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THE AIRY STRESS FUNCTION VECTOR IN 3 DIMENSIONS APPLIED TO FRAMES AND SHELLS AND ITS RELATIONSHIP WITH THE BELTRAMI-GÜNTHER TENSORS

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ABSTRACT

The relationship between the geometry of structures and their structural action is perhaps best understood using stress functions. In 2 dimensions the force in a pin jointed truss member is a discontinuity in slope or 'fold' in the Airy stress function surface and a moment in a member is discontinuity in value or 'cliff' in the surface.
We show that Airy stress function is actually only 1 component of a vector in 3 dimensions. Thus we can extend the ideas associated with the Airy stress function to fully 3-dimensional structures.
The Beltrami stress functions in 3 dimensions were extended by Wilhelm Günther to apply to Cosserat materials and we show that the Günther stress function tensors can be derived without introducing the concept of stress. We explain why it is inevitable that we can have both the Maxwell and Morera versions of the Beltrami stress function tensor.
We also develop Hermann Schaefer's concept of a Krustenschale or 'crust shell' in which the forces and moments in a shell are discontinuities in the Günther stress function tensors.

Keywords: *Graphic Statics and Graphical Methods, Gridshells, Masonry, Shell Structures, Space Structures*

1. INTRODUCTION

The aim of this paper is to try and gain a physical understanding of stress functions, in particular the Beltrami stress tensor. It is usual to introduce stress functions as mathematical entities aimed at dealing with the equations of equilibrium as expressed by differential equations in terms of stress. This requires us to make the continuum approximation in which the individual forces between the elements which make up the microstructure of a material are 'smoothed out' over some area which we hope is sufficiently small compared to the dimensions of our structure. We obtain stresses by differentiating the stress functions and therefore the stress functions are smoother than the stress and a concentrated force or infinite stress can be represented by a finite change in the stress function.

Stress functions represent the integral or sum of the forces and moments crossing an imaginary finite area within a structure. The structure might be a framework made of individual elements, a plate or shell of finite thickness or a fully 3 dimensional solid.

Figure 1 shows a model of a shell structure loaded by a whippletree[1], which represents the own weight of the structure and any applied load. Figure 2 shows a diagram of another shell, this time a gridshell, cut by an imaginary surface $\mathcal{O}$ bounded by a closed curve $\mathcal{C}$. $\mathcal{O}$ also cuts through the loading strings and possibly through some of the supporting structure. We have no body loads applied to the structure, we assume that the own weight of the structure itself can be included in the load from the strings or cables. Body loads, sometimes known as body forces, are external loads applied to the material inside a body by external agencies, such as gravity, inertial forces

[1] The word whippletree predates Squire Whipple.

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or electromagnetic forces. In this paper we shall only be concerned with statics and therefore we assume that all loads on a structure are known.

We take the expression 'body loads' to include non-equilibrated boundary stresses on internal voids. Non-equilibrated boundary stresses are stresses whose resultant force or moment are not zero. A uniform pressure within a void is equilibrated, but a hydrostatic pressure varying with depth would not be, since it is due to body loads on the fluid, which we can, of course consider to be part of the structure.

Now let us imagine that we have a second surface $\mathcal{O}'$ also bounded by the same curve $\mathcal{C}$ in figure 2. Thus together $\mathcal{O}$ and $\mathcal{O}'$ contain a certain volume of the shell and loading strings, and possibly also part of the loading frame, which we again assume is weightless. This volume must be in equilibrium, both in terms of forces and in terms of moment about some arbitrary point. Thus, we conclude that there must be the same force and moment passing through $\mathcal{O}$ and through $\mathcal{O}'$, and indeed through any surface bounded by $\mathcal{C}$.

This argument applies to any structure, even a 3-dimensional solid, provided that there are no body loads.

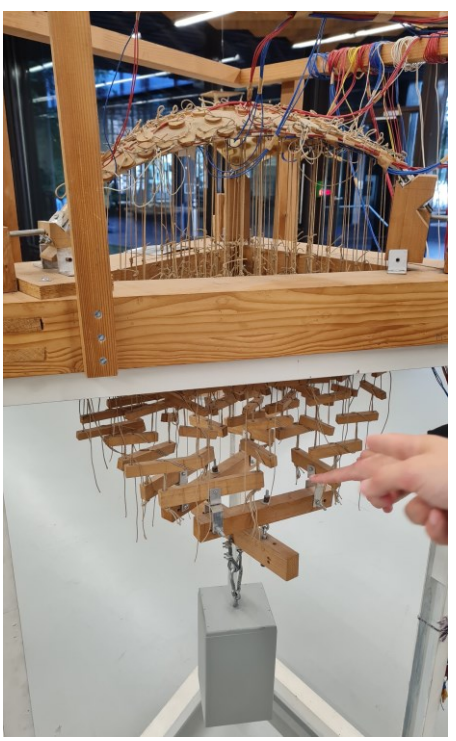

Figure 1: *Model by Heinz Isler at an exhibition organized by Giulia Boller of a shell structure loaded by a whippletree*

1.1. Body loads

As we have stated, in this paper we shall not be concerned with body loads. However, it is worth discussing them briefly to show how they can be dealt with relatively easily if they do exist. We shall use the symbol **S** for stress, and we can always write $\mathbf{S}_{\text{total}} = \mathbf{S}_{\text{body loads}} + \mathbf{S}_{\text{unloaded}}$ where $\mathbf{S}_{\text{body loads}}$ is any distribution of stress in equilibrium with the body loads and $\mathbf{S}_{\text{unloaded}}$ is in equilibrium with no body loads. $\mathbf{S}_{\text{total}}$ has to satisfy the boundary conditions and it also has to be consistent with the strain in the material and the stiffness and strength properties of the material, which may or may not be elastic.

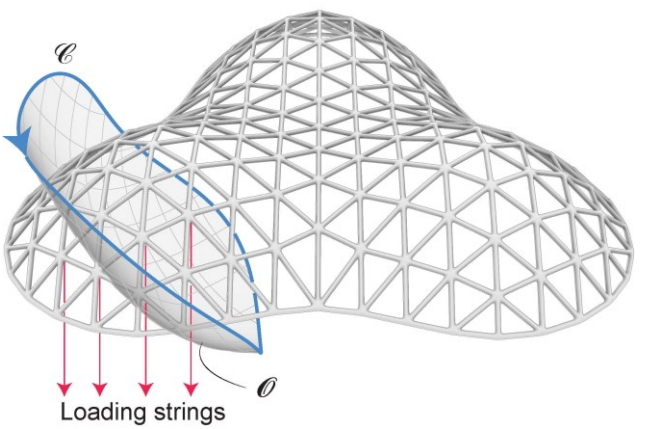

Figure 2: *Shell cut by surface $\mathcal{O}$ bounded by closed curve $\mathcal{C}$*

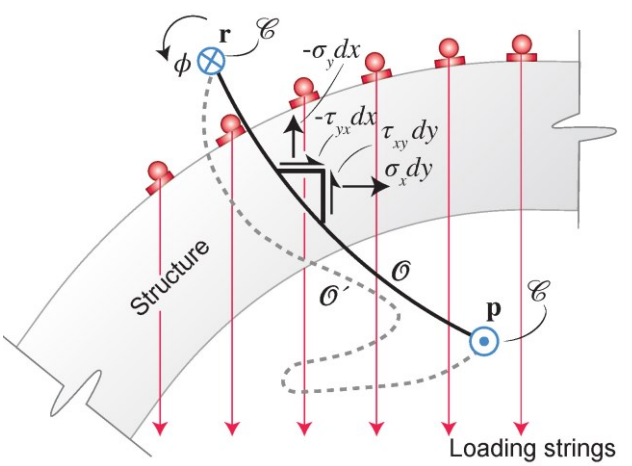

Figure 3: *The Airy stress function ϕ is the moment about* ***r***

However, $\mathbf{S}_{\text{body loads}}$ on its own does not have to obey these constraints, all it has to do is be in equilibrium with the body loads. We can leave looking after the boundary conditions and the material properties to $\mathbf{S}_{\text{unloaded}}$, including

correcting the effects of $\mathbf{S}_{\text{body loads}}$. If a boundary is unstressed, then $\mathbf{S}_{\text{unloaded}}$ is used to remove the unwanted stress due to $\mathbf{S}_{\text{body loads}}$. $\mathbf{S}_{\text{body loads}}$ is not unique, so that the own weight of a material can be resisted by a hydrostatic pressure or a uniaxial vertical stress. However, the difference between these two stress states simply causes a change in $\mathbf{S}_{\text{unloaded}}$. This means that finding some $\mathbf{S}_{\text{body loads}}$ which is in equilibrium with the body loads is a relatively trivial task. For example, the uniaxial vertical stress $S_{zz} = \int \rho g dz$ balances the own weight ρg of a material.

It is therefore important to remember that $\mathbf{S}_{\text{total}} = \mathbf{S}_{\text{body loads}} + \mathbf{S}_{\text{unloaded}}$ is unique, but the difference, $\mathbf{S}_{\text{body loads}} - \mathbf{S}_{\text{unloaded}}$ is not.

For a more general condition Gurtin [14] describes the Beltrami-Schaefer solution, which introduces a vector $\mathbf{h}$ to specify a part of the stress to resist any non-equilibrated boundary stresses applied to internal voids. In 2 dimensions having internal voids corresponds to the space not being simply connected. The surface of a torus is not a simply connected surface, but the torus does not contain any internal voids. On page 58 Gurtin describes the contribution to stress, which we shall write as

$$\mathbf{S} = \nabla\mathbf{h} + (\nabla\mathbf{h})^{\mathrm{T}} - (\nabla \cdot \mathbf{h})\mathbf{I}$$

using the notation that we shall describe below for $\nabla\mathbf{h}$, which is the gradient $\mathbf{h}$. Now this, of course, is very familiar since the stress in a linear elastic solid [25] corresponding to an infinitesimal displacement $\mathbf{h}$ is

$$\mathbf{S} = G\left(\nabla\mathbf{h} + (\nabla\mathbf{h})^{\mathrm{T}}\right) + \frac{2G\nu}{1-2\nu}(\nabla \cdot \mathbf{h})\mathbf{I}$$

where $G = E/\big(2(1+\nu)\big)$ is the shear modulus, E is Young's modulus and ν is Poisson's ratio. Thus the contribution to the stress described by Gurtin is equivalent to that in a material whose shear modulus is equal to 1 and whose Poisson's ratio, $\nu \to \pm\infty$. Such a material is physically impossible, however this is not a problem since we use $\mathbf{S}_{\text{unloaded}}$ to satisfy the material behaviour requirements.

We shall see below that the body load per unit volume $\mathbf{p}_{\text{body}}$ is given by

$$-\mathbf{p}_{\text{body}} = \nabla \cdot \mathbf{S} = G\nabla^2\mathbf{h} + \frac{G}{1-2\nu}\nabla(\nabla \cdot \mathbf{h})$$

which is 3 simultaneous linear partial differential equations in the 3 components of $\mathbf{h}$, if $\mathbf{p}_{\text{body}}$ is known. The Laplacian,

$$\nabla^2\mathbf{h} = \frac{\partial^2\mathbf{h}}{\partial x^2} + \frac{\partial^2\mathbf{h}}{\partial y^2} + \frac{\partial^2\mathbf{h}}{\partial z^2}$$

presents no problem since the x component of $\nabla^2\mathbf{h}$ depends only upon the x component of $\mathbf{h}$, and similarly in the y and z directions.

However, the components of $\nabla(\nabla \cdot \mathbf{h})$ each depend upon all the components of $\mathbf{h}$, making the 3 simultaneous partial differential equations difficult to solve, otherwise elasticity would not be difficult. But setting $\nu \to \pm\infty$ removes the $\nabla(\nabla \cdot \mathbf{h})$ term leaving us with

$$-\mathbf{p}_{\text{body}} = \nabla^2\mathbf{h}$$

if $G = 1$. Thus, if the body loads $\mathbf{p}_{\text{body}}$ are zero, then $\nabla^2\mathbf{h} = 0$, so that $\mathbf{h}$ is a harmonic vector field as described by Gurtin. If $\mathbf{p}_{\text{body}}$ are not zero, then we can still use $\mathbf{h}$ and we simply have to solve 3 independent Poisson equations in h_x, h_y and h_z.

