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# Assessing students' emerging awareness of algebraic structures

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*To solve linear equations, students need not only to understand algebraic procedures and concepts but also to identify relevant features of the equation and interpret the structure to be able to begin the solving process. This can be challenging if the equation is presented in an unfamiliar format, such as with non-natural numbers or new symbols. In this paper, we explore the features that 16-year-old students in vocational education focus on when sorting equations. The results show that students focus on a diversity of features when grouping equations together. We also discuss the potential of the task to assess students' algebraic structure sense.*

*Keywords: Algebra, linear equations, algebraic structure sense, sorting task.*

## Introduction

Students' awareness of structural properties of algebraic expressions, also known as algebraic structure sense, has been identified as an area in need of more research (Rojano, 2022a). This includes both investigating its importance for algebra learning and developing teaching methods that take it into account. This paper explores a sorting task to better understand its potential to reveal what features students consider relevant in linear equations, and its potential as a tool to assess students' awareness of structure.

The term *algebraic structure* is used to denote the internal order in an algebraic expression, referring to the relationship between the parts of the structure, that is either apparent or can be revealed by transformations (Hoch and Dreyfus, 2004). This algebraic structure can be interpreted in different ways, depending on how it is transformed or which parts are focused on; when parsing a linear equation we can either focus on the equality of two expressions  $(5x + 3) = (7x + 1)$ , or on the grouping of terms  $(5x) + (3) = (7x) + (1)$ , or on the relation between numbers, variables and operations  $(5) \cdot (x) + (3) = (7) \cdot (x) + (1)$  (cf. Rüede, 2013).

When students learn algebra, it is natural that they focus on solving one linear equation at a time. However, as they become more proficient in solving equations, they start to notice general aspects that are central to the solving process. Mason (1996) describes this algebraic awareness as the ability to shift attention – not only to see values and operations but also regard them as expressions and objects. In this process, students need to move their focus from features that might have initially been their primary concern (e.g., the numbers) to the observed generalities, such as the combination of symbols and the relations they represent (English & Sharry, 1996). Hence, as students develop their algebraic thinking, that is the analytical dealing with indeterminate quantities (Radford, 2018), they should also learn to notice features important for the algebraic context. This can be compared to the notion of “saming” described by Sfard (2008) as a central process in learning new mathematical discourses, which involves finding commonalities in what was not previously considered the same.

In line with this, Rüede (2013) found different interpretations of the structure of fractional expressions among 12 novices and 12 experts. Some of the novices looked for visual similarities (syntactical structuring), while others wanted to change the expression (operational structuring). Some saw an initial structure but re-interpreted it (first-order structuring), while all experts considered several structures from the start (second-order structuring).

It is natural that students' awareness of structure is influenced by their previous mathematical knowledge, for example their familiarity with the algebraic sign system and their ability to transform expressions to reveal structure (Rojano, 2022b). Moreover, it has been proposed that students' initial learning of algebra is significantly influenced by surface features they are already familiar with (English and Sharry, 1996). Furthermore, several aspects of equation solving that research has identified as conceptually challenging for students are related to concepts that students already have some understanding of. For example, there is considerable evidence showing that the equal sign is often perceived as something "to do" rather than a symbol for equality (Bush & Karp, 2013). This misunderstanding can make an equation with the equal sign at the beginning (e.g.,  $480 = 12 \cdot x$ ) confusing, as the "answer" appears first. Another example, which has been studied extensively (Bush & Karp, 2013), concerns the various ways students interpret letters in algebra (e.g., as labels for objects). Coefficients in linear equations can also serve as a visual feature that guides students toward seeing structure (Rojano & Palmas, 2022), but they also represent an aspect with which students have prior experience, potentially influencing their interpretation of structures (Holmlund, 2024).

This paper is part of a PhD project aiming at developing teaching on structural awareness through a research lesson, in the form of a learning study. The project involves three teachers and one researcher who collaboratively design and evaluate teaching strategies and improve it through three cycles based on variation theory (Marton, 2014). Variation theory explains learning as the discernment of aspects that constitute an object of learning. In this context, *identifying the structures of linear equations* is the object of learning. This is taught by varying other aspects of the equations, such as the coefficients, the letter used for the variable, and the position of the variable, while keeping the operations constant within a specific structure (e.g.,  $a = b \cdot x + c$ ,  $a = x + c$ , or  $a = b \cdot x$ ). One of the tasks used in the pre-test is analysed in this paper. The purpose of the task is to assess how students recognise equations as familiar structures – one of the abilities of structure sense according to the definition of Hoch and Dreyfus (2004) – by asking them to justify their sorting of linear equations. Sorting tasks have previously been used for both teaching and evaluating algebraic understanding. When used for assessment, sorting tasks can observe whether students notice irrelevant or relevant features in equations (English & Sharry, 1996; Rüede, 2009). English and Sharry (1996) used a sorting task to orally assess which features three secondary school students focused on in first- and second-degree equations. They found that, beyond features indicating the degree of the equations, students' attention was drawn to several other aspects: the use of different letters for the unknowns, the presence of one or more unknowns, whether one side contained only a number – particularly linking zero to second-degree equations – denominators with or without variables, and computational aspects, such as the need to group letters. In their study, extra

