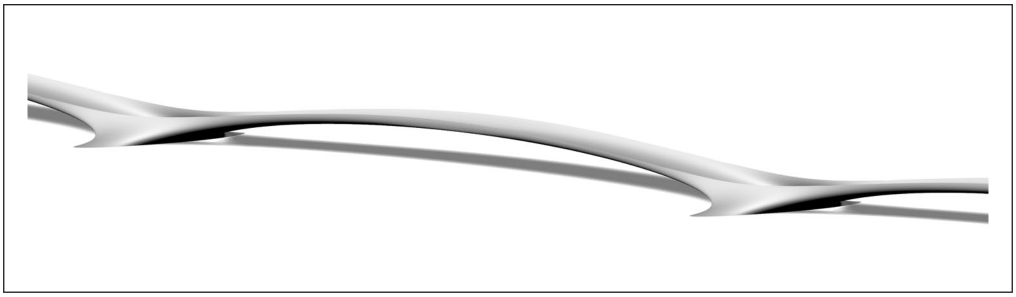

Figure 9. Inverted hanging bridge entirely in compression under own weight and a hydrostatic pressure to model internal fill supporting the roadway. The zero-length springs in the model all have the same constant tension coefficients.

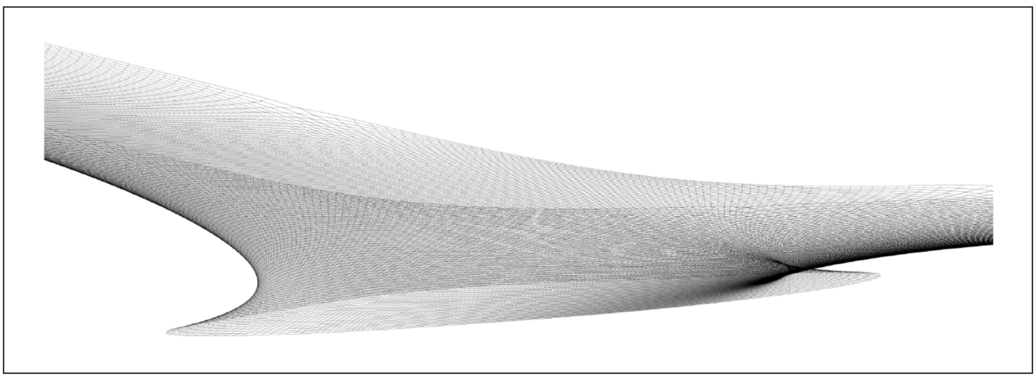

Figure 10. Detail of bridge in Figure 9 showing structural grid.

starting at some arbitrary point and then solving the eigenvalue problem to move in one of the principal directions. The arbitrary starting points mean that the spacing of the curves on the surface is not well controlled. We shall show that if we put conditions corresponding to those for a Weingarten surface on the state of membrane stress in a structure, then the principal stress trajectories become controlled, and the numerical procedure automatically produces principal stress trajectories as shown in Figures 3, 4, 9 and 10.

> The structural theory for membrane stress and the geometric theory for curvature are essentially the same. However one can do geometry with no knowledge of structures, but one can not do structures with no knowledge of geometry, so it would seem simpler to start with geometry and Weingarten surfaces.

Weingarten surfaces

A Weingarten surface is a surface for which there is a functional relationship between the mean and Gaussian curvature.[36,37] The mean curvature is the mean of the two principal curvatures and the Gaussian curvature is the product of the principal curvatures. Eisenhart[10] calls Weingarten surfaces W-surfaces and in Article 123 of Chapter VIII he starts his analysis with the Codazzi equations in principal curvature coordinates. Setting $b_{12} = 0$ in (A13) we have

$$b_{11,2} - \frac{a_{11,2}}{2}\left(\frac{b_{11}}{a_{11}} + \frac{b_{22}}{a_{22}}\right) = 0$$

$$b_{22,1} - \frac{a_{22,1}}{2}\left(\frac{b_{11}}{a_{11}} + \frac{b_{22}}{a_{22}}\right) = 0$$

or

$$\left(\frac{b_{11}}{a_{11}}\right)_{,2} - \frac{a_{11,2}}{2a_{11}}\left(-\frac{b_{11}}{a_{11}} + \frac{b_{22}}{a_{22}}\right) = 0$$

$$\left(\frac{b_{22}}{a_{22}}\right)_{,1} - \frac{a_{22,1}}{2a_{22}}\left(\frac{b_{11}}{a_{11}} - \frac{b_{22}}{a_{22}}\right) = 0.$$

Thus, writing

$$\kappa_I = \frac{b_{11}}{a_{11}} = H + \sqrt{H^2 - K}$$

$$\kappa_{II} = \frac{b_{22}}{a_{22}} = H - \sqrt{H^2 - K}$$

where H and K are the mean and Gaussian curvatures we have

$$\left(\log\sqrt{a_{11}}\right)_{,2} = \frac{\kappa_{I,2}}{-\kappa_I + \kappa_{II}}$$

$$\left(\log\sqrt{a_{22}}\right)_{,1} = \frac{\kappa_{II,1}}{\kappa_I - \kappa_{II}}$$

which is equation (45) in Article 123 of Chapter VIII of Eisenhart.[10] He then points out that if we have a functional relationship between κ_I and κ_{II} it follows that we can write

$$\begin{aligned}\sqrt{a_{11}} &= U\left(\theta^1\right)\exp\left(\int\frac{d\kappa_I}{-\kappa_I+\kappa_{II}}\right)\\ \sqrt{a_{22}} &= V\left(\theta^2\right)\exp\left(\int\frac{d\kappa_{II}}{\kappa_I-\kappa_{II}}\right)\end{aligned} \tag{1}$$

where $\exp(x)=e^x$ is the exponential function and $U\left(\theta^1\right)$ and $V\left(\theta^2\right)$ are arbitrary functions. Once we have made a choice of the functions U and V the spacing of the principal curvature lines is no longer arbitrary and the simplest choice is to write $U=V=constant$.

In the case of a minimal surface for which $\kappa_I=-\kappa_{II}$ or $\pm\kappa_I=\mp\kappa_{II}=1/\rho$, where ρ is the absolute value of the principal radii of curvature, we have

$$\begin{aligned}\sqrt{a_{11}} &= U\exp\left(-\frac{1}{2}\log\left|\kappa_I\right|\right)\\ &= \frac{U}{\sqrt{\left|\kappa_I\right|}}\end{aligned}$$

so that we obtain the well known result that

$$a_{11}=a_{22}=U^2\rho$$

which appears in Article 109 in Chapter VII of Eisenhart.[10]

Linear Weingarten surface

A linear Weingarten surface[10,36,37] is a surface which satisfies the relationship

$$2TH+2SK+p=0 \tag{2}$$

where T, S and p are constants and K and H are the Gaussian and mean curvature respectively, as given by equations (A9) and (A10) in Appendix A.

We have from (2) or (D4),

$$\begin{aligned}T\left(\kappa_I+\kappa_{II}\right)+2S\kappa_I\kappa_{II}+p &= 0\\ \kappa_{II} &= -\frac{T\kappa_I+p}{2S\kappa_I+T}\\ -\kappa_I+\kappa_{II} &= -\frac{2S\kappa_I^2+2T\kappa_I+p}{2S\kappa_I+T}\end{aligned}$$

so that

$$\begin{aligned}\log\sqrt{a_{11}} &= -\int\frac{\left(2S\kappa_I+T\right)d\kappa_I}{2S\kappa_I^2+2T\kappa_I+p}\\ &= -\frac{1}{2}\log\left(2S\kappa_I^2+2T\kappa_I+p\right)\\ &\quad + a\, constant\end{aligned}$$

and

$$\begin{aligned}a_{11} &= \pm\frac{L}{2S\kappa_I^2+2T\kappa_I+p}\\ &= \pm\frac{L}{\left(2S\kappa_I+T\right)\left(\kappa_I-\kappa_{II}\right)}\end{aligned}$$

where L is a constant length. The uncertainty as to sign is there because κ_I could be positive or negative, as could be p, S and T. a_{11} must be positive. Similarly,

$$a_{22}=\pm\frac{L}{\left(2S\kappa_{II}+T\right)\left(\kappa_I-\kappa_{II}\right)}$$

and again a_{22} must be positive. Thus if $\left(2S\kappa_I+T\right)$ and $\left(2S\kappa_{II}+T\right)$ both have the same sign,

$$\begin{aligned}a_{11}-a_{22} &= \pm\frac{2SL}{\left(2S\kappa_{II}+T\right)\left(2S\kappa_I+T\right)}\\ &= \pm\frac{SL}{2S\left(2SK+2TH\right)+T^2}\\ &= \pm\frac{SL}{T^2-2Sp}=constant.\end{aligned} \tag{3}$$

On the other hand, if $\left(2S\kappa_I+T\right)$ and $\left(2S\kappa_{II}+T\right)$ have opposite signs,

$$a_{11}+a_{22}=\pm\frac{SL}{T^2-2Sp}=constant$$

which means that the diagonals of the principal curvature coordinates form a Tchebychev net. Using 2, the product

$$\begin{aligned}\mathcal{P} &= \left(2S\kappa_I+T\right)\left(2S\kappa_{II}+T\right)\\ &= 4S^2K+8TSH+T^2\\ &= 4TSH+T^2-2Sp\\ &= -4S^2K+T^2-4Sp\end{aligned}$$

Thus, we arrive at the conclusion that principal curvature coordinates on a linear Weingarten surface can be constructed such that $a_{11}-a_{22}=constant$ if $\mathcal{P}$ is positive and $a_{11}+a_{22}=constant$ if $\mathcal{P}$ is negative.

If $S = 0$, then $\mathcal{P}$ must be positive and we have a constant mean curvature surface with curvilinear squares, $a_{11} = a_{22}$, and if in addition $p = 0$ we have a minimal surface.

If $T = 0$, we have a surface of constant Gaussian curvature and

$$a_{11} = \pm \frac{L}{2S\kappa_I(\kappa_I - \kappa_{II})}$$
$$a_{22} = \pm \frac{L}{2S\kappa_{II}(\kappa_I - \kappa_{II})}$$
$$b_{11} = a_{11}\kappa_I = \pm \frac{L}{2S(\kappa_I - \kappa_{II})} = \pm b_{22}.$$

The curvature and twist in the directions of the diagonals of the principal curvature net are

$$\kappa = \frac{b_{\alpha\beta} d\theta^\alpha d\theta^\beta}{a_{\alpha\beta} d\theta^\alpha d\theta^\beta} = \frac{b_{11} + b_{22}}{a_{11} + a_{22}}$$
$$\tau = \frac{b_{\alpha\lambda} a^{\lambda\mu} \epsilon_{\mu\beta} d\theta^\alpha d\theta^\beta}{a_{\alpha\beta} d\theta^\alpha d\theta^\beta}$$
$$= \frac{\left(b_{11}a^{11} - b_{22}a^{22}\right)\sqrt{a_{11}a_{22}}}{a_{11} + a_{22}}$$
$$= \frac{(\kappa_I - \kappa_{II})\sqrt{a_{11}a_{22}}}{a_{11} + a_{22}}$$
$$= \frac{\kappa_I - \kappa_{II}}{\sqrt{\left|\frac{\kappa_{II}}{\kappa_I}\right|} + \sqrt{\left|\frac{\kappa_I}{\kappa_{II}}\right|}}.$$

If K is negative $a_{11} - a_{22} = constant$ and $\kappa = 0$ so that we obtain the well known result that the asymptotic curves on a surface of constant negative Gaussian curvature form a Tchebychev net.[38] If K is positive $a_{11} + a_{22} = constant$ and

$$\kappa = \frac{2}{\frac{1}{\kappa_I} + \frac{1}{\kappa_{II}}} = \frac{K}{H}$$
$$\tau = \frac{\sqrt{H^2 - K}\sqrt{K}}{H}$$
$$\frac{\kappa}{\tau} = \frac{\sqrt{K}}{\sqrt{H^2 - K}}$$

so that a line through the origin is tangent to the Mohr's circle of curvature[39] at the point κ, τ.

We demonstrate in Appendix D how a linear Weingarten surface can be obtained by the minimisation of a surface integral, which could represent strain energy. We have relegated this to an appendix since it involves the bending theory, introduced in Appendix B and since we have introduced the bending theory, we describe the Willmore surface in Appendix E.

Boundary conditions at a free edge

Tellier et al.[37] discuss the boundary condition for a linear Weingarten surface attached to a cable, but here we will consider the case when we have a free edges with no forces applied to it. The normal shear force is automatically zero from (D1), leaving us with the membrane stresses in (D2) and (D3).

Consideration of Mohr's circle of stress[40] tells us that a free edge must be in a principal stress direction and that that principal stress must be zero. Thus, if we take $\sigma_{II} = 0$ along the edge, then (D3) gives us that

$$T + S\kappa_I = 0.$$

The membrane equilibrium equations

The equation of equilibrium of forces for shells in both the membrane and bending theories is (B2), repeated here

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{p} = 0$$

where $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{p}$ are the loads on the shell.

In the membrane theory there are no bending moments or normal shear forces and so (B3) become

$$\sigma^{\alpha\beta}|_\alpha + p^\beta = 0$$
$$\sigma^{\alpha\beta} b_{\alpha\beta} + p = 0$$

and (B4) become

$$\sigma^\beta = 0$$
$$\sigma^{\alpha\beta}\epsilon_{\alpha\beta} = 0$$

so that $\boldsymbol{\sigma}$ is a symmetric surface tensor with no normal components.

In the case when there are no tangential loads, then

$$\sigma^{\alpha\beta}|_\alpha = 0. \qquad (4)$$

Now we can rewrite the Codazzi equations (A12) as

$$\left(\epsilon^{\alpha\lambda}\epsilon^{\beta\mu} b_{\lambda\mu}\right)\Big|_\alpha = 0$$

which is exactly the same as (4) if we replace the tensor with components $\sigma^{\alpha\beta}$ with the tensor with components $\epsilon^{\alpha\lambda}\epsilon^{\beta\mu} b_{\lambda\mu}$.

The tensor with components $\epsilon^{\alpha\lambda}\epsilon^{\beta\mu} b_{\lambda\mu}$ has the same principal values and the same principal directions as the tensor with components $b_{\lambda\mu}$, except that the principal values, κ_I and κ_{II}, are interchanged.