This means that we can focus our attention on the much more difficult problem of finding $\mathbf{S}_{\text{unloaded}}$, and stress functions are used for finding $\mathbf{S}_{\text{unloaded}}$. In fact, as stated above, we can often include body loads as surface loads, so that we do not need $\mathbf{S}_{\text{body loads}}$ at all. We can imagine a 'virtual structures laboratory' freely floating in space, under the action of no gravity. In this virtual laboratory we have structural models and also loading machines with contraptions such as whippletrees (Figure 1) to apply loads and we have to include both the structure and the loading machine in our model.

1.2. The Airy stress function

Let us begin by considering 2-dimensional structures We shall return to 3 dimensions in §2. Figure 3 is a 2-dimensional version of figure 2. There is a concentrated stress in the loading strings, which pass through the shell and are anchored on the top of the shell. The surface $\mathcal{O}$ in figure 2 is replaced by the curve $\mathcal{O}$ in figure 3 and we also see a second curve $\mathcal{O}'$ between the same end points, $\mathbf{p}$ and $\mathbf{r}$. Using the same argument that we used before, there must be the same force and moment about some arbitrary point crossing $\mathcal{O}$ and $\mathcal{O}'$, provided that there are no body loads.

Let us now define ϕ to be the anticlockwise moment about the point **r** of the forces and moments crossing $\mathcal{O}$. ϕ only depends upon the position of **p** and **r** and not upon the shape of the curve itself.

Taking moments about **r** by integrating along $\mathcal{O}$,

$$\phi = \int_{\mathbf{p}}^{\mathbf{r}} (y_{\mathbf{r}} - y)(\sigma_x dy - \tau_{yx} dx) + \int_{\mathbf{p}}^{\mathbf{r}} (x - x_{\mathbf{r}})(\tau_{xy} dy - \sigma_y dx)$$

in which we have used standard engineering notations for the Cauchy stresses $\sigma_x, \tau_{xy}, \tau_{yx}$ and σ_y. The dx terms are negative because we move from right to left along $\mathcal{O}$. Now $(x - x_{\mathbf{r}}) = 0$ and $(y - y_{\mathbf{r}}) = 0$ at **r** so that we can differentiate ϕ to give

$$\frac{\partial \phi}{\partial y_{\mathbf{r}}} = \int_{\mathbf{p}}^{\mathbf{r}} (\sigma_x dy - \tau_{yx} dx)$$
$$\frac{\partial \phi}{\partial x_{\mathbf{r}}} = -\int_{\mathbf{p}}^{\mathbf{r}} (\tau_{xy} dy - \sigma_y dx)$$

in which $\partial\phi/\partial y_{\mathbf{r}}$ is the force in the x direction crossing $\mathcal{O}$ and $-\partial\phi/\partial x_{\mathbf{r}}$ is the force in the y direction.

If we differentiate again and drop the subscript r, we obtain

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \tau_{xy} = \tau_{yx} = -\frac{\partial^2 \phi}{\partial x \partial y} \text{ and } \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \quad (1)$$

so that we find that complementary shear stresses are equal, $\tau_{xy} = \tau_{yx}$. We can also see that ϕ is the Airy stress function [25]. The Airy stress function remains finite even when the curve $\mathcal{O}$ crosses concentrated forces and moments in structural members, and for that reason one could say that it is more fundamental than stress.

It follows directly from (1) that

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0 \text{ and } \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \quad (2)$$

which are the equations of static equilibrium in 2 dimensions in the absence of body forces [25]. Note that we have obtained (2) from (1), rather than the other way around, which is the way it is normally done.

Thus, the second partial derivatives or 'curvature' of the surface representing ϕ give the values of the stress components and so we can describe the state of stress using a geometrical analogue. A concentrated tension or compression in a pin jointed truss member is represented by a discontinuity in slope or 'fold' in the ϕ surface and a moment in a member is represented by a discontinuity in value or 'cliff' in the ϕ surface [27].

We use the word ‘cliff’ rather than jump because for a member of finite size we have a cliff of finite slope, but if we want to represent the member with a line, then the cliff becomes vertical. A cliff can be seen in figure 4 which shows a ring under stress with no applied loads. The stress could perhaps be caused by heating only parts of the ring. The moment in the ring about a vertical axis is caused by a tension towards the outside and a compression towards the inside on the left hand side of the diagram, and the other way around on the right hand side - using the convention of drawing the bending moment on the tension side. In areas of no stress the ϕ surface is a horizontal or sloping plane, which can be described by the value of ϕ at some chosen point and a slope vector $\boldsymbol{\psi}$ which will have 2 components, in the x and y directions. If the ring is sufficiently narrow for the Euler-Bernoulli beam theory to apply, giving a linear distribution of stress, then the shape of the cliff is a cubic.

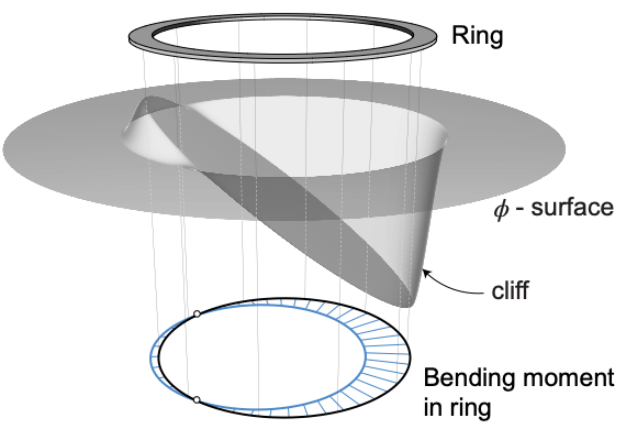

Figure 4: *Airy stress function for ring carrying internal bending moments. The height of the ‘cliff’ gives the value of moment and the narrower the ring the steeper the cliff*

Figure 5 shows the Airy stress function [25] for two equal and opposite point loads in an elastic plate under plane stress. The fold in the surface corresponds to the concentrated force in the loading contraption producing the two loads, perhaps a loop of elastic passing through 2 small holes in the plate. The curvature of the surface is the same on both sides of the fold, so the stress in the plate is continuous.

In the membrane theory of shells horizontal equilibrium is ensured by use of the Airy stress function and vertical equilibrium is satisfied by Pucher's equation,

$$w = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} - 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2}\frac{\partial^2 \phi}{\partial x^2} \qquad (3)$$

where w is the downwards load per unit plan area - see Art. 113 in Timoshenko & Woinowsky- Krieger [24]. Pucher's equation is used for the formfinding analysis of shells and vaults subject to vertical loads [22, 26]. It can be seen that (3) is symmetrical in the shape of the shell as described by z and the state of stress as described by ϕ.

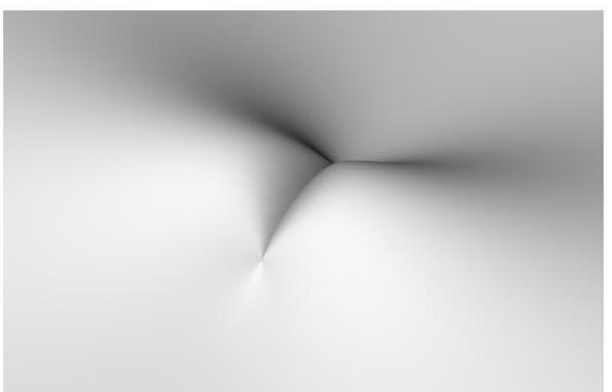

Figure 5: *Airy stress function for two equal and opposite point loads in an elastic plate under plane stress. The fold in the surface corresponds to the concentrated force in the loading contraption producing the two loads. The curvature of the surface is the same on both sides of the fold, so the stress in the plate is continuous*

2. THE BELTRAMI AND GÜNTHER STRESS FUNCTIONS

The Beltrami stress functions or Beltrami stress tensor [14] ensure equilibrium of an unloaded material in 3 dimensions and the Maxwell and Morera stress functions are special cases of the Beltrami stress functions. In 3 dimensions we have 6 components of stress, if we assume complementary shear stresses are equal in a non-Cosserat material. We have 3 equations of equilibrium and therefore we would expect to need only 3 stress functions and the Maxwell and Morera stress functions each provide 3 stress function components. We will see in §7 that if we include both the Maxwell and Morera stress functions, then we obtain all 6 terms of the symmetric Beltrami stress tensor.

It would seem most unfortunate that the Beltrami stress tensor has this built in redundancy. However, we shall see that this redundancy is a necessary product of the fact that we have to be able to extend our stress functions into the unstressed regions of space surrounding a structure, as seen in 2 dimensions in figure 4.

The Günther stress functions [13, 9] ensure equilibrium of an unloaded Cosserat [8, 19] continuum, that is a continuum which can carry couples per unit area as well as the usual Cauchy stresses. However, extra terms can be added to balance body loads such as own weight and this was done by Carlson [6] who also cites a lecture by Schaefer, corresponding to the discussion above regarding the paper by Gurtin [14].

We shall not be concerned with a Cosserat continuum because the behaviour of structural elements can almost always be described using only Cauchy stresses, and this includes such topics as the shear deformation of beams, even though it might be convenient to treat them as Cosserat or Timoshenko-Ehrenfest beams [23] in terms of their overall behaviour. Thus, a deep beam might be treated as a Timoshenko-Ehrenfest beam, in terms of its treatment as a line element, but the shear deformation is controlled by Cauchy stresses in the theory of elasticity.

However, we shall include the Cosserat couples in our derivation because it is easier to do that and then set the couples to zero. In addition, the Günther stress functions enable one to study equilibrium of forces and moments separately. This is quite a subtle point, it makes very little difference to the derivation whether we set the Cosserat moments equal to zero or not, so we might as well include them. On the other hand, they do complicate matters, and Cosserat - like behaviour can be described by saw-tooth discontinuities in the Cauchy stress.

As we have seen, in unstressed areas the Airy stress function corresponds simply to a sloping plane. In the application of the Beltrami-Günther stress functions in 3 dimensions we are similarly interested in what happens in the unstressed area surrounding a structure and following Günther [13, 9] we demonstrate that the stress functions in an unloaded region in 3 dimensions reduce to two vector fields to which we give the symbols $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$.

Of particular interest is Schaefer's [20, 21, 10] concept of a Krustenschale, that is a 'crust shell' in which a shell is treated as a crust across which there is a discontinuity in the stress functions. Schaefer considers a discontinuity in the Cosserat stress functions, however we shall see that moments in the shell can be included even if we set the Cosserat moments to zero.

3. TENSORS

A scalar is a physical quantity, such as density, which has a magnitude and no direction. Force is a vector which has a direction as well as a magnitude. An element of area is also a vector, it has a magnitude measured in square metres and a direction normal to the area.

When we first come across stress, we learn that it is 'force divided by area', but this presents a problem since it is difficult to divide one vector by another. Instead, it is better to define stress by stating that force = area × stress, and to write that in terms of components:

$$\begin{aligned} f_x &= A_x S_{xx} + A_y S_{yx} + A_z S_{zx} \\ f_y &= A_x S_{xy} + A_y S_{yy} + A_z S_{zy} \\ f_z &= A_x S_{xz} + A_y S_{yz} + A_z S_{zz} \end{aligned} \qquad (4)$$

In engineering literature the stress components S_{xx}, S_{yy} and S_{zz} are often written σ_x, σ_y and σ_z, and S_{xy} is often written τ_{xy} and so on for the other components of shear stress. We have used the letter S for stress to be consistent with Günther [13, 9].

(4) follows from consideration of a small tetrahedron, 3 of whose sides are perpendicular to the x, y, z Cartesian axes. It applies even in the presence of body forces or inertia forces since as the size of the tetrahedron tends to 0 its volume decreases faster than the areas of its faces.

