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An operator system approach to self-testing

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We develop a general framework for self-testing, in which bipartite correlations are described by states on the commuting tensor product of a pair of operator systems. We propose a definition of a local isometry between bipartite quantum systems in the commuting operator model, and define self-testing and abstract self-testing in the latter generality. We show that self-tests are in the general case always abstract self-tests and that, in some cases, the converse is also true. We apply our framework in a variety of instances, including to correlations with quantum inputs and outputs, quantum commuting correlations for the CHSH game, synchronous correlations, contextuality scenarios, quantum colourings and Schur quantum channels.

Contents

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1. Introduction

About two decades ago, a remarkable feature of quantum computation was uncovered in [39]: quantum devices can be certified either by a user or by an independent third party by only observing their outputs when completing specific quantum computational tasks. This feature became known as *self-testing*, and was intensively studied in the subsequent years, in particular in connection with device-independent quantum cryptography (see e.g. [6,7,47,48]). The notion was used in the celebrated solution of the Tsirelson Problem in [27], and a number of recent advances on the topic, from a variety of perspectives, has been achieved in [10,15,38,40,43,56]. A survey of the concept and associated results can be found in [50].

In its simplest form, namely, the (bipartite) Bell scenario, self-testing rests on the correlation between the behaviours of two non-communicating parties, Alice and Bob, performing a quantum experiment. Each of the two parties holds a quantum system, modelled by two finite dimensional Hilbert spaces H_A and H_B, and share an entangled state, modelled by a unit vector ξ in the tensor product $H_A \otimes H_B$. Alice (resp. Bob) has access to measurement devices, indexed by a finite set X (resp. Y), modelled by POVM's, say $(E_{x,a})_{a\in A}$, $x \in X$, (resp. $(F_{y,b})_{b\in B}$, $y \in Y$), where A (resp. B) is the set of outputs for Alice (resp. Bob). The tuple $M = (H_A, H_B, (E_{x,a})_{a\in A}, (F_{y,b})_{b\in B}, \xi)$ yields a *correlation*, that is, a family p_M of conditional probability distributions over $A \times B$, given by

$$p_M(a, b|x, y) = \langle (E_{x,a} \otimes F_{y,b})\xi, \xi\rangle, \quad x \in X, y \in Y, a \in A, b \in B. \tag{1}$$

We say that the tuple M is a model of p_M; naturally, different models may yield the same correlation p_M. In the Bell scenario, self-testing consists in the fact that, in some cases, the correlation p_M determines uniquely (up to a natural equivalence) the model M. Setups for self-testing, distinct from the Bell scenario, have also been considered; for example, it was shown in [3] that probabilistic assignments of some contextuality scenarios can be self-tested, thus enlarging the capabilities of self-testing beyond classical bipartite experiments.

Recent operator algebraic approaches to self-testing in the Bell scenario were made in [38], [43] and [56]. They rely on the fact that, given the index sets X and A for Alice, there exist an operator system $\mathcal{S}_{X,A}$ (resp. a C*-algebra $\mathcal{A}_{X,A}$) that is universal for families of POVM's $(E_{x,a})_{a\in A}$, indexed over X, in that the latter completely determine the unital completely positive maps (resp. unital *-homomorphisms) on the former. This led to an abstract view on self-testing, developed in [43] and pursued further in [56], which resides in the states of the commuting tensor product $\mathcal{S}_{X,A} \otimes_{\mathrm{c}} \mathcal{S}_{Y,B}$ [29] admitting a unique extension to a state on the maximal tensor product $\mathcal{A}_{X,A} \otimes_{\max} \mathcal{A}_{Y,B}$. It was shown in [43] that, in some cases, self-testing and abstract self-testing coincide.

In a parallel development, the *commuting operator model* of quantum mechanics has been gaining importance over the past decade as a genuinely different (due to [27]) and useful model in quantum information theory; see, for example, [17,18,45,46]. Abstract self-testing was studied in the commuting operator model in [43], but no advances have so far been made on providing a mathematical definition of self-testing in the latter context.

On the other hand, in connections with non-local games with quantum inputs and quantum outputs, no-signalling correlations were extended to the quantum setting in [53], and further used as strategies for such games, notably for games arising from quantum graphs, in [8] and [9]. The question of a suitable framework for self-testing of quantum-to-quantum, or classical-to-quantum, correlations has been, however, open.

The purpose of the present work is to fill both of these gaps. We provide a far-reaching framework for self-testing, which includes as a special case self-testing in the Bell scenario for classical bipartite no-signalling correlations, as well as known instances of self-testing for a single contextuality scenario, simultaneously developing a concept of self-testing in the commuting operator framework. Our setup includes self-testing of all correlation types of current interest, and is developed in a general case, where the correlations between Alice and Bob arise from an arbitrary pair of operator systems. Our approach covers a larger pool of correlations than the ones considered in [13] (e.g. ones not necessarily arising from Bell scenarios), as well as a larger pool of models (namely, quantum commuting ones). Our emphasis is on the mathematical workings of self-testing in the commuting operator framework, as opposed to [13], where the emphasis is on providing a rigorous operational framework of self-testing in the tensor product setting.

We apply our framework in a variety of new instances, including quantum commuting self-testing for the CHSH game, self-testing of classical correlations arising from representations of the Clifford relations, of synchronous correlations of quantum type, of perfect strategies for quantum colourings of classical complete graphs, and of quantum channels arising from Schur multipliers. We note that our results also capture self-testing of probabilistic assignments of contextuality scenarios, capturing in particular [3, Main Theorem].

We describe the content of the paper in more detail. In self-testing of no-signalling correlations of quantum type, a fundamental role for the identification of the uniqueness of the model M (see (1)) is played by *local isometries*; in this case the latter have the form $V_A \otimes V_B$, where $V_A : H_A \to K_A$ and $V_B : H_B \to K_B$ are isometries between the Alice's and Bob's systems, respectively. Local isometries allow to push forward correlation models via operations on each of the systems of Alice and Bob, separately. In the commuting operator model, the tensor splitting $H_A \otimes H_B$ of the joint quantum system is not available; instead, the latter is modelled by a single Hilbert space H. In Section 2 we therefore develop a concept of a local isometry between quantum commuting bipartite systems, which leads to a natural relation of dominance of one system by another. Along with establishing some basic properties, related to composing and tensoring, we show in Theorem 2.10 that the dominance relation is a pre-order.

In Section 3, we propose a notion of a model of a correlation for quantum commuting systems. The quantum commuting correlations between Alice and Bob are represented via states on the commuting tensor product $\mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B$ of operator systems $\mathcal{S}_A$ and $\mathcal{S}_B$, where the former captures the local degrees of freedom of Alice, while the latter – those of Bob. A quantum commuting model over the pair $(\mathcal{S}_A, \mathcal{S}_B)$ is defined as a tuple $S = ({}_{\mathcal{A}}H_{\mathcal{B}}, \varphi_A, \varphi_B, \xi)$, where $\mathcal{A}$ and $\mathcal{B}$ are the von Neumann algebras of local observables for Alice and Bob, respectively, acting on a Hilbert space H, $\xi \in H$ is a pure state, and $\varphi_A : \mathcal{S}_A \to \mathcal{B}(H)$ and $\varphi_B : \mathcal{S}_B \to \mathcal{B}(H)$ are unital completely positive maps with commuting ranges that are contained in the corresponding observable algebras. The model S determines a correlation f_S, given by

$$f_S(u) = \langle (\varphi_A \cdot \varphi_B)(u)\xi, \xi\rangle, \quad u \in \mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B, \tag{2}$$

and a state $\tilde{f}_S$ of the maximal tensor product $C_u^*(\mathcal{S}_A) \otimes_{\max} C_u^*(\mathcal{S}_B)$ of the corresponding universal C*-algebras, by a canonical extension. Local isometries now allow to define a *local dilation* relation $S \preceq \tilde{S}$ between models S and $\tilde{S}$ (of the same correlation); we show that the latter is a pre-order. In fact, it is natural to work in what appears to be a greater generality of the *approximate local dilation* order $S \preceq_{\mathrm{a}} \tilde{S}$, for which we require the existence of a sequence of local isometries that attain $\tilde{S}$ from S in the limit, and which continues to imply that $f_S = f_{\tilde{S}}$. (Approximate) local dilations between models lead naturally to the notion of (weak) self-testing of a given quantum commuting correlation by requiring the existence of a maximal (with respect to the (approximate) local dilation pre-order) model for the given correlation. The main result of Section 3 is Theorem 3.13, a result that extends [43, Proposition 4.10] to the commuting operator framework:

Theorem 1.1. *Let $\mathcal{S}_A$ and $\mathcal{S}_B$ be operator systems and $\mathfrak{M}$ be a family of quantum commuting models over $(\mathcal{S}_A, \mathcal{S}_B)$. Let $\mathcal{S} = \{\tilde{f}_S : S \in \mathfrak{M}\}$ and $f : \mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B \to \mathbb{C}$ be the restriction of an element of $\mathcal{S}$. Assume that $\mathfrak{M}$ contains a centrally supported Haag model of f. If f is a weak self-test for $\mathfrak{M}$ then f is an abstract self-test for $\mathcal{S}$.*

In Section 4, we study further abstract self-tests, providing a characterisation thereof in the general case. In Theorem 4.3, we extend [43, Theorem 3.5], thus establishing a partial converse of Theorem 1.1 for a given class of models $\mathfrak{M}$, where the observable algebras of Alice and Bob are of type I:

Theorem 1.2. *Let $\mathcal{S} = \{\tilde{f}_S : S \in \mathfrak{M}\}$, let $S \in \mathfrak{M}$, and let $f = f_S$. Assume that f is an extreme point of the state space of $\mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B$. Suppose that f has unique extension to a state on $C_u^*(\mathcal{S}_A) \otimes_{\min} C_u^*(\mathcal{S}_B)$. Then f is a self-test for $\mathfrak{M}$ and there exists an irreducible ideal model $S' \in \mathfrak{M}$. In particular, if f is an abstract self-test for $\mathcal{S}$ then f is a self-test for $\mathfrak{M}$ that admits an irreducible ideal model $S' \in \mathfrak{M}$.*

Section 5 is dedicated to applications and examples. In their majority, they are concerned with *finitary* quantum systems, that is, based on a pair of finite dimensional operator systems. In Subsection 5.1, we show how the proposed framework includes self-testing of QNS correlations introduced in [53]. Subsection 5.1 also places the self-testing of POVM correlations [43] in our setup, and explores the relation between the latter and QNS self-testing. En route, we record some new observations about the relation between the universal operator systems for POVM self-testing and QNS self-testing. Example 5.6 demonstrates that self-tests for classical no-signalling correlations cannot be automatically lifted to self-tests for quantum no-signalling correlations. Subsection 5.2 shows that the optimal quantum strategy for the CHSH game persists in being a self-test within the class of quantum commuting correlations.

In Subsection 5.3, we continue the study of self-testing for no-signalling correlations of quantum type. We single out a class such correlations (which we call Clifford correlations), arising from representations of the Clifford relations, and show that synchronous Clifford correlations are abstract self-tests. We employ the NPA hierarchy [41], showing that the latter fact leads to self-tests among the correlations with constraints on their order-two moments.

Our framework can host self-testing for a pair of contextuality scenarios [1]. We note that the framework of contextuality scenarios is designed to include as a special case no-signalling correlations; however, no-signalling is usually studied alongside contextuality. One can show that, even in the case where one of the scenarios consists of a single outcome, this setting may be non-trivial, by casting [3, Main Theorem] in our language.

In Subection 5.4, we show one yet new special case hosted within our framework, namely, classical-to-quantum no-signalling correlations. We show that perfect strategies for the quantum colouring game of classical complete graphs are self-tests. Finally, Subsection 5.5 provides self-testing examples among Schur quantum channels.

The relation between abstract self-testing – requiring a unique extension of a state to a tensor product of C*-covers – and self-testing – defined through the existence of a maximal dilation – is reminiscent of the relation between the unique extension property (UEP) for unital completely positive maps on operator systems and maximality in dilation order [2], wherein C*-envelopes play a crucial role. In fact, this similarity was one of our motivations for developing an operator system approach to self-testing. In order to elucidate this similarity, in Section 6 we show that many of the special cases studied in the literature as well as our new self-testing examples fall in the class of states that can be factored through the tensor product of C*-envelopes of the ground operator systems. We also show that every stochastic operator matrix admits a unitary dilation, allowing us to identify the C*-envelope of the universal operator system of QNS correlations. We leave a deeper investigation of the connections with UEP to future work.

In Section 7, we comment on some questions arising from our work.

Notation. We let $\mathcal{X} \otimes \mathcal{Y}$ be the algebraic tensor product of vector spaces $\mathcal{X}$ and $\mathcal{Y}$, unless the latter are Hilbert spaces, in which case the notation will be reserved for their Hilbertian tensor product. Given Hilbert spaces H and K, we let $\mathcal{B}(H, K)$ be the (Banach) space of all bounded linear operators from H into K; we set $\mathcal{B}(H) = \mathcal{B}(H, H)$. If $\mathcal{A}$ is a unital C*-algebra, we write $1_{\mathcal{A}}$ for its unit, and let $1_H = I_H = 1_{\mathcal{B}(H)}$. The opposite C*-algebra $\mathcal{A}^o$ of $\mathcal{A}$ has the same set-theoretic, linear and involutive structure as $\mathcal{A}$ and, writing its elements as a^o, where $a \in \mathcal{A}$, its multiplication is given by letting $a_1^o a_2^o = (a_2 a_1)^o$, $a_1, a_2 \in \mathcal{A}$. The maximal (resp. minimal) tensor product of C*-algebras $\mathcal{A}$ and $\mathcal{B}$ will be denoted by $\mathcal{A} \otimes_{\max} \mathcal{B}$ (resp. $\mathcal{A} \otimes_{\min} \mathcal{B}$). If $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras, their spatial weak* closed tensor product will be denoted by $\mathcal{A}\bar{\otimes}\mathcal{B}$.

For a finite set X, we let $\{e_x\}_{x\in X}$ denote the canonical orthonormal basis of $\mathbb{C}^X$. We write M_X for the algebra of all $X \times X$ complex matrices, and let $\{\epsilon_{x,x'}\}_{x,x'\in X}$ be the canonical matrix unit system in M_X. We let Tr (resp. tr) be the trace (resp. normalised trace) on M_X; we write $\mathrm{Tr}_{|X|}$ (resp. $\mathrm{tr}_{|X|}$) to improve clarity as needed.

2. Local isometries

In this section, we define a notion of local isometry in the commuting operator framework of quantum mechanics and establish some of its properties that will be used in the sequel. We first recall the tensor product model of composite quantum systems; it serves as a motivating example which we aim to generalise.

Let H_A, K_A, H_B and K_B be Hilbert spaces; they give rise to two (bipartite) quantum mechanical systems with observable algebras $\mathcal{B}(H_A\otimes H_B) = \mathcal{B}(H_A)\bar{\otimes}\mathcal{B}(H_B)$ and $\mathcal{B}(K_A\otimes K_B) = \mathcal{B}(K_A)\bar{\otimes}\mathcal{B}(K_B)$, respectively. An isometry $V : H_A \otimes H_B \to K_A \otimes K_B$ is called *local* if $V = V_A \otimes V_B$ for some isometries $V_A : H_A \to K_A$ and $V_B : H_B \to K_B$. We may view $V_A \otimes V_B$ as a composition in two ways, by introducing the intermediate Hilbert spaces $H_A \otimes K_B$ and $K_A \otimes H_B$ and writing

$$V_A \otimes V_B = (V_A \otimes I_{K_B}) \circ (I_{H_A} \otimes V_B) = (I_{K_A} \otimes V_B) \circ (V_A \otimes I_{H_B});$$

in other words, we have a commutative diagram

$$\begin{array}{ccc} H_A \otimes H_B & \xrightarrow{I_{H_A}\otimes V_B} & H_A \otimes K_B \\ \downarrow{\scriptstyle V_A\otimes I_{H_B}} & & \downarrow{\scriptstyle V_A\otimes I_{K_B}} \\ K_A \otimes H_B & \xrightarrow{I_{K_A}\otimes V_B} & K_A \otimes K_B. \end{array}$$

Towards formulating a suitable extension of the framework described in the previous paragraph, we fix von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$. The next definition recalls the standard notion of a bipartite quantum system that is commonly used (see, for example, [30,31,36,55]).

Definition 2.1. A *bipartite quantum system* over the pair $(\mathcal{A}, \mathcal{B})$ is a pair (H, π), where H is a Hilbert space and $\pi : \mathcal{A} \otimes_{\max} \mathcal{B}^o \to \mathcal{B}(H)$ is a unital $*$-representation, normal in each of the two variables (equivalently, π is a unital representation of the binormal tensor product $\mathcal{A} \otimes_{\text{bin}} \mathcal{B}^o$ [23]).

We will often write $\pi = \pi_H$ if we want to emphasise the underlying Hilbert space. We let $\pi_H(a) = \pi(a \otimes 1)$, $a \in \mathcal{A}$, and $\pi_H(b^o) = \pi(1 \otimes b^o)$, $b^o \in \mathcal{B}^o$. We set

$$a \cdot \xi := \pi_H(a)\xi \;\text{ and }\; \xi \cdot b := \pi_H(b^o)\xi, \quad a \in \mathcal{A},\; b \in \mathcal{B},\; \xi \in H;$$

thus, a bipartite system over $(\mathcal{A}, \mathcal{B})$ is, equivalently, a normal Hilbertian $\mathcal{A}$-$\mathcal{B}$-bimodule [52, §IX.3]. For clarity, we will use the notation ${}_{\mathcal{A}}H_{\mathcal{B}}$ in the place of H, and, by abuse of notation, will refer to ${}_{\mathcal{A}}H_{\mathcal{B}}$ as a biparite quantum system. Heuristically, a bipartite quantum system ${}_{\mathcal{A}}H_{\mathcal{B}}$ is a joint quantum system of parties Alice and Bob; the algebra $\mathcal{A}$ represents Alice's part of the system (namely, the observables accessible to Alice), while the algebra $\mathcal{B}$ – Bob's part of the system (namely, the observables accessible to Bob).

Remark 2.2. If ${}_{\mathcal{A}}H_{\mathcal{B}}$ and ${}_{\mathcal{C}}K_{\mathcal{D}}$ are biparite quantum systems then the tensor product $H \otimes K$ is a bipartite quantum system over the pair $(\mathcal{A}\bar{\otimes}\mathcal{C}, \mathcal{B}\bar{\otimes}\mathcal{D})$ in a canonical fashion. We write $H \otimes K = {}_{\mathcal{A}}H_{\mathcal{B}} \otimes {}_{\mathcal{C}}K_{\mathcal{D}}$.

Definition 2.3. Let $\mathcal{A}, \mathcal{A}_i, \mathcal{B}, \mathcal{B}_i$ be von Neumann algebras, $i = 1, 2$, and ${}_{\mathcal{A}_1}H_{\mathcal{B}}$, ${}_{\mathcal{A}_2}K_{\mathcal{B}}$, ${}_{\mathcal{A}}\tilde{H}_{\mathcal{B}_1}$ and ${}_{\mathcal{A}}\tilde{K}_{\mathcal{B}_2}$ be biparite quantum systems.

(i) An operator $T \in \mathcal{B}(H, K)$ is called $\mathcal{A}_1$-$\mathcal{A}_2$-*local with respect* $\mathcal{B}$ if
  (i') $T(\xi \cdot b) = (T\xi) \cdot b$, $\xi \in H$, $b \in \mathcal{B}$, and
  (i") $T\pi_H(\mathcal{A}_1)T^* \subseteq \pi_K(\mathcal{A}_2)$.
(ii) An operator $\tilde{T} \in \mathcal{B}(\tilde{H}, \tilde{K})$ is called $\mathcal{B}_1$-$\mathcal{B}_2$-*local with respect to* $\mathcal{A}$ if
  (ii') $\tilde{T}(a \cdot \tilde{\xi}) = a \cdot (\tilde{T}\tilde{\xi})$, $\tilde{\xi} \in \tilde{H}$, $a \in \mathcal{A}$, and
  (ii") $T\pi_{\tilde{H}}(\mathcal{B}_1^o)T^* \subseteq \pi_{\tilde{K}}(\mathcal{B}_2^o)$.

If no confusion arises, $\mathcal{A}_1$-$\mathcal{A}_2$-local with respect to $\mathcal{B}$ (resp. $\mathcal{B}_1$-$\mathcal{B}_2$-local with respect to $\mathcal{A}$) operators will be simply referred to as $\mathcal{A}$-*local* (resp. $\mathcal{B}$-*local*). We note that the $\mathcal{B}$-module condition (i') in Definition 2.3 is equivalent to the operator $T : {}_{\mathcal{A}_1}H_{\mathcal{B}} \to {}_{\mathcal{A}_2}K_{\mathcal{B}}$

being an intertwiner between the representations π_H and π_K of $\mathcal{B}$. Intuitively, such an intertwiner is $\mathcal{A}$-local if, in addition, the conjugation by T leaves Alice's algebras of observables globally invariant. Such module and invariance conditions are common features among notions of local operations in algebraic quantum theory (see e.g. [16,17,33,36,55]).

Definition 2.4. Let ${}_{\mathcal{A}_1}H_{\mathcal{B}_1}$ and ${}_{\mathcal{A}_2}K_{\mathcal{B}_2}$ be bipartite quantum systems over $(\mathcal{A}_i, \mathcal{B}_i)$, $i = 1, 2$. An operator $T : {}_{\mathcal{A}_1}H_{\mathcal{B}_1} \to {}_{\mathcal{A}_2}K_{\mathcal{B}_2}$ is called *local* if there exist bipartite quantum systems ${}_{\mathcal{A}_1}L_{\mathcal{B}_2}$ and ${}_{\mathcal{A}_2}\widetilde{L}_{\mathcal{B}_1}$, $\mathcal{A}_1$-$\mathcal{A}_2$ local maps

$$T_{1,1} : {}_{\mathcal{A}_1}H_{\mathcal{B}_1} \to {}_{\mathcal{A}_2}\widetilde{L}_{\mathcal{B}_1} \quad \text{and} \quad T_{2,2} : {}_{\mathcal{A}_1}L_{\mathcal{B}_2} \to {}_{\mathcal{A}_2}K_{\mathcal{B}_2},$$

and $\mathcal{B}_1$-$\mathcal{B}_2$ local maps

$$T_{1,2} : {}_{\mathcal{A}_1}H_{\mathcal{B}_1} \to {}_{\mathcal{A}_1}L_{\mathcal{B}_2} \quad \text{and} \quad T_{2,1} = {}_{\mathcal{A}_2}\widetilde{L}_{\mathcal{B}_1} \to {}_{\mathcal{A}_2}K_{\mathcal{B}_2},$$

for which the diagram

$$\begin{tikzcd} {}_{\mathcal{A}_1}H_{\mathcal{B}_1} \arrow[r, "T_{1,2}"] \arrow[d, "T_{1,1}"] & {}_{\mathcal{A}_1}L_{\mathcal{B}_2} \arrow[d, "T_{2,2}"] \\ {}_{\mathcal{A}_2}\widetilde{L}_{\mathcal{B}_1} \arrow[r, "T_{2,1}"] & {}_{\mathcal{A}_2}K_{\mathcal{B}_2} \end{tikzcd} \tag{3}$$

commutes, and $T = T_{2,2} \circ T_{1,2}$ $(= T_{2,1} \circ T_{1,1})$. If the local maps $T_{i,j}$ can be chosen to be isometric for all i and j, we say that T as a *local isometry*, and write ${}_{\mathcal{A}_1}H_{\mathcal{B}_1} \leq {}_{\mathcal{A}_2}K_{\mathcal{B}_2}$.

The commutative diagram (3) generalises commutativity of operators on a single Hilbert space as well as the natural factorisation of a tensor product of isometries; see Example 2.6. We note that the definition of a local isometry is modelled on the one in the tensor product case, where all analogous maps to $T_{i,j}$ are isometries; therefore we have required this property in the general case as well.

Henceforth, when discussing local isometries we will stick with the convention that the "vertical maps" $T_{1,1}$ and $T_{2,2}$ are $\mathcal{A}$-local and the "horizontal maps" are $\mathcal{B}$-local.

Remark 2.5. **(i)** Let $\mathcal{A}$ and $\mathcal{B}$ be commuting von Neumann algebras acting on a Hilbert space H. Then we can view $H =_{\mathcal{A}} H_{\mathcal{B}^o}$ as an $\mathcal{A}$-$\mathcal{B}^o$-bimodule in the canonical fashion. Let $u \in \mathcal{A}$ and $v \in \mathcal{B}$ be isometries. Then the product $uv : {}_{\mathcal{A}}H_{\mathcal{B}^o} \to {}_{\mathcal{A}}H_{\mathcal{B}^o}$ is a local isometry in the sense of Definition 2.4, with canonical factorisations.

(ii) In the context of item (i), assume that ${}_{\mathcal{A}}H_{\mathcal{B}} \leq {}_{\tilde{\mathcal{A}}}\tilde{H}_{\tilde{\mathcal{B}}}$, implemented by isometries $T_{i,j}$, $i, j = 1, 2$, as in (3). Writing ${}_{\mathcal{A}}L_{\tilde{\mathcal{B}}}$ for the upper right corner of (3), we have that there exists a normal *-representation $\sigma : \mathcal{A} \to \mathcal{B}(L)$ and a normal *-representation $\rho : \sigma(\mathcal{A}) \to \mathcal{B}(\tilde{H})$, such that $\mathrm{ran}(\rho) \subseteq \pi_{\tilde{H}}(\tilde{\mathcal{A}})$, and

$$T_{1,2}a = \sigma(a)T_{1,2} \ \text{ and } \ T_{2,2}\sigma(a) = \rho(\sigma(a))T_{2,2}, \quad a \in \mathcal{A}. \tag{4}$$

Indeed, the existence of σ and the first relation in (4) are a restatement of condition (ii') in Definition 2.3, while ρ can be defined by letting $\rho(\sigma(a)) = T_{2,2}\sigma(a)T_{2,2}^*$, $a \in \mathcal{A}$.

Example 2.6. For a Hilbert space H, let $\bar{H}$ be its dual Banach space, and let $\partial : H \to \bar{H}$ be the conjugate-linear isometry, given by $\partial(\xi)(\eta) = \langle \eta, \xi \rangle$, $\xi, \eta \in H$. We write $\bar{\xi} = \partial(\xi)$, $\xi \in H$. If K is a(nother) Hilbert space, let $\mathcal{S}_2(\bar{K}, H)$ be the Hilbert space of all Hilbert-Schmidt operators from $\bar{K}$ into H, and $\theta : H \otimes K \to \mathcal{S}_2(\bar{K}, H)$ be the unitary operator, given by $\theta(\xi \otimes \eta)(\bar{\zeta}) = \langle \eta, \zeta \rangle \xi$.

Let H_A, K_A, H_B and K_B be Hilbert spaces. We note the canonical identification $\mathcal{B}(H_B)^o = \mathcal{B}(\bar{H}_B)$, where an element $b^o \in \mathcal{B}(H_B)^o$ is identified with the dual operator $\bar{b} : \bar{H}_B \to \bar{H}_B$ of $b \in \mathcal{B}(H_B)$. Consider the normal *-representation $\pi_{H_A \otimes H_B} : \mathcal{B}(H_A) \otimes_{\max} \mathcal{B}(H_B)^o \to \mathcal{B}(H_A \otimes H_B)$, given by

$$\pi_H(a)(\xi) = \theta^{-1}(a\theta(\xi)), \quad \pi_H(b^o)(\xi) = \theta^{-1}(\theta(\xi)\bar{b}),$$

where $a \in \mathcal{B}(H_A), b^o \in \mathcal{B}(H_B)^o$ and $\xi \in H_A \otimes H_B$. We write b^{t} for the operator on H_B for which $\pi_H(b^o) = I_{H_A} \otimes b^{\mathrm{t}}$, $b \in \mathcal{B}(H_B)$. The *-representation $\pi_{H_A \otimes H_B}$ gives rise to a bipartite quantum system ${}_{\mathcal{B}(H_A)}(H_A \otimes H_B)_{\mathcal{B}(H_B)^o}$; we similarly obtain a bipartite quantum system ${}_{\mathcal{B}(K_A)}(K_A \otimes K_B)_{\mathcal{B}(K_B)^o}$. Quantum systems of the latter form will be called *quantum spacial systems*.

