The versal deformation of small resolutions of conic bundles over P1×P1 with two sections blown down

Citation for the original published paper (version of record):
Kreußler, B., Stevens, J. (2026). The versal deformation of small resolutions of conic bundles over P1×P1 with two sections blown
down. Journal of the London Mathematical Society, 113(2). http://dx.doi.org/10.1112/jlms.70443

Received: 14 November 2024 | Revised: 15 December 2025 | Accepted: 23 December 2025

DOI: 10.1112/jlms.70443

RESEARCH ARTICLE

Journal of the London Mathematical Society

The versal deformation of small resolutions of conic bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ with two sections blown down

Bernd Kreußler[1] | **Jan Stevens**[2]

[1]Mary Immaculate College, South Circular Road, Limerick, Ireland

[2]Matematiska vetenskaper, Göteborgs universitet och Chalmers tekniska högskola, Göteborg, Sweden

Correspondence
Bernd Kreußler, Mary Immaculate College, South Circular Road, Limerick, V94 VN26, Ireland.
Email: bernd.kreussler@mic.ul.ie

Dedicated to Herbert Kurke on the occasion of his 85th birthday.

Abstract

Twistor spaces are certain compact complex three-folds with an additional real fibre bundle structure. We focus here on twistor spaces over $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$. Such spaces are either small resolutions of double solids or they can be described as modifications of conic bundles. The last type is the more special one: they deform into double solids. We give an explicit description of this deformation, in a more general context.

MSC 2020

32L25 (primary), 32J17, 14J30, 32G05, 14D20 (secondary)

INTRODUCTION

This paper describes the deformation of a special non-projective complex manifold to another one, which is also Moishezon, but is given by different equations. Such manifolds firstly occurred as twistor spaces.

The original Penrose twistor correspondence interprets four-dimensional Minkowski space as a space of lines in complex projective three-space. The twistor construction in the context of Riemannian geometry was first worked out in detail by Atiyah, Hitchin and Singer [2]. They showed that the conformal structure of a self-dual four-dimensional Riemannian manifold determines a three-dimensional complex manifold, the twistor space, from which the conformal structure can be reconstructed with the aid of an anti-holomorphic involution (real structure) and a covering family of 'lines'. For an introduction from an algebro-geometric point of view, see LeBrun's survey [34].

Most twistor spaces are not projective: only $\mathbb{P}^3$ and the flag variety $\mathbb{F}(1,2)$ are Kähler twistor spaces, as shown by Hitchin [11], and independently by Friedrich and Kurke [9]. Indeed, the Kähler condition implies that the self-dual metric on the Riemannian four-manifold has positive scalar curvature; moreover, it can only be S^4, $\mathbb{P}^2$, $\mathbb{P}^2\#\mathbb{P}^2$ or $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$. The twistor space can only be projective 3-space, the flag variety, the intersection of two quadrics or a double solid with quartic branch surface, and the last two are excluded, because of their Euler characteristic. Poon [38] showed that a twistor space with real fibre bundle structure over $\mathbb{P}^2\#\mathbb{P}^2$ is a non-projective small resolution of a singular intersection of two quadrics; there is a 1-parameter family of such twistor spaces.

The existence of twistor spaces over $\mathbb{P}^2\# \cdots \#\mathbb{P}^2$, the connected sum of $n \geqslant 3$ copies of $\mathbb{P}^2$ considered as a real 4-manifold, was for all n shown by Donaldson–Friedman [7] with a deformation argument, using d-semistable reducible spaces. For $n \geqslant 4$, twistor spaces do no longer need to be Moishezon. LeBrun [32] constructed by more elementary methods explicit metrics for all values of n. The associated twistor spaces are called LeBrun twistor spaces. They are Moishezon, and are bimeromorphic to a small resolution of a certain conic bundle over the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ [32].

In the case $n = 3$, all twistor space are Moishezon, but the LeBrun twistor spaces are not the most general. Also small resolutions of double solids, branched over a singular quartic, are twistor spaces, by the work of Poon (see [39]) and of Kurke and the first Author [29, 30]. In the course of this work, Kurke realised that not only double solids but also modifications of conic bundles are candidates for twistor spaces (see [29, Theorem 2.3]). From a general deformation theory argument [7], it was clear that there must be a deformation of a LeBrun twistor space into a small resolution of a double solid. The purpose of this paper is to explain in detail how this can be done. This answers a question asked by Herbert Kurke at the Berlin–Hamburg–Oslo Seminar on Singularities which was hosted by Arnfinn Laudal on 25–27 October 1991 in Oslo.

During the past two decades, Honda has shed much light on the structure of twistor spaces over $\mathbb{P}^2\# \cdots \#\mathbb{P}^2$, see, for example, [12, 15–22]. For example, for $n > 4$, the general twistor space has algebraic dimension 0. Many examples of twistor spaces of algebraic dimension 1 or 3 have been described explicitly for all $n \geqslant 4$. Examples of algebraic dimension 2 are known [6, 12, 14] for $n = 4$, but remain mysterious for $n > 4$, see [24].

Recently, in [23], Honda proved the astonishing result that a Moishezon twistor space over $\mathbb{P}^2\# \cdots \#\mathbb{P}^2$, whose fundamental linear system is a pencil, either is bimeromorphic to a conic bundle over a possibly singular surface or to a branched double cover over a three-dimensional variety. This manifests in full generality the dichotomy we have seen in case $n = 3$. Our work is a first step towards understanding how Moishezon twistor spaces of one type deform into the other.

Before stating our results in more detail, we recall the description of LeBrun twistor spaces over $\mathbb{P}^2\# \cdots \#\mathbb{P}^2$. Such a space is bimeromorphic to a small resolution of a certain conic bundle over the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ [32], see also [31]. Let $\varphi_1, \dots, \varphi_n$ be bihomogeneous forms, defining smooth curves of type (1,1) on Q. Consider the hypersurface W in the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{O}_Q \oplus \mathcal{O}_Q(1-n,-1) \oplus \mathcal{O}_Q(-1,1-n))$ over Q with bihomogeneous equation

$$w_1 w_2 - \varphi_1 \cdots \varphi_n w_0^2 = 0\,. \tag{1}$$

The singular points of W lie over the intersection points of the curves φ_i in $w_1 = w_2 = 0$. Let $\widetilde{W}$ be a suitably chosen small resolution of W (the resolution has to be compatible with the real structure on W; only one choice leads to a twistor space, see [32] and [31] for details). The sections $E_1 : w_0 = w_1 = 0$ and $E_2 : w_0 = w_2 = 0$ in W do not pass through the singular locus, and therefore, they can also be considered as divisors in $\widetilde{W}$. Each can be blown down to a curve, along opposite rulings of the quadric, both in W and $\widetilde{W}$. As a result, we get a singular three-fold Z and its small resolution $\widetilde{Z}$, the LeBrun twistor space.

As the real structure and the twistor lines play no role in our main construction, we can consider more general discriminant curves, only subject to the condition that the singularities of W admit a small resolution. In particular, we can replace the product $\varphi_1 \cdots \varphi_n$ by any non-singular form of type (n, n). We call a smooth space obtained by blowing down the two sections E_1, E_2 an *SRCB manifold*.† In this Introduction, we mainly restrict ourselves to the case of the product $\varphi_1 \cdots \varphi_n$, with $n = 3$.

The general twistor space over $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$ is a small resolution of a double cover of $\mathbb{P}^3$, branched along a quartic surface with 13 ordinary double points [30, 39], with equation of the form

$$Q^2 - L_1L_2L_3L_4 = 0\,. \tag{2}$$

A similar phenomenon, of two families, occurs for $K3$-surfaces with a polarisation of degree 2 [41]. The general one is a double cover of $\mathbb{P}^2$, branched along a sextic curve, but it can also happen that the linear system only maps the surface to a conic. Twice the linear system exhibits such surfaces as double covers of the projective cone over the rational normal curve of degree 4. This cone deforms to the Veronese surface, which is an embedding of $\mathbb{P}^2$. Similarly, the Veronese embedding of $\mathbb{P}^3$ in $\mathbb{P}^9$ is a deformation of the projective cone over $\mathbb{P}^1 \times \mathbb{P}^1$, embedded with $\mathcal{O}(2,2)$ in $\mathbb{P}^8 \subset \mathbb{P}^9$.

The previous observation was the starting point for our investigations. The first problem one runs into is that the conic bundle involves three planes, whereas the double solid has four linear forms in its equation, each giving rise to two surfaces of degree 1 on the double cover (here degree is measured by intersecting with twistor lines), and these are the only surfaces of degree 1. The conic bundle twistor spaces have two pencils of surfaces of degree 1 (the inverse images of the rulings on the quadric), so most surfaces do not survive under the deformation. The solution to this problem is to take two surviving divisors as additional information, and consider the deformation theory of the manifold $\widetilde{Z}$ together with two divisors $\widetilde{S}_1, \widetilde{S}_2$, which are the strict transforms of the inverse image of two intersecting lines on Q. To specify the two surfaces, we just give a tangent plane to the quadric, thereby introducing a fourth plane; it intersects the quadric in two lines.

Instead of working with double covers, we extend the $\mathbb{P}^2$-bundle over $\mathbb{P}^3$. This is possible if we add the two lines to the discriminant curve, which then becomes a curve of degree (4,4). The bundle is

$$\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-2)) \to \mathbb{P}^3\,.$$

We connect conic bundles of the form (1) (with $n = 3$) and double solids of the form (2) by a family $\mathcal{Y}$ with equations

$$\begin{aligned} y_1y_2 - L_1L_2L_3L_4y_0^2 &= 0\,,\\ \alpha_2y_1 + \alpha_1y_2 - Qy_0 &= 0\,, \end{aligned} \tag{3}$$

where L_1, L_2 and L_3 are linear forms on $\mathbb{P}^3$, which restrict on the quadric to φ_1, φ_2 and φ_3, respectively, and L_4 depends on the product $\alpha_1\alpha_2$ in a way to be specified; for $\alpha_1\alpha_2 = 0$, it gives the

† SRCB stands for **s**mall **r**esolution of a **c**onic **b**undle.

mentioned tangent plane, whereas for $\alpha_1\alpha_2 \neq 0$, there should be a 13th singular point. The general fibre is indeed a double solid, for if $\alpha_1\alpha_2 \neq 0$ we can eliminate, say y_1, and find the double cover of $\mathbb{P}^3$, branched along the quartic with equation

$$Q^2 - 4\alpha_1\alpha_2 L_1 L_2 L_3 L_4 = 0 .$$

Equations (3) are the correct equations to write down, but to obtain a deformation of an SRCB manifold, we need to change the family, by birational transformations. The central fibre of the family $\mathcal{Y}$ is reducible, with the sections $y_0 = y_1 = 0$ and $y_0 = y_2 = 0$ as extra components. Furthermore, the remaining component is only birational to a conic bundle with discriminant curve of degree (3,3). Fortunately, we have here a case of two wrongs making one right.

The rational map to a conic bundle can be factored as a small resolution of the singularities introduced by the tangent plane L_4 (seven ordinary double points for a tangent plane in general position) and the fibre-wise blow down of two ruled surfaces. We realise this factorisation, and the subsequent map to an SRCB manifold, by a sequence of birational transformations of the total space:

$$\mathcal{Y} \longleftarrow \mathcal{Y}^- \dashrightarrow \mathcal{Y}^+ \longrightarrow \mathcal{Z} \longleftarrow \widetilde{\mathcal{Z}} .$$

The morphism $\mathcal{Y}^- \to \mathcal{Y}$ is the simultaneous small resolution of seven isolated singularities of the fibres of the family. Each ruled surface survives over one of the divisors $\alpha_i = 0$, with normal bundle in the total space restricting to each line of the ruling as $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Therefore, all the lines can be simultaneously flopped in the total space. The flop $\mathcal{Y}^- \dashrightarrow \mathcal{Y}^+$ induces a blowing down in the components in which the surfaces lie. On the other hand, it gives a blow-up of the extra components, which become relatively exceptional, and can be contracted to lines, by the Castelnuovo–Moishezon–Nakano criterion. This is achieved by the morphism $\mathcal{Y}^+ \to \mathcal{Z}$. Finally, one resolves the remaining singularities. The manifold $\widetilde{\mathcal{Z}}$ is the total space of a deformation of the original SRCB manifold. By varying the coefficients of L_1, L_2, L_3, L_4 in an appropriate way, we obtain a family $\widetilde{\mathcal{Z}} \to \Pi$. Our main result is the following.

Theorem. *The fibres of the so-constructed family $\widetilde{\mathcal{Z}} \to \Pi$ over $\alpha_1 = \alpha_2 = 0$ are SRCB manifolds, and the deformation $\widetilde{\mathcal{Z}} \to \Pi$ is for small α_1, α_2 versal for deformations of triples $(\widetilde{Z}, \widetilde{S}_1, \widetilde{S}_2)$.*

We prove the theorem (as stated) for general SRCB manifolds, that is, for general discriminant curves. Also, there are almost no restrictions on the position of the tangent plane. To prove versality, we need to know the spaces of infinitesimal deformations and obstructions. We compute them in the needed generality. With no extra effort, this can be done for general n. In the twistor case, the computations were performed by LeBrun [33], under the additional assumption that all singularities are of type A_1. We follow the same strategy.

We spend some time to define the family $\widetilde{\mathcal{Z}} \to \Pi$, by specifying a family $\mathcal{Y} \to \Pi$, making the above statement, that we vary the occurring coefficients in an appropriate way, more precise. The problem is that the base space Π can in general only be given implicitly. In the twistor case, we can give a rather explicit description. In two other cases, we have been able to give precise formulas, namely if the general fibre is a double solid, branched along a 14-nodal quartic, or along a Kummer surface.

It is known that a small real deformation of a LeBrun twistor space has again the structure of a twistor space. Our formulas simplify if the LeBrun twistor space has a torus action. Such twistor spaces and their deformations have been studied by Honda [15–17]. Therefore, our construction gives a new proof of a result of Honda [13, Theorem 2.1], which states the existence of degenerate double solids (double solids with branch surface with higher singularities) as twistor spaces. Finally, it is interesting to note that in our construction, we can interchange the rulings of the quadric. The effect on the double solid is a flop of all exceptional curves. Therefore, there are two small resolutions admitting a twistor structure.

The content of this paper is as follows. In the first section, we define and describe SRCB manifolds, for general n. Their infinitesimal deformations are studied in the next section. The third section contains the main result, the construction of a versal family. Some fibres, which cannot have a real structure, lie halfway between SRCB manifolds and double solids, and they are studied in more detail in a separate section. Finally, we specialise to the twistor case. We also give some other examples of our construction.

Notational convention. Often, we will have two objects, for example, E_1, E_2, under study, which behave in a similar way. To have efficient notation, we treat both cases at the same time by dropping all indices, that is, we write E, Δ, C, and so on, instead of E_i, Δ_i, C_i. But it is understood that all the new objects we introduce will pick up an index i if we return to deal with the global situation.

1 | SRCB MANIFOLDS

In this section, we define and describe the manifolds we consider. Although our deformation of SRCB manifolds works only for $n = 3$, we give the definition for arbitrary n. We remark that, on the one hand, Honda has constructed Moishezon twistor spaces, for arbitrary n, which are generalisations of the double solids for $n = 3$ [17, 21, 23]. On the other hand, Honda has also constructed Moishezon twistor spaces, for arbitrary n, which are generalisations of LeBrun twistor spaces [19]. It would be interesting to explore in which way our construction of a versal deformation could be extended to $n \geqslant 4$ and whether this could give a deformation of (generalised) LeBrun twistor spaces into Honda's generalised double solid twistor spaces.

We start with the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{E}_Q) := \mathbb{P}(\mathcal{O}_Q \oplus \mathcal{O}_Q(1-n,-1) \oplus \mathcal{O}_Q(-1,1-n))$ over the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$, in which we consider the hyper-surface W with bihomogeneous equation

$$w_1 w_2 - \varphi w_0^2 = 0 \,. \tag{4}$$

Here, $(w_0 : w_1 : w_2)$ are fibre coordinates corresponding to the direct summands in this order and $\varphi \in H^0(Q, \mathcal{O}_Q(n,n))$.

Remark 1.1. We shall consistently work with homogeneous coordinates. The bundle $\mathbb{P}(\mathcal{E}_Q)$ is obtained as the quotient of $\left(\mathbb{C}^2 \setminus 0\right) \times \left(\mathbb{C}^2 \setminus 0\right) \times \left(\mathbb{C}^3 \setminus 0\right)$ under the $(\mathbb{C}^*)^3$-action which is given by $(a,b,c) \cdot (s_0, s_1 \,;\, t_0, t_1 \,;\, w_0, w_1, w_2) = (as_0, as_1 \,;\, bt_0, bt_1 \,;\, cw_0, ca^{n-1}bw_1, cab^{n-1}w_2)$.

Obviously, Equation (4) defines a family of conic bundles which is contained in $\mathbb{P}(\mathcal{E}_Q) \times H^0(\mathcal{O}_Q(n,n))$. Because we get an isomorphic conic bundle, if we replace φ with a non-zero multiple of it, it is desirable to use the equation to define a family of conic bundles over $\mathbb{P}(H^0(\mathcal{O}_Q(n,n))^*)$. It turns out that we do not get a family of conic bundles in the product space $\mathbb{P}(\mathcal{E}_Q) \times \mathbb{P}(H^0(\mathcal{O}_Q(n,n))^*)$ but in a non-trivial bundle over $\mathbb{P}(H^0(\mathcal{O}_Q(n,n))^*)$. This bundle is most conveniently described as the quotient of $\left(\mathbb{C}^2 \setminus 0\right) \times \left(\mathbb{C}^2 \setminus 0\right) \times \left(\mathbb{C}^3 \setminus 0\right) \times \left(H^0(\mathcal{O}_Q(n,n)) \setminus 0\right)$

under the $(\mathbb{C}^*)^4$-action which is given by

$$(a, b, c, \lambda) \cdot (s_0, s_1 \,;\, t_0, t_1 \,;\, w_0, w_1, w_2 \,;\, \varphi)$$
$$= (as_0, as_1 \,;\, bt_0, bt_1 \,;\, cw_0, \lambda ca^{n-1}bw_1, cab^{n-1}w_2 \,;\, \lambda\varphi)\,.$$

The zero set of Equation (4) is invariant under this action, hence defines a family of conic bundles $\mathcal{W} \longrightarrow \mathbb{P}(H^0(\mathcal{O}_Q(n,n))^*)$. Note that if we evaluate Equation (4) at the point $(as \,;\, bt \,;\, cw_0, ca^{n-1}bw_1, cab^{n-1}w_2 \,;\, \varphi)$, the symbol φ is to be seen as $\varphi(as \,;\, bt) = a^n b^n \varphi(s \,;\, t)$. Briefly speaking, the family of $\mathbb{P}^2$-bundles constructed this way is the quotient of $\mathbb{P}(\mathcal{E}_Q) \times H^0(\mathcal{O}_Q(n,n))$ under the $\mathbb{C}^*$-action which is given by $\lambda \cdot (s \,;\, t \,;\, w_0, w_1, w_2 \,;\, \varphi) = (s \,;\, t \,;\, w_0, \lambda w_1, w_2 \,;\, \lambda\varphi)$. We could equally well let $\lambda \in \mathbb{C}^*$ act on the w_2-component rather than on w_1. The two choices are related by the $\mathbb{C}^*$-action on the conic bundle. In fact, there is a $(\mathbb{C}^*)^2$-action on $\mathbb{P}(\mathcal{E}_Q) \times H^0(\mathcal{O}_Q(n,n))$ (or a $(\mathbb{C}^*)^5$-action if we include the (a,b,c)), given by $(\lambda, \mu) \cdot (s \,;\, t \,;\, w_0, w_1, w_2 \,;\, \varphi) = (s \,;\, t \,;\, w_0, \lambda w_1, \mu w_2 \,;\, \lambda\mu\varphi)$. The subgroup $\lambda\mu = 1$ survives as $\mathbb{C}^*$-action on the conic bundle.

The quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ has two projections onto $\mathbb{P}^1$, the projection pr_1 on the first factor and pr_2 on the second factor. The section $E_1 \cong Q$ of the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{E}_Q)$, given by $w_0 = w_1 = 0$, is contained in W, and its normal bundle in W is $\mathcal{O}_Q(-1, 1-n)$, so it restricts to $\mathcal{O}_{\mathbb{P}^1}(-1)$ on each line $\mathrm{pr}_2^{-1}(t)$ of the second ruling. Therefore, E_1 can be contracted along the second ruling, according to the Castelnuovo–Moishezon–Nakano criterion [1, 6.11], [36, Theorem 3.1].

Theorem 1.2 (Castelnuovo–Moishezon–Nakano criterion). *Let Y be smooth of codimension 1 in the manifold X and let $Y \to Y'$ be a fibration with fibres projective spaces. If the normal bundle $N_{Y/X}$ restricts to each fibre as $\mathcal{O}(-1)$, then there exists a contraction $X \to X'$ which blows down Y to $Y' \subset X'$.*

In the case at hand, one easily can write down the contraction explicitly. On W, consider the chart $U_1 = W - (w_2) \supset E_1$. With the homogeneous coordinates $(s_0 : s_1 \,;\, t_0 : t_1)$ on Q introduced above, we define functions

$$v_{i;j} := \frac{w_0}{w_2} s_0^{1-i} s_1^i t_0^{n-1-j} t_1^j, \qquad i = 0, 1, \quad j = 0, \dots, n-1. \tag{5}$$

Together with the homogeneous coordinates $(t_0 : t_1)$, these functions map U_1 onto the subset V_1 of $\mathbb{C}^{2n} \times \mathbb{P}^1$ defined by the 2×2 minors of the matrix

$$\begin{pmatrix} t_0^{n-1} & t_0^{n-2}t_1 & \dots & t_1^{n-1} \\ v_{0;0} & v_{0;1} & \dots & v_{0;n-1} \\ v_{1;0} & v_{1;1} & \dots & v_{1;n-1} \end{pmatrix}.$$

As t_0 and t_1 are not both zero, two equations defining a subspace of $\mathbb{C}^4 \times \mathbb{P}^1$ suffice:

$$\begin{aligned} v_{0;0}t_1^{n-1} - v_{0;n-1}t_0^{n-1} &= 0\,, \\ v_{1;0}t_1^{n-1} - v_{1;n-1}t_0^{n-1} &= 0\,. \end{aligned} \tag{6}$$

This is the description of the blow-down given by Kurke [31]. The section E_1 is blown down to the curve $C_1 : \{0\} \times \mathbb{P}^1$. From the equations, it is immediate that its normal bundle is $\mathcal{O}(1-n) \oplus \mathcal{O}(1-n)$.

Likewise the section, $E_2 : w_0 = w_2 = 0$ can be contracted along the first ruling. An explicit contraction $U_2 = W - (w_1) \to V_2$ is constructed from the above formulas by reversing the role of the s_i and t_i and of w_1 and w_2. Altogether we obtain a blowing down map $\beta : W \to Z$.

The space Z may have singularities, but we require that they admit a small resolution. To study this condition, we first observe that W and Z have the same singularities, as the singular points of W lie in $w_1 = w_2 = 0$ over the singular points of the curve $D : \varphi = 0$, outside the charts U_1 and U_2. We could also resolve first and then blow down. This leads to the commutative diagram

$$\begin{array}{ccc} \widetilde{W} & \xrightarrow{\sigma_W} & W \\ \big\downarrow \widetilde{\beta} & & \big\downarrow \beta \\ \widetilde{Z} & \xrightarrow{\sigma_Z} & Z\,. \end{array}$$

The postulated existence of a small resolution limits the possible types of singularities of W and Z. Around each point, one can find local coordinates such that the singularity has an equation of the form $w_1 w_2 - g(x, y) = 0$. This is a so-called cA_k point, with general hyperplane section of type A_k, where $k + 1$ is the multiplicity of the plane curve singularity g. Such a singularity admits a small resolution if and only if g factors as a product of $k + 1$ factors, so defines a curve singularity Γ with $k + 1$ smooth branches [8, p. 676].

Definition. An *SRCB manifold* $\widetilde{Z}$ is a small resolution of the three-fold Z obtained by blowing down the sections $E_1 : w_0 = w_1 = 0$ and $E_2 : w_0 = w_2 = 0$ along opposite rulings in the hypersurface W in

$$\mathbb{P}\big(\mathcal{O}_Q \oplus \mathcal{O}_Q(1-n, -1) \oplus \mathcal{O}_Q(-1, 1-n)\big)$$

with bihomogeneous equation (4):

$$w_1 w_2 - \varphi w_0^2 = 0\,,$$

where φ is a section of $\mathcal{O}_Q(n, n)$, defining a curve, whose singularities have only smooth branches.