We can write the force vector as $\mathbf{f} = f_x\mathbf{i} + f_x\mathbf{j} + f_z\mathbf{k}$ in which $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are unit vectors in the directions of the Cartesian axes. It is preferable to discus $\mathbf{f}$, rather than its components whenever we can. Similarly, we can write the stress as

$$\begin{aligned} \mathbf{S} = S_{xx}\mathbf{ii} + S_{xy}\mathbf{ij} + S_{xz}\mathbf{ik} \\ + S_{yx}\mathbf{ji} + S_{yy}\mathbf{jj} + S_{yz}\mathbf{jk} \\ + S_{zx}\mathbf{ki} + S_{zy}\mathbf{kj} + S_{zz}\mathbf{kk}. \end{aligned}$$

If we introduce the word tensor, first used in a somewhat different way by Sir William Rowan Hamilton, we can say that a scalar is a zeroth order tensor, a vector is a first order tensor and stress is a second order tensor. In 3 dimensions an n^{th} order tensor has 3^n components.

Now we can write

$$\mathbf{f} = \mathbf{A} \cdot \mathbf{S} \qquad (5)$$

by introducing the tensor product of 2 vectors, $\mathbf{u}$ and $\mathbf{v}$, written as $\mathbf{uv}$ and defined by $(\mathbf{uv}) \cdot \mathbf{w} = \mathbf{u}(\mathbf{v} \cdot \mathbf{w})$ and $\mathbf{w} \cdot (\mathbf{uv}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{w}$ where $\mathbf{w}$ is any 3rd vector. The dot - indicates the scalar or inner product. $\mathbf{uv}$ is sometimes written $\mathbf{u} \otimes \mathbf{v}$ and is also described as a dyad or the outer product of $\mathbf{u}$ and $\mathbf{v}$.

In many situations the stress tensor $\mathbf{S}$ is symmetric, 'complementary shear stresses are equal', $S_{xy} = S_{yx}$ etc., or $\mathbf{S} = \mathbf{S}^{\mathrm{T}}$ where $\mathbf{S}^{\mathrm{T}}$ is the transpose of $\mathbf{S}$. If $\mathbf{S}$ is symmetric, then $\mathbf{A} \cdot \mathbf{S} = \mathbf{S} \cdot \mathbf{A}$, but in situations where a second order tensor is not symmetric it is important to get the order correct. In the same way we should always write terms such as $S_{yx}\mathbf{ji}$ with the y and the x in the order corresponding to the $\mathbf{j}$ and the $\mathbf{i}$, this leads also to $S_{xy}\mathbf{ij}$.

Equations such as (4) are rather long in Cartesian coordinates, and to get over this some authors use Cartesian tensors, in which case they would write (4) as $f_b = A_a S_{ab}$ where a and b are equal to 1 , 2 , or 3 corresponding to the Cartesian x, y and z directions. The repeated a implies that a is given the values 1, 2 or 3 and the resulting products are summed. In other words a $\sum_{a=1}^{3}$ is implied by the summation convention.

However, there it is often useful to be able to use other coordinate systems in which a point in space is defined as the intersection of 3 curved surfaces. These surfaces can in general be written $\theta^1 =$ constant, $\theta^2 =$ constant and $\theta^3 =$ constant, which

are 3 independent variables, rather than θ raised to the power 1, 2 , or 3 . We follow Green & Zerna [12] in using the symbols θ^i for the curvilinear coordinates.

In Euclidean space we can write the relationships between θ^i and the Cartesian x, y and z coordinates. For example in spherical polar coordinates $x = \theta^1 \cos\theta^2 \cos\theta^3$, $y = \theta^1 \sin\theta^2 \cos\theta^3$ and $z = \theta^1 \sin\theta^3$, and $\theta^1 = \sqrt{x^2 + y^2 + z^2}, \theta^2 = \tan^{-1}(y/x)$ and $\theta^3 = \sin^{-1}\left(z/\sqrt{x^2 + y^2 + z^2}\right)$.

Hence, we can define the covariant base vectors $\mathbf{g}_i$ and the contravariant base vectors $\mathbf{g}^a$ by

$$\mathbf{g}_i = \frac{\partial x}{\partial \theta^i}\mathbf{i} + \frac{\partial y}{\partial \theta^i}\mathbf{j} + \frac{\partial z}{\partial \theta^i}\mathbf{k}$$
$$\mathbf{g}^a = \frac{\partial \theta^a}{\partial x}\mathbf{i} + \frac{\partial \theta^a}{\partial y}\mathbf{j} + \frac{\partial \theta^a}{\partial z}\mathbf{k}$$

from which it follows that

$$\mathbf{g}_i \cdot \mathbf{g}^a = \frac{\partial \theta^a}{\partial x}\frac{\partial x}{\partial \theta^i} + \frac{\partial \theta^a}{\partial y}\frac{\partial y}{\partial \theta^i} + \frac{\partial \theta^a}{\partial z}\frac{\partial z}{\partial \theta^i}$$
$$= \delta_i^a \begin{cases} = 1 \text{ if } a = i \\ = 0 \text{ if } a \neq i \end{cases}$$

$\mathbf{g}^i$ is normal to the surface θ^i = constant and $\mathbf{g}_1$ is tangent to the intersection of the surfaces θ^2 = constant and θ^3 = constant.

In general, the surfaces θ^i = constant will not be orthogonal to each other. In the theory of structures it is often convenient to use convected coordinates which move with the material and so even if the coordinates surfaces are initially orthogonal, they will not remain so.

We can write $\mathbf{f} = f^j\mathbf{g}_j = f_j\mathbf{g}^j, f^j = \mathbf{f} \cdot \mathbf{g}^j$ and $f_j = \mathbf{f} \cdot \mathbf{g}_j$ from which we can see why it is useful to have both the covariant and contravariant base vectors. Note that the Einstein summation convention always involves a repeated subscript and superscript.
We also have relationships such as $f^i = g^{ij}f_j$ and $\mathbf{g}_i = g_{ij}\mathbf{g}^j$ where the components of the metric tensor, $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ and $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$. The metric tensor is so called because the distance between two adjacent points whose coordinates differ by $d\theta^i$ is equal to $\sqrt{g_{ij}d\theta^i d\theta^j}$. The coordinates always have superscripts and not subscripts. The metric tensor can also be called the unit tensor, $\mathbf{I} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} = \mathbf{g}_i\mathbf{g}^i = g_{ij}\mathbf{g}^i\mathbf{g}^j = g^{ij}\mathbf{g}_i\mathbf{g}_j$ because $\mathbf{I} \cdot \mathbf{v} = \mathbf{v}$ for any vector $\mathbf{v}$. Thus (5) becomes $f^j\mathbf{g}_j = A_k S^{ij}\mathbf{g}^k \cdot \mathbf{g}_i\mathbf{g}_j = A_k S^{ij}\delta_i^k\mathbf{g}_j = A_i S^{ij}\mathbf{g}_j$ or simply $f^j = A_i S^{ij}$.

We shall try and avoid using indices as far as possible, however there are times when they help with the manipulation but les débauches d'indices masquent une réaliteé géométrique souvent très simple [7].

4. DERIVATION OF THE GÜNTHER STRESS FUNCTIONS

As we saw in §1 for the Airy stress function, it is usual in the derivation of stress functions to start with the equations of static equilibrium involving stress, and possibly Cosserat couples, and then show that the equations are automatically satisfied by the stress functions.

However, we can again derive stress functions directly in 3 dimensions. Figure 6 shows the same surface $\mathcal{O}$ bounded by the curve $\mathcal{C}$ that we saw in figure 2.

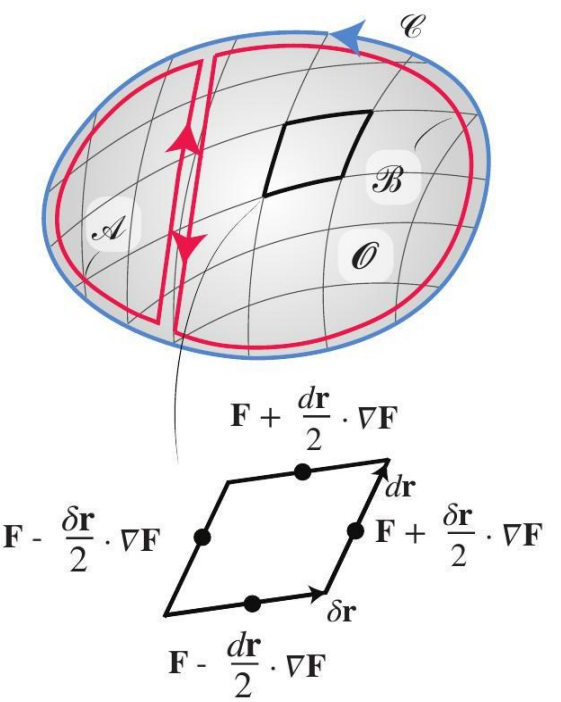

Figure 6: *Surface $\mathcal{O}$ bounded by the curve $\mathcal{C}$. F is a 2nd order tensor*

Using the argument in §1 we find that, in absence of body loads or non-equilibrated boundary stresses on internal voids, the force $\boldsymbol{f}_{\mathcal{C}}$ and moment $\mathbf{m}_{\mathbf{R}\mathcal{C}}$ about some arbitrary point **R** of all the forces and moments passing through $\mathcal{C}$ depend only upon $\mathcal{C}$ and not upon the shape of the surface $\mathcal{O}$ [18].

Also, the sum of the forces passing through curves $\mathcal{A}$ and $\mathcal{B}$ in figure 6 must equal the force passing through $\mathcal{C}$, that is

$$\mathbf{f}_{\mathcal{C}} = \mathbf{f}_{\mathcal{A}} + \mathbf{f}_{\mathcal{B}}. \tag{6}$$

This applies regardless of how we subdivide $\mathcal{O}$ and so we come to the conclusion that there must be some second order tensor **F** such that

$$\mathbf{f}_{\mathcal{C}} = \oint_{\mathcal{C}} d\mathbf{r} \cdot \mathbf{F} \tag{7}$$

where here **r** is a point on $\mathcal{C}$, but in general **r** could be any point in space. Similarly, **F** is on $\mathcal{C}$, but occurs everywhere in space.

We can see that (6) is satisfied since

$$\oint_{\mathcal{C}} d\mathbf{r} \cdot \mathbf{F} = \oint_{\mathcal{A}} d\mathbf{r} \cdot \mathbf{F} + \oint_{\mathcal{B}} d\mathbf{r} \cdot \mathbf{F}$$

because the integrals around $\mathcal{A}$ and $\mathcal{B}$ cancel out where $\mathcal{A}$ and $\mathcal{B}$ travel side by side, but in opposite directions, so that the vectors $d\mathbf{r}$ are equal and opposite.