This means that the results (1) from the Codazzi equations become

$$\log\sqrt{a_{11}} = \int \frac{d\sigma_{II}}{\sigma_I - \sigma_{II}}$$
$$\log\sqrt{a_{22}} = \int \frac{d\sigma_I}{-\sigma_I + \sigma_{II}}$$

if the coordinates follow the principal membrane stress directions and we have a functional relationship between σ_I and σ_{II}. The principal membrane stresses are

$$\sigma_I = \frac{\sigma^{11}}{a^{11}}$$
$$\sigma_{II} = \frac{\sigma^{22}}{a^{22}}.$$

Isotropic hyperelastic membrane

In Appendix C we derive (C8) which we repeat here,

$$\sigma^{\alpha\beta} = \frac{\partial}{\partial\mathcal{A}}(\mathcal{A}\phi)a^{\alpha\beta} + \frac{\partial\phi}{\partial\mathcal{B}}\mathcal{O}^{\alpha\beta}.$$

ϕ is the strain energy per unit area which is assumed to be a function of the strain invariants $\mathcal{A}$ and $\mathcal{B}$ in (C5) and (C6).

The tensor $\mathcal{O}^{\alpha\beta}\mathbf{a}_\alpha\mathbf{a}_\beta$ has components given in (C2) and it can be expressed in terms of the strain tensor.

The mean M and product P of the principal membrane stresses are given in (C9) and (C10) as

$$M = \frac{\partial}{\partial\mathcal{A}}(\mathcal{A}\phi) + \mathcal{B}\frac{\partial\phi}{\partial\mathcal{B}}$$

and

$$M^2 - P = \left(\frac{\partial\phi}{\partial\mathcal{B}}\right)^2(\mathcal{B}^2 - \mathcal{A}^2).$$

A soap film subject to its own weight

Minimal surfaces are of great interest to mathematicians[41] and to architects and engineers, notably Frei Otto[42] and Sergio Musmeci,[43,44] for the finding of beautiful and efficient structural forms. Minimal surfaces minimise the area of a surface, and this is the same as minimising the strain energy of a weightless surface with a constant strain energy per unit area, corresponding to a homogeneous isotropic surface tension. Soap films are not weightless, and so cannot be in equilibrium with uniform surface tension. It would seem reasonable to assume that the surface tension in a soap is isotropic, that is that it has the same value in all directions at a particular point, since otherwise there would be shear stresses in the film. However the surface tension clearly cannot be homogeneous since the tension at the top of a vertical soap film must be greater than that at the bottom to balance the weight of the film. In addition, for stability the surface tension must increase where the film is thinner to pull more fluid back into a thin region.[45] This is the reverse of gas pressure increasing with density. The variation in soap film thickness can be seen from the thin-film interference patterns in Figure 11.

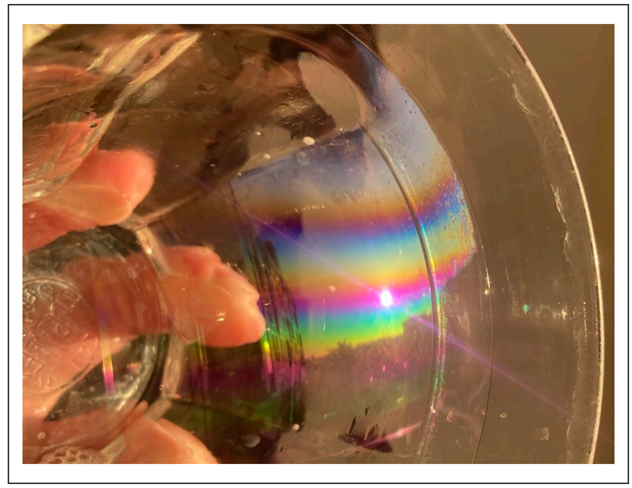

Figure 11. Thin-film interference on a soap film in a drinking glass

We are thus interested in the variation of stress in a soap film, which can be related to the strain energy per unit area stored in the film. If one watches a soap film one can see that the fluid is not stationary and is continuously in motion. 'Material points' are free to move on the surface and the surface tension can only depend on the thickness of the film. The film carries no memory of the shape it may have been in the past, it has no understanding of the concept of strain.

If the stress is isotropic,

$$\frac{\partial\phi}{\partial\mathcal{B}} = 0$$

in (C8) to give

$$\sigma^{\alpha\beta} = Ta^{\alpha\beta}$$
$$\boldsymbol{\sigma} = T(\mathbf{I} - \mathbf{nn})$$

where the surface tension,

$$T = \phi + \mathcal{A}\frac{\partial\phi}{\partial\mathcal{A}} = \frac{\partial}{\partial\mathcal{A}}(\mathcal{A}\phi).$$

If write w for the weight per unit area of the soap film, then the equilibrium equation (B2) becomes

$$\begin{aligned}0 &= \nabla\cdot\boldsymbol{\sigma} - w\mathbf{k}\\ &= \nabla\cdot\left(T\left(\mathbf{I}-\mathbf{nn}\right)\right) - w\mathbf{k}\\ &= T_{,\alpha}\mathbf{a}^{\alpha} - T\mathbf{a}^{\alpha}\cdot\mathbf{n}_{,\alpha}\mathbf{n} - w\mathbf{k}\\ &= T_{,\alpha}\mathbf{a}^{\alpha} + 2TH\mathbf{n} - w\mathbf{k}.\end{aligned}$$

where H is the mean curvature (A10).

Thus for equilibrium in the normal direction

$$2TH = w\mathbf{n}\cdot\mathbf{k}$$

and in the tangential direction,

$$\begin{aligned}0 &= T_{,\beta} - w\mathbf{k}\cdot\mathbf{a}_{\beta}\\ &= T_{,\beta} - wz_{,\beta}\end{aligned}$$

where z is the height above some datum.

Thus the gradient ∇T must be parallel to ∇z and so T and w must be a functions of z,

$$T(z) = T_0 + \int_{z_0}^{z} w(z)\,dz.$$

Zero-length spring surfaces

Let us write

$$\phi = T + k\frac{\mathcal{B}}{\mathcal{A}}$$

where T and k are independent of time. We could, if so desired allow T to vary with z, and therefore time if we need to balance the own weight as described in the previous section.

Then, from (C8)

$$\sigma^{\alpha\beta} = Ta^{\alpha\beta} + \frac{k}{\mathcal{A}}\mathcal{O}^{\alpha\beta}$$

and from (C9) and (C10),

$$\begin{aligned}M &= T + k\frac{\mathcal{B}}{\mathcal{A}}\\ P &= \left(T + k\frac{\mathcal{B}}{\mathcal{A}}\right)^2 - \frac{k^2}{\mathcal{A}^2}\left(\mathcal{B}^2 - \mathcal{A}^2\right)\\ &= T^2 + 2Tk\frac{\mathcal{B}}{\mathcal{A}} + k^2.\end{aligned}$$

Therefore

$$2TM - P = T^2 - k^2$$

giving us a relationship between the mean M and product P of the two principal membrane stresses. If k and T are constants, this is the same as (2) for a linear Weingarten surface if we substitute curvatures for stress, in other words K for P and H for M and change the constants. The constant T representing an isotropic membrane stress appears in both equations, and if the constant $S = 0$ then (2) becomes the equation of normal equilibrium where p is the pressure.

Now let us for simplicity examine the case when $T = 0$, giving

$$\begin{aligned}\phi &= k\frac{\mathcal{B}}{\mathcal{A}} = \frac{k}{2\mathcal{A}}a_{\alpha\beta}\mathcal{O}^{\alpha\beta}\\ &= \frac{k}{2}\sqrt{\frac{A}{a}}a_{\alpha\beta}\mathcal{O}^{\alpha\beta}.\end{aligned} \tag{5}$$

We have

$$\mathcal{O}^{11}\mathcal{O}^{22} - \mathcal{O}^{12^2} = \frac{1}{A}$$

and therefore ϕ depends only on the ratios between $\mathcal{O}^{11}$, $\mathcal{O}^{12} = \mathcal{O}^{21}$ and $\mathcal{O}^{22}$ and not on their values.

The components of membrane stress,

$$\sigma^{\alpha\beta} = k\mathcal{A}\epsilon^{\alpha\lambda}\epsilon^{\beta\mu}A_{\lambda\mu} \tag{6}$$

and the product of the two principal membrane stresses is equal to

$$P = k^2.$$

Thus as the membrane is deformed and strained, the product of the two principal membrane stresses remains constant at any point moving with the surface, although it may vary from point to point.

One might imagine that it is not possible to make a real physical surface with this interesting property, but we shall see that it can be done, at least in theory by making the surface from a fine grid of zero-length springs, each carrying a tension sufficient for the coils to separate. We described zero-length springs earlier in this paper and Figure 8(a) shows the tension / length relationship for a zero-length spring.

Perhaps the most important property of zero-length springs or constant tension coefficient members is that if the member is projected onto a plane, the member in the plane has the same tension coefficient. This is because the component of length and the component of force are both resolved in the same way.

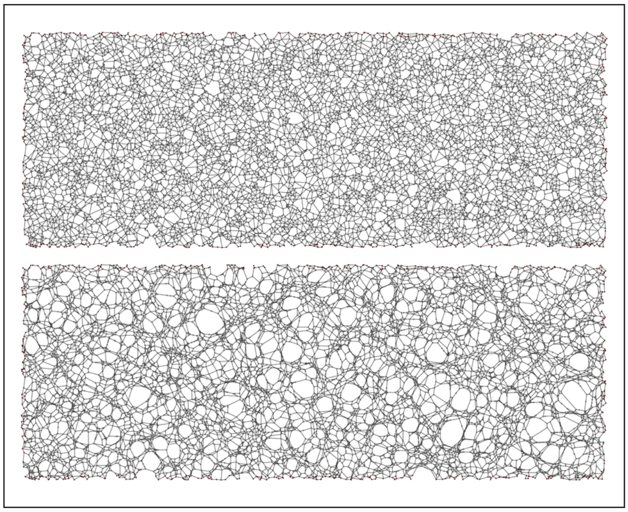

Figure 12. Random grid before and after relaxation. The initial grid consists of a random array of points joined by a Gabriel graph.[46]

The strain energy in a zero-length spring is equal to a the spring stiffness times half its length squared plus a constant. However we are not interested in this constant since we are always interested only in the change of strain energy. Now let us imagine that we have constructed a membrane from some fine grid of zero-length springs. The square of the distance between two points on a surface is equal to $a_{\alpha\beta} d\theta^{\alpha} d\theta^{\beta}$ and therefore the strain energy of the springs contained within a certain area of surface is

$$\sum_{springs} \frac{1}{2} k_{spring} \left(a_{\alpha\beta} d\theta^{\alpha} d\theta^{\beta} \right)_{spring}.$$

However from (5) the strain energy contained within a parallelogram of sides $\delta\theta^1$ and $\delta\theta^2$ is equal to

$$\phi\sqrt{a}\delta\theta^1\delta\theta^2 = \frac{k}{2}\sqrt{A}a_{\alpha\beta}\mathcal{O}^{\alpha\beta}\delta\theta^1\delta\theta^2.$$

These two expressions are identical in that they both involve a summation of a_{11}, $a_{12} = a_{21}$ and a_{22} multiplied by different factors.

Thus we can say that a surface made of a fine grid of zero-length springs produces a membrane in which the product of the principal stresses remains constant as the membrane is stretched in any direction. This product may vary with position as given by the surface coordinates, which are convected with the membrane. This applies regardless of the grid pattern of the springs, which might be triangular, quadratic, hexagonal or even a random array of springs as shown in Figure 12.

Properties of an unloaded homogeneous zero-length spring surface

If the surface is unloaded, then $\mathbf{p} = 0$ in (B2), and if the surface is homogeneous, then k is a constant in (5).

If we replace $A_{\alpha\beta}$ by $a_{\alpha\beta}$ in (6), then $\mathcal{A} = 1$, and in the reference configuration

$$\sigma^{\alpha\beta} = k\epsilon^{\alpha\lambda}\epsilon^{\beta\mu}A_{\lambda\mu} = ka^{\alpha\beta}$$

so that we have a constant isotropic surface tension. Let us choose a coordinate system in the reference configuration such that $A_{11} = A_{22}$ and $A_{12} = 0$ so that we have curvilinear squares on the reference surface. We know that we can always do this by solving Laplace's equation. So now if we return to the current deformed surface, we have

$$\sigma^{11} = k\sqrt{\frac{a}{A}}\frac{A_{22}}{a} = \frac{kA_{22}}{\sqrt{aA_{11}A_{22}}}$$
$$= \frac{k}{\sqrt{a}} = \sigma^{22}$$
$$\sigma^{12} = 0.$$

The fact that $\sqrt{a}\sigma^{11} = \sqrt{a}\sigma^{22} = k = constant$ does not mean that the membrane stress in the direction of $\mathbf{a}_1$, which is equal to $\sigma^{11}a_{11}$, is the same as that in the direction of $\mathbf{a}_2$, which is equal to $\sigma^{22}a_{22}$, since in general $a_{11} \neq a_{22}$.

We can write the equilibrium equation (B2) as

$$\left(\sigma^{\alpha\beta}\sqrt{a}\mathbf{a}_{\beta}\right)_{,\alpha} + \sqrt{a}\mathbf{p} = 0$$

so, if $\mathbf{p} = 0$ we have

$$\begin{aligned} \mathbf{a}_{1,1} + \mathbf{a}_{2,2} &= 0 \\ \mathbf{r}_{,11} + \mathbf{r}_{,22} &= 0 \end{aligned} \tag{7}$$

or in Cartesian coordinates

$$\begin{aligned} x_{,11} + x_{,22} &= 0 \\ y_{,11} + y_{,22} &= 0 \\ z_{,11} + z_{,22} &= 0. \end{aligned} \tag{8}$$

Thus the Cartesian coordinates satisfy Laplace's equation as a function of the surface coordinates.

In the case of radial symmetry, the solution to (8) is simply

$$\begin{aligned} r &= U\cosh\theta^2 + W\sinh\theta^2 \\ x &= r\cos\theta^1 \\ y &= r\sin\theta^1 \\ z &= V\theta^2 \end{aligned}$$

where U, V and W are constants. If $W=0$ and $U=V$ we obtain the catenoid of revolution, which is a minimal surface. If $U=0$ we obtain a surface with a cone-like peak. Returning to the general case, if we scalar multiply the first equation (7) by $\mathbf{a}_1$ we obtain

$$\frac{a_{11,1}}{2} + a_{12,2} - \frac{a_{22,1}}{2} = 0$$

and if we scalar multiplying the same equation by $\mathbf{a}_2$,

$$a_{12,1} - \frac{a_{11,2}}{2} + \frac{a_{22,2}}{2} = 0.$$

Thus

$$\begin{aligned} a_{12,2} + \frac{1}{2}\left(a_{11} - a_{22}\right)_{,1} &= 0 \\ a_{12,1} - \frac{1}{2}\left(a_{11} - a_{22}\right)_{,2} &= 0 \end{aligned}$$

so that

$$\left(a_{11} - a_{22}\right)_{,11} + \left(a_{11} - a_{22}\right)_{,22} = 0$$

and

$$a_{12,11} + a_{12,22} = 0.$$

Thus both $(a_{11} - a_{22})$ and a_{12} satisfy Laplace's equation. This means that if we arrange the boundary conditions such that $a_{12} = 0$ all around the boundary, by allowing nodes to slide, then a_{12} must be zero everywhere on the surface. This, in turn means that

$$\begin{aligned} a_{12} &= 0 \\ \frac{a_{11} - a_{22}}{2} &= D = constant. \end{aligned} \tag{9}$$

This is exactly the same as (3), and this comes as no surprise since we know that the Codazzi equations and the equations of equilibrium tangential to the surface are the same.