A cA_k singularity X of the form $w_1 w_2 - g(x, y) = 0$, where g defines a curve singularity Γ with $k + 1$ smooth branches, has $(k + 1)!$ small resolutions. Simultaneous small resolutions of such singularities have been studied by Friedman [8]. Each deformation of a given small resolution $\widetilde{X}$ of X blows down to a deformation of X, as $H^1(\mathcal{O}_{\widetilde{X}}) = 0$ [40]. This gives a map $\mathrm{Def}_{\widetilde{X}} \to \mathrm{Def}_X$ of deformation spaces (actually a map between the deformation functors), which is an inclusion of germs: the induced map $H^1(\Theta_{\widetilde{X}}) \to T^1_X$ of tangent spaces has as kernel the vanishing local cohomology group $H^1_C(\Theta_{\widetilde{X}})$, where $C \subset \widetilde{X}$ is the exceptional curve [8, Prop. 2.1].

The miniversal deformation of the three-fold singularity X, which is a double suspension of the curve singularity Γ, is obtained by double suspension of the miniversal deformation of Γ: it is of the form $w_1 w_2 - G(x, y; u_1, \dots, u_\tau) = 0$. For the infinitesimal deformations, one has the isomorphism $T^1_X = \mathbb{C}\{w_1, w_2, x, y\}/(w_2, w_1, g_x, g_y, w_1 w_2 - g) \cong \mathbb{C}\{x, y\}/(g_x, g_y, g) = T^1_\Gamma$.

The image of $\mathrm{Def}_{\widetilde{X}}$ in Def_X corresponds to the locus in the deformation space of Γ, where one still has the factorisation in $k+1$ branches. Such deformations are obtained by deforming the parametrisation of the curve Γ; this means that we deform the composed map germ $\nu : \widetilde{\Gamma} \to \mathbb{C}^2$, where $\widetilde{\Gamma}$ is the normalisation of Γ. These deformations are exactly the δ-constant deformations, where δ is the number of (virtual) double points of the singularity Γ. The locus of δ-constant deformations in Def_Γ is smooth of codimension δ, as Γ has smooth branches (a convenient reference is [10, Sect. II.2.7]).

Seen the other way round, the above discussion means that, given a small resolution $\widetilde{X}$ of X, and a small deformation of X, obtained from a δ-constant deformation of Γ, there is a unique simultaneous small resolution extending the given one.

But monodromy may prevent the existence of global simultaneous small resolutions: even if for a family $\mathcal{X} \to S$, all small resolutions of the singularities of the fibres X_s extend to a neighbourhood of $s \in S$, this does not mean that they extend to a simultaneous small resolution $\widetilde{\mathcal{X}} \to S$.

Example 1.3. Consider the 1-parameter deformation $w_1w_2 - x(x+y^2-\varepsilon) = 0$ of the cA_1 singularity $w_1w_2 - x(x+y^2) = 0$. The corresponding deformation of the A_3 curve singularity $\Gamma : x(x+y^2) = 0$ is the simplest δ-constant deformation one can imagine: the curve Γ consists of a parabola and its vertical tangent, and we move the parabola. So, the general fibre has two singular points, lying at $(w_1, w_2, x, y) = (0, 0, 0, \pm\sqrt{\varepsilon})$, which are ordinary A_1-singularities. Therefore, there are four small resolutions. The central fibre has only one singularity, and two small resolutions, both given as closure of the graph of a map to $\mathbb{P}^1$. These resolutions extend over the whole ε-axis. The maps are

$$\frac{w_1}{x} = \frac{x+y^2-\varepsilon}{w_2} = \frac{\sigma}{\tau} \qquad \text{and} \qquad \frac{w_1}{x+y^2-\varepsilon} = \frac{x}{w_2} = \frac{\sigma}{\tau}\,.$$

In both cases, the exceptional curve is a smooth $\mathbb{P}^1$ with normal bundle $\mathcal{O} \oplus \mathcal{O}(-2)$, which splits under deformation in two curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, see also [8, p. 678]. The extension of these two small resolutions to the general fibre gives us two of the four small resolutions of the general fibre. The other two are obtained by resolving the singularity at $y = \sqrt{\varepsilon}$ with one of the two resolutions above and the singularity at $y = -\sqrt{\varepsilon}$ with the other. But this cannot be done consistently for all $\varepsilon \neq 0$; one needs a base change. We consider for each ε in the unit disk the set of all possible small resolutions. By local extendibility, these fit together into three components, two disks corresponding to the two global small resolutions, and one punctured disk which is a $2:1$ covering of the punctured disk.

The description of the singularities translates into the following description of the discriminant curve $D : \varphi = 0$. Let $\widetilde{D}$ be the normalisation of D and consider the composed map $\nu : \widetilde{D} \to Q$. Then, ν is an immersion. We can deform the SRCB manifold Z by deforming $\widetilde{D}$ and the map ν. Let r be the number of components of D and g the sum of the geometric genera of these components. Let δ be the sum of the delta invariants of the singular points of D. As the arithmetic genus of D, a curve of type (n, d), is $(n-1)^2$, the cohomology exact sequence of the sequence

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{\widetilde{D}} \longrightarrow \bigoplus_p \mathcal{O}_{\widetilde{D},p}/\mathcal{O}_{D,p} \longrightarrow 0$$

gives the relation $1 + \delta + g = r + (n-1)^2$ between these invariants.

The space $\mathbb{P}(H^0\mathcal{O}_Q(n,n)^*)$ parametrising discriminant curves D splits naturally into different strata, according to the values of the total δ. These strata may have several components, which sometimes can be distinguished by the number r of components of D; for example, a curve of type (3,3) with $\delta = 4$ can be a rational curve with four double points, or the intersection of an elliptic curve (type (2,2)) with a conic (type (1,1)). In each stratum, we consider the open set where the normalisation ν is an immersion. As will be shown as a consequence of Lemma 2.2 in the next section, this open set is smooth of codimension δ. It constitutes a versal (but not miniversal) deformation at each of its points (versal for δ-const deformations). We prefer to work with this larger family and not with the miniversal deformation of a given curve D. In fact, as we shall see, the dimension of the miniversal deformation depends not only on δ, but also on the automorphism group of D, whereas the dimension of the stratum depends on δ only. Locally, one can always find a miniversal deformation by taking a suitable transverse slice to the orbit of the group of coordinate transformations.

Let $\overline{\Lambda}_0 \subset \mathbb{P}(H^0(\mathcal{O}_Q(n,n))^*)$ be a maximal open set in a stratum, parametrising possible discriminant curves D with smooth branches everywhere. For each fibre of the family of conic bundles over it, we consider all possible small resolutions. These fit together to a covering $\Lambda_0 \longrightarrow \overline{\Lambda}_0$ with finite fibres, which may not be locally trivial (see Example 1.3), but which still is a local homeomorphism. In this way, we get a family $W_{\Lambda_0} \to \Lambda_0$ with simultaneous small resolution $\widetilde{W}_{\Lambda_0} \to \Lambda_0$. Blowing down sections finally gives a family $\widetilde{Z}_{\Lambda_0} \to \Lambda_0$, where each fibre $\widetilde{Z}_{\lambda_0}$ is an SRCB manifold.

Remark 1.4. The largest stratum, where $\delta = 0$, parametrises smooth curves D, so no small resolution is needed. The next stratum ($\delta = 1$) gives irreducible curves with one ordinary double point or a cusp; we only consider the open set parametrising ordinary double points, so there are two small resolutions. Presumably, the double cover Λ_0 is globally irreducible. For $\delta = 2$, the local situation of Example 1.3 can occur. In the twistor case, we have $g = 0$ and $r = n$ and so $\delta = n(n-1)$, which is equal to 6 if $n = 3$, the case we specialise to later on.

2 | INFINITESIMAL DEFORMATIONS

To study the deformation theory of an SRCB manifold $\widetilde{Z}$, we first determine the cohomology of its tangent sheaf. We also consider infinitesimal deformations of $\widetilde{Z}$ together with two surfaces $\widetilde{S}_1, \widetilde{S}_2$. In the twistor case, such computations were done by LeBrun [33] under the assumption that all singularities are of type A_1. A closely related proof that $h^i(\Theta_{\widetilde{Z}}) = 0$ for $i \geqslant 2$ was given by Campana [4, 5], under the same assumptions. Honda [15] considered the case of extra symmetry. All these computations do not use the additional real structure and they generalise to our situation. We follow the same strategy. As our computations mostly take place on the singular spaces Z and W, we use the general theory of the cotangent complex, but for our purposes, the treatment in [35] suffices.

Let W be a conic bundle over $Q = \mathbb{P}^1 \times \mathbb{P}^1$ with Equation (4): $w_1 w_2 - \varphi w_0^2 = 0$. Let r be the number of components of the discriminant curve $D : \varphi = 0$ and g the sum of the geometric genera of these components. Let δ be the sum of the delta invariants of the singular points of D.

Proposition 2.1. *Let $\sigma : \widetilde{W} \to W$ be any small resolution of the three-fold W. Suppose $n \geqslant 3$. Then $h^0(\Theta_{\widetilde{W}}) = 2$, if D admits a 1-dimensional symmetry group, and $h^0(\Theta_{\widetilde{W}}) = 1$ otherwise, $h^1(\Theta_{\widetilde{W}}) - h^0(\Theta_{\widetilde{W}}) = n^2 + 2n - 7 - \delta = g - r + 4n - 7$ and $h^j(\Theta_{\widetilde{W}}) = 0$ for $j \geqslant 2$.*

Proof. We first study the tangent cohomology of W. It turns out that every deformation of W comes about by changing the equation inside the $\mathbb{P}^2$-bundle

$$\mathbb{P} := \mathbb{P}\big(\mathcal{O}_Q \oplus \mathcal{O}_Q(1-n,-1) \oplus \mathcal{O}_Q(-1,1-n)\big) \xrightarrow{\pi} Q\,.$$

To prove this, we look at global tangent cohomology. Consider the embedding $i\colon W \to \mathbb{P}$. The functor $\mathrm{Def}_{W/\mathbb{P}}$ describes the deformations of W and i inside $\mathbb{P}$. We want to compare it with the deformation functor of W. For the corresponding tangent cohomology sheaves, we have a long exact sequence [35, (4.3.1)]

$$0 \longrightarrow \mathcal{T}^0(W/\mathbb{P}) \longrightarrow \mathcal{T}^0_W \longrightarrow \mathcal{T}^0(\mathbb{P},\mathcal{O}_W) \longrightarrow \mathcal{T}^1(W/\mathbb{P}) \longrightarrow \mathcal{T}^1_W \longrightarrow \dots\,.$$

As i is an embedding, $\mathcal{T}^0(W/\mathbb{P}) = 0$ and $\mathcal{T}^1(W/\mathbb{P}) = N$, the normal sheaf of W in $\mathbb{P}$. The long exact sequence reduces to the familiar exact sequence

$$0 \longrightarrow \mathcal{T}^0_W \longrightarrow \Theta_{\mathbb{P}}|_W \longrightarrow N \longrightarrow \mathcal{T}^1_W \longrightarrow 0\,,$$

because $\mathcal{T}^i(\mathbb{P},\mathcal{O}_W) = 0$ for $i > 0$ ($\mathbb{P}$ is smooth) and $\mathcal{T}^i_W = 0$ for $i \geqslant 2$ as W has only hypersurface singularities. The local-to-global spectral sequence now gives $T^0(W/\mathbb{P}) = 0$ and $T^i(W/\mathbb{P}) = H^{i-1}(N)$. The global version of the long exact sequence is

$$0 \longrightarrow T^0(W/\mathbb{P}) \longrightarrow T^0_W \longrightarrow T^0(\mathbb{P},\mathcal{O}_W) \longrightarrow T^1(W/\mathbb{P}) \longrightarrow T^1_W \longrightarrow \dots$$

As (again by a local-to-global argument) $T^i(\mathbb{P},\mathcal{O}_W) = H^i(\Theta_{\mathbb{P}}|_W)$, one obtains

$$0 \longrightarrow T^0_W \longrightarrow H^0(\Theta_{\mathbb{P}}|_W) \longrightarrow H^0(N) \longrightarrow T^1_W \longrightarrow \dots \tag{7}$$

The relevant groups have been computed by Campana [5, Lemma 3.8] and LeBrun (in the proof of [33, Prop. 1]). The tangent bundle to $\mathbb{P}$ is described by the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow (\pi^*\mathcal{E}_Q^\vee)(1) \longrightarrow \Theta_{\mathbb{P}} \longrightarrow \pi^*\Theta_Q \longrightarrow 0\,,$$

where $\mathcal{E}_Q$, as before, is the vector bundle we have used to define $\mathbb{P} = \mathbb{P}(\mathcal{E}_Q) \xrightarrow{\pi} Q$. We first twist this sequence with the ideal sheaf $\mathcal{O}_{\mathbb{P}}(-W) = \mathcal{O}_{\mathbb{P}}(-2) \otimes \pi^*\mathcal{O}_Q(-n,-n)$ to compute $H^i(\Theta_{\mathbb{P}}(-W))$ using the Leray spectral sequence for the map π. As $H^i(\mathcal{O}_{\mathbb{P}^2}(-1)) = H^i(\mathcal{O}_{\mathbb{P}^2}(-2)) = 0$ for all i, we find that $H^i(\Theta_{\mathbb{P}}(-W)) = 0$. This implies that $H^i(\Theta_{\mathbb{P}}|_W) \cong H^i(\Theta_{\mathbb{P}})$, which is independent of W. We compute these groups using the exact sequence

$$0 \longrightarrow \Theta_\pi \longrightarrow \Theta_{\mathbb{P}} \longrightarrow \pi^*\Theta_Q \longrightarrow 0\,,$$

defining the sheaf Θ_π of vertical vector fields. Using $\pi_*\mathcal{O}_{\mathbb{P}} = \mathcal{O}_Q$, $R^1\pi_*\mathcal{O}_{\mathbb{P}} = 0$ and $\pi_*\mathcal{O}_{\mathbb{P}}(1) = \mathcal{E}_Q$, the exact sequence $0 \to \mathcal{O}_{\mathbb{P}} \to (\pi^*\mathcal{E}_Q^\vee)(1) \to \Theta_\pi \to 0$, obtained from the Euler sequence, implies $H^i(\Theta_\pi) = H^i(Q, \mathrm{End}_0(\mathcal{E}_Q))$, where $\mathrm{End}_0(\mathcal{E}_Q)$ is the sheaf of traceless endomorphisms of $\mathcal{E}_Q$. One finds $h^0(\Theta_{\mathbb{P}}) = 4n+8$, $h^1(\Theta_{\mathbb{P}}) = 2(n-1)(n-3)$ and $h^i(\Theta_{\mathbb{P}}) = 0$ for $i \geqslant 2$.

The cohomology of N can be computed from

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{P}}(2) \otimes \pi^*\mathcal{O}_Q(n,n) \longrightarrow N \longrightarrow 0\,.$$

The result is [33]: $h^0(N) = n^2 + 6n + 1$, $h^1(N) = 2(n-1)(n-3)$ and $h^i(N) = 0$ for $i \geqslant 2$. Here, one uses $\pi_*(\mathcal{O}_{\mathbb{P}}(k)) = \mathrm{Sym}^k(\mathcal{E}_Q)$ for $k \geqslant 0$.

All these groups occur in our long exact sequence (7), which now reads

$$0 \to T^0_W \longrightarrow H^0(\Theta_{\mathbb{P}}) \longrightarrow H^0(N) \longrightarrow T^1_W \longrightarrow H^1(\Theta_{\mathbb{P}}) \longrightarrow H^1(N) \longrightarrow T^2_W \to 0\,.$$

The map $H^1(\Theta_{\mathbb{P}}) \to H^1(N)$ is an isomorphism. For $n = 3$, both groups are zero. For $n > 3$, the proof is given in [5, 3.11] and [33]. One realises $H^1(\Theta_{\mathbb{P}})$ as $H^1(\mathcal{O}_Q(2-n, n-2))w_2\frac{\partial}{\partial w_1} \oplus H^1(\mathcal{O}_Q(n-2, 2-n))w_1\frac{\partial}{\partial w_2}$ and $H^1(N)$ as $H^1(\mathcal{O}_Q(2-n, n-2))w_1^2 \oplus H^1(\mathcal{O}_Q(n-2, 2-n))w_2^2$. The map is given by evaluating a vector field on the polynomial $w_1w_2 - \varphi w_0^2$ defining W; it clearly is an isomorphism. We note that it is independent of the precise form of the polynomial φ. We conclude that $T^i_W = 0$ for $i \geqslant 2$, so the deformation space of W is smooth. Furthermore, every deformation of W comes from deforming the equation of W inside $\mathbb{P}$ and we get the exact sequence

$$0 \longrightarrow T^0_W \longrightarrow H^0(\Theta_{\mathbb{P}}) \to H^0(N) \longrightarrow T^1_W \longrightarrow 0\,,$$

giving us a formula for the dimension of T^1_W, once we know the symmetry group of W. The automorphism group of $\mathbb{P}$ has dimension $4n + 8$. Automorphisms of the form

$$(w_0 : w_1 : w_2) \mapsto (w_0 : w_1 + \psi_1 w_0 : w_2 + \psi_2 w_0)\,,$$

with $\psi_1 \in H^0(\mathcal{O}_Q(n-1, 1))$ and $\psi_2 \in H^0(\mathcal{O}_Q(1, n-1))$, which form a $4n$-dimensional family, do not preserve the defining equation $w_1w_2 - \varphi w_0^2 = 0$ for W. The 1-parameter subgroup $\lambda \cdot (w_0, w_1, w_2) = (w_0, \lambda w_1, \lambda^{-1} w_2)$ of the two-dimensional diagonal action on $(w_0 : w_1 : w_2)$ leaves W invariant. Finally, one has the automorphisms of the base quadric Q. The dimension of the subgroup of automorphisms preserving the curve D is at most 1. A description of the occurring cases is given in remark 2.3 below. This implies $1 \leqslant \dim T^0_W \leqslant 2$.

To get to $\widetilde{W}$, we use the Leray spectral sequence $H^p(R^q\sigma_*\Theta_{\widetilde{W}}) \Rightarrow H^{p+q}(\Theta_{\widetilde{W}})$ and compare it to the local-to-global spectral sequence for T^i_W, as in [8, (3.4)]:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H^1(\mathcal{T}^0_W) & \longrightarrow & H^1(\Theta_{\widetilde{W}}) & \longrightarrow & H^0(R^1\sigma_*\Theta_{\widetilde{W}}) & \longrightarrow & H^2(\mathcal{T}^0_W) & \longrightarrow & H^2(\Theta_{\widetilde{W}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(\mathcal{T}^0_W) & \longrightarrow & T^1_W & \longrightarrow & H^0(\mathcal{T}^1_W) & \longrightarrow & H^2(\mathcal{T}^0_W) & \longrightarrow & T^2_W & \longrightarrow & 0
\end{array}$$

We have shown that $T^2_W = 0$. In general, the map $T^1_W \to H^0(\mathcal{T}^1_W)$ will not be surjective, that is, the singular points do not impose independent conditions. So, $H^2(\mathcal{T}^0_W)$ will be non-zero. We compare the image of the deformations of $\widetilde{W}$ in the deformations of W with the image of the δ-constant deformations of the curve $D : \varphi = 0$ (the discriminant of the conic fibration W) in all deformations of D.

Given a deformation of W, we get a deformation of the discriminant curve and conversely, given a deformation φ' of φ, we can write the equation $w_1w_2 - \varphi' w_0^2$. This shows that the natural map $\mathrm{Def}_{W/\mathbb{P}} \to \mathrm{Def}_{D/Q}$ is a smooth surjection of functors. On the tangent space level, the kernel is the space of global sections of vertical vector fields.

To study δ-constant deformations of D, we let $\widetilde{D}$ be the normalisation of D and we consider the composed map $\nu\colon \widetilde{D} \to Q$. We study the deformation functor $\mathrm{Def}_{\widetilde{D}/Q}$. The tangent cohomology $\mathcal{T}^1(\widetilde{D}/Q)$ is the normal bundle along the map ν occurring in the exact sequence

$$0 \longrightarrow \Theta_{\widetilde{D}} \longrightarrow \nu^*\Theta_Q \longrightarrow N_\nu \longrightarrow 0\,.$$

The assumption that all branches at the singular points of D are smooth gives that N_ν is a line bundle. The tangent space to $\mathrm{Def}_{\widetilde{D}/Q}$ is $H^0(N_\nu)$ and obstructions lie in $H^1(N_\nu)$. A straightforward calculation shows $\chi\left(\nu^*\Theta_Q\right) = 4n + 2(r-g)$, $\chi\left(\Theta_{\widetilde{D}}\right) = 3(r-g)$ and hence $\chi\left(N_\nu\right) = 4n - (r-g) = (n+1)^2 - 1 - \delta$.

Lemma 2.2. $H^1(N_\nu) = 0$.

Proof. As the degree of $\nu^*\Theta_Q$ is positive on every component $\widetilde{D}_i$ of $\widetilde{D}$, we have $\deg N_\nu|_{\widetilde{D}_i} > 2g(\widetilde{D}_i) - 2$ and $H^1(\widetilde{D}_i, N_\nu) = 0$. □

We conclude that $\mathrm{Def}_{\widetilde{D}/Q}$ is smooth of the expected codimension δ in $\mathrm{Def}_{D/Q}$, so $H^1(\Theta_{\widetilde{W}})$ also has the expected codimension in T^1_W. This is also the codimension of $H^0(R^1\sigma_*\Theta_{\widetilde{W}})$ in $H^0(\mathcal{T}^1_W)$. Therefore, $H^1(\Theta_{\widetilde{W}}) \to H^0(R^1\sigma_*\Theta_{\widetilde{W}})$ and $T^1_W \to H^0(\mathcal{T}^1_W)$ have the same cokernel. This shows that $H^2(\Theta_{\widetilde{W}}) = 0$. □

Remark 2.3. The symmetries of the curve D on $Q = \mathbb{P}^1 \times \mathbb{P}^1$ can be determined with the same methods as for plane curves [37]. We are grateful to C.T.C. Wall for help on this point. Only a few cases arise. If the symmetry group is semisimple, the action can be diagonalised. The monomials $s_0^i s_1^{n-i} t_0^j t_1^{n-j}$ occurring in the equation $\varphi = 0$ must be such that the points (i,j) lie on a straight line in $[0,n] \times [0,n]$. If this line joins two opposite corners, we get an equation of the form $\prod_{i=1}^n (a_i s_0 t_0 - b_i s_1 t_1)$ or with t_0 and t_1 interchanged; the twistor spaces with torus action studied by Honda [15] are of this type. The total δ increases if one or two of the conics degenerate, giving the cases $s_0 t_0 \prod_{i=1}^{n-1}(a_i s_0 t_0 - b_i s_1 t_1)$ and $s_0 s_1 t_0 t_1 \prod_{i=1}^{n-2}(a_i s_0 t_0 - b_i s_1 t_1)$. The line from (1,0) to $(n, n-1)$ leads to $s_0 t_1 \prod_{i=1}^{n-1}(a_i s_0 t_0 - b_i s_1 t_1)$, containing the degenerate conic $s_0 t_1$. For general n, these are the only possibilities to obtain a reduced curve with singularities with smooth branches, but for $n = 3$ and $n = 4$, one also has $t_0 t_1 (s_0^n t_0^{n-2} - s_1^n t_1^{n-2})$. For unipotent symmetry group, there are a priori two cases, but only $\lambda \cdot (s_0 : s_1\,; t_0 : t_1) = (s_0 : s_1 + \lambda s_0\,; t_0 : t_1 + \lambda t_0)$ gives reduced curves, namely tangent conics $\prod_{i=1}^n (s_0 t_1 - s_1 t_0 - c_i s_0 t_0)$, one of which may degenerate to the line pair $s_0 t_0$.