$\mathbf{f}_{\mathcal{C}}$ and $d\mathbf{r}$ are vectors and $d\mathbf{r} \cdot \mathbf{F}$ is also a vector, so that we can write

$$d\mathbf{r} \cdot \mathbf{F} = d\theta^i \mathbf{g}_i \cdot \left(F_a^{\cdot j} \mathbf{g}^a \mathbf{g}_j\right) = d\theta^i F_i^{jj} \mathbf{g}_j$$

The contributions of all the little areas in figure 6 cancel, except for those edges on the boundary. But the force crossing the little area that is enlarged on its own is

$$\begin{aligned} &d\mathbf{r} \cdot \left(\mathbf{F} + \frac{\delta\mathbf{r}}{2} \cdot \nabla\mathbf{F}\right) - d\mathbf{r} \cdot \left(\mathbf{F} - \frac{\delta\mathbf{r}}{2} \cdot \nabla\mathbf{F}\right) \\ &+\delta\mathbf{r} \cdot \left(\mathbf{F} - \frac{d\mathbf{r}}{2} \cdot \nabla\mathbf{F}\right) - \delta\mathbf{r} \cdot \left(\mathbf{F} + \frac{d\mathbf{r}}{2} \cdot \nabla\mathbf{F}\right) \\ &= (\delta\mathbf{r} d\mathbf{r} - d\mathbf{r}\delta\mathbf{r}) : \nabla\mathbf{F}. \end{aligned} \tag{8}$$

$\nabla\mathbf{F}$ is the gradient of **F**, which is defined by $d\mathbf{F} = d\mathbf{r} \cdot \nabla\mathbf{F}$ for any $d\mathbf{r}$. The gradient $\nabla\mathbf{F}$ is a tensor of one degree higher than **F**, so that the gradient of a scalar is a vector, and so on. In Cartesian coordinates

$$\nabla\mathbf{F} = \mathbf{i}\frac{\partial\mathbf{F}}{\partial x} + \mathbf{j}\frac{\partial\mathbf{F}}{\partial y} + \mathbf{k}\frac{\partial\mathbf{F}}{\partial z}$$

Note that order is important and because we put the $d\mathbf{r}$ in the front in $d\mathbf{r} \cdot \nabla\mathbf{F}$ we have to put the **i** in front of $\partial\mathbf{F}/\partial x$. In curvilinear coordinates

$$\begin{aligned} \nabla\mathbf{F} &= \mathbf{g}^i \frac{\partial\mathbf{F}}{\partial\theta^i} = \mathbf{g}^i \frac{\partial}{\partial\theta^i}\left(F_a^{\cdot j} \mathbf{g}^a \mathbf{g}_j\right) \\ &= \mathbf{g}^i \left(F_{a,i}^{\cdot j} \mathbf{g}^a \mathbf{g}_j - F_a^{\cdot j} \Gamma_{\cdot ki}^a \mathbf{g}^k \mathbf{g}_j + F_a^{\cdot j} \mathbf{g}^a \Gamma_{\cdot ij}^k \mathbf{g}_k\right) \\ &= \nabla_i F_a^{\cdot j} \mathbf{g}^i \mathbf{g}^a \mathbf{g}_j \end{aligned}$$

where the comma indicates partial differentiation, $\Gamma_{\cdot ij}^k = \mathbf{g}^k \cdot \mathbf{g}_{j,i}$ are the Christoffel symbols and $\nabla_i F_a^{\cdot j}$ is the covariant derivative [12, 11], sometimes written $F_{a:i}^{\cdot j}$ or $F_a^{\cdot j}\big|_i$. Because the unit tensor is a constant, the covariant derivatives of g_{ij} and g^{ij} are zero.

We can form the gradient of the gradient or the covariant derivative of the covariant derivative, and in Euclidean space the order of covariant differentiation is immaterial. This is not true of covariant differentiation on a curved surface or in curved space-time, however in this paper we shall always be differentiating in Euclidean space, even when discussing shells. The fact that the order of covariant differentiation does not matter is central to the use of stress functions, and otherwise they would not exist.

However, we can also calculate the force crossing the little area in (8) using the concept of stress and it must be equal to

$$\begin{aligned} (\delta\mathbf{r} \times d\mathbf{r}) \cdot \mathbf{S} &= (\delta\mathbf{r} d\mathbf{r}) : \boldsymbol{\epsilon} \cdot \mathbf{S} \\ &= \frac{1}{2}(\delta\mathbf{r} d\mathbf{r} - d\mathbf{r}\delta\mathbf{r}) : \boldsymbol{\epsilon} \cdot \mathbf{S} \end{aligned} \tag{9}$$

where **S** is the stress tensor and this effectively forms the definition of stress. $\boldsymbol{\epsilon}$ is the permutation tensor or Levi-Civita tensor with Cartesian components $\boldsymbol{\epsilon} = \mathbf{ijk} + \mathbf{jki} + \mathbf{kij} - \mathbf{kji} - \mathbf{ikj} - \mathbf{jik}$ which is used for performing the vector product. If **u** and **v** are any vectors

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \mathbf{v} \cdot \boldsymbol{\epsilon} \cdot \mathbf{u} = \mathbf{v} \cdot (\mathbf{u} \cdot \boldsymbol{\epsilon}) = (\mathbf{uv}) : \boldsymbol{\epsilon} \\ &= u_i v_j \epsilon^{ijk} \mathbf{g}_k = u^i v^j \epsilon_{ijk} \mathbf{g}^k \\ \epsilon_{ijk} &= (\mathbf{g}_i \times \mathbf{g}_j) \cdot \mathbf{g}_k. \end{aligned}$$

Note the use of the : for the double scalar product. Because $\boldsymbol{\epsilon}$ is a constant, the covariant derivatives of its components are zero.

Thus from (8) and (9)

$$(\delta\mathbf{r}d\mathbf{r} - d\mathbf{r}\delta\mathbf{r}):\nabla\mathbf{F} = \frac{1}{2}(\delta\mathbf{r}d\mathbf{r} - d\mathbf{r}\delta\mathbf{r}) : \boldsymbol{\epsilon}\cdot\mathbf{S}$$

which means that

$$\boldsymbol{\epsilon} : \nabla\mathbf{F} = \frac{1}{2}\boldsymbol{\epsilon} : \boldsymbol{\epsilon}\cdot\mathbf{S} = \mathbf{S} \qquad (10)$$

where we have used the 3rd of the identities

$$\begin{aligned}\epsilon^{ijk}\epsilon_{abc} &= \delta^i_a\delta^j_b\delta^k_c + \delta^i_b\delta^j_c\delta^k_a + \delta^i_c\delta^j_a\delta^k_b \\ &\quad -\delta^i_c\delta^j_b\delta^k_a - \delta^i_a\delta^j_c\delta^k_b - \delta^i_b\delta^j_a\delta^k_c \\ \epsilon^{ijk}\epsilon_{abk} &= \delta^i_a\delta^j_b - \delta^i_b\delta^j_a \\ \epsilon^{ijk}\epsilon_{ajk} &= \epsilon^{ijk}\epsilon_{jka} = 2\delta^i_a.\end{aligned}$$

In terms of components (10) can be written

$$S^{ij} = \epsilon^{ipa}\nabla_p F_a^{\cdot j} \qquad (11)$$

and this equation appears as $S^{kl} = \epsilon^{k\lambda\mu}\nabla_\lambda F_\mu^{\cdot l}$ numbered (3.15a) in Günther [13, 9].

Günther's equation numbering is rather complex and this equation appears on page 208 of [13] and page 14 of David Delphenich's translation [9] between equations (3.32a) and (3.33).

Note that we could have written down (11) from (7) using Stokes' theorem, however readers may not be familiar with the application of Stokes' theorem to a 2nd order tensor field, as we have here.

We can immediately see that **F** is not unique because if $\mathbf{F} = \nabla\mathbf{v}$ where **v** is any vector, then $\mathbf{S} = 0$.

The divergence of the stress tensor,

$$\nabla\cdot\mathbf{S} = \mathbf{g}^i\cdot\mathbf{S}_{,i} = \nabla_i S^{ij}\mathbf{g}_j = \epsilon^{ipa}\nabla_{ip}F_a^{\cdot j}\mathbf{g}_j = 0$$

and this is the equation of equilibrium of forces in the absence of body forces. The result is zero because ∇_{ip} is symmetric in i and p, whereas ϵ^{ipa} is antisymmetric in i and p. Note that we have derived the equation of equilibrium of forces from **F**, rather than the other way around.

Günther [13, 9] does not appear to give **F** a name, and we suggest calling it the Günther force tensor.

We can repeat the above exercise on the equilibrium of force with an equivalent exercise on the equilibrium of moments. If the moment about the arbitrary fixed point **R** due to the forces and moments passing through the curve $\mathcal{C}$ is $\mathbf{m}_{\mathbf{R}\mathcal{C}}$ and the moment about the point **X** due to the same forces and moments is $\mathbf{m}_{\mathbf{X}\mathcal{C}}$, then

$$\begin{aligned}\mathbf{m}_{\mathbf{R}\mathcal{C}} &= \mathbf{m}_{\mathbf{X}\mathcal{C}} + (\mathbf{X}-\mathbf{R})\times\mathbf{f}_{\mathcal{C}} \\ &= \mathbf{m}_{\mathbf{X}\mathcal{C}} + \mathbf{f}_{\mathcal{C}}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{X}-\mathbf{R}).\end{aligned}$$

And so, in the same way that we introduced **F**, let us introduce the tensor **G** such that

$$\mathbf{m}_{\mathbf{R}\mathcal{C}} = \oint_{\mathcal{C}} d\mathbf{r}\cdot(\mathbf{G} + \mathbf{F}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R})) \qquad (12)$$

in which **G** represents that part of $\mathbf{m}_{\mathbf{R}\mathcal{C}}$ and $\mathbf{m}_{\mathbf{X}\mathcal{C}}$ which is independent of **R** and **F**. **G** is not unique in the same way that **F** is not unique.

We have

$$\frac{d\mathbf{m}_{\mathbf{R}\mathcal{C}}}{d\mathbf{R}} = \frac{d}{d\mathbf{R}}(\mathbf{R}\cdot\boldsymbol{\epsilon}\cdot\mathbf{f}_{\mathcal{C}}) = \boldsymbol{\epsilon}\cdot\mathbf{f}_{\mathcal{C}}$$

so that $2\mathbf{f}_{\mathcal{C}} = \boldsymbol{\epsilon} : d\mathbf{m}_{\mathbf{R}\mathcal{C}}/d\mathbf{R}$. Thus, as was the case in 2 dimensions in §1, we can obtain forces by taking moments.

Now using the same argument that gave (8) from (7), the moment about **R** crossing the small parallelogram with sides $d\theta^r$ and $\delta\theta^s$ is

$$\begin{aligned}&(\delta\mathbf{r}d\mathbf{r} - d\mathbf{r}\delta\mathbf{r}):\nabla(\mathbf{G} + \mathbf{F}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R})) \\ &= (\delta\mathbf{r}d\mathbf{r} - d\mathbf{r}\delta\mathbf{r}):\begin{pmatrix}\nabla\mathbf{G} + \nabla\mathbf{F}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R}) \\ +\mathbf{g}^k\mathbf{F}\cdot\boldsymbol{\epsilon}\cdot\mathbf{g}_k\end{pmatrix}\end{aligned}$$

since $\mathbf{r}_{,k} = \mathbf{g}_k$. This must be equal to the moment due to the Cosserat [8, 19] couple per unit area, **T**, and Cauchy stress, **S**,

$$\begin{aligned}&(\delta\mathbf{r}\times d\mathbf{r})\cdot(\mathbf{T} + \mathbf{S}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R})) \\ &= (\delta\mathbf{r}d\mathbf{r} - d\mathbf{r}\delta\mathbf{r}):\left(\frac{1}{2}\boldsymbol{\epsilon}\cdot\mathbf{T} + \nabla\mathbf{F}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R})\right)\end{aligned}$$

We have used the symbol **T** to be consistent with Günther and it should not be confused with the superscript $^{\mathrm{T}}$ indicating the transpose of a second order tensor. Note that for our purposes we can set $\mathbf{T} = 0$, but it is useful to include it so that we can see in which ways the Cosserat theory differs from the non-Cosserat theory.

Thus

$$\begin{aligned}\mathbf{T} &= \frac{1}{2}\boldsymbol{\epsilon} : \boldsymbol{\epsilon}\cdot\mathbf{T} = \boldsymbol{\epsilon} : \left(\nabla\mathbf{G} + \mathbf{g}^k\mathbf{F}\cdot\boldsymbol{\epsilon}\cdot\mathbf{g}_k\right) \qquad (13)\\ &= \boldsymbol{\epsilon} : \nabla\mathbf{G} + \mathbf{g}_p\epsilon^{pka}F_a^{\cdot j}\epsilon_{jkk}\mathbf{g}^b \\ &= \boldsymbol{\epsilon} : \nabla\mathbf{G} + \mathrm{tr}(\mathbf{F})\mathbf{I} - \mathbf{F}^{\mathrm{T}} \\ T^i_{\cdot j} &= \epsilon^{ipa}\nabla_p G_{aj} + F_k^{\cdot k}\delta^i_j - F^{\cdot i}_j\end{aligned}$$

where $\mathbf{F}^{\mathrm{T}}$ is again the transpose of $\mathbf{F}$.

(13) appears as $T_{\cdot l}^{k} = \epsilon^{k\lambda\mu}\nabla_{\lambda}G_{\mu l} + \delta_{j}^{k}F_{\alpha}^{\cdot\alpha} - F_{l}^{\cdot k}$ numbered (3.16a) in Günther [13, 9] on the same line as (3.15a) on page 208 of [13] and page 14 of [9]. Thus our tensor $\mathbf{G}$ is the same as Günther's tensor with components $G_{\mu l}$. Again Günther [13, 9] does not appear to give $\mathbf{G}$ a name, and we suggest calling it the Günther moment tensor.

