THESIS FOR THE DEGREE OF LICENTIATE OF ENGINEERING

Photon Statistics in Waveguide Quantum Electrodynamics

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Göteborg, Sweden 2026

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ISSN 1652-0769
Technical Report MC2-477

Cover: Photon detection events, marked with circles, for antibunched, coherent, and bunched light, characterized by their second-order coherence functions $g^{(2)}(\tau)$. Antibunched light shows photon detection events separated in time, while coherent light exhibits randomly distributed events, and bunched light displays clustered events.

Chalmers digitaltryck
Göteborg, Sweden 2026

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Abstract

Nonlinear light-matter interactions have been studied for decades, leading to the discovery of various quantum phenomena, including generation of nonclassical states of light, antibunching, and superradiance. This thesis deals with light-matter interactions between atoms and propagating photonic fields in one-dimensional waveguides. Specifically, we study the scattering of a weak resonant coherent field by N identicial atoms in a waveguide. For atoms separated by the drive wavelength, increasing the number of atoms suppresses transmission while enhancing photon bunching. Transmission becomes a superbunched $(N + 1)$-photon scattering process that is predominantly incoherent. Remarkably, we find that transmission occurs through a process where all N atoms are excited, enabling heralded multi-photon state generation with applications in long-distance entanglement and quantum metrology.

Keywords: waveguide QED, photon statistics, superbunching

Acknowledgments

First and foremost, I would like to thank my supervisor, Göran Johansson, for giving me the chance to do research. I always look forward to having a discussion in your office, your excitement about solving a physics problem is contagious. I would also like to thank my examiner, Janine Splettstösser, for convincing me to write this thesis. It would otherwise not have seen the light of day.

I would also like to thank Maryam Khanahmadi, a coauthor whose mentorship and guidance have been just as valuable as the research we have done together. I am also grateful to Therese Karmstrand for the fruitful collaboration and stimulating discussions. A special thanks to my second supervisor, Laura García-Álvarez, whose guidance goes back to my Bachelor's and Master's theses. While we have yet to collaborate during my PhD, I look forward to working together again.

I am infinitely grateful to my office mates, I would not have had this much fun without you, although I might have gotten more work done. Isak Lyngfelt, we have been friends since the start of our Bachelor's studies and shared nearly every step of this academic path, from our studies and various jobs, to our Bachelor's and Master's theses, and now sharing an office during our PhD. I could not have asked for a better friend. Bruno Bertin-Johannet, although your time at the office did not last very long, you have left a lasting impression. Thank you to the rest of the people at the Applied Quantum Physics Laboratory for a friendly and stimulating environment.

A special thanks to New Yorkeshire Cityjaure, the fact that we all stayed to pursue a PhD at Chalmers is a blessing, I always look forward to all of our lunches. For my LADS, friends since we were kids, I would not be the same without you. Last but not least, thank you to my sisters and parents for your endless support.

Appended paper

This thesis is based on work presented in the following paper:

I **Superbunching from coherently driven atoms in a waveguide**
Zeidan Zeidan, Therese Karmstrand, Maryam Khanahmadi, and Göran Johansson
In review, June 2025, *arXiv:2506.05147* **[quant-ph]**

Contents

1 Introduction

The interaction between light and matter is one of the most fundamental processes in nature. While much of our everyday experience with light can be described classically, at the level of a few atoms and photons, quantum mechanics governs the interaction and gives rise to fundamentally new phenomena. Accessing this regime experimentally was once considered far out of reach. In 1952, Erwin Schrödinger, one of quantum mechanics' pioneers, captured this sentiment when he remarked "*...we never experiment with just one electron or atom or (small) molecule. ...In the first place it is fair to state that we are not experimenting with single particles, any more than we can raise Ichthyosauria in the zoo.*" [1] The idea of isolating and manipulating a single quantum system was, at the time, not much more than a thought experiment. However, over the following decades, experimental advances opened the door to new possibilities.

In 1989, Hans G. Dehmelt and Wolfgang Paul were awarded part of the Nobel Prize in Physics for their work on trapping ions and making it possible to study single ions or electrons [2, 3]. Two decades later, in 2012, Serge Haroche and David Wineland were awarded the Nobel Prize in Physics for their work on methods for observing and manipulating individual quantum systems without destroying them [4, 5]. Haroche trapped photons in microwave cavities and probed their state using atoms, while Wineland trapped ions and used light to control and measure their state. The experiments by Haroche fit under the field of cavity quantum electrodynamics (cavity QED), where atoms interact with photons confined to discrete cavity modes. The developments in cavity QED represent milestones in the broader field of quantum optics, where the quantum nature of light and its interaction with matter is studied. A natural extension of cavity QED is to replace the cavity with an open waveguide, where atoms instead couple to a continuum of propagating photonic modes.

1.1 Waveguide QED

Waveguide quantum electrodynamics (wQED) studies atoms coupled to propagating electromagnetic fields confined to one dimension, enabling strong light-matter coupling [6–8]. Experimentally, wQED has been demonstrated on various platforms, including cold atoms coupled to nanofibers or photonic crystals [9–17], quantum dots in photonic crystals [18], and superconducting circuits [19–26]. The first three platforms comprise photons at optical wavelengths (hundreds of nanometers), while superconducting circuits provide a platform for manipulating microwave photons (centimeter wavelengths).

An early experiment in wQED, demonstrating an effect unique to the one-dimensional geometry of a waveguide, showed a power-dependent reflection from an atom probed by a resonant coherent field [19]. For a strong coherent field, the atom saturates, and transmission approaches unity. Conversely, for a weak input field, the atom's emission in the transmitted field interferes destructively with the input field resulting in almost perfect reflection [27–30]. This effect was demonstrated using a superconducting qubit in a microwave transmission line [19], with up to 99.6% reflection [31, 32]. In contrast, at the time, experiments in free-space showed only up to 12% transmission extinction [7]. For a weak input field, the reflected field was shown to be antibunched, while the transmitted field exhibited bunching [21, 24, 32]. Antibunching, where photons tend to arrive separated in time, is a signature of nonclassical light, while bunching describes the tendency of photons to arrive in pairs or clusters. These photon statistics of the scattered fields reveal the quantum nature of the atom-light interaction and are a focus of this thesis.

When multiple atoms are coupled to a waveguide, the photon-mediated interaction between them gives rise to collective phenomena such as superradiance and subradiance, corresponding to enhanced and suppressed decay, respectively [7, 8, 12, 16, 23, 33]. In Paper I, we study the photon statistics of a weak resonant coherent field scattered by N atoms in a waveguide and show that transmission occurs through a process where all N atoms are excited and the field exhibits superbunching that grows with the number of atoms.

1.2 Outline of the thesis

The remainder of this thesis is organized as follows. Chapters 2 and 3 present the formalism that we use to describe atoms coupled to waveguides. Chapter 2 introduces the theoretical framework of open quantum systems, deriving the Lindblad master equation of an atom in a waveguide and the input-output relations that describe how input fields scatter on an atom and give rise to an output field. In Chapter 3, we present the SLH formalism for cascaded quantum systems and use it to derive the Lindblad master equation and output operators for N atoms in a waveguide. Chapter 4 presents the coherence functions used to characterize the properties of the output fields of light sources. Finally, in Chapter 5, a summary and outlook are given.

2 Open quantum systems

In the study of quantum systems, we distinguish between *closed systems*, which unitarily evolve according to the Schrödinger equation, and *open* quantum systems, which additionally interact with their surrounding environment. In practice, no physical quantum system exists in isolation and will inevitably couple to its environment, significantly modifying the system's dynamics. Establishing a theoretical framework for open quantum systems is thus essential to understand how a system is affected by its environment.