Example 2.4. In our discussion of Honda's deformation with torus action [15] in Example 5.2 in Section 5, we consider a discriminant curve with one node and two singularities of type $\widetilde{E}_7$ (four lines through one point). In suitable coordinates, its equation is

$$s_0 t_0 (a_1 s_0 t_0 - b_1 s_1 t_1)(a_2 s_0 t_0 - b_2 s_1 t_1)(a_3 s_0 t_0 - b_3 s_1 t_1) = 0\,.$$

The cross ratio of the four lines at the two singular points $(1:0;0:1)$ and $(0:1;1:0)$ is the same. This shows that the stratum does not form a versal δ-const deformation of each of the singular points separately. We have here an example where the singular points do not impose independent conditions.

Theorem 2.5. *Let $\widetilde{Z}$ be an SRCB manifold, that is, $\sigma : \widetilde{Z} \to Z$ is any small resolution of the three-fold Z, obtained by blowing down sections of W along opposite rulings. Suppose $n \geqslant 3$. Then, $h^0(\Theta_{\widetilde{Z}}) = 2$, if the discriminant curve D admits a one-dimensional symmetry group, and $h^0(\Theta_{\widetilde{Z}}) = 1$ otherwise, $h^1(\Theta_{\widetilde{Z}}) - h^0(\Theta_{\widetilde{Z}}) = n^2 + 6n - 15 - \delta = g - r + 8n - 15$ and $h^j(\Theta_{\widetilde{Z}}) = 0$ for $j \geqslant 2$. In particular, the deformation space is smooth.*

Proof (cf. [33, Prop. 2 and Cor. 1]). Let $\beta : W \to Z$ be the blowing down map. As the singular points of W are disjoint from the exceptional locus of the map β, we can choose a small resolution $\sigma : \widetilde{W} \to W$ such that $\widetilde{W}$ blows down to $\widetilde{Z}$. The image under β of the section E_1 is the curve C_1 and the curve C_2 is the image of E_2. These objects are not changed under the small resolution, but if we consider them on $\widetilde{W}$ or $\widetilde{Z}$, we call them $\widetilde{E}_i$ and $\widetilde{C}_i$. Let $(\Theta_{\widetilde{Z}})_{\widetilde{C}_1,\widetilde{C}_2}$ be the sheaf of those vector fields which are tangent to $\widetilde{C}_1$ and $\widetilde{C}_2$. Then $\widetilde{\beta}_*\Theta_{\widetilde{W}} = (\Theta_{\widetilde{Z}})_{\widetilde{C}_1,\widetilde{C}_2}$ and $R^i\widetilde{\beta}_*\Theta_{\widetilde{W}} = 0$ for $i > 0$. A local computation shows that on $\widetilde{Z}$, we have the exact sequence

$$0 \longrightarrow (\Theta_{\widetilde{Z}})_{\widetilde{C}_1,\widetilde{C}_2} \longrightarrow \Theta_{\widetilde{Z}} \longrightarrow N_{\widetilde{C}_1} \oplus N_{\widetilde{C}_2} \longrightarrow 0$$

[33, Proof of Prop. 2]. The normal bundle of the curves $\widetilde{C}_1$ and $\widetilde{C}_2$ is $\mathcal{O}(1-n) \oplus \mathcal{O}(1-n)$, so the result follows from Proposition 2.1: we have $H^j(\Theta_{\widetilde{Z}}) \cong H^j(\Theta_{\widetilde{W}}) = 0$ for $j \geqslant 2$ and the exact sequence

$$0 \longrightarrow H^1(\Theta_{\widetilde{W}}) \longrightarrow H^1(\Theta_{\widetilde{Z}}) \longrightarrow H^1(N_{\widetilde{C}_1}) \oplus H^1(N_{\widetilde{C}_2}) \longrightarrow 0\,. \tag{8}$$

□

For our construction, it is natural not only to consider the deformation theory of $\widetilde{Z}$ itself, but also deformations of $\widetilde{Z}$ together with two divisors $\widetilde{S}_1, \widetilde{S}_2$.

Let $l_1 = \mathrm{pr}_1^{-1}(s)$ be a line in first the ruling of Q, which is not blown down under the contraction $E_1 \to C_1$. We assume that l_1 is not a component of the discriminant curve D. Let $R_1 = \pi^{-1}(l_1) \subset W$ be its inverse image under the conic-bundle projection $\pi : W \to Q$. If l_1 does not pass through the singular points of D, we can consider R_1 also as surface $\widetilde{R}_1$ on a small resolution $\widetilde{W}$. In the opposite case, we denote by $\widetilde{R}_1$ the strict transform of our surface on $\widetilde{W}$. By blowing down, we obtain the divisors $S_1 := \beta_*(R_1)$ on Z and $\widetilde{S}_1 := \widetilde{\beta}_*(\widetilde{R}_1)$ on $\widetilde{Z}$. The surface $\widetilde{S}_1$ contains the curve $\widetilde{C}_1$. Choosing a line l_2 in the other ruling yields a surface $\widetilde{S}_2$. The choice of l_1 and l_2 is determined by giving a point on the quadric Q. Explicitly, with the coordinates of Equations (5), a point $(a_0 : a_1 ; b_0 : b_1)$ gives the line $l_1 : a_1 s_0 - a_0 s_1 = 0$ on the quadric. On $\widetilde{W}$, the surface $\widetilde{S}_1$ is then given by $a_1 v_{0;0} - a_0 v_{1;0} = 0$ and $a_1 v_{0;n-1} - a_0 v_{1;n-1} = 0$. It follows from Equations (6) that the normal bundle of $\widetilde{C}_1$ in $\widetilde{S}_1$ is $\mathcal{O}(1-n)$.

In general, $\widetilde{S}_1$ and $\widetilde{S}_2$ are smooth and intersect transversally, but they may have at most rational singularities if the line in question is tangent to the curve $D : \varphi = 0$.

We now look at deformations of the triple $(\widetilde{Z}, \widetilde{S}_1, \widetilde{S}_2)$. The relevant deformation theory is that of the map $f : \widetilde{S}_1 \amalg \widetilde{S}_2 \to \widetilde{Z}$. Here, it is allowed to deform all the spaces $\widetilde{S}_1, \widetilde{S}_2, \widetilde{Z}$ and the map f. Infinitesimal deformations are given by T^1_f, while obstructions land in T^2_f. These groups occur in the following long exact sequence, which relates them to deformations of $\widetilde{Z}$, cf. [35, (4.5.2)]:

$$\cdots \longrightarrow T^i\big(\widetilde{S}_1 \amalg \widetilde{S}_2/\widetilde{Z}\big) \longrightarrow T^i_f \longrightarrow H^i(\Theta_{\widetilde{Z}}) \longrightarrow \ldots \qquad .$$

The local tangent cohomology $\mathcal{T}^i(\widetilde{S}_1 \amalg \widetilde{S}_2/\widetilde{Z})$ is concentrated in degree 1 and is equal to $N_{\widetilde{S}_1} \oplus N_{\widetilde{S}_2}$. Therefore, $T^i(\widetilde{S}_1 \amalg \widetilde{S}_2/\widetilde{Z}) = H^{i-1}(N_{\widetilde{S}_1}) \oplus H^{i-1}(N_{\widetilde{S}_2})$. On $\widetilde{W}$, the normal bundle of $\widetilde{R}_i$ is trivial, so by blowing down the section E_i, we obtain $N_{\widetilde{S}_i} \cong \mathcal{O}_{\widetilde{S}_i}(\widetilde{C}_i)$, $i = 1, 2$.

Lemma 2.6. *The map $H^1(\Theta_{\widetilde{Z}}) \to H^1(N_{\widetilde{S}_1}) \oplus H^1(N_{\widetilde{S}_2})$ is surjective.*

Proof. We first study the normal bundle of the surfaces separately. According to our notational convention (from the end of the Introduction), we drop indices. The isomorphism $N_{\widetilde{S}} \cong \mathcal{O}_{\widetilde{S}}(\widetilde{C})$ gives that $H^i(N_{\widetilde{S}}(-\widetilde{C})) = H^i(\mathcal{O}_{\widetilde{S}}) = 0$ for $i > 0$ as $\widetilde{S}$ is a rational surface with at most rational singularities. From the exact sequence

$$0 \longrightarrow N_{\widetilde{S}}(-\widetilde{C}) \longrightarrow N_{\widetilde{S}} \longrightarrow N_{\widetilde{S}}|_{\widetilde{C}} \longrightarrow 0,$$

we conclude that $H^1(N_{\widetilde{S}}) = H^1(N_{\widetilde{S}}|_{\widetilde{C}})$. This last group also occurs in the normal bundle sequence of $\widetilde{C}$:

$$0 \longrightarrow N_{\widetilde{C}/\widetilde{S}} \longrightarrow N_{\widetilde{C}} \longrightarrow N_{\widetilde{S}}|_{\widetilde{C}} \longrightarrow 0\ .$$

Now we return to the global situation and consider the commutative diagram

$$\begin{array}{ccc} H^1(\Theta_{\widetilde{Z}}) & \longrightarrow & H^1(N_{\widetilde{S}_1}) \oplus H^1(N_{\widetilde{S}_2}) \\ \big\downarrow & & \big\downarrow \cong \\ H^1(N_{\widetilde{C}_1}) \oplus H^1(N_{\widetilde{C}_2}) & \longrightarrow & H^1(N_{\widetilde{S}_1}|_{\widetilde{C}_1}) \oplus H^1(N_{\widetilde{S}_2}|_{\widetilde{C}_2}) \end{array} \tag{9}$$

By the exact sequence (8), the first vertical map is surjective. □

Theorem 2.7. *The deformation space of the triple $(\widetilde{Z}, \widetilde{S}_1, \widetilde{S}_2)$ is smooth of dimension $n^2 + 4n - 8 - \delta = g - r + 6n - 8$, except when the curve $D \cup l_1 \cup l_2$ admits a one-dimensional symmetry group, in which case the dimension is $n^2 + 4n - 7 - \delta = g - r + 6n - 7$.*

Proof. We look at the long exact sequence containing T^1_f:

$$\begin{aligned} 0 \longrightarrow T^0_f \longrightarrow H^0(\Theta_{\widetilde{Z}}) \longrightarrow H^0(N_{\widetilde{S}_1}) \oplus H^0(N_{\widetilde{S}_2}) \longrightarrow \\ \longrightarrow T^1_f \longrightarrow H^1(\Theta_{\widetilde{Z}}) \longrightarrow H^1(N_{\widetilde{S}_1}) \oplus H^1(N_{\widetilde{S}_2}) \longrightarrow T^2_f \longrightarrow 0\ . \end{aligned}$$

If $h^0(\Theta_{\widetilde{Z}}) = 2$, then the map $H^0(\Theta_{\widetilde{Z}}) \longrightarrow H^0(N_{\widetilde{S}_1}) \oplus H^0(N_{\widetilde{S}_2})$ is the zero-map only if the symmetries of the curve D also preserve $l_1 \cup l_2$. Otherwise the image is one-dimensional. If $h^0(\Theta_{\widetilde{Z}}) = 1$, this map is the zero-map again. This shows that $\dim T^0_f = 1$, except when the curve $D \cup l_1 \cup l_2$ admits a one-dimensional symmetry group, in which case $\dim T^0_f = 2$.

The curve $\widetilde{C}_i$ has self-intersection $1 - n$ on $\widetilde{S}_i$, so also the degree of $N_{\widetilde{S}_i}|_{\widetilde{C}_i}$ is $1 - n$ ($i = 1, 2$). Therefore, $H^0(N_{\widetilde{S}_i}) \cong H^0(\mathcal{O}_{\widetilde{S}_i})$ is one-dimensional, corresponding to moving the line l_i, which obviously induces a deformation of f (which can be trivial if it comes from an automorphism of D). Furthermore, $\dim H^1(N_{\widetilde{S}_i}) = n-2$. The result now follows from Lemma 2.6 and Theorem 2.5. □

For $n \geqslant 4$, the dimension of the deformation space of the triples $(\widetilde{Z}, \widetilde{S}_1, \widetilde{S}_2)$ is less than the dimension of the deformation space of the SRCB manifold $\widetilde{Z}$ itself. In the absence of extra symmetry, these dimensions are the same for $n = 3$, but the map $T^1_f \to H^1(\Theta_{\widetilde{Z}})$ is not surjective, as it has always a non-trivial kernel (of dimension $2 + \dim T^0_f - h^0(\Theta_{\widetilde{Z}})$, which is either 1 or 2). In fact, for all n, the codimension of the image of T^1_f in $H^1(\Theta_{\widetilde{Z}})$ is $h^1(N_{\widetilde{C}_1}) + h^1(N_{\widetilde{C}_2}) = 2(n-2)$.

We now compute the image of T^1_f in $H^1(\Theta_{\widetilde{Z}})$. It is the kernel of the map $H^1(\Theta_{\widetilde{Z}}) \longrightarrow H^1(N_{\widetilde{S}_1}) \oplus H^1(N_{\widetilde{S}_2})$, which is the same as the kernel of the composed map $H^1(\Theta_{\widetilde{Z}}) \longrightarrow H^1(N_{\widetilde{C}_1}) \oplus H^1(N_{\widetilde{C}_2}) \longrightarrow H^1(N_{\widetilde{S}_1}|_{\widetilde{C}_1}) \oplus H^1(N_{\widetilde{S}_2}|_{\widetilde{C}_2})$, by the diagram (9). The kernel of the first surjection consists, by the exact sequence (8), of the deformations obtained by changing the equation of W. For the second map, we explicitly determine the restriction map $H^1(N_{\widetilde{C}_1}) \to H^1(N_{\widetilde{S}_1}|_{\widetilde{C}_1})$, the case with $i = 2$ being similar. We use the coordinates of Equations (5). We cover $\widetilde{C}_1$ with two coordinate patches $\{t_0 \neq 0\}$ and $\{t_1 \neq 0\}$ and compute Čech cocycles. In the chart $t_0 = 1$, we have coordinates $v_{0;0}$, $v_{1;0}$ and t_1, with $\widetilde{C}_1$ given by $v_{0;0} = v_{1;0} = 0$. A basis of $H^1(N_{\widetilde{C}_1})$ is conveniently represented by elements in $H^1(\Theta_{\widetilde{Z}}|_{\widetilde{C}_1})$. We take the following Čech cocycles:

$$\frac{1}{t_1^i}\frac{\partial}{\partial v_{0;0}}, \quad \frac{1}{t_1^i}\frac{\partial}{\partial v_{1;0}}, \qquad i = 1, \dots, n-2 \,.$$

The kernel of $H^1(N_{\widetilde{C}_1}) \to H^1(N_{\widetilde{S}_1}|_{\widetilde{C}_1})$ lies in $H^1(N_{\widetilde{C}_1/\widetilde{S}_1})$. It is represented by cocycles of vector fields, which are tangent to $\widetilde{S}_1$. This means that evaluation on a defining equation for $\widetilde{S}_1$ yields zero, in the appropriate cohomology group, which is $H^1(\mathcal{O}_{\widetilde{C}_1} \otimes \mathcal{O}_{\widetilde{S}_1}(\widetilde{C}))$. The choice of a pair of lines (l_1, l_2) is determined by a point $(a_0 : a_1 \,;\, b_0 : b_1)$ on Q. We start out from the line $a_1 s_0 - a_0 s_1 = 0$ on the quadric Q. In our chart, the surface $\widetilde{S}_1$ is then given by $a_1 v_{0;0} - a_0 v_{1;0} = 0$, as remarked above. We compute the action of a vector field:

$$\sum \left(\frac{\tau_{0;i}}{t_1^i}\frac{\partial}{\partial v_{0;0}} + \frac{\tau_{1;i}}{t_1^i}\frac{\partial}{\partial v_{1;0}} \right)(a_1 v_{0;0} - a_0 v_{1;0}) = \sum \frac{a_1\tau_{0;i} - a_0\tau_{1;i}}{t_1^i} \,,$$

where $\tau_{0;i}, \tau_{1;i}$ are coordinates on $H^1(N_{\widetilde{C}_1})$. The kernel of the map $H^1(N_{\widetilde{C}_1}) \to H^1(N_{\widetilde{S}_1}|_{\widetilde{C}_1})$ is therefore given by the equations $a_1\tau_{0;i} - a_0\tau_{1;i} = 0$. For l_2, we similarly find equations $b_1\sigma_{0;i} - b_0\sigma_{1;i} = 0$.

For $n = 3$, we have only one equation $a_1\tau_{0;1} - a_0\tau_{1;1} = 0$. Considered in $\widetilde{C}_1 \times H^1(N_{\widetilde{C}_1}) \cong \mathbb{P}^1 \times \mathbb{C}^2$, it describes the blow-up of the origin in $H^1(N_{\widetilde{C}_1})$. Likewise, the blow-up of the origin in $H^1(N_{\widetilde{C}_2})$ is given in $\widetilde{C}_2 \times H^1(N_{\widetilde{C}_2})$ by $b_1\sigma_{0;1} - b_0\sigma_{1;1} = 0$. Therefore, the union of the images of T^1_f for all choices of $\widetilde{S}_1 \cap \widetilde{S}_2$ is $H^1(\Theta_{\widetilde{Z}})$, provided that the discriminant curve D does not contain a line.

For $n \geqslant 4$, the union of all images for all pairs of lines (again under the assumption that D does not contain a line) is given by the equations

$$\operatorname{Rank}\begin{pmatrix} \sigma_{01} & \dots & \sigma_{0,n-2} \\ \sigma_{11} & \dots & \sigma_{1,n-2} \end{pmatrix} \leqslant 1 \,, \quad \operatorname{Rank}\begin{pmatrix} \tau_{01} & \dots & \tau_{0,n-2} \\ \tau_{11} & \dots & \tau_{1,n-2} \end{pmatrix} \leqslant 1 \,.$$

They describe the tangent cone to the image of the deformation space of triples.

For $n = 3$, the main result of this paper is the construction of a family, which is versal for deformations of triples $(\widetilde{Z}, \widetilde{S}_1, \widetilde{S}_2)$, along the locus of SRCB manifolds. In Section 5, we consider several examples in which we explicitly describe a global family. In these examples, the structure of the map to the deformation space of $\widetilde{Z}$, transverse to the locus of SRCB manifolds, is exactly what the above computation suggests: the map is $\mathrm{Bl}_0\mathbb{C}^2 \times \mathrm{Bl}_0\mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}^2$ with fibre over the origin $Q = \mathbb{P}^1 \times \mathbb{P}^1$. On the deformation spaces, we have a $\mathbb{C}^*$ action. The quotient of $\mathbb{C}^2 \times \mathbb{C}^2$ by this $\mathbb{C}^*$-action is a three-dimensional ordinary double point. Such singularities can be encountered on compactifications of moduli spaces of double solids, or of the branch quartics (e.g. for Kummer surfaces).

3 | A VERSAL FAMILY FOR $n = 3$

In this section, we construct the family, which gives the wanted deformation of SRCB manifolds with $n = 3$, or more precisely of triples $(\widetilde{Z}, \widetilde{S}_1, \widetilde{S}_2)$, and prove versality of the constructed family.

Let $\widetilde{Z}$ be an SRCB manifold. It is a fibre of a family $\widetilde{Z}_{\Lambda_0} \to \Lambda_0$, as constructed at the end of Section 1. It is defined from a conic bundle bundle W, lying in a family $W_{\Lambda_0} \to \Lambda_0$, with an equation of the form (4), that is, $w_1 w_2 - \varphi w_0^2 = 0$, in

$$\mathbb{P}\big(\mathcal{O}_Q \oplus \mathcal{O}_Q(-2,-1) \oplus \mathcal{O}_Q(-1,-2)\big) \; .$$

The singularities of the curve $D : \varphi = 0$ have only smooth branches. We also choose two surfaces $\widetilde{S}_1, \widetilde{S}_2$, by picking a pair of intersecting lines on Q, that is, by choosing a tangent plane. We do not allow[†] that one of these lines is a component of D. Then each line on Q intersects D in 3 points (counted with multiplicity).

We add the chosen lines to the discriminant curve, which therefore becomes a curve of type (4,4). The successful idea is, to consider the resulting equation in the bundle $\mathbb{P}\left(\mathcal{O}_Q \oplus \mathcal{O}_Q(-2,-2) \oplus \mathcal{O}_Q(-2,-2)\right)$, because this bundle can be extended to $\mathbb{P}^3$.

We therefore choose an embedding of $Q = \mathbb{P}^1 \times \mathbb{P}^1$ as smooth quadric in $\mathbb{P}^3$, whose equation again will be called Q. The section $\varphi \in H^0(Q, \mathcal{O}_Q(3,3))$ can be written as restriction of a cubic form $K \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$. Such a form is not unique, as it can be altered with a multiple of Q, but for now, we choose a lift; we will end up by considering all possible choices. A section $\varphi_1 \in H^0(Q, \mathcal{O}_Q(1,1))$ has a unique lift to a linear form $K_1 \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. If φ is of the form $\varphi_1\varphi_2$ with $\varphi_2 \in H^0(Q, \mathcal{O}_Q(2,2))$, we lift φ to a reducible form $K = K_1 K_2$. In particular, if φ is the product of three forms of type (1,1), we take K as a product of linear factors. Let L be an equation of the chosen tangent plane.

We now work with the $\mathbb{P}^2$-bundle

$$\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-2)) \to \mathbb{P}^3$$

over $\mathbb{P}^3$, with fibre coordinates $(y_0 : y_1 : y_2)$ corresponding to the three direct summands in this order. Let Y be the subspace given by the equations

$$\begin{aligned} y_1 y_2 - K L y_0^2 &= 0 \; , \\ Q &= 0 \; . \end{aligned} \tag{10}$$

[†] This assumption is made to exclude non-isolated singularities along such a line in our construction. Presumably, our arguments can be extended to this case.

The space Y is again a conic bundle over the quadric Q, but we have already written it in terms of $\mathbb{P}^3$.

Lemma 3.1. *The conic bundle Y is birational to the conic bundle W.*

Proof. Given the linear form L, we choose two sections $\psi_1 \in H^0(\mathcal{O}_Q(1,0))$ and $\psi_2 \in H^0(\mathcal{O}_Q(0,1))$ such that their product is the restriction of L to Q. We now define a rational map between bundles over Q by the formula

$$(y_0 : y_1 : y_2) = (w_0 : \psi_2 w_1 : \psi_1 w_2) .$$

Note that the map depends on the choice of ψ_1 and ψ_2, but different maps are connected by the $\mathbb{C}^*$-action on W. It is straightforward to check that this map is indeed birational. A detailed analysis of it can be found after Lemma 3.4. □

We introduce deformation variables α_1 and α_2 and consider the equations

$$\begin{aligned} y_1 y_2 - KLy_0^2 &= 0 , \\ \alpha_2 y_1 + \alpha_1 y_2 - Qy_0 &= 0 \end{aligned} \tag{11}$$

on the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-2)) \to \mathbb{P}^3$. From equations of this form, we eventually get a versal family for our SRCB manifold. This involves varying K and L in an appropriate way; in general, the linear form L does no longer define a tangent plane to the quadric Q.

We stress the fact that Equations (11) do not define a deformation of the space Y: the fibre over $\alpha_1 = \alpha_2 = 0$ (for a specific value of KL) is reducible, having Y as one of its components only. On the other hand, the general fibre is a double solid of the type we are after: if $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, we eliminate say y_1 and find $\alpha_1 y_2^2 - Qy_0 y_2 + \alpha_2 KLy_0^2 = 0$, which is a double cover of $\mathbb{P}^3$, branched along the quartic with equation

$$Q^2 - 4\alpha_1\alpha_2 KL = 0 .$$

3.1 | The family $\mathcal{Y} \to \Pi$

Our first task is to define a family, containing Y (as a component of a fibre), with equations of the form (11). The deformation $\widetilde{\mathcal{Z}} \to \Pi$ of $\widetilde{Z}$ will be constructed from it by birational transformations.