The divergence of $\mathbf{T}$,

$$\begin{aligned}\nabla\cdot\mathbf{T} &= \nabla_{i}T_{\cdot j}^{i}\mathbf{g}^{j}\\ &= \left(\epsilon^{ipa}\nabla_{ip}G_{aj} + \nabla_{i}F_{k}^{\cdot k}\delta_{j}^{i} - \nabla_{i}F_{j}^{\cdot i}\right)\mathbf{g}^{j}\\ &= \left(\nabla_{j}F_{k}^{\cdot k} - \nabla_{k}F_{j}^{\cdot k}\right)\mathbf{g}^{j}.\end{aligned}$$

However, from (11),

$$\epsilon_{rjk}S^{jk} = \epsilon_{rjk}\epsilon^{jpa}\nabla_{p}F_{a}^{\cdot k} = \nabla_{k}F_{j}^{\cdot k} - \nabla_{j}F_{k}^{\cdot k}$$

so that

$$\nabla\cdot\mathbf{T} + \boldsymbol{\epsilon} : \mathbf{S} = 0$$

which is the equation of equilibrium of moments.

Thus, again we have obtained the equation of equilibrium from the Günther moment tensor, rather than the other way around. We see that if the Cosserat moment $\mathbf{T} = 0$, then $\boldsymbol{\epsilon} : \mathbf{S} = 0$ so that $\mathbf{S}$ is symmetric and complementary shear stresses are equal.

We can obtain $\mathbf{F}$ in terms of $\mathbf{T}$ and $\mathbf{G}$ from (13) by first taking its trace, $\mathrm{tr}(\mathbf{T}) = \mathrm{tr}(\boldsymbol{\epsilon} : \nabla\mathbf{G}) + 3\mathrm{tr}(\mathbf{F}) - \mathrm{tr}(\mathbf{F})$ so that

$$\mathbf{F} = (\boldsymbol{\epsilon} : \nabla\mathbf{G} - \mathbf{T})^{\mathrm{T}} - \frac{1}{2}\mathrm{tr}(\boldsymbol{\epsilon} : \nabla\mathbf{G} - \mathbf{T})\mathbf{I} \qquad (14)$$

Therefore from (10)

$$\mathbf{S} = \boldsymbol{\epsilon} : \nabla\left((\boldsymbol{\epsilon} : \nabla\mathbf{G} - \mathbf{T})^{\mathrm{T}} - \frac{\mathrm{tr}(\boldsymbol{\epsilon} : \nabla\mathbf{G} - \mathbf{T})}{2}\mathbf{I}\right) \qquad (15)$$

which satisfies equilibrium of forces and moments in the absence of body loading forces or couples, regardless of the values of the Cosserat moments $\mathbf{T}$ or the Günther moment tensor $\mathbf{G}$.

5. ANALYSIS OF THE GÜNTHER MOMENT TENSOR G

$\mathbf{G}$ is a second order tensor in 3 dimensions and as such will have 9 components. Let us write

$$\begin{aligned}\mathbf{G} &= \mathbf{B} + \nabla\boldsymbol{\phi} - \boldsymbol{\epsilon}\cdot\boldsymbol{\psi} \qquad (16)\\ G_{ba} &= B_{ba} + \nabla_{b}\phi_{a} - \epsilon_{bac}\psi^{c}\end{aligned}$$

where $\mathbf{B} = \mathbf{B}^{\mathrm{T}}$ is a symmetric tensor and $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are vector fields. We shall see that the z component of $\boldsymbol{\phi}$ is effectively the Airy stress function and therefore $\boldsymbol{\phi}$ could termed the Airy stress function vector. We shall also see that $\mathbf{B}$ is the Beltrami stress tensor. $\mathbf{B}$ will have 6 components and $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ have 3 components each, so $\mathbf{B}$, $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are not unique for a given $\mathbf{G}$. Since $\boldsymbol{\epsilon}\cdot\boldsymbol{\psi}$ is an antisymmetric tensor, we could produce any $\mathbf{G}$ from just $\mathbf{B}$ and $\boldsymbol{\epsilon}\cdot\boldsymbol{\psi}$, but we shall see that nevertheless $\nabla\boldsymbol{\phi}$ has an important physical meaning. We can use either the Maxwell or the Morera form of $\mathbf{B}$ and adjust $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ to suit.

Our (16) corresponds to Günther's equation (3.18), although Günther uses $\boldsymbol{\phi}$ for our $\boldsymbol{\psi}$ and $\mathfrak{M}$ for our $\boldsymbol{\phi}$ [13, 9].

From (16)

$$\begin{aligned}\boldsymbol{\epsilon} : \nabla\mathbf{G} &= \boldsymbol{\epsilon} : \nabla\mathbf{B} + \boldsymbol{\epsilon} : \nabla\nabla\boldsymbol{\phi} - \boldsymbol{\epsilon} : (\nabla\boldsymbol{\psi}\cdot\boldsymbol{\epsilon})\\ &= \boldsymbol{\epsilon} : \nabla\mathbf{B} - \nabla\cdot\boldsymbol{\psi}\mathbf{I} + (\nabla\boldsymbol{\psi})^{\mathrm{T}}\end{aligned}$$

Thus, since $\mathbf{B}$ is symmetric,

$$\mathrm{tr}(\boldsymbol{\epsilon} : \nabla\mathbf{G}) = \mathrm{tr}(\boldsymbol{\epsilon} : \nabla\mathbf{B}) - 2\nabla\cdot\boldsymbol{\psi} = -2\nabla\cdot\boldsymbol{\psi}$$

so that (14) becomes

$$\mathbf{F} = (\boldsymbol{\epsilon} : \nabla\mathbf{B})^{\mathrm{T}} + \nabla\boldsymbol{\psi} - \mathbf{T}^{\mathrm{T}} + \frac{1}{2}\mathrm{tr}(\mathbf{T})\mathbf{I} \qquad (17)$$

and, again using (10), (15) becomes

$$\mathbf{S} = \boldsymbol{\epsilon} : \nabla\left((\boldsymbol{\epsilon} : \nabla\mathbf{B})^{\mathrm{T}} - \mathbf{T}^{\mathrm{T}} + \frac{1}{2}\mathrm{tr}(\mathbf{T})\mathbf{I}\right) \qquad (18)$$

$$S^{ij} = \epsilon^{ipa}\epsilon^{jqb}\nabla_{pq}B_{ba} - \epsilon^{ipa}\nabla_{p}\left(T_{\cdot a}^{j} - \frac{1}{2}T_{\cdot k}^{k}\delta_{a}^{j}\right)$$

Thus, of the 9 components of $\mathbf{G}$, the 6 components of $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ make no contribution to the stress $\mathbf{S}$.

Using (17) and the fact that $d\boldsymbol{\psi} = d\mathbf{r}\cdot\nabla\boldsymbol{\psi}$, equation (7) becomes

$$\begin{aligned}\mathbf{f}_{\mathcal{C}} &= \oint_{\mathcal{C}} d\mathbf{r}\cdot(\boldsymbol{\epsilon} : \nabla\mathbf{B})^{\mathrm{T}} + \oint_{\mathcal{C}} d\boldsymbol{\psi} \qquad (19)\\ &\quad -\oint_{\mathcal{C}} d\mathbf{r}\cdot\left(\mathbf{T}^{\mathrm{T}} - \frac{1}{2}\mathrm{tr}(\mathbf{T})\mathbf{I}\right)\end{aligned}$$

Even though $\boldsymbol{\psi}$ does not contribute to stress, it does contribute to the resultant force $\mathbf{f}_{\mathcal{C}}$, provided that there is at least one discontinuity in $\boldsymbol{\psi}$ in going around the closed curve $\mathcal{C}$.

Using (16) and (17) the integrand in (12) becomes

$$\begin{aligned}&\mathbf{G}+\mathbf{F}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R})\\&=\mathbf{B}+\nabla\boldsymbol{\phi}-\boldsymbol{\epsilon}\cdot\boldsymbol{\psi}\\&+\begin{pmatrix}(\boldsymbol{\epsilon}:\nabla\mathbf{B})^{\mathrm{T}}+\nabla\boldsymbol{\psi}\\-\mathbf{T}^{\mathrm{T}}+\frac{1}{2}\mathrm{tr}(\mathbf{T})\mathbf{I}\end{pmatrix}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R})\end{aligned}$$

in which

$$-\boldsymbol{\epsilon}\cdot\boldsymbol{\psi}+\nabla\boldsymbol{\psi}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R})=\nabla(\boldsymbol{\psi}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R}))$$

and

$$\begin{aligned}&\mathbf{B}+(\boldsymbol{\epsilon}:\nabla\mathbf{B})^{\mathrm{T}}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R})\\&=(\nabla(\mathbf{B}\cdot(\mathbf{r}-\mathbf{R})))^{\mathrm{T}}-(\mathbf{r}-\mathbf{R})\cdot\nabla\mathbf{B}\end{aligned}$$

so that

$$\begin{aligned}\mathbf{m}_{\mathbf{R}\mathcal{C}}=&\oint_{\mathcal{C}}\begin{pmatrix}\nabla(\mathbf{B}\cdot(\mathbf{r}-\mathbf{R}))\\-(\mathbf{r}-\mathbf{R})\cdot\nabla\mathbf{B}\end{pmatrix}\cdot d\mathbf{r}\\&+\oint_{\mathcal{C}}d\left(\boldsymbol{\phi}+\boldsymbol{\psi}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R})\right)\\&-\oint_{\mathcal{C}}d\mathbf{r}\cdot\left(\mathbf{T}^{\mathrm{T}}-\frac{1}{2}\mathrm{tr}(\mathbf{T})\mathbf{I}\right)\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R})\end{aligned}\qquad(20)$$

Again, even though $\boldsymbol{\phi}$ does not contribute to $\mathbf{F}$, it does contribute to the resultant moment $\mathbf{m}_{\mathbf{R}\mathcal{C}}$ about $\mathbf{R}$, provided that there is a discontinuity in $\boldsymbol{\phi}$.

6. THE RELATIONSHIP BETWEEN THE BELTRAMI STRESS FUNCTION TENSOR AND THE RATE OF STRAIN TENSOR

We stated in the introduction that we are interested in the unstressed space surrounding structural members or shell structures. If we set $\mathbf{T}=0$ and $\mathbf{S}=0$ in (18) we obtain

$$\boldsymbol{\epsilon}:\nabla(\boldsymbol{\epsilon}:\nabla\mathbf{B})^{\mathrm{T}}=0\text{ or }\epsilon^{ipa}\epsilon^{jqb}\nabla_{pq}B_{ba}=0\qquad(21)$$

and this differential equation in the Beltrami stress function tensor $\mathbf{B}$ is exactly the same as the well known compatibility equation for rate of strain in fluid mechanics, which can be derived as follows.

Consider a fluid undergoing a velocity, $\mathbf{v}$. The gradient of the velocity,

$$\nabla\mathbf{v}=\boldsymbol{\gamma}+\boldsymbol{\omega}$$

where the rate of strain tensor

$$\boldsymbol{\gamma}=\frac{1}{2}\left(\nabla\mathbf{v}+(\nabla\mathbf{v})^{\mathrm{T}}\right)\qquad(22)$$

and the vorticity tensor $\boldsymbol{\omega}=\left(\nabla\mathbf{v}-(\nabla\mathbf{v})^{\mathrm{T}}\right)/2=\boldsymbol{\epsilon}\cdot\boldsymbol{\Omega}$ in which $2\boldsymbol{\Omega}$ is the vorticity vector, the 2 is due to the fact that vorticity was defined as twice the mean angular velocity [4]. We have already come across a version of the infinitesimal strain tensor in §1when we discussed Gurtin's description [14] of the Beltrami-Schaefer solution.