Alternatively, if we arrange the boundary conditions such that $a_{11} = a_{22}$ all around the boundary, then $a_{11} - a_{22}$ must be zero everywhere on the surface. and so

$$\begin{aligned} a_{11} &= a_{22} = l^2 \\ a_{12} &= l^2\cos\lambda \\ &= l^2\left(\cos^2\frac{\lambda}{2} - \sin^2\frac{\lambda}{2}\right) = \frac{D}{2} \end{aligned} \tag{10}$$

where λ is the angle between the two sets of springs of length *l*. *D* is the difference between the squares of the diagonals of a rhombus formed by the springs. Thus (9) and (10) are effectively the same case with the one set of coordinate curves being the diagonals of the other, and they correspond to exactly the same state of stress.

If *k* is the stiffness of the zero-length springs, then the tension in the springs is $f = kl$. The shear force on a node on a boundary parallel to the edge of the rhombi is

$$f\cos\lambda = kl\cos\lambda = \frac{kD}{2l}. \tag{11}$$

If we can arrange things such that that $D = 0$ in (9) or in (10), then we have a minimal surface on which the coordinate curves form curvilinear squares.

In Figure 13 the edge cable tension is kept constant, so that the zero-length springs are free to slide along the cables. Thus the zero-length springs meet the edge cable at 90, and therefore $a_{12} = 0$ on the boundaries so that $a_{12} = 0$ everywhere and in the case of Figure 13 $D = 0$ by symmetry, so that we have a minimal surface. The edge cables must be asymptotic lines on the surface to satisfy equilibrium of forces normal to the surface, and it follows that the zero-length springs follow the asymptotic directions.

Figure 14 shows a surface with edge shear forces obeying (11), which means that we should find that we satisfy (10). Note that the spring lines will not be asymptotic directions.

If we rotate the spring grid by 45 the springs follow the principal curvature directions as shown in Figure 15.

Figures 16 and 17 show more applications of geometries controlled by tension coefficients, which are constant in both figures.

If a minimal surface or soap film is bounded by a rigid surface, it slides sideways so that it is normal to the surfaces. If the bounding surface is a sphere or plane, then all directions are principal curvature directions with no twist. It follows from the Joachimsthal theorem[14] that the boundary curve on the minimal surface must be a principal curvature direction, unlike a cable boundary which is an asymptotic direction. This fact was used in automatically generating the principal curvature lines in Figure 16 simply by using the constant tension coefficients.

Zero-length springs plus a variable isotropic membrane stress

Now let us imagine that we have a membrane stress with components

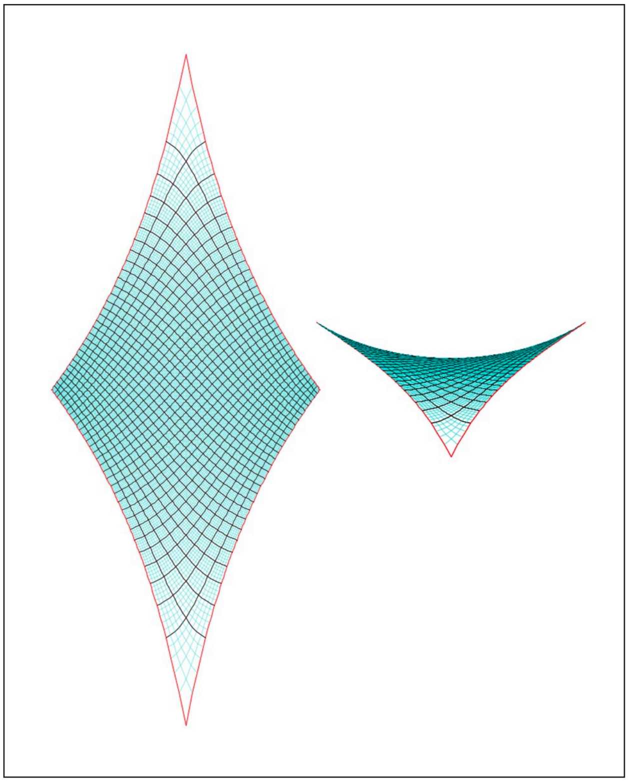

Figure 13. Minimal surface with asymptotic coordinates.

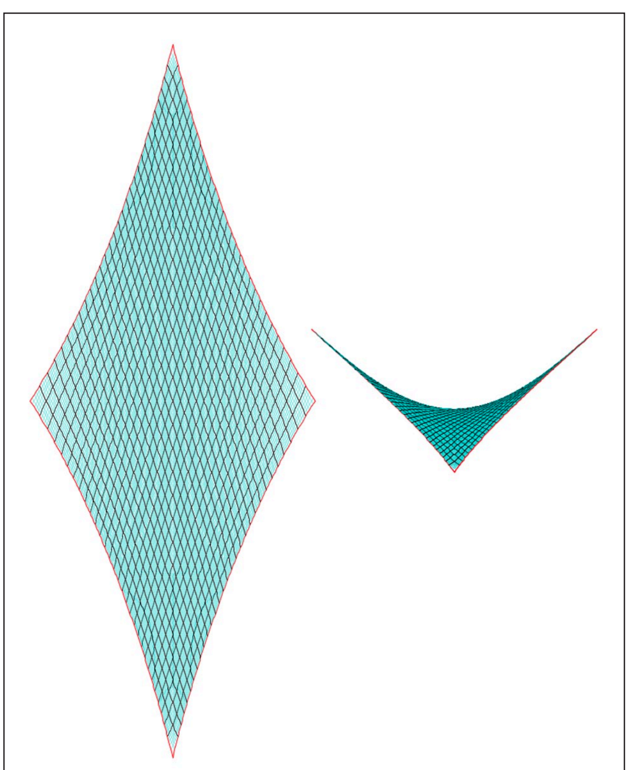

Figure 14. Surface with zero-length springs forming rhombi.

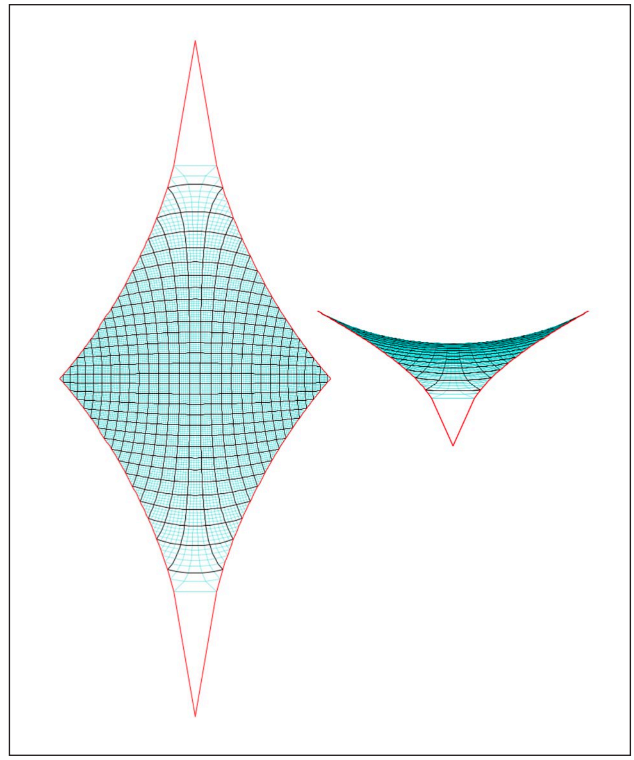

Figure 15. Minimal surface with principal curvature coordinates.

$$\sigma^{\alpha\beta} = \sigma^{\alpha\beta}_{springs} + T_{isotropic} a^{\alpha\beta} \tag{12}$$

in which $\sigma^{\alpha\beta}_{springs}$ is due to zero-length springs and $T_{isotropic}$ corresponds to the varying isotropic membrane stress in a soap film subject to own weight. The let us assume that

$$\sigma^{\alpha\beta}_{springs}\,|_{\alpha} = 0$$
$$T_{isotropic}\,|_{\alpha}\; a^{\alpha\beta} + p^{\beta} = 0$$

so that the tangential component of any load, particularly own weight is carried by $T_{isotropic}$. Now since $T_{isotropic}$ is isotropic it has no influence on the principal stress directions, which means that the direction and spacing of the principal stress trajectories is controlled solely by $\sigma^{\alpha\beta}_{springs}$. Thus in a numerical procedure all we have to do is to combine the zero-length springs with triangular soap film elements. This was done to produce Figures 1, 3, 4, 9 and 10. In the inverted formfinding models the springs are in tension but the soap film elements are in compression, with the compressive stress increasing moving downwards on the hanging model to resist the own weight, which is also downwards. The resulting principal stresses have to be tensile, putting a limit on the isotropic compression.

The mean membrane stress,

$$M = \frac{1}{2} a_{\alpha\beta}\left(\sigma^{\alpha\beta}_{springs} + T_{isotropic} a^{\alpha\beta}\right)$$
$$= M_{springs} + T_{isotropic}$$

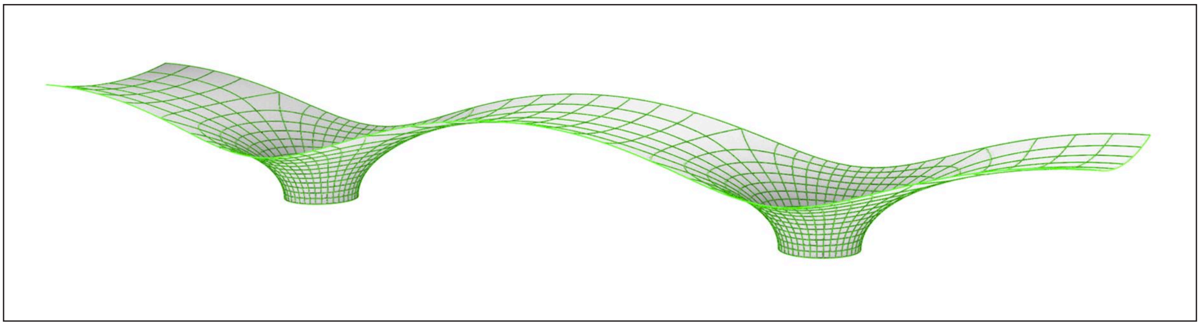

Figure 16. Minimal surface bridge, showing principal curvature lines, supported by a horizontal plane at the support, an inclined plane along the each long side and a vertical plane at each end.

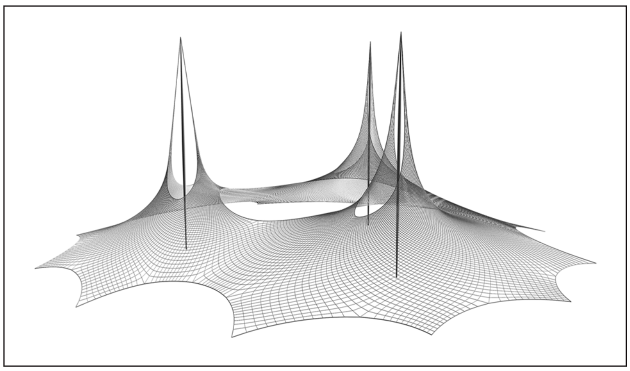

Figure 17. Prestressed cable net.

and the product of the principal stresses,

$$P = \frac{\epsilon_{\alpha\lambda}\epsilon_{\beta\mu}}{2}\left(\sigma^{\alpha\beta}_{springs} + T_{isotropic}a^{\alpha\beta}\right)\left(\sigma^{\lambda\mu}_{springs} + T_{isotropic}a^{\lambda\mu}\right) = P_{springs} + 2M_{springs}T_{isotropic} + T^2_{isotropic}.$$

Thus (12) explicitly gives us both M and P. This is more restrictive than the single condition on the mean curvature H and Gaussian curvature K of a Weingarten surface or a linear Weingarten surface. However the formfinding process requires us to have an explicit expression for stress in terms of the values of the first fundamental form, and the second fundamental form in the case of the bending theory.

Numerical strategy

The numerical models consist of nodes joined by linear and triangular elements

The linear elements exert forces on the nodes with components

$$f_x = \frac{T}{L}\delta x$$
$$f_y = \frac{T}{L}\delta y$$
$$f_z = \frac{T}{L}\delta z$$

where δx, δy and δz are the differences between the Cartesian components at the two ends of a member. $\frac{T}{L}$ is the tension coefficient in the member, that is the tension divided by the current length, and the tension coefficient in any one member remains constant. In some models all the tension coefficients are the same, and in others they vary.

Triangular constant strain elements with three nodes are used to model an isotropic stress. The force on each node acts in a direction perpendicular to the opposite side and has magnitude equal to half the length of the opposite side times the isotropic stress. The triangular elements are also used to exert forces on the nodes due to the own weight of the surface and any pressure load on the surface, as in the bridge in Figure 9. The value of isotropic stress varies due to the tangential component of the own weight, as described in the section *A soap film subject to its own weight* above.

Having calculated the forces on each node, they are moved using Verlet integration,[17] which is the same as dynamic relaxation.[18] In fact we do not need to store the nodal forces since we can increment the velocities directly when we calculate the forces from each element.

It is relatively easy, but tedious, to program the above using the graphics processing unit (GPU). However our experience is that it is not worth using the GPU for this type of work, unlike, for example, three dimensional particle simulations.

Conclusions

In this paper we have described Weingarten surfaces in which the relationship between the mean and Gaussian curvatures produces a controlled spacing of the principal curvature trajectories.

We have also described a formfinding procedure which automatically generates a relationship between the mean and product of the principal membrane stresses. This in turn means that the spacing of the principal stress trajectories is controlled and not arbitrary. The numerical procedure of zero-length springs with a constant tension coefficient or force density plus isotropic soap film-like triangles in compression is relatively simple, although the theory is rather more challenging.

It should be noted that in general the principal curvature and principal membrane stress directions will differ, but in principal they could be made to coincide by adjusting the 'cutting pattern' of the zero-length spring net.

As stated in the introduction, it is 'conventional' in the design of shell structures to have a relatively 'flimsy' shell supported on 'substantial' edge beams, arches and cables. One of the aims of this work was to formfind shapes in which the forces that would have been concentrated in the edge beams or arches are distributed into the shell itself, more like the shells that we see in nature,[28] and it can be seen in the figures that the shells and tension structures have indeed been able to avoid edge beams, arches and cables.

In Appendix D we describe the theory of how a linear Weingarten surface can be obtained by the minimisation of a surface integral. We do not develop the practicalities of doing this and this could form the basis for further research.