We start more generally from equations

$$\begin{aligned} y_1 y_2 - \Phi y_0^2 &= 0 , \\ \alpha_2 y_1 + \alpha_1 y_2 - Qy_0 &= 0 , \end{aligned} \tag{12}$$

with $\Phi \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$. These equations define a family of complete intersections. To have an explicit description of the base, we want to consider the coefficients of Φ as parameters. We therefore look at Equations (12) as a family over $\mathbb{C}^2 \times H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))$. In order to get a family involving $V = \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4))^*)$, we have to be careful. We shall not end up with a family over $\mathbb{C}^2 \times V$, but over a non-trivial vector bundle $\mathcal{V}$ over V. The family of $\mathbb{P}^2$-bundles, in which Equations (12) make sense, is obtained as the quotient of $(\mathbb{C}^4 \setminus 0) \times (\mathbb{C}^3 \setminus 0) \times \mathbb{C}^2 \times (H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \setminus 0)$ under the $(\mathbb{C}^*)^3$-action given by

$$(a, b, \lambda) \cdot (x_0, x_1, x_2, x_3 \,;\, y_0, y_1, y_2 \,;\, \alpha_1, \alpha_2 \,;\, \Phi)$$
$$= (ax_0, ax_1, ax_2, ax_3 \,;\, by_0, \lambda ba^2 y_1, ba^2 y_2 \,;\, \alpha_1, \lambda^{-1}\alpha_2 \,;\, \lambda\Phi)\,.$$

In this context, the equation $y_1y_2 - \Phi y_0^2 = 0$ at the point $(ax \,;\, y \,;\, \alpha \,;\, \Phi)$, for example, is to be read as $y_1y_2 - \Phi(ax)y_0^2 = 0$. With this in mind, it is easy to see that the zero set of Equations (12) is invariant under this action; hence, they define a family of singular spaces over the quotient of $\mathbb{C}^2 \times (H^0(\mathcal{O}_{\mathbb{P}^3}(4)) \setminus 0)$ under the $\mathbb{C}^*$-action given by $\lambda \cdot (\alpha_1, \alpha_2 \,;\, \Phi) = (\alpha_1, \lambda^{-1}\alpha_2 \,;\, \lambda\Phi)$. This quotient is the vector bundle $\mathcal{V}$ of rank two over V whose sheaf of sections is $\mathcal{O} \oplus \mathcal{O}(-1)$. Again, we may let $\lambda \in \mathbb{C}^*$ act on y_2 and α_1, as opposed to y_1 and α_2, but this produces an isomorphic family.

For $\Phi = KL$, we want to consider the coefficients of K and L as coordinates. Note that the multiplication map $\mathbb{P}(H^0(\mathcal{O}(3))^*) \times \mathbb{P}(H^0(\mathcal{O}(1))^*) \longrightarrow \mathbb{P}(H^0(\mathcal{O}(4))^*)$ is not an embedding, it factors as a Segre embedding followed by a linear projection. On the open set where the first component is an irreducible cubic, the multiplication map is an isomorphism onto its image. We denote this image by $V_{3;1} \subset V = \mathbb{P}(H^0(\mathcal{O}(4))^*)$, and by $\mathcal{V}_{3;1}$, the restriction of the vector bundle $\mathcal{V}$ to it.

On the open set $V_{2,1} \subset \operatorname{Im}\{\mathbb{P}(H^0(\mathcal{O}(2))^*) \times \mathbb{P}(H^0(\mathcal{O}(1))^*) \longrightarrow \mathbb{P}(H^0(\mathcal{O}(3))^*)\}$ where K splits as the product of a linear form and an irreducible quadric, the multiplication map $\mathbb{P}(H^0(\mathcal{O}(3))^*) \times \mathbb{P}(H^0(\mathcal{O}(1))^*) \longrightarrow \mathbb{P}(H^0(\mathcal{O}(4))^*)$ is a branched covering onto its image. As the linear form L plays a special role in our construction, we do not define $V_{2,1;1}$ as this image, but as the subset $V_{2,1} \times \mathbb{P}(H^0(\mathcal{O}(1))^*) \subset \mathbb{P}(H^0(\mathcal{O}(3))^*) \times \mathbb{P}(H^0(\mathcal{O}(1))^*)$, and $\mathcal{V}_{2,1;1}$ as the pull-back of the vector bundle $\mathcal{V}$ to it. Likewise, we define $V_{1^3;1}$ as the subset $V_{1^3} \times \mathbb{P}(H^0(\mathcal{O}(1))^*)$, where $V_{1^3} = \operatorname{Im}\{\operatorname{Sym}^3 \mathbb{P}(H^0(\mathcal{O}(1))^*) \longrightarrow \mathbb{P}(H^0(\mathcal{O}(3))^*)\}$ is the locus of the products of three linear forms, with vector bundle $\mathcal{V}_{1^3;1}$ over it.

Remark 3.2. The fibre over a general point of $\mathcal{V}_{3;1}$ has six double points, but the codimension of $\mathcal{V}_{3;1}$ in $\mathcal{V}$ is 12. There are trivial deformations of the fibre, which do not preserve the splitting $\Phi = KL$. To see this, observe that the defining ideal is not changed if we add a multiple of the second equation in (12) to the first one. We take an arbitrary quadratic form M, compute

$$y_1y_2 - \Phi y_0^2 - My_0(\alpha_2y_1 + \alpha_1y_2 - Qy_0)$$
$$= (y_1 - \alpha_1My_0)(y_2 - \alpha_2My_0) - (\Phi - M(Q - 2\alpha_1\alpha_2M) - \alpha_1\alpha_2M^2)y_0^2 = 0$$

and rewrite the second equation as

$$\alpha_2(y_1 - \alpha_1My_0) + \alpha_1(y_2 - \alpha_2My_0) - (Q - 2\alpha_1\alpha_2M)y_0 = 0\,.$$

For small $\alpha_1\alpha_2$, the quadric $Q - 2\alpha_1\alpha_2M$ is still non-degenerate and by a coordinate transformation, we get equations of the type (12). The upshot of this computation is that for fixed Φ, the family

$$\begin{aligned} y_1y_2 - (\Phi\circ h - MQ - \alpha_1\alpha_2M^2)y_0^2 &= 0 \\ \alpha_2y_1 + \alpha_1y_2 - Qy_0 &= 0 \end{aligned} \tag{13}$$

is trivial, where h is the coordinate transformation depending on the product $\alpha_1\alpha_2$, satisfying $Q\circ h = Q + 2\alpha_1\alpha_2 M$. The family lies in $\mathcal{V}_{3;1}$ if M is divisible by $L\circ h$.

In order to stratify according to singularities, we locate them in the fibres of the family (12).

Proposition 3.3. *Above $\alpha_1\alpha_2 = 0$, in $y_0 = 0$, lie non-isolated singularities, which are the intersections of the irreducible components of the fibres of $\mathcal{Y} \to \Pi$. The other singularities above $\Lambda:\ \alpha_1 = \alpha_2 = 0$ lie in $y_1 = y_2 = 0$. The isolated singularities above $\alpha_1 = 0$ and $\alpha_2 = 0$ are isomorphic to those above $\alpha_1 = \alpha_2 = 0$.*

Proof. The condition for a singular point is

$$\operatorname{Rank}\begin{pmatrix} y_2 & y_1 & -2y_0\Phi & -y_0^2\mathrm{d}\Phi \\ \alpha_2 & \alpha_1 & -Q & -y_0\mathrm{d}Q \end{pmatrix} \leqslant 1 \ . \tag{14}$$

We first consider a neighbourhood of $y_0 = 0$. We look at the chart $y_2 = 1$. The first equation of (12) gives $y_1 = \Phi y_0^2$. Equations (14) then reduce to $\alpha_1 = \alpha_2\Phi y_0^2$, $Q = 2\alpha_2\Phi y_0$ and $y_0(\mathrm{d}Q - \alpha_2 y_0 \mathrm{d}\Phi) = 0$. If $y_0 = 0$, then $\alpha_1 = 0$ and $y_1 = Q = 0$. Indeed, for $\alpha_1 = 0$, the section $y_0 = y_1 = 0$ of the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-2)) \to \mathbb{P}^3$ is a component of the fibre, and the intersection of the components is part of the singular locus. Likewise, for $\alpha_2 = 0$, the section $y_0 = y_2 = 0$ is a component.

The singularities outside $y_0 = 0$ satisfy $\mathrm{d}Q - \alpha_2 y_0 \mathrm{d}\Phi = 0$ (in the chart $y_2 = 1$). As Q is non-degenerate, none of these tend to $y_0 = 0$ if α_1 tends to zero. In fact, for $\alpha_1 = \alpha_2 = 0$, the singular points with $y_0 \neq 0$ lie in $y_1 = y_2 = 0$ on $Q = \Phi = 0$ and are given by the condition that the differential $\mathrm{d}\Phi$ is proportional to $\mathrm{d}Q$. So, they lie above the singular points of the curve $\Phi = 0$ on $Q \subset \mathbb{P}^3$.

For $\alpha_1 = 0, \alpha_2 \neq 0$, we find that $y_1 = 0$, as $\mathrm{d}Q \neq 0$. So, the singular points still satisfy $Q = \Phi = 0$ and lie above the singular points of the curve $\Phi = 0$ on $Q \subset \mathbb{P}^3$. The singularities are isomorphic to those over $\alpha_1 = \alpha_2 = 0$. This is best seen by a computation in local coordinates. We set $y_0 = 1$. As Q is non-degenerate, we can take Q as coordinate x_0. We write $\Phi(x_0, x_1, x_2) = \varphi(x_1, x_2) + x_0\psi(x_0, x_1, x_2)$. Equations (12) become $y_1y_2 - \varphi - x_0\psi = 0$, $\alpha_2 y_1 - x_0 = 0$, from which x_0 can be eliminated:

$$y_1(y_2 - \alpha_2\psi(\alpha_2 y_1, x_1, x_2)) - \varphi(x_1, x_2) = 0 \ . \qquad \square$$

The singularities outside $y_0 = 0$ for $\alpha_1\alpha_2 = 0$ are cA_k points (not necessarily isolated). Such singularities can be characterised as having corank at most two. An isolated cA_k singularity has the form $w_1w_2 - g(x, y) = 0$, with $g(x, y) = 0$ a reduced curve singularity, whose δ-invariant is an invariant of the cA_k singularity.

We stratify the parameter spaces $\mathcal{V}_{3;1}$, $\mathcal{V}_{2,1;1}$ and $\mathcal{V}_{1^3;1}$. One stratum contains the non-isolated singularities and the singularities of corank 3. The complementary open sets are stratified according to the sum of the δ-invariants over all isolated cA_k singularities outside $y_0 = 0$. We consider only the open sets of connected components of strata, where all isolated singularities admit small resolutions, or equivalently that the corresponding plane curve singularities have smooth branches.

Let $\overline{\Pi}$ be such an open set of a stratum in $\mathcal{V}_{3;1}$. It can be defined locally at a point $p \in \overline{\Pi}$ using the induced map of germs $\kappa_{3;1} : (\mathcal{V}_{3;1}, p) \to \prod \operatorname{Def}(\mathcal{Y}_p, x_i)$, where the x_i are the isolated singularities of the fibre $\mathcal{Y}_p$. Let $S^\delta = \prod S_i^\delta$ be the product of the δ-constant strata in the deformation spaces $\mathrm{Def}(\mathcal{Y}_p, x_i)$. Then, $\overline{\Pi} = \kappa_{3;1}^{-1}(S^\delta)$. If it is possible to choose K reducible, we consider open sets of strata in $\mathcal{V}_{2,1;1}$ or $\mathcal{V}_{1^3;1}$, definable by $\kappa_{2,1;1}$ or $\kappa_{1^3;1}$, respectively.

As in Section 1 for SRCB manifolds, for each fibre in the family over a maximal open set $\overline{\Pi}$ in a stratum in $\mathcal{V}_{3;1}$, $\mathcal{V}_{2,1;1}$ or $\mathcal{V}_{1^3;1}$, we consider all possible small resolutions of the isolated singularities. Again, these fit together to a covering $\Pi \to \overline{\Pi}$ with finite fibres. Therefore, we have a simultaneous small resolution of all isolated singularities in a family $\mathcal{Y} \to \Pi$ of complete intersections in our $\mathbb{P}^2$-bundle over $\mathbb{P}^3$.

Let $\widetilde{Z}$ be an SRCB manifold, defined by an equation of the form (4). The given choice of small resolutions defines a point in a stratum Λ_0, which is a covering of a stratum $\overline{\Lambda}_0$ in $\mathbb{P}(H^0(\mathcal{O}_Q(3,3))^*)$.

To get to Equations (10), we note that restriction to Q defines a rational map $\mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(3))^*) \dashrightarrow \mathbb{P}(H^0(\mathcal{O}_Q(3,3))^*)$ which is not defined along the projective subspace $\mathbb{P}((Q \cdot H^0(\mathcal{O}(1)))^*)$, because $Q \cdot H^0(\mathcal{O}(1))$ is the kernel of the surjective restriction map $H^0(\mathcal{O}(3)) \longrightarrow H^0(\mathcal{O}_Q(3,3))$. This also implies that the fibres of the morphism $\mathbb{P}(H^0(\mathcal{O}(3))^*) \setminus \mathbb{P}((Q \cdot H^0(\mathcal{O}(1)))^*) \longrightarrow \mathbb{P}(H^0(\mathcal{O}_Q(3,3))^*)$ are isomorphic to $H^0(\mathcal{O}(1))$. We are forced to enlarge our family again, but we avoid this as much as possible by considering three cases according to the way $\varphi \in H^0(\mathcal{O}_Q(3,3))$ splits.

Given Λ_0, we first consider a stratum $\overline{\Lambda}_1 \subset \mathbb{P}(H^0(\mathcal{O}(3))^*)$, which maps surjectively onto $\overline{\Lambda}_0$ under the restriction map $\mathbb{P}(H^0(\mathcal{O}(3))^*) \setminus \mathbb{P}((Q \cdot H^0(\mathcal{O}(1)))^*) \longrightarrow \mathbb{P}(H^0(\mathcal{O}_Q(3,3))^*)$. In general, $\overline{\Lambda}_1$ is the inverse image of $\overline{\Lambda}_0$, but if all members of $\overline{\Lambda}_0$ split as $\varphi_1\varphi_2\varphi_3$ or as $\varphi_1\psi$ with $\varphi_i \in H^0(\mathcal{O}_Q(1,1)), \psi \in H^0(\mathcal{O}_Q(2,2))$, then we restrict the inverse image to $V_{2,1}$, resp. V_{1^3}. In the case of three linear forms, which occurs in the twistor case, we have $\overline{\Lambda}_1 \cong \overline{\Lambda}_0$.

Now we add the linear form L defining a tangent plane to Q. We get a stratum $\overline{\Lambda}_1 \times Q^* \subset \mathbb{P}(H^0(\mathcal{O}(3))^*) \times \mathbb{P}(H^0(\mathcal{O}(1))^*)$ over $\overline{\Lambda}_0 \times Q^* \subset \mathbb{P}(H^0(\mathcal{O}_Q(3,3))^*) \times \mathbb{P}(H^0(\mathcal{O}_Q(1,1))^*)$. We require that the restriction of KL to Q has only isolated singularities: we have to exclude that the tangent plane L and the curve $K = Q = 0$ both contain the same line. This can only happen if a curve D in the stratum $\overline{\Lambda}_0$ (and therefore each curve D) contains a line of one of the rulings. In this case, we replace $\overline{\Lambda}_1 \times Q^*$ by its open subset which is obtained by removing those pairs (K, L) for which the tangent plane L and the curve $K = Q = 0$ both contain the same line. In all cases, we denote the resulting stratum by $\overline{\Lambda}$. According to our choices, it lies entirely in $V_{3;1}$, $V_{2,1;1}$ or $V_{1^3;1}$.

We now go over to Equations (11), where we have the deformation parameters α_1 and α_2. We identify each stratum $\overline{\Lambda} \subset V_{3;1}$ (resp. $V_{2,1;1}$ or $V_{1^3;1}$) with its image in the zero section of $\mathcal{V}_{3;1}$ (resp. $\mathcal{V}_{2,1;1}$ or $\mathcal{V}_{1^3;1}$). The fibres over $\overline{\Lambda}$ are now reducible, consisting of Y and the sections $\mathcal{E}_i : y_0 = y_i = 0$ of the $\mathbb{P}^2$-bundle. The stratum $\overline{\Lambda}$ lies in a unique stratum $\overline{\Pi}$, and inside $\overline{\Pi}$, it is given by the equations $\alpha_1 = \alpha_2 = 0$.

The covering $\Lambda_0 \to \overline{\Lambda}_0$ induces a covering $\Lambda_1 \to \overline{\Lambda}_1$ from which we obtain a covering $\Lambda \to \overline{\Lambda}$. The linear form L introduces new singularities, which are to be resolved in a specific way, governed by the birational isomorphism between Y and W (Lemma 3.1). We describe it in detail in Remark 3.8. The resolution can be done simultaneously over the whole stratum Λ. Therefore, we pick out a specific small resolution, and we get an embedding $\Lambda \subset \Pi$.

Lemma 3.4. *The base space* Π *is smooth in a neighbourhood of* Λ.

Before embarking on the proof in full generality, we treat the simpler case that all isolated singularities are ordinary double points. As the question is local, it suffices to prove that $\overline{\Pi}$ is smooth at the points of $\overline{\Lambda}$. Of the three cases to be considered, we take the one relevant for twistor spaces. We therefore suppose that the stratum $\overline{\Pi}$ lies in $\mathcal{V}_{1^3;1}$. We fix a point $p \in \overline{\Lambda} \subset \overline{\Pi}$. Locally, around p, the stratum is the fibre over 0 of the map $\kappa_{1^3;1} : (\mathcal{V}_{1^3;1}, p) \to \prod \mathrm{Def}(\mathcal{Y}_p, x_i)$, as the δ-const stratum of an ordinary double point consists of the origin only. The map $\kappa_{1^3;1}$ is not surjective, as all fibres over $\mathcal{V}_{1^3;1}$ with only ordinary double points have at least 12 such points: they lie above the intersection of Q with the edges of the tetrahedron $L_1L_2L_3L_4$. We therefore divide the set Σ of isolated singular points into two, the first, Σ_1, consisting of these 12 points, while the remaining $\delta - 5$ double points make up Σ_2 (here δ is the number of double points of the SRCB manifold we started with). The image of $\mathcal{V}_{1^3;1}$ is contained in $0 \times \prod_{\Sigma_2} \mathrm{Def}(\mathcal{Y}_p, x_i) \subset \prod_{\Sigma_1} \mathrm{Def}(\mathcal{Y}_p, x_i) \times \prod_{\Sigma_2} \mathrm{Def}(\mathcal{Y}_p, x_i)$. To prove smoothness of $\overline{\Pi}$ at p, we establish that $\kappa_{1^3;1} : (\mathcal{V}_{1^3;1}, p) \to 0 \times \prod_{\Sigma_2} \mathrm{Def}(\mathcal{Y}_p, x_i)$ is a submersion. In fact, this already holds for the restriction of $\kappa_{1^3;1}$ to the zero section $(V_{1^3;1}, p)$ of $(\mathcal{V}_{1^3;1}, p)$. We factor this restriction through $(W_{1^3;1}, [p]) \subset (\mathbb{P}(H^0(\mathcal{O}_Q(3,3))^*) \times \mathbb{P}(H^0(\mathcal{O}_Q(1,1))^*), [p])$, where $W_{1^3;1}$ is defined analogously to $V_{1^3;1}$, and $[p]$ is the image of p. The map $W_{1^3;1} \to \mathbb{P}(H^0(\mathcal{O}_Q(4,4))^*)$ is a branched covering onto its image. But ramification occurs exactly when L coincides with one of the factors of K. By our assumption on isolatedness of the newly introduced singularities, this does not occur at the point $[p]$. We can therefore identify the germ with its image germ. We get the following diagram:

$$\begin{array}{ccccc}
(\overline{\Lambda}_1 \times Q^*, p) & \longrightarrow & (\overline{\Lambda}_0 \times Q^*, [p]) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
(V_{1^3;1}, p) & \longrightarrow & (W_{1^3;1}, [p]) & \longrightarrow & 0 \times \prod_{\Sigma_2} \mathrm{Def}(\mathcal{Y}_p, x_i) \\
 & & \downarrow & & \downarrow \\
 & & (\mathbb{P}(H^0(\mathcal{O}_Q(4,4))^*), [p]) & \longrightarrow & \prod_{\Sigma_1} \mathrm{Def}(\mathcal{Y}_p, x_i) \times \prod_{\Sigma_2} \mathrm{Def}(\mathcal{Y}_p, x_i)
\end{array}$$

Note that we do not put $(V, p) = (\mathbb{P}(H^0(\mathcal{O}(3))^*), p)$ in the lower left corner of the diagram, as $(V_{1^3;1}, p)$ is not the inverse image of $0 \times \prod_{\Sigma_2} \mathrm{Def}(\mathcal{Y}_p, x_i)$ in (V, p): the trivial deformations of type (13) do not respect the splitting of Φ.

The two horizontal maps on the left are submersions. According to Lemma 2.2 and the remarks following it, the stratum $\overline{\Lambda}_0 \times Q^*$ is smooth at $[p]$, of codimension $\delta + 7 = (\delta - 5) + 12$ in $(\mathbb{P}(H^0(\mathcal{O}_Q(4,4))^*), [p])$. As it is the inverse image of $0 \in \prod_{\Sigma_1} \mathrm{Def}(\mathcal{Y}_p, x_i) \times \prod_{\Sigma_2} \mathrm{Def}(\mathcal{Y}_p, x_i)$, the map $(\mathbb{P}(H^0(\mathcal{O}_Q(4,4))^*), [p]) \longrightarrow \prod_{\Sigma_1} \mathrm{Def}(\mathcal{Y}_p, x_i) \times \prod_{\Sigma_2} \mathrm{Def}(\mathcal{Y}_p, x_i)$ is a submersion. The dimension of $\mathbb{P}(H^0(\mathcal{O}_Q(4,4))^*)$ is 24 and that of the smooth space $W_{1^3;1}$ is 12, so the codimension of $(W_{1^3;1}[p])$ in $(\mathbb{P}(H^0(\mathcal{O}_Q(4,4))^*), [p])$ is also 12. We conclude that $(W_{1^3;1}, [p]) \to 0 \times \prod_{\Sigma_2} \mathrm{Def}(\mathcal{Y}_p, x_i)$ is a submersion and therefore also $(V_{1^3;1}, p) \to 0 \times \prod_{\Sigma_2} \mathrm{Def}(\mathcal{Y}_p, x_i)$.

The proof in the general case is based on the same ideas. To describe the subspace in which the image of $(V_{3;1}, p)$, $(V_{2,1;1}, p)$ or $(V_{1^3;1}, p)$ lands, we need some more facts about deformations of plane curve singularities. For instance, in $V_{3;1}$, we change K and L separately. This can also be done for components of reducible curve singularities. We borrow the term *equi-intersectional* for the resulting stratum from [10]; the theory can be modelled on the careful treatment of the δ-const stratum in [10].

We consider a curve singularity $C = \bigcup_{i=1}^k C_i$, where the C_i might themselves be reducible. We look at the deformation theory of the map $\amalg C_i \to C$.

Definition. The equi-intersectional stratum S^{ei} of the curve $C = \bigcup_{i=1}^{k} C_i$ is the image of the deformation space of the map $\amalg C_i \to C$ in the deformation space of C.

Lemma 3.5. *The equi-intersectional stratum S^{ei} is smooth of codimension $\sum_{i<j}(C_i \cdot C_j)$.*

Proof. One first shows that the deformation functors of the map $\amalg C_i \to C$ and that of $\amalg C_i \to \mathbb{C}^2$ are isomorphic, with the reasoning of the proof [10, Prop. II.2.23]; it uses [10, Prop. II.2.9], which is formulated in great generality and also applies to our situation. The vector space T^2 for the second deformation problem vanishes (this is not true for the first problem) by computations similar to those in [10, Sect. II.2.4]. Therefore, the deformation space of $\amalg C_i \to C$ is smooth.

Let $C = \bigcup_{i=1}^{k} C_i$ be given by $f_1 \cdots f_k = 0$. The image in $\mathbb{C}^2$ of a deformation of $\amalg C_i \to C$ is given by a product $F_1 \cdots F_k = 0$. If this is a trivial deformation of C, then necessarily, the C_i are deformed trivially and also the map. Therefore, the natural forgetful map from the deformation space of $\amalg C_i \to C$ to the deformation space of C is an immersion.

To compute the codimension, it suffices by openness of versality to look at a general point of the stratum, for which the deformed curves C_i intersect transversally. The number of intersection points is $\sum_{i<j}(C_i \cdot C_j)$. □

Proof of Lemma 3.4. As the question is local, it suffices to prove that $\overline{\Pi}$ is smooth at the points of $\overline{\Lambda}$. We fix a point $p \in \overline{\Lambda} \subset \overline{\Pi}$. We prove that $\overline{\Pi}$ is smooth at p by realising it as the inverse image of the δ-const stratum S^{δ} under a submersion. Let S^{ei} be the product of the equi-intersectional strata of the isolated singularities $(\mathcal{Y}_p, x_i)$.

