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Fiducial inference framework for restricted parameter spaces: poisson mean with background

Chao Chen¹, Shimin Chen¹, Shishi Wang¹, Dongsheng Wang¹, Yanting Chen^{1,2*} and Zhirong Zeng^{1,3*}

Abstract

Objective To address the challenge of constructing valid confidence intervals (CIs) for Poisson means in biomedical low-count experiments (e.g., radiation or molecular counting) with known background signals, where existing methods yield overly conservative intervals due to constraints in parameter space.

Methods We propose a fiducial framework that redefines the fiducial distribution by adjusting for conditional probability within the restricted parameter space. This computationally efficient approach eliminates empty intervals and leverages parameter constraints to ensure frequentist validity.

Results Numerical simulations demonstrate that the proposed CIs are narrower than conventional methods while maintaining nominal coverage probabilities, particularly near boundary conditions. The method was validated using three real-world biomedical/physics datasets.

Conclusion The fiducial approach provides a robust, statistically efficient solution for Poisson mean inference in restricted spaces. It offers improved precision without compromising coverage, making it highly suitable for analyzing low-count data in biomedical and physical sciences.

Keywords restricted space, fiducial, Poisson mean, background parameter

Introduction

Inference to unknown parameters constrained to restricted spaces is a recurring challenge in statistical practice, particularly in fields such as epidemiology and high-energy physics. A canonical example involves estimating the mean λ of a Poisson process in the presence

of a known background rate b , where $\lambda \geq 0$ is required. In radiopharmaceutical dosimetry, where b quantifies inherent device radiation and λ isolates drug-induced exposure; in vaccine safety monitoring, where b reflects historical adverse event baselines and λ captures post-vaccination excess risks; and in cancer screening, where b accounts for false-positive rates in healthy populations while λ identifies true biomarker signals; and in community obesity interventions, where b represents the expected number of new obesity cases under usual conditions and λ estimates the reduction attributable to a targeted health promotion program. Such cross-disciplinary consistency underscores the versatility of Poisson models with background parameters in separating targeted signals from noise across physics and healthcare domains.

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Considerable research and discussion have been conducted on the confidence interval (CI) of the Poisson mean in the presence of background. This includes the work of Feldman and Cousins [1], Giunti [2], Roe and Woodroffe [3, 4], Mandelkern and Schultz [5, 6], Mandelkern [7], Fraser et al. [8], Zhu [9], Coakley et al. [10], Ermini Leaf and Liu [11], Lokhov and Tkachov [12]. The unified method proposed by Feldman and Cousins [1] is simpler and more computationally efficient than other classical methods. Giunti [2] demonstrated that the upper limit of the unified method decreases when b increases and proposed a new ordering principle to modify the unified method. Mandelkern and Schultz [5] found that the CI based on the ordering principle does not adequately account for the known bounds. They therefore suggested using a modified likelihood function to construct a non-negative maximum likelihood estimator in case involving bounded parameters. Roe and Woodroffe [4] used a Bayesian procedure with a uniform improper prior to construct a CI. Zhang and Woodroffe [13] demonstrated that the Bayesian CIs based on a uniform priors remain conservative. Ermini and Liu [11] proposed an elastic belief (EB) method. However, this method cannot improve conservative performance. Some minor improvements can be found in the work of Mandelkern [7], Fraser et al. [8] and Zhu [9].

A critical unresolved issue lies in the inadequate utilization of constraint information during interval construction. Traditional CI constructions often fail to adequately incorporate constraints, leading to overly conservative coverage probabilities and unnecessarily wide intervals that hinder scientific interpretation. When estimators approach parameter space boundaries, conventional methods discard directional information implied by the constraint, artificially inflating interval widths to maintain over-conservative coverage. This problem is exacerbated in low-count regimes, where some methods even yield empty intervals, a statistically invalid and medically implausible outcome.

This paper proposes a fiducial inference framework that systematically integrates parameter space constraints into CI construction. Unlike Bayesian methods, which require prior specification, or frequentist approaches, which rely on ad hoc adjustments, our method redefines the fiducial distribution directly through conditional probability truncation within the restricted space. Fiducial inference has a long history of use in making decisions about model parameters. It can take into account valid observational information and avoid model uncertainties. The fiducial approach is a useful tool for finding solutions to many complex problems with satisfactory frequentist properties (see, for example, Wang and Hannig [14], Hannig

[15], Krishnamoorthy et al. [16], Chen and Pan [17] and Chen et al. [18]).

The rest of the paper is structured as follows. **Fiducial confidence interval for the parameter in a restricted space** formalizes the fiducial framework for restricted parameters, with detailed derivation for the Poisson background case. **Numerical comparison** compares empirical performance against six available methods. **Practical application examples** applies the method to real-world particle data. We conclude with discussions on broader applications in **Discussion**.

Fiducial confidence interval for the parameter in a restricted space

Brief review of fiducial inference

Fiducial inference, pioneered by Fisher [19–21], offers a paradigm for deriving parameter distributions directly from data without requiring prior specifications. Its core philosophy lies in transferring randomness from observations to parameters through data-generating equations (DGEs), which encode the stochastic relationship between data X and parameters θ . A DGE is expressed as:

$$X = A(U, \theta) \quad (1)$$

where U is a pivotal quantity with a fully known distribution P_U , independent of θ . X and $A(U, \theta)$ are identically distributed.