Declaration of conflicting interests

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The authors disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: Emil Adiels gratefully acknowledges the funding received from Lars Erik Lundberg Scholarship Foundation (grant ID no. 1001904) and Åke och Greta Lissheds Stiftelse (grant ID no. 2025-029), both of which have provided essential support for his contributions to this research.

ORCID iD

Chris JK Williams https://orcid.org/0000-0002-5089-3340

References

1. Boller G and Schwartz J. Modelling the form. Heinz Isler, Frei Otto and their approaches to form-finding. In Campbell JWP, Baker N, Draper K, et al. (eds.) *Iron, steel and buildings: studies in the history of construction the proceedings of the seventh conference of the construction history society*. Cambridge: Construction History Society, 2020, pp.565–576.
2. Barnes MR. Form finding and analysis of tension structures by dynamic relaxation. *Int J Space Struct* 1999; 14(2): 89–104.
3. Tomlow J. Gaudí's reluctant attitude towards the inverted catenary. *Proc Inst Civil Eng Hist Heritage* 2011; 164(4): 219–233.
4. Boller G, Block P and Schwartz J. Heinz Isler's physical form finding of the HIB tennis shells. *Structures* 2024; 65: 106559.
5. Happold E and Liddell WI. Timber lattice roof for the Mannheim Bundesgartenschau. *Struct Eng* 1975; 53(3): 99–135.
6. Hayman B. Creep buckling — a general view of the phenomena. In Ponter ARS and Hayhurst DR (eds) *Creep in structures*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1981, pp.289–307.
7. Timoshenko S and Woinowsky-Krieger S. *Theory of plates and shells*. 2nd ed. New York: McGraw-Hill, 1959.
8. Adiels E and Williams CJK. The geometry of the equilibrium of forces and moments in shells. *Meccanica* 2025; 60: 499–523.
9. Leroy CFA. *Traité de stéréotomie, Tome I Texte & Tome II Dessins*. cinquième ed. Paris: Gauthier-Villars, 1870.
10. Eisenhart LP. *A treatise on the differential geometry of curves and surfaces*. Boston: Ginn & Company, 1909.
11. Green AE and Zerna W. *Theoretical elasticity*. 2nd ed. Oxford: Oxford University Press, 1968.
12. Dirac PAM. *General theory of relativity*. New York: Wiley, 1975.
13. Eisenhart LP. *An introduction to differential geometry: with use of the tensor calculus*. Princeton, NJ: Princeton University Press, 1947.
14. Struik DJ. *Lectures on classical differential geometry*. Reading, MA: Addison-Wesley, 1961.
15. Tchebychev PL. Sur la coupe des vêtements. In: *Association français pour l'avancement des sciences, Congrès de Paris*. Paris: Imprimerie central des chemins de fer, 1878, pp.154–155.
16. Clough RW. Original formulation of the finite element method. *Finite Elem Anal Des* 1990; 7(2): 89–101.
17. Verlet L. Computer "experiments" on classical fluids. I. Thermodynamical properties of Lennard-Jones molecules. *Phys Rev* 1967; 159: 98–103.
18. Day AS. An introduction to dynamic relaxation. *Engineer* 1965; 219: 218–221.
19. Ogden RW. *Non-linear elastic deformations*. Mineola, New York: Dover, 2013.
20. Green AE and Adkins JE. *Large elastic deformations*. 2nd ed. Oxford: Clarendon Press, 1970.
21. Spencer AJM. George Green and the foundations of the theory of elasticity. *J Eng Math* 2015; 95(1): 5–6.
22. French MJ and Widden MB. The spring-and-lever balancing mechanism, George Carwardine and the Anglepoise lamp. *Proc Inst Mech Eng C J Mech Eng Sci* 2000; 214(3): 501–508.

23. LaCoste LJB, Jr. A new type long period vertical seismograph. *Physics* 1934; 5(7): 178–180.
24. Southwell RV. On the calculation of stresses in braced frameworks. *Proc R Soc Lond Ser A* 1933; 139(839): 475–507.
25. Schek HJ. The force density method for form finding and computation of general networks. *Comput Methods Appl Mech Eng* 1974; 3(1): 115–134.
26. Linkwitz K. About formfinding of double-curved structures. *Eng Struct* 1999; 21(8): 709–718.
27. Linkwitz K and Schek HJ. Einige bemerkungen zur berechnung von vorgespannten seilnetzkonstruktionen. *Ingenieur-Archiv* 1971; 40(3): 145–158.
28. D'Arcy WT. *On Growth and Form*. 2nd ed. Cambridge: Cambridge University Press, 1942.
29. Pinkall U and Gross O. *Differential geometry: from elastic curves to Willmore surfaces*. Cham: Birkhäuser, 2024.
30. Block P and Ochsendorf J. Thrust network analysis: a new methodology for three-dimensional equilibrium. *J Int Assoc Shell Spat Struct* 2007; 48: 167–173.
31. Olivieri C, Angelillo M, Gesualdo A, et al. Parametric design of purely compressed shells. *Mech Mater* 2021; 155: 103782.
32. Olivieri C, Sibille L, Cocking S, et al. Design of purely compressive shells under vertical and horizontal loads through Machine Learning-driven form-finding. *Mech Res Commun* 2025; 148: 104491.
33. Cottrell JA, Hughes TJR and Basilievs Y. *Isogeometric analysis: toward integration of CAD and FEA*. Hoboken, NJ: Wiley, 2009,
34. Williams CJK. The use of modified Bézier triangles for the form finding and analysis of cable net structures. *Struct Eng Rev* 1994; 6(3–4): 245–253.
35. Southwell RV. *Relaxation methods in engineering science*. Oxford: Oxford University Press, 1940.
36. Pellis D, Kilian M, Pottmann H, et al. Computational design of Weingarten surfaces. *ACM Trans Graph* 2021; 40(4): 114.
37. Tellier X, Douthe C, Hauswirth L, et al. Linear-Weingarten membranes with funicular boundaries. *Struct Conc* 2020; 21(6): 2293–2306.
38. Mansouri M, Abdelmagid A, Tošić Z, et al. Corresponding principal and asymptotic patches for negatively-curved gridshell designs. In Dörfler K, Knippers J, Menges A, et al. (eds) *Advances in architectural geometry 2023*. Berlin, Boston: De Gruyter, pp.55–68.
39. Nutbourne AW. A circle diagram for local differential geometry. In: *The mathematics of surfaces: the proceedings of a conference organized by the Institute of Mathematics and its Applications and held at the University of Manchester* (ed Gregory JA), University of Manchester, 17–19 September 1984, pp.59–71. Oxford: Oxford University Press.
40. Timoshenko S. *Strength of materials. Part I elementary theory and problems*. 2nd ed. New York: D. Van Nostrand, 1940.
41. Hilbert D and Cohn-Vossen S. *Geometry and the imagination*. New York: Chelsea Publishing Company, 1956.
42. Burkhardt B. Natural structures - the research of Frei Otto in natural sciences. *Int J Space Struct* 2016; 31(1): 9–15.
43. Ingold L and Rinke M. Sergio Musmeci's search for new forms of concrete structures. In: *5th international conference on construction history*, Chicago, IL, USA, 3–7 June 2015, pp.339–346. Construction History Society of America.
44. Marmo F, Demartino C, Candela G, et al. On the form of the Musmeci's bridge over the Basento river. *Eng Struct* 2019; 191: 658–673.
45. Sane A, Mandre S and Kim I. Surface tension of flowing soap films. *J Fluid Mech* 2018; 841: R2.
46. Gabriel KR and Sokal RR. A new statistical approach to geographic variation analysis. *Syst Zool* 1969; 18(3): 259–278.
47. Cosserat E and Cosserat F. *Théorie des corps déformables*. Paris: Hermann et Fils, 1909.
48. Wright DT. Membrane forces and buckling in reticulated shells. *J Struct Div* 1965; 91(1): 173–201.
49. Timoshenko SP. *History of strength of materials*. New York: McGraw-Hill, 1953.
50. Adiels E, Ander M, Hörteborn E, et al. The use of virtual work for the formfinding of fabric, shell and gridshell structures. In: Hesselgren L, Kilian A, Malek S, et al. (eds) *Advances in architectural geometry 2018*. Vienna: Klein Publishing GmbH. 2018, pp.286–315.
51. Willmore TJ. Note on embedded surfaces. *Al I Cuza Iaşi Mat* 1965; 11B: 493–496.
52. Timoshenko SP and Goodier JN. *Theory of elasticity*. 3rd ed. New York: McGraw-Hill, 1970.
53. Williams C. Use of structural analogy in generation of smooth surfaces for engineering purposes. *Comput Aided Des* 1987; 19(6): 310–322.

Appendix A

Differential geometry and differentiation on a surface

We cannot use Cartesian coordinates on a surface with double curvature. Instead let us use the curvilinear coordinates θ^1 and θ^2 in which θ^1 and θ^2 are two separate parameters, and not θ raised to the power 1 and θ squared. The reason for using θ^1 and θ^2 instead of u and v is that then we can use the Einstein summation convention to write the displacement from the point $\mathbf{r}$ on a surface to an adjacent point $\mathbf{r} + d\mathbf{r}$, also on the surface as

$$\begin{aligned} d\mathbf{r} &= \mathbf{a}_1 d\theta^1 + \mathbf{a}_2 d\theta^2 \\ &= \mathbf{a}_\alpha d\theta^\alpha \end{aligned} \tag{A1}$$

where the covariant base vectors

$$\mathbf{a}_\alpha = \mathbf{r}_{,\alpha} = \frac{\partial \mathbf{r}}{\partial \theta^\alpha}. \tag{A2}$$

In (A1) the repeated α means that α has to be given the values 1 and 2 and the results summed. In the summation convention the repeated index is always written as a superscript and a subscript, unless one is using Cartesian tensors, which we cannot use here.

(A1) is one vector equation, whereas (A2) is two vector equations, applying for $\alpha = 1$ and $\alpha = 2$.

$\mathbf{a}_\alpha$ lie in the tangent plane to the surface, and in general they will not be unit vectors, nor will the be perpendicular to each other.

We only have two coordinates or parameters on our curved membrane surface, but in 3 dimensions we would need 3 coordinates, θ^1, θ^2 and θ^3 and (A1) and (A2) would become

$$d\mathbf{r} = \mathbf{g}_i d\theta^i$$
$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial \theta^i}$$

using the notation in Green and Zerna.[11] Greek indices have the values 1 or 2, whereas Latin indices have the values 1, 2 or 3. Dirac[12] uses Greek indices with the values 0, 1, 2 and 3 for the 4 dimensional space-time of general relativity. The unit normal,

$$\mathbf{n} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{\sqrt{a}}$$

where

$$\sqrt{a} = |\mathbf{a}_1 \times \mathbf{a}_2| = \sqrt{a_{11}a_{22} - a_{12}^2} \tag{A3}$$

and

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta.$$

$a_{\alpha\beta}$ are called the components of the metric tensor or coefficients of the first fundamental form[14] because, if $ds = |d\mathbf{r}|$

$$\begin{aligned} ds^2 = d\mathbf{r} \cdot d\mathbf{r} &= \left(\mathbf{a}_\alpha d\theta^\alpha\right) \cdot \left(\mathbf{a}_\beta d\theta^\beta\right) \\ &= a_{\alpha\beta} d\theta^\alpha d\theta^\beta. \end{aligned} \tag{A4}$$

The area of an elemental parallelogram with sides $d\theta^1$ and $d\theta^2$ is $\sqrt{a} d\theta^1 d\theta^2$. However a is not a scalar because its value depends upon the coordinate system $\left(\theta^1, \theta^2\right)$ that we use. Despite this, the quantities

$$\begin{aligned} \epsilon_{\alpha\beta} &= \left(\mathbf{a}_\alpha \times \mathbf{a}_\beta\right) \cdot \mathbf{n} \\ \epsilon_{11} &= 0 \\ \epsilon_{12} &= -\epsilon_{21} = \sqrt{a} \\ \epsilon_{22} &= 0 \end{aligned}$$

are the components of a tensor, the surface permutation tensor. In order to differentiate on a curved surface it is useful to introduce a second set of base vectors in the local tangent plane to the surface, the contravariant base vectors $\mathbf{a}^\alpha$ defined by

$$d\theta^\alpha = d\mathbf{r} \cdot \mathbf{a}^\alpha.$$

Thus

$$d\theta^\alpha = d\theta^\beta \mathbf{a}_\beta \cdot \mathbf{a}^\alpha$$

and so

$$\begin{aligned} \mathbf{a}_\beta \cdot \mathbf{a}^\alpha = \delta_\beta^\alpha &= 1 \, if \, \alpha = \beta \\ &= 0 \, if \, \alpha \neq \beta. \end{aligned} \tag{A5}$$

δ_β^α are the Kronecker deltas.

It may seem unnecessary to have two sets of base vectors, but they are extremely useful. If $\mathbf{v}$ is any vector on the surface with tangential and normal components,

$$\begin{aligned} \mathbf{v} &= v^\alpha \mathbf{a}_\alpha + v\mathbf{n} \\ &= v_\alpha \mathbf{a}^\alpha + v\mathbf{n} \\ v^\alpha &= \mathbf{v} \cdot \mathbf{a}^\alpha = a^{\alpha\beta} v_\beta \\ v_\alpha &= \mathbf{v} \cdot \mathbf{a}_\alpha = a_{\alpha\beta} v^\beta \end{aligned}$$

in which

$$a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta.$$

In addition we have

$$\begin{aligned} \boldsymbol{\epsilon} &= \epsilon^{\alpha\beta} \mathbf{a}_\alpha \mathbf{a}_\beta = \epsilon_{\alpha\beta} \mathbf{a}^\alpha \mathbf{a}^\beta \\ \epsilon_{\alpha\beta} \epsilon^{\lambda\mu} &= \delta_\alpha^\lambda \delta_\beta^\mu - \delta_\alpha^\mu \delta_\beta^\lambda \\ \epsilon_{\alpha\beta} \epsilon_{\lambda\mu} &= a_{\alpha\lambda} a_{\beta\mu} - a_{\alpha\mu} a_{\beta\lambda} \\ a^{\alpha\beta} &= a^{\alpha\lambda} a_{\lambda\mu} a^{\beta\mu} = \epsilon^{\alpha\lambda} a_{\lambda\mu} \epsilon^{\beta\mu}. \end{aligned} \tag{A6}$$

We can now define the gradient $\nabla \mathbf{Q}$ of a scalar, vector or higher order tensor $\mathbf{Q}$ by

$$\begin{aligned} d\mathbf{Q} &= d\mathbf{r} \cdot \nabla \mathbf{Q} \\ &= d\theta^\alpha \mathbf{Q}_{,\alpha} \end{aligned}$$

where

$$\mathbf{Q}_{,\alpha} = \frac{\partial \mathbf{Q}}{\partial \theta^\alpha}.$$

Thus

$$\nabla \mathbf{Q} = \mathbf{a}^\alpha \mathbf{Q}_{,\alpha}$$

and $\nabla \mathbf{Q}$ is a tensor of order one greater than $\mathbf{Q}$.