Suppose that $V_A : H_A \to K_A$ and $V_B : H_B \to K_B$ are isometries. Let $L = H_A \otimes K_B$ and $\widetilde{L} = K_A \otimes H_B$, and consider the bipartite quantum systems ${}_{\mathcal{B}(H_A)}L_{\mathcal{B}(K_B)^o}$ and ${}_{\mathcal{B}(K_A)}\widetilde{L}_{\mathcal{B}(H_B)^o}$, arising as described in the previous paragraph. Setting $T_{1,1} = V_A \otimes I_{H_B}$, $T_{1,2} = I_{H_A} \otimes V_B$, $T_{2,1} = I_{K_A} \otimes V_B$ and $T_{2,2} = V_A \otimes I_{K_B}$, we obtain a commutative diagram

$$\begin{tikzcd} {}_{\mathcal{B}(H_A)}(H_A \otimes H_B)_{\mathcal{B}(H_B)^o} \arrow[r, "T_{1,2}"] \arrow[d, "T_{1,1}"] & {}_{\mathcal{B}(H_A)}L_{\mathcal{B}(K_B)^o} \arrow[d, "T_{2,2}"] \\ {}_{\mathcal{B}(K_A)}\widetilde{L}_{\mathcal{B}(H_B)^o} \arrow[r, "T_{2,1}"] & {}_{\mathcal{B}(K_A)}(K_A \otimes K_B)_{\mathcal{B}(K_B)^o}, \end{tikzcd} \tag{5}$$

for which $V_A \otimes V_B = T_{2,2}T_{1,2} = T_{2,1}T_{1,1}$. We will call the local isometries of the latter form *split*.

Proposition 2.7. *Let H_A, K_A, H_B and K_B be Hilbert spaces. Every local isometry from ${}_{\mathcal{B}(H_A)}(H_A \otimes H_B)_{\mathcal{B}(H_B)^o}$ to ${}_{\mathcal{B}(K_A)}(K_A \otimes K_B)_{\mathcal{B}(K_B)^o}$ is split.*

Proof. Suppose that $V : H_A \otimes H_B \to K_A \otimes K_B$ is a local isometry in the sense of Definition 2.4, and let ${}_{\mathcal{B}(H_A)}L_{\mathcal{B}(K_B)^o}$ and ${}_{\mathcal{B}(K_A)}\widetilde{L}_{\mathcal{B}(H_B)^o}$ be bipartite quantum systems, while $T_{1,1}, T_{1,2}, T_{2,1}$ and $T_{2,2}$ are local isometries in the diagram (5), such that $V =$

$T_{2,2}T_{1,2} = T_{2,1}T_{1,1}$. There exists a normal *-representation $\rho : \mathcal{B}(H_A) \to \mathcal{B}(L)$ such that $T_{1,2}a = \rho(a)T_{1,2}$, $a \in \mathcal{B}(H_A)$. Up to unitary conjugation, we may thus assume that $L = H_A \otimes L'$ for some Hilbert space L' and $\rho(a) = a \otimes I_{L'}$, $a \in \mathcal{B}(H_A)$. Further, there exists a normal *-representation $\theta : \mathcal{B}(H_B)^o \to \mathcal{B}(H_A \otimes L')$, such that $\theta(b)\rho(a^o) = \rho(a^o)\theta(b)$ for all $a \in \mathcal{B}(H_A)$ and all $b \in \mathcal{B}(H_B)$. It follows that, up to unitary conjugation, $L' = H_B \otimes L_0$ and $\theta(b^o) = I_{H_A} \otimes b^{\mathrm{t}} \otimes I^{\mathrm{t}}_{L_0}$, $b \in \mathcal{B}(H_B)$.

We may similarly assume that $\widetilde{L} = K_A \otimes H_B \otimes \widetilde{L}_0$ for some Hilbert spaces L_0 and $\widetilde{L}_0$, and that

$$\pi_L(a) = a \otimes I_{K_B} \otimes I_{L_0} \quad \text{and} \quad \pi_L(b^o) = I_{H_A} \otimes b^{\mathrm{t}} \otimes I^{\mathrm{t}}_{L_0},$$

for $a \in \mathcal{B}(H_A)$ and $b \in \mathcal{B}(K_B)$, and

$$\pi_{\widetilde{L}}(a) = a \otimes I_{H_B} \otimes I_{\widetilde{L}_0} \quad \text{and} \quad \pi_{\widetilde{L}}(b^o) = I_{K_A} \otimes b^{\mathrm{t}} \otimes I^{\mathrm{t}}_{\widetilde{L}_0},$$

for $a \in \mathcal{B}(K_A)$ and $b \in \mathcal{B}(H_B)$. Condition (ii') in Definition 2.3 implies that, for every $h_1, h_2 \in H_A$, $\xi \in H_B$, $\eta \in K_B \otimes L_0$ and $a \in \mathcal{B}(H_A)$, we have

$$\langle T_{1,2}(ah_1 \otimes \xi), h_2 \otimes \eta\rangle = \langle T_{1,2}(h_1 \otimes \xi)), a^* h_2 \otimes \eta\rangle.$$

It follows that

$$(\mathrm{id}_{H_A} \otimes \omega_{\xi,\eta})(T_{1,2}) \in \mathcal{B}(H_A)' = \mathbb{C} I_{H_A}.$$

Letting $\lambda_{\xi,\eta} \in \mathbb{C}$ with $(\mathrm{id}_{H_A} \otimes \omega_{\xi,\eta})(T_{1,2}) = \lambda_{\xi,\eta} I_{H_A}$, it is straightforward to verify that the map $(\xi, \eta) \to \lambda_{\xi,\eta}$ is conjugate-linear and bounded, therefore yielding a bounded operator $W_B : H_B \to K_B \otimes L_0$, such that

$$\lambda_{\xi,\eta} = \langle W_B \xi, \eta\rangle, \quad \xi \in H_B, \eta \in K_B \otimes L_0.$$

Thus $T_{1,2} = I_{H_A} \otimes W_B$ and, since $T_{1,2}$ is an isometry, so is W_B.

Condition (ii") in Definition 2.3 now reads

$$(I_{H_A} \otimes W_B)(I_{H_A} \otimes \mathcal{B}(H_B)^o)(I_{H_A} \otimes W_B^*) \subseteq I_{H_A} \otimes \mathcal{B}(K_B)^o \otimes I_{L_0},$$

that is,

$$W_B \mathcal{B}(H_B)^o W_B^* \subseteq \mathcal{B}(K_B)^o \otimes I_{L_0}. \tag{6}$$

If $b^o \in \mathcal{B}(H_B)^o$ is a rank one operator, then the left hand side in (6) is a rank one operator, implying that $\dim L_0 = 1$. Similarly, $\dim \widetilde{L}_0 = 1$, and there exist isometries $\widetilde{W_B} : H_B \to K_B$, $W_A : H_A \to K_A$ and $\widetilde{W_A} : H_A \to K_A$ such that

$$T_{2,1} = I_{K_A} \otimes \widetilde{W_B}, \quad T_{1,1} = W_A \otimes I_{H_B}, \quad \text{and} \quad T_{2,2} = \widetilde{W_A} \otimes I_{K_B}.$$

It follows that $V = \widetilde{W_A} \otimes W_B = W_A \otimes \widetilde{W_B}$, implying that $W_A = \widetilde{W_A}$ and $W_B = \widetilde{W_B}$. The proof is complete. $\square$

Example 2.8. Let ${}_{\mathcal{A}}H_{\mathcal{C}}$, ${}_{\mathcal{A}}\widetilde{H}_{\mathcal{C}}$, ${}_{\mathcal{C}}K_{\mathcal{B}}$ and ${}_{\mathcal{C}}\widetilde{K}_{\mathcal{B}}$ be bipartite quantum system, for some von Neumann algebras $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Suppose that $V : {}_{\mathcal{A}}H_{\mathcal{C}} \to {}_{\mathcal{A}}\widetilde{H}_{\mathcal{C}}$ is an $\mathcal{A}$-local isometry and $W : {}_{\mathcal{C}}K_{\mathcal{B}} \to {}_{\mathcal{C}}\widetilde{K}_{\mathcal{B}}$ is a $\mathcal{B}$-local isometry. Fix a faithful normal semi-finite weight φ on $\mathcal{C}$ (or a faithful normal state if $\mathcal{C}$ is σ-finite), and let $\boxtimes$ denote Connes' fusion product of bimodules relative to φ [52, Definition IX.3.16]. Then the commuting diagram

$$\begin{tikzcd}
{}_{\mathcal{A}}H_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}K_{\mathcal{B}} \arrow[r, "I_H \otimes W"] \arrow[d, "V \otimes I_K"] & {}_{\mathcal{A}}H_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\widetilde{K}_{\mathcal{B}} \arrow[d, "V \otimes I_{\widetilde{K}}"] \\
{}_{\mathcal{A}}\widetilde{H}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}K_{\mathcal{B}} \arrow[r, "I_{\widetilde{H}} \otimes W"] & {}_{\mathcal{A}}\widetilde{H}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\widetilde{K}_{\mathcal{B}}
\end{tikzcd}$$

gives rise to a local isometry $V \boxtimes W : {}_{\mathcal{A}}H_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}K_{\mathcal{B}} \to_{\mathcal{A}} \widetilde{H}_{\mathcal{C}} \boxtimes {}_{\mathcal{C}}\widetilde{K}_{\mathcal{B}}$.

Remark 2.9. (i) It is straightforward to see that, if $\mathcal{A}_i, \mathcal{B}$ (resp. $\mathcal{A}, \mathcal{B}_i$) are von Neumann algebras, ${}_{\mathcal{A}_i}H^{(i)}_{\mathcal{B}}$ (resp. ${}_{\mathcal{A}}K^{(i)}_{\mathcal{B}_i}$) are biparite quantum systems, $i = 1, 2, 3$, and $S_1 \in \mathcal{B}(H^{(1)}, H^{(2)})$ and $S_2 \in \mathcal{B}(H^{(2)}, H^{(3)})$ (resp. $T_1 \in \mathcal{B}(K^{(1)}, K^{(2)})$ and $T_2 \in \mathcal{B}(K^{(2)}, K^{(3)})$) are $\mathcal{A}$-local (resp. $\mathcal{B}$-local) operators, then the operator S_2S_1 (resp. T_2T_1) is $\mathcal{A}$-local (resp. $\mathcal{B}$-local).

(ii) Let ${}_{\mathcal{A}_1}H_{\mathcal{B}_1}$ and ${}_{\mathcal{A}_2}K_{\mathcal{B}_2}$ (resp. ${}_{\tilde{\mathcal{A}}_1}\tilde{H}_{\tilde{\mathcal{B}}_1}$ and ${}_{\tilde{\mathcal{A}}_2}\tilde{K}_{\tilde{\mathcal{B}}_2}$) be bipartite quantum systems over $(\mathcal{A}_i, \mathcal{B}_i)$ (resp. $(\tilde{\mathcal{A}}_i, \tilde{\mathcal{B}}_i)$), $i = 1, 2$. If $T : {}_{\mathcal{A}_1}H_{\mathcal{B}_1} \to {}_{\mathcal{A}_2}K_{\mathcal{B}_2}$ and $\tilde{T} : {}_{\tilde{\mathcal{A}}_1}\tilde{H}_{\tilde{\mathcal{B}}_1} \to {}_{\tilde{\mathcal{A}}_2}\tilde{K}_{\tilde{\mathcal{B}}_2}$ are local maps (resp. isometries) then $T \otimes \tilde{T} : {}_{\mathcal{A}_1}H_{\mathcal{B}_1} \otimes {}_{\tilde{\mathcal{A}}_1}\tilde{H}_{\tilde{\mathcal{B}}_1} \to {}_{\mathcal{A}_2}K_{\mathcal{B}_2} \otimes {}_{\tilde{\mathcal{A}}_2}\tilde{K}_{\tilde{\mathcal{B}}_2}$ is a local map (resp. isometry). Indeed, suppose that $T = T_{2,2} \circ T_{1,2} = T_{2,1} \circ T_{1,1}$ and $\tilde{T} = \tilde{T}_{2,2} \circ \tilde{T}_{1,2} = \tilde{T}_{2,1} \circ \tilde{T}_{1,1}$ as in diagram (3). Setting $\hat{T}_{i,j} = T_{i,j} \otimes \tilde{T}_{i,j}$, we witness the locality of $T \otimes \tilde{T}$ through the identity

$$T \otimes \tilde{T} = \hat{T}_{2,2} \circ \hat{T}_{1,2} = \hat{T}_{2,1} \circ \hat{T}_{1,1}.$$

We now extend Remark 2.9 (i) by showing that the relation $\leq$ is transitive.

Theorem 2.10. *Let $\mathcal{A}_i$ and $\mathcal{B}_i$ be von Neumann algebras, $i = 1, 2, 3$. If ${}_{\mathcal{A}_1}H_{\mathcal{B}_1} \leq {}_{\mathcal{A}_2}K_{\mathcal{B}_2}$ and ${}_{\mathcal{A}_2}K_{\mathcal{B}_2} \leq {}_{\mathcal{A}_3}L_{\mathcal{B}_3}$, then ${}_{\mathcal{A}_1}H_{\mathcal{B}_1} \leq {}_{\mathcal{A}_3}L_{\mathcal{B}_3}$.*

Proof. We need to complete the diagram

$$\begin{tikzcd}
{}_{\mathcal{A}_1}H_{\mathcal{B}_1} \arrow[r, "V_{1,2}"] \arrow[d, "V_{1,1}"] & {}_{\mathcal{A}_1}\widetilde{H}_{\mathcal{B}_2} \arrow[d, "V_{2,2}"] & \\
{}_{\mathcal{A}_2}\widetilde{\widetilde{H}}_{\mathcal{B}_1} \arrow[r, "V_{2,1}"] & {}_{\mathcal{A}_2}K_{\mathcal{B}_2} \arrow[r, "V_{2,3}"] \arrow[d, "\widetilde{V_{2,2}}"] & {}_{\mathcal{A}_2}\widetilde{K}_{\mathcal{B}_3} \arrow[d, "V_{3,3}"] \\
& {}_{\mathcal{A}_3}\widetilde{\widetilde{K}}_{\mathcal{B}_2} \arrow[r, "V_{3,2}"] & {}_{\mathcal{A}_3}L_{\mathcal{B}_3}
\end{tikzcd}$$

to a commuting diagram of the form

$$\begin{tikzcd}
{}_{\mathcal{A}_1}H_{\mathcal{B}_1} \arrow[r, "V_{1,2}"] \arrow[d, "V_{1,1}"] & {}_{\mathcal{A}_1}\widetilde{H}_{\mathcal{B}_2} \arrow[r, dashed, "W_{1,3}"] \arrow[d, "V_{2,2}"] & {}_{\mathcal{A}_1}\widetilde{L}_{\mathcal{B}_3} \arrow[d, dashed, "W_{2,3}"] \\
{}_{\mathcal{A}_2}\widetilde{\widetilde{H}}_{\mathcal{B}_1} \arrow[r, "V_{2,1}"] \arrow[d, dashed, "W_{3,1}"] & {}_{\mathcal{A}_2}K_{\mathcal{B}_2} \arrow[r, "V_{2,3}"] \arrow[d, "\widetilde{V_{2,2}}"] & {}_{\mathcal{A}_2}\widetilde{K}_{\mathcal{B}_3} \arrow[d, "V_{3,3}"] \\
{}_{\mathcal{A}_3}\widetilde{\widetilde{L}}_{\mathcal{B}_1} \arrow[r, dashed, "W_{3,2}"] & {}_{\mathcal{A}_3}\widetilde{\widetilde{K}}_{\mathcal{B}_2} \arrow[r, "V_{3,2}"] & {}_{\mathcal{A}_3}L_{\mathcal{B}_3},
\end{tikzcd} \tag{7}$$

for some quantum systems ${}_{\mathcal{A}_1}\widetilde{L}_{\mathcal{B}_3}$ and ${}_{\mathcal{A}_3}\widetilde{\widetilde{L}}_{\mathcal{B}_1}$, some $\mathcal{B}$-local isometries $W_{1,3}$ and $W_{3,2}$, and some $\mathcal{A}$-local isometries $W_{2,3}$ and $W_{3,1}$.

Let $f_2 \in \mathcal{A}_2$ be a central projection, such that $\ker \pi_K|_{\mathcal{A}_2} = f_2^{\perp}\mathcal{A}_2$. We have that $\pi_K|_{f_2\mathcal{A}_2} : f_2\mathcal{A}_2 \to \pi_K(\mathcal{A}_2)$ is a normal unital *-isomorphism; in the rest of the proof, we denote by π_K^{-1} its inverse, from $\pi_K(\mathcal{A}_2)$ onto $f_2\mathcal{A}_2$. By the $\mathcal{A}$-locality of $V_{2,2}$, there exists a (unique) projection $e_2 \in f_2\mathcal{A}_2$ such that $p_2 := V_{2,2}V_{2,2}^* = \pi_K(e_2)$. Let $p_{\mathcal{A}} = \pi_{\widetilde{K}}(e_2) \in \mathcal{B}(\widetilde{K})$, and set $\widetilde{L} := p_{\mathcal{A}}\widetilde{K}$.

Recalling that $\pi_K^{-1}(V_{2,2}\pi_{\widetilde{H}}(a_1)V_{2,2}^*) \in f_2\mathcal{A}_2$, define $\pi_{\widetilde{L}}^{\mathcal{A}} : \mathcal{A}_1 \to \mathcal{B}(\widetilde{L})$ by

$$\pi_{\widetilde{L}}^{\mathcal{A}}(a_1) = \pi_{\widetilde{K}}(\pi_K^{-1}(V_{2,2}\pi_{\widetilde{H}}(a_1)V_{2,2}^*)), \quad a_1 \in \mathcal{A}_1.$$

Note that the $\mathcal{A}$-locality of $V_{2,2}$ implies $V_{2,2}\pi_{\widetilde{H}}(a_1)V_{2,2}^* \in \pi_K(\mathcal{A}_2)$, so by the fact that $V_{2,2}$ is an isometry, $\pi_{\widetilde{L}}^{\mathcal{A}}$ is a well-defined normal $*$-homomorphism. It is also unital as $\pi_{\widetilde{L}}^{\mathcal{A}}(1_{\mathcal{A}_1}) = p_{\mathcal{A}} = 1_{\mathcal{B}(p_{\mathcal{A}}\widetilde{K})}$.

Next, define $\pi_{\widetilde{L}}^{\mathcal{B}} : \mathcal{B}_3^o \to \mathcal{B}(\widetilde{L})$ by letting

$$\pi_{\widetilde{L}}^{\mathcal{B}}(b_3^o) = p_{\mathcal{A}}\pi_{\widetilde{K}}(b_3^o)p_{\mathcal{A}}, \quad b^o \in \mathcal{B}_3^o.$$

Since $\pi_{\widetilde{K}}(\mathcal{A}_2) \subseteq \pi_{\widetilde{K}}(\mathcal{B}_3^o)'$, we have that $p_{\mathcal{A}} \in \pi_{\widetilde{K}}(\mathcal{B}_3^o)'$, and hence the map $\pi_{\widetilde{L}}^{\mathcal{B}}$ is a normal unital $*$-homomorphism. The inclusion $\pi_{\widetilde{K}}(\mathcal{A}_2) \subseteq \pi_{\widetilde{K}}(\mathcal{B}_3^o)'$ implies that $\pi_{\widetilde{L}}^{\mathcal{A}}$ and $\pi_{\widetilde{L}}^{\mathcal{B}}$ have commuting ranges, and therefore define a unital, separately weak* continuous, $*$-homomorphism

$$\pi_{\widetilde{L}} := \pi^{\mathcal{A}}_{\widetilde{L}} \times \pi^{\mathcal{B}}_{\widetilde{L}} : \mathcal{A}_1 \otimes_{\max} \mathcal{B}_3^o \to \mathcal{B}(\widetilde{L}),$$

turning $\widetilde{L}$ into an $\mathcal{A}_1$-$\mathcal{B}_3$-bimodule ${}_{\mathcal{A}_1}\widetilde{L}_{\mathcal{B}_3}$.

Let $W_{1,3} = V_{2,3}V_{2,2}$. Since

$$V_{2,3}V_{2,2} = V_{2,3}\pi_K(e_2)V_{2,2} = \pi_{\widetilde{K}}(e_2)V_{2,3}V_{2,2} = p_{\mathcal{A}}V_{2,3}V_{2,2},$$

we have that $W_{1,3} : {}_{\mathcal{A}_1}H_{\mathcal{B}_2} \to {}_{\mathcal{A}_1}\widetilde{L}_{\mathcal{B}_3}$ is a well-defined isometry. Given $a_1 \in \mathcal{A}_1$, let a_2 be the unique element of $f_2\mathcal{A}_2$ such that $V_{2,2}\pi_{\widetilde{H}}(a_1)V_{2,2}^* = \pi_K(a_2)$. We have

$$\begin{aligned} W_{1,3}\pi_{\widetilde{H}}(a_1) &= V_{2,3}V_{2,2}\pi_{\widetilde{H}}(a_1) = V_{2,3}V_{2,2}\pi_{\widetilde{H}}(a_1)V_{2,2}^*V_{2,2} \\ &= V_{2,3}\pi_K(a_2)V_{2,2} = \pi_{\widetilde{K}}(a_2)V_{2,3}V_{2,2} \\ &= \pi_{\widetilde{K}}(\pi_K^{-1}(V_{2,2}\pi_{\widetilde{H}}(a_1)V_{2,2}^*))V_{2,3}V_{2,2} = \pi_{\widetilde{L}}(a_1)W_{1,3}. \end{aligned}$$

Given $b_2 \in \mathcal{B}_2$, we have $V_{2,3}\pi_K(b_2^o)V_{2,3}^* = \pi_{\widetilde{K}}(b_3^o)$ for some $b_3 \in \mathcal{B}_3$. Let $e_3 \in \mathcal{B}_3$ be a projection for which $V_{2,3}V_{2,3}^* = \pi_{\widetilde{K}}(e_3^o)$. Then

$$\begin{aligned} W_{1,3}\pi_{\widetilde{H}}(b_2^o)W_{1,3}^* &= V_{2,3}V_{2,2}\pi_{\widetilde{H}}(b_2^o)V_{2,2}^*V_{2,3}^* = V_{2,3}\pi_K(b_2^o)V_{2,2}V_{2,2}^*V_{2,3}^* \\ &= V_{2,3}\pi_K(b_2^o)\pi_K(e_2)V_{2,3}^* = V_{2,3}\pi_K(b_2^o)V_{2,3}^*V_{2,3}\pi_K(e_2)V_{2,3}^* \\ &= \pi_{\widetilde{K}}(b_3^o)V_{2,3}\pi_K(e_2)V_{2,3}^* = \pi_{\widetilde{K}}(b_3^o)p_{\mathcal{A}}V_{2,3}V_{2,3}^* \\ &= \pi_{\widetilde{K}}(b_3^o)p_{\mathcal{A}}\pi_{\widetilde{K}}(e_3^o) = \pi_{\widetilde{K}}(b_3^oe_3^o)p_{\mathcal{A}} = \pi_{\widetilde{L}}(b_3^oe_3^o). \end{aligned}$$

Hence, $W_{1,3}$ is a $\mathcal{B}$-local map. By Remark 2.9, the composition $W_{1,3}V_{1,2}$ is a $\mathcal{B}$-local map.

Let $W_{2,3} : \widetilde{L} \to \widetilde{K}$ be the inclusion map. We have that $W_{2,3}$ is $\mathcal{A}$-local. In fact, condition (i') from Definition 2.3 is tautological because of the definition of $W_{2,3}$, while, if $a_1 \in \mathcal{A}_1$, there is unique $a_2 \in f_2\mathcal{A}_2$ such that

$$\begin{aligned} W_{2,3}\pi_{\widetilde{L}}(a_1)W_{2,3}^* &= \pi_{\widetilde{K}}(\pi_K^{-1}(V_{2,2}\pi_{\widetilde{H}}(a_1)V_{2,2}^*))p_{\mathcal{A}} \\ &= \pi_{\widetilde{K}}(a_2)\pi_{\widetilde{K}}(e_2) = \pi_{\widetilde{K}}(a_2e_2), \end{aligned}$$

implying $W_{2,3}\pi_{\widetilde{L}}(a_1)W_{2,3}^* \in \pi_{\widetilde{K}}(\mathcal{A}_2)$, and thus condition (i") from Definition 2.3.

We similarly define a bipartite quantum system ${}_{\mathcal{A}_3}\widetilde{\widetilde{L}}_{\mathcal{B}_1}$, and isometries $W_{3,1} : \widetilde{\widetilde{H}} \to \widetilde{\widetilde{L}}$ and $W_{3,2} : \widetilde{\widetilde{L}} \to \widetilde{\widetilde{K}}$. Using symmetric arguments, we have that the operator $W_{3,1}$ is $\mathcal{A}$-local, while the operator $W_{3,2}$ is $\mathcal{B}$-local. By Remark 2.9, the composition $V_{3,2}W_{3,2}$ is $\mathcal{B}$-local, while the compositions $W_{3,1}V_{1,1}$ and $V_{3,3}W_{2,3}$ are $\mathcal{A}$-local.

The relation $W_{2,3}W_{1,3} = V_{2,3}V_{2,2}$ follows from the definition of the operators $W_{1,3}$ and $W_{2,3}$; similarly, $W_{3,2}W_{3,1} = \widetilde{V}_{2,2}V_{2,1}$. The commutativity of the diagram (7) is now immediate, and the proof is complete. □

3. Models and self-tests

Our generalisation of self-testing is based on the notion of an operator system; we recall some basic facts and concepts, and refer the reader to [44] for details. If H is a Hilbert space and $\mathcal{S} \subseteq \mathcal{B}(H)$ is a linear subspace, the space $M_n(\mathcal{S})$ of all n by n matrices with entries in $\mathcal{S}$ can be viewed as a subspace of $\mathcal{B}(H^n)$ after identifying $M_n(\mathcal{B}(H))$ with $\mathcal{B}(H^n)$. If $\mathcal{S} \subseteq \mathcal{B}(H)$ and $\mathcal{T} \subseteq \mathcal{B}(K)$ (where K is a(nother) Hilbert space) are subspaces and $\phi : \mathcal{S} \to \mathcal{T}$ is a linear map, we let $\phi^{(n)} : M_n(\mathcal{S}) \to M_n(\mathcal{T})$ be the (linear) map, given by $\phi^{(n)}((a_{i,j})_{i,j}) = (\phi(a_{i,j}))_{i,j}$. An *operator system* is a subspace $\mathcal{S} \subseteq \mathcal{B}(H)$, such that $I_{\mathcal{H}} \in \mathcal{S}$ and $s \in \mathcal{S} \Rightarrow s^* \in \mathcal{S}$. We note that every operator system is an operator space in a canonical fashion. Every operator system $\mathcal{S}$ is an *abstract operator system* in the sense that (a) $\mathcal{S}$ is a linear *-space; (b) the real vector space $M_n(\mathcal{S})_h$ of all hermitian elements in the *-space $M_n(\mathcal{S})$ is equipped with a proper cone $M_n(\mathcal{S})^+$; (c) $T^* M_n(\mathcal{S})^+ T \subseteq M_m(\mathcal{S})^+$ for all $n, m \in \mathbb{N}$ and all $T \in M_{n,m}$, and (d) the cone family $(M_n(\mathcal{S})^+)_{n\in\mathbb{N}}$ admits an Archimedean matrix order unit. If $\mathcal{S}$ and $\mathcal{T}$ are abstract operator systems, a linear map $\phi : \mathcal{S} \to \mathcal{T}$ is called *positive* if $\phi(\mathcal{S}^+) \subseteq \mathcal{T}^+$, *completely positive* if $\phi^{(n)}$ is positive for every $n \in \mathbb{N}$, *unital* if $\phi(I) = I$, and a *complete order isomorphism* if ϕ is completely positive, bijective, and its inverse ϕ^{-1} is completely positive. By virtue of the Choi-Effros Theorem [44, Theorem 13.1], every abstract operator system is completely order isomorphic to an operator system and hence carries a canonical operator space structure. We note that every unital completely positive map between abstract operator systems is automatically contractive. A *state* of an operator system $\mathcal{S}$ is a positive unital linear functional; we denote by $S(\mathcal{S})$ the (convex) set of all states of $\mathcal{S}$.

Let $\mathcal{S}$ be an operator system. Recall that a pair $(C^*_u(\mathcal{S}), \iota)$ is called a universal cover of $\mathcal{S}$, if $C^*_u(\mathcal{S})$ is a unital C^*-algebra, $\iota : \mathcal{S} \to C^*_u(\mathcal{S})$ is a unital complete order embedding such that $\iota(\mathcal{S})$ generates $C^*_u(\mathcal{S})$ and, whenever H is a Hilbert space and $\phi : \mathcal{S} \to \mathcal{B}(H)$ is a unital completely positive map, there exists a $*$-representation $\pi_\phi : C^*_u(\mathcal{S}) \to \mathcal{B}(H)$ such that $\pi_\phi \circ \iota = \phi$ (see e.g. [29]). It is clear that the universal cover is unique up to a canonical $*$-isomorphism.