As mentioned, for a closed quantum system, the time evolution of the system is governed by the *Schrödinger equation*:

$$i\hbar \frac{d}{dt} |\psi\rangle = H |\psi\rangle , \tag{2.1}$$

where the state $|\psi\rangle$ describes the system which evolves under the Hamiltonian H. From now on we set $\hbar = 1$ for convenience. The Schrödinger equation describes the evolution of *pure* states. More generally, a quantum state can be described by a density operator. For a pure state $|\psi\rangle$, the density operator is $\rho = |\psi\rangle\langle\psi|$, while for a *mixed* state, the density operator is a statistical mixture of pure states,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| , \tag{2.2}$$

where p_i is the probability of being in the state $|\psi_i\rangle$. The density operator evolves according to the *Liouville–von Neumann equation*:

$$\dot{\rho} = -i[H, \rho]. \tag{2.3}$$

For an open quantum system, in addition to the unitary evolution given by the Hamiltonian, terms are added to Eq. (2.3) to describe evolution of the system due to coupling to its surrounding environment.

There are several ways to model an open quantum system. We will only concern ourselves with the Markovian regime where open quantum systems are governed by the *Lindblad master equation*:

$$\dot{\rho} = -i[H, \rho] + \sum_i \mathscr{D}(L_i)[\rho] \tag{2.4}$$

where $\mathscr{D}(L)[\rho] = L\rho L^\dagger - \frac{1}{2}\{L^\dagger L, \rho\}$ is the Lindblad dissipator. The Lindblad jump operators L_i typically represent dissipative processes whose specific form depends on the system under consideration. In Section 2.1.1, we present the Lindblad master equation for an atom undergoing spontaneous emission.

2.1 Lindblad master equation

The Lindblad master equation may be derived in several ways, e.g. using Kraus operators [34], the Nakajima–Zwanzig projector method [35], or a system-reservoir microscopic approach [34, 36]. Below we follow the microscopic approach [34, 36], where the goal is to understand the assumptions made in deriving the Lindblad master equation. Further inspiration for the derivation has been taken from [37, 38].

We start by considering our system of interest S and environment B, also known as a bath or reservoir, as a closed system with the total Hamiltonian

$$H = H_S + H_B + H_I, \tag{2.5}$$

where H_S and H_B are the Hamiltonians of S and B respectively, and H_I is the interaction Hamiltonian for the system and bath. We let ρ_{SB} be the density operator describing the composite system $S \otimes B$. The reduced density operator, solely describing S is then

$$\rho_S = \mathrm{Tr}_B(\rho_{SB}), \tag{2.6}$$

where we have taken the partial trace over the bath degrees of freedom. As we are not interested in the state of the bath, we aim to derive a master equation merely for the system density operator ρ where bath properties are present/appear as parameters.

We begin our quest in finding a master equation by first moving to an interaction picture by transforming all operators O such that

$$\tilde{O}(t) = e^{i(H_S+H_B)t} O e^{-i(H_S+H_B)t}. \tag{2.7}$$

The time evolution of our system and bath in the interaction picture is then

$$\dot{\tilde{\rho}}_{SB} = -i\left[\tilde{H}_I, \tilde{\rho}_{SB}\right], \tag{2.8}$$

with the formal solution

$$\tilde{\rho}_{SB}(t) = \tilde{\rho}_{SB}(0) - i\int_0^t dt' \left[\tilde{H}_I(t'), \tilde{\rho}_{SB}(t')\right]. \tag{2.9}$$

Substituting this solution back into Eq. (2.8) gives

$$\dot{\tilde{\rho}}_{SB}(t) = -i\left[\tilde{H}_I(t), \tilde{\rho}_{SB}(0)\right] - \int_0^t dt' \left[\tilde{H}_I(t), \left[\tilde{H}_I(t'), \tilde{\rho}_{SB}(t')\right]\right]. \quad (2.10)$$

Up until now all equations are exact as no approximations have been made.

Born approximation: We now assume that there initially exist no correlations between the bath and system and our initial state can be written as a separable state $\tilde{\rho}_{SB}(0) = \tilde{\rho}_S(0) \otimes \tilde{\rho}_B(0)$. Furthermore, we assume the bath to be large and the coupling to be weak such that bath B is not affected by its interaction with the system S. This means that to first order in $\tilde{H}_I$, our state is separable at all times

$$\rho_{SB}(t) = \tilde{\rho}_S(t) \otimes \tilde{\rho}_B(0). \quad (2.11)$$

Tracing over the bath degrees of freedom in Eq. (2.10) and applying the Born approximation gives

$$\dot{\tilde{\rho}}_S(t) = -\int_0^t dt' \,\mathrm{tr}_B\left(\left[\tilde{H}_I(t), \left[\tilde{H}_I(t'), \tilde{\rho}_S(t') \otimes \tilde{\rho}_B(0)\right]\right]\right). \quad (2.12)$$

It is worth noting that the term $\mathrm{tr}_B\left(\left[\tilde{H}_I(t), \tilde{\rho}_{SB}(0)\right]\right)$ has been neglected as any nonzero initial bath operators coupling to the system can be absorbed into the system Hamiltonian [36].

Markov approximation The time evolution of $\tilde{\rho}(t)$ in Eq. (2.12) depends on its past state. However, since the bath is so large and the coupling so weak, any change the system induces on the bath at a time t' quickly leaves the bath's memory and does not return to influence the system at a later time t. This memoryless property of the bath means that we can replace $\tilde{\rho}_S(t')$ with $\tilde{\rho}_S(t)$ in Eq. (2.12) to obtain

$$\dot{\tilde{\rho}}_S(t) = -\int_0^t dt' \,\mathrm{tr}_B\left(\left[\tilde{H}_I(t), \left[\tilde{H}_I(t'), \tilde{\rho}_S(t) \otimes \tilde{\rho}_B(0)\right]\right]\right). \quad (2.13)$$

The time evolution in Eq. (2.13) is referred to as a Redfield equation [34]. Although this equation is local in time, it does not guarantee that the system's density operator remains positive [34, 39–41] and a final approximation must be made.

The final step to arrive at a Lindblad equation requires the secular approximation, which removes rapidly oscillating terms that average to zero on the timescale of system dynamics. To perform this approximation and arrive at a Lindblad type equation as in Eq. (2.4), we will continue the derivation with an illustrative example of a two-level system coupled to a bath of an infinite number of harmonic oscillators in Section 2.1.1. Completing this derivation in detail would be fatiguing and is out of scope for this thesis, the interested reader is instead referred to [34, 36].

2.1.1 Two-level system coupled to a bath

The full derivation of this illustrative example can be found in [37]. Our system of interest S is now an atom modeled as a two-level system with ground and excited states $\{|g\rangle, |e\rangle\}$ and transition frequency ω. The atom is coupled to a bath B of an infinite number of harmonic oscillators such as a waveguide. The Hamiltonian of the composite system is

$$H = H_S + H_B + H_I, \tag{2.14a}$$
$$H_S = \frac{\omega}{2}\sigma_z, \tag{2.14b}$$
$$H_B = \sum_k \omega_k b_k^\dagger b_k, \tag{2.14c}$$
$$H_I = \sum_k g_k(b_k + b_k^\dagger)(\sigma_- + \sigma_+), \tag{2.14d}$$

where $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$ is the Pauli-Z operator, $\sigma_- = |g\rangle\langle e|$ and $\sigma_+ = |e\rangle\langle g|$ are the lowering and raising operators of the atom, and $b_k^\dagger$ (b_k) is the bosonic creation (annihilation) operator of bath mode k with frequency ω_k. The interaction Hamiltonian is given by the dipole approximation where g_k is the coupling strength between atom and bath.