Given $X = x$, solving $X = A(U, \theta)$ for θ yields a fiducial distribution $F_x(\theta)$, which effectively inverts the DGE. To construct the fiducial cumulative distribution function (CDF) of θ without restrictions denoted as $F_x(\theta)$, one must consider X as the parameters of the function and θ as a random variable. Fisher proposed substituting an independent copy $U^* \sim P_U$ into the inverse DGE:

$$\theta = A^{-1}(U^*, x)$$

where $A^{-1}(\bullet)$ represents the algebraic solution for θ , it may be an implicit equation. Repeated sampling of U^* generates realizations from P_U , which encapsulates post-data uncertainty about θ .

We will now illustrate the concept of fiducial inference using a specific example. Consider $X \sim N(\theta, 1)$ with $U \sim N(0, 1)$. The DGE for linking the data to the parameter is $X = \theta + U$. Given $X = x$, solving for θ yields $\hat{\theta} = x - U^*$, and the fiducial distribution for θ is $\theta = x - U^* \sim N(x, 1)$. Here, the fiducial distribution coincides with the Bayesian posterior under a flat prior, yet avoids prior dependence. For more details on fiducial

inference, see Hannig [22], Hannig et al. [14] and Li et al. [23].

Constrained fiducial inference: methodology and implementation

The CI of a restricted parameter can be constructed simply by considering the intersection of the CI and the restricted parameter space [24]. The fiducial CIs for unrestricted parameters may violate constraints (e.g., yielding negative values for Poisson means), necessitating ad hoc truncation. To address this, we propose a systematic adjustment of the fiducial distribution through conditional probability reweighting within the restricted parameter space Θ' . The method ensures intervals inherently satisfy $\theta \in \Theta'$ while optimally utilizing constraint-induced information.

According to Fisher’s argument, the two-sided $1 - \alpha$ fiducial CI for θ is given as $[F_{x;\alpha/2}(\theta), F_{x;1-\alpha/2}(\theta)]$, where $F_{x;\alpha}(\theta)$ denotes the α quantiles of $F_x(\theta)$. Given the restricted condition $\theta \in \Theta'$, the intersection of the CI and the restricted parameter space is

$$[\min\{\max(F_{x;\alpha/2}(\theta), \theta_{min}), \theta_{max}\}, \max\{\min(F_{x;1-\alpha/2}(\theta), \theta_{max}), \theta_{min}\}]$$

where θ_{min} satisfy $\theta_{min} = \min\{\theta : \theta \in \Theta'\}$, and θ_{max} satisfy $\theta_{max} = \max\{\theta : \theta \in \Theta'\}$. If either $F_{x;\alpha/2}(\theta) > \theta_{max}$ or $F_{x;1-\alpha/2}(\theta) < \theta_{min}$, this CI is empty. Although constructing CIs for constrained parameters by intersecting with restricted parameter spaces provides basic feasibility, this approach fundamentally fails to leverage the full informational value of parameter constraints. The critical limitation stems from the post hoc application of constraints rather than their intrinsic integration into the interval construction mechanism, thereby preventing optimal utilization of the a priori structural information embedded in the parameter space restrictions.

We proposed a modified fiducial (MF) approach based on a new fiducial distribution. For the natural parameter space $\Theta = \{\theta : x = A(U^*, \theta)\}$, for all $U^* \in \mathcal{U}$, given $X = x$, there is a $\theta \in \Theta$ that satisfies $x = A(U^*, \theta)$ for any U^* copy of U . For a restricted parameter space Θ' , not all $U^* \in \mathcal{U}$ satisfy that $x = A(U^*, \theta)$. Removing the value of U^* for which there is no solution from the restricted parameter space and then renormalizing the probabilities. The restricted space \mathcal{U}' is $\mathcal{U}'|_{\theta \in \Theta'} = \{U^* : x = A(U^*, \theta) \text{ for all } \theta \in \Theta'\}$. The fiducial distribution of θ is constructed in the restricted space \mathcal{U}' , ensuring that the CI falls within the restricted space Θ' . The CDF of the restricted fiducial distribution $F'_x(\theta)$ is defined as:

$$F'_x(\theta) = \frac{F_x(\theta) - \min(F_x(\theta))}{\max(F_x(\theta)) - \min(F_x(\theta))}, \theta \in \Theta'$$

with truncation outside Θ' . This adjusts probabilities to reflect both data evidence and prior knowledge of Θ' . The quantiles of this truncated distribution are derived by solving $F'_x(L) = \alpha/2$ and $F'_x(U) = 1 - \alpha/2$.

Case 1: Lower-bounded parameter

For $\Theta' = (\theta_0, \infty)$ the $F'_x(\theta)$ is

$$F'_x(\theta) = \begin{cases} 0 & \theta \leq \theta_0 \\ \frac{F_x(\theta) - F_x(\theta_0)}{1 - F_x(\theta_0)} & \theta > \theta_0 \end{cases}$$

Let $[L, U]$ denotes the unrestricted two-sided $1 - \alpha$ CI for θ . If $L \geq \theta_0$, the constrained CI for θ is

$$[L_\theta, U_\theta] = [F'_{x;\alpha/2}(\theta), F'_{x;1-\alpha/2}(\theta)] = [F_{x;\gamma_1}(\theta), F_{x;\gamma_2}(\theta)]$$

where $F'_{x;\alpha}(\theta)$ denotes the α quantiles of $F'_x(\theta)$, $\gamma_1 = F_x(\theta_0) + \alpha/2(1 - F_x(\theta_0))$, and $\gamma_2 = 1 - \alpha/2(1 - F_x(\theta_0))$. If $L < \theta_0$, adjust the lower bound to θ_0 and solve for the upper bound U_θ satisfying:

$$F'_x(U_\theta) = 1 - \alpha \Rightarrow U_\theta = F_x^{-1}(F_x(\theta_0) + (1 - \alpha)(1 - F_x(\theta_0))).$$

We have

$$[L_\theta, U_\theta] = [\theta_0, F_x^{-1}(F_x(\theta_0) + (1 - \alpha)(1 - F_x(\theta_0)))] = [\theta_0, F_{x;\gamma_3}(\theta)].$$

where $\gamma_3 = 1 - \alpha(1 - F_x(\theta_0))$.