We now need to introduce the Christoffel symbols $\Gamma^\eta_{\cdot\alpha\beta}$ and $b_{\alpha\beta}$ the components of the shape operator tensor $\mathbf{b}$ which are defined by

$$\begin{aligned} \mathbf{a}_{\beta,\alpha} &= \mathbf{a}_{\alpha,\beta} = \Gamma^\eta_{\cdot\alpha\beta} \mathbf{a}_\eta + b_{\alpha\beta} \\ \Gamma^\eta_{\cdot\alpha\beta} &= \Gamma^\eta_{\cdot\beta\alpha} = \mathbf{a}^\eta \cdot \mathbf{a}_{\alpha,\beta} \\ &= \frac{a^{\eta\nu}}{2} \left(a_{\nu\alpha,\beta} + a_{\beta\nu,\alpha} - a_{\alpha\beta,\nu}\right) \\ b_{\alpha\beta} &= b_{\beta\alpha} = \mathbf{n} \cdot \mathbf{a}_{\alpha,\beta} = -\mathbf{n}_{,\beta} \cdot \mathbf{a}_\alpha. \end{aligned} \tag{A7}$$

$b_{\alpha\beta}$ are also known as the coefficients of the second fundamental form[14] and they are the components of the tensor,

$$\mathbf{b} = -\nabla\mathbf{n}. \tag{A8}$$

$\Gamma^{\eta}_{\cdot\alpha\beta}$ are not the components of a tensor because they are at least partly a property of the coordinate system. If we change from the coordinates θ^{α} to coordinates $\theta^{\lambda'}$, then

$$\begin{aligned}\Gamma^{\zeta'}_{\cdot\lambda'\mu'} &= \mathbf{a}^{\zeta'}\cdot\mathbf{a}_{\lambda',\mu'} = \theta^{\zeta'}_{,\eta}\mathbf{a}^{\eta}\cdot\left(\theta^{\alpha}_{,\lambda'}\mathbf{a}_{\alpha}\right)_{,\beta}\theta^{\beta}_{,\mu'}\\ &= \Gamma^{\eta}_{\cdot\beta\alpha}\theta^{\zeta'}_{,\eta}\theta^{\alpha}_{,\lambda'}\theta^{\beta}_{,\mu'} + \theta^{\zeta'}_{,\alpha}\theta^{\alpha}_{,\lambda'\mu'}\end{aligned}$$

and it is the presence of $\theta^{\alpha}_{,\lambda'\mu'}$ which breaks the rule for the change of coordinates. It is interesting that if a surface is being deformed so that $\Gamma^{\eta}_{\cdot\beta\alpha}$ are a function of time as well as θ^{α}, then $\dfrac{\partial\Gamma^{\eta}_{\cdot\beta\alpha}}{\partial t}$ are the components of a tensor because the offending $\theta^{\zeta'}_{,\alpha}\theta^{\alpha}_{,\lambda'\mu'}$ term is removed. If $\mathbf{v}$ is any vector on the surface with both tangential and normal components, we can write

$$\nabla\mathbf{v} = \mathbf{a}^{\alpha}\begin{pmatrix}\left(v^{\beta}\,|_{\alpha} - v b^{\beta}_{\alpha}\right)\mathbf{a}_{\beta}\\ +\left(v^{\beta}b_{\alpha\beta} + v\,|_{\alpha}\right)\mathbf{n}\end{pmatrix}$$

where the covariant derivatives,

$$\begin{aligned}v\,|_{\alpha} &= v_{,\alpha}\\ v^{\beta}\,|_{\alpha} &= v^{\beta}_{,\alpha} + v^{\eta}\Gamma^{\beta}_{\cdot\eta\alpha}\end{aligned}$$

which can be extended to tensors of any order. $v^{\beta}\,|_{\alpha}$ are the components of a tensor, whereas $v^{\beta}_{,\alpha}$ are not. Green and Zerna[11] write the covariant derivative as $v^{\beta}\,|_{\alpha}$, whereas Dirac[12] would write it as $v^{\beta}_{:\alpha}$. The covariant derivatives of $a_{\alpha\beta}$, $a^{\alpha\beta}$, $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ are all zero.

$\mathbf{b}$ is a symmetric second order surface tensor, and as such has two principal values and principal directions. $d\mathbf{n}$ and $d\mathbf{r}$ are parallel if

$$\begin{gathered}d\mathbf{n} = \kappa d\mathbf{r}\\ \mathbf{a}^{\alpha}\left(b_{\alpha\beta} - \kappa a_{\alpha\beta}\right)d\theta^{\beta} = 0\end{gathered}$$

and so the determinant $\left|b_{\alpha\beta} - \kappa a_{\alpha\beta}\right| = 0$ and therefore

$$\begin{aligned}&\left(b_{11} - \kappa a_{11}\right)\left(b_{22} - \kappa a_{22}\right)\\ &\quad -\left(b_{12} - \kappa a_{12}\right)^2 = 0.\end{aligned}$$

This quadratic will have two roots, κ_I and κ_{II}, known as the principal curvatures and they occur in orthogonal directions, unless $\kappa_I = \kappa_{II}$ when all directions are principal curvature directions. The product of the principal curvatures is the Gaussian curvature,

$$\begin{aligned}K = \kappa_I\kappa_{II} &= \frac{b_{11}b_{22} - b_{12}^{\ 2}}{a_{11}a_{22} - a_{12}^{\ 2}}\\ &= \frac{1}{2}\epsilon^{\alpha\lambda}\epsilon^{\beta\mu}b_{\alpha\beta}b_{\lambda\mu}\end{aligned} \tag{A9}$$

and the mean of the principal curvatures,

$$\begin{aligned}H &= \frac{\kappa_I + \kappa_{II}}{2}\\ &= \frac{a_{22}b_{11} + a_{11}b_{22} - 2a_{12}b_{12}}{2\left(a_{11}a_{22} - a_{12}^{\ 2}\right)}\\ &= \frac{a^{\alpha\beta}b_{\alpha\beta}}{2} = \frac{b^{\alpha}_{\alpha}}{2}.\end{aligned} \tag{A10}$$

The Gauss-Codazzi equationsIf we form the gradient of the gradient of any tensor $\mathbf{Q}$, we obtain

$$\begin{aligned}\nabla\nabla\mathbf{Q} &= \mathbf{a}^{\alpha}\left(\mathbf{a}^{\beta}\mathbf{Q}_{,\beta}\right)_{,\alpha}\\ &= \mathbf{a}^{\alpha}\left(\mathbf{a}^{\beta}_{,\alpha}\mathbf{Q}_{,\beta} + \mathbf{a}^{\beta}\mathbf{Q}_{,\alpha\beta}\right)\end{aligned}$$

in which, using (A5),

$$\mathbf{a}^{\beta}_{,\alpha} = -\Gamma^{\beta}_{\cdot\eta\alpha}\mathbf{a}^{\eta} + b^{\beta}_{\alpha}\mathbf{n}$$

- note that we can write b^{β}_{α} rather that $b^{\beta}_{\cdot\alpha}$ or $b^{\cdot\beta}_{\alpha}$ because $\mathbf{b}$ is symmetric. Thus

$$\begin{aligned}&\boldsymbol{\epsilon}:\nabla\nabla\mathbf{Q}\\ &= \epsilon^{\lambda\mu}\left(\mathbf{a}_{\lambda}\mathbf{a}_{\mu}\right):\\ &\left(\mathbf{a}^{\alpha}\begin{pmatrix}\left(-\Gamma^{\beta}_{\cdot\eta\alpha}\mathbf{a}^{\eta} + b^{\beta}_{\alpha}\mathbf{n}\right)\mathbf{Q}_{,\beta}\\ +\mathbf{a}^{\beta}\mathbf{Q}_{,\alpha\beta}\end{pmatrix}\right)\\ &= \epsilon^{\lambda\mu}\left(-\Gamma^{\beta}_{\cdot\mu\lambda}\mathbf{Q}_{,\beta} + \mathbf{a}^{\beta}\mathbf{Q}_{,\lambda\beta}\right)\\ &= 0.\end{aligned}$$

This applies to the gradient of the gradient of the normal, so that from (A8),

$$\boldsymbol{\epsilon}:\nabla\mathbf{b} = -\boldsymbol{\epsilon}:\nabla\nabla\mathbf{n} = 0. \tag{A11}$$

These are the Peterson-Mainardi-Codazzi equations, often known as simply the Codazzi equations. (A11) is a vector equation, however it is identically satisfied in the normal direction, so that there are only 2 Codazzi equations. In terms of components,

$$\begin{aligned}0 &= \epsilon^{\alpha\beta}b_{\alpha\zeta}\,|_{\beta}\\ &= \epsilon^{\alpha\beta}\left(b_{\alpha\zeta,\beta} - b_{\eta\zeta}\Gamma^{\eta}_{\cdot\alpha\beta} - b_{\alpha\eta}\Gamma^{\eta}_{\cdot\zeta\beta}\right)\\ &= \epsilon^{\alpha\beta}\left(b_{\alpha\zeta,\beta} - b_{\alpha\eta}\Gamma^{\eta}_{\cdot\zeta\beta}\right)\end{aligned} \tag{A12}$$

so that

$$b_{1\zeta,2} - b_{1\eta}\Gamma^{\eta}_{\cdot\zeta 2} = b_{2\zeta,1} - b_{2\eta}\Gamma^{\eta}_{\cdot\zeta 1}$$
$$b_{11,2} - b_{11}\Gamma^{1}_{\cdot 12} - b_{12}\Gamma^{2}_{\cdot 12} = b_{21,1} - b_{21}\Gamma^{1}_{\cdot 11} - b_{22}\Gamma^{2}_{\cdot 11}.$$

We will find the Codazzi equations useful when the coordinate curves are orthogonal, so that $a_{12} = 0$ and

$$b_{11,2} - b_{11}\frac{a_{11,2}}{2a_{11}} - b_{12}\frac{a_{22,1}}{2a_{22}} = b_{21,1} - b_{21}\frac{a_{11,1}}{2a_{11}} + b_{22}\frac{a_{11,2}}{2a_{22}}$$

or

$$b_{11,2} - \frac{a_{11,2}}{2}\left(\frac{b_{11}}{a_{11}} + \frac{b_{22}}{a_{22}}\right) = b_{12,1} - \frac{b_{12}}{2}\left(\frac{a_{11,1}}{a_{11}} - \frac{a_{22,1}}{a_{22}}\right) = b_{12,1} - b_{12}\left(\log\left(\frac{\sqrt{a_{11}}}{\sqrt{a_{22}}}\right)\right)_{,1} \quad \text{(A13)}$$

and similarly with the indices 1 and 2 interchanged.

If the coordinate curves follow the principal curvature directions the Codazzi equations become[14]

$$b_{11,2} = Ha_{11,2}$$
$$b_{22,1} = Ha_{22,1}$$

where H is the mean curvature (A10).

The third equation relating $a_{\alpha\beta}$ and $b_{\alpha\beta}$ is obtained from

$$\mathbf{a}_{\mu}\cdot\mathbf{a}_{\alpha,\beta\lambda} = \left(\mathbf{a}_{\mu}\cdot\mathbf{a}_{\alpha,\beta}\right)_{,\lambda} - \mathbf{a}_{\mu,\lambda}\cdot\mathbf{a}_{\alpha,\beta} = \Gamma_{\mu\alpha\beta,\lambda} - \left(\Gamma_{\eta\lambda\mu}\mathbf{a}^{\eta} + b^{\mu}_{\lambda}\mathbf{n}\right)\cdot\left(\Gamma_{\zeta\alpha\beta}\mathbf{a}^{\zeta} + b_{\alpha\beta}\mathbf{n}\right) = \Gamma_{\mu\alpha\beta,\lambda} - a^{\eta\zeta}\Gamma_{\eta\lambda\mu}\Gamma_{\zeta\alpha\beta} - b_{\lambda\mu}b_{\alpha\beta}.$$

which must be symmetric in λ and β. Thus, interchanging λ and β, and subtracting,

$$b_{\lambda\mu}b_{\alpha\beta} - b_{\beta\mu}b_{\alpha\lambda} = \Gamma_{\mu\alpha\beta,\lambda} - a^{\eta\zeta}\Gamma_{\eta\lambda\mu}\Gamma_{\zeta\alpha\beta} - \left(\Gamma_{\mu\alpha\lambda,\beta} - a^{\eta\zeta}\Gamma_{\eta\beta\mu}\Gamma_{\zeta\alpha\lambda}\right)$$

so that

$$b_{11}b_{22} - b_{12}^2 = \Gamma_{122,1} - a^{\eta\zeta}\Gamma_{\eta 11}\Gamma_{\zeta 22} - \left(\Gamma_{121,2} - a^{\eta\zeta}\Gamma_{\eta 21}\Gamma_{\zeta 21}\right). \quad \text{(A14)}$$

Thus we can express the Gaussian curvature K in (A9) in terms of $a_{\alpha\beta}$ and their first and second derivatives. This is Gauss's Theorema Egregium, which tells us that we can discover the Gaussian curvature purely by measuring lengths and angles on a surface.

> The Gauss-Codazzi equations, that is Gauss's Theorema Egregium and the Peterson-Mainardi-Codazzi equations, are three equations relating a_{11}, $a_{12} = a_{21}$ and a_{22}, and b_{11}, $b_{12} = b_{21}$ and b_{22} which ensure that the surface fits together. Bonnet's theorem or the fundamental theorem of surface theory states that if $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are known and satisfy the Gauss-Codazzi equations, then the shape of a surface is determined, up to a rigid body translation and rotation.