We fix throughout this section operator systems $\mathcal{S}_A$ and $\mathcal{S}_B$. Their commuting tensor product $\mathcal{S}_A \otimes_{\rm c} \mathcal{S}_B$ is the operator system with underlying vector space the algebraic tensor product $\mathcal{S}_A \otimes \mathcal{S}_B$, and matrix order structure inherited from the inclusion $\mathcal{S}_A \otimes_{\rm c} \mathcal{S}_B \subseteq C^*_u(\mathcal{S}_A) \otimes_{\max} C^*_u(\mathcal{S}_B)$; thus, $\mathcal{S}_A \otimes_{\rm c} \mathcal{S}_B$ sits completely order isomorphically in $C^*_u(\mathcal{S}_A) \otimes_{\max} C^*_u(\mathcal{S}_B)$ (we note that we are using an equivalent definition of the commuting tensor product to the original one, see [29, Theorem 6.4]). Given a Hilbert space H and unital completely positive maps $\varphi_A : \mathcal{S}_A \to \mathcal{B}(H)$ and $\varphi_B : \mathcal{S}_B \to \mathcal{B}(H)$ with commuting ranges, there exists a (unique) unital completely positive map $\varphi_A \cdot \varphi_B : \mathcal{S}_A \otimes_{\rm c} \mathcal{S}_B \to \mathcal{B}(H)$, such that

$$(\varphi_A \cdot \varphi_B)(s \otimes t) = \varphi_A(s)\varphi_B(t), \quad s \in \mathcal{S}_A, t \in \mathcal{S}_B$$

(see [29, Corollary 6.5]).

Definition 3.1. A *quantum commuting model* over the pair $(\mathcal{S}_A, \mathcal{S}_B)$ is a tuple

$$S = ({}_{\mathcal{A}}H_{\mathcal{B}}, \varphi_A, \varphi_B, \xi),$$

where H is a Hilbert space, $\xi \in H$ is a unit vector, $\varphi_A : \mathcal{S}_A \to \mathcal{B}(H)$ and $\varphi_B : \mathcal{S}_B \to \mathcal{B}(H)$ are unital completely positive maps with commuting ranges, $\mathcal{A}$ is von Neumann subalgebra of $\mathcal{B}(H)$, $\mathcal{B}$ is a von Neumann algebra with $\mathcal{B}^o \subseteq \mathcal{B}(H)$, $H = {}_{\mathcal{A}}H_{\mathcal{B}}$ is a bipartite quantum system over $(\mathcal{A}, \mathcal{B})$, and the inclusions $\varphi_A(\mathcal{S}_A) \subseteq \mathcal{A}$ and $\varphi_B(\mathcal{S}_B) \subseteq \mathcal{B}^o$ hold.

We say that S is a *Haag model* if $\mathcal{B}^o = \mathcal{A}'$.

We note that every pair (φ_A, φ_B) of unital completely positive maps with commuting ranges, say, $\varphi_A : \mathcal{S}_A \to \mathcal{B}(H)$ and $\varphi_B : \mathcal{S}_B \to \mathcal{B}(H)$, and a choice of a unit vector $\xi \in H$, give rise to a canonical quantum commuting model over the pair $(\mathcal{S}_A, \mathcal{S}_B)$ by letting $\mathcal{A}$ (resp. $\mathcal{B}^o$) be the von Neumann algebra, generated by $\varphi_A(\mathcal{S}_A)$ (resp. $\varphi_B(\mathcal{S}_B)$). To each model $S = ({}_{\mathcal{A}}H_{\mathcal{B}}, \varphi_A, \varphi_B, \xi)$, we associate a linear functional $f_S : \mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B \to \mathbb{C}$ by letting

$$f_S(u) = \langle (\varphi_A \cdot \varphi_B)(u)\xi, \xi \rangle, \quad u \in \mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B,$$

and a linear functional $\tilde{f}_S : C_u^*(\mathcal{S}_A) \otimes_{\max} C_u^*(\mathcal{S}_B) \to \mathbb{C}$ by letting

$$\tilde{f}_S(u) = \langle (\pi_A \cdot \pi_B)(u)\xi, \xi \rangle, \quad u \in C_u^*(\mathcal{S}_A) \otimes_{\max} C_u^*(\mathcal{S}_B).$$

We note that f_S and $\tilde{f}_S$ depend only on the tuple $(H, \varphi_A, \varphi_B, \xi)$ and not on the von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$.

We note that Definition 3.1 is motivated by the commuting operator model for no-signalling correlations; for the convenience of the reader, we recall here the relevant definitions. Let X, Y, A and B be finite sets. A family $p = ((p(a, b|x, y)_{a,b} : (x, y) \in X \times Y\}$ of conditional probability distributions over $A \times B$ is called a no-signalling correlation *of quantum commuting type* [45,46], if

$$p(a, b|x, y) = \langle E_{x,a}F_{y,b}\xi, \xi \rangle, \quad x \in X, y \in Y, a \in A, b \in B,$$

for some POVM's $(E_{x,a})_{a \in A}$, $x \in X$ (resp. $(F_{y,b})_{b \in B}$, $y \in Y$) on a Hilbert space H such that $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$ for all x, y, a, b, and a unit vector $\xi \in H$.

Let $\mathcal{A}_{X,A}$ be the universal C*-algebra, generated by projections $e_{x,a}$, $x \in X$, $a \in A$, satisfying the relations $\sum_{a \in A} e_{x,a} = 1$, $x \in X$, and let $\mathcal{S}_{X,A} = \mathrm{span}\{e_{x,a} : x \in X, a \in A\}$, viewed as an operator subsystem of $\mathcal{A}_{X,A}$ (see e.g. [37]). As it will be seen in greater detail in Subsection 5.1, the tuple $(H, (E_{x,a})_{x,a}, (F_{y,b})_{y,b}, \xi)$ as in the previous paragraph gives rise in a canonical fashion to a quantum commuting model S over $(\mathcal{S}_{X,A}, \mathcal{S}_{Y,B})$, and the family $p = ((p(a, b|x, y)_{a,b} : (x, y) \in X \times Y\}$, defined by $p(a, b|x, y) = f_S(e_{x,a} \otimes e_{y,b})$, is a quantum commuting no-signalling correlation.

We now return to the discussion of general quantum commuting models and introduce the fundamental partial order between such models that will lead to the notion of self-testing in the commuting operator framework (Definition 3.6 below).

Definition 3.2. Let $\mathcal{S}_A$ and $\mathcal{S}_B$ be operator systems, and suppose that $S = ({}_{\mathcal{A}}H_{\mathcal{B}}, \varphi_A, \varphi_B, \xi)$ and $\widetilde{S} = ({}_{\widetilde{\mathcal{A}}}\widetilde{H}_{\widetilde{\mathcal{B}}}, \widetilde{\varphi}_A, \widetilde{\varphi}_B, \widetilde{\xi})$ are quantum commuting models over the pair $(\mathcal{S}_A, \mathcal{S}_B)$.

(i) We say that $\widetilde{S}$ is a *local dilation* of S, and write $S \preceq \widetilde{S}$, if there exist a bipartite quantum system ${}_{\mathcal{A}_{\text{aux}}}(H_{\text{aux}})_{\mathcal{B}_{\text{aux}}}$ over a pair of von Neumann algebras $(\mathcal{A}_{\text{aux}}, \mathcal{B}_{\text{aux}})$, a unit vector $\xi_{\text{aux}} \in H_{\text{aux}}$, and a local isometry $V : {}_{\mathcal{A}}H_{\mathcal{B}} \to {}_{\widetilde{\mathcal{A}}}\widetilde{H}_{\widetilde{\mathcal{B}}} \otimes {}_{\mathcal{A}_{\text{aux}}}(H_{\text{aux}})_{\mathcal{B}_{\text{aux}}}$ such that

$$V\varphi_A(s)\varphi_B(t)\xi = (\tilde{\varphi}_A(s)\tilde{\varphi}_B(t)\tilde{\xi}) \otimes \xi_{\text{aux}}, \quad s \in \mathcal{S}_A, t \in \mathcal{S}_B. \tag{8}$$

(ii) We say that $\widetilde{S}$ is an *approximate local dilation* of S, and write $S \preceq_{\text{a}} \widetilde{S}$, if there exist bipartite quantum systems ${}_{\mathcal{A}_{i,\text{aux}}}(H_{i,\text{aux}})_{\mathcal{B}_{i,\text{aux}}}$ over a pair of von Neumann algebras $(\mathcal{A}_{i,\text{aux}}, \mathcal{B}_{i,\text{aux}})$, unit vectors $\xi_{i,\text{aux}} \in H_{i,\text{aux}}$, and local isometries $V_i : {}_{\mathcal{A}}H_{\mathcal{B}} \to {}_{\widetilde{\mathcal{A}}}\widetilde{H}_{\widetilde{\mathcal{B}}} \otimes {}_{\mathcal{A}_{i,\text{aux}}}(H_{i,\text{aux}})_{\mathcal{B}_{i,\text{aux}}}$, $i \in \mathbb{N}$, such that

$$\left\|V_i\varphi_A(s)\varphi_B(t)\xi - (\tilde{\varphi}_A(s)\tilde{\varphi}_B(t)\tilde{\xi}) \otimes \xi_{i,\text{aux}}\right\| \to_{i\to\infty} 0, \quad s \in \mathcal{S}_A, t \in \mathcal{S}_B. \tag{9}$$

We note that the reverse notation $\widetilde{S} \preceq S$ was used in [43] to designate that $\widetilde{S}$ is a local dilation of S, but we have decided to employ the one specified in Definition 3.2 as it agrees with the usual conventions in operator algebra theory (see e.g. [19]). More precisely, even though in self-testing examples, an ideal model $\widetilde{S}$ is typically smaller (in the sense of Hilbert space dimension) than a given model S satisfying $S \preceq \widetilde{S}$ (in line with the reverse notation from [43]), assistance from an auxiliary system together with a (typically entangled) auxiliary state is needed to witness the local dilation property $S \preceq \widetilde{S}$:

$$V^*(\tilde{\varphi}_A(s)\tilde{\varphi}_B(t) \otimes I)V\xi = \varphi_A(s)\varphi_B(t)\xi.$$

We therefore view this property as a form of entanglement assisted local dilation.

Remark 3.3. By linearity, condition (8) is equivalent to

$$V(\varphi_A \cdot \varphi_B)(u)\xi = (\tilde{\varphi}_A \cdot \tilde{\varphi}_B)(u)\tilde{\xi} \otimes \xi_{\text{aux}}$$

being fulfilled for every u in the completion of the tensor product $\mathcal{S}_A \otimes_{\text{c}} \mathcal{S}_B$. In addition, choosing $u = 1$, we have that $V\xi = \tilde{\xi} \otimes \xi_{\text{aux}}$.

An $\epsilon/3$-argument, together with the uniform boundedness of the sequence $(V_i)_{i\in\mathbb{N}}$ appearing in (9), shows that (9) is equivalent to the condition

$$\left\|V_i(\varphi_A\cdot\varphi_B)(u)\xi-(\tilde\varphi_A\cdot\tilde\varphi_B)(u)\tilde\xi\otimes\xi_{i,\mathrm{aux}}\right\|\to_{i\to\infty}0, \tag{10}$$

for all u in the completion of the tensor product $\mathcal{S}_A\otimes_{\mathrm{c}}\mathcal{S}_B$. Choosing $u=1$, we have, in particular,

$$\left\|V_i\xi-\tilde\xi\otimes\xi_{i,\mathrm{aux}}\right\|\to_{i\to\infty}0.$$

It is clear that

$$S\preceq\widetilde{S}\implies S\preceq_{\mathrm{a}}\widetilde{S}.$$

Proposition 3.4. *The relations $\preceq$ and $\preceq_{\mathrm{a}}$ are preorders.*

Proof. Assume that $S\preceq_{\mathrm{a}}\widetilde{S}$ and $\widetilde{S}\preceq_{\mathrm{a}}\widetilde{\widetilde{S}}$, realised via sequences $(V_i)_{i\in\mathbb{N}}$ and $(\widetilde{V}_i)_{i\in\mathbb{N}}$ of local isometries, where $V_i:{}_{\mathcal{A}}H_{\mathcal{B}}\to{}_{\widetilde{\mathcal{A}}}\widetilde{H}_{\widetilde{\mathcal{B}}}\otimes{}_{\mathcal{A}_{i,\mathrm{aux}}}(H_{i,\mathrm{aux}})_{\mathcal{B}_{i,\mathrm{aux}}}$ and $\widetilde{V}_i:{}_{\widetilde{\mathcal{A}}}\widetilde{H}_{\widetilde{\mathcal{B}}}\to{}_{\widetilde{\widetilde{\mathcal{A}}}}\widetilde{\widetilde{H}}_{\widetilde{\widetilde{\mathcal{B}}}}\otimes{}_{\widetilde{\mathcal{A}}_{i,\mathrm{aux}}}(\widetilde{H}_{i,\mathrm{aux}})_{\widetilde{\mathcal{B}}_{i,\mathrm{aux}}}$, $i\in\mathbb{N}$. By Remark 2.9, $\widetilde{V}_i\otimes I_{H_{i,\mathrm{aux}}}$ is a local isometry from ${}_{\widetilde{\mathcal{A}}}\widetilde{H}_{\widetilde{\mathcal{B}}}\otimes{}_{\mathcal{A}_{i,\mathrm{aux}}}(H_{\mathrm{aux}})_{\mathcal{B}_{i,\mathrm{aux}}}$ to ${}_{\widetilde{\widetilde{\mathcal{A}}}}\widetilde{\widetilde{H}}_{\widetilde{\widetilde{\mathcal{B}}}}\otimes{}_{\widetilde{\mathcal{A}}_{i,\mathrm{aux}}}(\widetilde{H}_{i,\mathrm{aux}})_{\widetilde{\mathcal{B}}_{i,\mathrm{aux}}}\otimes{}_{\mathcal{A}_{i,\mathrm{aux}}}(H_{i,\mathrm{aux}})_{\mathcal{B}_{i,\mathrm{aux}}}$. It follows now from Theorem 2.10 that $(\widetilde{V}_i\otimes I_{H_{i,\mathrm{aux}}})V_i$ is a local isometry from ${}_{\mathcal{A}}H_{\mathcal{B}}$ to ${}_{\widetilde{\widetilde{\mathcal{A}}}}\widetilde{\widetilde{H}}_{\widetilde{\widetilde{\mathcal{B}}}}\otimes{}_{\widetilde{\mathcal{A}}_{i,\mathrm{aux}}}(\widetilde{H}_{i,\mathrm{aux}})_{\widetilde{\mathcal{B}}_{i,\mathrm{aux}}}\otimes{}_{\mathcal{A}_{i,\mathrm{aux}}}(H_{i,\mathrm{aux}})_{\mathcal{B}_{i,\mathrm{aux}}}$, $i\in\mathbb{N}$. Further, if $s\in\mathcal{S}_A$ and $t\in\mathcal{S}_B$ then

$$\begin{aligned}
&\left\|(\widetilde{V}_i\otimes I_{H_{i,\mathrm{aux}}})V_i\varphi_A(s)\varphi_B(t)\xi-(\widetilde{\widetilde{\varphi}}_A(s)\widetilde{\widetilde{\varphi}}_B(t)\widetilde{\widetilde{\xi}})\otimes\widetilde{\xi}_{i,\mathrm{aux}}\otimes\xi_{i,\mathrm{aux}}\right\|\\
&\le\left\|(\widetilde{V}_i\otimes I_{H_{i,\mathrm{aux}}})V_i\varphi_A(s)\varphi_B(t)\xi-(\widetilde{V}_i\otimes I_{H_{i,\mathrm{aux}}})(\tilde\varphi_A(s)\tilde\varphi_B(t)\tilde\xi)\otimes\xi_{i,\mathrm{aux}}\right\|\\
&+\left\|(\widetilde{V}_i\otimes I_{H_{i,\mathrm{aux}}})\tilde\varphi_A(s)\tilde\varphi_B(t)\tilde\xi\otimes\xi_{i,\mathrm{aux}}-(\widetilde{\widetilde{\varphi}}_A(s)\widetilde{\widetilde{\varphi}}_B(t)\widetilde{\widetilde{\xi}})\otimes\widetilde{\xi}_{i,\mathrm{aux}}\otimes\xi_{i,\mathrm{aux}}\right\|\\
&\le\left\|V_i\varphi_A(s)\varphi_B(t)\xi-\tilde\varphi_A(s)\tilde\varphi_B(t)\tilde\xi\otimes\xi_{i,\mathrm{aux}}\right\|\\
&+\left\|(\widetilde{V}_i\otimes I_{H_{i,\mathrm{aux}}})\tilde\varphi_A(s)\tilde\varphi_B(t)\tilde\xi-(\widetilde{\widetilde{\varphi}}_A(s)\widetilde{\widetilde{\varphi}}_B(t)\widetilde{\widetilde{\xi}})\otimes\widetilde{\xi}_{i,\mathrm{aux}}\right\|.
\end{aligned}$$

It follows that

$$\left\|(\widetilde{V}_i\otimes I_{H_{i,\mathrm{aux}}})V_i\varphi_A(s)\varphi_B(t)\xi-(\widetilde{\widetilde{\varphi}}_A(s)\widetilde{\widetilde{\varphi}}_B(t)\widetilde{\widetilde{\xi}})\otimes\widetilde{\xi}_{i,\mathrm{aux}}\otimes\xi_{i,\mathrm{aux}}\right\|\to_{i\to\infty}0.$$

Thus the family $((\widetilde{V}_i\otimes I_{H_{i,\mathrm{aux}}})V_i)_{i\in\mathbb{N}}$ implements an approximate dilation yielding the relation $S\preceq_{\mathrm{a}}\widetilde{\widetilde{S}}$, and showing that the relation $\preceq_{\mathrm{a}}$ is a preorder. The fact that $\preceq$ is a preorder now follows after noticing that it is a special case of $\preceq_{\mathrm{a}}$, implemented by constant sequences of local isometries. $\square$

It is straightforward that if $S\preceq\widetilde{S}$ then $f_S=f_{\widetilde{S}}$. In fact, the same implication holds for approximate dilations.

Proposition 3.5. *If* $S \preceq_{\mathrm{a}} \widetilde{S}$ *then* $f_S = f_{\widetilde{S}}$.

Proof. Let ${}_{\mathcal{A}_{i,\mathrm{aux}}}(H_{i,\mathrm{aux}})_{\mathcal{B}_{i,\mathrm{aux}}}$ be bipartite quantum systems, $\xi_{i,\mathrm{aux}} \in H_{i,\mathrm{aux}}$ be unit vectors, and $V_i : {}_{\mathcal{A}}H_{\mathcal{B}} \to {}_{\widetilde{\mathcal{A}}}\widetilde{H}_{\widetilde{\mathcal{B}}} \otimes {}_{\mathcal{A}_{i,\mathrm{aux}}}(H_{i,\mathrm{aux}})_{\mathcal{B}_{\mathrm{aux}}}$ be local isometries, $i \in \mathbb{N}$, for which condition (9) holds. Given $\epsilon > 0$ and $u \in \mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B$, in view of (10), let $i \in \mathbb{N}$ be such that

$$\left\|V_i(\varphi_A \cdot \varphi_B)(u)\xi - (\tilde{\varphi}_A \cdot \tilde{\varphi}_B)(u)\tilde{\xi} \otimes \xi_{i,\mathrm{aux}}\right\| < \frac{\epsilon}{2}$$

and

$$\left\|V_i\xi - \widetilde{\xi} \otimes \xi_{i,\mathrm{aux}}\right\| < \frac{\epsilon}{2\|u\|}.$$

We have

$$\begin{aligned}
&\left|f_{\widetilde{S}}(u) - f_S(u)\right| = \left|\left\langle(\tilde{\varphi}_A \cdot \tilde{\varphi}_B)(u)\tilde{\xi}, \tilde{\xi}\right\rangle - \langle(\varphi_A \cdot \varphi_B)(u)\xi, \xi\rangle\right| \\
&= \left|\left\langle(\tilde{\varphi}_A \cdot \tilde{\varphi}_B)(u)\tilde{\xi} \otimes \xi_{i,\mathrm{aux}}, \tilde{\xi} \otimes \xi_{i,\mathrm{aux}}\right\rangle - \langle V_i(\varphi_A \cdot \varphi_B)(u)\xi, V_i\xi\rangle\right| \\
&\leq \left|\left\langle(\tilde{\varphi}_A \cdot \tilde{\varphi}_B)(u)\tilde{\xi} \otimes \xi_{i,\mathrm{aux}}, \tilde{\xi} \otimes \xi_{i,\mathrm{aux}}\right\rangle - \left\langle V_i(\varphi_A \cdot \varphi_B)(u)\xi, \tilde{\xi} \otimes \xi_{i,\mathrm{aux}}\right\rangle\right| \\
&+ \left|\left\langle V_i(\varphi_A \cdot \varphi_B)(u)\xi, \tilde{\xi} \otimes \xi_{i,\mathrm{aux}}\right\rangle - \langle V_i(\varphi_A \cdot \varphi_B)(u)\xi, V_i\xi\rangle\right| \\
&\leq \frac{\epsilon}{2}\left\|\tilde{\xi} \otimes \xi_{i,\mathrm{aux}}\right\| + \frac{\epsilon}{2\|u\|}\|V_i(\varphi_A \cdot \varphi_B)(u)\xi\| \leq \epsilon.
\end{aligned}$$

As ϵ is an arbitrary positive real, we have that $f_{\widetilde{S}}(u) = f_S(u)$ for every $u \in \mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B$. □

Definition 3.6. Let $\mathcal{C}$ be a class of quantum commuting models over the pair $(\mathcal{S}_A, \mathcal{S}_B)$, and $\mathcal{S}$ be a subset of states of the C*-algebra $C_u^*(\mathcal{S}_A) \otimes_{\max} C_u^*(\mathcal{S}_B)$.

(i) We say that a state $f : \mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B \to \mathbb{C}$ is a *self-test* for the class $\mathcal{C}$ if there exists $\widetilde{S} \in \mathcal{C}$ such that $f = f_{\widetilde{S}}$ and, whenever $S \in \mathcal{C}$ is such that $f_S = f$, we have that $S \preceq \widetilde{S}$. In this case, we say that $\widetilde{S}$ is an *ideal model* for f.

(ii) We say that a state $f : \mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B \to \mathbb{C}$ is a *weak self-test* for the class $\mathcal{C}$ if there exists $\widetilde{S} \in \mathcal{C}$ such that $f = f_{\widetilde{S}}$ and, whenever $S \in \mathcal{C}$ is such that $f_S = f$, we have that $S \preceq_{\mathrm{a}} \widetilde{S}$. In this case, we say that $\widetilde{S}$ is an *weak ideal model* for f.

(iii) We say that a state $f : \mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B \to \mathbb{C}$ is an *abstract self-test* for $\mathcal{S}$ if there exists a unique state $g \in \mathcal{S}$ such that $g|_{\mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B} = f$.

It is clear that, in the notation of Definition 3.6, the state $f : \mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B \to \mathbb{C}$ is a self-test for the class $\mathcal{C}$ if and only if it is a self-test for the class $\mathcal{C}_f := \{S \in \mathcal{C} : f_S = f\}$.

Remark 3.7. The setup of Definition 3.6 will be applied in several different contexts in Section 5. The primary motivation behind it is the self-testing paradigm for no-signalling correlations of quantum type [38,43]. Letting X, Y, A, B be finite sets, $\mathcal{S}_{X,A}$ (resp. $\mathcal{S}_{Y,B}$)

be the universal operator system of $|X|$ (resp. $|Y|$) POVM's, each of cardinality $|A|$ (resp. $|B|$, see (27)), Definition 3.6 (i) provides a commuting operator generalisation of the usually employed notion of self-testing for quantum spacial systems (see e.g. [50]). We refer the reader to Subsection 5.1 for an extended discussion.

Note that *robust self-testing* for no-signalling correlations arising from finite dimensional models (see e.g. [56, Definition 3.3]) is a priori stronger than the corresponding notion of weak self-testing from Definition 3.6 (ii).

The remainder of this section is devoted to the proof of our main Theorem 3.13. To that end, we require some notation, terminology and a number of preparatory lemmas.

We recall that, if H is a Hilbert space, $\mathcal{M} \subseteq \mathcal{B}(H)$ is a von Neumann algebra and $\omega : \mathcal{M} \to \mathbb{C}$ is a normal positive functional, the *support* of ω [28, Definition 7.2.4] can be defined as the smallest projection $p \in \mathcal{M}$ such that $\omega(x) = \omega(pxp)$ for all $x \in \mathcal{M}$. For $\xi \in H$, write ω_ξ for the vector state, given by $\omega_\xi(x) = \langle x\xi, \xi\rangle$, $x \in \mathcal{B}(H)$. Note that, if p is the support of the functional $\omega_\xi|_{\mathcal{M}}$ then

$$p\xi = \xi. \tag{11}$$

Indeed,

$$\|\xi - p\xi\|^2 = \langle \xi, \xi\rangle - \langle \xi, p\xi\rangle - \langle p\xi, \xi\rangle + \langle p\xi, p\xi\rangle = \omega_\xi(1) - \omega_\xi(p) = 0,$$

and (11) follows.

Given a quantum commuting model $S = ({}_{\mathcal{A}}H_{\mathcal{B}^o}, \varphi_A, \varphi_B, \xi)$, where $\mathcal{A}, \mathcal{B}^o \subseteq \mathcal{B}(H)$, we let $\epsilon_A \in \mathcal{A}$ (resp. $\epsilon_B \in \mathcal{B}^o$) be the support projection of the state $\omega_\xi|_{\mathcal{A}}$ (resp. $\omega_\xi|_{\mathcal{B}^o}$) of $\mathcal{A}$ (resp. $\mathcal{B}^o$). Following [43], we say that S is *centrally supported* if

$$\varphi_A(s)\epsilon_A = \epsilon_A\varphi_A(s) \ \text{ and } \ \varphi_B(t)\epsilon_B = \epsilon_B\varphi_B(t), \quad s \in \mathcal{S}_A, t \in \mathcal{S}_B.$$

Lemma 3.8. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and $\xi \in H$ be a unit vector. Assume that ξ is separating for $\mathcal{A}$. Then, given $x \in \mathcal{A}$ and $\epsilon > 0$, there exists $y \in \mathcal{A}'$ such that $\|x\xi - y\xi\| < \epsilon$.*

Proof. Since ξ is separating for $\mathcal{A}$, it is cyclic for its commutant $\mathcal{A}'$. The statement is now immediate. $\square$

In what follows, for $\xi, \eta \in H$ and $\epsilon > 0$, we write $\xi \sim^\epsilon \eta$ if $\|\xi - \eta\| < \epsilon$.

Lemma 3.9. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and $\xi \in H$ be a unit vector. Let $\epsilon_A \in \mathcal{A}$ be the support of the state $\omega_\xi|_{\mathcal{A}}$ of $\mathcal{A}$, and $p_A \in \mathcal{B}(H)$ be the projection onto $\overline{\mathcal{A}\xi}$. Then $\xi \in \epsilon_A p_A H$, and ξ is cyclic and separating for the von Neumann subalgebra $\epsilon_A p_A \mathcal{A} \epsilon_A p_A$ of $\mathcal{B}(\epsilon_A p_A H)$.*

Proof. Note that, since $p_A \in \mathcal{A}'$ and $\epsilon_A \in \mathcal{A}$, we have that $\epsilon_A p_A \mathcal{A} \epsilon_A p_A$ is indeed a von Neumann algebra acting on $\epsilon_A p_A H$. By (11), $\epsilon_A p_A \xi = \xi$ and hence $\omega_{\epsilon_A p_A \xi}(\epsilon_A p_A a \epsilon_A p_A) = \omega_\xi(a)$, $a \in \mathcal{A}$.

Assume that $a \in \mathcal{A}$, $\epsilon_A p_A a \epsilon_A p_A \geq 0$ and $\omega_\xi(p_A \epsilon_A a \epsilon_A p_A) = 0$. Then

$$0 = p_A \epsilon_A a \epsilon_A p_A \xi = \epsilon_A a \epsilon_A p_A \xi = \epsilon_A a \epsilon_A \xi;$$

thus $\epsilon_A a^* \epsilon_A a \epsilon_A \xi = 0$ and hence $\omega_\xi(\epsilon_A a^* \epsilon_A a \epsilon_A) = 0$. As ω_ξ is faithful on $\epsilon_A \mathcal{A} \epsilon_A$, we get $\epsilon_A a^* \epsilon_A a \epsilon_A = 0$ and hence $\epsilon_A a \epsilon_A = 0$. This implies that ω_ξ is faithful on $\epsilon_A p_A \mathcal{A} \epsilon_A p_A$ and therefore, ξ is separating for $\epsilon_A p_A \mathcal{A} \epsilon_A p_A$.