We can write (22) as $\gamma_{ba}=(\nabla_b v_a+\nabla_a v_b)/2$ and so

$$\epsilon^{ipa}\epsilon^{jqb}\nabla_{pq}\gamma_{ba}=0\text{ or }\boldsymbol{\epsilon}:\nabla(\boldsymbol{\epsilon}:\nabla\boldsymbol{\gamma})^{\mathrm{T}}=0\qquad(23)$$

which is exactly the same as (21).

Now, if we have a second order tensor $\boldsymbol{\gamma}$ which satisfies (23), it follows that there exists a velocity field $\mathbf{v}$ such that (22) is satisfied. $\mathbf{v}$ is unique, except for a rigid body motion of the fluid.

It therefore follows that if we have a state of zero stress we can always write $\mathbf{B}$ in terms of a vector field $\boldsymbol{\xi}$ where

$$\mathbf{B}=\frac{1}{2}\left(\nabla\boldsymbol{\xi}+(\nabla\boldsymbol{\xi})^{\mathrm{T}}\right)\qquad(24)$$

If we can construct the right hand side of (24) from $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ then we can set $\mathbf{B}=0$ in an area of zero stress.

If we set the Cosserat moments $\mathbf{T}=0$ and the Beltrami stress function tensor $\mathbf{B}=0$ in (19) we obtain

$$\mathbf{f}_{\mathcal{C}}=\oint_{\mathcal{C}}d\boldsymbol{\psi}\qquad(25)$$

and if we set $\mathbf{T}=0$ and $\mathbf{B}=0$ in (20)

$$\mathbf{m}_{\mathbf{R}\mathcal{C}}=\oint_{\mathcal{C}}d(\boldsymbol{\phi}+\boldsymbol{\psi}\cdot\boldsymbol{\epsilon}\cdot(\mathbf{r}-\mathbf{R}))\qquad(26)$$

Thus if the curve $\mathcal{C}$ passes though unstressed space surrounding one or more structural elements containing stress but no body loads, the force $\mathbf{f}_{\mathcal{C}}$ and moment $\mathbf{m}_{\mathbf{R}\mathcal{C}}$ about $\mathbf{R}$ are determined by discontinuities in $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$.

7. STANDARD FORMS OF THE BELTRAMI STRESS FUNCTION TENSOR

We can write the Maxwell and Morera stress functions [17] in tensor form using the Cartesian base vectors as $\mathbf{B}=A\mathbf{ii}+B\mathbf{jj}+C\mathbf{kk}$ for Maxwell and $\mathbf{B}=Z(\mathbf{ij}+\mathbf{ji})+X(\mathbf{jk}+\mathbf{kj})+Y(\mathbf{ki}+\mathbf{ik})$ for Morera. Since both the Maxwell and Morera stress functions must produce the same stress, if we subtract one from the other, we obtain a state of zero

stress. We saw in §6, that we can produce this state of zero stress with a suitable choice of the vector fields $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$.

Let us consider specifying $\mathbf{B}$ by the equation

$$\mathbf{B}\cdot\mathbf{k}=0 \tag{27}$$

which we shall use in the following section.

Now we can write $\mathbf{B}=U\mathbf{ii}+W(\mathbf{ij}+\mathbf{ji})+V\mathbf{jj}$. Then, using engineering notation,

$$\begin{aligned}\sigma_x &= V_{,zz} & \tau_{xy} &= -W_{,zz}\\ \sigma_y &= U_{,zz} & \tau_{yz} &= W_{,zx}-U_{,yz}\\ \sigma_z &= U_{,yy}-2W_{,xy}+V_{,xx} & \tau_{zx} &= W_{,yz}-V_{,zx}\end{aligned}$$

We can see that, as is the case with the Maxwell and Morera stress functions, we can arbitrarily specify 3 stress components and solve for the coefficients in the stress function, which then give the other stress components. However, we cannot arbitrarily specify 3 stress components which share the same subscript, because, for example, equilibrium in the x direction is controlled only by σ_x, τ_{zx} and τ_{yx}, so that we cannot specify all 3 of them.

8. A STRAIGHT CIRCULAR BAR UNDER TENSION

Consider a circular bar of radius a under a uniform axial stress σ. We can use the Beltrami stress function tensor satisfying (27) where $\mathbf{k}$ is along the axis of the bar. From (7), if $\mathcal{C}$ is a circle radius r centred on the axis of the bar,

$$\begin{aligned}\mathbf{f}_{\mathcal{C}} &= \int_{\theta=0}^{2\pi} r d\theta(-\sin\,\theta\mathbf{i}+\cos\,\theta\mathbf{j})\cdot\mathbf{F}\\ &= \pi r^2\sigma\mathbf{k} \text{ when } r\le a\\ &= \pi a^2\sigma\mathbf{k} \text{ when } r\ge a\end{aligned}$$

This is satisfied by

$$\begin{aligned}\mathbf{F} &= \frac{r\sigma}{2}(-\sin\,\theta\mathbf{i}+\cos\,\theta\mathbf{j})\mathbf{k} \text{ when } r\le a\\ &= \frac{a^2\sigma}{2r}(-\sin\,\theta\mathbf{i}+\cos\,\theta\mathbf{j})\mathbf{k} \text{ when } r\ge a\end{aligned}$$

Setting $\mathbf{T}=0$ in (17)

$$\mathbf{F}=(\boldsymbol{\epsilon}:\nabla\mathbf{B})^{\mathrm{T}}+\nabla\boldsymbol{\psi} \tag{28}$$

so that when $r\ge a$

$$\begin{aligned}\mathbf{B} &= 0\\ \boldsymbol{\psi} &= \frac{a^2\sigma\theta}{2}\mathbf{k}\\ \nabla\boldsymbol{\psi} &= \frac{a^2\sigma}{2r}(-\sin\,\theta\mathbf{i}+\cos\,\theta\mathbf{j})\mathbf{k}\end{aligned}$$

and when $r\le a$ we can either have

$$\begin{aligned}\mathbf{B} &= \sigma\left(\frac{(r^2-a^2)}{4}-a^2\log\frac{r}{a}\right)(\mathbf{I}-\mathbf{kk})\\ (\boldsymbol{\epsilon}:\nabla\mathbf{B})^{\mathrm{T}} &= \frac{\sigma}{2}\left(r-\frac{a^2}{r}\right)(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})\mathbf{k}\\ \nabla\boldsymbol{\psi} &= \frac{a^2\sigma}{2r}(-\sin\,\theta\mathbf{i}+\cos\,\theta\mathbf{j})\mathbf{k}\end{aligned} \tag{29}$$

or, if we prefer

$$\begin{aligned}\mathbf{B} &= \sigma\frac{(r^2-a^2)}{4}(\mathbf{I}-\mathbf{kk})\\ (\boldsymbol{\epsilon}:\nabla\mathbf{B})^{\mathrm{T}} &= \frac{\sigma r}{2}(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})\mathbf{k}\\ \nabla\boldsymbol{\psi} &= 0\end{aligned} \tag{30}$$

which both produce the same value of $\mathbf{F}$ in (28). (30) is simpler, but it does have discontinuities in $\nabla\boldsymbol{\psi}$ and $\nabla\mathbf{B}$ at $r=a$, but no discontinuity in $\mathbf{B}$ itself. (29) is more complicated, but it avoids the discontinuities in $\nabla\boldsymbol{\psi}$ and $\nabla\mathbf{B}$ at $r=a$. On the other hand, $\mathbf{B}\to\infty$ as $r\to 0$, which might be construed as a disadvantage.

9. A SINGLE UNLOADED LOOP OF STRUCTURE

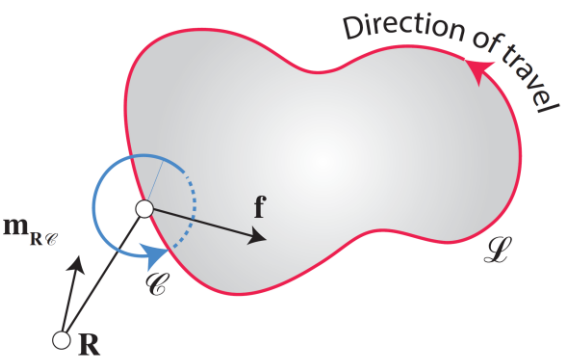

Figure 7: *$\mathcal{L}$ is a physical unloaded loop carrying force and moment. $\mathcal{C}$ is an imaginary curve encircling $\mathcal{L}$*

Figure 7 shows a structure consisting of a closed loop, $\mathcal{L}$. The loop might be a bent and twisted wire coat hanger [18]. The loop is freely floating in space contains a purely internal force and moment. We can represent the loop with some closed axis curve which

we think best represents the loop, and the loop is stressed in the material around the axis, but there is no stress outside this material.

Figure 7 also shows a closed curve $\mathcal{C}$ which encircles $\mathcal{L}$ in such a way that $\mathcal{C}$ only passes through unstressed space. However, while $\mathcal{L}$ is a real physical object made of wire or whatever, $\mathcal{C}$ is a figment of our imagination. There is a direction of travel around both $\mathcal{L}$ and $\mathcal{C}$ such that we move clockwise around $\mathcal{C}$ looking along the direction of travel around $\mathcal{L}$.

Both $\mathcal{L}$ and $\mathcal{C}$ form the boundaries of arbitrary surfaces, which again are figments of our imagination. The surface bounded by $\mathcal{L}$ is there to act as a surface of discontinuity in our vector fields $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ and the surface bounded by $\mathcal{C}$ is there because all the forces and moments in the loop $\mathcal{L}$ pass through this surface.

The force in the loop passing through the surface bounded by $\mathcal{C}$ is $\mathbf{f}_{\mathcal{C}}$, and $\mathbf{f}_{\mathcal{C}}$ has the same value regardless of where we draw $\mathcal{C}$ and the shape of the surface bounded by $\mathcal{C}$. This assumes that $\mathcal{C}$ is entirely in unstressed space and that it travels once around $\mathcal{L}$. If we reverse the direction of travel around $\mathcal{L}$ or $\mathcal{C}$, then the force $\mathbf{f}_{\mathcal{C}}$ reverses its direction.

Thus (25) tells us that as $\mathcal{C}$ passes around $\mathcal{L}$ it must encounter 1 or more discontinuities in $\boldsymbol{\psi}$ whose total value is $\mathbf{f}_{\mathcal{C}}$. However, since we have only a single loop we only need one surface of discontinuity and we can use our surface bounded by $\mathcal{L}$ as the location of the discontinuity. If $\delta\boldsymbol{\psi}$ is the discontinuity in $\boldsymbol{\psi}$

$$\mathbf{f}_{\mathcal{C}} = \delta\boldsymbol{\psi}$$

and $\delta\boldsymbol{\psi}$ has the same value at all points of the surface of discontinuity.

The discontinuity in the vector $\boldsymbol{\psi}$ is very much like the discontinuity in the magnetic vector potential in moving around a current carrying wire [2].

Similarly, the moment $\mathbf{m}_{\mathbf{R}\mathcal{C}}$ about the fixed point $\mathbf{R}$ of the stresses in $\mathcal{L}$ passing through the surface bounded by $\mathcal{C}$ is constant and independent of where we draw $\mathcal{C}$ or the shape of the surface bounded by $\mathcal{C}$. (26) tells us that discontinuities in both $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ contribute to $\mathbf{m}_{\mathbf{R}\mathcal{C}}$. Not only that but the location of the discontinuity is important. If we have just one set of discontinuities of magnitude $\delta\boldsymbol{\phi}$ and $\delta\boldsymbol{\psi}$ at the location $\mathbf{r}$, where $\mathcal{C}$ crosses the surface of discontinuity, then from (26)

$$\mathbf{m}_{\mathbf{R}\mathcal{C}} = \delta\boldsymbol{\phi} + \delta\boldsymbol{\psi} \cdot \boldsymbol{\epsilon} \cdot (\mathbf{r} - \mathbf{R}).$$

Thus since $\delta\boldsymbol{\psi}$ has the same value at all points on the surface of discontinuity bounded by $\mathcal{L}$, it follows that $\delta\boldsymbol{\phi}$ has to vary linearly with position such that

$$\begin{aligned} \delta\phi_x + \delta\psi_y(z - Z) - \delta\psi_z(y - Y) &= \text{constant} \\ \delta\phi_y + \delta\psi_z(x - X) - \delta\psi_x(z - Z) &= \text{constant} \\ \delta\phi_z + \delta\psi_x(y - Y) - \delta\psi_y(x - X) &= \text{constant} \end{aligned} \quad (31)$$

In 2 dimensions $\delta\phi_z$ becomes simply the Airy stress function ϕ and the 3rd equation (31) tells us that $-\psi_x$ is the ϕ slope in the y direction and ψ_y is the ϕ slope in the x direction. ψ_y and $-\psi_x$ are the coordinates of the Cremona diagram [16]. The Cremona diagram is usually used for pin-jointed frames, in which case the lines on the diagram are parallel to the structural members, or sometimes rotated by 90°. However, it can also be applied to rigid jointed frames, to include both the axial force and shear force in a member, and then the discontinuity in the Airy stress function gives the bending moment.