Secular approximation In the interaction picture given by Eq. (2.7), the interaction Hamiltonian H_I consists of both slow and fast oscillating terms. We will assume the rotating wave approximation (RWA) where only the slow oscillating terms depending on $\omega - \omega_k$ are considered while fast oscillating terms depending on $\omega + \omega_k$ are neglected. This approximation is valid when the bath frequencies ω_k are close to the system frequency ω as well the coupling strength g_k being weak compared to these frequencies. Continuing with the derivation from Eq. (2.13) using Eq. (2.14) and applying the RWA gives us a Lindblad type master equation. This step is also called the secular approximation because we only keep the secular, i.e., slowly varying, terms in the evolution. After transforming back from the interaction picture, we obtain the Lindblad master equation for the dynamics of the atom at zero temperature:

$$\dot{\rho}_S = -i\left[\frac{\omega}{2}\sigma_z, \rho_S\right] + \gamma\mathcal{D}(\sigma_-)[\rho_S]. \tag{2.15}$$

This equation now gives us a description of an atom with unitary Hamiltonian evolution and a dissipative term describing spontaneous emission into the environment. The atomic relaxation rate is given by $\gamma = 2\pi J(\omega)g^2$ where $J(\omega)$ is the bath density of states and g is the coupling strength at the atom transition frequency ω. Note that we have ignored the Lamb shift in ω, a small shift in the atom's frequency due to the atom coupling with the bath [35, 37]. The type of Lindblad master equation in Eq. (2.15) forms the basis for describing the atomic dynamics in Paper I.

2.1.2 Approximation validity in the microwave regime

Most commonly in the literature, the approximations leading to the Lindblad master equation are discussed in the optical regime [34, 36, 42]. We will for the sake of variety and personal bias towards superconducting circuits discuss these approximations in the microwave regime.

The validity of the approximations made is governed by the energy scales of the system, the bath, and their coupling. For superconducting waveguides, the system is an artificial atom coupled to a continuum of electromagnetic modes in a transmission line acting as the bath with typical atom transition frequencies $\omega/2\pi \sim 1 - 10$ GHz and coupling rates $\gamma/2\pi \sim 10 - 100$ MHz [7, 8]. The Born approximation requires the coupling to be weak relative to the atom's transition energy, i.e., $\gamma/\omega \sim 10^{-2} - 10^{-3} \ll 1$. The Markov approximation requires the bath correlation time, i.e., the timescale over which the bath retains information of its interaction with the system, to be much shorter than the system's relaxation time $1/\gamma$. The bath correlation time is inversely proportional to its bandwidth and specifically for a superconducting transmission line, the bath is considered to be Ohmic [23] with spectral density that varies linearly up to a cutoff frequency ω_c, which characterizes the bandwidth of the bath. A reasonable choice for ω_c is the superconducting gap of the transmission line material, giving a cutoff frequency on the order of $\omega_c \sim 100$ GHz [43]. The requirement then becomes $\gamma/\omega_c \ll 1$. Finally, the rotating wave approximation is valid when the coupling rate is much smaller than the atom's transition frequency as the fast rotating terms oscillating at 2ω then average to zero on the timescale of the system's dynamics. In this thesis we work in the regime where these approximations are valid, however, they break down in the so-called ultrastrong coupling regime, where $\gamma/\omega \gtrsim 0.1$, which gives rise to new and rich physics [44, 45].

In Section 2.1.1 we presented a master equation at zero temperature. An artificial atom in a superconducting circuit is in practice placed in a dilution refrigerator with temperature $T \sim 10$ mK. For a transition frequency of $\omega/2\pi = 5$ GHz, the ratio between the atom's transition energy and the thermal energy of the environment $\hbar\omega/k_B T \sim (240\,\text{mK})/(10\,\text{mK}) \approx 25$, giving a thermal occupation $\bar{n} \approx e^{-\hbar\omega/k_B T} \approx 10^{-11} \ll 1$, making the zero temperature assumption valid.

2.2 Input-output theory

The master equation presented in Section 2.1 gives us a description of a system coupled to an environment. To complement this, we would like to formulate a theory on how input fields scatter on our system and relate to an output field. The input-output theory for quantum systems was formulated by Gardiner and Collet in [46] and the derivations presented here follow their work. We will once again consider a system, although not specified, coupled to a bath of harmonic oscillators with Hamiltonian

$$\begin{aligned} H &= H_S + H_B + H_I, \\ H_B &= \int_{-\infty}^{+\infty} d\omega \omega b^\dagger(\omega) b(\omega), \\ H_I &= i \int_{-\infty}^{+\infty} d\omega \kappa(\omega) \left(b^\dagger(\omega) a - a^\dagger b(\omega) \right), \end{aligned} \tag{2.16}$$

where $b(\omega)$ are bosonic annihilation operators of the bath satisfying the commutation relation $[b(\omega), b^\dagger(\omega')] = \delta(\omega - \omega')$ and a is a system operator coupling it to the bath. The bath is now considered to be of a continuum of modes where the interaction with the system consists of linear coupling. The RWA has been performed on the interaction Hamiltonian and the frequency-dependent coupling strength is given by $\kappa(\omega)$. The Heisenberg equations of motion for our bath and system are then

$$\dot{b}(\omega) = -i\omega b(\omega) + \kappa(\omega) a, \tag{2.17a}$$

$$\dot{A} = i[H_S, A] - \int_{-\infty}^{+\infty} d\omega \kappa(\omega) \left(b^\dagger(\omega)[a, A] - [a^\dagger, A] b(\omega) \right) \tag{2.17b}$$

where $A(t)$ is an arbitrary system operator. We will now assume the Markov approximation where $\kappa(\omega) = \sqrt{\gamma/2\pi}$, i.e., a flat bath spectrum such that the coupling strength is independent of frequency. The solution for $b(\omega)$ is

$$b(\omega) = e^{-i\omega(t-t_0)} b_0(\omega) + \sqrt{\frac{\gamma}{2\pi}} \int_{t_0}^{t} dt' a(t') e^{-i\omega(t-t')} \tag{2.18}$$

where $b_0(\omega) = b(\omega, t = t_0)$ is the bath operator $b(\omega)$ at time $t_0 < t$. We now define the *input* operator

$$b_{\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega e^{-i\omega(t-t_0)} b_0(\omega), \tag{2.19}$$

which describes the input field interacting with S at time t and obeys the commutation relation $[b_{\text{in}}(t), b_{\text{in}}^\dagger(t')] = \delta(t - t')$. Substituting with Eqs. (2.18) and (2.19) in Eq. (2.17b) gives us the *quantum Langevin equation*:

$$\begin{aligned} \dot{A}(t) = {}& i[H_S, A(t)] - \left(\sqrt{\gamma} b_{\text{in}}^\dagger(t) + \frac{\gamma}{2} a^\dagger(t) \right) [a(t), A(t)] \\ &+ [a^\dagger(t), A(t)] \left(\sqrt{\gamma} b_{\text{in}}(t) + \frac{\gamma}{2} a(t) \right). \end{aligned} \tag{2.20}$$

Furthermore, we are able to solve Eq. (2.17a) backwards in time such that

$$b(\omega) = e^{-i\omega(t-t_1)} b_1(\omega) + \sqrt{\frac{\gamma}{2\pi}} \int_{t}^{t_1} dt' a(t') e^{-i\omega(t-t')} \tag{2.21}$$

where $b_1(\omega) = b(\omega, t = t_1)$ is the bath operator $b(\omega)$ at time $t < t_1$. We then define the *output* operator

$$b_{\text{out}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega e^{-i\omega(t-t_1)} b_1(\omega), \tag{2.22}$$

and proceed as previously to obtain the *time-reversed Langevin equation*:

$$\begin{aligned} \dot{A}(t) = i[H_S, A(t)] &- \left(\sqrt{\gamma} b^\dagger_{\text{out}}(t) - \frac{\gamma}{2} a^\dagger(t)\right) [a(t), A(t)] \\ &+ [a^\dagger(t), A(t)] \left(\sqrt{\gamma} b_{\text{out}}(t) - \frac{\gamma}{2} a(t)\right). \end{aligned} \tag{2.23}$$

The output operator in Eq. (2.22) represents the output field leaving S at time t. The Langevin equation and its time-reversed counterpart from Eqs. (2.20) and (2.23) both give us the solution for the system operator A at time t, and by equating them we find the following relation between the input and output operators

$$b_{\text{out}}(t) = b_{\text{in}}(t) + \sqrt{\gamma} a(t). \tag{2.24}$$

This relation has a simple physical interpretation, the output field is the sum of the input field and the field radiated by the system with coupling rate γ.