proficient algebra students identified more features related to the equations' structures compared to less skilled students.

Identifying the features students focus on at an early stage can assist teachers in guiding students towards a more algebraic understanding of equations and helping students to engage with the algebraic context. In this paper, we will discuss the features students attend to in a sorting task and explore the potential of this task as an indicator of students' awareness of structure in the previously mentioned learning study. The research question guiding the study is: What can we learn about students' awareness of structure regarding linear equations from a sorting task?

## Method

In this paper we focus on a sorting task in the pre-test of the learning study, answered by students in their first year of upper secondary school in Sweden. 40 students signed a written consent, and the results displayed concern their answers. The students are approximately 16 years old (age was however not controlled in the study). They attend three different schools in two cities and are therefore taught by separate teachers. The students are all in vocational education, focusing on electricity and will likely use linear equations with both decimal and negative numbers as coefficients in their future profession as electricians. They have previously, in lower secondary school, learnt to solve linear equations, but mainly using natural numbers as coefficients. In the present study, the sorting task was adapted to focus specifically on linear equations (see Figure 1). Twelve equations were designed to vary in features that might be relevant for students' attention. Some equations were inspired by equations that had proved useful in probing students' understanding for structure in previous studies, for example  $0.12 = 0.4 \cdot x$  and  $0.657 = 0.045 \cdot x$  (Holmlund, 2024) and  $-24 = 6 \cdot x$  (Vlassis, 2002). Inspiration was also taken from the task by English and Sharry (1996), for example including a zero on one side and varying the letter used for the unknown. Other features were varied among the equations, based on previous research indicating them as central to students in their understanding of linear equations, for example the position of variables on one or both sides of the equal sign (Vlassis, 2002). The task instructed students to "sort the equations you think 'belong together' and resemble each other. Encircle the ones you think belong together and mark the first group as A, the second as B and so on." After sorting, students were asked to justify why they grouped the equations as they did. The response field was divided into four sections, so while students were encouraged to create as many groups as they wanted, the task did not support more than four groups. Also, the task did not encourage students to encircle one equation several times. The free sorting asked for in the task was intended as a tool to learn about the features that students immediately look for in the equations – without there being any wrong answers – and thus not intended to display their understanding of the structures in more depth. However, the open formulation of the task was mentioned as confusing by two students.

A qualitative content analysis was conducted on the output of the sorting task in the pre-test, using students' justifications as the unit of analysis (Graneheim & Lundman, 2004). To identify categories that describe the features students focus on, all answers were carefully reviewed, looking for similarities and differences, and a list of commonalities in students' justifications was compiled. The analysis focused on the meanings conveyed in the students' answers.

Sometimes, a student provided multiple reasons for considering equations as similar, such as noting both the letter used and the operation (e.g., “they have  $y$  in the equation and a multiplication sign”). These features were categorised based on shared characteristics. The goal was to identify categories that encompassed all the justifications while remaining mutually exclusive. The first analysis was conducted by the first author. The second author reviewed all student answers and categorizations. A few disagreements were discussed with a third researcher until consensus was reached. The intention was to let the students’ perspectives guide the categorization, minimising the influence of the researchers’ perspectives.