Let $\eta \in H$ be such that $\langle \epsilon_A p_A a \epsilon_A p_A \xi, \epsilon_A p_A \eta \rangle = 0$ for all $a \in \mathcal{A}$. Then $\langle a\xi, p_A \epsilon_A \eta \rangle = 0$ and, since $p_A \epsilon_A \eta \in p_A H$, while $p_A H = \overline{\mathcal{A}\xi}$, we have $p_A \epsilon_A \eta = 0$, showing that ξ is cyclic. □

Lemma 3.10. *Let H be a Hilbert space, $\mathcal{A} \subseteq \mathcal{B}(H)$ be a von Neumann algebra, $\xi \in H$, and ϵ_A be the support projection of $\omega_\xi|_{\mathcal{A}}$. Suppose that for every selfadjoint $a \in \mathcal{A}$ and every $\epsilon > 0$, there exists $\hat{a} \in \mathcal{A}'$ such that $a\xi \sim^\epsilon \hat{a}\xi$. Then $[\epsilon_A, a] = 0$ for every $a \in \mathcal{A}$.*

Proof. By (11), $\epsilon_A \xi = \xi$. Thus

$$\epsilon_A a \xi \sim^\epsilon \epsilon_A \hat{a} \xi = \hat{a} \epsilon_A \xi = \hat{a}\xi \sim^\epsilon a\xi.$$

Therefore $\omega_\xi(a(1-\epsilon_A)a) = 0$ and, as ϵ_A is the support projection of $\omega_\xi|_{\mathcal{A}}$ and $a(1-\epsilon_A)a$ is positive, we obtain $\epsilon_A a(1-\epsilon_A)a\epsilon_A = 0$, and hence $(1-\epsilon_A)a\epsilon_A = 0$, that is, $a\epsilon_A = \epsilon_A a \epsilon_A$. As a is selfadjoint, this implies $\epsilon_A a = a\epsilon_A$. □

The next Lemma and Proposition are the main steps in the proof of Theorem 3.13.

Lemma 3.11. *Let $S = ({}_{\mathcal{A}}H_{\mathcal{B}}, \varphi_A, \varphi_B, \xi)$ and $\widetilde{S} = ({}_{\widetilde{\mathcal{A}}}\widetilde{H}_{\widetilde{\mathcal{B}}}, \widetilde{\varphi}_A, \widetilde{\varphi}_B, \widetilde{\xi})$ be quantum commuting models over $(\mathcal{S}_A, \mathcal{S}_B)$. Suppose that $S \preceq_{\mathrm{a}} \widetilde{S}$ and that $\widetilde{S}$ is centrally supported. Then the states $\omega_\xi \circ \pi_{\varphi_A \cdot \varphi_B}$ and $\omega_{\widetilde{\xi}} \circ \pi_{\widetilde{\varphi}_A \cdot \widetilde{\varphi}_B}$ of $C_u^*(\mathcal{S}_A) \otimes_{\max} C_u^*(\mathcal{S}_B)$ coincide.*

Proof. We write $\varphi = \varphi_A \cdot \varphi_B$ and $\widetilde{\varphi} = \widetilde{\varphi}_A \cdot \widetilde{\varphi}_B$ for brevity. It suffices to show that $(\omega_\xi \circ \pi_\varphi)(a \otimes b) = (\omega_{\widetilde{\xi}} \circ \pi_{\widetilde{\varphi}})(a \otimes b)$ holds for all elements $a \in C_u^*(S_A)$ (resp. $b \in C_u^*(S_B)$) of the form $a = s_1 \dots s_n$ (resp. $b = t_1 \dots t_n$), where $s_i \in \mathcal{S}_A$ (resp. $t_i \in \mathcal{S}_B$), $i = 1, \dots, n$. Write $\pi_A(a) = \pi_\varphi(a \otimes 1)$, $\widetilde{\pi}_A(a) = \pi_{\widetilde{\varphi}}(a \otimes 1)$, $\pi_B(b) = \pi_\varphi(1 \otimes b)$ and $\widetilde{\pi}_B(b) = \pi_{\widetilde{\varphi}}(1 \otimes b)$, $a \in C_u^*(\mathcal{S}_A)$, $b \in C_u^*(\mathcal{S}_B)$.

Let $(V_i)_{i \in \mathbb{N}}$ be a sequence of local isometries, where $V_i : H \to \widetilde{H} \otimes H_{i,\mathrm{aux}}$, $i \in \mathbb{N}$, implementing the preorder assumption $S \preceq_{\mathrm{a}} \widetilde{S}$ (see Definition 3.2 (ii)), and write $V_i = V_{i,2,2} \circ V_{i,1,2} = V_{i,2,1} \circ V_{i,1,1}$, where $V_{i,1,2}$, $V_{i,2,1}$ are $\mathcal{B}$-local (and $V_{i,1,1}$, $V_{i,2,2}$ are $\mathcal{A}$-local).

We have that

$$V_{i,1,2}\pi_A(a) = \sigma_{A,i}(\pi_A(a))V_{i,1,2} \text{ and } V_{i,2,2}\sigma_{A,i}(\pi_A(a)) = \rho_{A,i}(\pi_A(a))V_{i,2,2}, \tag{12}$$

for $a \in C^*_u(\mathcal{S}_A)$ and for some normal *-representations $\sigma_{A,i}$ and $\rho_{A,i}$ of $\mathcal{A}$, the latter mapping into $\pi_{\widetilde{H}}(\widetilde{\mathcal{A}})\bar{\otimes}\pi_{H_{i,\mathrm{aux}}}(\mathcal{A}_{i,\mathrm{aux}})$. Let $\widetilde{\epsilon}_A$ be the support projection of the restriction of $\omega_{\widetilde{\xi}}$ to $\pi_{\widetilde{H}}(\widetilde{\mathcal{A}})$ and $\widetilde{p}_A$ be the projection onto $\overline{\pi_{\widetilde{H}}(\widetilde{\mathcal{A}})\widetilde{\xi}}$. Set $\widetilde{q}_A = \widetilde{\epsilon}_A\widetilde{p}_A$. As $\widetilde{S}$ is centrally supported,

$$\widetilde{\epsilon}_A\widetilde{\pi}_A(a) = \widetilde{\pi}_A(a)\widetilde{\epsilon}_A, \quad a \in C^*_u(\mathcal{S}_A), \tag{13}$$

and hence

$$\widetilde{q}_A\widetilde{\pi}_A(a) = \widetilde{\pi}_A(a)\widetilde{q}_A, \quad a \in C^*_u(\mathcal{S}_A). \tag{14}$$

We claim that for every $n \in \mathbb{N}$, contractions $s_1, \dots, s_n \in \mathcal{S}_A$, $\epsilon > 0$, there exists i_0 such that, setting $a = s_1 \cdots s_n$, if $i \geq i_0$ then

$$\widetilde{\pi}_A(a)\widetilde{\xi} \otimes \xi_{i,\mathrm{aux}} \sim^{\epsilon} (\widetilde{\epsilon}_A \otimes I_{H_{i,\mathrm{aux}}})\rho_{A,i}(\pi_A(a))(\widetilde{\xi} \otimes \xi_{i,\mathrm{aux}}). \tag{15}$$

Let $n = 1$, $s \in \mathcal{S}_A$ be contractive and $\epsilon > 0$. By the relation $S \preceq_{\mathrm{a}} \widetilde{S}$ and (12), there exists i_0 such that, if $i \geq i_0$ then

$$\begin{aligned}\widetilde{\pi}_A(s)\widetilde{\xi} \otimes \xi_{i,\mathrm{aux}} &\sim^{\epsilon/2} V_{i,2,2}V_{i,1,2}\pi_A(s)\xi = V_{i,2,2}\sigma_{A,i}(\pi_A(s))V_{i,1,2}\xi \\ &= \rho_{A,i}(\pi_A(s))V_i\xi \\ &\sim^{\epsilon/2} \rho_{A,i}(\pi_A(s))(\widetilde{\xi} \otimes \xi_{i,\mathrm{aux}}).\end{aligned}$$

Claim (15) now follows from (11) and (13).

Assume the claim holds for $n > 1$. Consider $a = s_1 \dots s_n s_{n+1}$, where each s_i is a contraction in $\mathcal{S}_A$, and let $\epsilon > 0$. By hypothesis (and the case $n = 1$), there is an i_0 such that for every $i \geq i_0$,

$$\widetilde{\pi}_A(a')\widetilde{\xi} \otimes \xi_{i,\mathrm{aux}} \sim^{\epsilon/5} (\widetilde{\epsilon}_A \otimes 1_{H_{i,\mathrm{aux}}})\rho_{A,i}(\pi_A(a'))(\widetilde{\xi} \otimes \xi_{i,\mathrm{aux}}), \tag{16}$$

where $a' = s_1 \dots s_n$ or $a' = s_{n+1}$. By Lemmas 3.8 and 3.9, there exists $\tilde{a} \in (\widetilde{q}_A\pi_{\widetilde{H}}(\widetilde{\mathcal{A}})\widetilde{q}_A)' \subseteq \mathcal{B}(\widetilde{q}_A\widetilde{H})$ such that

$$\widetilde{\pi}_A(s_{n+1})\widetilde{\xi} = \widetilde{q}_A\widetilde{\pi}_A(s_{n+1})\widetilde{q}_A\widetilde{\xi} \sim^{\epsilon/5} \tilde{a}\widetilde{\xi}.$$

Using (12), (13), (14), (16) and the fact that

$$\rho_{A,i}(\pi_A(a')) \in \pi_{\widetilde{H}}(\widetilde{\mathcal{A}})\bar{\otimes}\pi_{H_{i,\mathrm{aux}}}(\mathcal{A}_{i,\mathrm{aux}}),$$

for $i \geq i_0$, we have

$$
\begin{aligned}
&\widetilde{\pi}_A(a')\widetilde{\pi}_A(s_{n+1})\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}} \sim^{\epsilon/5} \widetilde{\pi}_A(a')\tilde{a}\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}}\\
&\quad = \widetilde{q}_A\widetilde{\pi}_A(a')\widetilde{q}_A\tilde{a}\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}} = (\tilde{a}\otimes 1_{i,\mathrm{aux}})(\widetilde{\pi}_A(a')\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}})\\
&\quad \sim^{\epsilon/5} (\tilde{a}\otimes 1_{i,\mathrm{aux}})(\widetilde{\epsilon}_A\otimes 1_{H_{i,\mathrm{aux}}})\rho_{A,i}(\pi_A(a'))(\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}})\\
&\quad = (\widetilde{q}_A\otimes 1_{H_{i,\mathrm{aux}}})\rho_{A,i}(\pi_A(a'))(\tilde{a}\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}})\\
&\quad \sim^{\epsilon/5} (\widetilde{q}_A\otimes 1_{H_{i,\mathrm{aux}}})\rho_{A,i}(\pi_A(a'))(\widetilde{\pi}_A(s_{n+1})\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}})\\
&\quad \sim^{\epsilon/5} (\widetilde{q}_A\otimes 1_{H_{i,\mathrm{aux}}})\rho_{A,i}(\pi_A(a'))V_{i,2,2}V_{i,1,2}\pi_A(s_{n+1})\xi\\
&\quad = (\widetilde{q}_A\otimes 1_{H_{i,\mathrm{aux}}})\rho_{A,i}(\pi_A(a'))(V_{i,2,2}\sigma_{A,i}(\pi_A(s_{n+1}))V_{i,1,2}\xi\\
&\quad = (\widetilde{q}_A\otimes 1_{H_{i,\mathrm{aux}}})\rho_{A,i}(\pi_A(a's_{n+1}))V_{i,2,2}V_{i,1,2}\xi\\
&\quad = (\widetilde{q}_A\otimes 1_{H_{i,\mathrm{aux}}})\rho_{A,i}(\pi_A(a))V_i\xi\\
&\quad \sim^{\epsilon/5} (\widetilde{\epsilon}_A\otimes 1_{H_{i,\mathrm{aux}}})\rho_{A,i}(\pi_A(a))(\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}}).
\end{aligned}
$$

Thus,

$$\widetilde{\pi}_A(a)\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}} \sim^{\epsilon} (\widetilde{\epsilon}_A\otimes 1_{H_{i,\mathrm{aux}}})\rho_{A,i}(\pi_A(a))(\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}}).$$

A similar induction argument shows that, if $\widetilde{\epsilon}_B$ is the support projection of the restriction of the functional $\omega_{\widetilde{\xi}}$ on $\pi_{\widetilde{H}}(\mathcal{B}^o)$, then for all monomials $b\in C_u^*(\mathcal{S}_B)$ whose individual terms are contractions in $\mathcal{S}_B$, and $\epsilon>0$, there is an i_0 such that for $i\geq i_0$,

$$\widetilde{\pi}_B(b)\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}} \sim^{\epsilon} (\widetilde{\epsilon}_B\otimes 1_{H_{i,\mathrm{aux}}})\rho_{B,i}(\pi_B(b))(\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}}), \tag{17}$$

where $\rho_{B,i}$ is the normal $*$-representation of $\mathcal{B}^o$ constructed from the locality of V_i, analogous to $\rho_{A,i}$. We have that $\widetilde{\epsilon}_A\in\pi_{\widetilde{H}}(\widetilde{\mathcal{A}})$, $\widetilde{\epsilon}_B\in\pi_{\widetilde{H}}(\widetilde{\mathcal{B}}^o)$ and $[\pi_{\widetilde{H}}(x),\pi_{\widetilde{H}}(y^o)]=0$, $x\in\widetilde{\mathcal{A}}$, $y\in\widetilde{\mathcal{B}}$. Using (11), (13), (15), (17), the relations

$$[\widetilde{\pi}_A(a)\otimes 1_{H_{i,\mathrm{aux}}},\rho_{B,i}(\pi_B(b))]=0$$

and the fact that $V_{i,2,2}V_{i,1,2}=V_{i,2,1}V_{i,1,1}$, for monomials $a\in C_u^*(\mathcal{S}_A)$ and $b\in C_u^*(\mathcal{S}_B)$, whose individual terms are contractions in $\mathcal{S}_A$ and $\mathcal{S}_B$, respectively and $\epsilon>0$, there is an i_0 such that for $i\geq i_0$, we obtain

$$
\begin{aligned}
&(\omega_{\widetilde{\xi}}\circ\pi_{\widetilde{\varphi}})(a\otimes b) = \langle\widetilde{\pi}_A(a)\widetilde{\pi}_B(b)\widetilde{\xi},\widetilde{\xi}\rangle = \langle\widetilde{\pi}_A(a)\widetilde{\pi}_B(b)\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}},\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}}\rangle\\
&\sim^{\epsilon/4} \langle(\widetilde{\pi}_A(a)\widetilde{\epsilon}_B\otimes 1_{H_{i,\mathrm{aux}}})\rho_{B,i}(\pi_B(b))(\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}}),\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}}\rangle\\
&= \langle\rho_{B,i}(\pi_B(b))(\widetilde{\pi}_A(a)\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}}),\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}}\rangle\\
&\sim^{\epsilon/4} \langle\rho_{B,i}(\pi_B(b))(\widetilde{\epsilon}_A\otimes 1_{H_{i,\mathrm{aux}}})\rho_{A,i}(\pi_A(a))(\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}}),\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}}\rangle\\
&\sim^{\epsilon/4} \langle\rho_{B,i}(\pi_B(b))\rho_{A,i}(\pi_A(a))V_{i,2,2}V_{i,1,2}\xi,\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}}\rangle\\
&= \langle\rho_{B,i}(\pi_B(b))V_{i,2,2}V_{i,1,2}\pi_A(a)\xi,\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}}\rangle
\end{aligned}
$$

$$= \langle \rho_{B,i}(\pi_B(b))V_{i,2,1}V_{i,1,1}\pi_A(a)\xi, \widetilde{\xi} \otimes \xi_{i,\mathrm{aux}}\rangle$$
$$= \langle V_{i,2,1}V_{i,1,1}\pi_B(b)\pi_A(a)\xi, \widetilde{\xi} \otimes \xi_{i,\mathrm{aux}}\rangle$$
$$= \langle \pi_B(b)\pi_A(a)\xi, V_i^*(\widetilde{\xi} \otimes \xi_{i,\mathrm{aux}})\rangle$$
$$\sim^{\epsilon/4} \langle \pi_B(b)\pi_A(a)\xi, \xi\rangle = (\omega_\xi \circ \pi_\varphi)(a \otimes b).$$

Since ϵ is an arbitrary positive real, we conclude that

$$(\omega_{\widetilde{\xi}} \circ \pi_{\widetilde{\varphi}})(a \otimes b) = (\omega_\xi \circ \pi_\varphi)(a \otimes b).$$

By linearity, density and continuity it follows that $\omega_\xi \circ \pi_{\varphi_A \cdot \varphi_B} = \omega_{\widetilde{\xi}} \circ \pi_{\widetilde{\varphi}_A \cdot \widetilde{\varphi}_B}$. $\square$

Proposition 3.12. *Let S and $\widetilde{S}$ be models of a functional $f : \mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B \to \mathbb{C}$ such that $S \preceq_{\mathrm{a}} \widetilde{S}$. If S is a centrally supported Haag model then the model $\widetilde{S}$ is centrally supported.*

Proof. Let $S = ({}_{\mathcal{A}}H_{\mathcal{B}}, \varphi_A, \varphi_B, \xi)$ and $\widetilde{S} = ({}_{\widetilde{\mathcal{A}}}\widetilde{H}_{\widetilde{\mathcal{B}}}, \widetilde{\varphi}_A, \widetilde{\varphi}_B, \widetilde{\xi})$, and suppose that

$$V_i = V_{i,2,2}V_{i,1,2} = V_{i,2,1}V_{i,1,1}$$

is a local isometry from H to $\widetilde{H} \otimes H_{i,\mathrm{aux}}$, $i \in \mathbb{N}$, such that the sequence $(V_i)_{i\in\mathbb{N}}$ implements the relation $S \preceq_{\mathrm{a}} \widetilde{S}$; here $V_{i,1,2} : H \to L_i$ and $V_{i,2,2} : L_i \to \widetilde{H} \otimes H_{i,\mathrm{aux}}$, for some Hilbert space L_i, $i \in \mathbb{N}$. Let $\epsilon > 0$ and $s \in \mathcal{S}_A$ be self-adjoint. Since S is centrally supported, $\epsilon_A p_A \in \varphi_A(\mathcal{S}_A)'$, and thus

$$\varphi_A(s)\xi = \epsilon_A p_A \varphi_A(s)\epsilon_A p_A \xi.$$

Noting the equality $(\epsilon_A p_A \pi_H(\mathcal{A})\epsilon_A p_A)' = \epsilon_A p_A \pi_H(\mathcal{A})'\epsilon_A p_A$, by Lemmas 3.8 and 3.9, there exists $x \in \pi_H(\mathcal{A})'$, $x \neq 0$, such that

$$\varphi_A(s)\xi \sim^\epsilon \epsilon_A p_A x \epsilon_A p_A \xi = p_A x \xi.$$

Choose $i \in \mathbb{N}$ such that

$$\left\| V_i \varphi_A(s)\xi - \tilde{\varphi}_A(s)\tilde{\xi} \otimes \xi_{i,\mathrm{aux}} \right\| < \min\left\{\epsilon, \frac{\epsilon}{\|x\|}\right\}$$

and

$$\left\| V_i \xi - \tilde{\xi} \otimes \xi_{i,\mathrm{aux}} \right\| < \frac{\epsilon}{\|x\|}.$$

We have

$$\begin{aligned} \widetilde{\varphi}_A(s)\widetilde{\xi} \otimes \xi_{i,\mathrm{aux}} &\sim^\epsilon V_{i,2,2}V_{i,1,2}\varphi_A(s)\xi \sim^\epsilon V_{i,2,2}V_{i,1,2}p_A x \xi \\ &\sim^\epsilon V_{i,2,2}V_{i,1,2}p_A x V_{i,1,2}^* V_{i,2,2}^* \widetilde{\xi} \otimes \xi_{i,\mathrm{aux}}. \end{aligned}$$

By Haag duality, $p_A x \in \pi_H(\mathcal{A})' = \pi_H(\mathcal{B}^o)$; let $b^o \in \mathcal{B}^o$ be such that $p_A x = \pi_H(b^o)$, and note that

$$V_{i,2,2}V_{i,1,2}\pi_H(b^o)V_{i,1,2}^*V_{i,2,2}^* \in (\pi_H(\widetilde{\mathcal{B}}^o)\bar{\otimes}\pi_{H_{i,\mathrm{aux}}}(\mathcal{B}_{i,\mathrm{aux}}^o))V_{i,2,2}V_{i,2,2}^*.$$

Let $y \in \pi_{\widetilde{H}}(\widetilde{\mathcal{B}}^o)\bar{\otimes}\pi_{H_{i,\mathrm{aux}}}(\mathcal{B}_{i,\mathrm{aux}}^o)$ be such that

$$V_{i,2,2}V_{i,1,2}\pi_H(b^o)V_{i,1,2}^*V_{i,2,2}^* = yV_{i,2,2}V_{i,2,2}^*.$$

We have

$$\widetilde{\varphi}_A(s)\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}} \sim^{3\epsilon} yV_{i,2,2}V_{i,2,2}^*(\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}}) \sim^{2\epsilon} y(\widetilde{\xi}\otimes\xi_{i,\mathrm{aux}}).$$

If $\epsilon_{i,\mathrm{aux}}^A$ is the support projection of $\omega_{\xi_{i,\mathrm{aux}}}|_{\mathcal{A}_{i,\mathrm{aux}}}$, by Lemma 3.10, we obtain $[\widetilde{\varphi}_A(s)\otimes 1_{H_{i,\mathrm{aux}}}, \widetilde{\epsilon}_A\otimes\epsilon_{i,\mathrm{aux}}^A] = 0$ and hence $[\widetilde{\varphi}_A(s),\widetilde{\epsilon}_A] = 0$. Similar arguments applied to $\widetilde{\varphi}_B$ show that $\widetilde{S}$ is centrally supported. □

We are now ready to prove the main theorem of this section, relating weak self-test with abstract self-test.

Theorem 3.13. *Let $\mathcal{S}_A$ and $\mathcal{S}_B$ be operator systems and $\mathfrak{M}$ be a family of quantum commuting models over $(\mathcal{S}_A,\mathcal{S}_B)$. Let $\mathcal{S} = \{\tilde{f}_S : S \in \mathfrak{M}\}$ and $f : \mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B \to \mathbb{C}$ be the restriction of an element of $\mathcal{S}$. Assume that $\mathfrak{M}$ contains a centrally supported Haag model of f. If f a weak self-test for $\mathfrak{M}$ then f is an abstract self-test for $\mathcal{S}$.*

Proof. Let $\widetilde{S}$ be a weak ideal model for $\mathfrak{M}$. If $S \in \mathfrak{M}$ is a centrally supported Haag model of f then, by Proposition 3.12, $\widetilde{S}$ is centrally supported. The statement now follows from Lemma 3.11. □

In the finite dimensional tensor product setting of Example 2.6, the authors of [43] study the notion of a full rank model and its relevance to self-testing. In the sequel, we comment on a natural commuting operator version of this notion. Let $S = ({}_{\mathcal{A}}H_{\mathcal{B}}, \varphi_A, \varphi_B, \xi)$ be a quantum commuting model of $(\mathcal{S}_A,\mathcal{S}_B)$, and let ϵ_A (resp. ϵ_B) be the support of the functional $\omega_\xi|_{\mathcal{A}}$ (resp. $\omega_\xi|_{\mathcal{B}}$). Write $r = \epsilon_A\epsilon_B$, and let $\varphi_{A,r} : \mathcal{S}_A \to \mathcal{B}(rH)$ (resp. $\varphi_{B,r} : \mathcal{S}_B \to \mathcal{B}(rH)$) be the unital completely positive map, given by $\varphi_{A,r}(a) = r\varphi_A(a)r$ (resp. $\varphi_{B,r}(b) = r\varphi_B(b)r$). We note that, if $a \in \mathcal{S}_A$ and $b \in \mathcal{S}_B$ then

$$\begin{aligned}
\varphi_{A,r}(a)\varphi_{B,r}(b) &= \epsilon_A\epsilon_B\varphi_A(a)\epsilon_A\epsilon_B\varphi_B(b)\epsilon_A\epsilon_B \\
&= \epsilon_A\epsilon_B\varphi_A(a)\epsilon_B\epsilon_A\varphi_B(b)\epsilon_A\epsilon_B = \epsilon_A\epsilon_B\varphi_A(a)\varphi_B(b)\epsilon_A\epsilon_B \\
&= \epsilon_A\epsilon_B\varphi_B(b)\varphi_A(a)\epsilon_A\epsilon_B = \epsilon_B\epsilon_A\varphi_B(b)\varphi_A(a)\epsilon_B\epsilon_A \\
&= \epsilon_A\epsilon_B\varphi_B(b)\epsilon_A\epsilon_B\varphi_A(a)\epsilon_A\epsilon_B = \varphi_{B,r}(b)\varphi_{A,r}(a).
\end{aligned}$$

Taking into account (11), we see that $S_r := ({}_{r\mathcal{A}r}(rH)_{r\mathcal{B}r}, \varphi_{A,r}, \varphi_{B,r}, \xi)$ is a quantum commuting model. It is straightforward to check that $f_{S_r} = f_S$. We call the model S_r the *reduced model* of S.

A quantum commuting model $S = ({}_{\mathcal{A}}H_{\mathcal{B}}, \varphi_A, \varphi_B, \xi)$ is said to be of *full rank* if $\epsilon_A = \epsilon_B = I$.

Corollary 3.14. *Let $\mathfrak{M}$ be a family of quantum commuting models over $(\mathcal{S}_A, \mathcal{S}_B)$, closed under passing to reduced models, that contains a Haag model. Then every weak self-test for $\mathfrak{M}$ is an abstract self-test.*

Proof. Let S be a Haag model in $\mathfrak{M}$. The reduction S_r is a full rank Haag model. Since every full rank model is trivially centrally supported, the claim follows from Theorem 3.13. □

4. Abstract self-tests

In this section, we examine the property of being an abstract self-test and show that in some cases it is equivalent to the property of being a self-test. We fix some notation that will be used subsequently. Let $\mathcal{S}_A$ and $\mathcal{S}_B$ be operator systems, $\mathcal{S} \subseteq S(C_u^*(\mathcal{S}_A) \otimes_{\max} C_u^*(\mathcal{S}_B))$ and $\tilde{\mathcal{S}} = \{f|_{\mathcal{S}_A \otimes_c \mathcal{S}_B} : f \in \mathcal{S}\}$. For technical simplicity, we impose the restriction that all models we consider have associated Hilbert spaces that are separable.

The specific algebras $\mathcal{A}$ and $\mathcal{B}$ participating in a model $({}_{\mathcal{A}}H_{\mathcal{B}}, \varphi_A, \varphi_B, \xi)$ will not play a role in the next result, so we will temporarily omit them from the notation. The models $(H, \varphi_A, \varphi_B, \xi)$ and $(K, \psi_A, \psi_B, \eta)$ will be called *unitarily equivalent* if there exists a unitary operator $U : H \to K$ such that

$$U\xi = \eta \quad \text{and} \quad U\varphi_A(a)\varphi_B(b)U^* = \psi_A(a)\psi_B(b), \quad a \in \mathcal{S}_A, b \in \mathcal{S}_B.$$

Given an operator system $\mathcal{T}$, a Hilbert space H, and a unital completely positive map $\varphi : \mathcal{T} \to \mathcal{B}(H)$, for $\kappa \in \mathbb{N} \cup \{\infty\}$, we let $\ell^2(\kappa) = \oplus_{i=1}^{\kappa}\mathbb{C}$ (as usual, we set $\ell^2 = \ell^2(\infty)$), write $\varphi \otimes 1_\kappa : \mathcal{T} \to \mathcal{B}(H \otimes \ell^2(\kappa))$ for the map, given by $(\varphi \otimes 1_\kappa)(u) = \varphi(u) \otimes I_{\ell^2(\kappa)}$ and let $\varphi^{(\infty)} = \varphi \otimes 1_\infty$.

The following theorem is a version of [43, Theorem 7.5] in the case of arbitrary bipartite quantum systems. For a convex set C, we denote by $\mathrm{Ext}(C)$ the set of extreme points of C.