2.2.1 Input coherent state

A particularly relevant case for this thesis is the input field being in a coherent state, representing a classical drive, with complex amplitude $\langle b_{\text{in}}(t)\rangle = \alpha(t)$ such that the input operator satisfies

$$\begin{aligned} b_{\text{in}}(t)\rho_{\text{in}} &= \alpha(t)\rho_{\text{in}}, \\ \rho_{\text{in}} b^\dagger_{\text{in}}(t) &= \alpha^*(t)\rho_{\text{in}} \end{aligned} \tag{2.25}$$

where ρ_{in} is the density operator of the input field. To exemplify, we will consider the two-level atom coupled to a bath from Section 2.1.1. The atom is coupled to the bath through the lowering operator $a(t) = \sigma_-(t)$ and the system's Lindblad master equation is reiterated here for clarity:

$$\dot{\rho}_S = -i\left[\frac{\omega}{2}\sigma_z, \rho_S\right] + \gamma\mathcal{D}(\sigma_-)[\rho_S]. \tag{2.26}$$

By considering an input coherent state driving the atom, from the Langevin equation in Eq. (2.20), we can extract a modified system Hamiltonian with an added driving term such that

$$\frac{\omega}{2}\sigma_z \longrightarrow \frac{\omega}{2}\sigma_z - i\sqrt{\gamma}(\alpha\sigma_+ - \alpha^*\sigma_-) \tag{2.27}$$

and the output operator in the atomic Hilbert space is then

$$b_{\text{out}}(t) = \alpha(t) + \sqrt{\gamma}\sigma_-(t). \tag{2.28}$$

This example will be generalized to a system of N atoms in a waveguide in Chapter 3. The output operators will be used in Chapter 4 when computing the photon statistics of output fields which is the basis of Paper I.

3

Cascaded quantum systems

In the previous chapter, we developed a framework for describing a single quantum system coupled to a bath. However, modern experimental setups allow for more complex connected systems where the output of one system becomes the input of another, giving rise to a *cascaded* quantum system. Deriving a master equation and input-output relations for such a system from a microscopic Hamiltonian quickly becomes cumbersome. The SLH framework [47–49] addresses this by providing a mathematical toolset for modeling networks of open quantum systems. It offers a systematic approach to deriving a Lindblad master equation and input-output relations for a cascaded system.

3.1 SLH formalism

In this chapter, the notation of [48] will be adopted. In the SLH framework, each component of a quantum network, whether an atom, coherent drive, or a beam splitter, is represented by a triple. A system with n input-output ports is described by the SLH triple $G = (\mathbf{S}, \mathbf{L}, H)$, where $\mathbf{S}$ is the $n \times n$ scattering matrix of the system, $\mathbf{L}$ is a list of output operators with dimension $n \times 1$, and H is the Hamiltonian of the system. The system with the SLH triple G is then governed by the Lindblad master equation:

$$\dot{\rho} = -i[H, \rho] + \sum_{L \in \mathbf{L}} \mathscr{D}(L)[\rho]. \tag{3.1}$$

The SLH framework then provides algebraic composition rules for combining these components into arbitrarily complex networks or systems.

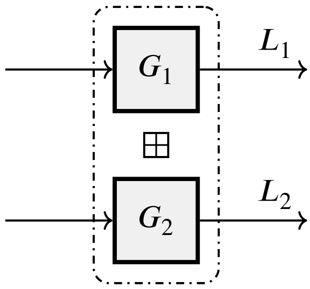

Figure 3.1: Concatenation of two independent systems G_1 and G_2 with output operators $\mathbf{L}_1$ and $\mathbf{L}_2$ respectively.

The first and simplest composition rule in the SLH framework is the *concatenation* of two systems in parallel $G_1 = (\mathbf{S}_1, \mathbf{L}_1, H_1)$ and $G_2 = (\mathbf{S}_2, \mathbf{L}_2, H_2)$:

$$G_1 \boxplus G_2 = \left(\begin{bmatrix} \mathbf{S}_1 & 0 \\ 0 & \mathbf{S}_2 \end{bmatrix}, \begin{bmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \end{bmatrix}, H_1 + H_2 \right). \tag{3.2}$$

The concatenation rule is visualized in Fig. 3.1. One could also consider a direct interaction between the two systems, defined by some interaction Hamiltonian H_I which is added to the total Hamiltonian of the system.

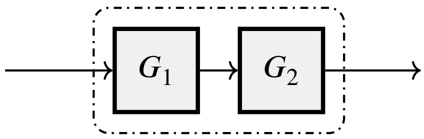

Figure 3.2: Systems G_1 and G_2 in series.

The second rule defines two systems in *series*, i.e., the output of system G_1 is fed in as input of system G_2:

$$G_2 \lhd G_1 = \left(\mathbf{S}_2\mathbf{S}_1, \mathbf{L}_2 + \mathbf{S}_2\mathbf{L}_1, H_1 + H_2 + \frac{1}{2i}\left(\mathbf{L}_2^\dagger \mathbf{S}_2 \mathbf{L}_1 - \mathbf{L}_1^\dagger \mathbf{S}_2^\dagger \mathbf{L}_2 \right) \right). \tag{3.3}$$

Here we see that the series rule is not invariant under the interchange of system 1 and 2. Here we assume both systems have the same number of input-output ports, with the k-th output of G_1 fed into the k-th input of G_2. If we are connecting systems with a different number of ports, then padding elements can be used to ensure compatibility. Furthermore, if we want to change the order of input-outputs, then permutation elements can be used, see [49] for more details.

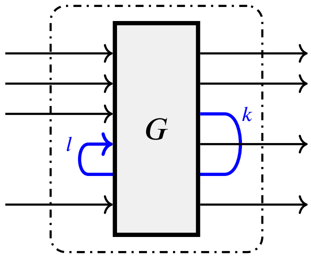

Figure 3.3: The k-th output of a system G is fed back into the l-th input, the feedback is highlighted in blue.

Finally, we will also define the rule of *feedback*, see Fig. 3.3, when the k-th output of a system is fed back into the l-th input:

$$[(\mathbf{S}, \mathbf{L}, H)]_{k\to l} = (\tilde{\mathbf{S}}, \tilde{\mathbf{L}}, \tilde{H}), \tag{3.4}$$

where

$$\begin{aligned}
\tilde{\mathbf{S}} &= \mathbf{S}_{[\not k,\not l]} + \begin{pmatrix} S_{1,l} \\ \vdots \\ S_{k-1,l} \\ S_{k+1,l} \\ \vdots \\ S_{n,l} \end{pmatrix} (1 - S_{k,l})^{-1} \begin{pmatrix} S_{k,1} & \cdots & S_{k,l-1} & S_{k,l+1} & \cdots & S_{k,n} \end{pmatrix}, \\
\tilde{\mathbf{L}} &= \mathbf{L}_{[\not k]} + \begin{pmatrix} S_{1,l} \\ \vdots \\ S_{k-1,l} \\ S_{k+1,l} \\ \vdots \\ S_{n,l} \end{pmatrix} (1 - S_{k,l})^{-1} L_k, \\
\tilde{H} &= H + \frac{1}{2i}\left(\left(\sum_j L_j^\dagger S_{j,l}\right)(1 - S_{k,l})^{-1} L_k - H.C.\right).
\end{aligned} \tag{3.5}$$

The diagonal strike in the subscripts denote that row k and column l has been removed from the original matrix or vector. Although the rule of feedback will not be used in this thesis, it is neat enough to justify mentioning it and can be used to for example derive the SLH triple of an atom placed in front of a mirror [50–52]. The rules of concatenation and composition will be used in Section 3.3 to derive the SLH triple for an ensemble of atoms in a waveguide.