A complication is that in general the map from $(\mathcal{V}, p) \to \prod \mathrm{Def}(\mathcal{Y}_p, x_i)$ is not surjective. Nevertheless, by Lemma 2.2, the stratum $\overline{\Lambda}_0 \times Q^*$ is smooth of codimension $\delta + 7$ in $\mathbb{P}(H^0(\mathcal{O}_Q(4,4))^*)$. We therefore map down to a smooth space with tangent space isomorphic to $\prod T^1_{(\mathcal{Y}_p, x_i)}/TS^{\delta}$. As the strata $S^{\delta} \subset S^{\mathrm{ei}}$ are smooth, we can choose a transversal slice $S^{\perp}$ and a projection $\sigma : \prod \mathrm{Def}(\mathcal{Y}_p, x_i) \to S^{\perp}$ with $S^{\delta} = \sigma^{-1}(0)$ and $\sigma^{-1}(\sigma(S^{\mathrm{ei}})) = S^{\mathrm{ei}}$.

We treat the three cases, that $\overline{\Pi}$ lies entirely in $\mathcal{V}_{3;1}$, $\mathcal{V}_{2,1;1}$ or $\mathcal{V}_{1^3;1}$, at the same time by writing $\mathcal{V}_{\alpha;1}$ with α standing for a partition. The stratum S^{ei} depends on the chosen partition.

Smoothness of $\overline{\Pi}$ at p follows when we show that $\sigma \circ \kappa_{\alpha;1} : (\mathcal{V}_{\alpha;1}, p) \to \sigma(S^{\mathrm{ei}})$ is a submersion. In fact, this already holds for the restriction of $\sigma \circ \kappa_{\alpha;1}$ to the zero section $(V_{\alpha;1}, p)$ of $(\mathcal{V}_{\alpha;1}, p)$. We factor this map through $(W_{\alpha;1}, [p]) \subset (\mathbb{P}(H^0(\mathcal{O}_Q(3,3))^*) \times \mathbb{P}(H^0(\mathcal{O}_Q(1,1))^*), [p])$, where $W_{\alpha;1}$ is defined analogously to $V_{\alpha;1}$, and $[p]$ is the image of p. In the two cases $\alpha = (2,1)$ and $\alpha = (1^3)$, the map $W_{\alpha;1} \to \mathbb{P}(H^0(\mathcal{O}_Q(4,4))^*)$ is a branched covering onto its image. But ramification does not occur at the point $[p]$. We can therefore identify the germ with its image germ.

In this way, we get immersions $(\overline{\Lambda}_0 \times Q^*, [p]) \subset (W_{\alpha;1}, [p]) \subset (\mathbb{P}(H^0(\mathcal{O}_Q(4,4))^*), [p])$. According to Lemma 2.2 and the remarks following it, the stratum $\overline{\Lambda}_0 \times Q^*$ is smooth at $[p]$, of codimension $\delta + 7$ in $(\mathbb{P}(H^0(\mathcal{O}_Q(4,4))^*), [p])$. As this stratum is the inverse image of $0 \in S^{\perp}$, the map $(\mathbb{P}(H^0(\mathcal{O}_Q(4,4))^*), [p]) \longrightarrow (S^{\perp}, 0)$ is a submersion. The dimension of $\mathbb{P}(H^0(\mathcal{O}_Q(4,4))^*)$ is 24. The dimensions of the smooth spaces $W_{3;1}$, $W_{2,1;1}$ and $W_{1^3;1}$ are 18, 14 and 12, respectively, so the codimensions of the $(W_{\alpha;1}[p])$ in $(\mathbb{P}(H^0(\mathcal{O}_Q(4,4))^*), [p])$ are 6, 10 and 12, respectively. These numbers are the respective global intersection numbers of the components of KL, so they coincide with the codimensions of S^{ei} in $\prod \mathrm{Def}(\mathcal{Y}_p, x_i)$ and therefore also of $\sigma(S^{\mathrm{ei}})$ in $S^{\perp}$. We conclude that $(W_{\alpha;1}, [p]) \to \sigma(S^{\mathrm{ei}})$ is a submersion and therefore also $(V_{\alpha;1}, p) \to \sigma(S^{\mathrm{ei}})$. □

Example 3.6. We can give a more explicit description of $\overline{\Pi}$, when this stratum is given by only one equation in $\mathcal{V}_{\alpha;1}$. This is the case if K lies in the open stratum inside the space of forms of given splitting type. Our aim is to describe L depending on K and $\alpha_1\alpha_2$. We use homogeneous coordinates on $\mathbb{P}(H^0(\mathcal{O}(3))^*) \times \mathbb{P}(H^0(\mathcal{O}(1))^*)$. On the total space, we have the $(\mathbb{C}^*)^4$-action given by

$$(a,b,\lambda,\mu)\cdot(x_0,x_1,x_2,x_3\ ;\ y_0,y_1,y_2\ ;\ \alpha_1,\alpha_2\ ;\ K,L)$$
$$= (ax_0,ax_1,ax_2,ax_3\ ;\ by_0,\lambda ba^2y_1,\mu ba^2y_2\ ;\ \mu^{-1}\alpha_1,\lambda^{-1}\alpha_2\ ;\ \lambda K,\mu L)\ .$$

In particular, the equation $Q^2 - 4\alpha_1\alpha_2 LK$ is invariant under the (λ,μ)-action. We compute the condition that this quartic has a singular point P outside the quadric Q. We find the condition

$$2Q(P)\,\mathrm{d}Q(P) - 4\alpha_1\alpha_2 L(P)\,\mathrm{d}K(P) - 4K(P)\,\mathrm{d}(\alpha_1\alpha_2 L) = 0\ ,$$

so

$$\mathrm{d}(\alpha_1\alpha_2 L) = \frac{Q(P)}{2K(P)}\mathrm{d}Q(P) - \frac{Q^2(P)}{4K^2(P)}\mathrm{d}K(P)\ .$$

Note that the right-hand side of this equation is homogeneous of degree 0 in P. This shows that $\alpha_1\alpha_2 L$ is determined by the position of the point P. As the formula does not work if P lies on $\{K=0\}$, we blow up $\mathbb{P}^3$ in the intersection of Q and K and look at a neighbourhood of the strict transform of the quadric Q. We realise the blow-up as subset of a $\mathbb{P}^1$-bundle over $\mathbb{P}^3$. The singular points lie above the singular points of $Q \cap K$, but away from the strict transform $\widetilde{Q}$ of the quadric. Note that the normal bundle of $\widetilde{Q}$ is of type $(-1,-1)$, as we have blown up a curve of type (3,3) on Q. It is natural to take the product $\alpha_1\alpha_2$ as inhomogeneous fibre coordinate, so that the blow-up is given by

$$Q - 2\alpha_1\alpha_2 K = 0\ . \tag{15}$$

But we rather consider this equation in a rank 2 vector bundle over $\mathbb{P}^3$, described by the action

$$\mu\cdot(x_i\ ;\ \alpha_1,\alpha_2) = (\mu x_i\ ;\ \mu^{-1}\alpha_1,\alpha_2)\ .$$

Now we have all the ingredients to describe L in Equations (11). As a linear form is determined by its differential, we can define L by

$$\mathrm{d}L = \mathrm{d}Q(P) - \alpha_1\alpha_2\mathrm{d}K(P)\ .$$

We consider the coefficients of K, α_1, α_2 and the point P (related by Equation (15)) as coordinates. In this way, we obtain the normalisation of the stratum $\overline{\Pi}$ (or rather of a closure). We have an isomorphism in a neighbourhood of $\alpha_1 = \alpha_2 = 0$, but, in general, several singular points can lie on one and the same quartic branch surface. We will see this phenomenon again in Example 5.2.

3.2 | The family $\widetilde{\mathcal{Z}} \to \Pi$

Having constructed the family $\mathcal{Y} \to \Pi$, we use birational transformations of the total space to obtain a family, which is a deformation of SRCB manifolds. We also give divisors $\widetilde{S}_1, \widetilde{S}_2$ in $\widetilde{\mathcal{Z}}$.

To see what has to be done, we first describe how the rational map $W \dashrightarrow Y$ of Lemma 3.1 can be factored. To be consistent with later notation, we call W for Y^+ and factor the rational map $Y^+ \dashrightarrow Y$ as composition $Y^+ \longleftarrow Y^- \longrightarrow Y$ of a blow-up and a small contraction. In the process, all objects on W (as defined in previous sections) pick up a plus sign as upper index. On $W = Y^+$, we therefore have the divisors

$$E_i^+ : \quad w_0 = w_i = 0 \qquad \text{and} \qquad R_i^+ : \quad \psi_i = 0\,.$$

The rational map is not defined on the following two disjoint curves:

$$B_1^+ = R_1^+ \cap E_1^+ : \quad \psi_1 = 0\,, \ \ w_0 = w_1 = 0\,,$$
$$B_2^+ = R_2^+ \cap E_2^+ : \quad \psi_2 = 0\,, \ \ w_0 = w_2 = 0\,.$$

To describe the blow-up of B_1^+, we first describe its blow-up in the ambient $\mathbb{P}^2$-bundle over Q. Inside the fibred product

$$\mathbb{P}(\mathcal{O}_Q \oplus \mathcal{O}_Q(-2,-1) \oplus \mathcal{O}_Q(-1,-2)) \times_Q \mathbb{P}(\mathcal{O}_Q \oplus \mathcal{O}_Q(-2,-1) \oplus \mathcal{O}_Q(-1,0))$$

with fibre coordinates $(w_0 : w_1 : w_2)$ and $(z_0 : z_1 : z_2)$ this blow-up can be described by the equations

$$\operatorname{Rank}\begin{pmatrix} w_0 & w_1 & \psi_1 \\ z_0 & z_1 & z_2 \end{pmatrix} \leqslant 1\,.$$

The strict transform Y^- of Y^+ is given by the five maximal minors of the matrix

$$\begin{pmatrix} w_0 & w_1 & \psi_1 \\ z_0 & z_1 & z_2 \\ w_2 & \varphi w_0 & \end{pmatrix}.$$

The exceptional divisor is a ruled surface F_1^-, given by $\psi_1 = w_0 = w_1 = z_1 = 0$. We can extend our rational map by

$$(y_0 : y_1 : y_2) = (z_0 : \psi_2 z_1 : z_2 w_2)\,.$$

It is everywhere defined in a neighbourhood of F_1^-. The blow-up of B_2^+ introduces a ruled surface $F_2^- \subset Y^-$.

The morphism $Y^- \to Y$ blows down several curves, which lie in the strict transforms R_1^- and R_2^- of R_1^+ and R_2^+. This fact is most easily seen from the given formula for the rational map $Y^+ \dashrightarrow Y$. On Y^+, these curves form the fibre over the point P in which L is the tangent plane to Q, given by $R_1^+ \cap R_2^+ : \psi_1 = \psi_2 = 0$, and the lines $\psi_1 = w_1 = \varphi = 0$, $\psi_2 = w_2 = \varphi = 0$. If P is in general position exactly, seven lines are blown down, introducing seven new A_1 singularities. Otherwise the given

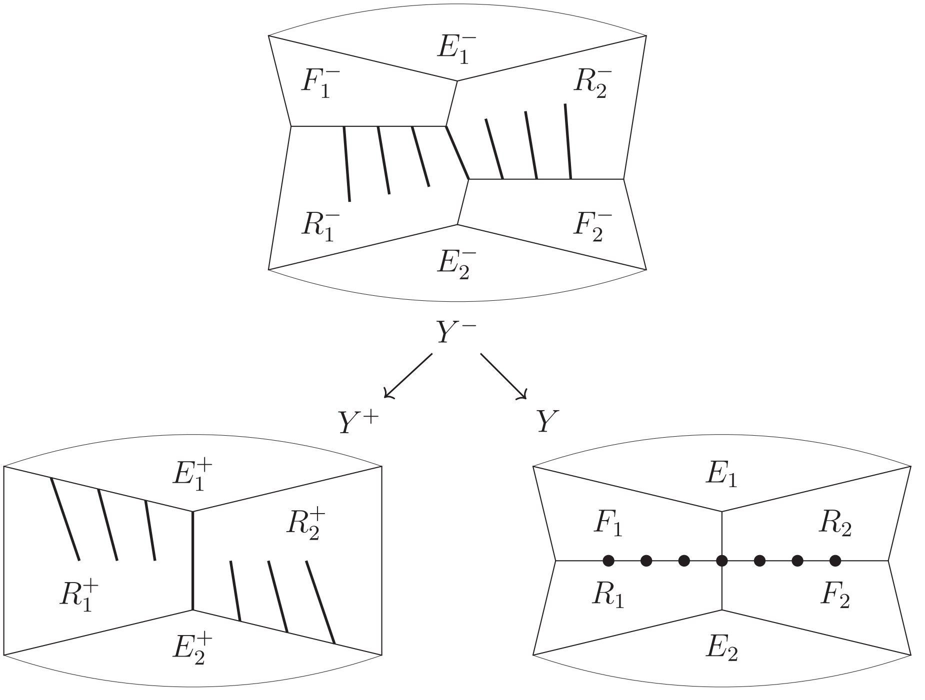


FIGURE 1 Factorisation of the map $Y^+ \dashrightarrow Y$.

equations define the blown down curves with possibly non-reduced structure and we get less but higher singularities. Mostly this will not influence our constructions, but it is important to distinguish whether or not the point P lies on the discriminant curve $\varphi = 0$.

Figure 1 shows, in the general case, the divisors involved and the exceptional curves. In the case where the point P lies on the discriminant, both curves in the fibre are contracted to a cA_2 singularity. We also give a general picture for this situation, see Figure 2.

We now consider the family $\mathcal{Y} \to \Pi$. In the base space Π, we have two divisors Δ_i, given by $\alpha_i = 0$ $(i = 1, 2)$. Over these, the fibres of the family are reducible. In fact, the section $\mathcal{E}_i :\ y_0 = y_i = 0$ of the $\mathbb{P}^2$-bundle over Δ_i is a component. The remaining component of $\mathcal{Y}|_{\Delta_i}$ will be denoted by Y_i. The intersection E_i of $\mathcal{E}_i$ and Y_i lies over Q and is given by $\alpha_i = y_0 = y_i = Q = 0$. Over $\Lambda = \Delta_1 \cap \Delta_2$, there are three components, $\mathcal{E}_1|_\Lambda$, $\mathcal{E}_2|_\Lambda$ and Y_Λ, which is the intersection of Y_1 and Y_2. The space Y_Λ over Λ is given by Equations (10).

Remark 3.7. It is crucial for the construction of the birational transformations to observe that the plane L is tangent to Q for all parameters in $\Delta_1 \cup \Delta_2$. This follows from Proposition 3.3. It is interesting to note that away from $\Delta_1 \cup \Delta_2$, that is, if $\alpha_1\alpha_2 \neq 0$, where L in general is not tangent to Q, even if L happens to be tangent to Q, over a point of tangency, there is no singularity of the fibre of $\mathcal{Y}$, provided that L was not tangent to Q at a point on K.

Some of the divisors we encountered in describing Y (see Figure 1) also extend over Δ_1 and others over Δ_2. If over Δ_1, where $\alpha_1 = 0$, also $Q = L = 0$, then the lines $\{y_1 = 0\}$ in the fibres of the $\mathbb{P}^2$-bundle lie on Y_1. As $Q = L = 0$ consists of two lines $l_1 = \mathrm{pr}_1^{-1}(a)$ and $l_2 = \mathrm{pr}_2^{-1}(b)$, one from each ruling, we have two ruled surfaces in each fibre of our family over Δ_1. As the ruled surfaces

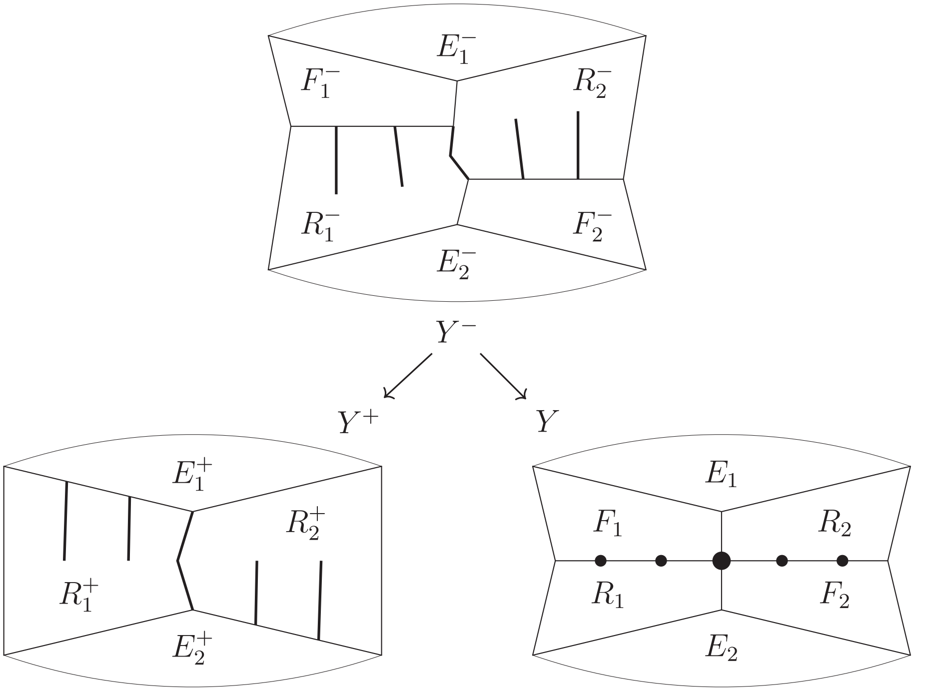


FIGURE 2 Factorisation in case P lies on the discriminant curve.

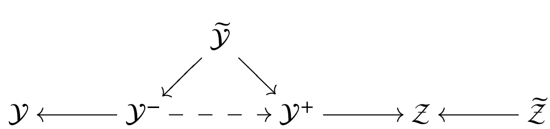
are given by $y_1 = 0$, they are the restriction of the bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2)) \to \mathbb{P}^3$ to a line, so each is a Hirzebruch surface $\mathbb{F}_2$. We set $F_1 = \pi^{-1}(l_1) \cap \{y_1 = 0\}$ and $R_2 = \pi^{-1}(l_2) \cap \{y_1 = 0\}$, consistent with our earlier notation. We consider F_1 as a family of ruled surfaces over Δ_1. Correspondingly, we have F_2 and R_1 over Δ_2. Note that $F_1 \cap F_2 = \pi^{-1}(l_1 \cap l_2) \cap \{y_1 = y_2 = 0\}$ is a single point in each fibre over Λ. This point is a singularity of the corresponding fibre of $\mathcal{Y}$.

The deformation of the divisors $\widetilde{S}_1$ and $\widetilde{S}_2$ will be constructed from the following divisors in $\mathcal{Y}$:

$$\begin{aligned} \mathcal{R}_1 &= \{L = 0,\ y_2 = 0,\ \alpha_2 y_1 - y_0 Q = 0\} \\ \mathcal{R}_2 &= \{L = 0,\ y_1 = 0,\ \alpha_1 y_2 - y_0 Q = 0\}\,. \end{aligned}$$

Over each point of $\Pi \setminus \Delta_2$, we have $\alpha_2 \neq 0$, thus $\mathcal{R}_1$ is isomorphic to the plane defined by L in $\mathbb{P}^3$. Similarly, over each point of $\Pi \setminus \Delta_1$, $\mathcal{R}_2$ is isomorphic to the same plane. Over Δ_2, however, $\mathcal{R}_1$ splits into three components: F_2, R_1 and $\{L = 0, y_0 = y_2 = 0\}$, a family of planes in $\mathcal{E}_2$. Similarly, over Δ_1, $\mathcal{R}_2$ splits into F_1, R_2 and $\{L = 0, y_0 = y_1 = 0\}$. Only the components R_i survive our construction, the other components will be contracted, so that we shall eventually have a family of irreducible divisors.

The construction of the deformation $\widetilde{\mathcal{Z}} \to \Pi$ proceeds by constructing the following maps, which are maps of families over the base space Π:



Let us give a brief overview over these constructions before we enter a detailed description.

(1) The morphism $\mathcal{Y}^- \to \mathcal{Y}$ is the simultaneous small partial resolution of the extra singularities in each fibre, introduced by the plane L.
(2) The rational map $\mathcal{Y}^- \dashrightarrow \mathcal{Y}^+$ is the simultaneous flop of all lines on the transforms F_1^- and F_2^- of F_1 and F_2. We construct the flop explicitly as blow-up followed by blowing down in the other direction.
(3) The morphism $\mathcal{Y}^+ \to \mathcal{Z}$ contracts the strict transforms $\mathcal{E}_i^+$ of the components $\mathcal{E}_i$.
(4) The final step $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ is the simultaneous small resolution of the remaining singularities in the fibres.

In the second and third steps, the family is modified only over $\Delta_1 \cup \Delta_2$. For $p \notin \Delta_1 \cup \Delta_2$, the fibres of $\widetilde{\mathcal{Z}}$ are small resolutions of the fibres of $\mathcal{Y}$.

3.3 | $\mathcal{Y}^- \longrightarrow \mathcal{Y}$

By construction of Π, a small simultaneous resolution of all isolated singular points exists. In the first step, we only want to resolve the new singularities, introduced by the plane L; we take care of the remaining singularities in the final step. The reason is that we want to stay as close as possible to the spaces defined by our equations. In particular, after the second step, we want the space $Y^+ = W$ as component of a fibre, and not a small resolution of it. This also means that the construction works over the base space $\overline{\Pi}$: the covering $\Pi \to \overline{\Pi}$ is only needed for the final step. Therefore, we consider only a partial resolution.

Over Λ, the partial small resolution is the partial resolution Y^-, described above in factoring the rational map $W = Y^+ \dashrightarrow Y$ (cf. Figures 1 and 2). The divisors $\mathcal{R}_i \subset \mathcal{Y}$ will be blown up in each fibre to become divisors $\mathcal{R}_i^- \subset \mathcal{Y}^-$.

Remark 3.8. We give a more explicit description of the partial resolution, although this is not necessary for the proof. We first locate the singularities. Then, we use that, over Λ, which small resolution of the extra singularities has to be chosen is determined by $W \dashrightarrow Y_\Lambda$. The condition for a singular point in a fibre of $\mathcal{Y} \to \Pi$ is given by (14). Above Λ, that is, for $\alpha_1 = \alpha_2 = 0$, the isolated singular points lie in $y_1 = y_2 = 0$ on $Q = KL = 0$ and are given by the condition that the differential $\mathrm{d}(KL)$ is proportional to $\mathrm{d}Q$.

On the intersection of $K = 0$ and $L = 0$, one has $\mathrm{d}(KL) = 0$, so also for $\alpha_1, \alpha_2 \neq 0$, there are isolated singular points at $y_1 = y_2 = 0$ on $Q = KL = 0$. In general, this gives six of the seven ordinary double points introduced by L. Over Λ, in general, the seventh singularity is the point of intersection of F_1 and F_2.

The first equation $y_1y_2 - KLy_0^2 = 0$ of (11) leads to two globally defined partial small resolutions. Namely, we can consider the closure $\widehat{\mathcal{Y}}_1$ of the graph of the rational map $\mathcal{Y} \dashrightarrow \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1))$ given by $(z_0 : z_1) = (y_1 : Ky_0) = (Ly_0 : y_2)$, or $\widehat{\mathcal{Y}}_2$ coming from the map $(z_0 : z_1) = (y_1 : Ly_0) = (Ky_0 : y_2)$. The space $\widehat{\mathcal{Y}}_1$ is given by the equations

$$\operatorname{Rank}\begin{pmatrix} y_1 & Ly_0 & z_0 \\ Ky_0 & y_2 & z_1 \end{pmatrix} \leqslant 1 ,$$

$$\alpha_2 y_1 + \alpha_1 y_2 - Qy_0 = 0 .$$

The projection $\widehat{\mathcal{Y}}_1 \to \mathcal{Y}$ is an isomorphism outside the set given by $K = Q = L = 0$ and $y_1 = y_2 = 0$. In the chart $z_0 = 1 = y_0$, we have the equations $y_2 = Lz_1$ and $K = y_1 z_1$ (and $\alpha_2 y_1 + \alpha_1 L z_1 - Q = 0$), so for a singular point of K, we have only a partial resolution, and we indeed keep exactly the singularity we want to leave until the final step.