Theorem 4.1. *Let $f \in \mathrm{Ext}(S(\mathcal{S}_A \otimes_c \mathcal{S}_B))$. The following are equivalent:*

(i) *f is an abstract self-test for $\mathcal{S} = S(C_u^*(\mathcal{S}_A) \otimes_{\max} C_u^*(\mathcal{S}_B))$;*
(ii) *there exists a model $\tilde{S} = (\tilde{H}, \tilde{\varphi}_A, \tilde{\varphi}_B, \tilde{\xi})$ of f such that, for every model S of f, there exists a unit vector $\xi_{\mathrm{aux}} \in \ell^2$ such that S is unitarily equivalent to the model $(\tilde{H} \otimes \ell^2, \tilde{\varphi}_A^{(\infty)}, \tilde{\varphi}_B^{(\infty)}, \tilde{\xi} \otimes \xi_{\mathrm{aux}})$.*

Proof. (i)⇒(ii) We set $\mathcal{A} = C_u^*(\mathcal{S}_A)$ and $\mathcal{B} = C_u^*(\mathcal{S}_B)$ for brevity. Let $\tilde{f} \in \mathcal{S}$ be the unique extension of f to a state on $\mathcal{A} \otimes_{\max} \mathcal{B}$, and note that $\tilde{f} \in \mathrm{Ext}(S(\mathcal{A} \otimes_{\max} \mathcal{B}))$. Indeed, if $g_1, g_2 \in S(\mathcal{A} \otimes_{\max} \mathcal{B})$ are such that $\tilde{f} = \lambda g_1 + (1-\lambda) g_2$ for some $\lambda \in (0,1)$, then

$$f = \lambda g_1|_{\mathcal{S}_A \otimes_c \mathcal{S}_B} + (1-\lambda) g_2|_{\mathcal{S}_A \otimes_c \mathcal{S}_B}$$

and, by the extremality of f, we have that

$$f = g_1|_{\mathcal{S}_A \otimes_c \mathcal{S}_B} = g_2|_{\mathcal{S}_A \otimes_c \mathcal{S}_B}.$$

Since f is an abstract self-test for $\mathcal{S}$, we have that $g_1 = g_2 = \tilde{f}$.

Let $(\tilde{H}, \tilde{\pi}, \tilde{\xi})$ be the GNS triple associated with $\tilde{f}$; thus, $\tilde{H}$ is a Hilbert space, $\tilde{\pi} : \mathcal{A} \otimes_{\max} \mathcal{B} \to \mathcal{B}(\tilde{H})$ is a unital *-representation and $\tilde{\xi} \in \tilde{H}$ is a unit vector, cyclic for $\tilde{\pi}$, such that

$$\tilde{f}(u) = \langle \tilde{\pi}(u)\tilde{\xi}, \tilde{\xi} \rangle, \quad u \in \mathcal{A} \otimes_{\max} \mathcal{B}. \tag{18}$$

Using the inclusion $\mathcal{S}_A \otimes_c \mathcal{S}_B \subseteq \mathcal{A} \otimes_{\max} \mathcal{B}$, let $\tilde{\varphi}_A = \tilde{\pi}|_{\mathcal{S}_A}$, $\tilde{\varphi}_B = \tilde{\pi}|_{\mathcal{S}_B}$, and $\tilde{S} = (\tilde{H}, \tilde{\varphi}_A, \tilde{\varphi}_B, \tilde{\xi})$; by (18), $\tilde{S}$ is a model of f.

Let $S = (H, \varphi_A, \varphi_B, \xi)$ be a model of f, and write π_A (resp. π_B) for the unique extension of φ_A (resp. φ_B) to a unital *-homomorphism from $\mathcal{A}$ (resp. $\mathcal{B}$) into $\mathcal{B}(H)$. Since the ranges of φ_A and φ_B commute, so do the ranges of π_A and π_B; let $\pi = \pi_A \cdot \pi_B$, viewed as a unital *-representation of $\mathcal{A} \otimes_{\max} \mathcal{B}$ on H. Since f is an abstract self-test for $\mathcal{S}$, we have that $\tilde{f}(u) = \langle \pi(u)\xi, \xi \rangle$, $u \in \mathcal{A} \otimes_{\max} \mathcal{B}$. Write $H = \oplus_{i \in \mathbb{N}} H_i$ and $\xi = (\xi_i)_{i \in \mathbb{N}}$, where $\xi_i \in H_i$, $i \in \mathbb{N}$, H_i is invariant for π, and ξ_i is a cyclic vector for the representation $\pi_i := \pi|_{H_i}$. Let $\lambda_i = \|\xi_i\|^2$, $i \in \mathbb{N}$; thus, $\sum_{i=1}^{\infty} \lambda_i = 1$. Setting $\eta_i = \frac{1}{\sqrt{\lambda_i}} \xi_i$, we have that

$$\tilde{f}(u) = \sum_{i=1}^{\infty} \lambda_i \langle \pi_i(u)\eta_i, \eta_i \rangle, \quad u \in \mathcal{A} \otimes_{\max} \mathcal{B}.$$

Since $\tilde{f}$ is an extreme point of $S(\mathcal{A} \otimes_{\max} \mathcal{B})$, we have that

$$\tilde{f}(u) = \langle \pi_i(u)\eta_i, \eta_i \rangle, \quad u \in \mathcal{A} \otimes_{\max} \mathcal{B}, i \in \mathbb{N}.$$

Since the vector η_i is cyclic for π_i, there exists a unitary operator $U_i : H_i \to \tilde{H}$, such that

$$U_i \pi_i(u) U_i^* = \tilde{\pi}(u) \text{ and } U_i \eta_i = \tilde{\xi}, \quad i \in \mathbb{N}. \tag{19}$$

Set $\xi_{\mathrm{aux}} = (\sqrt{\lambda_i})_{i \in \mathbb{N}}$, viewed as a (unit) vector in ℓ^2. Write $U = \oplus_{i \in \mathbb{N}} U_i$; thus, $U : H \to \oplus_{i \in \mathbb{N}} \tilde{H}$ is a unitary operator. Identifying $\oplus_{i \in \mathbb{N}} \tilde{H}$ with $\tilde{H} \otimes \ell^2$ canonically, we have that

$$U\xi = U\left((\sqrt{\lambda_i}\eta_i)_{i\in\mathbb{N}}\right) = \left(\sqrt{\lambda_i}U_i\eta_i\right)_{i\in\mathbb{N}} = \left(\sqrt{\lambda_i}\tilde{\xi}\right)_{i\in\mathbb{N}} = \tilde{\xi}\otimes\xi_{\text{aux}}.$$

Equation (19) now implies that S is unitarily equivalent, via U, to the model $(\tilde{H}\otimes\ell^2, \tilde{\varphi}_A^{(\infty)}, \tilde{\varphi}_B^{(\infty)}, \tilde{\xi}\otimes\xi_{\text{aux}})$.

(ii)⇒(i) In the notation of (ii), let $\tilde{f}:\mathcal{A}\otimes_{\max}\mathcal{B}\to\mathbb{C}$ be the state, given by (18). Let $g\in\mathcal{S}$ be an extension of f, (H,π,ξ) be the GNS triple associated with g, and φ_A (resp. φ_B) be the restrictions of π to $\mathcal{S}_A$ (resp. $\mathcal{S}_B$). By assumption, there exists a unit vector $\xi_{\text{aux}}\in\ell^2$ and a unitary operator $U: H\to\tilde{H}\otimes\ell^2$, such that $U\xi = \tilde{\xi}\otimes\xi_{\text{aux}}$, and

$$U\varphi_A(a)\varphi_B(b)U^* = \tilde{\pi}(a\otimes b)\otimes I, \quad a\in\mathcal{S}_A, b\in\mathcal{S}_B. \tag{20}$$

Let π_A (resp. π_B) be the unique extension of φ_A (resp. φ_B) to a unital *-representation of $\mathcal{A}$ (resp. $\mathcal{B}$). We have that

$$\pi(a\otimes b) = (\pi_A\cdot\pi_B)(a\otimes b), \quad a\in\mathcal{S}_A, b\in\mathcal{S}_B;$$

since the elementary tensors of the form $a\otimes b$, $a\in\mathcal{S}_A$, $b\in\mathcal{S}_B$, generate $\mathcal{A}\otimes_{\max}\mathcal{B}$ as a C*-algebra, we have that $\pi = \pi_A\cdot\pi_B$. Now (20) implies that

$$U\pi(u)U^* = \tilde{\pi}(u)\otimes I, \quad u\in\mathcal{A}\otimes_{\max}\mathcal{B}.$$

It follows that, if $u\in\mathcal{A}\otimes_{\max}\mathcal{B}$, then

$$\begin{aligned} g(u) &= \langle\pi(u)\xi,\xi\rangle = \langle U\pi(u)U^*U\xi, U\xi\rangle \\ &= \langle(\tilde{\pi}(u)\otimes I)(\tilde{\xi}\otimes\xi_{\text{aux}}), \tilde{\xi}\otimes\xi_{\text{aux}}\rangle = \langle\tilde{\pi}(u)\tilde{\xi},\tilde{\xi}\rangle; \end{aligned}$$

Thus, $g = \tilde{f}$, and the proof is complete. □

Remark 4.2. In the notation of Theorem 4.1, suppose that a state $f\in\tilde{\mathcal{S}}$ is an abstract self-test for $\mathcal{S}$. By Theorem 4.1, f has a model $\tilde{S}$, such that every other model of f is unitarily equivalent to an ampliation of $\tilde{S}$. Let

$$\mathfrak{M} = \left\{S = ({}_{\mathcal{A}}H_{\mathcal{B}}, \varphi_A, \varphi_B, \xi) : f_S = f, \mathcal{A} = (\varphi_A(S_A))'', \mathcal{B} = ((\varphi_B(S_B))'')^o\right\}. \tag{21}$$

The proof of Theorem 4.1 shows that the unitary operator U, constructed therein, implements the dilation relation ${}_{\mathcal{A}}H_{\mathcal{B}} \leq_{\tilde{\mathcal{A}}\otimes I} (\tilde{H}\otimes\ell^2)_{\tilde{\mathcal{B}}\otimes I}$. Thus, the problem in reversing the implication established in Theorem 3.13 resides in allowing the use of models, whose algebras of observables are more general than the ones indicated in (21). Indeed, often the interest lies in *Haag models*, where $\mathcal{B}^o = \mathcal{A}'$. We next exhibit cases where such a reversal can be achieved more generally. Our next result, Theorem 4.3, is an extension of [43, Theorem 4.12].

In the next theorem (Theorem 4.3), we establish a partial converse to Theorem 3.13. We first gather some brief preliminary background on type I representations of C*-algebras. Let $\mathcal{A}$ be a C^*-algebra. Recall that a representation $\pi : \mathcal{A} \to \mathcal{B}(K)$ is called a type I representation if the bicommutant $\pi(\mathcal{A})''$ in $\mathcal{B}(K)$ is a von Neumann algebra of type I. Let $\{H(\gamma) : \gamma \in \Gamma\}$ be a measurable field of Hilbert spaces on a Borel space Γ equipped with a σ-finite Borel measure μ. Suppose that, to each $\gamma \in \Gamma$, there corresponds a representation π_γ of $\mathcal{A}$ on $H(\gamma)$ such that for every $a \in \mathcal{A}$, the operator field $\gamma \mapsto \pi_\gamma(a) \in \mathcal{B}(H(\gamma))$ is measurable, in which case we say that the field $\{\pi_\gamma\}_{\gamma\in\Gamma}$ is a measurable field of representations. Set $H = \int_\Gamma^\oplus H(\gamma)d\mu(\gamma)$ and $\pi(a) = \int_\Gamma^\oplus \pi_\gamma(a)d\mu(\gamma) \in \mathcal{B}(H)$. Then π is a representation of $\mathcal{A}$ on H, called the direct integral of $\{\pi_\gamma\}_{\gamma\in\Gamma}$ and written $\pi = \int_\Gamma^\oplus \pi_\gamma d\mu(\gamma)$. We refer the reader to [51, §IV.8] for further details concerning measurable fields of Hilbert spaces and operators. By [20], if π is a representation of type I on a separable Hilbert space, there is a standard σ-finite measure space (Γ, μ) a measurable field of irreducible representations $\{\pi_\gamma\}_{\gamma\in\Gamma}$, measurable function $\gamma \mapsto n(\gamma) \in \mathbb{N}_\infty = \{1, 2, \ldots, \infty\}$ such that

$$\pi \simeq \int_\Gamma^\oplus \pi_\gamma \otimes 1_{n(\gamma)} d\mu(\gamma).$$

For the remainder of this section, $\mathcal{C}$ will denote the class of quantum commuting models defined by the canonical bimodules ${}_{\mathcal{B}(H_A)}H_{\mathcal{B}(H_B)^o} = H_A \otimes H_B$ over $(\mathcal{B}(H_A), \mathcal{B}(H_B)^o)$ and unital completely positive maps $\varphi = \varphi_A \otimes \varphi_B$, where $\varphi_A : \mathcal{S}_A \to \mathcal{B}(H_A)$ and $\varphi_B : \mathcal{S}_B \to \mathcal{B}(H_B)$ extend to type I representations π_A and π_B of $\mathcal{A} := C_u^*(\mathcal{S}_A)$ and $\mathcal{B} := C_u^*(\mathcal{S}_B)$ on H_A and H_B, respectively. We note that, if $S \in \mathcal{C}$ then f_S is in fact a state (not only on the commuting tensor product $\mathcal{S}_A \otimes_{\rm c} \mathcal{S}_B$ but also) on the minimal tensor product $\mathcal{S}_A \otimes_{\min} \mathcal{S}_B$. We say that a model is irreducible if the corresponding representations π_A and π_B are irreducible. We note that, if a model $S \in \mathcal{C}$ is irreducible then $\pi_A(\mathcal{S}_A)'' = \mathcal{B}(H_A)$ and $\pi_B(\mathcal{S}_B)'' = \mathcal{B}(H_B)$.

Theorem 4.3. *Let $\mathcal{S} = \{\tilde{f}_S : S \in \mathcal{C}\}$, let $\mathfrak{M} \in \mathcal{C}$, and let $f = f_{\mathfrak{M}}$. Assume that $f \in \mathrm{Ext}(S(\mathcal{S}_A \otimes_{\rm c} \mathcal{S}_B))$. Suppose that f has unique extension to a state on $C_u^*(\mathcal{S}_A) \otimes_{\min} C_u^*(\mathcal{S}_B)$. Then f is a self-test for $\mathcal{C}$ and there exists an irreducible ideal model $S \in \mathcal{C}$. In particular, if f is an abstract self-test for $\mathcal{S}$ then f is a self-test for $\mathcal{C}$ that admits an irreducible ideal model $S \in \mathcal{C}$.*

Proof. Write $\mathfrak{M} = ({}_{\mathcal{B}(H_A)}(H_A \otimes H_B)_{\mathcal{B}(H_B)^o}, \varphi_A, \varphi_B, \xi)$, so that

$$f(u) = \langle (\varphi_A \otimes \varphi_B)(u)\xi, \xi \rangle, \quad u \in \mathcal{S}_A \otimes_{\min} \mathcal{S}_B.$$

By assumption, the *-representations π_A and π_B extending φ_A and φ_B, respectively, are type I. Consider the direct integral decompositions of π_A and π_B into irreducible representations:

$$\pi_A = \int_X (\pi_x^A \otimes I_{m(x)}) d\mu(x) \text{ and } \pi_B = \int_Y (\pi_y^A \otimes I_{n(y)}) d\nu(y),$$

acting on

$$H_A = \int_X^{\oplus} H_A(x) \otimes \ell^2(m(x)) d\mu(x) \text{ and } H_B = \int_Y^{\oplus} H_B(y) \otimes \ell^2(n(y)) d\nu(y),$$

respectively. Let $\{e_x^i\}_{i=1}^{m(x)}$ and $\{e_y^j\}_{j=1}^{n(y)}$ be the standard bases in $\ell^2(m(x))$ and $\ell^2(n(y))$, respectively, and write δ for the counting measure on $\mathbb{N}_\infty \times \mathbb{N}_\infty$ (we set $\mathbb{N}_k = \{1, 2, \dots, k\}$, so that $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$). As $\xi \in H_A \otimes H_B$, there exist measurable families $(\xi_{x,y}^{i,j})_{x,y}^{i,j}$, where $\xi_{x,y}^{i,j} \in H_A(x) \otimes H_B(y)$, $\|\xi_{x,y}^{i,j}\| = 1$, and $(\lambda_{x,y}^{i,j})_{x,y}^{i,j} \subseteq \mathbb{C}$, such that $\lambda_{x,y}^{i,j} = 0$ if $(i, j) \in (\mathbb{N}_{m(x)} \times \mathbb{N}_{n(y)})^c$,

$$\int_{X \times Y} \int_{\mathbb{N} \times \mathbb{N}} |\lambda_{x,y}^{i,j}|^2 d\delta(i, j) d(\mu \times \nu)(x, y) = 1,$$

and

$$\xi = \int_{X \times Y} \int_{\mathbb{N} \times \mathbb{N}} \xi_{x,y}^{i,j} \otimes \lambda_{x,y}^{i,j} (e_x^i \otimes e_y^j) d\delta(i, j) d(\mu \times \nu)(x, y).$$

Write

$$S_{x,y}^{i,j} = (H_A(x) \otimes H_B(y), \pi_x^A|_{\mathcal{S}_A}, \pi_y^B|_{\mathcal{S}_B}, \xi_{x,y}^{i,j}), \quad x, y \in X, i, j \in \mathbb{N}_\infty.$$

As π_A and π_B are type I representations, so are π_x^A and π_y^B for almost all x and y [20, Proposition 8.4.8]. Thus, for almost all x, y, the model $S_{x,y}^{i,j}$ belongs to the class $\mathcal{C}$. Moreover,

$$f(u) = \int_{X \times Y} \int_{\mathbb{N} \times \mathbb{N}} \langle (\pi_x^A \otimes \pi_y^B)(u) \xi_{x,y}^{i,j}, \xi_{x,y}^{i,j} \rangle |\lambda_{x,y}^{i,j}|^2 d\delta(i, j) d(\mu \times \nu)(x, y).$$

Consider the probability measure α on $X \times Y \times \mathbb{N}_\infty \times \mathbb{N}_\infty$, given by

$$\alpha(E) = \int_E |\lambda_{x,y}^{i,j}|^2 d\delta(i, j) d(\mu \times \nu)(x, y),$$

where $E \subseteq X \times Y \times \mathbb{N}_\infty \times \mathbb{N}_\infty$ is measurable. Suppose that $\alpha(E) \neq 0$ and $\alpha(E^c) \neq 0$, set

$$f_1(u) = \frac{1}{\alpha(E)} \int_E \langle (\pi_x^A \otimes \pi_y^B)(u) \xi_{x,y}^{i,j}, \xi_{x,y}^{i,j} \rangle |\lambda_{x,y}^{i,j}|^2 d\delta(i, j) d(\mu \times \nu)(x, y),$$

considered as a state on $C_u^*(\mathcal{S}_A) \otimes_{\min} C_u^*(\mathcal{S}_B)$, and let f_2 be defined similarly, using the set E^c in the place of E. Then

$$f(u) = \alpha(E) f_1(u) + \alpha(E^c) f_2(u), \quad u \in \mathcal{S}_A \otimes_{\min} \mathcal{S}_B.$$

Since f is an extreme point,

$$f(u) = f_1(u) = f_2(u), \quad u \in \mathcal{S}_A \otimes_{\min} \mathcal{S}_B.$$

It follows that

$$\int_E \langle (\pi_x^A \otimes \pi_y^B)(u) \xi_{x,y}^{i,j}, \xi_{x,y}^{i,j} \rangle |\lambda_{x,y}^{i,j}|^2 d\delta(i,j) d(\mu \times \nu)(x,y)$$
$$= \int_E f(u) |\lambda_{x,y}^{i,j}|^2 d\delta(i,j) d(\mu \times \nu)(x,y).$$

As the equality trivially holds for E of full measure, we obtain that

$$f(u) = \left\langle (\pi_x^A \otimes \pi_y^B)(u) \xi_{x,y}^{i,j}, \xi_{x,y}^{i,j} \right\rangle \quad \alpha\text{-almost everywhere,} \tag{22}$$

showing in particular that there is an irreducible model in $\mathcal{C}$ that gives rise to f.

Fix irreducible representations $\tilde{\pi}^A$ and $\tilde{\pi}^B$ of $C_u^*(\mathcal{S}_A)$ and $C_u^*(\mathcal{S}_B)$, acting on Hilbert spaces $\tilde{H}_A$ and $\tilde{H}_B$, respectively, and a unit vector $\tilde{\xi} \in \tilde{H}_A \otimes \tilde{H}_B$, such that $f(u) = \langle (\tilde{\pi}^A \otimes \tilde{\pi}^B)(u) \tilde{\xi}, \tilde{\xi} \rangle$, $u \in \mathcal{S}_A \otimes_{\min} \mathcal{S}_B$, and let

$$N = \{(x, y, i, j) \in X \times Y \times \mathbb{N}_\infty \times \mathbb{N}_\infty : (22) \text{ holds}\}.$$

Then $\alpha(N^c) = 0$. Since f has a unique extension to $C_u^*(\mathcal{S}_A) \otimes_{\min} C_u^*(\mathcal{S}_B)$, we have that

$$\left\langle (\tilde{\pi}^A \otimes \tilde{\pi}^B)(u) \tilde{\xi}, \tilde{\xi} \right\rangle = \left\langle (\pi_x^A \otimes \pi_y^B)(u) \xi_{x,y}^{i,j}, \xi_{x,y}^{i,j} \right\rangle, \quad (x, y, i, j) \in N,$$

for every $u \in C_u^*(\mathcal{S}_A) \otimes_{\min} C_u^*(\mathcal{S}_B)$. As π_x^A and π_y^B are irreducible, so is $\pi_x^A \otimes \pi_y^B$ and hence $(\pi_x^A \otimes \pi_y^B, \xi_{x,y}^{i,j})$ is a GNS representation for the state f, $(x, y, i, j) \in N$. We have, in particular, $\xi_{x,y}^{i,j} = \alpha_{x,y}^{i,j,i',j'} \xi_{x,y}^{i',j'}$ for some $\alpha_{x,y}^{i,j,i',j'} \in \mathbb{T}$.

Let $\leq$ be the lexicographic order on $\mathbb{N}_\infty \times \mathbb{N}_\infty$, that is, $(n_1, n_2) < (m_1, m_2)$ if $n_1 < m_1$ or $n_1 = m_1$ and $n_2 < m_2$. Define $\tau : X \times Y \to \mathbb{N}_\infty \times \mathbb{N}_\infty \cup \{\infty\}$ by letting $\tau(\omega) = \min\{(i, j) : (\omega, i, j) \in N\}$, for $\omega = (x, y) \in X \times Y$, where we have set $\min \emptyset = \infty$. Let $\pi_{X \times Y} : (X \times Y) \times (\mathbb{N}_\infty \times \mathbb{N}_\infty) \to X \times Y$ be the projection map. Clearly, $(\omega, \tau(\omega)) \in N$ for every $\omega \in \pi_{X \times Y}(N)$. We claim that τ is measurable. Indeed, writing $N_{i,j}$ for the slice of N along $(i, j) \in \mathbb{N}_\infty \times \mathbb{N}_\infty$, note that $N \cap ((X \times Y) \times \{(i, j)\}) = N_{i,j} \times \{(i, j)\}$ and hence $N_{i,j}$ is measurable. This shows that $\tau^{-1}(\{(i, j)\}) = N_{i,j} \setminus \cup_{(i',j') < (i,j)} N_{i',j'}$ is measurable. In addition, the set $\pi_{X \times Y}(N) = \cup_{i,j} N_{i,j}$ is measurable. Set $\zeta_{x,y} = \xi_{x,y}^{\tau(x,y)} \in H_A(x) \otimes H_B(y)$. For $(x, y) \in X \times X$, let

$$\tilde{f}_{x,y} = \sum_{(i,j):(x,y,i,j)\in N} \alpha_{x,y}^{(i,j),\tau(x,y)} \lambda_{x,y}^{i,j} (e_x^i \otimes e_y^j).$$

Then

$$\begin{aligned}\xi &= \int_N \xi_{x,y}^{i,j} \otimes \lambda_{x,y}^{i,j} (e_x^i \otimes e_y^j) d\delta(i,j) d(\mu\times\nu)(x,y) \\ &= \int_{\pi_{X\times Y}(N)} \zeta_{x,y} \otimes \tilde{f}_{x,y} d(\mu\times\nu)(x,y).\end{aligned}$$

Let $r_{x,y} = \|\tilde{f}_{x,y}\|$. If $\tilde{f}_{x,y} \neq 0$, set $f_{x,y} = \frac{\tilde{f}_{x,y}}{\|\tilde{f}_{x,y}\|}$; otherwise, let $f_{x,y} = 0$. We thus have that $\int_{X\times Y} |r_{x,y}|^2 d(\mu\times\nu)(x,y) = 1$ and $\{f_{x,y}\}_{(x,y)\in X\times Y}$ is a measurable field of unit vectors in $\ell^2(m(x)) \otimes \ell^2(n(y))$ with

$$r_{x,y} f_{x,y} = \int \alpha_{x,y}^{(i,j),\tau(x,y)} \lambda_{x,y}^{i,j} (e_x^i \otimes e_y^j) d\delta(i,j).$$

Consider now the set $\Lambda = \{(x,y) \in \pi_{X\times Y}(N) : r_{x,y} \neq 0\}$. Then for $(x,y) \in \Lambda$, $(\pi_x^A \otimes \pi_y^B, \zeta_{x,y})$ is a GNS representation of f. Since π_A and π_B are irreducible, we obtain that $\pi_x^A \otimes \pi_y^B \sim \tilde{\pi}^A \otimes \tilde{\pi}^B$ and hence $\pi_x^A \sim \tilde{\pi}^A$ and $\pi_y^B \sim \tilde{\pi}^B$ whenever $(x,y) \in \Lambda$ (we use the symbol $\sim$ to denote unitary equivalence). Let $\Lambda_A = \pi_X(\Lambda)$ and $\Lambda_B = \pi_Y(\Lambda)$, where π_X and π_Y are the corresponding projections in the Cartesian product $X \times Y$. We have that Λ_A and Λ_B are analytic sets and there exist subsets $M_A \subseteq \Lambda_A$ and $M_B \subseteq \Lambda_B$, such that $\mu(M_A) = \nu(M_B) = 0$ and $\widetilde{\Lambda}_A := \Lambda_A \setminus M_A$ and $\widetilde{\Lambda}_B := \Lambda_B \setminus M_B$ are measurable (see [51, Appendix]). By [21, p. 166, Lemme 2], there exist measurable $U_x : H_A(x) \to \tilde{H}_A$ and $U_y : H_B(y) \to \tilde{H}_B$ such that

$$U_x \pi_x^A(a) U_x^* = \tilde{\pi}^A(a) \ \text{ and } \ U_y \pi_y^B(b) U_y^* = \tilde{\pi}^B(b), \quad a \in C_u^*(\mathcal{S}_A), b \in C_u^*(\mathcal{S}_B).$$

Then $(U_x \otimes U_y)\zeta_{x,y} = \beta_{x,y}\tilde{\xi}$ for $\beta_{x,y} \in \mathbb{T}$. Hence $\xi = \tilde{\xi} \otimes \psi_{\mathrm{aux}}$, where

$$\psi_{\mathrm{aux}} = \int_{\widetilde{\Lambda}_A \times \widetilde{\Lambda}_B} r_{x,y} \beta_{x,y} f_{x,y} d(\mu\times\nu)(x,y).$$

For $x \in \widetilde{\Lambda}_A$ let $V_x : H_A(x) \otimes \ell^2(m(x)) \to \tilde{H}_A \otimes \ell^2(m(x))$ be given by $V_x = U_x \otimes 1_{m(x)}$; if $x \notin \widetilde{\Lambda}_A$ let $V_x : H_A(x) \otimes \ell^2(m(x)) \to \tilde{H}_A \otimes (H_A(x) \otimes \ell^2(m(x)))$ be given by $V_x(v) = w \otimes v$ for a fixed $w \in \tilde{H}_A$, and set $V_A = \int_X^\oplus V_x d\mu(x)$. Define an isometry V_B in a similar way. Let

$$H_A^{\mathrm{aux}} = \int_{\widetilde{\Lambda}_A}^{\oplus} \ell^2(m(x)) d\mu(x) \oplus \int_{\widetilde{\Lambda}_A^c}^{\oplus} H_A(x) \otimes \ell^2(m(x)) d\mu(x)$$

and

$$H_B^{\mathrm{aux}} = \int\limits_{\tilde{\Lambda}_B}^{\oplus} \ell^2(n(y))d\nu(y) \oplus \int\limits_{\tilde{\Lambda}_B^c}^{\oplus} H_B(y) \otimes \ell^2(n(y))d\nu(y).$$

Then $\psi_{\mathrm{aux}} \in H_A^{\mathrm{aux}} \otimes H_B^{\mathrm{aux}}$ and

$$(V_A \otimes V_B)(\pi_A \otimes \pi_B)(a \otimes b)\xi = (\tilde{\pi}_A \otimes \tilde{\pi}_B)(a \otimes b)(\tilde{\xi} \otimes \psi_{\mathrm{aux}})$$

for $a \in C_u^*(\mathcal{S}_A)$, $b \in C_u^*(\mathcal{S}_B)$. $\square$

5. Applications and examples

In this section, we apply the general operator system framework developed in the previous sections to several special cases, including those of QNS correlations, quantum graph homomorphisms, synchronous correlations and positive definite functions defined on groups. The special cases we consider are based at pairs $(\mathcal{S}_A, \mathcal{S}_B)$ of finitely generated operator systems, say

$$\mathcal{S}_A = \mathrm{span}\{e_1, \ldots, e_k\} \text{ and } \mathcal{S}_B = \mathrm{span}\{f_1, \ldots, f_l\},$$

so that the pair (φ_A, φ_B) of unital completely positive maps, where $\varphi_A : \mathcal{S}_A \to \mathcal{B}(H)$ and $\varphi_B : \mathcal{S}_B \to \mathcal{B}(H)$, is determined by the mutually commuting families $(E_i)_{i=1}^k$ and $(F_j)_{j=1}^l$ of operators on H vis the assignments $\varphi_A(e_i) = E_i$, $i \in [k]$ and $\varphi_B(f_j) = F_j$, $j \in [l]$. Thus, we will consider a commuting operator model over $(\mathcal{S}_A, \mathcal{S}_B)$ as a tuple $S = (H, (E_i)_{i=1}^k, (F_j)_{j=1}^l, \xi)$, where $\xi \in H$ is a unit vector. The tuple S gives rise to the *correlation* $p_S : [k] \times [l] \to \mathbb{C}$, given by

$$p_S(i,j) = \langle E_i F_j \xi, \xi \rangle, \quad i \in [k], j \in [l]; \tag{23}$$

we say that S is a *model* of p_S. The correlations of the form p_S correspond precisely to states $s : \mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B \to \mathbb{C}$ via the assignment $s(e_i \otimes f_j) = p_S(i,j)$, $i \in [k]$, $j \in [l]$.