If $\mathbf{R}$ is located at some point on the axis of $\mathcal{L}$, then $\mathbf{m}_{\mathbf{R}\mathcal{C}}$ is the moment in $\mathcal{L}$ about its axis at that point. If the cross-sectional dimensions of $\mathcal{L}$ are sufficiently small we can allow the point of discontinuity, $\mathbf{r}$, to approach $\mathbf{R}$ so that $\mathbf{r} \approx \mathbf{R}$ and $\mathbf{m}_{\mathbf{R}\mathcal{C}} \approx \delta\boldsymbol{\phi}$.

10. COMBINATIONS OF STRUCTURAL LOOPS

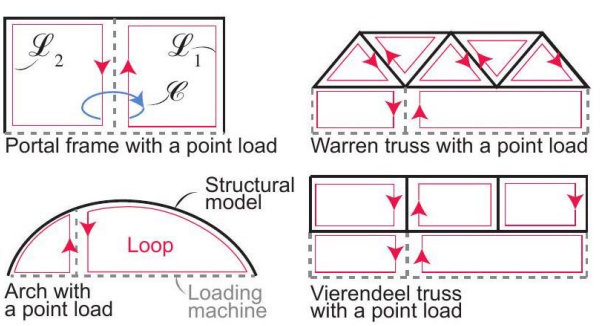

Figure 8: *Structures formed of loops, with the loading contraptions shown dotted*

We can construct framed structures in 3 dimensions by combining loops. In figure 8 the lines representing the structure are solid black lines and the loading apparatus is represented by dotted grey lines. The loops are in red. Thus, the figure on the

top left represents a portal frame loaded by a point load in the centre of the beam. The loading apparatus is a horizontal beam and a vertical jack in tension. The force in this jack is obtained by going around the curve $\mathcal{C}$ in the figure which will encounter discontinuities in $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ on the surfaces bounded by loops $\mathcal{L}_1$ and $\mathcal{L}_2$. If there is only a tension in the jack, then there will only be a combined discontinuity in $\boldsymbol{\psi}$ and not in $\boldsymbol{\phi}$.

There is no need for the structure to lie in a plane or for it to have a planar graph [3]. It may not be immediately obvious how many loops are required and where to put them. Figure 9 shows a triangular bipyramid, that is two tetrahedra put face to face. The two vertices with 3 edges are then joined by a loading jack shown dotted. In this case it would appear that we need 6 loops, that is any 5 of the 6 exterior faces plus the interior loop shown in red.

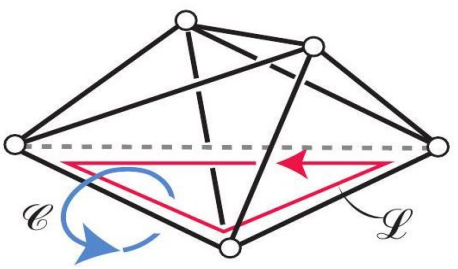

Figure 9: *Triangular bipyramid. The loop $\mathcal{L}$ in red is added for the dotted grey the loading jack*

Let us imagine that we want to consider the bipyramid as pin-jointed. 6 loops means that we have $6 \times 6 = 36$ quantities corresponding to discontinuities in $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ that can be chosen. There are 10 members, including the jack, so one might assume that we can put $10 \times 5 = 50$ releases in to make the structure pinned - we have to stop each member rotating about its own axis. However then the 5 nodes would be free to rotate, and we must stop this, so $5 \times 3 = 15$ constraints. Then $50 - 15 = 35$, and $36 - 35 = 1$, corresponding to the fact that specifying the jack tension gives all the other member tensions [5].

11. SHELLS

Figure 10 shows a section through a shell structure, and we have drawn our curve $\mathcal{C}$ so that it is partly above the shell and partly below the shell, passing through the shell at two points separated by the vector $d\mathbf{r}$. We can also see part of two adjacent curves which also encircle the shell section. The directions of travel around these two curves are in the opposite direction to that of $\mathcal{C}$ where they pass through the surface, and therefore their contribution to forces via (7) cancel out in this region.

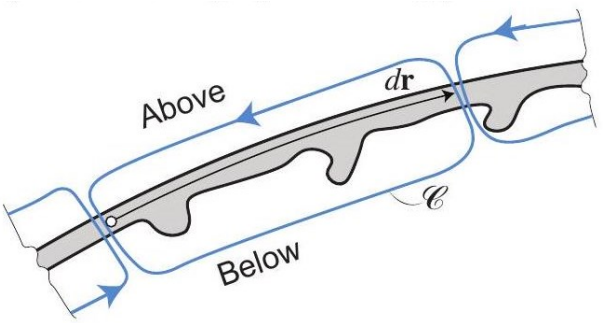

Figure 10: *Section through a shell structure*

$\mathring{\boldsymbol{F}} = \boldsymbol{F}_{\text{above}} - \boldsymbol{F}_{\text{below}}$ and $\mathring{\boldsymbol{G}} = \boldsymbol{G}_{\text{above}} - \boldsymbol{G}_{\text{below}}$ are the discontinuities in the Günther force and moment tensors across the shell reference surface.

Thus, dropping the subscript $\mathcal{C}$, if $d\mathbf{f}$ is the element of force associated with $\mathcal{C}$

$$d\mathbf{f} = -d\mathbf{r} \cdot \mathring{\mathbf{F}} \tag{32}$$

where $\mathring{\mathbf{F}} = \mathbf{F}_{\text{above}} - \mathbf{F}_{\text{below}}$ and $\mathbf{F}_{\text{above}}$ and $\mathbf{F}_{\text{below}}$ are the values of $\mathbf{F}$ above and below the shell.

Similarly, again dropping the subscript $\mathcal{C}$, we have from (12) that the element of moment about some point $\mathbf{R}$

$$d\mathbf{m}_{\mathbf{R}} = -d\mathbf{r} \cdot (\mathring{\mathbf{G}} + \mathring{\mathbf{F}} \cdot \boldsymbol{\epsilon} \cdot (\mathbf{r} - \mathbf{R}))$$

where $\mathring{\mathbf{G}} = \mathbf{G}_{\text{above}} - \mathbf{G}_{\text{below}}$.

If $\mathbf{R}$ is on the surface representing the shell, we have $\mathbf{r} - \mathbf{R} = 0$ so that

$$d\mathbf{m} = -d\mathbf{r} \cdot \mathring{\mathbf{G}} \tag{33}$$

Following (16) we can also write

$$\mathring{\mathbf{G}} = \mathring{\mathbf{B}} + \nabla\mathring{\boldsymbol{\phi}} - \boldsymbol{\epsilon} \cdot \mathring{\boldsymbol{\psi}} \tag{34}$$

The difference in the Beltrami stress function tensor above and below the shell, $\mathring{\mathbf{B}}$, represents the loading on the shell, which can be adjusted to include the own weight of the shell. The stresses in a shell are many orders of magnitude greater than the applied load or the weight per unit area and therefore it makes little difference is we apply the weight as a

body load or a load pushing down on the top or pulling down on the bottom.

If we assume that the Cosserat moments $\mathbf{T}$ above and below the shell are both zero, in other words that there are no loading couples applied to the shell, then from (17),

$$\overset{\circ}{\mathbf{F}} = (\boldsymbol{\epsilon}\colon \nabla\overset{\circ}{\mathbf{B}})^{\mathrm{T}} + \nabla\overset{\circ}{\boldsymbol{\psi}} \qquad (35)$$

We also have

$$d\mathbf{f} = (d\mathbf{r} \times \mathbf{n}) \cdot \boldsymbol{\sigma} = d\mathbf{r} \cdot (\mathbf{n} \cdot \boldsymbol{\epsilon}) \cdot \boldsymbol{\sigma}$$
$$d\mathbf{m} = (d\mathbf{r} \times \mathbf{n}) \cdot \boldsymbol{\mu} = d\mathbf{r} \cdot (\mathbf{n} \cdot \boldsymbol{\epsilon}) \cdot \boldsymbol{\mu}$$

in which $\mathbf{n} \cdot \boldsymbol{\sigma} = 0$ and $\mathbf{n} \cdot \boldsymbol{\mu} = 0$. These equations are the definitions of the tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ that describe the internal forces and internal moments in the shell per unit width. $\mathbf{n}$ is the unit normal to the reference surface describing the shell on which we have measured $d\mathbf{r}$.

If the shell is not 'thin' we have to imagine that it is represented by the reference surface and that the curve $\mathcal{C}$ follows the surface closely above and below. Otherwise $d\mathbf{r}$ would not be equal and opposite above and below the surface due to the curvature of the surface. Thus, the definition of 'thin' is in relation to the principal radii of curvature of the reference surface.

Thus from (32) and (33)
$(\mathbf{n} \cdot \boldsymbol{\epsilon}) \cdot \boldsymbol{\sigma} = -(\mathbf{I} - \mathbf{nn}) \cdot \overset{\circ}{\mathbf{F}}$ and
$(\mathbf{n} \cdot \boldsymbol{\epsilon}) \cdot \boldsymbol{\mu} = -(\mathbf{I} - \mathbf{nn}) \cdot \overset{\circ}{\mathbf{G}}$, and since
$(\mathbf{n} \cdot \boldsymbol{\epsilon}) \cdot (\mathbf{n} \cdot \boldsymbol{\epsilon}) = -(\mathbf{I} - \mathbf{nn})$ we have from (35)

$$\boldsymbol{\sigma} = \mathbf{n} \cdot \boldsymbol{\epsilon} \cdot \overset{\circ}{\mathbf{F}} = \mathbf{n} \cdot \boldsymbol{\epsilon} \cdot \left((\boldsymbol{\epsilon} : \nabla\overset{\circ}{\mathbf{B}})^{\mathrm{T}} + \nabla\overset{\circ}{\boldsymbol{\psi}}\right) \qquad (36)$$

and using (34)

$$\boldsymbol{\mu} = \mathbf{n} \cdot \boldsymbol{\epsilon} \cdot \overset{\circ}{\mathbf{G}}$$
$$= \mathbf{n} \cdot \boldsymbol{\epsilon} \cdot (\overset{\circ}{\mathbf{B}} + \nabla\overset{\circ}{\boldsymbol{\phi}}) + \overset{\circ}{\boldsymbol{\psi}} \cdot (\mathbf{nI} - \mathbf{In}). \qquad (37)$$

The contributions of $\overset{\circ}{\boldsymbol{\phi}}$ and $\overset{\circ}{\boldsymbol{\psi}}$ to (36) and (37) are identical with the contributions of $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ to equations (8) and (9) in Adiels & Williams [1] which were derived in a completely different way. In that paper the tensor $\boldsymbol{\epsilon}$ is equal to $\mathbf{n} \cdot \boldsymbol{\epsilon}$ in this paper.

Since $\mathbf{n} \cdot \boldsymbol{\epsilon} \cdot \mathbf{n} = 0$ it follows that the components $\mathbf{n} \cdot \nabla\overset{\circ}{\boldsymbol{\psi}}$ and $\mathbf{n} \cdot \nabla\overset{\circ}{\boldsymbol{\phi}}$ in (36) and (37) make no contribution to $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ and so we are only interested in how $\overset{\circ}{\boldsymbol{\psi}}$ and $\overset{\circ}{\boldsymbol{\phi}}$ vary in the tangential direction.