The preimage of each of the points given by $K = Q = L = 0$ and $y_1 = y_2 = 0$ is a $\mathbb{P}^1$. These curves lie in the strict transform of $y_2 = L = 0$, so over Λ in $F_2^- \cup R_1^-$. Therefore, $\widehat{\mathcal{Y}}_1$ is not the correct resolution for all singularities. We use it to resolve only those singularities which are near those which are contained in F_1 over Δ_1. For the singularities close to those in F_2 over Δ_2, we use $\widehat{\mathcal{Y}}_2$. This gives a simultaneous small resolution of (in general) six of the extra singularities for fibres over a neighbourhood of Λ in Π.

For the last singular point, introduced by L, we only describe the resolution over Λ. We consider two cases, depending on whether the point of tangency P lies on the discriminant curve or not. We treat here only the more difficult case. The assumption that L is a tangent plane implies that we can write, at least locally on the base space, $Q = LM - N_1 N_2$, where M, N_1 and N_2 are linear forms. The equations $L = N_i = 0$ define the line l_i on Q, so F_i is now given by $N_i = y_i = 0$, and R_i by $N_i = y_{3-i} = 0$. We derive the equation

$$M y_1 y_2 - K N_1 N_2 y_0^2 = 0 \,.$$

The small resolution comes about as the graph of a map to the $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over $\mathbb{P}^3$

$$\mathbb{P}(\mathcal{O}(-2) \oplus \mathcal{O}) \times_{\mathbb{P}^3} \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1)) \,.$$

On the target, we have fibre coordinates $(z_0 : z_1 \,;\, z_0' : z_1')$. We set

$$\frac{z_0}{z_1} = \frac{M y_1}{N_2 y_0} = \frac{K N_1 y_0}{y_2} \,, \qquad \frac{z_0'}{z_1'} = \frac{M y_1}{N_2 K y_0} = \frac{N_1 y_0}{y_2} \,.$$

The exceptional curve of the small resolution consists of two intersecting rational curves. We look at the chart centred at their intersection point; it is given by $z_0' = 1$, $z_1 = 1$. Then $y_2 = z_1' N_1 y_0$, $M y_1 = z_0 N_2 y_0$ (note that $M \neq 0$ if $L = N_1 = N_2 = 0$) and $K = z_0 z_1'$. If P is a singular point of K, we have only a partial resolution. The exceptional curves lie in the intersection of the strict transforms R_i^+ of the R_i.

What we have achieved now is that the strict transforms F_i^- of F_i, being isomorphic to F_i, are disjoint ruled surfaces with smooth neighbourhoods, cf. Figure 1 and 2.

The remaining singularities are not contained in one of the subvarieties $\mathcal{E}_i^-, F_i^-$, as over Λ, they satisfy $y_1 = y_2 = 0$ and are mapped to the singular points of the curve $Q = K = 0$ in $\mathbb{P}^3$. Therefore, they do not affect our constructions until the final step.

3.4 | $\mathcal{Y}^- \dashrightarrow \mathcal{Y}^+$

We flop all the lines in the disjoint union of F_1^- and F_2^-. Because of their disjointness, we can study the flop for each component separately. We study both cases at the same time using our notational convention of dropping all indices, that is, we write Δ, E^-, F^-, and so on, instead of Δ_i, E_i^-, F_i^-.

We start the construction of the flop by blowing up F^-. To describe the result, we first compute the normal bundle $N_{F^-/\mathcal{Y}^-}$ of $F^- \subset \mathcal{Y}^-$. More precisely, we compute the restriction of it to the fibre over any point $p \in \Delta$. We drop the index p from the more correct way of writing. We recall that $F^- \cong \mathbb{F}_2$, and so, its Picard group is generated by the classes of its negative section $B^- = F^- \cap E^-$ and of a fibre f of the projection $F^- \to B^-$.

Proposition 3.9.

$$N_{F^-/\mathcal{Y}^-} \cong \mathcal{O}_{F^-}(-B^- - f)^{\oplus 2} .$$

Proof. The restriction $\mathcal{Y}^-|_\Delta$ consists of the two components $\mathcal{E}^-$ and Y^-. The fibres of $Y^- \to \Delta$ over Λ consist of two components but all other fibres are irreducible. The inclusions $F^- \subset Y^- \subset \mathcal{Y}^-$ equip us with an exact sequence of normal bundles:

$$0 \longrightarrow N_{F^-/Y^-} \longrightarrow N_{F^-/\mathcal{Y}^-} \longrightarrow N_{Y^-/\mathcal{Y}^-}|_{F^-} \longrightarrow 0 .$$

Because this is a sequence of locally free sheaves (in a neighbourhood of F^-), its restriction to a fibre over p is still exact, justifying our abuse of notation. To compute $N_{Y^-/\mathcal{Y}^-}|_{F^-}$, we observe that Y^- is a divisor in $\mathcal{Y}^-$, hence $N_{Y^-/\mathcal{Y}^-}|_{F^-} \cong \mathcal{O}_{\mathcal{Y}^-}(Y^-) \otimes \mathcal{O}_{F^-}$. Using $\mathcal{O}_{\mathcal{Y}^-}(\mathcal{Y}^-|_\Delta) \otimes \mathcal{O}_{F^-} \cong \mathcal{O}_{F^-}$, the equality $Y^- = \mathcal{Y}^-|_\Delta - \mathcal{E}^-$ in $\mathrm{Pic}(\mathcal{Y}^-)$ and the fact that $\mathcal{E}^-$ and F^- intersect transversally along B^-, we obtain $\mathcal{O}_{\mathcal{Y}^-}(Y^-) \otimes \mathcal{O}_{F^-} \cong \mathcal{O}_{F^-}(-\mathcal{E}^- \cdot F^-) \cong \mathcal{O}_{F^-}(-B^-)$.

We write $N_{F^-/Y^-} \cong \mathcal{O}_{F^-}(aB^- + bf)$ with certain integers a, b. To compute these numbers, we observe that they appear as intersection numbers in Y^- as follows: $(F^- \cdot f) = (N_{F^-/Y^-} \cdot f) = ((aB^- + bf) \cdot f)_{F^-} = a$ and $(F^- \cdot B^-) = (N_{F^-/Y^-} \cdot B^-) = ((aB^- + bf) \cdot B^-)_{F^-} = b - 2a$.

Using $F^- \cdot E^- = B^-$, we see that $(F^- \cdot B^-) = (F^- \cdot F^- \cdot E^-) = (B^- \cdot B^-)_{E^-} = 0$, because B^-, as a curve in E^-, is a fibre of one of the projections of $E^- \cong \mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$. This gives $b = 2a$.

Next, we compute $a = (F^- \cdot f)$. For all $p \in \Delta$, we choose a fibre by specifying a point in B^-. As B^- is a line on the image of the quadric Q in the section $\mathcal{E}^- \cong \mathbb{P}^3$, it intersects the image of a general plane (not tangent to Q) transversally at one point. As we now have two flat families over Δ, the surfaces F^- and the fibres f, the intersection number $(F^- \cdot f)$ does not depend on p (here we use that Δ is connected). So, we can restrict our attention to a general point $p \in \Lambda$. The map $\mathcal{Y}^- \to \mathcal{Y}$ is an isomorphism in the neighbourhood of the fibre f, so we compute $(F \cdot f)$ on Y. Now Y is reducible. The component isomorphic to $\mathbb{P}^3$ does not meet F, so we can compute inside the component Y_0, being a fibre of $Y_\Delta \to \Delta$. Because F is defined by $y_i = 0$ over one of the lines given by $Q = L = 0$, the fibre of $Y_0 \to Q$ which contains $f \subset F$ has a second component, which intersects F transversally at one point. The intersection number of F with any fibre of the projection of the conic bundle is zero, as F is disjoint to generic fibres. Therefore, $a = (F \cdot f) = -1$.

Our exact sequence has therefore the following form:

$$0 \longrightarrow \mathcal{O}_{F^-}(-B^- - 2f) \longrightarrow N_{F^-/\mathcal{Y}^-} \longrightarrow \mathcal{O}_{F^-}(-B^-) \longrightarrow 0 . \tag{16}$$

To see that this sequence does not split, we restrict it to the curve B^-. As F^- intersects $\mathcal{E}^-$ transversally in B^-, we have $N_{F^-/\mathcal{Y}^-}|_{B^-} \cong N_{B^-/\mathcal{E}^-}$. If the sequence splits, then $N_{F^-/\mathcal{Y}^-}|_{B^-} \cong \mathcal{O}_{F^-}(-B^- - 2f)|_{B^-} \oplus \mathcal{O}_{F^-}(-B^-)|_{B^-} \cong \mathcal{O}_{B^-} \oplus \mathcal{O}_{B^-}(2)$, which is not the case as B^- is an ordinary line in $\mathcal{E}^- \cong \mathbb{P}^3$ with normal bundle $\mathcal{O}_{B^-}(1)^{\oplus 2}$. The statement follows. □

Let $\widetilde{\mathcal{Y}}$ be the blow-up of F^- in $\mathcal{Y}^-$ (or more precisely, of F_1^- and F_2^-). Note that F^- is a family of surfaces over Δ, so that its codimension in $\mathcal{Y}^-$ is equal to 2. We denote the exceptional locus of the blow-up by $\widetilde{F}$. From our computation of the normal bundle, we know $\widetilde{F} \cong \mathbb{P}(N^\vee_{F^-/\mathcal{Y}^-}) \cong F^- \times \mathbb{P}^1$ and $N_{\widetilde{F}/\widetilde{\mathcal{Y}}} \cong \mathcal{O}_{\widetilde{F}}(-1) \cong \sigma^*\mathcal{O}_{F^-}(-B^- - f) \otimes \tau^*\mathcal{O}_{\mathbb{P}^1}(-1)$, where σ, τ are the two projections of $\widetilde{F} \cong F^- \times \mathbb{P}^1$ onto its factors.

We are now going to contract $\widetilde{F}$ inside $\widetilde{\mathcal{Y}}$. To define this contraction, we note that the isomorphism $\widetilde{F} \cong F^- \times \mathbb{P}^1$ makes $\widetilde{F} \to \widetilde{B}$ into a $\mathbb{P}^1$-bundle, where $\widetilde{B} \subset \widetilde{\mathcal{Y}}$ is the preimage of $B^- \subset \mathcal{Y}^-$. We have a $\mathbb{P}^1$-bundle $\widetilde{F} \to \widetilde{B}$ with fibre $\widetilde{f}$. We compute $N_{\widetilde{F}/\widetilde{\mathcal{Y}}}|_{\widetilde{f}} \cong \sigma^*\mathcal{O}_{F^-}(-B^- - f) \otimes \tau^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\widetilde{f}} \cong \mathcal{O}_{\widetilde{f}}(-1)$, because $(f \cdot f)_{F^-} = 0$ and $(B^- \cdot f)_{F^-} = 1$. By the Castelnuovo–Moishezon–Nakano criterion (Theorem 1.2), $\widetilde{F}$ can be contracted.

This shows that there exists a morphism $\widetilde{\mathcal{Y}} \to \mathcal{Y}^+$ which contacts both $\widetilde{F}_i$. The image of $\widetilde{F}$ in $\mathcal{Y}^+$ is denoted as F^+. This surface no longer lies in Y^+, it is contained in $\mathcal{E}^+$. It intersects Y^+ along the curve $B^+ := F^+ \cap E^+$. The pair (E^-, B^-) is isomorphic to (E^+, B^+) under the flop. The component Y^+ is obtained from Y^- by contracting F^- along the fibration $F^- \to B^-$.

We describe the subvarieties involved in the flop:

$$\begin{array}{ccccc} \mathcal{E}^- & \longleftarrow & E^- & \longrightarrow & Y^- \\ & & \uparrow & & \uparrow \\ & & B^- & \longrightarrow & F^- \end{array} \qquad \leftarrow - - \rightarrow \qquad \begin{array}{ccccc} \mathcal{E}^+ & \longleftarrow & E^+ & \longrightarrow & Y^+ \\ \uparrow & & \uparrow & & \\ F^+ & \longleftarrow & B^+ & & \end{array}$$

All arrows on both sides are inclusions.

We consider the effect of the flop on the divisor $\mathcal{R}$. We write $\mathcal{R}_i$ because both indices occur. As $F^-_{3-i} \subset \mathcal{R}^-_i$, the blow-up does not change $\mathcal{R}^-_i$. We obtain divisors $\widetilde{\mathcal{R}}_i$ in $\widetilde{\mathcal{Y}}$ which are isomorphic to $\mathcal{R}^-_i$. During the contraction to $\mathcal{Y}^+$, however, the component F_{3-i} of $\widetilde{\mathcal{R}}_i$ over Δ_{3-i} will be contracted. The resulting divisor $\mathcal{R}^+_i \subset \mathcal{Y}^+$ still has irreducible fibres over $\Pi \setminus \Delta_{3-i}$, but consists of two components only over Δ_{3-i}.

3.5 | $\mathcal{Y}^+ \longrightarrow \mathcal{Z}$

We continue to omit the subscripts i to deal with two disjoint subvarieties at the same time. The goal of this step is the contraction of the strict transform $\mathcal{E}^+$ of $\mathcal{E}^-$ inside $\mathcal{Y}^+$ to a $\mathbb{P}^1$-bundle over Δ, thereby making the fibres of the obtained family $\mathcal{Z} \to \Pi$ irreducible.

The first useful observation is that $\mathcal{E}^+$ is the blow-up of $\mathcal{E}^-$ in B^-. Over each point $p \in \Delta$, the curve B^- is an ordinary line in $\mathcal{E}^- \cong \mathbb{P}^3$, so blowing up makes $\mathcal{E}^+$ into a $\mathbb{P}^2$-bundle over a $\mathbb{P}^1$-bundle over Δ.

To contract $\mathcal{E}^+$ inside $\mathcal{Y}^+$, we first construct a map $c : \mathcal{E}^+ \to C$. We define

$$C := \mathbb{P}(\delta_*(\mathcal{O}_{\mathcal{E}^-}(1) \otimes \mathcal{I}_{B^-})) ,$$

where $\delta : \mathcal{E}^- \to \Delta$ is the projection, $\mathcal{O}_{\mathcal{E}^-}(1)$ denotes the pull back of $\mathcal{O}_{\mathbb{P}^3}(1)$ under the projection $\mathcal{E}^- \cong \mathbb{P}^3 \times \Delta \to \mathbb{P}^3$ and $\mathcal{I}_{B^-} \subset \mathcal{O}_{\mathcal{E}^-}$ denotes the ideal sheaf of $B^- \subset \mathcal{E}^-$. Because B^- is a family of lines, C is a $\mathbb{P}^1$-bundle over Δ. There exists a natural morphism $c : \mathcal{E}^+ \to C$. For any $p \in \Delta$, the fibres are the strict transforms of the planes in $\mathcal{E}^-$ which contain the line B^-.

To show that we can contract $\mathcal{E}^+$ inside $\mathcal{Y}^+$ along the morphism c, we have to show that the normal bundle $N_{\mathcal{E}^+/\mathcal{Y}^+}$ restricts to $\mathcal{O}_{\mathbb{P}^2}(-1)$ on the fibres of c.

As a divisor in $\mathcal{Y}^+$, we can write $\mathcal{E}^+$ as the difference $\mathcal{Y}^+|_\Delta - Y^+$ and, as before, we obtain $N_{\mathcal{E}^+/\mathcal{Y}^+} \cong \mathcal{O}_{\mathcal{Y}^+}(-Y^+) \otimes \mathcal{O}_{\mathcal{E}^+}$. Each plane in a fibre intersects Y^+ along a line, so indeed, the normal bundle restricts to $\mathcal{O}_{\mathbb{P}^2}(-1)$ on the fibres of c.

The above implies the existence of a morphism $\mathcal{Y}^+ \to \mathcal{Z}$ contracting $\mathcal{E}_1^+$ and $\mathcal{E}_2^+$ to $\mathbb{P}^1$-bundles C_1 and C_2 over Δ_1 and Δ_2, respectively. This morphism contracts the additional component of $\mathcal{R}_i^+$ over Δ_{3-i}, so that the image $\mathcal{S}_i$ of $\mathcal{R}_i^+$ in $\mathcal{Z}$ is a family of irreducible divisors.

3.6 | $\widetilde{\mathcal{Z}} \longrightarrow \mathcal{Z}$

In the final step, we resolve the remaining fibre singularities. Over Λ, the small resolution is given by construction of the family of SRCB manifolds we started out with. We end up with a family of smooth manifolds $\widetilde{\mathcal{Z}} \to \Pi$. The strict transforms of the divisors $\mathcal{S}_i \subset \mathcal{Z}$ are divisors $\widetilde{\mathcal{S}}_i \subset \widetilde{\mathcal{Z}}$.

3.7 | The main theorem

Theorem 3.10. *Let $\widetilde{\mathcal{Z}} \to \Pi$ be a family, constructed as above from a given stratum of SRCB manifolds. The fibres over $\Lambda \subset \Pi$ are SRCB manifolds, and the family $\widetilde{\mathcal{Z}} \to \Pi$ together with the two divisors $\widetilde{\mathcal{S}}_1$ and $\widetilde{\mathcal{S}}_2$ is a deformation, which locally around Λ is versal for deformations of triples $(\widetilde{Z}, \widetilde{S}_1, \widetilde{S}_2)$.*

Proof. The construction of $\widetilde{\mathcal{Z}}$ and Lemma 3.1 guarantee that the fibres over Λ are indeed SRCB manifolds. Let $\widetilde{Z}_p$ be an SRCB manifold over the point $p \in \Lambda \subset \Pi$ in the family $\widetilde{\mathcal{Z}} \to \Pi$. The space T^1_f of infinitesimal deformations for our deformation problem is described in the proof of Theorem 2.7. We have to show that the Kodaira-Spencer map $T_p\Pi \to T^1_f$ is surjective. Changing the lines in our construction, that is, changing L, gives surjectivity on the image of $H^0(N_{\widetilde{S}_{1,p}}) \oplus H^0(N_{\widetilde{S}_{2,p}})$. Therefore, we have to study the image of T^1_f in $H^1(\Theta_{\widetilde{Z}_p})$. On the subspace of deformations coming from deforming the conic bundle, we again have surjectivity, as the stratum Λ_0 gives a versal (but not miniversal) deformation of the conic bundle. We are left with showing that the image of the α_1, α_2 deformations under the Kodaira–Spencer map span the two-dimensional kernel of $H^1(N_{\widetilde{C}_{1,p}}) \oplus H^1(N_{\widetilde{C}_{2,p}}) \to H^1(N_{\widetilde{S}_{1,p}}|_{\widetilde{C}_{1,p}}) \oplus H^1(N_{\widetilde{S}_{2,p}}|_{\widetilde{C}_{2,p}})$. Here, the notation $\widetilde{C}_{i,p}$ means that we consider the fibre over p of the $\mathbb{P}^1$-bundle $\widetilde{C}_i \subset \widetilde{\mathcal{Z}}$, which is the isomorphic pre-image under the small resolution of the bundle $C_i \subset \mathcal{Z}$, defined in Section 3.5.

Given a 1-parameter deformation $\widetilde{\mathcal{Z}}_T \to T$ of $\widetilde{Z}_p$, the image of the Kodaira–Spencer map in $H^1(\Theta_{\widetilde{Z}_p})$ is given by the connecting homomorphism in the cohomology sequence of the sequence

$$0 \longrightarrow \Theta_{\widetilde{Z}_p} \longrightarrow \Theta_{\widetilde{\mathcal{Z}}_T}|_{\widetilde{Z}_p} \longrightarrow N_{\widetilde{Z}_p/\widetilde{\mathcal{Z}}_T} \cong \mathcal{O}_{\widetilde{Z}_p} \longrightarrow 0\,. \tag{17}$$

We consider a 1-parameter deformation with base space $T \subset \Delta_2$, given in terms of the equation of the form (11) by

$$y_1y_2 - KLy_0^2 = 0\,,$$

$$\alpha_1 y_2 - Qy_0 = 0\,,$$

where KL is unchanged. We look at the image of the Kodaira–Spencer map under the surjection $H^1(\Theta_{\widetilde{Z}_p}) \to H^1(N_{\widetilde{C}_{1,p}}) \oplus H^1(N_{\widetilde{C}_{2,p}})$. As the curves in question are disjoint from the singular points, we can do the computation on Z_p, and consequently drop all tildes from the notation. The exact sequence (17) gives two normal bundle sequences,

$$0 \longrightarrow N_{C_{1,p}/Z_p} \longrightarrow N_{C_{1,p}/\mathcal{Z}_T} \longrightarrow N_{Z_p/\mathcal{Z}_T}|_{C_{1,p}} \cong \mathcal{O}_{C_{1,p}} \longrightarrow 0\,, \tag{18}$$

and

$$0 \longrightarrow N_{C_{2,p}/Z_p} \longrightarrow N_{C_{2,p}/\mathcal{Z}_T} \longrightarrow N_{Z_p/\mathcal{Z}_T}|_{C_{2,p}} \cong \mathcal{O}_{C_{2,p}} \longrightarrow 0\,. \tag{19}$$

The restriction map $H^0(\mathcal{O}_{Z_p}) \cong H^0(N_{Z_p/\mathcal{Z}_T}) \to H^0(N_{Z_p/\mathcal{Z}_T}|_{C_{1,p}}) \oplus H^0(N_{Z_p/\mathcal{Z}_T}|_{C_{2,p}}) \cong H^0(\mathcal{O}_{C_{1,p}}) \oplus H^0(\mathcal{O}_{C_{2,p}})$ is the diagonal embedding. Furthermore, we know that $N_{C_{i,p}/Z_p} \cong \mathcal{O}_{C_{i,p}}(-2) \oplus \mathcal{O}_{C_{i,p}}(-2)$ for $i = 1, 2$.

We first look at the second sequence (19). In our construction, C_2 is a $\mathbb{P}^1$-bundle over Δ_2. So, the curve $C_{2,p}$ lies in a family $C_{2,T} \to T$. We can compute $N_{C_{2,p}/\mathcal{Z}_T}$ from the sequence

$$0 \longrightarrow N_{C_{2,p}/C_{2,T}} \longrightarrow N_{C_{2,p}/\mathcal{Z}_T} \longrightarrow N_{C_{2,T}/\mathcal{Z}_T}|_{C_{2,p}} \longrightarrow 0\,.$$

As $N_{C_{2,T}/\mathcal{Z}_T}|_{C_{2,p}} \cong N_{C_{2,p}/Z_p} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, we conclude that the sequence splits, and that $N_{C_{2,p}/\mathcal{Z}_T} \cong \mathcal{O} \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-2)$. Therefore, also the sequence (19) splits and the connecting homomorphism $H^0(\mathcal{O}_{C_{2,p}}) \to H^1(N_{C_{2,p}/Z_p})$ is the zero map.

Secondly, we look at the first sequence (18). The map $Y_2^+|_T \to \mathcal{Z}_T$ contracts the $\mathbb{P}^2$ bundle $\mathcal{E}_1^+|_p$ to the curve $C_{1,p}$. As $\mathcal{E}_1^+|_p$ is the blow-up of $\mathbb{P}^3$ in an ordinary line, it is isomorphic to $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1))$ as bundle over $\mathbb{P}^1$. This shows that the normal bundle $N_{C_{1,p}/\mathcal{Z}_T}$ is a twist of $\mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$, and as its degree is -4, we find that $N_{C_{1,p}/\mathcal{Z}_T} \cong \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Therefore, the connecting homomorphism $H^0(\mathcal{O}_{C_{1,p}}) \to H^1(N_{C_{1,p}/Z_p})$ is non-trivial.

The Kodaira–Spencer map for the analogous deformation in Δ_1 is non-trivial on the other factor. So, indeed, the images span the two-dimensional kernel in question. □

4 | THE FIBRES OVER Δ_i

In this section, we investigate the structure of the fibres of the family $\widetilde{Z}$ over Δ_1 and Δ_2. These spaces are halfway between modifications of conic bundles (over $\Lambda = \Delta_1 \cap \Delta_2$) and double solids. Of the two pencils of surfaces on the conic bundle one survives. As this is not compatible with the real structure, these manifolds do not figure in the twistor literature.