If the coordinates curves are orthogonal (A14) becomes

$$b_{11}b_{22} - b_{12}^2 = \Gamma_{122,1} - a^{\eta\zeta}\Gamma_{\eta 11}\Gamma_{\zeta 22} - \left(\Gamma_{121,2} - a^{\eta\zeta}\Gamma_{\eta 21}\Gamma_{\zeta 21}\right) = -\frac{a_{22,11}}{2} + \frac{1}{4a_{11}}a_{11,1}a_{22,1} + \frac{1}{4a_{22}}a_{22,2}a_{11,2} - \left(\frac{a_{11,22}}{2} - \frac{1}{4a_{11}}a_{11,2}^2 - \frac{1}{4a_{22}}a_{22,1}^2\right)$$

from which it follows that[14]

$$K = -\frac{1}{\sqrt{a_{11}a_{22}}}\frac{\partial}{\partial\theta^1}\left(\frac{1}{\sqrt{a_{11}}}\frac{\partial\sqrt{a_{22}}}{\partial\theta^1}\right) + \frac{1}{\sqrt{a_{11}a_{22}}}\frac{\partial}{\partial\theta^2}\left(\frac{1}{\sqrt{a_{22}}}\frac{\partial\sqrt{a_{11}}}{\partial\theta^1}\right) = -\frac{1}{\sqrt{a_{11}a_{22}}}\frac{\partial}{\partial\theta^1}\left(\frac{a_{22,1}}{2\sqrt{a_{11}a_{22}}}\right) + \frac{1}{\sqrt{a_{11}a_{22}}}\frac{\partial}{\partial\theta^2}\left(\frac{a_{11,2}}{2\sqrt{a_{11}a_{22}}}\right).$$

Proof of divergence theorem

Consider

$$\int_R \nabla\cdot\mathbf{Q}\,dR = \int_R \mathbf{a}^{\alpha}\cdot\mathbf{Q}_{,\alpha}\sqrt{a}\,d\theta^1 d\theta^2 = \int_R \left(\mathbf{a}^{\alpha}\cdot\mathbf{Q}\sqrt{a}\right)_{,\alpha} d\theta^1 d\theta^2 - \int_R \left(\mathbf{a}^{\alpha}\sqrt{a}\right)_{,\alpha}\cdot\mathbf{Q}\,d\theta^1 d\theta^2$$

where dR is an element of area on the region of surface R. But

$$\left(\mathbf{a}^{\alpha}\sqrt{a}\right)_{,\alpha} = \left(-\Gamma^{\alpha}_{\alpha\beta}\mathbf{a}^{\beta} + \frac{\left(\sqrt{a}\right)_{,\alpha}}{\sqrt{a}}\mathbf{a}^{\alpha}\right) + 2H\sqrt{a}\mathbf{n} = 2H\sqrt{a}\mathbf{n}$$

and

$$\begin{aligned}
&\int_R \left(\mathbf{a}^{\alpha}\cdot\mathbf{Q}\sqrt{a}\right)_{,\alpha} d\theta^1 d\theta^2 \\
&= \int\left(\int\left(\mathbf{a}^1\cdot\mathbf{Q}\sqrt{a}\right)_{,1} d\theta^1\right)d\theta^2 \\
&+ \int\left(\int\left(\mathbf{a}^2\cdot\mathbf{Q}\sqrt{a}\right)_{,2} d\theta^2\right)d\theta^1 \\
&= \oint_{\partial R}\mathbf{a}^1\cdot\mathbf{Q}\sqrt{a}d\theta^2 - \oint_{\partial R}\mathbf{a}^2\cdot\mathbf{Q}\sqrt{a}d\theta^1 \\
&= \oint_{\partial R}\epsilon_{\alpha\beta}\mathbf{a}^{\alpha}\cdot\mathbf{Q}d\theta^{\beta} \\
&= \oint_{\partial R}\left(d\mathbf{r}\times\mathbf{n}\right)\cdot\mathbf{Q}
\end{aligned}$$

in which the minus sign appears because in going anticlockwise around the boundary ∂R $d\theta^1$ is negative at the larger value of θ^2.

Thus

$$\int_R \nabla\cdot\mathbf{Q}dR - 2\int_R H\mathbf{n}\cdot\mathbf{Q}dR = \oint_{\partial R}\left(d\mathbf{r}\times\mathbf{n}\right)\cdot\mathbf{Q}$$

or

$$\int_R \nabla\cdot\mathbf{Q}dR = \oint_{\partial R}\left(d\mathbf{r}\times\mathbf{n}\right)\cdot\mathbf{Q} \qquad \text{(A15)}$$

if $\mathbf{n}\cdot\mathbf{Q}=0$, which does not mean that $\mathbf{Q}$ contains no normal components, so that for example $\mathbf{Q}=\mathbf{vn}$ satisfies $\mathbf{n}\cdot\mathbf{Q}=0$ if the vector $\mathbf{v}$ satisfies $\mathbf{n}\cdot\mathbf{v}=0$.

(A15) is the divergence theorem for a surface.

Appendix B

Strain energy including bending

Consider a surface which is moving and deforming such that the velocity of a typical point is $\mathbf{v}=\mathbf{v}\left(\theta^1,\theta^2,t\right)$ in which t is time. $\mathbf{N}=\mathbf{N}\left(\theta^1,\theta^2,t\right)$ is minus the rate of change of the unit normal and $\mathbf{N}$ and $\mathbf{v}$ are related by

$$\begin{aligned}
&\mathbf{v} = \frac{\partial\mathbf{r}}{\partial t} = v_{\alpha}\mathbf{a}^{\alpha} + v\mathbf{n} \\
&\mathbf{a}_{\alpha}\cdot\mathbf{n} = 0 \\
&\mathbf{a}_{\alpha}\cdot\frac{\partial\mathbf{n}}{\partial t} + \mathbf{v}_{,\alpha}\cdot\mathbf{n} = 0 \\
&\mathbf{N} = -\frac{\partial\mathbf{n}}{\partial t} = \nabla\mathbf{v}\cdot\mathbf{n} \\
&= \mathbf{b}\cdot\mathbf{v} + \nabla v \\
&\mathbf{N}\cdot\mathbf{n} = 0.
\end{aligned}$$

Thus, introducing

$$\mathbf{J} = \mathbf{I} - \mathbf{nn}$$

we can write

$$\begin{aligned}
\nabla\mathbf{v} &= \nabla\mathbf{v}\cdot\mathbf{J} + \nabla\mathbf{v}\cdot\mathbf{nn} \\
&= \Upsilon + \Omega\epsilon + \mathbf{Nn} \\
\Upsilon &= \frac{1}{2}\left(\nabla\mathbf{v}\cdot\mathbf{J} + \left(\nabla\mathbf{v}\cdot\mathbf{J}\right)^T\right) \\
\Omega\epsilon &= \frac{1}{2}\left(\nabla\mathbf{v}\cdot\mathbf{J} - \left(\nabla\mathbf{v}\cdot\mathbf{J}\right)^T\right) \\
\Upsilon_{\alpha\beta} &= \frac{1}{2}\frac{\partial a_{\alpha\beta}}{\partial t}.
\end{aligned}$$

Υ is the rate of membrane strain tensor and Ω is the mean angular velocity of the surface about the normal. If we differentiate (A4) with respect to time,

$$\begin{aligned}
2ds\frac{\partial\left(ds\right)}{\partial t} &= 2\Upsilon_{\alpha\beta}d\theta^{\alpha}d\theta^{\beta} \\
\frac{1}{ds}\frac{\partial\left(ds\right)}{\partial t} &= \frac{\Upsilon_{\alpha\beta}d\theta^{\alpha}d\theta^{\beta}}{a_{\lambda\mu}d\theta^{\lambda}d\theta^{\mu}}.
\end{aligned}$$

Let us introduce the tensor $\boldsymbol{\beta}$ whose components are the rate of change of the components of the shape operator $\mathbf{b}$,

$$\begin{aligned}
\beta_{\alpha\beta} &= \frac{\partial b_{\alpha\beta}}{\partial t} = -\frac{\partial}{\partial t}\left(\mathbf{n}_{,\alpha}\cdot\mathbf{a}_{\beta}\right) \\
&= -\left(\frac{\partial\mathbf{n}}{\partial t}\right)_{,\alpha}\cdot\mathbf{a}_{\beta} - \mathbf{n}_{,\alpha}\cdot\mathbf{v}_{,\beta} \\
&= \mathbf{N}_{,\alpha}\cdot\mathbf{a}_{\beta} + \mathbf{a}_{\beta}\cdot\nabla\mathbf{v}\cdot\mathbf{b}\cdot\mathbf{a}_{\alpha} \\
&= \mathbf{a}_{\alpha}\cdot\nabla\mathbf{N}\cdot\mathbf{a}_{\beta} \\
&+ \mathbf{a}_{\beta}\cdot\left(\Upsilon + \Omega\epsilon\right)\cdot\mathbf{b}\cdot\mathbf{a}_{\alpha} \\
&= \mathbf{a}_{\alpha}\cdot\left(\nabla\mathbf{N} + \mathbf{b}\cdot\left(\Upsilon - \Omega\epsilon\right)\right)\cdot\mathbf{a}_{\beta} \\
\boldsymbol{\beta} &= \nabla\mathbf{N} + \mathbf{b}\cdot\left(\Upsilon - \Omega\epsilon\right).
\end{aligned}$$

The rate of work being done by the boundary forces and moments and the loading forces and moments is equal to,

$$W = \oint_{\partial R} (d\mathbf{r}\times\mathbf{n})\cdot(\boldsymbol{\sigma}\cdot\mathbf{v}+\mathbf{m}\cdot\mathbf{N}) + \int_R (\mathbf{p}\cdot\mathbf{v}+\mathbf{C}\cdot\mathbf{N})\,dR$$

where $\mathbf{p}$ is the load per unit area and $\mathbf{C}$ loading couple per unit area, expressed in such a way that $\mathbf{C}\cdot\mathbf{N}$ is the rate of work being done per unit area. This means that $\mathbf{C}$ has no normal component.

This equation effectively forms the definition of the stress tensor $\boldsymbol{\sigma} = \sigma^{\alpha\beta}\mathbf{a}_\alpha\mathbf{a}_\beta + \sigma^\alpha\mathbf{n}$ which contains both tangential and normal components and has dimensions force per unit length. It is also the definition of the bending momenttensor $\mathbf{m} = m^{\alpha\beta}\mathbf{a}_\alpha\mathbf{a}_\beta$ which only contains tangential components. We are assuming that the rate of work being done by $\mathbf{m}$ depends only upon the angular velocity of the normal, which means that we are assuming no shear deformation, as would occur with a Cosserat shell.[47] We are also assuming that there is no moment about an axis normal to the surface as would occur due to the geodesic curvature of gridshell laths. If the shell has ribs or a grid, we are assuming that the grid is sufficiently fine to be treated as a continuum.[48]

Thus using the divergence theorem (A15),

$$W = \int_R P\,dR$$

where P is the rate of work being absorbed by the surface per unit area and

$$\begin{aligned} P &= \nabla\cdot(\boldsymbol{\sigma}\cdot\mathbf{v}+\mathbf{m}\cdot\mathbf{N})+\mathbf{p}\cdot\mathbf{v} \\ &\quad +\mathbf{C}\cdot\mathbf{N} \\ &= (\nabla\cdot\boldsymbol{\sigma}+\mathbf{p})\cdot\mathbf{v}+\boldsymbol{\sigma}:\nabla\mathbf{v} \\ &\quad +(\nabla\cdot\mathbf{m}+\mathbf{C})\cdot\mathbf{N}+\mathbf{m}:\nabla\mathbf{N} \\ &= (\nabla\cdot\boldsymbol{\sigma}+\mathbf{p})\cdot\mathbf{v}+\boldsymbol{\sigma}:(\Upsilon+\Omega\epsilon) \\ &\quad +(\nabla\cdot\mathbf{m}+\boldsymbol{\sigma}\cdot\mathbf{n}+\mathbf{C})\cdot\mathbf{N} \\ &\quad +\mathbf{m}:(\beta-\mathbf{b}\cdot(\Upsilon-\Omega\epsilon)) \\ &= (\nabla\cdot\boldsymbol{\sigma}+\mathbf{p})\cdot\mathbf{v}+\boldsymbol{\sigma}:\Upsilon \\ &\quad +(\nabla\cdot\mathbf{m}+\boldsymbol{\sigma}\cdot\mathbf{n}+\mathbf{C})\cdot\mathbf{N} \\ &\quad +\mathbf{m}:(\beta-\mathbf{b}\cdot\Upsilon) \\ &\quad +(\boldsymbol{\sigma}:\epsilon+\mathbf{m}:(\mathbf{b}\cdot\epsilon))\Omega \end{aligned} \tag{B1}$$

This equation applies for an velocity $\mathbf{v}$ and angular velocity of an element of surface as specified by $\boldsymbol{\omega}$ and $\mathbf{N}$. They may be 'virtual' and not correspond to the velocity and angular velocity in any real situation. Thus (B1) is the virtual work equation in English, or to be more precise the rate of virtual work or virtual power equation, le principe des puissances virtuelles in French.

It therefore follows that

$$\nabla\cdot\boldsymbol{\sigma}+\mathbf{p} = 0 \tag{B2}$$

which is the equation of equilibrium of forces and

$$\begin{aligned} (\nabla\cdot\mathbf{m})\cdot\mathbf{J}+\boldsymbol{\sigma}\cdot\mathbf{n}+\mathbf{C} &= 0 \\ (\boldsymbol{\sigma}+\mathbf{b}\cdot\mathbf{m}):\epsilon &= 0 \end{aligned}$$

which are the equations of equilibrium of moments. In terms of components they are

$$\begin{aligned} \sigma^{\alpha\beta}|_\alpha - \sigma^\alpha b_\alpha^\beta + p^\beta &= 0 \\ \sigma^{\alpha\beta} b_{\alpha\beta} + \sigma^\alpha|_\alpha + p &= 0 \end{aligned} \tag{B3}$$

and

$$\begin{aligned} m^{\alpha\beta}|_\alpha + \sigma^\beta + C^\beta &= 0 \\ (\sigma^{\alpha\beta} + b_\eta^\alpha m^{\eta\beta})\epsilon_{\alpha\beta} &= 0 \end{aligned} \tag{B4}$$

respectively. (B3) are equations (10.4.4) and (10.4.5) in Green and Zerna[11] and (B4) are (10.4.6) and (10.4.7), except their sign convention for moment takes 'hogging positive' whereas we take 'sagging positive' to go with positive $\mathbf{b}$. Green & Zerna derive their equations in a very different way to that which we have used here. So now (B1) becomes

$$\begin{aligned} P &= \boldsymbol{\sigma}:\Upsilon+\mathbf{m}:(\beta-\mathbf{b}\cdot\Upsilon) \\ &= (\boldsymbol{\sigma}-\mathbf{b}\cdot\mathbf{m}):\Upsilon+\mathbf{m}:\beta \\ &= \frac{1}{2}\frac{\partial a_{\alpha\beta}}{\partial t}(\sigma^{\alpha\beta}-b_\eta^\alpha m^{\eta\beta}) \\ &\quad + m^{\alpha\beta}\frac{\partial b_{\alpha\beta}}{\partial t}. \end{aligned} \tag{B5}$$

This equation applies regardless of whether the shell is made from an elastic material. However, if it is made from an elastic material then

$$P = \frac{1}{\sqrt{a}}\frac{\partial}{\partial t}(\phi\sqrt{a})$$

where ϕ is the strain energy per unit area. The reason why we write this, rather than $P = \frac{\partial\phi}{\partial t}$ is that we have to consider the work done on a particular element of surface, whose current area is $\sqrt{a}\,d\theta^1 d\theta^2$.