The terms containing the Beltrami stress function tensor $\overset{\circ}{\mathbf{B}}$ in (36) and (37) representing the contribution of the load on the shell, replace the vector fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in Adiels & Williams [1].

If the vector $\mathbf{p}$ is the load per unit area applied to the shell, then, setting $\mathbf{T} = 0$ in (18),

$$\mathbf{p} = \mathbf{n} \cdot \overset{\circ}{\mathbf{S}} = \mathbf{n} \cdot (\mathbf{S}_{\text{above}} - \mathbf{S}_{\text{below}})$$
$$= \mathbf{n} \cdot \boldsymbol{\epsilon} : \nabla(\boldsymbol{\epsilon} : \nabla\overset{\circ}{\mathbf{B}})^{\mathrm{T}} \qquad (38)$$

and $\overset{\circ}{\boldsymbol{\phi}}$ and $\overset{\circ}{\boldsymbol{\psi}}$ represent internal statically indeterminate internal forces and moments in equilibrium with zero loads because only $\overset{\circ}{\mathbf{B}}$ contributes to $\mathbf{p}$. The covariant derivatives of $\overset{\circ}{\mathbf{B}}$ in (38) are in 3 dimensional space, and so could be done using Cartesian coordinates.

12. CONVENTIONAL SHELL STRUCTURES

In the conventional theory of shell structures [24] it is usual to assume that the normal component of $\boldsymbol{\mu}$, that is $\boldsymbol{\mu} \cdot \mathbf{n}$, is zero. This might be termed a Cosserat moment and in the case of the sieve in figure 11 would correspond to the moment in the wires due to their geodesic curvature.

Figure 11: *A kitchen sieve*

Setting $\boldsymbol{\mu} \cdot \mathbf{n} = 0$ in (37),

$$\mathbf{n} \cdot \boldsymbol{\epsilon} \cdot \left(\overset{\circ}{\mathbf{B}} + \nabla\overset{\circ}{\boldsymbol{\phi}}\right) \cdot \mathbf{n} - \overset{\circ}{\boldsymbol{\psi}} \cdot (\mathbf{I} - \mathbf{nn}) = 0 \qquad (39)$$

and it follows that we can introduce the internal moment tensor such that

$$\boldsymbol{\mu} = \mathbf{M} \cdot \boldsymbol{\epsilon} \cdot \mathbf{n}$$

and hence

$$\begin{aligned}\mathbf{M} &= -\boldsymbol{\mu} \cdot \boldsymbol{\epsilon} \cdot \mathbf{n} \\ &= -\mathbf{n} \cdot \boldsymbol{\epsilon} \cdot (\overset{\circ}{\mathbf{B}} + \nabla \overset{\circ}{\boldsymbol{\phi}}) \cdot \boldsymbol{\epsilon} \cdot \mathbf{n} - (\overset{\circ}{\boldsymbol{\psi}} \cdot \mathbf{n}) \boldsymbol{\epsilon} \cdot \mathbf{n}\end{aligned}$$

which can be divided into symmetric and skew-symmetric parts,

$$\begin{aligned}&\frac{1}{2}\left(\mathbf{M} + \mathbf{M}^{\mathrm{T}}\right) = -\mathbf{n} \cdot \boldsymbol{\epsilon} \cdot \overset{\circ}{\mathbf{B}} \cdot \boldsymbol{\epsilon} \cdot \mathbf{n} \\ &-\frac{1}{2}\mathbf{n} \cdot \boldsymbol{\epsilon} \cdot \left(\nabla \overset{\circ}{\boldsymbol{\phi}} + (\nabla \overset{\circ}{\boldsymbol{\phi}})^{\mathrm{T}}\right) \cdot \boldsymbol{\epsilon} \cdot \mathbf{n} \qquad (40)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\left(\mathbf{M} - \mathbf{M}^{\mathrm{T}}\right) = &-\frac{1}{2}\mathbf{n} \cdot \boldsymbol{\epsilon} \cdot \left(\nabla \overset{\circ}{\boldsymbol{\phi}} - (\nabla \overset{\circ}{\boldsymbol{\phi}})^{\mathrm{T}}\right) \cdot \boldsymbol{\epsilon} \cdot \mathbf{n} \\ &- \left(\overset{\circ}{\boldsymbol{\psi}} \cdot \mathbf{n}\right) \boldsymbol{\epsilon} \cdot \mathbf{n}.\end{aligned}$$

The shell only carries load through $\overset{\circ}{\mathbf{B}}$ and therefore only the symmetric part of $\mathbf{M}$ contributes to load carrying.

It is common in the theory of shell to assume that $\mathbf{M}$ is symmetric, particularly in the elastic theory and if this is done $\mathbf{M} = \mathbf{M}^{\mathrm{T}}$ and

$$\begin{aligned}\overset{\circ}{\boldsymbol{\psi}} = &\mathbf{n} \cdot \boldsymbol{\epsilon} \cdot (\overset{\circ}{\mathbf{B}} + \nabla \overset{\circ}{\boldsymbol{\phi}}) \cdot \mathbf{n} \\ &-\frac{1}{2}\mathbf{n} \cdot \boldsymbol{\epsilon} \cdot \left(\nabla \overset{\circ}{\boldsymbol{\phi}} - (\nabla \overset{\circ}{\boldsymbol{\phi}})^{\mathrm{T}}\right) \cdot (\mathbf{I} - \mathbf{nn}).\end{aligned}$$

(40) now gives the moments in the shell in terms of the load as represented by $\overset{\circ}{\mathbf{B}}$ and the purely internal statically indeterminate or 'redundant' vector $\boldsymbol{\phi}$.

In the case of gridshells there is no reason why $\mathbf{M}$ should be symmetric, we could have torsionally stiff tubes running one way and torsionally flexible open sections running the other. This would require us to treat the shell as a Cosserat shell. Wright [28] describes how a gridshell or reticulated (meaning net-like) shell can be approximated by a continuum.

13. THE MEMBRANE THEORY OF SHELLS

In the membrane theory we assume that the moments $\left(\mathbf{M} + \mathbf{M}^{\mathrm{T}}\right)/2$ in (40) are zero. Thus, (40) represents 3 simultaneous differential equations in the 3 components of $\overset{\circ}{\boldsymbol{\phi}}$. If the loading is in a vertical direction only it is possible to reduce these 3 equations to 1 equation in 1 unknown, the z component of $\overset{\circ}{\boldsymbol{\phi}}$, which we shall see in §15 is the Airy stress function. This produces Pucher's equation (3) in §1, which can be derived using rather simpler methods. Alternatively, it is possible to reduce the equations to one equation in only the normal component of $\overset{\circ}{\boldsymbol{\phi}}$ [1]. Regardless of how we reduce the equations to one equation in one unknown, the single partial differential equation is elliptic if the Gaussian curvature is positive and hyperbolic if the Gaussian curvature is negative. This is consistent with the fact that shells with positive and negative Gaussian curvature behave very differently.

Following (22), the rate of membrane strain in a shell undergoing a velocity $\mathbf{v}$ is

$$\frac{1}{2}(\mathbf{I} - \mathbf{nn}) \cdot \left(\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}}\right) \cdot (\mathbf{I} - \mathbf{nn})$$

and comparison with the contribution of $\overset{\circ}{\boldsymbol{\phi}}$ in (40) shows that we have to solve the same vector differential equation to find membrane stresses in equilibrium as we do to prevent inextensional deformation [15], although the boundary conditions are very different.

In essence membrane shells are statically determinate because we have the same number of differential equations of equilibrium as we have unknown membrane stresses, 3 in each case. However, depending upon its shape and boundary support, the shell might be a mechanism, allowing inextensional deformation, in which case we have a solution for velocity, but not stresses, unless the load is 'funicular'. Alternatively, it might be statically determinate in which case the velocity is zero and the stresses are determined, or it might be statically indeterminate, in which case it can be prestressed.

14. BENDING OF A FLAT PLATE

If $\mathbf{k}$ is the constant unit normal to the plate, and $\mathbf{M}$ is assumed to be symmetric, (40) becomes

$$\begin{aligned}\mathbf{M} = &-\mathbf{k} \cdot \boldsymbol{\epsilon} \cdot \overset{\circ}{\mathbf{B}} \cdot \boldsymbol{\epsilon} \cdot \mathbf{k} \\ &-\frac{1}{2}\mathbf{k} \cdot \boldsymbol{\epsilon} \cdot \left(\nabla \overset{\circ}{\boldsymbol{\phi}} + (\nabla \overset{\circ}{\boldsymbol{\phi}})^{\mathrm{T}}\right) \cdot \boldsymbol{\epsilon} \cdot \mathbf{k}\end{aligned}$$

Thus, if we know the loading as represented by $\overset{\circ}{\mathbf{B}}$, the difference between $\mathbf{B}$ above and below the plate,

we can write down the moment $\mathbf{M}$ in terms of the internal statically indeterminate or 'redundant' moments $\overset{\circ}{\boldsymbol{\phi}}$.

It follows from (18), setting the Cosserat moments $\mathbf{T} = 0$, that the normal load on the plate is $\mathbf{k} \cdot \mathbf{p} = \nabla \cdot (\nabla \cdot \mathbf{M})$ or in Cartesian coordinates,

$$p_z = \frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2}$$

since we are assuming that $M_{xy} = M_{yx}$. This is equation (100) in Art. 21 of Timoshenko & Woinowsky-Krieger [24] except for the sign convention, which in Timoshenko & Woinowsky-Krieger is not compatible with the tensor notation.

15. THE AIRY STRESS FUNCTION

In §14 we considered normal loading on a flat plate. Now let us examine equilibrium in the plane.

Let us consider an unloaded flat plate and replacing the variable unit normal $\mathbf{n}$ by the constant unit vector $\mathbf{k}$ and setting $\overset{\circ}{\mathbf{B}} = 0$ in (39), so that the plate is unloaded,

$$\mathbf{k} \cdot \boldsymbol{\epsilon} \cdot \nabla \overset{\circ}{\boldsymbol{\phi}} \cdot \mathbf{k} - \overset{\circ}{\boldsymbol{\psi}} \cdot (\mathbf{I} - \mathbf{k}\mathbf{k}) = 0$$

and so if $\overset{\circ}{\boldsymbol{\phi}} = \overset{\circ}{\phi}_z \mathbf{k}$ and $\overset{\circ}{\boldsymbol{\psi}} \cdot \mathbf{k} = 0$ then (36) becomes

$$\boldsymbol{\sigma} = \mathbf{k} \cdot \boldsymbol{\epsilon} \cdot \nabla \overset{\circ}{\boldsymbol{\psi}} = -\mathbf{k} \cdot \boldsymbol{\epsilon} \cdot \nabla \nabla \overset{\circ}{\phi}_z \cdot \boldsymbol{\epsilon} \cdot \mathbf{k} \qquad (41)$$

and thus $\overset{\circ}{\phi}_z$, which is the z component in the discontinuity in $\boldsymbol{\phi}$ across the plate, is the Airy stress function. If we think of $\mathbf{k}$ as being the vertical direction so that the plate lies in a horizontal plane, then moving from 3 dimensions to 2 dimensions requires $\overset{\circ}{\boldsymbol{\psi}}$ to be horizontal and $\overset{\circ}{\boldsymbol{\phi}}$ to be vertical in the same way that forces are horizontal and moments act about a vertical axis.

16. CONCLUSIONS

Our original aim was to extend the relatively simple and useful application of the Airy stress function in 2 dimensions, including its application in Putcher's equation, to fully 3 dimensional problems.

To this end we used the Günther force and moment tensors to demonstrate that the Beltrami stress function tensor in the unstressed regions surrounding a structural frame or shell can be replaced by 2 vector fields, $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$, containing discontinuities that describe the state of stress within the structure.

However, the application is mathematically complex and the vectors, $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$, are difficult to visualise, except in special cases such as the bending of a flat plate.

The vector fields $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ 'exist' in the unstressed space surrounding a structural member such as a column, vault or flying buttress in a Gothic cathedral. However, there would seem to be no way in which to detect their presence, and so perhaps they do not really exist.

ACKNOWLEDGEMENTS

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