3.2 Basic SLH elements

We now introduce some basic SLH elements that might be useful when constructing a cascaded quantum system. A coherent drive with amplitude α is given by the triple

$$G_\alpha = \left(1, \alpha, 0\right). \tag{3.6}$$

A phase shift is given by

$$G_\phi = \left(e^{i\phi}, 0, 0\right), \tag{3.7}$$

which is particularly useful when connecting spatially separated systems. However, to ensure Markovianity, the travel time between the systems must be shorter than the timescale of the system's evolution. A padding element of dimension k is given by the triple

$$G_I = \left(\mathbf{I}_k, \mathbf{0}, 0\right), \tag{3.8}$$

and is used to match the number of input-output ports when connecting systems of different sizes. Finally, although not used in this thesis, a beam splitter with unitary scattering matrix $\mathbf{B}$ is given by the triple

$$G_B = \left(\mathbf{B}, \mathbf{0}, 0\right). \tag{3.9}$$

3.3 Ensemble of non-homogeneous atoms in a waveguide

In [49], the SLH triple for a pair of atoms in a waveguide supporting counter-propagating modes was derived, and in this section, we follow their method to derive the SLH triple for an ensemble of N non-homogeneous atoms. The distance between atoms will be taken into account by an accumulated phase shift for waves propagating in the waveguide. This is an approximation valid in the Markov regime where the propagation delay is small compared to the systems response time.

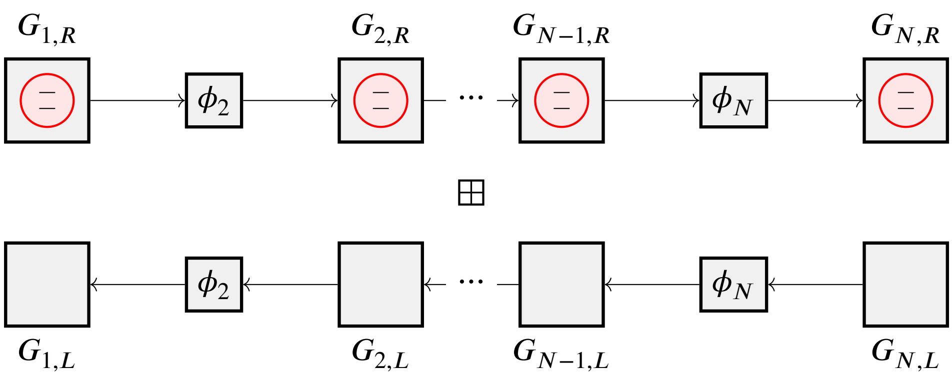

Figure 3.4: Ensemble of atoms in a waveguide

The left and right counter-propagating modes in the waveguide will be taken into account by constructing the SLH triple of the system with two input-output ports according to Fig. 3.4. Notice how we have placed inputs/outputs on both sides to emphasize the physical direction of the propagating light waves, however, typically one would place inputs on one side and outputs on the other as in Fig. 3.1. The SLH triple for the right-propagating modes is then

$$G_R = G_{N,R} \lhd G_{\phi_N} \lhd G_{N-1,R} \lhd \cdots \lhd G_{2,R} \lhd G_{\phi_2} \lhd G_{1,R}. \tag{3.10}$$

The SLH triple for atom i, modelled as a two-level system with transition frequency ω_i and spontaneous decay rate γ_i, is $G_{i,R} = \left(1, \sqrt{\gamma_i}\sigma_-^{(i)}, \frac{\omega_i}{2}\sigma_z^{(i)}\right)$, where $\sigma_-^{(i)}$ is the lower operator and $\sigma_z^{(i)}$ is the Pauli-Z operator. The SLH triple for the phase shift of a wave propagating between atom $j-1$ and j is $G_{\phi_j} = \left(e^{i\phi_j}, 0, 0\right)$. First we use the series rule to incorporate the phase shifts between atoms

$$\begin{aligned} G_R = \left(e^{i\phi_N}, \sqrt{\gamma_N}\sigma_-^{(N)}, \frac{\omega_N}{2}\sigma_z^{(N)}\right) \lhd \left(e^{i\phi_{N-1}}, \sqrt{\gamma_{N-1}}\sigma_-^{(N-1)}, \frac{\omega_{N-1}}{2}\sigma_z^{(N-1)}\right) \lhd \dots \\ \lhd \left(e^{i\phi_2}, \sqrt{\gamma_2}\sigma_-^{(2)}, \frac{\omega_2}{2}\sigma_z^{(2)}\right) \lhd \left(1, \sqrt{\gamma_1}\sigma_-^{(1)}, \frac{\omega_1}{2}\sigma_z^{(1)}\right). \end{aligned} \tag{3.11}$$

Next we use the series rule one step at a time from right to left,

$$\begin{aligned} G_R = &\left(e^{i\phi_N}, \sqrt{\gamma_N}\sigma_-^{(N)}, \frac{\omega_N}{2}\sigma_z^{(N)}\right) \lhd \left(e^{i\phi_{N-1}}, \sqrt{\gamma_{N-1}}\sigma_-^{(N-1)}, \frac{\omega_{N-1}}{2}\sigma_z^{(N-1)}\right) \lhd \dots \\ &\lhd \left(e^{i\phi_4}, \sqrt{\gamma_4}\sigma_-^{(4)}, \frac{\omega_4}{4}\sigma_z^{(4)}\right) \lhd \left(e^{i\phi_3}, \sqrt{\gamma_3}\sigma_-^{(3)}, \frac{\omega_3}{3}\sigma_z^{(3)}\right) \\ &\lhd \left(e^{i\phi_2}, \sqrt{\gamma_2}\sigma_-^{(2)} + e^{i\phi_2}\sqrt{\gamma_1}\sigma_-^{(1)}, \frac{\omega_1}{2}\sigma_z^{(1)} + \frac{\omega_2}{2}\sigma_z^{(2)} + \frac{\sqrt{\gamma_1\gamma_2}}{2i}\left(\sigma_+^{(2)}e^{i\phi_2}\sigma_-^{(1)} - H.C.\right)\right). \end{aligned} \tag{3.12}$$