Case 2: Upper-bounded parameter

For $\Theta' = (-\infty, \theta_0)$, the $F'_x(\theta)$ is

$$F'_x(\theta) = \begin{cases} \frac{F_x(\theta)}{F_x(\theta_0)} & \theta < \theta_0 \\ 1 & \theta \geq \theta_0 \end{cases}$$

If $U < \theta_0$, $[L_\theta, U_\theta] = [F'_{x;\alpha/2}(\theta), F'_{x;1-\alpha/2}(\theta)] = [F_{x;\gamma_1}(\theta), F_{x;\gamma_2}(\theta)]$. If $U > \theta_0$, fix the upper bound to θ_0 and solve for the lower bound L_θ satisfying:

$$F'_x(L_\theta) = \alpha \Rightarrow L_\theta = F_x^{-1}(\alpha F_x(\theta_0)).$$

Then, $[L_\theta, U_\theta] = [F'_{x;\alpha}(\theta), \theta_0] = [F_{x;\gamma_4}(\theta), \theta_0]$, where $\gamma_4 = \alpha F_x(\theta_0)$.

Case 3: Interval-bounded parameter

For $\Theta' = (\theta_1, \theta_2)$ then

$$F'_x(\theta) = \begin{cases} 0 & \theta \leq \theta_1 \\ \frac{F_x(\theta) - F_x(\theta_1)}{F_x(\theta_2) - F_x(\theta_1)} & \theta_1 < \theta < \theta_2 \\ 1 & \theta \geq \theta_2 \end{cases}$$

If $L < \theta_1$ and $U < \theta_2$, set $[L_\theta, U_\theta] = [\theta_1, F_{x;\gamma_3}(\theta)]$.

If $U > \theta_2$ and $L > \theta_1$, set $[L_\theta, U_\theta] = [F_{x;\gamma_4}(\theta), \theta_2]$.

If $L > \theta_1$ and $U < \theta_2$, set $[L_\theta, U_\theta] = [F_{x;\gamma_1}(\theta), F_{x;\gamma_2}(\theta)]$.

Remark The MF's validity stems from two principles: (1) Probability Redistribution: By conditioning $F_x(\theta)$ on Θ' , we exclude impossible parameter values while preserving relative likelihoods within the valid space. (2) Frequentist Calibration: Implementing this involves four steps (i) compute the unconstrained fiducial distribution $F_x(\theta)$; (ii) evaluate $\min(F_x(\theta))$ and $\max(F_x(\theta))$; (iii) solve adjusted quantiles using the constrained CDF; (iv) truncate results to Θ' . This workflow bypasses the computational complexities associated with Markov chain Monte Carlo or likelihood profiling, necessitating only inverse CDF evaluations.

The proposed method for the Poisson parameter with background

In this section, we propose a CI for the Poisson parameter within a restricted space. Let $X = X_1 + X_2$, where $X_1 \sim \text{Poisson}(\lambda)$, $X_2 \sim \text{Poisson}(b)$, $\lambda > 0$ is the parameter of interest, and $b > 0$ is the known background parameter. Then $X \sim \text{Poisson}(\lambda + b)$, where X is a known observed data and both X_1 and X_2 are unknown. The Poisson model only requires $\theta > 0$, equivalently $\lambda > -b$. However, a negative value of λ is invalid, so the constraint $\theta > b$ is required. Most literature focuses on a small number of observed events, which is an important issue given that the CI for λ is empty.

Simulation studies conducted by Wang [24] demonstrated that satisfactory CIs for parameter θ can be obtained through the intersection of CIs with the restricted parameter space. However, this methodology exhibits a critical limitation: when the observed statistic X falls below threshold b ($X < b$), the derived interval tends to yield an empty set. To address this methodological deficiency, our research focuses on developing a CI that guarantees non-empty estimation under the $X < b$ regime while also maintaining desirable statistical properties such as an appropriate coverage probability and interval precision.

In traditional fiducial approaches, the total mean θ is approximated using a chi-squared distribution, $0.5\chi^2(2x + 1)$. Given $X = x$, the two-sided $1 - \alpha$ CI for θ is $[0.5\chi^2_{\alpha/2}(2x + 1), 0.5\chi^2_{1-\alpha/2}(2x + 1)]$ [25], where $\chi^2_\alpha(df)$ denotes the α quantile of the Chi-square distribution with degrees of freedom df . The fiducial CI for λ is simply constructed by considering the intersection of the CI and the restricted parameter space and is given as

$$[L_f, U_f] = [\max(0, 0.5\chi^2_{\alpha/2}(2x + 1) - b), \max(0, 0.5\chi^2_{1-\alpha/2}(2x + 1) - b)] \quad (2)$$

The naive CI for λ , derived by subtracting b from the bounds of θ , may collapse to an empty set when the lower bound falls below b , particularly for small observed counts X . This limitation underscores the need for a method that explicitly incorporates the parameter space restriction $\theta \geq b$ into the fiducial framework.