A tuple $\tilde{S} = (\tilde{H}, (\tilde{E}_i)_{i=1}^k, (\tilde{F}_j)_{j=1}^l, \tilde{\xi})$ is an *ideal model* of a correlation $p : [k] \times [l] \to \mathbb{C}$ if $p = p_{\tilde{S}}$ and, whenever $S = (H, (E_i)_{i=1}^k, (F_j)_{j=1}^l, \xi)$ is a model of p then there exists a Hilbert space H_{aux}, a unit vector $\xi_{\mathrm{aux}} \in H_{\mathrm{aux}}$ and a local isometry $V : H \to \tilde{H} \otimes H_{\mathrm{aux}}$ such that

$$V E_i F_j \xi = \tilde{E}_i \tilde{F}_j \tilde{\xi} \otimes \xi_{\mathrm{aux}}, \quad i \in [k], j \in [l].$$

The framework of self-testing described above will be referred to as *finitary*; we will refer to the pair $(\mathcal{S}_A, \mathcal{S}_B)$ as a *finitary context*. Quantum models of correlations $p : [k] \times [l] \to \mathbb{C}$ are similarly described in the finitary framework by replacing the operator products E_iF_j in (23) by tensor products of finite dimensionally acting families $(E_i)_{i=1}^k$ and $(F_j)_{j=1}^l$.

5.1. Self-testing for QNS correlations

The purpose of this subsection is to introduce self-testing for quantum no-signalling correlations using our general (finitary) framework, and to show how POVM self-testing is hosted within it.

Let X and A be finite sets and H be a Hilbert space. A *quantum channel* from M_X into M_A is a completely positive trace preserving map $\Phi : M_X \to M_A$. A *stochastic operator matrix (SOM)* over (X, A) acting on H is a positive block operator matrix $E = (E_{x,x',a,a'})_{x,x',a,a'}$, where $E_{x,x',a,a'} \in \mathcal{B}(H)$ for all $x, x' \in X$ and all $a, a' \in A$, such that $\mathrm{Tr}_A E = I_X \otimes I_H$ (as usual, Tr_A denotes the partial trace along M_A). It was shown in [53] that there exists a unital C*-algebra $\mathcal{C}_{X,A}$, generated by elements $e_{x,x',a,a'}$, $x, x' \in X$, $a, a' \in A$, such that the matrix $(e_{x,x',a,a'})_{x,x',a,a'}$ is positive as an element of $M_X \otimes M_A \otimes \mathcal{C}_{X,A}$,

$$\sum_{a\in A} e_{x,x',a,a} = \delta_{x,x'}1, \quad x, x' \in X,$$

and possessing the universal property that for every stochastic operator matrix $E = (E_{x,x',a,a'})_{x,x',a,a'}$, acting on a Hilbert space H, there exists a unique *-homomorphism $\pi_E : \mathcal{C}_{X,A} \to \mathcal{B}(H)$, such that $\pi_E(e_{x,x',a,a'}) = E_{x,x',a,a'}$, $x, x' \in X$, $a, a' \in A$. Let

$$\mathcal{T}_{X,A} = \mathrm{span}\{e_{x,x',a,a'} : x, x' \in X, a, a' \in A\},$$

viewed as an operator subsystem of $\mathcal{C}_{X,A}$. By [53, Corollaries 5.3 and 5.4], $C_u^*(\mathcal{T}_{X,A}) = \mathcal{C}_{X,A}$ and the stochastic operator matrices $(E_{x,x',a,a'})_{x,x',a,a'}$ are in one-to-one correspondence with the unital completely positive maps $\phi_E : \mathcal{T}_{X,A} \to \mathcal{B}(H)$ via the assignment $\phi_E(e_{x,x',a,a'}) = E_{x,x',a,a'}$.

Letting Y and B be further finite sets, the pair $(\mathcal{T}_{X,A}, \mathcal{T}_{Y,B})$ of operator systems determines a finitary framework for self-testing. A commuting operator model $S = (H, (E_{x,x',a,a'}), (F_{y,y',b,b'}), \xi)$ (referred to later as a *SOM qc-model*) gives rise to a correlation p_S via (23) which, in its own turn, determines a linear map $\Gamma_S : M_{XY} \to M_{AB}$, given by

$$\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = \sum_{a,a'\in A} \sum_{b,b'\in B} \langle E_{x,x',a,a'} F_{y,y',b,b'} \xi, \xi\rangle \, \epsilon_{a,a'} \otimes \epsilon_{b,b'}, \tag{24}$$

for all $x, x' \in X$ and all $y, y' \in Y$. The map $\Gamma = \Gamma_S$ is a *quantum no-signalling (QNS) correlation* over (X, Y, A, B) [22] in that

$$\mathrm{Tr}_A \Gamma(\rho_X \otimes \rho_Y) = 0 \;\text{ whenever }\; \mathrm{Tr}(\rho_X) = 0 \tag{25}$$

and

$$\mathrm{Tr}_B \Gamma(\rho_X \otimes \rho_Y) = 0 \text{ whenever } \mathrm{Tr}(\rho_Y) = 0. \tag{26}$$

QNS correlations $\Gamma : M_{XY} \to M_{AB}$ admitting a representation of the form (24) are said to be of *quantum commuting type* [53]. One defines QNS correlations of *quantum type* by replacing the operator product in (24) by tensor products of finite dimensionally acting SOM's. We write $\mathcal{Q}_{\mathrm{qc}}$ (resp. $\mathcal{Q}_{\mathrm{q}}$) for the (convex) set of all quantum commuting (resp. quantum) QNS correlations, and note the inclusion $\mathcal{Q}_{\mathrm{q}} \subseteq \mathcal{Q}_{\mathrm{qc}}$. We refer to Γ_S being an self-test (resp. abstract self-test) if the corresponding correlation p_S is a self-test (resp. an abstract self-test). We distinguish between *qc-self-tests* (self-tests among QNS correlations of quantum commuting type) and *q-self-tests* (self-tests among QNS correlations of quantum type).

Given a set $\mathcal{M}$ of quantum commuting models (resp. quantum models) for the finitary context $(\mathcal{T}_{X,A}, \mathcal{T}_{Y,B})$, let $\tilde{\mathcal{M}} = \{\Gamma_S : S \in \mathcal{M}\}$. It is clear from the preceding discussion that if $\mathcal{S}_{\mathcal{M}} = \{s_{\Gamma_S} : S \in \mathcal{M}\}$ then a quantum commuting QNS correlation $\Gamma \in \tilde{\mathcal{M}}$ is a (abstract) qc-self-test for $\mathcal{M}$ if and only if its corresponding state $s_\Gamma : \mathcal{T}_{X,A} \otimes_{\mathrm{c}} \mathcal{T}_{Y,B} \to \mathbb{C}$ is an (abstract) self-test for $\tilde{\mathcal{M}}$. The abstract q-self-tests have a convenient characterisation which we now state; we omit the argument as it relies on similar techniques from the (latter portion of the) proof of Theorem 4.3.

Proposition 5.1. *Let $\mathcal{F}$ be the set of quantum commuting models whose underlying Hilbert space is finite dimensional. Then $\Gamma \in \tilde{\mathcal{F}}$ is an abstract self-test for $\mathcal{Q}_{\mathrm{q}}$ if and only if s_Γ is an abstract self-test for $\mathcal{S}_{\mathcal{F}}$.*

In the remainder of this subsection we show how POVM self-testing considered in [43] fits into the general framework of Section 3. We start by introducing the relevant finitary context.

Let X and A be finite sets. The C*-algebra $\mathcal{A}_{\mathrm{POVM}}$ was introduced in [43] as the universal C*-algebra of a family of POVM's over the set A with $|X|$ elements, that is, the unital C*-algebra generated by positive elements $\tilde{e}_{x,a}$, $x \in X$, $a \in A$, satisfying the relations $\sum_{a \in A} \tilde{e}_{x,a} = 1$, $x \in X$, such that whenever $(P_{x,a})_{a \in A}$ is a POVM acting on the Hilbert space H, $x \in X$, there exists a unique *-representation $\pi : \mathcal{A}_{\mathrm{POVM}} \to \mathcal{B}(H)$ such that $\pi(\tilde{e}_{x,a}) = P_{x,a}$, $x \in X$, $a \in A$. In the next proposition, we identify a concrete description of $\mathcal{A}_{\mathrm{POVM}}$. Recall the C*-algebra $\mathcal{C}_{X,A}$ from Subsection 5.1 and let $\tilde{\mathcal{A}}_{X,A}$ be its C*-subalgebra, generated by the elements $e_{x,x,a,a}$, $x \in X$, $a \in A$. Let

$$\tilde{\mathcal{S}}_{X,A} = \mathrm{span}\{e_{x,x,a,a} : x \in X, a \in A\},$$

viewed as an operator subsystem of $\tilde{\mathcal{A}}_{X,A}$.

Recall from Section 3 that $\mathcal{A}_{X,A}$ is the universal C*-algebra, generated by projections $e_{x,a}$, $x \in X$, $a \in A$, satisfying the relations $\sum_{a \in A} e_{x,a} = 1$, $x \in X$; the C*-algebra $\mathcal{A}_{X,A}$ satisfies the analogous universal property to the one described in the previous paragraph for $\tilde{\mathcal{A}}_{X,A}$ but with $(P_{x,a})_{a\in A}$ being PVM's as opposed to POVM's. Recall, further, the operator subsystem

$$\mathcal{S}_{X,A} = \mathrm{span}\{e_{x,a} : x \in X, a \in A\}, \tag{27}$$

of $\mathcal{A}_{X,A}$.

Proposition 5.2.

(i) *There exists a *-isomorphism* $\rho : \mathcal{A}_{\mathrm{POVM}} \to \tilde{\mathcal{A}}_{X,A}$*, such that* $\rho(\tilde{e}_{x,a}) = e_{x,x,a,a}$*,* $x \in X$*,* $a \in A$*.*
(ii) *The map* $e_{x,a} \mapsto \tilde{e}_{x,a}$ *defines a unital complete order isomorphism* $\mathcal{S}_{X,A} \cong \tilde{\mathcal{S}}_{X,A}$*.*
(iii) *Up to a canonical *-isomorphism,* $C_u^*(\mathcal{S}_{X,A}) = \tilde{\mathcal{A}}_{X,A}$*.*

Proof. (i) We show that the C*-algebra $\tilde{\mathcal{A}}_{X,A}$ satisfies the universal property of $\mathcal{A}_{\mathrm{POVM}}$. Clearly, $\{e_{x,x,a,a}\}_{a\in A}$ is a POVM in $\tilde{\mathcal{A}}_{X,A}$, $x \in X$. Suppose that $(P_{x,a})_{a\in A}$, $x \in X$, are POVM's acting on the Hilbert space H. Let $E_{x,x',a,a'} := \delta_{x,x'} P_{x,a}$, $x, x' \in X$, $a, a' \in A$; then $E := (E_{x,x',a,a'})_{x,x',a,a'}$ is a stochastic operator matrix and, by the universal property of $\mathcal{C}_{X,A}$, there exists a unital *-homomorphism $\pi : \mathcal{C}_{X,A} \to \mathcal{B}(H)$, such that $\pi(e_{x,x',a,a'}) = E_{x,x',a,a'}$, $x, x' \in X$, $a, a' \in A$. The restriction $\rho = \pi|_{\tilde{\mathcal{A}}_{X,A}}$ of π to $\tilde{\mathcal{A}}_{X,A}$ is a *-representation with the property that $\rho(e_{x,x,a,a}) = P_{x,a}$, $x \in X$, $a \in A$. Since the elements $e_{x,x,a,a}$ generate $\tilde{\mathcal{A}}_{X,A}$, such a representation is unique.

(ii) By (i), the families $E = \{(E_{x,a})_{a\in A} : x \in X\}$ of POVM's acting on a Hilbert space H are in bijective correspondence with the unital completely positive maps $\phi_E : \tilde{\mathcal{S}}_{X,A} \to \mathcal{B}(H)$ via the assignment $\phi_E(\tilde{e}_{x,a}) = E_{x,a}$. A combination of Arveson's Extension Theorem and Stinespring's Dilation Theorem shows that the same universal property holds for $\mathcal{S}_{X,A}$ (see e.g. [46, p. 680]). The conclusion follows.

(iii) Let H be a Hilbert space and $\phi : \mathcal{S}_{X,A} \to \mathcal{B}(H)$ be a completely positive map. Then $(\phi(e_{x,a}))_{a\in A}$ is a POVM, $x \in X$. By (i), there exists a unital *-homomorphism $\pi : \tilde{\mathcal{A}}_{X,A} \to \mathcal{B}(H)$, such that $\pi(e_{x,a}) = \phi(e_{x,a})$, $x \in X$, $a \in A$. The proof is complete. □

It was shown in [46, Lemma 2.8] that

$$\mathcal{S}_{X,A} \otimes_{\mathrm{c}} \mathcal{S}_{Y,B} \subseteq \mathcal{A}_{X,A} \otimes_{\max} \mathcal{A}_{Y,B}$$

as an operator subsystem. The next corollary complements this fact.

Corollary 5.3. *Up to a canonical complete order embedding,* $\mathcal{S}_{X,A} \otimes_{\mathrm{c}} \mathcal{S}_{Y,B} \subseteq \mathcal{T}_{X,A} \otimes_{\mathrm{c}} \mathcal{T}_{Y,B}$*.*

Proof. Using Proposition 5.2 (ii), let $\iota_{X,A} : \mathcal{S}_{X,A} \to \mathcal{T}_{X,A}$ be the inclusion map, and $\gamma_{X,A} : \mathcal{T}_{X,A} \to \mathcal{S}_{X,A}$ be the unital completely positive map, given by

$$\gamma_{X,A}(e_{x,x',a,a'}) = \delta_{x,x'} \delta_{a,a'} e_{x,a}, \quad x \in X, a \in A;$$

note that $\gamma_{X,A} \circ \iota_{X,A} = \mathrm{id}_{\mathcal{S}_{X,A}}$. We have that $\iota := \iota_{X,A} \otimes \iota_{Y,B}$ is a unital completely positive map from $\mathcal{S}_{X,A} \otimes_{\mathrm{c}} \mathcal{S}_{Y,B}$ into $\mathcal{T}_{X,A} \otimes_{\mathrm{c}} \mathcal{T}_{Y,B}$. We show that ι is a complete order isomorphism onto its range. Suppose that $u \in M_n(\mathcal{S}_{X,A} \otimes \mathcal{S}_{Y,B})$ is such that $\iota^{(n)}(u) \in M_n(\mathcal{T}_{X,A} \otimes_{\mathrm{c}} \mathcal{T}_{Y,B})^+$. Let H be a Hilbert space, and $\phi : \mathcal{S}_{X,A} \to \mathcal{B}(H)$ and $\psi : \mathcal{S}_{Y,B} \to \mathcal{B}(H)$ be unital completely positive maps with commuting ranges. Then the maps $\phi \circ \gamma_{X,A} : \mathcal{T}_{X,A} \to \mathcal{B}(H)$ and $\psi \circ \gamma_{Y,B} : \mathcal{T}_{Y,B} \to \mathcal{B}(H)$ are unital and completely positive, and have commuting ranges. It follows that

$$(\phi \cdot \psi)^{(n)}(u) = ((\psi \circ \gamma_{Y,B}) \cdot (\psi \circ \gamma_{Y,B}))^{(n)} \left(\iota^{(n)}(u)\right) \in M_n(\mathcal{B}(H))^+.$$

The proof is complete. $\square$

Fix further finite sets Y and B. A quantum commuting model for the finitary context $(\mathcal{S}_{X,A}, \mathcal{S}_{Y,B})$ is thus a tuple $S = (H, (E_{x,a})_{x,a}, (F_{y,b})_{y,b}, \xi)$, where H is a Hilbert space, $\xi \in H$ is a unit vector, and $(E_{x,a})_{a\in A}$ (resp. $(F_{y,b})_{b\in B}$) is a POVM on H for every $x \in X$ (resp. $y \in Y$) such that $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$ for all x, y, a, b; such a model will be referred to as a *POVM qc-model*. The model S gives rise, via (23), to the correlation p_S *of quantum commuting type*, given by

$$p_S(a, b|x, y) = \langle E_{x,a}F_{y,b}\xi, \xi\rangle, \quad x \in X, y \in Y, a \in A, b \in B;$$

we note that $p = p_S$ is a *no-signalling correlation* over the quadruple (X, Y, A, B) in that $(p(a, b|x, y))_{a,b}$ is a probability distribution over $A \times B$ for every $(x, y) \in X \times Y$, and

$$\sum_{b\in B} p(a, b|x, y) = \sum_{b\in B} p(a, b|x, y'), \quad x \in X, y, y' \in Y, a \in A,$$

and

$$\sum_{a\in A} p(a, b|x, y) = \sum_{a\in A} p(a, b|x', y), \quad x, x' \in X, y \in Y, b \in B$$

(see e.g. [37,45]). *POVM q-models* of no-signalling correlations are defined analogously, using tensor products of POVM's acting on finite dimensional Hilbert spaces. We denote the (convex) set of all NS correlations by $\mathcal{C}_{\mathrm{ns}}$. With a correlation $p \in \mathcal{C}_{\mathrm{ns}}$, we associate the classical information channel $\mathcal{N}_p : \mathcal{D}_{XY} \to \mathcal{D}_{AB}$, given by

$$\mathcal{N}_p(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \sum_{a\in A} \sum_{b\in B} p(a, b|x, y)\epsilon_{a,a} \otimes \epsilon_{b,b}, \tag{28}$$

and the quantum information channel $\Gamma_p : M_{XY} \to M_{AB}$, given by $\Gamma_p = \iota_{AB} \circ \mathcal{N}_p \circ \Delta_{XY}$, where $\iota_{AB} : \mathcal{D}_{AB} \to M_{AB}$ is the inclusion map and $\Delta_{XY} : M_{XY} \to \mathcal{D}_{XY}$ is the canonical conditional expectation; it can be easily verified that Γ_p is a QNS correlation. We let $\mathcal{C}_{\mathrm{qc}} = \{p_S : S \text{ a POVM qc-model}\}$ (resp. $\mathcal{C}_{\mathrm{q}} = \{p_S : S \text{ a POVM q-model}\}$) be the set of *quantum commuting* (resp. *quantum*) NS correlations, and note the inclusions $\mathcal{C}_{\mathrm{q}} \subseteq \mathcal{C}_{\mathrm{qc}} \subseteq \mathcal{C}_{\mathrm{ns}}$.

In view of Corollary 5.3, a state $f : \mathcal{T}_{X,A} \otimes_{\mathrm{c}} \mathcal{T}_{Y,B} \to \mathbb{C}$ gives rise, via restriction, to a state $f_{\mathrm{cl}} : \mathcal{S}_{X,A} \otimes_{\mathrm{c}} \mathcal{S}_{Y,B} \to \mathbb{C}$. In the reverse direction, a state $g : \mathcal{S}_{X,A} \otimes_{\mathrm{c}} \mathcal{S}_{Y,B} \to \mathbb{C}$ gives rise to the state $g_{\mathrm{q}} : \mathcal{T}_{X,A} \otimes_{\mathrm{c}} \mathcal{T}_{Y,B} \to \mathbb{C}$ by letting $g_{\mathrm{q}} = g \circ (\gamma_{X,A} \otimes_{\mathrm{c}} \gamma_{Y,B})$.

A model $S = (H, \psi_A, \psi_B, \xi)$ of the state $f : \mathcal{T}_{X,A} \otimes_{\mathrm{c}} \mathcal{T}_{Y,B} \to \mathbb{C}$ for the bipartite quantum system $(\mathcal{T}_{X,A}, \mathcal{T}_{Y,B})$ gives rise to the model $S_{\mathrm{cl}} = (H, \varphi_A, \varphi_B, \xi)$ of the state $f_{\mathrm{cl}} : \mathcal{S}_{X,A} \otimes_{\mathrm{c}} \mathcal{S}_{Y,B} \to \mathbb{C}$ for the bipartite quantum system $(\mathcal{S}_{X,A}, \mathcal{S}_{Y,B})$, by letting $\varphi_A = \psi_A \circ \iota_{X,A}$ and $\varphi_B = \psi_B \circ \iota_{Y,B}$. In the reverse direction, a model $N = (H, \varphi_A, \varphi_B, \xi)$ for a state $g : \mathcal{S}_{X,A} \otimes_{\mathrm{c}} \mathcal{S}_{Y,B} \to \mathbb{C}$, gives rise to the model $N_{\mathrm{q}} = (H, \psi_A, \psi_B, \xi)$ for the state g_{q} of $\mathcal{T}_{X,A} \otimes_{\mathrm{c}} \mathcal{T}_{Y,B}$. We note that $(N_{\mathrm{q}})_{\mathrm{cl}} = N$. The following proposition is straightforward; we omit its proof.

Proposition 5.4. *Let $\mathfrak{M}$ be a set of SOM qc-models for the finitary context $(\mathcal{T}_{X,A}, \mathcal{T}_{Y,B})$, set $\mathfrak{M}_{\mathrm{cl}} = \{S_{\mathrm{cl}} : S \in \mathfrak{M}\}$ and assume that $(\mathfrak{M}_{\mathrm{cl}})_{\mathrm{q}} \subseteq \mathfrak{M}$. If a state f of $\mathcal{T}_{X,A} \otimes_{\mathrm{c}} \mathcal{T}_{Y,B}$ is a self-test for $\mathfrak{M}$ with the property that $f = f \circ \iota_{X,A} \circ \gamma_{X,A}$ then the state f_{cl} of $\mathcal{S}_{X,A} \otimes_{\mathrm{c}} \mathcal{S}_{Y,B}$ is a self-test for $\mathfrak{M}_{\mathrm{cl}}$.*

Remark 5.5. A special case of POVM self-testing is PVM self-testing [43]. In the latter setup, the pool of models of a given no-signalling correlation of quantum commuting type is restricted in that the participating measurements are PVM's. We note that PVM self-testing also fits in our general framework. Indeed, here the families of states $s : C_u^*(\mathcal{S}_{X,A}) \otimes_{\max} C_u^*(\mathcal{S}_{Y,B})$ to be self-tested are restricted to ones that factor from the quotient map

$$C_u^*(\mathcal{S}_{X,A}) \otimes_{\max} C_u^*(\mathcal{S}_{Y,B}) \mapsto \mathcal{A}_{X,A} \otimes_{\max} \mathcal{A}_{Y,B},$$

whose existence is guaranteed by Proposition 5.2.

In view of Proposition 5.4, it is natural to ask if, in general, every PVM self-test canonically gives rise to a SOM self-test. In the next example, we show that the standard self-test within the family $\mathcal{C}_{\mathrm{q}}$, arising from the CHSH game, does not canonically give rise to a self-test within the family $\mathcal{Q}_{\mathrm{q}}$.

Example 5.6. Recall the CHSH game [14]; here $X = Y = A = B = \mathbb{Z}_2 = \{0, 1\}$ and a quadruple $(x, y, a, b) \in X \times Y \times A \times B$ belongs to the support of the rule function precisely when $xy = (a + b) \pmod 2$. Recall the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

let

$$A_0 = \frac{\sigma_x + \sigma_z}{\sqrt{2}}, \quad A_1 = \frac{\sigma_x - \sigma_z}{\sqrt{2}}, \quad B_0 = \sigma_x \quad \text{and} \quad B_1 = \sigma_z,$$

and write $(E_{x,a})_{a\in A}$ (resp. $(F_{y,b})_{b\in B}$) for the spectral resolution of the operator A_x (resp. B_y). Write $\Omega_2 = \frac{1}{\sqrt{2}}(e_0 \otimes e_0 + e_1 \otimes e_1)$ for the maximally entangled vector in $\mathbb{C}^2 \otimes \mathbb{C}^2$ (here $\{e_0, e_1\}$ denotes the standard basis of $\mathbb{C}^2$). It is well-known that the model

$$\widetilde{S} = (\mathbb{C}^2 \otimes \mathbb{C}^2, (E_{x,a})_{x\in X,a\in A}, (F_{y,b})_{y\in Y,b\in B}, \Omega_2) \tag{29}$$

yields a correlation $p_{\widetilde{S}} \in \mathcal{C}_{\mathrm{q}}$ that is an optimal strategy of quantum type for the CHSH game and a self-test for the class $\mathcal{C}_{\mathrm{q}}$ with ideal model $\tilde{S}$ (see e.g. [50] and [38]).

Let $\tilde{E} = (\delta_{x,x'}\delta_{a,a'}E_{x,a})_{x,x',a,a'}$ and $\tilde{F} = (\delta_{y,y'}\delta_{b,b'}F_{y,b})_{y,y',b,b'}$ be the canonical extensions of the families $\{(E_{x,a})_{a\in A} : x \in X\}$ (resp. $\{(F_{y,b})_{b\in B} : y \in Y\}$ to stochastic operator matrices and $\widetilde{S}_{\mathrm{q}} = (\mathbb{C}^2 \otimes \mathbb{C}^2, \tilde{E}, \tilde{F}, \Omega_2)$; thus, $\widetilde{S}_{\mathrm{q}}$ is a model over $(\mathcal{T}_{X,A}, \mathcal{T}_{Y,B})$, giving rise to a canonical element $f_{\widetilde{S}}$ of $\mathcal{Q}_{\mathrm{q}}$, namely,

$$f_{\widetilde{S}}(e_{x,x',a,a'} \otimes e_{y,y',b,b'}) = \delta_{x,x'}\delta_{a,a'}\delta_{y,y'}\delta_{b,b'}\langle(E_{x,a} \otimes E_{y,b})\Omega_2, \Omega_2\rangle.$$

We claim that $f_{\widetilde{S}}$ is not a self-test for $\mathcal{Q}_{\mathrm{q}}$. Indeed, let $\xi_{x,0}$, $\xi_{x,1}$, $\eta_{y,0}$ and $\eta_{y,1}$ be unit vectors in the range of $E_{x,0}$, $E_{x,1}$, $F_{y,0}$ and $F_{y,1}$, respectively, and let V_x, $W_y : \mathbb{C}^2\otimes\mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$, be isometries, satisfying

$$V_x(\xi_{x,0} \otimes e_0) = \xi_{x,1} \otimes e_1 \quad \text{and} \quad W_y(\eta_{y,0} \otimes e_0) = \eta_{y,1} \otimes e_1,$$

and

$$V_x(\xi_{x,k} \otimes e_m) = 0 \quad \text{and} \quad W_y(\eta_{y,k} \otimes e_m) = 0, \quad \text{if } (k,m) \neq (0,0).$$

Further, let

$$(E_{x,x',a,b})_{a,b} = \delta_{x,x'} \begin{pmatrix} E_{x,0} \otimes I_2 & V_x^* \\ V_x & E_{x,1} \otimes I_2 \end{pmatrix}, \quad x, x' \in X,$$

$$(F_{y,y',a,b})_{a,b} = \delta_{y,y'} \begin{pmatrix} F_{y,0} \otimes I_2 & W_y^* \\ W_y & F_{y,1} \otimes I_2 \end{pmatrix}, \quad y, y' \in Y,$$

and observe that $E := (E_{x,x',a,a'})_{x,x',a,a'}$ and $F := (F_{y,y',b,b'})_{y,y',b,b'}$ are stochastic operator matrices. Set $\xi = \Omega_2 \otimes e_0 \otimes e_0$ and $S = \big(\mathbb{C}^2 \otimes \mathbb{C}^2, E, F, \big)$. We have that

$$\langle(E_{x,x',a,a'} \otimes F_{y,y',b,b'})\xi, \xi\rangle = \delta_{x,x'}\delta_{y,'y}\delta_{a,a'}\delta_{b,b'}\langle(E_{x,a} \otimes F_{y,b})\Omega_2, \Omega_2\rangle,$$

that is, S is a model of $f_{\widetilde{S}}$.