If $\alpha_1 \neq 0$, but $\alpha_2 = 0$, we can eliminate y_2 and have only one equation in the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ over $\mathbb{P}^3$. As before, we deal with a single fibre for fixed $\alpha_1 \neq 0, \alpha_2 = 0$, but continue to simplify notation by not introducing additional subscripts. After dividing by y_0, which cuts away the component $\mathcal{E}_2$, we have the following equation for Y_2:

$$y_1 Q - \alpha_1 KL y_0 = 0\,. \tag{20}$$

It describes the blow-up of $\mathbb{P}^3$ in the curve $Q = KL = 0$. The singularities of the blow-up occur at the singularities of the curve. These have to be resolved with a small resolution. For the singularities on $L = 0$, it is determined by our construction. To describe it in terms of a modification of $\mathbb{P}^3$, we first study the local description of the blow-up of a curve with ordinary double point in a smooth three-fold.

Let the curve be given by $z = xy = 0$. The blow-up $sz - txy = 0$ can be covered by two affine charts, the first one $s = 1$, which is smooth, containing the strict transform of $z = 0$, the second one $(t = 1)$ having an A_1-singularity: $sz - xy = 0$. A small resolution of this A_1-singularity is given by $(s : y) = (x : z) = (u : v)$. We have in total three charts, with $(x, y, z) = (x, y, txy) = (zu, y, z) = (x, vs, xv)$. The same manifold is obtained by first blowing up the branch $z = x = 0$, setting $(x : z) = (u : v)$, and then the strict transform $y = v = 0$ of $y = z = 0$, setting $(y : v) = (s : t)$. Interchanging the role of x and y gives the other small resolution.

We now return to the space Y_2, given as subspace of a $\mathbb{P}^1$-bundle over $\mathbb{P}^3$ by Equation (20) for $\alpha_1 \neq 0$. Over $l_1 \subset Q$ lies R_1, over $l_2 \subset Q$ lies F_2. The surfaces F_1 lie only over $\alpha_1 = 0$. The planes R_2, given by $y_1 = L = 0$ are contained in Y_2. The singular point of Y_2 above the point of tangency of the quadric and its tangent plane $L = 0$ has in general non-zero y_1-coordinate. The correct small resolution is determined by the condition that no exceptional curve is contained in the strict transform F_2^-.

This small resolution of Y_2 is obtained from $\mathbb{P}^3$ by first blowing up l_1, then the curve $K = 0$ on the strict transform of Q, introducing singularities coming from the singular points of the curve and finally blowing up l_2, as lying on the strict transform of Q. The exceptional surface F_2^- is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. The flop $\mathcal{Y}^- \dashrightarrow \mathcal{Y}^+$ has on Y_2^- the effect of contracting F_2^- again. So, after the second step, blowing up the curve $K = 0$ on the quadric, we have already obtained the space Y_2^+ (over $\alpha_1 \neq 0$). This discussion proves the following proposition.

Proposition 4.1. *Let $\widetilde{Z}$ be an SRCB manifold, which is a small resolution of a space Z defined by Equation (4) with $n = 3$. Fix a line $l_1 = \mathrm{pr}_1^{-1}(s)$ of the first ruling of the quadric Q. The fibres of the family $\widetilde{\mathcal{Z}} \to \Pi$ over $\Delta_2 \setminus \Lambda$, constructed from lines l_2 of the other ruling, such that L in Equations (11) satisfies $\{Q = L = 0\} = l_1 \cup l_2$, are isomorphic for all l_2. Such a manifold $\widetilde{Z}_2$ is a modification of $\mathbb{P}^3$, which is the composition of the following birational maps*

$$\mathbb{P}^3 \longleftarrow Y_2' \longleftarrow Y_2^+ \longrightarrow Z_2 \longleftarrow \widetilde{Z}_2\,,$$

where the first map is the blow-up of l_1, the second the blow-up of the curve $K = 0$ on the strict transform Q' of Q and the map to Z_2 is the blow-down of the strict transform Q^+ along the first ruling. The singularities of Z_2 are the same as those of Z, and the map $\widetilde{Z}_2 \longrightarrow Z_2$ resolves them in the same way as $\widetilde{Z} \longrightarrow Z$ does.

We can use this direct description of the three-fold Z_2 to study the surfaces $\widetilde{S}_1$ and $\widetilde{S}_2$. The surface $\widetilde{S}_2$ moves in a pencil, which contains some singular elements; these are described in the proof of the following proposition.

Proposition 4.2. *Let $\widetilde{Z}_2$ be a modification of $\mathbb{P}^3$ as in the previous proposition. It contains one pencil of surfaces $\widetilde{S}_2$, of which the smooth elements are the blow-up of $\mathbb{P}^2$ in three (possibly infinitely near) collinear points. Each surface intersects a surface $\widetilde{S}_1$, which does not move in a pencil, and is the blow-up of $\mathbb{P}^2$ in three (possibly infinitely near) non-collinear points, if non-singular.*

Proof. The pencil of planes through the line $l_1 \subset \mathbb{P}^3$ gives rise to a pencil of planes R_2' on Y_2'. The blow-up introduces the exceptional surface $R_1' \cong \mathbb{P}^1 \times \mathbb{P}^1$. The strict transform Q' of the quadric intersects R_1' in the diagonal of R_1', and each surface R_2' in the line l_2' lying in it. The curve $K = 0$ on Q' intersects l_2' with multiplicity 3.

First suppose that the intersection consists of three distinct points. Then, the strict transform R_2^+ is the blow-up of $\mathbb{P}^2$ in three collinear points. The image S_2 under the contraction map $Y_2^+ \to Z_2$ is isomorphic to R_2^+, and the small resolution does not change it, as S_2 does not pass through any singular point in this case.

If l_2' is tangent to the curve $K = 0$ on Q' at a smooth point, then R_2^+ is the blow-up of $\mathbb{P}^2$ in a rectilinear scheme of length 3 (supported on one or two points) and therefore has a singularity which remains under the subsequent rational maps. A local model for the simplest situation is that one blows up (x, y, z)-space in the curve $x = z - y^2$, which is tangent to the plane $z = 0$. The strict transform of the plane has an ordinary double point. It can also be obtained by blowing up the plane in the ideal (x, y^2).

If the line l_2' passes through a singular point of the curve $K = 0$ on Q', the blow-up R_2^+ is also singular, but the small resolution $\widetilde{Z}_2 \to Z_2$ resolves it, and $\widetilde{S}_2$ is the blow-up of $\mathbb{P}^2$ in three collinear points, some of which are infinitely near, unless the line is also tangent to a branch of the curve.

The surface R_1' is also blown up in a scheme of length three, which lies on the diagonal. In the next step, the strict transform of the diagonal is blown down. If the three points are in general position, then it is easy to see that the surface S_1 has six (-1)-curves, the strict transforms of both lines through the three points, which form a cycle, and the surface is a Del Pezzo surface of degree 6. Analysing the other possibilities is left to the reader. □

5 | EXAMPLES

5.1 | Twistor spaces

We specialise to the case of LeBrun twistor spaces, as considered in the Introduction. So, K is the product of three linear factors, defining smooth curves of type (1,1) on the quadric.

Via the embedding $(x_0 : x_1 : x_2 : x_3) = (s_0t_0 : s_0t_1 : s_1t_0 : s_1t_1)$, the real structure $(s_0 : s_1 ; t_0 : t_1) \mapsto (\bar{t}_0 : \bar{t}_1 ; \bar{s}_0 : \bar{s}_1)$ on Q gives rise to a real structure on our family (11), given by

$$(x_0 : x_1 : x_2 : x_3 ; y_0 : y_1 : y_2 ; \alpha_1, \alpha_2) \mapsto (\bar{x}_0 : \bar{x}_2 : \bar{x}_1 : \bar{x}_3 ; \bar{y}_0 : \bar{y}_2 : \bar{y}_1 ; \bar{\alpha}_2, \bar{\alpha}_1) .$$

Furthermore, we have a real structure on the space of forms $L_1L_2L_3$, and the conditions we impose are compatible with the real structure. As our constructions are also compatible with the real structure, we get a real structure on our versal deformation. The fibres over Δ_i do not occur in the real deformation, which lies over $(\alpha_1, \alpha_2) = (\alpha, \bar{\alpha})$. In particular, $\alpha\bar{\alpha} = |\alpha|^2 \neq 0$ for $\alpha \neq 0$.

A well-known argument of Donaldson and Friedman [7, proof of Theorem 4.1], see also [33, §5] and [4, Proposition 2.1], shows that the fibres over real points of our versal deformation are indeed twistor spaces, at least in a neighbourhood of Λ in Π. As a consequence, we obtain another proof of a result of Honda [13, Theorem 2.1], which states the existence of degenerate double solids as twistor spaces, see Example 5.2 below.

We indicate how the twistor lines are deformed in the general case. By [31], the general twistor line on a LeBrun twistor space lies above the intersection of $Q \subset \mathbb{P}^3$ with a real hyperplane, in fact, one has a whole S^1 of twistor lines over the same curve. The curve can be given by a positive hermitian matrix ψ with determinant 1. It intersects each of the three distinguished planes L_i, given by a positive-definite hermitian matrix φ_i, at two points. One of them has to be lifted to $w_1 = 0$, the other to $w_2 = 0$. The choice is made in the following way. The secular equation $\det(\lambda\psi - \varphi_i)$ has two real solutions $\lambda_i > 1 > \mu_i$ and one can write $\varphi_i = \lambda_i\psi + f_i(s)\bar{f}_i(t) = \mu_i\psi - g_i(s)\bar{g}_i(t)$ with f_i, g_i unique up to a factor from $S^1 \subset \mathbb{C}^*$. Let (s,t) and $(\bar{t},\bar{s})$ be the two intersection points. Then, either $f_i(s) = \bar{g}_i(t) = 0$ or $\bar{f}_i(t) = g_i(s) = 0$ and $\frac{\varphi_i}{\psi}(s,\bar{s}) - \frac{\varphi_i}{\psi}(\bar{t},t) = \lambda_i - \mu_i$ in the first case, while being equal to $\mu_i - \lambda_i$ in the second case. One chooses a fixed sign; by looking at a suitable degeneration, it turns out that one needs always the negative sign.

We consider a double solid, which is a small real deformation of a LeBrun twistor space. The same plane in $\mathbb{P}^3$ (assumed to be general) now contains a smooth quartic branch curve and the twistor lines lie over the real contact conics in one of the 63 such systems. We have to choose the correct one. Each system is determined by one of six pairs of bitangents. We note that four bitangents are already given, the intersections of the planes L_i, $L_4 = L$ with the given plane. We can write the equation $L_1L_2L_3L_4 - Q^2$ in three ways as symmetric determinantal, like

$$\begin{vmatrix} L_1L_2 & Q \\ Q & L_3L_4 \end{vmatrix} .$$

From this matrix, one gets a system of contact conics, namely $\lambda^2L_1L_2 + 2\lambda\mu Q + \mu^2L_3L_4$. To find the four other pairs of bitangents, one computes when the conic degenerates. This gives an equation of degree 4 in λ and μ. In this way, we get all bitangents.

As the double solid degenerates to the conic bundle, the quartic curve degenerates to the conic section counted twice, with eight marked points (the intersection of the conic with the four planes L_i), so to a hyperelliptic curve of genus 3. This degeneration was already studied by Felix Klein [28]. The bitangents degenerate to the 28 lines joining the eight points. Kurke's construction divides the points into two groups of four. The diagonals of the quadrangles thus found are the limits of the six pairs of bitangents in the system of contact conics we are looking for.

Given a double solid, there is no preferred choice of a ruling of Q. But by fixing coordinates as we did, we have made a choice. Suppose that $\widetilde{Z}$ is a double solid twistor space, occurring in our family. Reversing the roles of the rulings means flopping all exceptional curves. By degenerating to $\alpha_1 = \alpha_2 = 0$, we obtain a small resolution of the isolated singularities, which does not lie over Λ. But we can interchange the rulings by an automorphism of $\mathbb{P}^3$, to obtain an isomorphic double solid (with in general a different KL), which does occur in our family. Seen from another perspective, given a deformation of a LeBrun twistor space, we can interchange the rulings in the whole construction and get the same general fibre (with 13 singular points). The difference is seen in the small resolution of the 13th singular point.

Proposition 5.1. *A double solid twistor space, which is a small deformation of a LeBrun twistor space, is transformed into a twistor space by flopping all exceptional curves.*

We do not know if the double solid can be degenerated in more ways, maybe due to some additional symmetries. The results of Honda [16], who finds only two small twistor resolutions for spaces with extra symmetry, indicate that this is not the case.

Example 5.2 (Honda's deformation with torus action). A LeBrun twistor space with torus action has an equation $w_1w_2 + \varphi_1\varphi_2\varphi_3w_0^2 = 0$, where the φ_i define three conics with common intersection points. These are real for our real structure (note that it differs from Honda's [15]) if we take $\varphi_i = a_i s_0 t_0 + b_i s_1 t_1$. The singular points are $(1:0;0:1)$ and $(0:1;1:0)$. If we now choose two lines, one in each ruling, invariant under the $\mathbb{C}^*$-action on the quadric, they have to pass through the singular points. Invariance under the involution dictates that we take a degenerate conic in the pencil, either s_0t_0 or s_1t_1. Both give an equivariant deformation, in accordance with Honda's results. We proceed to describe the first one explicitly. The general fibre is a double solid, the branch quartic of which has an equation of the form $Q^2 + 4\alpha_1\alpha_2 KL$, where $Q = x_0x_3 - x_1x_2$ and KL is a polynomial of degree 4 in x_0 and x_3 only, depending on $\alpha_1\alpha_2$. We therefore take Equations (11) in the form

$$\begin{aligned} y_1y_2 + (a_1x_0 + b_1x_3)(a_2x_0 + b_2x_3)(a_3x_0 + b_3x_3)(x_0 + sx_3)y_0^2 &= 0 \\ \alpha_2 y_1 + \alpha_1 y_2 - Qy_0 &= 0\,, \end{aligned} \tag{21}$$

where s is a function of $\alpha_1\alpha_2$, to be determined. We now consider $\beta = 1/\alpha_1\alpha_2$ and s as independent variables, and ask when the double solid has an A_1-singularity, besides the two singular points at $(0:1:0:0)$ and $(0:0:1:0)$ of type $\widetilde{E}_7$. The corresponding A_1-singularity of the branch quartic lies outside $Q = 0$, so the vanishing of the derivatives w.r.t. x_1 and x_2 gives $x_1 = x_2 = 0$. The condition for a singular point is then that $\beta x_0^2x_3^2 + 4K(x_0,x_3)L(x_0,x_3\,;s) = 0$ has a multiple root. This means that the discriminant of this binary form has to vanish. This condition defines a curve in a $\mathbb{P}^2$ with affine coordinates $(\beta : s : 1)$. As Zariski observed [42], it is a rational sextic curve with six cusps (the maximal number), being the dual of the rational quartic parametrised by

$$\psi(x_0,x_3) = (x_0^2x_3^2 : x_0K : x_3K)\,.$$

Indeed, the locus $\beta x_0^2x_3^2 + 4x_0K + 4sx_3K = 0$ is the incidence correspondence between $\mathbb{P}^2$ and its dual, restricted to the curve defined by ψ. Requiring a multiple root means that the line corresponding to the point $(\beta : 4 : 4s)$ in the dual plane is tangent to the curve given by ψ.

The dual curve can be parametrised by the cross product $\psi_0 \times \psi_3$, where we denote the partial derivative w.r.t. x_i by the subscript i. We obtain

$$(\beta : s : 1) = \left(-4K^2 : x_0^2x_3(2K - x_3K_3) : x_0x_3^2(2K - x_0K_0)\right)\,.$$

From this formula, we find the curve in the $(s,\alpha_1\alpha_2)$-plane which describes the dependence of s on $\alpha_1\alpha_2$. We write $(x_0 : x_3) = (t : 1)$ and $K(t,1) = A_0 + A_1t + A_2t^2 + A_0t^3$, so that $A_3 = a_1a_2a_3, A_2 = a_1a_2b_3 + a_1b_2a_3 + b_1a_2a_3, A_1 = a_1b_2b_3 + b_1a_2b_3 + b_1b_2a_3$ and $A_0 = b_1b_2b_3$. A straightforward calculation now shows that

$$s = \frac{-t(A_0 - A_2t^2 - 2A_3t^3)}{2A_0 + A_1t - A_3t^3} \qquad \text{and} \qquad \alpha_1\alpha_2 = \frac{-t(2A_0 + A_1t - A_3t^3)}{4(A_0 + A_1t + A_2t^2 + A_0t^3)^2}\,.$$

In the $(s,\alpha_1\alpha_2)$-plane, this is a curve of degree 10, with 6 cusps, 3 double points and two very singular points at infinity.

For example, if $K = (x_0 + x_3)(x_0 + 2x_3)(x_0 + \frac{1}{2}x_3)$, we obtain

$$\begin{aligned} s &= -t(2 - 7t^2 - 4t^3)/(4 + 7t - 2t^3)\,, \\ \alpha_1\alpha_2 &= -t(4 + 7t - 2t^3)/2(1 + t)^2(1 + 2t)^2(2 + t)^2\,. \end{aligned}$$

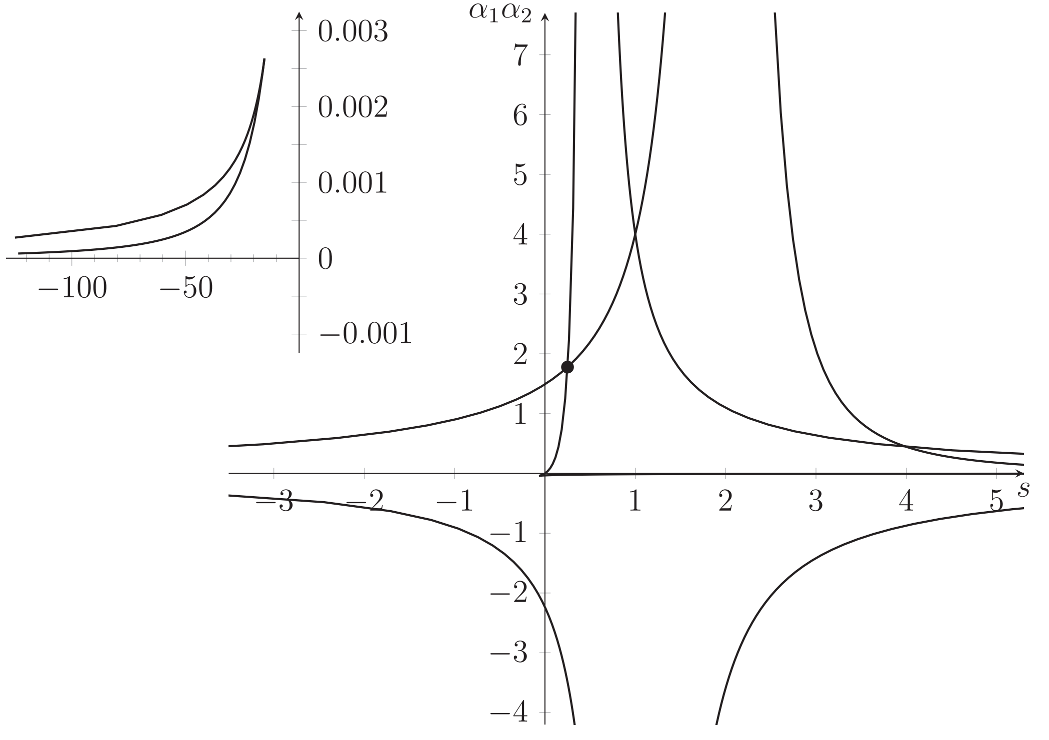


FIGURE 3 The dependence of s on $\alpha_1\alpha_2$ with a far away component at another scale.

For $(-15+\sqrt{97})/16 < t < 0$, we find that $\alpha_1\alpha_2$ is positive, and that the three-fold has no real points besides the extra singular point. For $t = (-15+\sqrt{97})/16$, the rational curve has a double point, with $(s, \alpha_1\alpha_2) = (\frac{1}{4}, \frac{16}{9})$; the corresponding double solid has not one, but two extra double points. There are two real cusps in this example, one at $t \approx 0.25$ and the other at $t \approx 3.996$. We include a picture of the curve, and an enlarged view around the origin in the $(s, \alpha_1\alpha_2)$-plane (Figures 3 and 4).

We remark that the simultaneous flop of all singularities is induced by the global involution permuting x_1 and x_2, showing that in this case, the two twistor spaces are isomorphic, see also [16, Thm. 8.1].

5.2 | Quartics with 14 double points

A quartic, which is given by an equation of the form $Q^2 - L_1L_2L_3L_4 = 0$, in general, has 12 double points. The condition that it has a 13th singular point, is a rather complicated one, at least in terms of the coefficients of the planes L_i, as can be seen from the explicit example with additional symmetry above.

With one additional singular point, the situation becomes much easier. A quartic surface with 14 double points has six tropes, and its equation can be written in terms of them in the irrational form

$$\sqrt{x_1x_2} + \sqrt{y_1y_2} + \sqrt{z_1z_2} = 0\,, \tag{22}$$

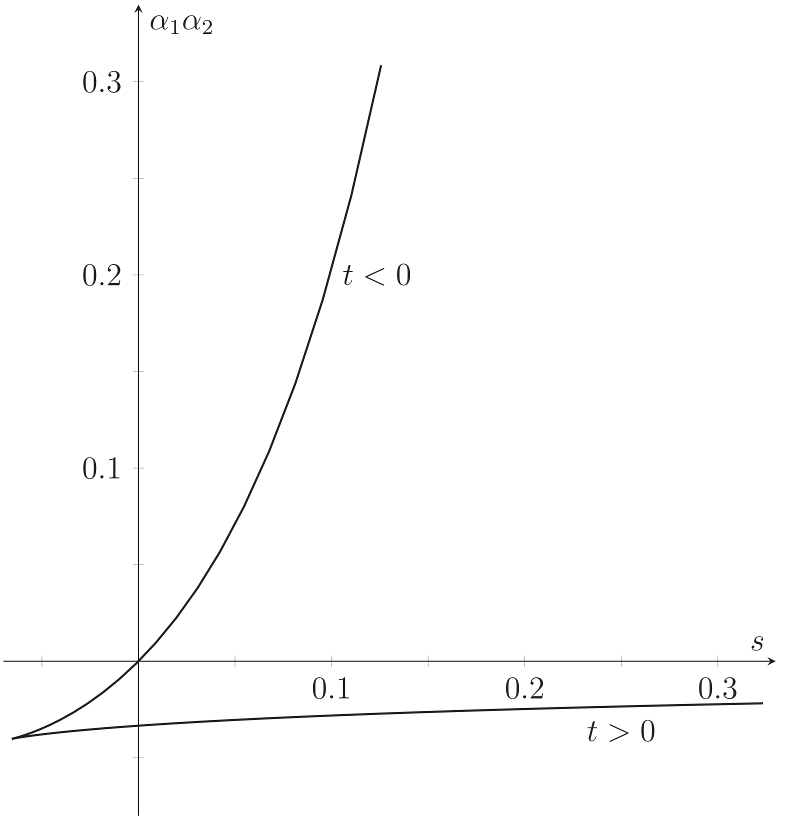


FIGURE 4 Enlarged view around the origin.

where x_i, y_i, z_i are linear forms defining the tropes, see, for example, [25, No. 55], or in rational form as $(x_1x_2)^2 + (y_1y_2)^2 + (z_1z_2)^2 - 2y_1y_2z_1z_2 - 2x_1x_2z_1z_2 - 2x_1x_2y_1y_2 = 0$. By singling out four planes, we can write it in our preferred form $(x_1x_2 + y_1y_2 - z_1z_2)^2 - 4x_1x_2y_1y_2 = 0$. The singular points come in two types, eight being given by $x_i = y_j = z_k = 0$, and six by equations of the type $x_1 = x_2 = y_1y_2 - z_1z_2 = 0$. If the product x_1x_2 tends to zero, the quartic degenerates to a double quadric. But this is exactly the quadric in our construction.