To address this issue, we construct a constrained fiducial distribution that reweights the probabilities within the valid parameter space. Let $G_x(\theta)$ denote the CDF of the unrestricted fiducial distribution $0.5\chi^2(2x + 1)$. By truncating θ to the restricted space ($\theta \geq b$) and renormalizing the probability mass, the adjusted fiducial CDF is given as:

$$G'_x(\theta) = \begin{cases} 0 & \theta \leq b \\ \frac{G_x(\theta) - G_x(b)}{1 - G_x(b)} & \theta > b \end{cases}$$

where $G_x(b) = P(0.5\chi^2(2x + 1) < b)$. The quantiles of this truncated distribution are derived by solving $G'_x(L_\theta) = \alpha/2$ and $G'_x(U_\theta) = 1 - \alpha/2$ or $1 - \alpha$, leading to adjusted probability levels:

$$\alpha_1 = G_x(b) + \alpha/2(1 - G_x(b))$$

$$\alpha_2 = 1 - \alpha/2(1 - G_x(b))$$

$$\alpha_3 = 1 - \alpha(1 - G_x(b))$$

If $G_x(b) < \alpha/2$, the corresponding CI for λ is computed as $L_\lambda = 0$ and $U_\lambda = 0.5\chi^2_{\alpha_3}(2x + 1) - b$. If $G_x(b) \geq \alpha/2$, $[L_\lambda, U_\lambda] = [0.5\chi^2_{\alpha_1}(2x + 1) - b, 0.5\chi^2_{\alpha_2}(2x + 1) - b]$. This adjustment ensures non-empty intervals by design, as the lower bound for λ is explicitly truncated at zero. For a given set (x, b, α) , the following R code can be used to fine the $1 - \alpha$ fiducial CI for λ .

```

ci_poisson <- function(x, b, alpha) {
  df <- 2 * x + 1
  two_b <- 2 * b
  chi2_alpha_half <- qchisq(alpha / 2, df)
  if (chi2_alpha_half <= two_b) {
    L <- 0
    U <- qchisq(1 - alpha * (1 - pchisq(two_b, df)), df) / 2 - b
  } else {
    L <- qchisq(pchisq(two_b, df) + alpha / 2 * (1 - pchisq(two_b, df)), df) / 2 - b
    U <- qchisq(1 - alpha / 2 * (1 - pchisq(two_b, df)), df) / 2 - b
  }
  return(c(L, U))
}
ci_poisson(x = 0, b = 3, alpha = 0.1)

```

The method's efficacy stems from its ability to redistribute fiducial probability mass toward the boundary $\theta = b$ when the observed data X suggests proximity to the constraint. Unlike post hoc truncation, this approach leverages the restricted parameter space to refine interval widths without sacrificing coverage probability. For instance, when the unconstrained fiducial distribution assigns significant mass below b , the renormalization process amplifies the relative likelihood of values above b , naturally tightening the interval. The theoretical justification for this redistribution is rooted in the conditional probability framework of Hannig et al. [22], which ensures frequentist validity through calibrated fiducial distributions.

Numerical comparison

To evaluate the performance of the proposed fiducial CI, we conducted Monte Carlo simulations comparing it with seven established methods: the Feldman-Cousins (FC) unified approach [1], Giunti's modified ordering principle [2], Roe-Woodrooffe's (RW) [3] method, the Wald method, the score method [23], and the Elastic Belief (EB) plausibility intervals [11], and the standard fiducial (SF) intervals.

Following Leaf and Liu [11], we adopted the parameter configuration, setting the background mean to $b = 3$ and varying the signal mean λ over a grid of 0.01 increments

within [0,3.8]. The exact coverage probability (ECP) and expected width (EW) were calculated as follow:

$$ECP = \sum_{x=0}^{\infty} \frac{(\lambda + b)^x e^{-(\lambda + b)}}{x!} I(L_{\lambda} \leq \lambda \leq U_{\lambda})$$

$$EW = \sum_{x=0}^{\infty} \frac{(\lambda + b)^x e^{-(\lambda + b)}}{x!} (U_{\lambda} - L_{\lambda})$$

where $I(*)$ is an indicator function.

Figures 1 and 2 illustrates the ECP and EW for eight methods at a 0.9 nominal level, respectively. The oscillatory pattern of ECPs observed in Fig. 1 stems from the discrete nature of the Poisson data and the adjustment mechanism of interval construction under parameter constraints. This is a common phenomenon in interval estimation for constrained parameters within discrete distributions. The simulation analysis reveals distinct performance profiles across the compared methods. The RW and EB methods yield the highest ECPs, indicating a tendency toward conservatism. In contrast, the FC, Giunti, and the proposed MF methods achieve coverage closer to the nominal 0.9 level while being less conservative than RW and EB. The SF intervals show median ECPs nearest to the nominal level; however, they exhibit occasional under-coverage, with the minimum ECP falling to 0.8553. Conversely, the Wald and score intervals