Using the series rule once more gives,

$$\begin{aligned} G_R = &\left(e^{i\phi_N}, \sqrt{\gamma_N}\sigma_-^{(N)}, \frac{\omega_N}{2}\sigma_z^{(N)}\right) \lhd \left(e^{i\phi_{N-1}}, \sqrt{\gamma_{N-1}}\sigma_-^{(N-1)}, \frac{\omega_{N-1}}{2}\sigma_z^{(N-1)}\right) \lhd \dots \\ &\lhd \left(e^{i\phi_4}, \sqrt{\gamma_4}\sigma_-^{(4)}, \frac{\omega_4}{2}\sigma_z^{(4)}\right) \lhd \Bigg(e^{i\phi_2}e^{i\phi_3}, \sqrt{\gamma_3}\sigma_-^{(3)} + e^{i\phi_3}\sqrt{\gamma_2}\sigma_-^{(2)} + e^{i\phi_3}e^{i\phi_2}\sqrt{\gamma_1}\sigma_-^{(1)}, \\ &\frac{\omega_1}{2}\sigma_z^{(1)} + \frac{\omega_2}{2}\sigma_z^{(2)} + \frac{\omega_3}{2}\sigma_z^{(3)} + \frac{1}{2i}\Big(\sqrt{\gamma_1\gamma_2}\sigma_+^{(2)}e^{i\phi_2}\sigma_-^{(1)} + \sqrt{\gamma_2\gamma_3}\sigma_+^{(3)}e^{i\phi_3}\sigma_-^{(2)} \\ &+ \sqrt{\gamma_2\gamma_3}\sigma_+^{(3)}e^{i\phi_2}e^{i\phi_3}\sigma_-^{(1)} - H.C.\Big)\Bigg). \end{aligned} \tag{3.13}$$

By thinking really hard we arrive at,

$$G_R = \Bigg(\prod_{k=2}^{N} e^{i\phi_k}, \sum_{j=1}^{N} \sqrt{\gamma_j}\sigma_-^{(j)} \prod_{k=j+1}^{N} e^{i\phi_k}, \frac{1}{2}\sum_{j=1}^{N} \omega_j \sigma_z^{(j)} + \frac{1}{2i}\Bigg(\sum_{j=1}^{N}\sum_{l=j+1}^{N} \sqrt{\gamma_j\gamma_l}\sigma_+^{(l)}\sigma_-^{(j)} \prod_{k=j+1}^{l} e^{i\phi_k} - H.C.\Bigg)\Bigg). \tag{3.14}$$

Now we move on to evaluating the SLH triple for the left-propagating modes

$$G_L = G_{1,L} \lhd G_{\phi_2} \lhd G_{2,L} \lhd \cdots \lhd G_{N-1,L} \lhd G_{\phi_N} \lhd G_{N,L}, \tag{3.15}$$

where the SLH triple for atom i is $G_{i,L} = \left(1, \sqrt{\gamma_i}\sigma_-^{(i)}, 0\right)$. Notice that the Hamiltonian is now zero as the atoms' energy is already considered in the SLH triple of the right-propagating mode. It is also worth noting that we have assumed symmetrical emission in the waveguide with the total spontaneous decay rate of each atom being $2\gamma_i$. We proceed with Eq. (3.15) by first incorporating the phase shifts between atoms

$$\begin{aligned} G_L = \left(1, \sqrt{\gamma_1}\sigma_-^{(1)}, 0\right) &\lhd \left(e^{i\phi_2}, e^{i\phi_2}\sqrt{\gamma_2}\sigma_-^{(2)}, 0\right) \lhd \dots \\ &\lhd \left(e^{i\phi_{N-1}}, e^{i\phi_{N-1}}\sqrt{\gamma_{N-1}}\sigma_-^{(N-1)}, 0\right) \lhd \left(e^{i\phi_N}, e^{i\phi_N}\sqrt{\gamma_N}\sigma_-^{(N)}, 0\right). \end{aligned} \tag{3.16}$$

Next we use the series rule from right to left,

$$\begin{aligned} G_L = \left(1, \sqrt{\gamma_1}\sigma_-^{(1)}, 0\right) &\lhd \left(e^{i\phi_2}, e^{i\phi_2}\sqrt{\gamma_2}\sigma_-^{(2)}, 0\right) \lhd \cdots \lhd \left(e^{i\phi_{N-2}}, e^{i\phi_{N-2}}\sqrt{\gamma_{N-2}}\sigma_-^{(N-2)}, 0\right) \\ &\lhd \Bigg(e^{i\phi_{N-1}}e^{i\phi_N}, e^{i\phi_{N-1}}\sqrt{\gamma_{N-1}}\sigma_-^{(N-1)} + e^{i\phi_{N-1}}e^{i\phi_N}\sqrt{\gamma_N}\sigma_-^{(N)}, \\ &\qquad \frac{\sqrt{\gamma_{N-1}\gamma_N}}{2i}\left(\sigma_+^{(N-1)}e^{i\phi_N}\sigma_-^{(N)} - H.C.\right)\Bigg). \end{aligned} \tag{3.17}$$

Using the series rule once more gives us

$$\begin{aligned} G_L = \left(1, \sqrt{\gamma_1}\sigma_-^{(1)}, 0\right) &\lhd \left(e^{i\phi_2}, e^{i\phi_2}\sqrt{\gamma_2}\sigma_-^{(2)}, 0\right) \lhd \cdots \lhd \left(e^{i\phi_{N-3}}, e^{i\phi_{N-3}}\sqrt{\gamma_{N-3}}\sigma_-^{(N-3)}, 0\right) \\ \lhd \Bigg(& e^{i\phi_{N-2}}e^{i\phi_{N-1}}e^{i\phi_N}, e^{i\phi_{N-2}}\sqrt{\gamma_{N-2}}\sigma_-^{(N-2)} + e^{i\phi_{N-2}}e^{i\phi_{N-1}}\sqrt{\gamma_{N-1}}\sigma_-^{(N-1)} \\ &+ e^{i\phi_{N-2}}e^{i\phi_{N-1}}e^{i\phi_N}\sqrt{\gamma_N}\sigma_-^{(N)}, \frac{1}{2i}\Bigg(\sqrt{\gamma_{N-1}\gamma_N}\sigma_+^{(N-1)}e^{i\phi_N}\sigma_-^{(N)} \\ &+ \sqrt{\gamma_{N-2}\gamma_{N-1}}\sigma_+^{(N-2)}e^{i\phi_{N-1}}\sigma_-^{(N-1)} + \sqrt{\gamma_{N-2}\gamma_N}\sigma_+^{(N-2)}e^{i\phi_{N-1}}e^{i\phi_N}\sigma_-^{(N)} - H.C.\Bigg)\Bigg). \end{aligned} \tag{3.18}$$

Following the same pattern, we obtain

$$G_L = \left(\prod_{k=2}^{N} e^{i\phi_k}, \sum_{j=1}^{N} \sqrt{\gamma_j}\sigma_-^{(j)} \prod_{k=2}^{j} e^{i\phi_k}, \frac{1}{2i}\left(\sum_{j=1}^{N} \sum_{l=j+1}^{N} \sqrt{\gamma_j\gamma_l}\sigma_+^{(j)}\sigma_-^{(l)} \prod_{k=j+1}^{l} e^{i\phi_k} - H.C. \right)\right). \tag{3.19}$$

Finally we concatenate the right- and left-propagating modes to obtain the SLH triple of the atomic ensemble

$$\begin{aligned} G_A = G_R \boxplus G_L = &\left(\begin{bmatrix} \prod_{k=2}^{N} e^{i\phi_k} & 0 \\ 0 & \prod_{k=2}^{N} e^{i\phi_k} \end{bmatrix}, \begin{bmatrix} \sum_{j=1}^{N} \sqrt{\gamma_j}\sigma_-^{(j)} \prod_{k=j+1}^{N} e^{i\phi_k} \\ \sum_{j=1}^{N} \sqrt{\gamma_j}\sigma_-^{(j)} \prod_{k=2}^{j} e^{i\phi_k} \end{bmatrix}, \right. \\ &\left. \frac{1}{2}\sum_{j=1}^{N} \omega_j \sigma_z^{(j)} + \frac{1}{2i}\left(\sum_{j=1}^{N} \sum_{l=j+1}^{N} \sqrt{\gamma_j\gamma_l} \left(\sigma_+^{(l)}\sigma_-^{(j)} + \sigma_+^{(j)}\sigma_-^{(l)} \right) \prod_{k=j+1}^{l} e^{i\phi_k} - H.C. \right)\right) \\ &= \left(\mathbf{I}_2 \prod_{k=2}^{N} e^{i\phi_k}, \begin{bmatrix} \sum_{j=1}^{N} \sqrt{\gamma_j}\sigma_-^{(j)} \prod_{k=j+1}^{N} e^{i\phi_k} \\ \sum_{j=1}^{N} \sqrt{\gamma_j}\sigma_-^{(j)} \prod_{k=2}^{j} e^{i\phi_k} \end{bmatrix}, \right. \\ &\left. \frac{1}{2}\sum_{j=1}^{N} \omega_j \sigma_z^{(j)} + \sum_{j=1}^{N} \sum_{l=j+1}^{N} \sqrt{\gamma_j\gamma_l} \left(\sigma_+^{(l)}\sigma_-^{(j)} + \sigma_+^{(j)}\sigma_-^{(l)} \right) \sin\left(\sum_{k=j+1}^{l} \phi_k \right)\right) \end{aligned} \tag{3.20}$$