We start therefore with the quadric $Q: \ x_0x_3 - x_1x_2 = 0$, parametrised by $(x_0 : x_1 : x_2 : x_3) = (s_0t_0 : s_0t_1 : s_1t_0 : s_1t_1)$. On Q, we take a discriminant curve consisting of two smooth conics and a pair of lines. Two lines intersecting at a point $P = (a_0 : a_1 ; b_0 : b_1)$ on the quadric are given by the intersection of Q and the tangent plane $a_1b_1x_0 - a_1b_0x_1 - a_0b_1x_2 + a_0b_0x_3 = 0$ at the point P. We write $K = K_1K_2L_0$ with K_1, K_2 general linear forms, and we take L_0 as tangent plane $\lambda\mu x_0 - \lambda x_1 - \mu x_2 + x_3 = 0$ defined by the point $(1 : \lambda ; 1 : \mu)$. These data define an SRCB manifold. For our construction, we add a tangent plane $L_3 = a_1b_1x_0 - a_1b_0x_1 - a_0b_1x_2 + a_0b_0x_3$. In general, we get 14 double points, as Figure 5 illustrates. If the two conics are not tangent, as in the picture, we may normalise the equation, and take $K = K_1K_2L_0 = (x_0 - px_3)(x_1 - qx_2)(\lambda\mu x_0 - \lambda x_1 - \mu x_2 + x_3)$. In general, we can also fix the intersection point of the two lines, making $\lambda = \mu = 0$, but if the curve has additional symmetry, which happens if the intersection point of the lines coincides with one of the other two points, or if the lines pass through both points, then we have to consider a larger family.

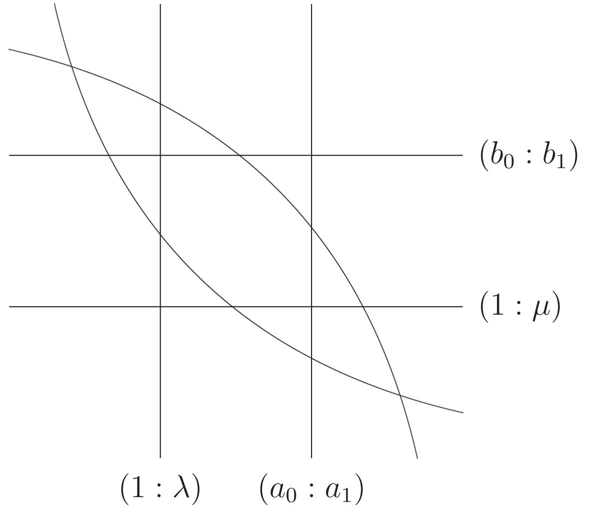


FIGURE 5 Discriminant curve for a 14-nodal conic bundle.

After choosing the extra point on the quadric, we have two pairs of lines. The intersection points of the two lines also determine tangent planes, $L_1 = \lambda b_1 x_0 - \lambda b_0 x_1 - b_1 x_2 + b_0 x_3$ and $L_2 = a_1 \mu x_0 - a_1 x_1 - a_0 \mu x_2 + a_0 x_3$. The quadric $L_0 L_3 - L_1 L_2 = 0$ passes through the four lines, so it is an element of the pencil of quadrics through these lines. We compute that its equation is, in fact, a multiple of Q:

$$L_0 L_3 - L_1 L_2 = (a_1 - \lambda a_0)(b_1 - \mu b_0)(x_0 x_3 - x_1 x_2)\ .$$

So, if the point $P = (a_0 : a_1 ; b_0 : b_1)$ lies on one of the two lines of L_0, the formula breaks down. Actually, we excluded such cases in the construction to avoid non-isolated singularities. Otherwise we can write a deformation of the form $Q^2 - 4\alpha_1\alpha_2 K_1 K_2 L_0 L_3$, with $Q = L_0 L_3 - L_1 L_2 + \alpha_1 \alpha_2 K_1 K_2$. In the description of our family, we insisted that we kept the quadric unchanged. This can, of course, be achieved by a coordinate transformation, as long as the quadric is non-singular, but it is easier to work with the present formula. To get rid of the term $(a_1 - \lambda a_0)(b_1 - \mu b_0)$, we replace α_i by $(a_1 - \lambda a_0)(b_1 - \mu b_0)\alpha_i$ and divide. Then, we obtain formulas which always hold.

We now write down the formula defining a versal deformation of SRCB manifolds on the given stratum:

$$y_1 y_2 - K_1 K_2 L_0 L_3 y_0^2 = 0\ ,$$
$$\alpha_2 y_1 + \alpha_1 y_2 - Q y_0 = 0\ ,$$

where

$$Q = x_0 x_3 - x_1 x_2 + (a_1 - \lambda a_0)(b_1 - \mu b_0)\alpha_1 \alpha_2 K_1 K_2,$$
$$K_1 = c_{10} x_0 + c_{11} x_1 + c_{12} x_2 + c_{13} x_3,$$
$$K_2 = c_{20} x_0 + c_{21} x_1 + c_{22} x_2 + c_{23} x_3,$$

$$L_0 = \lambda\mu x_0 - \lambda x_1 - \mu x_2 + x_3,$$
$$L_1 = \lambda b_1 x_0 - \lambda b_0 x_1 - b_1 x_2 + b_0 x_3,$$
$$L_2 = a_1\mu x_0 - a_1 x_1 - a_0\mu x_2 + a_0 x_3,$$
$$L_3 = a_1 b_1 x_0 - a_1 b_0 x_1 - a_0 b_1 x_2 + a_0 b_0 x_3 \ .$$

The general fibre is a double solid with branch surface $Q^2 - 4\alpha_1\alpha_2 K_1 K_2 L_0 L_3 = 0$, with in general 14 double points.

If the point $P = (a_0 : a_1 ; b_0 : b_1)$ lies on one of the two lines of L_0, the conic bundle has a singular line. In this case, the line $L_0 = L_3 = 0$ lies on the quadric Q for all $\alpha_1\alpha_2$: if say $a_1 = \lambda a_0$, then the line $L_0 = L_3 = 0$ is given by $\lambda x_0 - x_2 = \lambda x_1 - x_3 = 0$. The line is then a singular line of the quartic surface $Q^2 - 4\alpha_1\alpha_2 K_1 K_2 L_0 L_3 = 0$. So, presumably also in this case, our construction goes through, with appropriate changes, but we have not checked details.

We note that the equations for the double solids only involve the four products $\alpha_1 a_0$, $\alpha_1 a_1$, $\alpha_2 b_0$ and $\alpha_2 b_1$, which can be considered as coordinates on our versal deformation of the SRCB manifold. To describe the other fibres, which are not double solids, and the total family, one needs the choice of the extra plane, which gives the structure of the product $\mathrm{Bl}_0\mathbb{C}^2 \times \mathrm{Bl}_0\mathbb{C}^2$ transversal to Λ.

5.3 | Kummer surfaces

A similar description is possible if the general fibre is a double solid, branched over a Kummer surface. We shall relate our family to the moduli space of Kummer surfaces.

In this case, there is a finite number of SRCB manifolds, which all are small modifications of a unique conic bundle. The discriminant of the bundle consists of three reducible conics, given as the intersection of the quadric with three tangent planes in general position. The versal deformation has a four-dimensional base space B, with a $\mathbb{C}^*$-action. The quotient $B/\mathbb{C}^*$ under the $\mathbb{C}^*$-action is a three-dimensional A_1-singularity.

A modern presentation of classical results on the moduli space of Kummer surfaces with level two structure can be found in [26, Ch. 3]. The moduli space in question is nowadays known as Igusa quartic $\mathcal{I}_4$. It is a compactification of the Siegel modular three-fold for $\Gamma_2(2)$, and a resolution is the non-singular Satake compactification. The Igusa quartic is a rational variety, which is dual to the 10-nodal Segre cubic $\mathcal{S}_3$. The duality map $\mathcal{S}_3 \dashrightarrow \mathcal{I}_4$ blows up the 10 A_1 singularities, but blows down 15 quadric surfaces to singular lines on the quartic.

The Igusa quartic not only parametrises Kummer surfaces, but also gives a direct construction of the parametrised objects: the tangent plane at a (general) point intersects $\mathcal{I}_4$ in a 16-nodal surface (15 as intersection with the singular lines, plus one extra from the point of tangency), a fact already mentioned by Hudson [25, § 80]. For special sections, corresponding to boundary points of the Satake compactification, one obtains Plücker surfaces, quartics with a double line and 8 isolated double points in general (for a description of these surfaces, see [27, Art. 83]).

Dually, projecting the Segre cubic $\mathcal{S}_3$ from a general point onto a linear space gives a birationally equivalent double solid, branched along a Kummer surface. The 16 nodes arise from the 10 nodes of $\mathcal{S}_3$ and from six lines of the cubic lying in the tangent cone at the projection point. More details can be found in Baker's book [3]. The construction degenerates if the projection point becomes one of the 10 nodes. A neighbourhood of each of them is isomorphic to the quotient $B/\mathbb{C}$ alluded to above. Our family yields deformations of the SRCB manifold, together with two divisors, specified by giving a point of the quadric Q. We obtain this quadric by blowing up the node of $B/\mathbb{C}$. We therefore find a neighbourhood of the quadric Q on a common resolution of $\mathcal{I}_4$ and $\mathcal{S}_3$ (which is provided by the Satake compactification).

We proceed to give explicit equations, similar to Equation (22) above, which will allow us to describe our family with $\mathbb{C}^*$-action. We follow Baker [3]. We start by parametrising the Segre cubic $\mathcal{S}_3$. The linear system of quadrics through the five points in $\mathbb{P}^3$ (for which we take the vertices of the coordinate tetrahedron and the point $(1:1:1:1)$) defines a birational map to $\mathcal{S}_3$. It blows up the five points, but the 10 lines connecting the points are blown down to the singular points.

We take as basis of the linear system

$$\begin{aligned} x &= z_1(z_0 - z_2)\,, & x' &= z_2(z_1 - z_0)\,, \\ y &= z_2(z_0 - z_3)\,, & y' &= z_3(z_2 - z_0)\,, \\ z &= z_3(z_0 - z_1)\,, & z' &= z_1(z_3 - z_0)\,, \end{aligned}$$

with coordinates $(z_0 : z_1 : z_2 : z_3)$ on $\mathbb{P}^3$. In this way, we embed $\mathcal{S}_3$ in a hyperplane in $\mathbb{P}^5$. The equations of the image are

$$\begin{aligned} x + y + z + x' + y' + z' &= 0\,, \\ xyz + x'y'z' &= 0\,. \end{aligned}$$

To determine the dual variety $\mathcal{I}_4$, we compute the Jacobian matrix of these two polynomials. We set $\xi = yz$, $\xi' = y'z'$, and so on. Using the parametrisation of $\mathcal{S}_3$, one checks that the following equation in irrational form holds:

$$\sqrt{(\eta - \zeta')(\eta' - \zeta)} + \sqrt{(\zeta - \xi')(\zeta' - \xi)} + \sqrt{(\xi - \eta')(\xi' - \eta)} = 0\,.$$

Writing $a = \eta - \zeta'$, and so on, we find the Igusa quartic $\mathcal{I}_4$ embedded in the hyperplane $a + b + c + a' + b' + c' = 0$ in $\mathbb{P}^5$, with equation in irrational form:

$$\sqrt{aa'} + \sqrt{bb'} + \sqrt{cc'} = 0\,.$$

We remark that equations with even more symmetry are known, but they are not suitable for our purpose. In the coordinates (a, b, c, a', b', c'), the parametrisation of $\mathcal{I}_4$ is given by

$$\begin{aligned} a &= (z_3 - z_1)z_2(z_0 - z_1)(z_0 - z_3)\,, \\ a' &= (z_1 - z_3)z_1z_3(z_0 - z_2)\,, \end{aligned}$$

and the other variables by cyclic permutation (on (123)).

The tangent space of $\mathcal{I}_4$ at the corresponding point can be computed from the given parametrisation of the Segre cubic by a suitable coordinate transformation, or directly from the defining equations. The result is

$$z_1z_3a + z_2z_1b + z_2z_3c + z_2(z_1+z_3-z_0)a' + z_3(z_2+z_1-z_0)b' + z_1(z_3+z_2-z_0)c' = 0\ ,$$
$$a+b+c+a'+b'+c' = 0\ .$$

We now look at a neighbourhood of 1 of the 10 quadrics on a common resolution of the Segre cubic and the Igusa quartic. Let $\widetilde{S} \to S_3$ the resolution obtained by blowing up the singular points, so the exceptional divisors are quadric surfaces. We also get $\widetilde{S}$ by first blowing up the five points in $\mathbb{P}^3$ (which already gives a small resolution) and then blowing up the strict transforms of the 10 lines.

Specifically, we choose the quadric obtained by blowing up the strict transform of the line $(z_0 : z_1 : 0 : 0)$ in $\mathbb{P}^3$. We first blow up the points $(1:0:0:0)$ and $(0:1:0:0)$. In $\mathbb{P}^3 \times \mathbb{P}^2$, the blow-up is given by

$$\operatorname{Rank}\begin{pmatrix} z_0z_1 & z_2 & z_3 \\ a_{01} & a_2 & a_3 \end{pmatrix} \leqslant 1\ .$$

We take the chart $a_{01} = 1$. Next, we blow up the strict transform $a_2 = a_3 = 0$ of the line. We describe it with homogeneous coordinates, that is, with a $\mathbb{C}^*$-action. We explain this for the blow-up of the origin in $\mathbb{C}^2$. Consider coordinates (u, v, t) with action $\lambda \cdot (u, v, t) = (\lambda u, \lambda v, \lambda^{-1} t)$. The quotient $\mathbb{C}^3//\mathbb{C}^*$ is just $\mathbb{C}^2$: the invariant functions $x = ut$ and $y = vt$ are the coordinates. The quotient only parametrises closed orbits. We take out the t-axis, consisting of the origin, which lies in the closure of all non-closed orbits, and a non-closed orbit: consider the set $B:\ u = v = 0$. The quotient $(\mathbb{C}^3 \setminus B)/\mathbb{C}^*$ is the blow-up of $\mathbb{C}^2$.

At this point, we can introduce our four-dimensional base space, as the other ruling of the quadric can be treated in the same way. This means also replacing z_0 and z_1. We therefore set

$$z_0 = \alpha_1 a_0\ , \qquad a_2 = \alpha_2 b_0\ ,$$
$$z_1 = \alpha_1 a_1\ , \qquad a_3 = \alpha_2 b_1\ .$$

The coordinates $(a_0 : a_1\ ;\ b_0 : b_1\ ;\ \alpha_1, \alpha_2)$ describe the product $\mathrm{Bl}_0\mathbb{C}^2 \times \mathrm{Bl}_0\mathbb{C}^2$ of blow-ups of the origin in $\mathbb{C}^2$. We substitute the above values in the first equation for the tangent space, add a multiple of the second equation and divide out common factors to obtain

$$\alpha_1\alpha_2\big(a_1b_0a + a_1b_1b + (a_1b_0 + a_1b_1 - \alpha_1\alpha_2a_0a_1b_0b_1)c$$
$$+ (a_0b_0 + a_1b_1 - \alpha_1\alpha_2a_0a_1b_0b_1)a' + (a_0b_1 + a_1b_0 - \alpha_1\alpha_2a_0a_1b_0b_1)b'\big) + c' = 0\ ,$$

from which we conclude that $c' = -\alpha_1\alpha_2c''$ for some expression c'' not involving c'. A rational form of the quartic equation for the hyperplane section is

$$(-\alpha_1\alpha_2cc'' + aa' - bb')^2 + 4\alpha_1\alpha_2aa'cc'' = 0\ .$$

We can therefore write a formula for the deformation of the SRCB manifold as

$$\begin{aligned} y_1y_2 + aa'cc''y_0^2 &= 0, \\ \alpha_2 y_1 + \alpha_1 y_2 - (-\alpha_1\alpha_2 cc'' + aa' - bb')y_0 &= 0, \\ a + b + c + a' + b' - \alpha_1\alpha_2 c'' &= 0. \end{aligned} \tag{23}$$

The variable c can be eliminated, if its coefficient does not vanish, so certainly in a neighbourhood of the quadric (for $\alpha_1\alpha_2$ small). Equations (23) describe therefore a family over a neighbourhood of the exceptional locus of $\mathrm{Bl}_0\mathbb{C}^2 \times \mathrm{Bl}_0\mathbb{C}^2$.

From the other nine quadrics on $\mathcal{I}_4$, only four are visible in our chart: the strict transforms of the lines through $(1:1:1:1)$ and the vertices of the coordinate tetrahedron in $\mathbb{P}^3$. For the line $z_0 - z_3 = z_1 - z_3 = 0$, we find the locus $1 - \alpha_1\alpha_2 a_1 b_1 = 1 - \alpha_1\alpha_2 a_0 b_1 = 0$ (so $a_0 = a_1$) in our base space. Using the explicit expression for the last equation of (23)

$$\begin{aligned} &(1 - \alpha_1\alpha_2 a_1 b_0)a + (1 - \alpha_1\alpha_2 a_1 b_1)b + (1 - \alpha_1\alpha_2 a_1(b_0 + b_1) + \alpha_1^2\alpha_2^2 a_0 a_1 b_0 b_1)c \\ &\qquad + (1 - \alpha_1\alpha_2 a_0 b_0)(1 - \alpha_1\alpha_2 a_1 b_1)a' + (1 - \alpha_1\alpha_2 a_0 b_1)(1 - \alpha_1\alpha_2 a_1 b_0)b' = 0\,, \end{aligned}$$

we obtain that it reduces to $a = 0$. The Kummer surface degenerates to the double quadric $(\alpha_1\alpha_2 cc'' + bb')^2 = 0$, but the fibre in the family of Equations (23) is reducible: $y_1y_2 = 0$ and $\alpha_2 y_1 + \alpha_1 y_2 + (\alpha_1\alpha_2 cc'' + bb')y_0 = 0$. These are two sections of the $\mathbb{P}^2$-bundle over $\mathbb{P}^3$, tangent along a quadric. So, only above the originally chosen quadric, at $\alpha_1 = \alpha_2 = 0$, we have SRCB manifolds.

JOURNAL INFORMATION

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

Bernd Kreußler https://orcid.org/0009-0003-4977-8585
Jan Stevens https://orcid.org/0000-0001-9099-419X

REFERENCES

1. M. Artin, *Algebraization of formal moduli. II. Existence of modifications*, Ann. of Math. **91** (1970), 88–135.
2. M. F. Atiyah, N. J. Hitchin, and I. M. Singer, *Self-duality in four-dimensional Riemannian Geometry*, Proc. R. Soc. London A **362** (1978), 452–461.
3. H. F. Baker, *Principles of geometry*, vol. IV, Cambridge University Press, Cambridge, 1925.
4. F. Campana, *The class C is not stable by small deformations*, Math. Ann. **290** (1991), 19–30.
5. F. Campana, *The class C is not stable by small deformations. II*, Classification of algebraic varieties (L'Aquila, 1992), Contemp. Math., vol. 162, Amer. Math. Soc., Providence, RI, 1994, pp. 65–76.
6. F. Campana and B. Kreussler, *Existence of twistor spaces of algebraic dimension two over the connected sum of four complex projective planes*, Proc. Amer. Math. Soc. **127** (1999), 2633–2642.
7. S. Donaldson and R Friedman, *Connected sums of self-dual manifolds and deformations of singular spaces*, Nonlinearity **2** (1989), 197–239.
8. R. Friedman, *Simultaneous resolution of threefold double points*, Math. Ann. **274** (1986), 671–689.

9. Th. Friedrich and H. Kurke, *Compact four-dimensional self-dual Einstein manifolds with positive scalar curvature*, Math. Nachr. **106** (1982), 271–299.
10. G.-M. Greuel, C. Lossen, and E. Shustin, *Introduction to Singularities and Deformations*, Springer, Berlin, 2007.
11. N. J. Hitchin, *Kählerian twistor spaces*, Proc. London Math. Soc., III Ser. **43** (1981), 133–150.
12. N. Honda, *On some twistor spaces over* $4\mathbb{CP}^2$, Compositio Math. **122** (2000), 323–336.
13. N. Honda, *Degenerate double solids as twistor spaces*, Comm. Anal. Geom. **10** (2002), 985–998.
14. N. Honda, *Non-Moishezon twistor spaces of* $4\mathbf{CP}^2$ *with non-trivial automorphism group*, Trans. Amer. Math. Soc. **358** (2006), 1897–1920.
15. N. Honda, *Equivariant deformations of LeBrun's self-dual metric with torus action*, Proc. Amer. Math. Soc. **135** (2007), 495–505.
16. N. Honda, *Self-dual metrics and twenty-eight bitangents*, J. Differential Geom. **75** (2007), 175–258.
17. N. Honda, *Double solid twistor spaces: the case of arbitrary signature*, Invent. math. **174** (2008), 463–504.
18. N. Honda, *Explicit construction of new Moishezon twistor spaces*, J. Differential Geom. **82** (2009), 411–444.
19. N. Honda, *A new series of compact minitwistor spaces and Moishezon twistor spaces over them*, J. reine angew. Math. **642** (2010), 197–235.
20. N. Honda, *Moishezon twistor spaces on* $4\mathbb{CP}^2$, J. Algebraic Geom. **23** (2014), 471–538.
21. N. Honda, *Double solid twistor spaces II: general case*, J. reine angew. Math. **698** (2015), 181–220.
22. N. Honda, *Geometry of some twistor spaces of algebraic dimension one*, Complex Manifolds **2** (2015), 105–130.
23. N. Honda, *Twistors, quartics, and del Pezzo fibrations*, Mem. Amer. Math. Soc. **1414** (2023), 1–134.
24. N. Honda and B. Kreussler, *Algebraic dimension of twistor spaces whose fundamental system is a pencil*, J. London Math. Soc. **95** (2017), 989–1010.
25. R. W. H. T. Hudson, *Kummer's quartic surface*, Cambridge University Press, Cambridge, 1905.
26. B. Hunt, *The geometry of some special arithmetic quotients*, Lect. Notes in Math., vol. 1637, Springer, Berlin, 1996.
27. C. M. Jessop, *Quartic surfaces with singular points*, Cambridge University Press, Cambridge, 1916.
28. F. Klein, *Ueber den Verlauf der Abel'schen Integrale bei den Curven vierten Grades. (Zweiter Aufsatz.)*, Math. Ann. **10** (1877), 365–397.
29. B. Kreussler, *Small resolutions of double solids, branched over a 13-nodal quartic surface*, Ann. Global Anal. Geom. **7** (1989), 227–267.
30. B. Kreussler and H. Kurke, *Twistor spaces over the connected sum of 3 projective planes*, Compos. Math. **82** (1992), 25–55.
31. H. Kurke, *Classification of twistor spaces with a pencil of surfaces of degree 1. I*, Math. Nachr. **158** (1992), 67–85.
32. C. LeBrun, *Explicit self-dual metrics on* $\mathbf{C}\mathrm{P}_2\# \cdots \#\mathbf{C}\mathrm{P}_2$, J. Differential Geom. **34** (1991), 223–253.
33. C. LeBrun, *Twistors, Kähler manifolds, and bimeromorphic geometry. I*, J. Amer. Math. Soc. **5** (1992), 289–316.
34. C. LeBrun, *Twistors for tourists: a pocket guide for algebraic geometers*, Algebraic geometry, Proceedings of the Summer Research Institute (Santa Cruz, 1995), Proc. Symp. Pure Math., vol. 62 Part 2, American Mathematical Society, Providence, RI, 1997, pp. 361–385.
35. V. P. Palamodov, *Deformations of complex spaces, Several complex variables. IV. Algebraic aspects of complex analysis*, Encycl. Math. Sci. **10** (1990), 105-194; translation from Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya **10** (1986), 123–221.
36. Th. Peternell, *Modifications, Several complex variables, VII*, Encyclopaedia Math. Sci. **74**, 285–317.
37. A. A. du Plessis and C. T. C. Wall, *Curves in* $\mathrm{P}^2(C)$ *with 1-dimensional symmetry*, Rev. Mat. Complut. **12** (1999), 117–132.
38. Y. S. Poon, *Compact self-dual manifolds with positive scalar curvature*, J. Differential Geom. **24** (1986), 97–132.
39. Y. S. Poon, *On the algebraic structure of twistor spaces*, J. Differential Geom. **36** (1992), 451–491.
40. O. Riemenschneider, *Familien komplexer Räume mit streng pseudokonvexer spezieller Faser*, Comment. Math. Helv. **51** (1976), 547–565.
41. J. Shah, *A complete moduli space for* $K3$ *surfaces of degree 2*, Ann. of Math. **112** (1980), 485–510.
42. O. Zariski, *On the Poincaré group of rational plane curves*, Amer. J. Math. **58** (1936), 607–619.