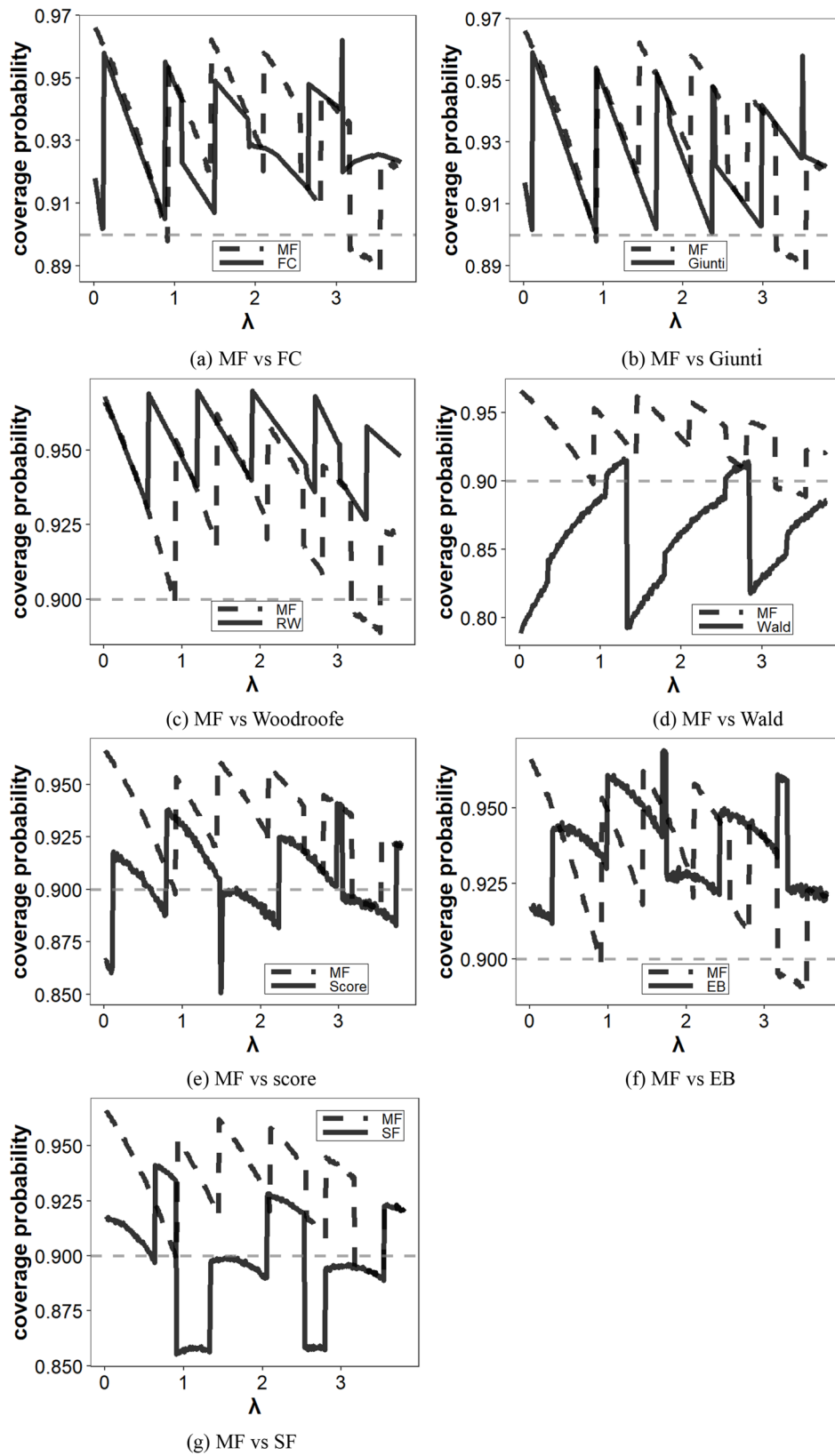


Fig. 1 Coverage probabilities comparisons for the nominal 90% modified fiducial interval against various CIs for $\lambda \in [0, 3.8]$, with $b = 3$

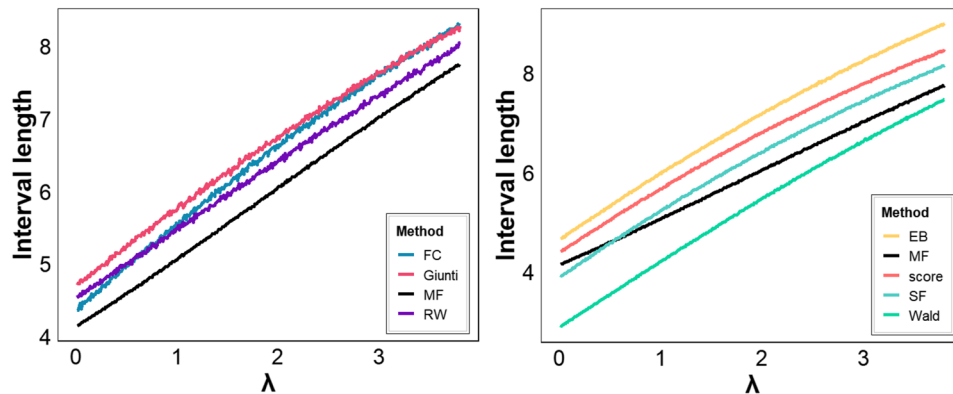


Fig. 2 Interval widths comparisons for the nominal 90% modified fiducial interval against various CIs for $\lambda \in [0, 3.8]$, with $b = 3$

demonstrate liberal behavior, consistently producing coverage below the nominal level.

Figure 2 displays the EWs of the eight methods under identical simulation conditions ($b=3$, $\lambda \in [0, 3.8]$, nominal level 90%). The results further substantiate the advantages of the proposed MF method. The intervals produced by the FC, Giunti, RW, EB, and score methods are consistently wider than those of the MF method. Considering their coverage performance shown earlier, the MF method strikes a more favorable balance between interval precision and coverage reliability than these five methods. Although the Wald interval yields the shortest EW, this apparent gain in precision is achieved at the cost of substantial under-coverage, as evident in Fig. 1. Compared to the SF method, the MF interval is slightly wider for small λ ($\lambda < 0.53$), which is attributable to the fact that the SF method often produces empty intervals in this region. For $\lambda \geq 0.53$, however, the MF method provides narrower intervals than SF while maintaining better coverage properties, demonstrating its overall improvement.