Suppose that $f_{\widetilde{S}}$ were a self-test for $\mathcal{Q}_{\mathrm{q}}$ with an ideal model $S^{\mathrm{id}} = \Big(\mathbb{C}^2\otimes\mathbb{C}^2, (E^{\mathrm{id}}_{x,x',a,a'}, (F^{\mathrm{id}}_{y,y',b,b'}), \xi^{\mathrm{id}}\Big)$. Then, necessarily,

$$(E^{\rm id}_{x,x',a,a'} \otimes F^{\rm id}_{y,y',b,b'})\xi^{\rm id} = \delta_{x,x'}\delta_{y,y'}\delta_{a,a'}\delta_{b,b'}(E^{\rm ideal}_{x,x',a,a'} \otimes F^{\rm id}_{y,y',b,b'})\xi^{\rm id}.$$

As $(E_{x,x,a,a'} \otimes F_{y,y,b,b'})\xi \neq 0$ for $a \neq a'$, $b \neq b'$, there are no isometries V_A, V_B such that $(V_A \otimes V_B)(E_{x,x,a,a'} \otimes F_{y,y',b,b'})\xi = (E^{\rm id}_{x,x',a,a'} \otimes F^{\rm id}_{y,y',b,b'})\xi^{\rm id} \otimes \xi_{\rm aux}$ for a unit vector $\xi_{\rm aux}$, giving the statement.

5.2. The CHSH game: quantum commuting self-tests

It is known that the quantum value $\omega_{\rm q}(\mathrm{CHSH})$ of the CHSH game coincides with its quantum commuting value $\omega_{\rm qc}(\mathrm{CHSH})$ and that these are equal to $\frac{1}{2} + \frac{1}{2\sqrt{2}}$, with an optimal quantum strategy underlying the model $\tilde{S}$ given by (29). As pointed out in Example 5.6, the corresponding correlation $p_{\tilde{S}}$ is a self-test for the quantum PVM models with $\tilde{S}$ being an ideal model. In this subsection, we extend this by showing that $p_{\tilde{S}}$ determines an abstract self-test, as well as a self-test, for the class $\mathcal{C}_{\rm qc}$ of quantum commuting POVM models. While Theorem 3.13 can be used to deduce the former fact from the latter, we include a direct argument, showing how the algebraic relations lying at the core of the fact that $p_{\tilde{S}}$ is a quantum abstract self-test can be extended to the commuting operator framework. We follow a well-established route for the quantum case, generalising some of the constructions that appear in [50].

5.2.1. The correlation $p_{\tilde{S}}$ determines an abstract quantum commuting self-test

Let $X = Y = A = B = \mathbb{Z}_2$, H be a Hilbert space, $\xi \in H$ be a unit vector, and $\{A_x\}_{x\in X}$ $\{B_y\}_{y\in Y}$ be families of selfadjoint unitary operators on H. Set $\mathcal{A} = \{A_x : x \in X\}''$ and $\mathcal{B} = \{B_y : y \in Y\}''$. Let $\{E_{x,a}\}_{a\in A}$ (resp. $\{F_{y,b}\}_{b\in B}$) be the spectral family of A_x (resp. B_y), so that the relations $A_x = E_{x,0} - E_{x,1}$ (resp. $B_y = F_{y,0} - F_{y,1}$) are satisfied and, writing

$$S = ({}_{\mathcal{A}}H_{\mathcal{B}^o}, (E_{x,a})_{x\in X, a\in A}, (F_{y,b})_{y\in Y, b\in B}, \xi),$$

assume that $p := p_S = p_{\tilde{S}}$, that is,

$$p_{\tilde{S}}(a, b|x, y) = \langle E_{x,a}F_{y,b}\xi, \xi\rangle, \quad x, y, a, b \in \mathbb{Z}_2. \tag{30}$$

Let β_S be the *bias operator* of the model S, defined by letting

$$\beta_S = A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1;$$

using a straightforward calculation, (30) implies

$$\frac{1}{8}\langle \beta_S\xi, \xi\rangle + \frac{1}{2} = \frac{1}{4}\sum_{xy=a+b} p_{\tilde{S}}(a, b|x, y) = \frac{1}{2} + \frac{1}{2\sqrt{2}}.$$

Set

$$Z_A = \frac{A_0 + A_1}{\sqrt{2}} \text{ and } X_A = \frac{A_0 - A_1}{\sqrt{2}} \tag{31}$$

and observe that

$$Z_A X_A + X_A Z_A = 0. \tag{32}$$

Moreover,

$$2\sqrt{2} - \beta_S = \frac{1}{2}\left(\left[\frac{A_0 + A_1}{\sqrt{2}} - B_0\right]^2 + \left[\frac{A_0 - A_1}{\sqrt{2}} - B_1\right]^2\right).$$

As $\langle \beta_S \xi, \xi \rangle = 2\sqrt{2}$, we conclude that

$$Z_A \xi = B_0 \xi \text{ and } X_A \xi = B_1 \xi, \tag{33}$$

giving, by (32),

$$\begin{aligned}(B_0 B_1 + B_1 B_0)\xi &= \frac{(A_0 - A_1)B_0 + (A_0 + A_1)B_1}{\sqrt{2}}\xi \\ &= \frac{((A_0 - A_1)(A_0 + A_1) + (A_0 + A_1)(A_0 - A_1)))}{\sqrt{2}}\xi = 0.\end{aligned}$$

By symmetry, we also have the relation $(A_0 A_1 + A_1 A_0)\xi = 0$.

Let $\mathfrak{A} = C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \otimes C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ as a C*-algebraic tensor product (since the C*-algebra $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ is nuclear, the C*-tensor product is unambiguously defined); we have that $\mathfrak{A}$ is the universal C^*-algebra, generated by selfadjoint unitaries a_x, b_y, satisfying the property $a_x b_y = b_y a_x$, $x, y \in \mathbb{Z}_2$. Thus, the families $(A_x)_{x\in X}$ and $(B_y)_{y\in Y}$ determine a (unique) *-representation π of $\mathfrak{A}$ via the relations

$$\pi(a_x \otimes b_y) = A_x B_y, \quad x, y \in \mathbb{Z}_2.$$

Moreover, any representation π of $\mathfrak{A}$ is determined by commuting pairs of selfadjoint unitaries $(A_x)_{x\in\mathbb{Z}_2}$, $(B_y)_{y\in\mathbb{Z}_2}$, by letting $\pi(a_x \otimes 1) = A_x$ and $\pi(1 \otimes b_y) = B_y$.

The irreducible representations of $\mathfrak{A}$ are given by $\pi_A \otimes \pi_B$, where π_A and π_B are irreducible representations of $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$; the latter are unitarily equivalent to one of the following (see e.g. [42]):

(1) A continuum of two dimensional *-representations:

$$\pi_\varphi(a_0) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \text{ and } \pi_\varphi(a_1) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ -\sin\varphi & -\cos\varphi \end{pmatrix},$$

where $\varphi \in (0, \pi/2)$;

(2) Four one-dimensional *-representations:

$$\pi_{i,j}(a_0) = (-1)^i, \quad \pi_{i,j}(a_1) = (-1)^j, \quad \text{where } i,j \in \{0,1\}.$$

We note the identities

$$\pi_\varphi(a_0a_1 + a_1a_0) = 2(\cos^2\varphi - \sin^2\varphi) \text{ and } \pi_{i,j}(a_0a_1 + a_1a_0) = 2(-1)^{i+j}. \tag{34}$$

Write $\widehat{\mathfrak{A}}$ for the dual space of classes of irreducible representations of $\mathfrak{A}$; thus,

$$\begin{aligned}\widehat{\mathfrak{A}} = \{\pi_\varphi \otimes \pi_\psi, \pi_\varphi \otimes \pi_{i,j}, \pi_{i,j} \otimes \pi_\psi, \pi_{i,j} \otimes \pi_{s,t} :\\ \varphi, \psi \in (0,\pi/2), i,j,s,t \in \{0,1\}\}.\end{aligned}$$

As the pairs (A_0, A_1) and (B_0, B_1) determine a $*$-representation of $\mathfrak{A}$ on H, by the structure result [20, Theorem 8.6.6], we obtain the existence of a measure μ on the Borel space $\widehat{\mathfrak{A}}$ such that, up to unitary equivalence, $H = \int_{\widehat{\mathfrak{A}}}^{\oplus} H_\pi \otimes \ell^2(m(\pi))d\mu(\pi)$, where $H_\pi = \mathbb{C}^2$ is the Hilbert space on which the irreducible representation π acts,

$$A_x = \int_{\widehat{\mathfrak{A}}}^{\oplus} \pi(a_x \otimes 1) \otimes I_{m(\pi)} d\mu(\pi) \text{ and } B_y = \int_{\widehat{\mathfrak{A}}}^{\oplus} \pi(1 \otimes b_y) \otimes I_{m(\pi)} d\mu(\pi).$$

Write $\xi = \int_{\widehat{\mathfrak{A}}}^{\oplus} \xi(\pi)d\mu(\pi)$, where, $\xi(\pi) \in H_\pi \otimes \ell^2(m(\pi))$ for each $\pi \in \widehat{\mathfrak{A}}$, and $\pi \mapsto \xi(\pi)$ is a measurable field of vectors over $\widehat{\mathfrak{A}}$. Since $(A_0A_1 + A_1A_0)\xi = 0$ and $(B_0B_1 + B_1B_0)\xi = 0$, identities (34) imply that, if $\tau := \pi_{\pi/4} \otimes \pi_{\pi/4}$ then the measure μ has $\{\tau\}$ as an atom, the function $\pi \mapsto \xi(\pi)$ is supported in the singleton $\{\tau\}$, and $H(\tau) := H_\tau \otimes \ell^2(m(\tau))$ is a direct summand in H, invariant with respect to (A_0, A_1) and (B_0, B_1). Furthermore, up to unitary equivalence,

$$\tau(a_i) = \pi_{\pi/4}(a_i) \otimes I_2, \ \tau(b_0) = I_2 \otimes \sigma_x \text{ and } \tau(b_1) = I_2 \otimes \sigma_z,$$

so that, up to unitary equivalence,

$$\begin{aligned} A_0|_{H(\tau)} &= \frac{\sigma_x + \sigma_z}{\sqrt{2}} \otimes I_2 \otimes I_{m(\tau)}, \quad A_1|_{H(\tau)} = \frac{\sigma_z - \sigma_x}{\sqrt{2}} \otimes I_2 \otimes I_{m(\tau)} \\ B_0|_{H(\tau)} &= I_2 \otimes \sigma_z \otimes I_{m(\tau)} \text{ and } B_1|_{H(\tau)} = I_2 \otimes \sigma_x \otimes I_{m(\tau)}. \end{aligned}$$

Recall that $\mathcal{S}_{X,A}$ is the operator subsystem of the C*-algebra $\mathcal{A}_{X,A}$, with canonical generators the projections $e_{x,a}$, $x \in X$, $a \in A$ (see Subsection 5.1); in the case under consideration, $\mathcal{A}_{X,A} = \mathcal{A}_{Y,B} \simeq C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$. Let $f : \mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B} \to \mathbb{C}$ be a state such that $f(e_{x,a} \otimes e_{y,b}) = p_{\tilde{\mathcal{S}}}(a,b|x,y)$, $x,y,a,b \in \mathbb{Z}_2$, and let g be a state on $\mathcal{A}_{X,A} \otimes_{\max} \mathcal{A}_{Y,B}$, given by $g(u) = \langle \pi(u)\xi, \xi \rangle$ such that $g|_{\mathcal{S}_{X,A} \otimes_{\rm c} \mathcal{S}_{Y,B}} = f$. By the previous paragraph,

$$g(u)=\langle(\tau(u)\otimes I_{m(\tau)})\xi(\tau),\xi(\tau)\rangle,\quad u\in\mathcal{A}_{X,A}\otimes_{\max}\mathcal{A}_{Y,B}.$$

We will show that $\xi(\tau)=\Omega_2\otimes\xi_{\text{aux}}$, where Ω_2 is the maximal entangled vector and $\xi_{\text{aux}}\in\ell^2(m(\tau))$. A straightforward calculation shows that the identity

$$f(a_0\otimes b_0+a_0\otimes b_1+a_1\otimes b_0-a_1\otimes b_1)=2\sqrt{2}$$

implies that

$$\frac{1}{2}\left\langle((\sigma_x\otimes\sigma_x+\sigma_z\otimes\sigma_z)\otimes I_{m(\tau)})\xi(\tau),\xi(\tau)\right\rangle=1.$$

The largest eigenvalue of the operator $\frac{1}{2}(\sigma_x\otimes\sigma_x+\sigma_z\otimes\sigma_z)$ is equal to 1 and has corresponding eigenspace the one-dimensional subspace spanned by Ω_2. This implies that $\xi(\tau)=\Omega_2\otimes\xi_{\text{aux}}$ for some $\xi_{\text{aux}}\in\ell^2(m(\tau))$. Thus,

$$g(u)=\langle\tau(u)\Omega_2,\Omega_2\rangle,\quad u\in\mathcal{A}_{X,A}\otimes_{\max}\mathcal{A}_{Y,B}.$$

This also finishes the proof that f (equivalently p) is an abstract self-test for the class of states that factor through $\mathcal{A}_{X,A}\otimes_{\max}\mathcal{A}_{Y,B}$.

5.2.2. The correlation $p_{\tilde{S}}$ determines a quantum commuting self-test

We show that p is a self-test for the class of quantum commuting PVM models. We start with some preparations. Let P_A be the kernel of the operator Z_A defined in (31) and set $\hat{Z}_A=(Z_A+P_A)|Z_A+P_A|^{-1}$ (a regularization of Z_A in terms of [50, A.2]), where $|T|:=(T^*T)^{1/2}$ is the absolute value of an operator T. Note that Z_A+P_A is an injective selfadjoint operator and hence $\hat{Z}_A$ is a unitary operator. We similarly let $\hat{X}_A=(X_A+Q_A)|X_A+Q_A|^{-1}$, where Q_A be the projection onto the kernel of the operator X_A defined in (31). Write

$$z_A(\pi)=\begin{cases}\sqrt{2}\cos\varphi\begin{pmatrix}1&0\\0&-1\end{pmatrix}\otimes I, & \pi=\pi_\varphi\otimes\pi_\psi \text{ or } \pi_\varphi\otimes\pi_{i,j}\\ ((-1)^i+(-1)^j)/\sqrt{2}\otimes I, & \pi=\pi_{i,j}\otimes\pi_\psi \text{ or } \pi_{i,j}\otimes\pi_{s,t},\end{cases}$$

$$x_A(\pi)=\begin{cases}\sqrt{2}\sin\varphi\begin{pmatrix}0&1\\1&0\end{pmatrix}\otimes I, & \pi=\pi_\varphi\otimes\pi_\psi \text{ or } \pi_\varphi\otimes\pi_{i,j}\\ ((-1)^i-(-1)^j)/\sqrt{2}\otimes I, & \pi=\pi_{i,j}\otimes\pi_\psi \text{ or } \pi_{i,j}\otimes\pi_{s,t},\end{cases}$$

$$\hat{z}_A(\pi)=\begin{cases}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\otimes I, & \pi=\pi_\varphi\otimes\pi_\psi \text{ or } \pi_\varphi\otimes\pi_{i,j}\\ \pm1\otimes I, & \pi=\pi_{i,j}\otimes\pi_\psi \text{ or } \pi_{i,j}\otimes\pi_{s,t},\end{cases}$$

and

$$\hat{x}_A(\pi) = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I, & \pi = \pi_\varphi \otimes \pi_\psi \text{ or } \pi_\varphi \otimes \pi_{i,j} \\ \pm 1 \otimes I, & \pi = \pi_{i,j} \otimes \pi_\psi \text{ or } \pi_{i,j} \otimes \pi_{s,t}, \end{cases}$$

so that

$$Z_A = \int_{\widehat{\mathfrak{A}}}^{\oplus} z_A(\pi) \otimes I_{m(\pi)} d\mu(\pi), \quad X_A = \int_{\widehat{\mathfrak{A}}}^{\oplus} x_A(\pi) \otimes I_{m(\pi)} d\mu(\pi)$$

and

$$\hat{Z}_A = \int_{\widehat{\mathfrak{A}}}^{\oplus} \hat{z}_A(\pi) \otimes I_{m(\pi)} d\mu(\pi), \quad \hat{X}_A = \int_{\widehat{\mathfrak{A}}}^{\oplus} \hat{x}_A(\pi) \otimes I_{m(\pi)} d\mu(\pi).$$

In particular, as $\pi \mapsto \xi(\pi)$ is supported in $\{\tau\}$, by (33) we obtain

$$\hat{Z}_A\xi = Z_A\xi = B_0\xi, \quad \hat{X}_A\xi = X_A\xi = B_1\xi, \quad Z_A^2\xi = X_A^2\xi = \xi, \tag{35}$$

and

$$(\hat{Z}_A\hat{X}_A + \hat{X}_A\hat{Z}_A)\xi = 0, \quad (B_0B_1 + B_1B_0)\xi = 0. \tag{36}$$

We write $\{e_k^*\}_{k=0,1}$ for the dual basis of $\mathbb{C}^2$, considered as linear functionals on $\mathbb{C}^2$, given by $e_k^*(e_i) = \delta_{i,k}$. It will further be convenient we use the notation e_k for the linear map $\mathbb{C} \to \mathbb{C}^2$, $1 \mapsto e_k$, and note that e_k is the adjoint e_k^*, $k = 0, 1$. Let $V_{1,2} : H \mapsto \mathbb{C}^2 \otimes H$ be the operator, given by

$$V_{1,2} = \frac{1}{2}(e_0 \otimes (I + \hat{Z}_A) + e_1 \otimes \hat{X}_A(I - \hat{Z}_A)). \tag{37}$$

Observe that, as $(1 + \hat{Z}_A)/2$ is a projection and $\hat{X}_A$ is a unitary, for $\eta \in H$ we have

$$\begin{aligned} \|V_{1,2}\eta\|^2 &= \frac{1}{4}\left(\left\|(I + \hat{Z}_A)\eta\right\|^2 + \left\|\hat{X}_A(I - \hat{Z}_A)\eta\right\|^2\right) \\ &= \left\|\frac{1}{2}(I + \hat{Z}_A)\eta\right\|^2 + \left\|\frac{1}{2}(I - \hat{Z}_A)\eta\right\|^2 = \|\eta\|^2, \end{aligned}$$

that is, $V_{1,2}$ is an isometry. Equipping $\mathbb{C}^2 \otimes H$ with the natural $M_2(\mathbb{C}) \otimes \mathcal{A}$-$(M_2(\mathbb{C}) \otimes \mathcal{B})^o$-bimodule action, we claim that $V_{1,2}$ is $\mathcal{A}$-$(M_2(\mathbb{C}) \otimes \mathcal{A})$-local.

Set $P_{i,A} = \hat{X}_A^i(I + (-1)^i\hat{Z}_A)/2$. Then $P_{i,A} \in \mathcal{A}$. For $\eta \in \mathbb{C}^2 \otimes H$ and $\psi \in H$, we have

$$\langle V_{1,2}^*\eta, \psi\rangle = \langle \eta, V_{1,2}\psi\rangle = \left\langle \eta, \sum_{i=0}^{1}(e_i \otimes P_{i,A})\psi \right\rangle = \left\langle \sum_{i=0}^{1}(e_i^* \otimes P_{i,A}^*)\eta, \psi \right\rangle,$$

giving $V_{1,2}^* = \sum_{i=0}^{1} e_i^* \otimes P_{i,A}^*$. Thus,

$$V_{1,2}TV_{1,2}^* = \sum_{i,j=0}^{1} e_j e_i^* \otimes P_{j,A}TP_{i,A}^*, \quad T \in \mathcal{A},$$

showing that $V_{1,2}\mathcal{A}V_{1,2}^* \subseteq M_2(\mathbb{C}) \otimes \mathcal{A}$. Using the fact that $\mathcal{A} \subseteq \mathcal{B}'$, we also have that

$$(I \otimes R)V_{1,2} = V_{1,2}R, \quad R \in \mathcal{B}.$$

Let now $V_{2,2} : \mathbb{C}^2 \otimes H \to \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes H$ be given by

$$V_{2,2} = \sum_{i=0}^{1} I \otimes e_i \otimes P_{i,B}, \tag{38}$$

where $P_{i,B}$ is defined similarly, replacing Z_A (resp. X_A) by $Z_B := B_0$ (resp. $X_B := B_1$). Using the previous arguments, we can see that $V_{2,2}$ is an $M_2(\mathbb{C})\otimes\mathcal{B}$-$(M_2(\mathbb{C})\otimes M_2(\mathbb{C})\otimes\mathcal{B}$-local isometry and

$$V_{2,2}V_{1,2} = \sum_{i,j=0}^{1} e_i \otimes e_j \otimes P_{j,B}P_{i,A}.$$

Make natural modifications in (37) and (38), we further define local isometries $V_{1,1} : H \to \mathbb{C}^2 \otimes H$ and $V_{2,1} : \mathbb{C}^2 \otimes H \to \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes H$ by letting $V_{1,1} = \sum_{j=0}^{1} e_j \otimes P_{j,B}$ and $V_{2,1} = \sum_{i=0}^{1} e_i \otimes I \otimes P_{i,A}$, noting that $V_{2,1}V_{1,1} = V_{2,2}V_{1,2}$.

We show that there exists a unit vector $\xi_{\mathrm{aux}} \in H$ such that $V_{2,2}V_{1,2}\xi = \Omega_2 \otimes \xi_{\mathrm{aux}}$, and that

$$V_{2,2}V_{1,2}\pi(u)\xi = \tau(u)\Omega_2 \otimes \xi_{\mathrm{aux}}, \quad u \in \mathcal{S}_{X,A} \otimes \mathcal{S}_{Y,B}. \tag{39}$$

By (35), $\hat{Z}_A\xi = Z_A\xi = Z_B\xi$, giving

$$(I \mp \hat{Z}_B)(I \pm \hat{Z}_A)\xi = 0,$$

and hence

$$P_{j,B}P_{i,A}\xi = 0 \ \text{ if } i \neq j. \tag{40}$$

Next we observe that, by (35) and (36),

$$\hat{X}_A\hat{Z}_A\xi = -\hat{Z}_A\hat{X}_A\xi, \ X_BZ_B\xi = -Z_BX_B\xi \text{ and } X_B\xi = X_A\xi = \hat{X}_A\xi,$$

and therefore, using the fact that $\mathcal{A} \subset \mathcal{B}'$, we have

$$P_{1,B}P_{1,A}\xi = X_B\hat{X}_A(I - Z_B)(I - \hat{Z}_A)\xi = (I + Z_B)(I + \hat{Z}_A)X_B\hat{X}_A\xi$$
$$= (I + Z_B)(I + \hat{Z}_A)\xi = P_{0,B}P_{0,A}\xi.$$

Setting $\xi_{\text{aux}} = \sqrt{2}P_{0,B}P_{0,A}\xi$, using (40) we thus have

$$V_{2,2}V_{1,2}\xi = (e_0 \otimes e_0 + e_1 \otimes e_1) \otimes P_{0,B}P_{0,A}\xi = \Omega_2 \otimes \xi_{\text{aux}}.$$

Using the fact that $\hat{Z}_A\xi = Z_A\xi$ and $Z_A^2\xi = \xi$ (see relations (35)), we get

$$P_{j,B}P_{i,A}Z_A\xi = X_B^j\hat{X}_A^i(I + (-1)^jZ_B)(I + (-1)^i\hat{Z}_A)Z_A\xi$$
$$= X_B^j\hat{X}_A^i(I + (-1)^jZ_B)(\hat{Z}_A + (-1)^iI)\xi = (-1)^iP_{j,B}P_{i,A}\xi$$

and hence, taking into account (40), we obtain

$$V_{2,2}V_{1,2}Z_A\xi = (e_0 \otimes e_0 - e_1 \otimes e_1) \otimes P_{0,B}P_{0,A}\xi = (\sigma_z \otimes 1)\Omega_2 \otimes \xi_{\text{aux}}.$$

Since $\mathcal{A} \subseteq \mathcal{B}'$, we furthermore have

$$P_{j,B}P_{i,A}X_A\xi = X_B^j\hat{X}_A^i(I + (-1)^jZ_B)(I + (-1)^i\hat{Z}_A)X_A\xi$$
$$= X_B^j\hat{X}_A^{i+1}(I + (-1)^jZ_B)(I + (-1)^{i+1})\hat{Z}_A)\xi$$
$$= P_{j,B}P_{i+1,A}\xi$$

and hence

$$V_{2,2}V_{1,2}X_A\xi = (e_1 \otimes e_0 + e_0 \otimes e_1) \otimes P_{0,B}P_{0,A}\xi = (\sigma_x \otimes I)\Omega_2 \otimes \xi_{\text{aux}}.$$

In a similar way we show that

$$V_{2,2}V_{1,2}Z_B\xi = (e_0 \otimes e_0 - e_1 \otimes e_1) \otimes P_{0,B}P_{0,A}\xi = (I \otimes \sigma_z)\Omega_2 \otimes \xi_{\text{aux}}$$

and

$$V_{2,2}V_{1,2}X_B\xi = (e_0 \otimes e_1 + e_1 \otimes e_0) \otimes P_{0,B}P_{0,A}\xi = (I \otimes \sigma_x)\Omega_2 \otimes \xi_{\text{aux}}.$$

Equation (39) is therefore established.

Finally, we can remove the condition of being quantum commuting PVM model. It relies on the following statement which is similar to [43, Proposition 5.5].