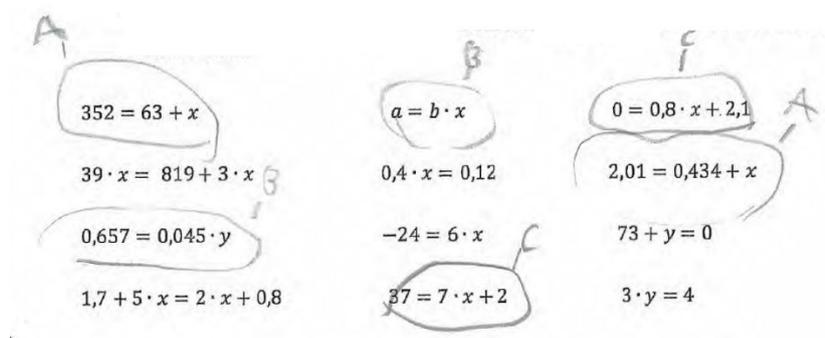


Figure 1: The analysed task with a student’s encircled equations

## Results

The qualitative content analysis of students’ justifications resulted in seven categories, which give an overview of the structural properties of algebraic expressions that the students attend to. The properties, inherent in the equations, are what students notice in their search for characteristics. In parallel with the presentation of each category below, examples from students’ answers will illustrate the potential and shortfalls of the sorting task to indicate and describe students’ awareness of structure.

### Separating different operations

Similarities of operations is a frequent feature that students use to group equations together. Making the type of operations explicit in their justification is distinctive of this category. As an example, one student explains the group of equations as “containing” either addition, multiplication or both addition and multiplication. It is possible to group equations in different ways based on the operations: for example, seeing equations with both addition and multiplication as a third structure or as belonging to the structure of multiplication. Generally, this feature occurs in students’ answers with at least the distinction between equations with (only) addition and equations with (only) multiplication. One student explains  $a = b \cdot x$  as a formula for multiplicative structure that other equations with multiplication match with.

### Separating number of steps

A related feature, that some students mention, is the number of steps required to solve the equation. As an example, one student describes the common feature in group A in Figure 2 as the need for only one calculation on one of the sides to find  $x$  or  $y$ . In contrast, for group B “two calculations are required on one of the sides if there is no negative number on one of the

sides which can complicate the solution somewhat". Operations are thereby noticed but sorted according to how many they are, rather than by their type (as in the previous category).

A	$352 = 63 + x$	D	$a = b \cdot x$	B	$0 = 0,8 \cdot x + 2,1$
C	$39 \cdot x = 819 + 3 \cdot x$	A	$0,4 \cdot x = 0,12$	A	$2,01 = 0,434 + x$
A	$0,657 = 0,045 \cdot y$	B	$-24 = 6 \cdot x$	A	$73 + y = 0$
C	$1,7 + 5 \cdot x = 2 \cdot x + 0,8$	B	$37 = 7 \cdot x + 2$	A	$3 \cdot y = 4$

**Figure 2: Encircling with justifications based on the number of steps**

### Variable on one or both sides

A feature considered by some students is whether  $x$  is on one or both sides of the equality sign. For example, one student describes one of his groups by stating "There are equations [expressions] on both sides". Looking at Figure 3, this student explains his choice of group C as "Because you have to calculate on both sides" which also focuses on the variable being on both sides, rather than on the number of calculations (as in the preceding category).

### Specific positions

As in the example in Figure 1, several students identify the position of components within the equations as a defining feature. This feature appears in two distinct ways. For some, the position is described, with a focus on either the right or left side of the equation, mentioning whether the side in focus contains only a number, an expression involving a variable, or an expression involving a particular operation. In other cases, students focus not only on one side of the equation but on a specific position within it, such as the final position, sorting equations based on the symbols present at this position (e.g., a number or an unknown). One student places  $1.7 + 5 \cdot x = 2 \cdot x + 0.8$  in the same group as  $3 \cdot y = 4$  because "they end with numbers".

### Letters

For some students, features concerning the unknown play a central role in sorting the equations. Their justifications involve the specific letter used to denote the unknown, leading them to categorize equations with  $y$  separately from those with  $x$ . For example, one student groups  $0.657 = 0.045 \cdot y$ ,  $73 + y = 0$ , and  $3 \cdot y = 4$  together, noting "they have  $y$  instead of  $x$ ". Another aspect of using the letter is to look at the number of unknowns. It is not uncommon for the students to see the equation  $a = b \cdot x$  as a category on its own, observing that this equation features multiple letters and "contains only variables".

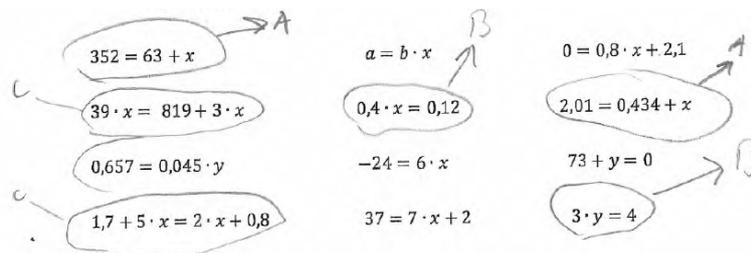
### Numbers

Some students base their grouping of equations on numerical features. These justifications sometimes involve certain numbers. For instance, some students sort equations that contain only decimal numbers separately from those with whole numbers. One student sorted equations involving negative numbers in one group, explaining that "then you need to calculate and think in a different manner than in the other groups". Another student identified odd numbers as a common characteristic of one group of equations. Other students focus on more specific

numerical features, such as one noting that “they have very similar digits 3, 5, and 7” or another observing “they are equal to 0”. In one instance, the presence of many digits was used as a justification for grouping or excluding equations.