To further illustrate the advantages of the proposed method, we calculated the ECPs and EWs of the SF and MF for various b ($b=1, 5, 8$) and α ($\alpha=0.01, 0.05, 0.1$). The results are shown in Figs. 3 and 4, respectively. Figure 3 demonstrates that, in most scenarios, the MF achieves higher ECPs than the SF. In most cases, particularly when λ is near zero, the MF delivers ECPs that are either greater than or close to the nominal level. In contrast, the coverage performance of the SF method is less satisfactory, as it frequently falls below the nominal level, particularly in certain settings.

Figure 4 compares the interval widths of the SF and MF. A consistent pattern emerges in all scenarios: when the signal mean λ is near zero, the SF method produces slightly shorter EWs, whereas for larger values of λ , the MF method yields narrower EWs. This behavior is directly attributable to the tendency of the SF to generate empty intervals when λ is small. In contrast, the MF method systematically avoids empty intervals and,

as λ increases, provides more precise estimates (shorter widths) while maintaining nominal coverage. This demonstrates that the apparent advantage of SF near zero is a methodological artifact rather than a genuine gain in precision, whereas MF offers improved interval precision in practically relevant regions where the signal is detectable.

The construction of CIs that simultaneously achieve an enhanced ECP and a reduced EW poses significant theoretical challenges. Remarkably, the MF approach demonstrates concurrent improvements in both metrics compared to the SF method across multiple scenarios. Through systematic evaluation under the configuration $b=3$ and $\lambda=0.5$, the SF method yields an ECP of 0.9044 with an EW of 4.6042, while the MF approach achieves superior performance with an ECP 0.9346 (+0.0302) and reduced EW of 4.5874 (-0.0168). As summarized in Table 1, these empirical findings are consistent with our theoretical analysis. The MF enhancement ECP primarily originates from its effective containment of $\lambda = 0.5$ ($CI=[0, 2.08]$) at $X=0$ observations, where the SF produces an empty interval, while maintaining identical coverage patterns to the SF when $X \neq 0$. Furthermore, the EW reduction advantage is particularly pronounced for $X \geq 3$, with the MF method consistently producing more precise interval estimates.

Practical application examples

The practical utility of the proposed fiducial method is demonstrated by applying it to the KARMEN neutrino oscillation experiment. This landmark study in high-energy physics is aimed at detecting rare particle interactions. In this experiment, the observed neutrino events comprise both a signal (neutrino oscillations) and a background (cosmic ray interactions) component. The background mean is estimated to be $b=3$ events. A critical challenge arises when the observed total count X is equal to or less than the background mean. This scenario demands rigorous statistical inference to distinguish potential signals from background fluctuations.