The SLH triple of the system consists of two output operators, representing the right- and left-propagating modes respectively. We observe that the total phase shifts accumulated by the outgoing light are in reverse order between the two modes, demonstrating the counter-propagating nature of light in the waveguide. Furthermore, the Hamiltonian contains an exchange interaction term between atoms which is proportional to the phase accumulated by light travelling between them. For a microscopic derivation of inhomogeneous atoms in a superconducting transmission line, the interested reader is referred to [22].

3.4 Collective atomic system in a waveguide

For an ensemble of homogenous atoms, the SLH triple in Eq. (3.20) simplifies to

$$\begin{aligned} G_A = &\left(\mathbf{I}_2 \prod_{k=2}^{N} e^{i\phi_k}, \begin{bmatrix} \sqrt{\gamma} \sum_{j=1}^{N} \sigma_-^{(j)} \prod_{k=j+1}^{N} e^{i\phi_k} \\ \sqrt{\gamma} \sum_{j=1}^{N} \sigma_-^{(j)} \prod_{k=2}^{j} e^{i\phi_k} \end{bmatrix}, \right. \\ &\left. \frac{\omega}{2}\sum_{j=1}^{N} \sigma_z^{(j)} + \gamma \sum_{j=1}^{N} \sum_{l=j+1}^{N} \left(\sigma_+^{(l)}\sigma_-^{(j)} + \sigma_+^{(j)}\sigma_-^{(l)} \right) \sin\left(\sum_{k=j+1}^{l} \phi_k \right)\right) \end{aligned} \tag{3.21}$$

where γ is the spontaneous decay rate into the left- and right-going modes, and ω is the transition frequency of the atoms. If we assume the atoms to be separated by wavelength distances, such that $\phi_k = 2\pi \;\forall k \in \{2, \ldots, N\}$, the exchange interaction between the atoms vanishes and the SLH triple further simplifies to

$$G_A = \left(\mathbf{I}_2, \begin{bmatrix} \sqrt{\gamma} S_- \\ \sqrt{\gamma} S_- \end{bmatrix}, \frac{\omega}{2} S_z \right). \tag{3.22}$$

Here we have used the collective atomic dipole operators $S_\pm = \sum_{j=1}^{N} \sigma_\pm^{(j)}$ and the collective atomic inversion operator $S_z = \sum_{i=1}^{N} \sigma_z^{(j)}$. Furthermore, we introduce a coherent input with rate amplitude $\alpha = i\Omega/\sqrt{\gamma}$ for the right-propagating modes, i.e. the atomic ensemble is driven from the left, as shown in Fig. 3.5. We assume the drive to be resonant with the atoms and move to a frame rotating with this frequency, the SLH triple is then

$$\begin{aligned} G_{A,\alpha} &= G_A \lhd \left(\mathbf{I}_2, \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, 0 \right) \\ &= \left(\mathbf{I}_2, \begin{bmatrix} \alpha + \sqrt{\gamma} S_- \\ \sqrt{\gamma} S_- \end{bmatrix}, \frac{\sqrt{\gamma}}{2i} \left(S_+ \alpha - S_- \alpha^* \right) \right) \\ &= \left(\mathbf{I}_2, \begin{bmatrix} \alpha + \sqrt{\gamma} S_- \\ \sqrt{\gamma} S_- \end{bmatrix}, \frac{\Omega}{2} \left(S_+ + S_- \right) \right), \end{aligned} \tag{3.23}$$

where Ω is the Rabi frequency. Notice how we used a padding element for the coherent drive to ensure compatibility when concatenating with the triple for the atoms.

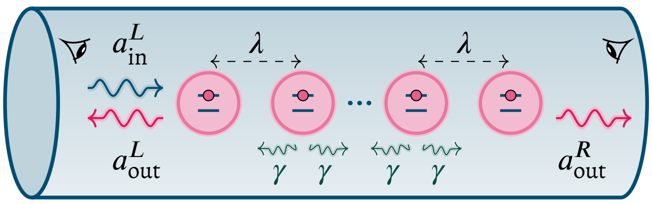

Figure 3.5: N identical atoms, modeled as two-level systems, coupled to a waveguide. The atoms are spaced apart by wavelength distance λ and decay to the left and right propagating fields with spontaneous emission rate γ. The operator a_{in}^L refers to the annihilation operator of the incoming coherent drive from the left and $a_{\text{out}}^{R/L}$ is the operator of the outgoing field propagating toward the right/left.

The Lindblad master equation for the density matrix ρ of the atomic system is then

$$\dot{\rho} = -i\frac{\Omega}{2} \left[S_+ + S_-, \rho \right] + \mathscr{D}(a_{\text{out}}^R)[\rho] + \mathscr{D}(a_{\text{out}}^L)[\rho], \tag{3.24}$$

where $a_{\text{out}}^{R/L}$ is the annihilation operator of the outgoing field propagating toward the right/left. The outgoing field operators are related to the annihilation operators of the incoming field from the right/left, $a_{\text{in}}^{R/L}$, through the input-output relations

$$\begin{aligned} a_{\text{out}}^{L} &= a_{\text{in}}^{R} + \sqrt{\gamma} S_- = \sqrt{\gamma} S_-, \\ a_{\text{out}}^{R} &= a_{\text{in}}^{L} + \sqrt{\gamma} S_- = \alpha + \sqrt{\gamma} S_-, \end{aligned} \tag{3.25}$$

where the last equality gives the form in the atomic system's Hilbert space used in Eq. (3.24). Generally, the Hilbert space of N two-level systems scales as 2^N, however, when initialized in the ground state, the collective atomic system governed by the master equation in Eq. (3.24) evolves within the symmetric $(N+1)$-dimensional Dicke subspace [33, 53]. This is because the coherent drive and collective decay operators only connect states within the symmetric subspace, so starting from the ground state, which belongs to this subspace, the system never leaves it. The steady state of this system has been solved analytically [53–58] and is used in Paper I to calculate the scattered photon rates, providing initial insight into the system's dynamics.

4 Photon statistics

Photon statistics provide a powerful framework for characterizing the properties of light sources. Glauber laid the foundation for quantum coherence functions in [59–61], and in this chapter we follow the treatment in [42, 62, 63]. Having established the output field operators of an atomic ensemble coupled to a waveguide in Chapter 3, we now define the correlation functions that characterize the statistical properties of these output fields.

We restrict ourselves to correlation functions of stationary states, where all expectation values depend only on the time difference τ. Furthermore, we consider correlations of a single output field evaluated at a fixed spatial point, such as a detector, rather than the more general case of correlations between fields at different spatial coordinates. For a comprehensive treatment of correlation functions, see [42, 62, 63].