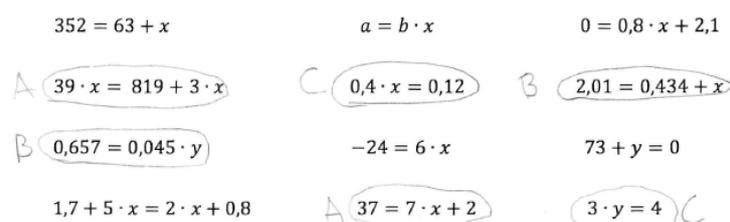
### Length of equation

Some students use the length of the equations as a criterion for categorization. Usually, longer equations are those that involve multiple operations, have the variable  $x$  on both sides of the equality sign or contain numbers with many digits. For example, one student sorts  $0 = 0.8 \cdot x + 2.1$  and  $2.01 = 0.434 + x$  together because they are “very long compared to the other [equations]”. Another student sorted  $352 = 63 + x$  and  $3 \cdot y = 4$  in the same group with “short” as the only justification.



**Figure 3: An example where a student focuses on different features**

A limitation of the sorting task is occasional weak alignment between students’ encircling of equations and the accompanying justifications. For example, some students mention operations as a criterion for the groups but do not encircle all relevant equations, sometimes due to conflict with other features they noticed. This can be seen in Figure 3 where one student justifies groups A and B by mentioning the specific operations without encircling all additive equations for group A, nor all multiplicative equations for group B, as positions of “solutions” and “answers” (which he also mentions) seems to be an important feature to him. On the contrary, the student behind Figure 1 does not mention operations in the justifications but examining the encircling reveals that this is a relevant feature to him. It appears that additional features, such as the positioning of the variable in relation to the equal sign, are significant to this student. Another example is given by a student in Figure 4. He describes group B as “has decimal numbers as an answer and in solution and has one variable” underscoring the importance of numerical features to the student. Group C is characterized as “they look the same, the only difference is that one is integer and the other decimal number”. Comparing groups B and C indicates that positioning likely plays a central role in the students’ awareness of structure, which is not clear from the justifications.



**Figure 4: An example where specific numbers guide the sorting**

## Discussion

To learn more about students' awareness of different features in linear equation, and to be able to use this when assessing how they improve in recognising equations as familiar structures, students in the study were asked to sort a set of equations. When reviewing the seven categories found in students' justifications, we see three features that indicate an emerging focus on algebraic structures. Based on Radford's description of algebraic thinking as analytically handling unknown quantities, it is reasonable to consider the feature of "Variable on one or both sides" as algebraic: as it makes a distinction between equations that requires handling the unknown and those that do not. Focus on the first two features, operations and number of steps, indicates that the students noticed the different parts of the equation. This does not necessarily imply algebraic thinking but could at least show that they are on the way. Focusing on these features can be compared to the operational focus and first-order structuring found in the study by Rüede (2013). However, we interpret the other features mentioned by students (i.e., specific positions, letters, numbers and length of the equation) as not showing algebraic thinking or an emerging awareness of structure. These categories are visually salient, and they are of relevance to students' previous understanding of familiar mathematical concepts. For example, justifications related to specific positions can be partly related to students' view of the equality sign as indicating something to compute, often placing the answer on the right side, without seeing the equality as symmetric. Many of these justifications are similar to those described by English and Sharry (1996). References to the length of the equations generally indicate that the student struggles with these equations, as they rarely solved many equations elsewhere in the pre-test. Students' focus on numbers is supported by earlier research suggesting that numbers are a key feature that students attend to in equation solving (Holmlund, 2024). Thus, regarding saming as a central activity to enter the discourse of algebra (Sfard, 2008), students also need to be guided into what is *not* the same.

To summarize, the analysis of the sorting task has revealed that aligning students' encircling (or lack thereof) with their justifications can be challenging. However, it can display signs of emerging awareness of structure and to effectively use the sorting task to assess students' awareness of structure, it is important to consider both the encircling and the justifications.

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