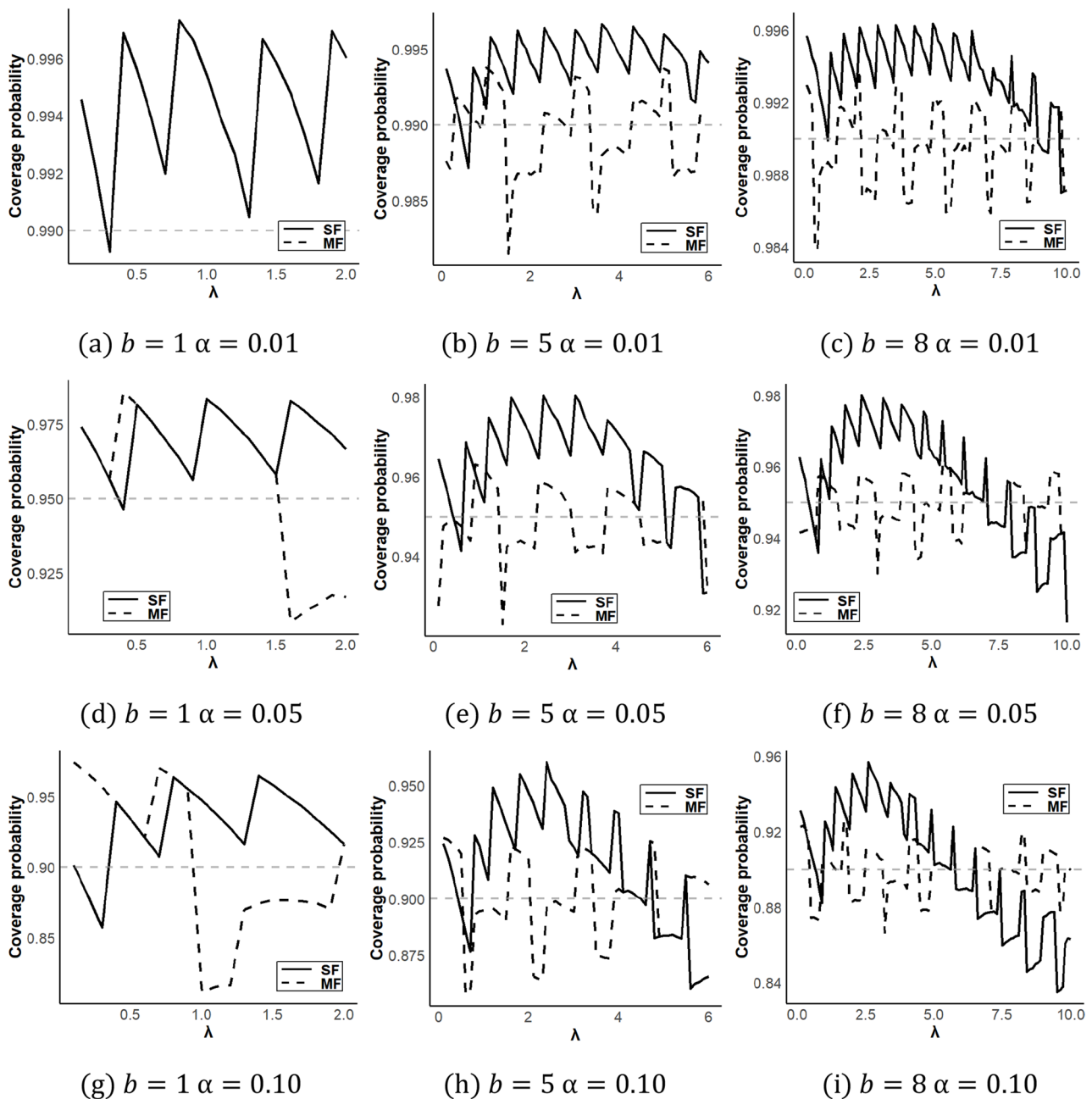


Fig. 3 Coverage probabilities comparisons for modified fiducial CI against standard fiducial CI

For example, when two or less events are observed, traditional methods such as FC ($X = 0, 1, 2$) and Giunti ($X = 0, 1$), as well as standard asymptotic methods (e.g., Wald, score, and EB), tend to produce notably narrow or, in some cases, empty intervals. Such intervals often provide limited practical utility in these settings. In contrast, the Roe–Woodroffe method generally yields considerably wider CIs. This overall trend is clearly illustrated in Table 2.

Applying the fiducial method to this case of $X=2$ yields a 90% CI of $[0, 3.16]$ for the signal mean λ . This

interval is narrower than three CIs, excluding FC ($[0, 3.04]$), and avoids the unphysical empty intervals produced by Wald, score and SF. The precision of the MF interval stems from its adaptive redistribution of probability mass towards the boundary $\theta=b$. This leverages the constraint to refine estimates without inflating ECPs. This example highlights the robustness of the MF method in real-world scenarios where small signal counts are obscured by background noise. By providing statistically valid, non-empty CIs with enhanced precision, the fiducial approach addresses a long-standing

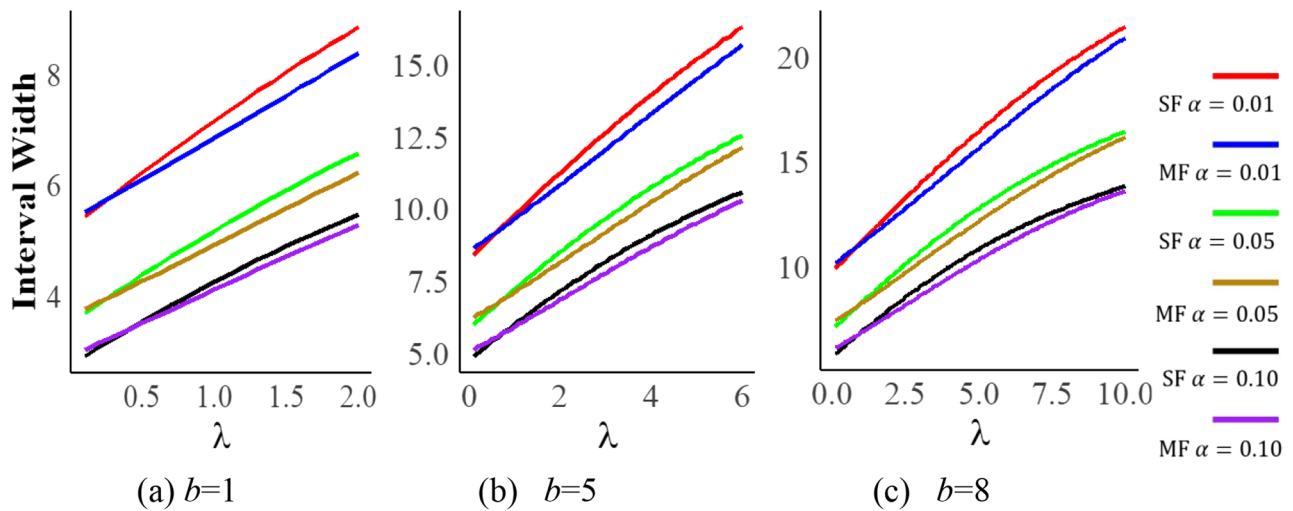


Fig. 4 Interval widths comparisons for modified fiducial CI against standard fiducial CI

Table 1 Probability distribution, 90% confidence intervals (SF vs. MF), and interval widths for $X \sim \text{Poisson}(b + \lambda)$ ($b = 3, \lambda = 0.5$)

X	0	1	2	3	4	5	6	7
P	0.0302	0.1057	0.1850	0.2158	0.1888	0.1322	0.0771	0.0385
SF-CI	0,0	0,0.91	0,2.54	0,4.03	0,5.46	0,6.83	0,8.18	0.63,9.50
MF-CI	0,2.08	0,2.55	0,3.16	0,3.92	0,4.84	0,5.88	0,7.01	0.91,9.54
SF-interval widths	0.00	0.91	2.54	4.03	5.46	6.84	8.18	8.87
MF-interval widths	2.08	2.55	3.16	3.92	4.84	5.88	7.01	8.63