We now assume that ϕ is a scalar function of the current values of $a_{\alpha\beta}$ and $b_{\alpha\beta}$, which are functions of time and other quantities containing elastic properties and also initial or reference values of $a_{\alpha\beta}$ and $b_{\alpha\beta}$ *which are independent* *of time, and for which we need some other symbols, such as* $A_{\alpha\beta}$ *and* $B_{\alpha\beta}$. Then

$$\frac{1}{\sqrt{a}}\frac{\partial}{\partial t}\left(\phi\sqrt{a}\right)=\frac{1}{\sqrt{a}}\frac{\partial}{\partial a_{\alpha\beta}}\left(\phi\sqrt{a}\right)\frac{\partial a_{\alpha\beta}}{\partial t}+\frac{\partial\phi}{\partial b_{\alpha\beta}}\frac{\partial b_{\alpha\beta}}{\partial t}$$

since a is independent of $b_{\alpha\beta}$. Now (B5) gives us that

$$\begin{aligned}P&=\frac{1}{\sqrt{a}}\frac{\partial}{\partial a_{\alpha\beta}}\left(\phi\sqrt{a}\right)\frac{\partial a_{\alpha\beta}}{\partial t}+\frac{\partial\phi}{\partial b_{\alpha\beta}}\frac{\partial b_{\alpha\beta}}{\partial t}\\&=\frac{1}{2}\frac{\partial a_{\alpha\beta}}{\partial t}\left(\sigma^{\alpha\beta}-b^{\alpha}_{\eta}m^{\eta\beta}\right)+m^{\alpha\beta}\frac{\partial b_{\alpha\beta}}{\partial t}.\end{aligned}$$

This applies for any $\frac{\partial a_{\alpha\beta}}{\partial t}$ and $\frac{\partial b_{\alpha\beta}}{\partial t}$, and therefore because $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are symmetric, we can only write the symmetric relationships

$$\begin{aligned}&\frac{1}{2}\left(\sigma^{\alpha\beta}-b^{\alpha}_{\eta}m^{\eta\beta}\right)+\frac{1}{2}\left(\sigma^{\beta\alpha}-b^{\beta}_{\eta}m^{\eta\alpha}\right)\\&=\frac{1}{\sqrt{a}}\frac{\partial}{\partial a_{\alpha\beta}}\left(\phi\sqrt{a}\right)+\frac{1}{\sqrt{a}}\frac{\partial}{\partial a_{\beta\alpha}}\left(\phi\sqrt{a}\right)\end{aligned}$$

and

$$\frac{1}{2}\left(m^{\alpha\beta}+m^{\beta\alpha}\right)=\frac{1}{2}\left(\frac{\partial\phi}{\partial b_{\alpha\beta}}+\frac{\partial\phi}{\partial b_{\beta\alpha}}\right).$$

However, if we are careful and only write ϕ so that it is symmetric in $a_{12}=a_{21}$ and $b_{12}=b_{21}$, then

$$\begin{aligned}&\frac{1}{2}\left(\sigma^{\alpha\beta}-b^{\alpha}_{\eta}m^{\eta\beta}\right)+\frac{1}{2}\left(\sigma^{\beta\alpha}-b^{\beta}_{\eta}m^{\eta\alpha}\right)\\&=\frac{2}{\sqrt{a}}\frac{\partial}{\partial a_{\alpha\beta}}\left(\phi\sqrt{a}\right)\end{aligned}$$

and

$$\frac{1}{2}\left(m^{\alpha\beta}+m^{\beta\alpha}\right)=\frac{\partial\phi}{\partial b_{\alpha\beta}}. \quad \text{(B6)}$$

For example, we should write a as $a_{11}a_{22}-a_{12}a_{21}$ and not as $a_{11}a_{22}-a_{12}^2$ and thus Thus

$$\begin{aligned}\frac{\partial a}{\partial a_{11}}&=a_{22}=aa^{11}\\\frac{\partial a}{\partial a_{12}}&=-a_{21}=aa^{12}\\\frac{\partial a}{\partial a_{21}}&=-a_{12}=aa^{21}\\\frac{\partial a}{\partial a_{22}}&=a_{11}=aa^{22}\end{aligned}$$

so that

$$\frac{\partial a}{\partial a_{\alpha\beta}}=aa^{\alpha\beta}. \quad \text{(B7)}$$

From this we also have

$$\begin{aligned}\frac{\partial a^{\lambda\mu}}{\partial a_{\alpha\beta}}&=\frac{\partial}{\partial a_{\alpha\beta}}\left(\epsilon^{\lambda\xi}\epsilon^{\mu\zeta}a_{\xi\zeta}\right)\\&=-a^{\alpha\beta}\epsilon^{\lambda\xi}\epsilon^{\mu\zeta}a_{\xi\zeta}+\epsilon^{\lambda\alpha}\epsilon^{\mu\beta}\\&=-a^{\alpha\beta}a^{\lambda\mu}+\epsilon^{\lambda\alpha}a^{\mu\xi}a^{\beta\zeta}\epsilon_{\xi\zeta}\\&=-a^{\alpha\beta}a^{\lambda\mu}+a^{\mu\lambda}a^{\beta\alpha}-a^{\mu\alpha}a^{\beta\lambda}\\&=-a^{\mu\alpha}a^{\beta\lambda}.\end{aligned} \quad \text{(B8)}$$

We have no more information regarding whether $\mathbf{m}$ is symmetric, and it may not be for a gridshell or ribbed shell which we are treating as a continuum.[48] However for our purposes in this paper we can assume that $\mathbf{m}$ is symmetric so that (B6) becomes

$$m^{\alpha\beta}=\frac{\partial\phi}{\partial b_{\alpha\beta}}. \quad \text{(B9)}$$

We can write the second equation (B4) as

$$\frac{1}{2}\left(\sigma^{\alpha\beta}-\sigma^{\beta\alpha}+b^{\alpha}_{\eta}m^{\eta\beta}-b^{\beta}_{\eta}m^{\eta\alpha}\right)=0$$

and so

$$\sigma^{\alpha\beta}-b^{\beta}_{\eta}m^{\eta\alpha}=2\frac{\partial\phi}{\partial a_{\alpha\beta}}+\phi a^{\alpha\beta}. \quad \text{(B10)}$$

If the loading couple $\mathbf{C}=0$, as is almost invariably the case, then

$$\sigma^{\beta}=-m^{\alpha\beta}\mid_{\alpha}. \quad \text{(B11)}$$

Appendix C

Finite strain

When we are considering a finite strain, which we will have in the formfinding of some membrane structure using some stretchy material, then we have to imagine two separate states, the current state and some previous reference state, which may or may not correspond to an unstressed state.

In fact we are only interested in the components of the metric tensor in the reference state, not in the base vectors or any other aspect of its geometry, it could be folded up in a box. If $a_{\alpha\beta}$ are the components of the metric tensor in the current state and $A_{\alpha\beta}$ are the components of the metric tensor in the reference state, then (A4) gives us,

$$ds^2-dS^2=\left(a_{\alpha\beta}-A_{\alpha\beta}\right)d\theta^{\alpha}d\theta^{\beta}$$

where dS is the distance between two adjacent points in the reference state. We are assuming the the coordinate curves are convected with the membrane, as if they were drawn on the surface. Thus we can write

$$\gamma_{\alpha\beta} = \frac{1}{2}\left(a_{\alpha\beta} - A_{\alpha\beta}\right) \tag{C1}$$

so that

$$\begin{aligned} \frac{1}{2}\left(ds^2 - dS^2\right) &= \gamma_{\alpha\beta} d\theta^\alpha d\theta^\beta \\ \frac{ds^2 - dS^2}{2dS^2} &= \frac{\left(ds - dS\right)}{dS}\frac{\left(ds + dS\right)}{2dS} \\ &= \frac{\gamma_{\alpha\beta} d\theta^\alpha d\theta^\beta}{A_{\lambda\mu} d\theta^\lambda d\theta^\mu} \end{aligned}$$

and if we know $\gamma_{\alpha\beta}$ and $a_{\alpha\beta}$ we know how all lengths on the surface have changed. Thus $\gamma_{\alpha\beta}$ is a measure of strain and from it we can calculate all other strain measures, including logarithmic strain. If the strain is small, then $\frac{\left(ds + dS\right)}{2dS} \approx 1$ so that we arrive at the engineering definition of strain, that is increase in length divided by original length.

If we start with a Cartesian coordinate system $A_{11} = 1$, $A_{12} = 0$ and $A_{22} = 1$, then, if the strain is small

$$\begin{aligned} \gamma_{12} &= \frac{1}{2}\left(a_{12} - 0\right) \\ &= \frac{1}{2}\sqrt{a_{11}a_{22}}\cos\left(\frac{\pi}{2} - \gamma_{xy}\right) \\ &\approx \frac{\gamma_{xy}}{2}. \end{aligned}$$

$\gamma_{xy} = \gamma_{yx}$ the 'engineering shear strain' and is twice the 'mathematical shear strain' $\gamma_{12} = \gamma_{21}$. Tensor equations simply do not work with engineering shear strain, and the mathematical shear strain has to be used for Mohr's circle of strain. The shear modulus and the viscosity are defined in terms of engineering shear strain and engineering shear strain rate, and this means that unexpected factors of 2 appear in equations containing the shear modulus or the viscosity.

We also have

$$\gamma_{\alpha\beta} = \frac{\partial \gamma_{\alpha\beta}}{\partial t}$$

since $A_{\alpha\beta}$ are independent of time.

$\gamma_{\alpha\beta}$ are the components of a tensor, since they obey the rule under a change of coordinates, but to turn them into a tensor we need some base vectors, and the only sensible ones are the current ones, producing the tensor

$$\boldsymbol{\gamma} = \gamma_{\alpha\beta}\mathbf{a}^\alpha\mathbf{a}^\beta.$$

$A_{\alpha\beta}$ are the covariant components of the metric tensor in the reference configuration, and there are also the contravariant components, which one could denote by $A^{\alpha\beta}$, however we will give them the symbol $\mathcal{O}^{\alpha\beta}$ since

$$\mathcal{O}^{\alpha\beta} \neq a^{\alpha\lambda}a^{\beta\mu}A_{\lambda\mu}$$

but

$$\mathcal{O}^{\alpha\beta} = \frac{a}{A}\epsilon^{\alpha\lambda}\epsilon^{\beta\mu}A_{\lambda\mu} \tag{C2}$$

where corresponding to (A3),

$$A = A_{11}A_{22} - A_{12}^2.$$

The quantities $A_{\alpha\beta}$ $\mathcal{O}^{\alpha\beta}$, and A on the undeformed surface are independent of time, but will in general be functions of the coordinates θ^α.

Since $\gamma_{\alpha\beta} = \gamma_{\beta\alpha}$, the tensor $\boldsymbol{\gamma}$ will have principal values corresponding to those in (A9) and (A10),

$$\begin{aligned} \mathcal{Y} &= \gamma_I\gamma_{II} \\ &= \frac{\gamma_{11}\gamma_{22} - \gamma_{12}{}^2}{a_{11}a_{22} - a_{12}{}^2} \\ &= \frac{\begin{pmatrix} a + A - A_{11}a_{22} - a_{11}A_{22} \\ +2a_{12}A_{12} \end{pmatrix}}{a} \\ &= \frac{a + A}{a} \\ &\quad -\left(A_{11}a^{11} + a^{22}A_{22} - 2a^{12}A_{12}\right) \\ &= 1 + \frac{A}{a} - a^{\alpha\beta}A_{\alpha\beta} \end{aligned} \tag{C3}$$

and

$$\begin{aligned} \mathcal{X} &= \frac{\gamma_I + \gamma_{II}}{2} \\ &= \frac{a^{\alpha\beta}\gamma_{\alpha\beta}}{2} = \frac{a^{\alpha\beta}}{4}\left(a_{\alpha\beta} - A_{\alpha\beta}\right) \\ &= \frac{1}{4}\left(2 - a^{\alpha\beta}A_{\alpha\beta}\right). \end{aligned} \tag{C4}$$

(C1) occurs in equation (4.3.3) of Green and Adkins,[20] except that the $a_{\alpha\beta}$ and $A_{\alpha\beta}$ are reversed. We use $a_{\alpha\beta}$ for the current deformed state, whereas they use if for the reference 'undeformed' state. Green and Zerna[11] use $a_{\alpha\beta}$ for the current deformed state of a shell and we follow this precedent.

We could use $\mathcal{Y}$ and $\mathcal{X}$ in (C3) and (C4) as out two strain invariants, however we will find it more convenient to instead use

$$\begin{aligned}\mathcal{A} &= \sqrt{\frac{a}{A}} \\ &= \frac{\text{current area of an element}}{\text{area of an element in reference state}} \\ &= \frac{1}{\sqrt{\frac{1}{2}\epsilon^{\alpha\lambda}\epsilon^{\beta\mu}\left(a_{\alpha\beta}-2\gamma_{\alpha\beta}\right)\left(a_{\lambda\mu}-2\gamma_{\lambda\mu}\right)}} \\ &= \frac{1}{\sqrt{1-2a^{\alpha\beta}\gamma_{\alpha\beta}+2\epsilon^{\alpha\lambda}\epsilon^{\beta\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu}}} \\ &= \frac{1}{\sqrt{1-4\mathcal{X}+4\mathcal{Y}}}\end{aligned} \tag{C5}$$

and

$$\begin{aligned}\mathcal{B} &= \frac{1}{2}\frac{a}{A}a^{\alpha\beta}A_{\alpha\beta} = \frac{1}{2}\frac{a}{A}a_{\alpha\beta}\epsilon^{\alpha\lambda}\epsilon^{\beta\mu}A_{\lambda\mu} \\ &= \frac{1}{2}a_{\alpha\beta}\mathcal{O}^{\alpha\beta} \\ &= \frac{1}{2}\mathcal{A}^2a^{\alpha\beta}\left(a_{\alpha\beta}-2\gamma_{\alpha\beta}\right) \\ &= \mathcal{A}^2\left(1-2\mathcal{X}\right).\end{aligned} \tag{C6}$$

Note that $\sqrt{\frac{a}{A}}$ is a scalar, even though neither a nor A is a scalar.

An invariant is a scalar whose value is independent of the coordinate system. Any function $\mathcal{A}$ of and $\mathcal{B}$ is also an invariant and equation (4.3.4) of Green and Adkins[20] shows their choice. They include a 3rd invariant λ which is the strain in the direction across the membrane, giving the change in thickness. They include this because they want to obtain the properties of the membrane from the properties of a 3 dimensional specimen. However, we are only concerned with the membrane itself, which could, for example, be a knitted fabric.