We begin with the *first-order correlation function* defined as

$$G^{(1)}(\tau) = \langle a^\dagger(t)a(t+\tau)\rangle\,, \tag{4.1}$$

which tells us how the field amplitude at time t relates to the field amplitude at time $t+\tau$. Here a is an operator of the output field of interest, e.g. the output operators $a_{\text{out}}^{R/L}$ from Eq. (3.25). If $\tau = 0$ then $G^{(1)}(0)$ is proportional to the intensity of the light or more precisely the rate of photons in the outgoing field. The normalized first-order correlation function, also called the *first-order coherence function*, is

$$g^{(1)}(\tau) = \frac{\langle a^\dagger(t)a(t+\tau)\rangle}{\langle a^\dagger(t)a(t)\rangle} \tag{4.2}$$

which quantifies the coherence of the outgoing field, i.e., the degree to which the field maintains a phase relationship between the times t and $t+\tau$, and satisfies $|g^{(1)}(\tau)| \leq 1$. Both classical and quantum light fields satisfy these limits.

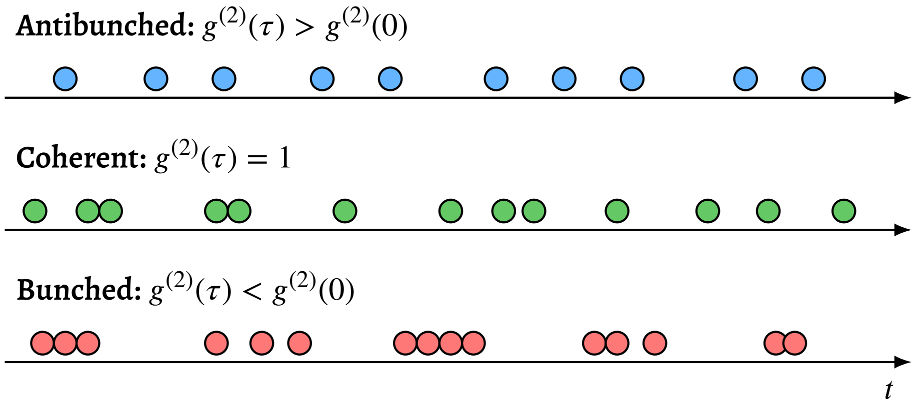

Figure 4.1: Photon detection events, marked with circles, for antibunched, coherent, and bunched light, characterized by their second-order coherence functions $g^{(2)}(\tau)$. Antibunched light shows photon detection events separated in time, while coherent light exhibits randomly distributed events, and bunched light displays clustered events.

To distinguish between classical and quantum light, we turn to the *second-order correlation function*, defined as

$$G^{(2)}(\tau) = \langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle\,, \tag{4.3}$$

with the *second-order coherence function* being

$$g^{(2)}(\tau) = \frac{\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle}{\langle a^\dagger(t)a(t)\rangle^2}. \tag{4.4}$$

The coherence function $g^{(2)}(\tau)$ quantifies the likelihood of detecting a photon at time $t+\tau$ given that a photon was detected at time t relative to independent detections. For a classical light source, the second-order coherence function satisfies $g^{(2)}(0) \geq 1$ and $g^{(2)}(\tau) \leq g^{(2)}(0)$, while a quantum light source can violate both of these conditions. A light source satisfying $g^{(2)}(0) \geq 1$ exhibits super-Poissonian statistics and is called *bunched* if it also satisfies $g^{(2)}(\tau) < g^{(2)}(0)$. Conversely, a light source satisfying $g^{(2)}(0) \leq 1$ exhibits sub-Poissonian statistics and is called *antibunched* if it also satisfies $g^{(2)}(\tau) > g^{(2)}(0)$. It is worth noting that although antibunching and sub-Poissonian statistics are often used interchangeably, they are distinct concepts as a sub-Poissonian light source can still exhibit bunching [64]. The middle ground between antibunched and bunched light is coherent light with $g^{(2)}(\tau) = 1\ \forall\tau$. A coherent light source exhibits Poissonian statistics, where photon detections in a fixed time interval occur at a constant mean rate and are independent of each other. A sub-(super-)Poissonian source then follows a distribution with a smaller (larger) variance than a Poisson distribution with the same mean. Fig. 4.1 illustrates the photon detection patterns for these three light sources. A typical example of an antibunched light source is a single-photon source where $g^{(2)}(0) = 0$ in an ideal case. A laser and a microwave source are typical coherent light sources, and thermal light, also known as chaotic light, is bunched with $g^{(2)}(0) = 2$. A light field with a zero-delay second-order coherence function exceeding that of thermal light, $g^{(2)}(0) > 2$, is referred to as superbunched [65–67]. In Paper I, we compute $g^{(2)}(0)$ for an arbitrary number of atoms in the collective atomic system introduced in Section 3.4 and show that the transmitted field exhibits superbunching that becomes more pronounced with an increasing number of atoms. Though in Paper I we only compute zero-delay correlation functions, the quantum regression formula can be applied to compute correlation functions with a time delay and the interested reader is referred to [36].

Generalizing to arbitrary order, for a zero-delay stationary field, we define the *n-th order correlation function*

$$G^{(n)}(0) = \langle a^\dagger(t)^n a(t)^n \rangle \,, \tag{4.5}$$

and the *n-th order coherence function*

$$g^{(n)}(0) = \frac{\langle a^\dagger(t)^n a(t)^n \rangle}{\langle a^\dagger(t) a(t) \rangle^n}. \tag{4.6}$$

This coherence function quantifies the relative likelihood of simultaneously detecting n photons and gives insight into higher order detection events. In Paper I, we compute $g^{(n)}(0)$ for the output fields of the collective atomic system introduced in Section 3.4.

5 Summary and outlook

In this thesis, we have presented the theoretical framework for studying photon statistics of light scattered by atoms coupled to a one-dimensional waveguide. In Chapter 2, we derived the Lindblad master equation for an open quantum system from a microscopic system-bath Hamiltonian. We complemented the master equation with input-output theory, which relates the output field to the input field and the system operators, providing a description of the scattered fields. In Chapter 3, the SLH formalism for cascaded quantum systems was introduced and used to derive the output field operators for an ensemble of N atoms in a waveguide driven by a coherent field. Finally in Chapter 4, we defined the coherence functions that characterize the statistical properties of the output fields, with particular focus on the second-order coherence function that distinguishes bunched, coherent, and antibunched light.

5.1 Overview of Paper I

In Paper I, we study the photon statistics of a weak resonant coherent field scattered by N identical two-level systems, separated by the drive wavelength, in a waveguide. Using the collective atomic model derived in Section 3.4 and the analytical steady-state solution of the driven Dicke system [53–58], we compute the scattered photon rates and coherence functions of the scattered fields. We find that increasing the number of atoms suppresses transmission while enhancing photon bunching. We show that transmission in the weak-drive regime is a superbunched $(N + 1)$-photon scattering process that is predominantly incoherent, with no phase relation to the input drive. Furthermore, we analytically find that this transmission occurs through a process where all N atoms are collectively excited where the first detection of a transmitted photon in a bunch heralds the atomic system most likely in a fully excited state. We verify our analytical findings with numerical simulations. The bidirectional spontaneous emission from the fully excited atoms produces an entangled two-mode binomial state between the left- and right-propagating modes, enabling a route to heralded multi-photon state generation with applications in entanglement distribution and quantum metrology.

5.2 Outlook

The results of Paper I open several directions for future investigation. It would be valuable to study how imperfections in realistic setups affect the robustness of the super-bunching and the heralded multi-photon states. As we envision experimental implementation on superconducting waveguides, it would be particularly relevant to extend the work to transmon systems and study how anharmonicity, dephasing, and detuning affect the photon bunching robustness. Finally, investigating larger atomic ensembles, where the Markov approximation is no longer valid and where retardation effects become relevant, is highly interesting.

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