*The interval containing λ is indicated in bold font

Table 2 The 90% CIs for λ when $b = 3$

X	FC	G	RW	Wald	Score	EB	SF	MF
0	0, 1.08	0, 1.82	0, 2.53	00	00	0, 0	0, 0	0, 2.08
1	0, 1.88	0, 2.42	0, 3.09	00	0, 1.48	0, 1.74	0, 0.91	0, 2.55
2	0, 3.04	0, 3.52	0, 3.82	0, 1.33	0, 3.04	0, 3.30	0, 2.54	0, 3.16
3	0, 4.42	0, 4.76	0, 4.17	0, 2.85	0, 4.51	0, 4.75	0, 4.03	0, 3.92
4	0, 5.60	0, 5.69	0, 5.74	0, 4.29	0, 5.91	0, 6.15	0, 5.46	0, 4.84
5	0, 6.99	0, 7.10	0, 6.85	0, 5.68	0, 7.27	0, 7.51	0, 6.84	0, 5.88
6	0.15, 8.47	0.15, 8.54	0, 8.07	0, 7.03	0.10, 8.60	0, 8.84	0, 8.18	0, 7.01
7	0.89, 9.53	0.90, 9.65	0.55, 9.29	0, 8.35	0.80, 9.91	0.29, 10.15	0.63, 9.50	0.91, 9.54
8	1.51, 10.99	1.66, 11.03	0.55, 10.62	0.35, 9.65	1.51, 11.20	0.98, 11.43	1.34, 10.79	1.45, 10.81
9	1.88, 12.30	2.38, 12.30	1.21, 11.91	1.07, 10.93	2.24, 12.47	1.70, 12.71	2.06, 12.07	2.10, 12.08
10	2.63, 13.50	2.98, 13.53	1.90, 13.24	1.80, 12.20	2.98, 13.73	2.43, 13.96	2.80, 13.34	2.81, 13.34
11	3.04, 14.81	3.52, 14.81	2.64, 14.47	2.54, 13.46	3.73, 14.97	3.17, 15.21	3.55, 14.59	3.55, 14.59
12	4.01, 16.00	4.36, 16.03	4.14, 15.69	3.30, 14.70	4.50, 16.21	3.92, 16.44	4.31, 15.83	4.31, 15.83

*Interval widths shorter than those of MF are highlighted in bold

limitation in particle physics and related fields where traditional methods struggle to balance reliability and practicality.

In medical applications of Poisson signal-background models, radiopharmaceutical dosimetry in positron emission tomography imaging accounts for intrinsic device radiation (e.g. $b = 10$ counts/min) to isolate drug-induced signals (λ), where observed counts $X \sim \text{Poisson}(\lambda + 10)$. Similarly, vaccine safety monitoring compares myocarditis incidence in 500,000 vaccinated individuals (e.g.

$X = 8$ cases) against a baseline rate (e.g. $b = 0.1/100,000$ person-months), modeling $X \sim \text{Poisson}(\lambda N + bN)$ to estimate excess risk λ . For prostate cancer screening, age-related prostate-specific antigen fluctuations (e.g. $b = 5$ false positives/1,000 tests) are distinguished from true cancer signals (λ) using $X \sim \text{Poisson}(\lambda + 5)$. These cases exemplify the unified framework $X \sim \text{Poisson}(\lambda + b)$, where constrained intervals for λ avoid non-physical estimates while enhancing precision across physics and healthcare contexts.

Discussion

The proposed fiducial framework addresses a critical issue in constrained parameter inference by systematically integrating constraint information into CI construction. Unlike conventional methods, our approach redefines the fiducial distribution through conditional probability truncation, ensuring that intervals inherently respect parameter bounds while maintaining frequentist validity. This methodology resolves two persistent issues in low-count scenarios: the elimination of empty intervals and the reduction of over-conservatism through adaptive redistribution of probability mass toward constraint boundaries. For example, in the Poisson background problem, the SF method appears to achieve the nominal level $1 - \alpha$. However, this apparent validity arises from the frequent occurrence of empty intervals in SF results when λ approaches zero. In contrast, the MF method not only eliminates empty intervals but also attains comparable ECP while significantly reducing the EW of intervals, a statistical improvement achieved by fully leveraging the background constraint information. The key innovation lies in renormalizing the fiducial distribution to concentrate probability mass within the restricted parameter space. This adjustment not only ensures non-empty intervals but also tightens intervals near boundaries by exploiting constraint-induced information. Computationally, the method requires only chi-square quantile adjustments, avoiding the complexity of Markov Chain Monte Carlo sampling or iterative optimization. Its generality extends beyond Poisson models to other constrained inference problems, such as non-negative normal means and binomial proportions near boundaries. The fiducial approach offers a principled alternative to existing methods, balancing coverage accuracy, interval economy, and computational efficiency. These advantages position the method as a versatile tool for modern statistical challenges, from rare-event detection in particle physics to bounded parameter estimation in public health studies. Future work should focus on adapting the framework to nonparametric constraints and developing open-source software implementations to enhance accessibility.

Our proposed fiducial method exhibits certain limitations in its scope of application. It demonstrates clear superiority in low-count regimes, where conventional intervals often fail or are excessively conservative. However, when the observed count is sufficiently large, the naive CIs performs satisfactorily, and the improvement offered by our method becomes less pronounced. In such settings with larger counts, the naive CI may be a preferable and computationally simpler alternative.

Supplementary Information

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Supplementary Material 1

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Authors' contributions

Z.Z., Y.C. and C.C conceived the study, developed the methodology, performed formal analysis, and wrote the main manuscript text. Y.C. and S.C. implemented the software and curated data. S.W. and D.W. validated the results. C.C. supervised the research, acquired resources, and administered the project. All authors contributed to manuscript review and editing, and approved the final version.

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Data availability

All R codes used in this article is available from the corresponding author upon reasonable request.

Declarations

Ethics approval and contest to participate

No live human or animal was involved in this study. Therefore, consent to participate was not applicable. Moreover, given the exclusively computational/non-interventional nature of this methodology research involving secondary data analysis, no ethics approval was necessary under institutional guidelines.

Consent for publication

Not applicable.

Competing interests

The authors declare no competing interests.

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