Isotropic membranes

An isotropic membrane is one which has the same elastic properties in all directions, but which may vary with position. This means that the strain energy per unit area ϕ must be a function of the strain invariants $\mathcal{A}$ and $\mathcal{B}$ in equations (C5) and (C6). Thus if we set the components of moment $m^{\eta\alpha}$ to zero (B10) becomes

$$\begin{aligned}\sigma^{\alpha\beta} &= 2\frac{\partial\phi}{\partial a_{\alpha\beta}}+\phi a^{\alpha\beta} = \frac{2}{\sqrt{a}}\frac{\partial}{\partial a_{\alpha\beta}}\left(\sqrt{a}\phi\right) \\ &= 2\frac{\partial\phi}{\partial\mathcal{A}}\frac{\partial\mathcal{A}}{\partial a_{\alpha\beta}}+2\frac{\partial\phi}{\partial\mathcal{B}}\frac{\partial\mathcal{B}}{\partial a_{\alpha\beta}}+\phi a^{\alpha\beta}.\end{aligned} \tag{C7}$$

From (B7), we have

$$\frac{\partial\mathcal{A}}{\partial a_{\alpha\beta}} = \frac{1}{2\sqrt{aA}}\frac{\partial a}{\partial a_{\alpha\beta}} = \frac{1}{2}\mathcal{A}a^{\alpha\beta}$$

and

$$\frac{\partial\mathcal{B}}{\partial a_{\alpha\beta}} = \frac{\mathcal{O}^{\alpha\beta}}{2}.$$

Thus (C7) becomes

$$\begin{aligned}\sigma^{\alpha\beta} &= \left(\phi+\mathcal{A}\frac{\partial\phi}{\partial\mathcal{A}}\right)a^{\alpha\beta}+\frac{\partial\phi}{\partial\mathcal{B}}\mathcal{O}^{\alpha\beta} \\ &= \frac{\partial}{\partial\mathcal{A}}\left(\mathcal{A}\phi\right)a^{\alpha\beta}+\frac{\partial\phi}{\partial\mathcal{B}}\mathcal{O}^{\alpha\beta}.\end{aligned} \tag{C8}$$

The mean M and product P of the principal membrane stresses are

$$\begin{aligned}M &= \frac{1}{2}\sigma^{\alpha\beta}a_{\alpha\beta} = \frac{\partial}{\partial\mathcal{A}}\left(\mathcal{A}\phi\right) \\ &\quad+\frac{1}{2}\frac{\partial\phi}{\partial\mathcal{B}}a_{\alpha\beta}\mathcal{O}^{\alpha\beta} \\ &= \frac{\partial}{\partial\mathcal{A}}\left(\mathcal{A}\phi\right)+\mathcal{B}\frac{\partial\phi}{\partial\mathcal{B}}\end{aligned} \tag{C9}$$

and, using (A6),

$$\begin{aligned}P &= \frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\lambda\mu}\sigma^{\alpha\lambda}\sigma^{\beta\mu} \\ &= \frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\lambda\mu}\left(\frac{\partial}{\partial\mathcal{A}}\left(\mathcal{A}\phi\right)\right)^2a^{\alpha\lambda}a^{\beta\mu} \\ &\quad+\frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\lambda\mu}\frac{\partial}{\partial\mathcal{A}}\left(\mathcal{A}\phi\right) \\ &\quad\frac{\partial\phi}{\partial\mathcal{B}}\left(a^{\alpha\lambda}\mathcal{O}^{\beta\mu}+\mathcal{O}^{\alpha\lambda}a^{\beta\mu}\right) \\ &\quad+\frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\lambda\mu}\left(\frac{\partial\phi}{\partial\mathcal{B}}\right)^2\mathcal{O}^{\alpha\lambda}\mathcal{O}^{\beta\mu} \\ &= \left(\frac{\partial}{\partial\mathcal{A}}\left(\mathcal{A}\phi\right)\right)^2 \\ &\quad+\frac{\partial}{\partial\mathcal{A}}\left(\mathcal{A}\phi\right)\frac{\partial\phi}{\partial\mathcal{B}}a_{\beta\mu}\mathcal{O}^{\beta\mu} \\ &\quad+\left(\frac{\partial\phi}{\partial\mathcal{B}}\right)^2\frac{a}{A} \\ &= \left(\frac{\partial}{\partial\mathcal{A}}\left(\mathcal{A}\phi\right)\right)^2 \\ &\quad+2\mathcal{B}\frac{\partial}{\partial\mathcal{A}}\left(\mathcal{A}\phi\right)\frac{\partial\phi}{\partial\mathcal{B}}+\left(\frac{\partial\phi}{\partial\mathcal{B}}\right)^2\mathcal{A}^2.\end{aligned}$$

Therefore

$$M^2 - P = \left(\frac{\partial \phi}{\partial \mathcal{B}}\right)^2 \left(\mathcal{B}^2 - \mathcal{A}^2\right). \tag{C10}$$

However, from (C5) and (C6),

$$\begin{aligned}\mathcal{B}^2 - \mathcal{A}^2 &= \mathcal{A}^4 (1 - 2\mathcal{X})^2 \\ &\quad - \mathcal{A}^4 (1 - 4\mathcal{X} + 4\mathcal{Y}) \\ &= 4\mathcal{A}^4 \left(\mathcal{X}^2 - \mathcal{Y}\right) \\ &= \mathcal{A}^4 (\gamma_I - \gamma_{II})^2.\end{aligned}$$

so that[49]

$$\frac{\text{The square of the radius of the Mohr's circle of membrane stress}}{\text{The square of the radius of the Mohr's circle of membrane strain}} = 4\mathcal{A}^4 \left(\frac{\partial \phi}{\partial \mathcal{B}}\right)^2.$$

Appendix D

Linear Weingarten surface obtained by minimisation of a surface integral

A linear Weingarten surface is defined by the equation (2), which is a differential equation containing the curvature of the surface. We will now show that we can also define a linear Weingarten surface by minimising the surface integral

$$\int_R \phi dR$$

over the region R of surface subject to the constraint of the surface enclosing a fixed volume which is closed by a second fixed surface through the same boundary as R. The strain energy per unit area ϕ is equal to a constant times the mean curvature[50] plus a second constant,

$$\phi = T + 2SH = T + Sa^{\lambda\mu} b_{\lambda\mu}$$

where T and S are constants.

The constraint of constant volume is assured by applying a constant pressure to the surface, the pressure is simply the Lagrange multiplier.

Elsewhere in this paper we are only concerned with strain energy that is a function of lengths on a surface, and hence the membrane theory. However here we include the strain energy due to bending. But this is a very special case where strain energy due to bending produces bending moments and normal shear forces which are automatically in equilibrium and can therefore be considered to be 'virtual'. The equations of equilibrium and strain energy due to bending and stretching are included in Appendix B, derived using the divergence theorem and virtual work. For comparison, Appendix E describes the Willmore[51] surface in which the strain energy per unit area is the square of the mean curvature and Appendix F describes minimisation of the surface integral of the Gaussian curvature, which as expected produces a surface which is automatically in equilibrium because of the Gauss-Bonnet theorem.[14]

Then from (B9),

$$m^{\alpha\beta} = \frac{\partial \phi}{\partial b_{\alpha\beta}} = Sa^{\alpha\beta}$$

and from (B11), the components of normal shear force

$$\sigma^{\beta} = -m^{\alpha\beta}\big|_{\alpha} = 0. \tag{D1}$$

We also have, using (B8),

$$\begin{aligned}\frac{\partial \phi}{\partial a_{\alpha\beta}} &= S\frac{\partial a^{\lambda\mu}}{\partial a_{\alpha\beta}} b_{\lambda\mu} \\ &= -Sa^{\mu\alpha} a^{\beta\lambda} b_{\lambda\mu} \\ &= -Sb^{\alpha\beta}\end{aligned}$$

and therefore from (B10)

$$\begin{aligned}\sigma^{\alpha\beta} &= 2\frac{\partial \phi}{\partial a_{\alpha\beta}} + \phi a^{\alpha\beta} + b^{\beta}_{\eta} m^{\eta\alpha} \\ &= -2Sb^{\alpha\beta} + (T + 2SH) a^{\alpha\beta} + Sb^{\alpha\beta} \qquad \text{D2)} \\ &= (T + 2SH) a^{\alpha\beta} - Sb^{\alpha\beta} \\ &= Ta^{\alpha\beta} + S\epsilon^{\alpha\lambda} \epsilon^{\beta\mu} b_{\lambda\mu}\end{aligned}$$

so that we only have symmetric membrane stresses, as in the membrane theory. The principal membrane stresses are in the same direction as the principal curvatures,[37]

$$\begin{aligned}\sigma_I &= T + S\kappa_{II} \\ \sigma_{II} &= T + S\kappa_I\end{aligned} \tag{D3}$$

and this relationship between stress and curvature is essentially the same as with the Airy stress function.[52] The mean and product of the principal membrane stresses are

$$\begin{aligned} M &= \frac{\sigma_I + \sigma_{II}}{2} \\ &= T + S\frac{\left(\kappa_I + \kappa_{II}\right)}{2} = T + SH \end{aligned}$$

and

$$\begin{aligned} P &= \sigma_I \sigma_{II} \\ &= T^2 + TS\left(\kappa_I + \kappa_{II}\right) + S^2 \kappa_I \kappa_{II} \\ &= T^2 + 2TSH + S^2 K. \end{aligned}$$

Using the Codazzi equations in the form of (A12), and remembering that S and T are constants we have

$$\sigma^{\alpha\beta}\,|_{\alpha} = 0$$

and so the surface is in equilibrium with zero tangential loads. However, the tangential component of own weight could be included if we allow T to vary, as we did for soap films.

Equilibrium in the normal direction becomes

$$\begin{aligned} -p &= \sigma^{\alpha\beta} b_{\alpha\beta} \\ &= 2TH + 2SK \end{aligned} \qquad \text{(D4)}$$

which is the equation of a linear Weingarten surface (2), if the normal pressure p is constant.

If we model a surface with flat triangles, then the nodal forces can be calculated as follows. The uniform surface tension produces nodal forces equal to T times one half the opposite side length in a direction in the plane of the triangle perpendicular to the opposite side. The forces due to S are along the folds between the triangles and have a magnitude equal to S times the fold angle in radians.[50] For stability σ_I and σ_{II} have to be positive, corresponding to tension. This gives us a condition on T, S, κ_I and κ_{II} from (D3).

Appendix E

Willmore surface

Let us now minimise the surface integral of the Willmore energy, which is the square of the mean curvature,

$$\phi = H^2$$

so that

$$\begin{aligned} m^{\alpha\beta} &= \frac{\partial \phi}{\partial b_{\alpha\beta}} = H\frac{\partial b_{\lambda\mu}}{\partial b_{\alpha\beta}} a^{\lambda\mu} \\ &= H a^{\alpha\beta} \end{aligned}$$

and from (D8)

$$\begin{aligned} \frac{\partial \phi}{\partial a_{\alpha\beta}} &= H\frac{\partial a^{\lambda\mu}}{\partial a_{\alpha\beta}} b_{\lambda\mu} \\ &= -H a^{\mu\alpha} a^{\beta\lambda} b_{\lambda\mu} \\ &= -H b^{\alpha\beta}. \end{aligned}$$

Thus

$$\begin{aligned} \sigma^{\alpha\beta} &= -2Hb^{\alpha\beta} + H^2 a^{\alpha\beta} + b^{\beta}_{\eta} H a^{\eta\alpha} \\ &= H^2 a^{\alpha\beta} - Hb^{\alpha\beta} \end{aligned}$$

and

$$\sigma^{\beta} = -H\,|_{\alpha}\; a^{\alpha\beta}.$$

We therefore have

$$\begin{aligned} \sigma^{\alpha\beta}\,|_{\alpha} - \sigma^{\alpha} b^{\beta}_{\alpha} &= 2HH\big|_{\alpha} a^{\alpha\beta} - H\big|_{\alpha} b^{\alpha\beta} \\ &\quad - Hb^{\alpha\beta}\,|_{\alpha} + H\big|_{\eta}\, a^{\eta\alpha} b^{\beta}_{\alpha} \\ &= 0 \end{aligned}$$

as expected and

$$\begin{aligned} &\sigma^{\alpha\beta} b_{\alpha\beta} + \sigma^{\alpha}\,|_{\alpha} \\ &= \left(H^2 a^{\alpha\beta} - Hb^{\alpha\beta}\right) b_{\alpha\beta} - H\,|_{\alpha\beta}\; a^{\alpha\beta} \\ &= 2H^3 + H\left(b^{\alpha}_{\alpha} b^{\beta}_{\beta} - b^{\alpha\beta} b_{\alpha\beta}\right) \\ &\quad - Hb^{\alpha}_{\alpha} b^{\beta}_{\beta} - a^{\alpha\beta} H\,|_{\alpha\beta} \\ &= 2H^3 + 2HK - 4H^3 - a^{\alpha\beta} H\,|_{\alpha\beta} \\ &= -\left(a^{\alpha\beta} H\,|_{\alpha\beta} - 2H\left(H^2 - K\right)\right) \\ &= -\left(\nabla^2 H - 2H\left(H^2 - K\right)\right). \end{aligned}$$

Thus, if there is no normal load,

$$\nabla^2 H - 2H\left(H^2 - K\right) = 0$$

which is the equation of a Willmore surface, Theorem 13.10 in Pinkall and Gross[29] and equation (14) in Williams.[53]

Appendix F

Minimising surface integral of the Gaussian curvature

Now let is minimise the integral of the Gaussian curvature,

$$\phi = K$$

so that

$$\begin{aligned} m^{\alpha\beta} &= \frac{1}{2}\frac{\partial}{\partial b_{\alpha\beta}}\left(\epsilon^{\xi\lambda}\epsilon^{\zeta\mu}b_{\xi\zeta}b_{\lambda\mu}\right) \\ &= \epsilon^{\alpha\lambda}\epsilon^{\beta\mu}b_{\lambda\mu} \end{aligned}$$

and (B7)

$$\frac{\partial\phi}{\partial a_{\alpha\beta}} = aK\frac{\partial}{\partial a_{\alpha\beta}}\left(\frac{1}{a}\right) = -Ka^{\alpha\beta}.$$

$$\frac{\partial a}{\partial a_{\alpha\beta}} = aa^{\alpha\beta}.$$

Thus from (B10),

$$\begin{aligned} \sigma^{\alpha\beta} &= -Ka^{\alpha\beta} + b^{\beta}_{\eta}\epsilon^{\eta\lambda}\epsilon^{\alpha\mu}b_{\lambda\mu} \\ &= -Ka^{\alpha\beta} + a^{\beta\rho}b_{\rho\eta}\epsilon^{\eta\lambda}\epsilon^{\alpha\mu}b_{\lambda\mu} \\ &= -Ka^{\alpha\beta} + a^{\beta\rho}\delta^{\alpha}_{\rho}K = 0 \end{aligned}$$

and from (B11) and using the Codazzi equations (A12),

$$\sigma^{\beta} = -\epsilon^{\alpha\lambda}\epsilon^{\beta\mu}b_{\lambda\mu}|_{\alpha} = 0.$$

Thus minimising the surface integral of K produces a shell in equilibrium with no load. This, of course is to be expected because of the Gauss-Bonnet theorem[14] which tells us that the surface integral of the Gaussian curvature depends only upon the Boundary integral of the geodesic curvature. Nevertheless it is instructive that the bending theory is consistent with Gauss-Bonnet.