Proposition 5.7. *Let X and Y be finite sets and suppose that the quantum commuting correlation $p = \{p(a,b|x,y) : x \in X, y \in Y, a,b \in \mathbb{Z}_2\}$ is an extreme point in $\mathcal{C}_{\text{qc}}$. If S is a quantum commuting model for p then there exists a projective quantum commuting model $\widetilde{S}$ such that $S \preceq \widetilde{S}$.*

Proof. Let $S = (H, (E_{x,a})_{x\in X,a\in\mathbb{Z}_2}, (F_{y,b})_{y\in Y,b\in\mathbb{Z}_2}, \xi)$ be a quantum commuting model for p, that is, $p(a,b|x,y) = \langle E_{x,a}F_{y,b}\xi, \xi\rangle$ and $(E_{x,a})_{a\in\mathbb{Z}_2}$ and $(F_{y,b})_{b\in\mathbb{Z}_2}$ are POVMs for each $x \in X$ and $y \in Y$. Let $E_{x,0} = \int_{[0,1]} \lambda dE_x(\lambda)$ be the spectral decomposition of $E_{x,0}$, $x \in X$. Fix $x \in X$ and $0 < c < 1/2$, and set $\Omega_c = [c, 1]$. Write

$$E_{x,0} = cE_x(\Omega_c) + (1 - c)T_{x,c},$$

where $T_{x,c} = \frac{1}{1-c}\left(\int_{[0,1]} \lambda E_x(\lambda) - cE_x(\Omega_c)\right)$. Then

$$T_{x,c} = \frac{1}{1-c}\left(\int\limits_{\Omega_c} (\lambda - c)dE_x(\lambda) + \int\limits_{\Omega_c^c} \lambda dE_x(\lambda)\right) \geq 0,$$

and

$$I - T_{x,c} = \frac{1}{1-c}\left(\int\limits_{\Omega_c} (1 - \lambda)dE_x(\lambda) + \int\limits_{\Omega_c^c} (1 - c - \lambda)dE_x(\lambda)\right) \geq 0.$$

It gives $p = cp_1 + (1 - c)p_2$, where p_1 and p_2 are the quantum commuting correlations corresponding to the POVMs, where $(E_{x,0},\ I - E_{x,0})$ is replaced by $(E_x(\Omega_c),\ 1 - E_x(\Omega_c))$ and $(T_{x,c},\ I - T_{x,c})$, respectively. As p is extreme, $p_1 = p_2$ and hence $\langle E_x([c, 1])\xi, \xi\rangle = \langle E_{x,0}\xi, \xi\rangle$. Letting $c \to 0$, we obtain that $\langle E_x((0, 1])\xi, \xi\rangle = \langle E_{x,0}\xi, \xi\rangle$. Let $P_{x,0} = E_x(\{1\})$ and note that if $P_{x,0} \neq 0$, it is the projection onto non-zero eigenspace of $E_{x,0}$ corresponding to the eigenvalue 1. We have

$$0 = \langle (E_{x,0} - E_x((0, 1])\xi, \xi\rangle = \int\limits_{(0,1)} (\lambda - 1)d\langle E_x(\lambda)\xi, \xi\rangle,$$

showing that $E_x((0, 1))\xi = 0$ and hence $E_{x,0}\xi = P_{x,0}\xi$. Note that $P_{x,0}$ commute with each $F_{y,b}$. Similarly, we find projections $Q_{y,0}$ such that $F_{y,0}\xi = Q_{y,0}\xi$ and commute with each $P_{x,0}$. Set

$$\widetilde{S} = (H, (P_{x,a})_{x\in X,a\in\mathbb{Z}_2}, (Q_{y,b})_{y\in Y,b\in\mathbb{Z}_2}, \xi).$$

Then $S \preceq \widetilde{S}$. $\square$

Corollary 5.8. *The correlation $p_{\tilde{S}}$ (where $\tilde{S}$ is the model given in (29)) is a self-test for the class of all quantum commuting models.*

Proof. Since $\mathcal{C}_{\mathrm{q}}(2, 2) = \mathcal{C}_{\mathrm{qc}}(2, 2)$, the uniqueness of the optimal quantum strategy for the CHSH game implies the uniqueness of the optimal quantum commuting strategy for this game. Assuming that $p_{\tilde{S}} = cp_1 + (1-c)p_2$, $c \in (0,1)$, for some quantum commuting correlations p_1 and p_2, we obtain that

$$\omega_{\mathrm{qc}}(\mathrm{CHSH}, p_1) = \omega_{\mathrm{qc}}(\mathrm{CHSH}, p_2) = \omega_{\mathrm{qc}}(\mathrm{CHSH}, p_{\tilde{S}}).$$

As the optimal strategy is unique, $p_1 = p_2 = p_{\tilde{S}}$ which show that $p_{\tilde{S}}$ is extreme in $\mathcal{C}_{\mathrm{qc}}$. The statement now follows from Proposition 5.7. □

5.3. *Clifford correlations*

In this subsection, we introduce the class of Clifford correlations as a class of correlations that factor through the tensor product of two copies of the Clifford algebra, and show that synchronous Clifford correlations are abstract self-tests (see Theorem 5.11).

Let X be a finite set. We note that the C*-algebra $\mathcal{A}_{X,\mathbb{Z}_2}$ is identical with the universal unital C*-algebra generated by $|X|$ projections e_x, $x \in X$, via the isomorphism given by letting $e_{x,0} = e_x$ and $e_{x,1} = 1 - e_x$, $x \in X$. Let

$$\mathcal{J}_{\mathrm{C}} = \left\langle \left\{ e_{x,0}e_{y,0} + e_{y,0}e_{x,0} - e_{x,0} - e_{y,0} + \frac{1}{2} \cdot 1 : x, y \in X, x \neq y \right\} \right\rangle$$

as a closed ideal of $\mathcal{A}_{X,\mathbb{Z}_2}$. Recall, further, that the *Clifford algebra* $\mathfrak{C}_X$ over X is the (unital) C*-algebra, generated by a family $\{u_x\}_{x\in X}$ of self-adjoint unitaries, satisfying the anticommutation relations

$$u_x u_y + u_y u_x = 0, \quad x, y \in X,\ x \neq y.$$

For the following fact, see, for example, [54].

Theorem 5.9. *If $|X|$ is even then, up to unitary equivalence, $\mathfrak{C}_X$ has a unique irreducible representation π_{C}, which is also faithful, and whose image is $M_{2^{\frac{n}{2}}}$.*

Proposition 5.10. *We have that $\mathfrak{C}_X = \mathcal{A}_{X,\mathbb{Z}_2}/\mathcal{J}_{\mathrm{C}}$, up to a canonical *-isomorphism.*

Proof. Let p_x be the eigen-projection of u_x corresponding to the eigenvalue 1; thus, $u_x = 2p_x - 1$, $x \in X$. For $x \neq y$, we have that

$$\begin{aligned} u_x u_y + u_y u_x &= (2p_x - 1)(2p_y - 1) + (2p_y - 1)(2p_x - 1) \\ &= 4p_x p_y - 2p_y - 2p_x + 1 + 4p_y p_x - 2p_x - 2p_y + 1 \\ &= 4p_x p_y + 4p_y p_x - 4p_x - 4p_y + 2. \end{aligned}$$

The claim follows from the fact that $\mathcal{A}_{X,\mathbb{Z}_2}$ is the universal C*-algebra of $|X|$ self-adjoint unitaries. □

We introduce two classes of no-signalling correlations, which will prove to be amenable to self-testing. For the first subclass, write $q_{\mathrm{C}} : \mathcal{A}_{X,\mathbb{Z}_2} \to \mathfrak{C}_X$ for the quotient map arising from Proposition 5.10. Let $\mathfrak{S}_X = q_{\mathrm{C}}(\mathcal{S}_{X,[2]})$; thus, $\mathfrak{S}_X = \mathrm{span}\{1, u_x : x \in X\}$, an operator subsystem of $\mathfrak{C}_X$. An NS correlation p over $(X, X, \mathbb{Z}_2, \mathbb{Z}_2)$ will be called a *Clifford correlation* if there exists a state $s : \mathfrak{S}_X \otimes_{\min} \mathfrak{S}_X \to \mathbb{C}$ such that

$$p_s(a, b|x, y) := s(q_{\mathrm{C}}(e_{x,a}) \otimes q_{\mathrm{C}}(f_{y,b})), \quad x, y \in X, a, b \in \mathbb{Z}_2$$

(for clarity, we denote by $f_{y,b}$, $y \in X$, $b \in \mathbb{Z}_2$, the canonical generators of the second copy of $\mathcal{A}_{X,\mathbb{Z}_2}$). Let $\mathcal{S}_{\mathrm{C}}$ be the set of all states on $\mathcal{A}_{X,\mathbb{Z}_2} \otimes_{\min} \mathcal{A}_{X,\mathbb{Z}_2}$ that factor through $\mathfrak{C}_X \otimes_{\min} \mathfrak{C}_X$.

For the second class, let $\Sigma = \{e_{x,a}, f_{y,b} : x, y \in X, a, b \in \mathbb{Z}_2\}$, Σ^* be the set of all finite words on the alphabet Σ, reduced under the relations $e_{x,a}^2 = e_{x,a}$, $f_{y,b}^2 = f_{y,b}$ and $e_{x,a}f_{y,b} = f_{y,b}e_{x,a}$, equipped with a canonical involution (each element $w \in \Sigma^*$ can be equivalently considered as an element of $\mathcal{A}_{X,\mathbb{Z}_2} \otimes \mathcal{A}_{X,\mathbb{Z}_2}$, where $e_{x,a}$ (resp. $f_{y,b}$) are identified with the canonical generators of $\mathcal{A}_{X,\mathbb{Z}_2}$ (resp. $\mathcal{A}_{Y,\mathbb{Z}_2}$)). We note that the empty word ε is considered to be an element of Σ^*, and let $\Sigma' = \Sigma \cup \{\varepsilon\}$. We identify a quantum commuting correlation $p = ((p(a, b|x, y))_{x,y,a,b}$ with the matrix $M^{(p)} = [m_{\alpha,\beta}]_{\alpha,\beta\in\Sigma'}$, where

$$m_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha = \beta = \varepsilon \\ p(a|x) & \text{if } \alpha = \varepsilon \ \&\ \beta = e_{x,a}, \text{ or } \alpha = e_{x,a} \ \&\ \beta = \varepsilon \\ p(b|y) & \text{if } \alpha = \varepsilon \ \&\ \beta = f_{y,b}, \text{ or } \alpha = f_{y,b} \ \&\ \beta = \varepsilon \\ p(a, b|x, y) & \text{if } \alpha = e_{x,a} \ \&\ \beta = f_{y,b}, \text{ or } \alpha = f_{y,b} \ \&\ \beta = e_{x,a}. \end{cases}$$

Every linear functional $s : \mathcal{A}_{X,\mathbb{Z}_2} \otimes \mathcal{A}_{X,\mathbb{Z}_2} \to \mathbb{C}$ gives rise to the matrix $M^{(s)} = [s(\beta^*\alpha)]_{\alpha,\beta\in\Sigma^*}$; we call the matrices arising in this way *admissible*. We denote the set of all admissible positive semi-definite matrices over $\Sigma^* \times \Sigma^*$ (that is, admissible matrices M whose every finite minor is positive semi-definite) by $\mathbb{A}$. According to the NPA hierarchy [41], a no-signalling correlation $p = ((p(a, b|x, y))_{x,y,a,b}$ is of quantum commuting type if and only if the matrix $M^{(p)}$ over $\Sigma' \times \Sigma'$ can be completed to a matrix $M = (M_{\alpha,\beta})_{\alpha,\beta}$ that lies in the set $\mathbb{A}$.

Let $w_{x,y} = e_{x,0}e_{y,0}$; we view $w_{x,y}$ as elements of Σ^*. We write

$$\mathbb{A}_{\mathrm{C}} = \left\{ [m_{\alpha,\beta}]_{\alpha,\beta\in\Sigma^*} \in \mathbb{A} : m_{e_{x,0},e_{y,0}} - m_{w_{x,y},w_{y,x}} = \frac{1}{8}, \ \ x, y \in X \right\}.$$

Theorem 5.11. *Let X be a finite set of even cardinality, $s : \mathfrak{S}_X \otimes_{\min} \mathfrak{S}_X \to \mathbb{C}$ be a state, and let $\tilde{s} = s \circ (q_{\mathrm{C}} \otimes q_{\mathrm{C}})$. The following hold true:*

(i) *if $p_{\tilde{s}}$ is a synchronous Clifford correlation then $\tilde{s}$ is an abstract self-test for $\mathcal{S}_{\mathrm{C}}$;*
(ii) *if $p_{\tilde{s}}$ is a synchronous correlation that admits a positive completion to an element of $\mathbb{A}_{\mathrm{C}}$ then $\tilde{s}$ is an abstract self-test for $\mathcal{S}_{\mathrm{C}}$.*

Proof. (i) Suppose that p_s is a synchronous Clifford correlation, and let, as in (i), $\phi : \mathfrak{C}_X \otimes_{\min} \mathfrak{C}_X \to \mathbb{C}$ be an extension of s. Let $\tilde{u}_x = e_{x,0} - e_{x,1}$, $x \in X$. By the synchronicity of p_s, we have

$$\tilde{s}(e_{x,1} \otimes e_{x,0}) = \tilde{s}(e_{x,0} \otimes e_{x,1}) = 0, \quad x \in X.$$

Note that

$$\begin{aligned} |X|1 &= \sum_{x\in X} (e_{x,0} + e_{x,1}) \otimes (e_{x,0} + e_{x,1}) \\ &= \sum_{x\in X} e_{x,0} \otimes e_{x,0} + e_{x,1} \otimes e_{x,0} + e_{x,0} \otimes e_{x,1} + e_{x,1} \otimes e_{x,1}, \end{aligned}$$

while

$$\begin{aligned} \sum_{x\in X} \tilde{u}_x \otimes \tilde{u}_x &= \sum_{x\in X} (e_{x,0} - e_{x,1}) \otimes (e_{x,0} - e_{x,1}) \\ &= \sum_{x\in X} e_{x,0} \otimes e_{x,0} - e_{x,0} \otimes e_{x,1} - e_{x,1} \otimes e_{x,0} + e_{x,1} \otimes e_{x,1}. \end{aligned}$$

Thus,

$$|X|1 - \sum_{x\in X} \tilde{u}_x \otimes \tilde{u}_x = 2 \sum_{x\in X} e_{x,0} \otimes e_{x,1} + e_{x,1} \otimes e_{x,0}.$$

It follows that

$$|X|1 - \sum_{x\in X} u_x \otimes u_x \geq 0 \tag{41}$$

and that

$$\tilde{s}(|X|1 - \sum_{x\in X} \tilde{u}_x \otimes \tilde{u}_x) = 0. \tag{42}$$

Write

$$\phi(u) = \langle \pi(u)\xi, \xi \rangle, \quad u \in \mathfrak{C}_X \otimes_{\min} \mathfrak{C}_X, \tag{43}$$

arising from the GNS representation of ϕ. By (42),

$$\left\langle \pi\left(|X|1 - \sum_{x\in X} u_x \otimes u_x\right)\xi, \xi \right\rangle = 0$$

which, together with (41) implies that $\pi\left(|X|1 - \sum_{x\in X} u_x \otimes u_x\right)\xi = 0$. By [54, Lemma 1.3], π is irreducible, and since $|X|$ is even, Theorem 5.9 implies that, up to unitary equivalence, $\pi = \pi_{\mathrm{C}}$. Furthermore, from the proof of [54, Lemma 1.3], the vector ξ is determined uniquely up to a scalar multiple. The relation (43) shows that $\tilde{\phi}$ is the unique extension of $\tilde{s}$.

(ii) Let $\Theta : \mathcal{A}_{X,\mathbb{Z}_2} \otimes \mathcal{A}^o_{X,\mathbb{Z}_2} \to \mathcal{A}_{X,\mathbb{Z}_2}$ be the map, given by $\Theta(u \otimes v^o) = uv$, and let $\partial : \mathcal{A}_{X,\mathbb{Z}_2} \to \mathcal{A}^o_{X,\mathbb{Z}_2}$ be the *-isomorphism, given by $\partial(e_{x,a}) = e^o_{x,a}$ [32]. By the synchronicity of $p_{\tilde{s}}$ and [45], there exists a tracial state $\tau : \mathcal{A}_{X,\mathbb{Z}_2} \to \mathbb{C}$, such that

$$\tilde{s}(w) = \tau\left((\Theta \circ \partial)(w)\right), \quad w \in \mathcal{S}_{X,\mathbb{Z}_2} \otimes \mathcal{S}_{X,\mathbb{Z}_2}.$$

Write $r_x = e_{x,0}$ for brevity, and let $c = r_x r_y + r_y r_x - r_x - r_y + \frac{1}{2}$, as an element of $\mathcal{A}_{X,\mathbb{Z}_2}$. A direct calculation shows that

$$c^2 = r_x r_y r_x r_y + r_y r_x r_y r_x - r_x r_y r_x - r_y r_x r_y + \frac{1}{4} \cdot 1.$$

It follows that

$$\tau(c^2) = 2\tau(r_x r_y r_x r_y) - 2\tau(r_x r_y) + \frac{1}{4} \cdot 1.$$

Since $p_{\tilde{s}} \in \mathbb{A}_{\mathrm{C}}$, we have that $\tau(c^2) = 0$, and hence τ annihilates the ideal $\mathcal{J}_{\mathrm{C}}$ generated by c. By Proposition 5.10, τ induces a trace $\tilde{\tau} : \mathfrak{C}_X \to \mathbb{C}$, and

$$s(u) = \tilde{\tau}\left(q_{\mathrm{C}}((\Theta \circ \partial)(u))\right), \quad u \in \mathcal{S}_{X,\mathbb{Z}_2} \otimes \mathcal{S}_{X,\mathbb{Z}_2}. \tag{44}$$

Let $\tilde{\phi} : \mathcal{A}_{X,\mathbb{Z}_2} \otimes_{\max} \mathcal{A}_{X,\mathbb{Z}_2} \to \mathbb{C}$ be an extension of $\tilde{s}$ to an element of $S(\mathcal{A}_{X,\mathbb{Z}_2} \otimes_{\max} \mathcal{A}_{X,\mathbb{Z}_2})$. Then $\tilde{\phi}$ annihilates $\mathcal{J}_{\mathrm{C}} \otimes \mathcal{A}_{X,\mathbb{Z}_2} + \mathcal{A}_{X,\mathbb{Z}_2} \otimes \mathcal{J}_{\mathrm{C}}$ and, by the projectivity of the maximal tensor product, induces a state $\phi : \mathfrak{C}_X \otimes_{\max} \mathfrak{C}_X \to \mathbb{C}$. By the nuclearity of $\mathfrak{C}_X$, we may consider ϕ as a state on $\mathfrak{C}_X \otimes_{\min} \mathfrak{C}_X$. By (44), ϕ extends s, and hence $p_{\tilde{s}}$ is a Clifford correlation. The claim now follows from (ii). $\square$

5.4. A self-test for full graph colourings

In this subsection we consider self-testing for classical-to-quantum no-signalling (CQNS) correlations, exhibiting an example for the classical-to-quantum game of complete graph colouring. In particular, we show that a perfect quantum (commuting) strategy for such colouring game is an abstract self-test (Corollary 5.15) and a self-test for the class of quantum models (Proposition 5.16).

Let $d \in \mathbb{N}$ with $d \geq 2$, and

$$\mathcal{B}_{d^2,d} := \underbrace{M_d *_1 \cdots *_1 M_d}_{d^2 \text{ times}},$$

where the free product is amalgamated over the units. Let $\{\epsilon_{x,a,a'} : a, a' \in [d]\}$ be the standard matrix units of the x-th copy of M_d in $\mathcal{B}_{d^2,d}$. Further, let

$$\mathcal{R}_{d^2,d} := \operatorname{span}\{\{\epsilon_{x,a,a'} : x \in [d]^2, a, a' \in [d]\}, \tag{45}$$

viewed as an operator subsystem of $\mathcal{B}_{d^2,d}$, thus obtaining a finitary context $(\mathcal{R}_{d^2,d}, \mathcal{R}_{d^2,d})$. Each state $s : \mathcal{R}_{d^2,d} \otimes_{\mathrm{c}} \mathcal{R}_{d^2,d} \to \mathbb{C}$ gives rise to a trace preserving completely positive map $\Gamma_s : \mathcal{D}_d \otimes \mathcal{D}_d \to M_d \otimes M_d$, defined by

$$\Gamma_s(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \sum_{a,a'=1}^{d} \sum_{b,b'=1}^{d} s(e_{x,a,a'} \otimes e_{y,b'b}))\epsilon_{a,a'} \otimes \epsilon_{b,b'}, \quad x, y \in [d^2],$$

which is a *classical-to-quantum no-signalling (CQNS) correlation* in that it satisfies a natural version of the no-signalling conditions (25) and (26) and which is further of *quantum commuting type*, denoted $\mathcal{CQ}_{\mathrm{qc}}$ (see [53, Section 7] for the precise definitions).

Let $\mathcal{K}_d$ be the complete classical graph on d vertices, and $\mathcal{Q}_d$ be the complete quantum graph on d vertices, that is, $\mathcal{Q}_d = \{\Omega_d\}^\perp$, as a subspace of the Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$, where $\Omega_d = \frac{1}{\sqrt{d}} \sum_{a=1}^d e_a \otimes e_a$ is the maximally entangled unit vector in dimension d. The *graph homomorphism game* $\mathcal{K}_{d^2} \mapsto \mathcal{Q}_d$ was defined in [53]. Its *game algebra* is the universal C*-algebra $\operatorname{Hom}(\mathcal{K}_{d^2}, \mathcal{Q}_d)$, generated by elements $e_{x,a,a'}$, $x \in [d^2]$, $a, a' \in [d]$, satisfying the relations

(i) $e_{x,a,a'} e_{x,b',b} = \delta_{a',b'} e_{x,a,b}$, $\sum_{a=1}^d e_{x,a,a} = 1$ for all $x \in [d^2]$;
(ii) $\sum_{a,b=1}^d e_{x,a,b} e_{y,b,a} = 0$ if $x \neq y$.

We note that $\operatorname{Hom}(\mathcal{K}_{d^2}, \mathcal{Q}_d)$ is a quotient of $\mathcal{B}_{d^2,d}$, realised via the map $\epsilon_{x,a,a'} \mapsto e_{x,a,a'}$.

Remark 5.12. Suppose that H is a finite dimensional Hilbert space and let $\pi : \operatorname{Hom}(\mathcal{K}_{d^2}, \mathcal{Q}_d) \to \mathcal{B}(H)$ be a unital *-representation. As the subalgebra generated by $e_{x,a,a'}$ is isomorphic to M_d, for each $x \in [d^2]$ there exists a (finite dimensional) Hilbert space K_x and unitaries $V_x : H \to \mathbb{C}^d \otimes K_x$ such that

$$\pi(e_{x,a,a'}) = V_x^*(\epsilon_{a,a'} \otimes I_{K_x})V_x, \quad x \in [d^2], a, a' \in [d].$$

Since π is unital, $\dim(K_x)$ is constant across $x \in [d^2]$. Applying a further unitary, we may assume that there exists a Hilbert space K with $K_x = K$ for all $x \in [d^2]$.

By relation (ii) (see the paragraph before the formulation of Remark 5.12), $\sum_{a,b=1}^d \pi(e_{x,a,b})\pi(e_{y,b,a}) = 0$ whenever $x \neq y$. It follows that

$$\sum_{a,b=1}^{d} (\epsilon_{a,b} \otimes 1)V_x V_y^*(\epsilon_{b,a} \otimes 1) = 0, \quad x, y \in [d^2],\ x \neq y,$$

and hence, taking partial trace Tr_{M_d} along M_d, we obtain

$$(\operatorname{tr}_d \otimes \operatorname{id}_{\mathcal{B}(K)})(V_x V_y^*) = 0, \quad x, y \in [d^2],\ x \neq y. \tag{46}$$

Remark 5.13. Suppose that $\mathcal{A}$ is a finite dimensional C*-algebra and $\pi : \mathrm{Hom}(\mathcal{K}_{d^2}, \mathcal{Q}_d) \to \mathcal{A}$ is a unital *-homomorphism. Then, up to a unital *-isomorphism, $\mathcal{A} = \oplus_{i=1}^k M_{dn_i}$ for some $k \in \mathbb{N}$ and some $n_i \in \mathbb{N}$, $i \in [k]$. Indeed, since $\mathrm{Hom}(\mathcal{K}_{d^2}, \mathcal{Q}_d)$ is a quotient of $\mathcal{B}_{d^2,d}$, the *-homomorphism π gives rise to a unital *-homomorphism $\tilde{\pi} : \mathcal{B}_{d^2,d} \to \mathcal{A}$, and therefore to a unital *-homomorphism $\tilde{\pi}_0 : M_d \to \mathcal{A}$. Assume, without loss of generality, that $\mathcal{A} = \oplus_{i=1}^k M_{m_i}$ for some $k \in \mathbb{N}$ and some $m_i \in \mathbb{N}$, $i \in [k]$. The projection $\mathrm{proj}_i : \mathcal{A} \to M_{m_i}$ is a unital *-homomorphism and hence $\mathrm{proj}_i \circ \tilde{\pi}_0 : M_d \to M_{m_i}$ is a unital *-homomorphism. It follows that $d | m_i$, $i \in [k]$.

Following [53], a *perfect quantum commuting strategy* (resp. *perfect quantum strategy*) for the graph homomorphism game $\mathcal{K}_{d^2} \to \mathcal{Q}_d$ is a CQNS correlation of quantum commuting (resp. quantum) type $\Gamma : \mathcal{D}_{d^2} \otimes \mathcal{D}_{d^2} \to M_d \otimes M_d$, such that $\Gamma(\epsilon_{x,x} \otimes \epsilon_{y,y})$ is supported in $\mathcal{Q}_d$ whenever $x \neq y$, and $\Gamma(\epsilon_{x,x} \otimes \epsilon_{x,x}) = \Omega_d \Omega_d^*$ for every $x \in [d^2]$. By [8,53], a quantum CQNS correlation $\Gamma : \mathcal{D}_{d^2} \otimes \mathcal{D}_{d^2} \to M_d \otimes M_d$ is a perfect quantum commuting strategy for $\mathcal{K}_{d^2} \to \mathcal{Q}_d$ if and only if there exist a tracial C*-algebra $(\mathcal{A}, \tau)$ and a unital *-homomorphism $\pi : \mathrm{Hom}(K_{d^2}, \mathcal{Q}_d) \to \mathcal{A}$, such that

$$\Gamma(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \sum_{a,a'=1}^{d} \sum_{b,b'=1}^{d} \tau(\pi(e_{x,a,a'} e_{y,b'b}))\epsilon_{a,a'} \otimes \epsilon_{b,b'}, \quad x, y \in [d^2];$$

we write $\Gamma = \Gamma^{\pi,\tau}$.

Recall that a *unitary error basis* in M_d is a basis of M_d with respect to the trace inner product that consists of unitaries, in other words, a collection $\{u_i\}_{i=1}^{d^2}$ of unitaries in M_d, such that $\mathrm{tr}_d(u_i u_j) = \delta_{i,j}$, $i, j \in [d^2]$. In the rest of this subsection, we restrict to the case $d = 2$. Recall the set

$$\mathcal{P} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right\} \tag{47}$$

of Pauli matrices in M_2; it will be convenient to temporarily denote them by U_x, $x \in [4]$, in the order they appear in (47). We note that, by [34], if $\mathcal{E}$ is a unitary error basis in M_2 then there exist unitary matrices R, $T \in M_2$ and constants c_V, $V \in \mathcal{P}$, such that $\mathcal{E} = \{c_V RVT : V \in \mathcal{P}\}$.

Let $\pi_{\mathcal{K}_4} : \mathcal{B}_{d^2,d} \to M_2$ be the *-homomorphism, given by $\pi_{\mathcal{K}_4}(e_{x,a,a'}) = U_x^* \epsilon_{a,a'} U_x$, and $\Gamma_{\mathcal{K}_4} : \mathcal{D}_4 \otimes \mathcal{D}_4 \to M_2 \otimes M_2$ be the CQNS correlation, given by

$$\Gamma_{\mathcal{K}_4}(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \sum_{a,a'=1}^{2} \sum_{b,b'=1}^{2} \mathrm{tr}_2(\pi_{\mathcal{K}_4}(e_{x,a,a'})\pi_{\mathcal{K}_4}(e_{y,b',b}))\epsilon_{a,a'} \otimes \epsilon_{b,b'},$$

where $x, y \in [4]$, and note that $\Gamma_{\mathcal{K}_4}$ is a perfect quantum strategy for the game $\mathcal{K}_4 \to \mathcal{Q}_2$.

The following theorem is the main result of this subsection.

Theorem 5.14. *Let $(\mathcal{A}, \tau)$ be a tracial von Neumann algebra with a faithful trace τ, and $\pi : \mathrm{Hom}(\mathcal{K}_4, \mathcal{Q}_2) \to \mathcal{A}$ be a unital *-homomorphism, such that $\Gamma^{\pi,\tau} = \Gamma_{\mathcal{K}_4}$. Then there exist a tracial von Neumann algebra $(\mathcal{N}, \tau_{\mathcal{N}})$, a trace preserving $*$-isomorphism $\rho : \mathcal{A} \to M_2 \otimes \mathcal{N}$ and a unitary $V \in M_2 \otimes \mathcal{N}$ such that*

$$V^* \rho(\pi(e_{x,a,a'}))V = U_x^* \epsilon_{a,a'} U_x \otimes 1_{\mathcal{N}}, \quad x \in [4], \; a, a' \in [2]. \tag{48}$$

Proof. Fix $x \in [4]$. Then $\{\pi(e_{x,a,a'})\}_{a,a'}$ is a system of matrix units in $\mathcal{A}$. Let $q = \pi(e_{x,1,1})$. Then for $m \in \mathcal{A}$ we have that $m_{i,j} := \pi(e_{x,1,i}) m \pi(e_{x,j,1})$ is in $q\mathcal{A}q$ and the map

$$\rho : \mathcal{A} \to M_2 \otimes q\mathcal{A}q, \quad m \mapsto \sum_{i,j=1}^{2} \epsilon_{i,j} \otimes m_{i,j}$$

is a normal $*$-isomorphism. Let $\mathcal{N} = q\mathcal{A}q$ and $\tau_{\mathcal{N}}$ be the restriction of τ to $q\mathcal{A}q$. Then

$$(\mathrm{Tr}_2 \otimes \tau_{\mathcal{N}})(\rho(m)) = \sum_{i=1}^{2} \tau_{\mathcal{N}}(m_{i,i}) = \sum_{i=1}^{2} \tau(m\pi(e_{x,i,1})\pi(e_{x,1,i})) = \tau(m).$$

Thus ρ is a trace preserving $*$-isomorphism and hence $\Gamma^{\pi,\tau} = \Gamma^{\rho\circ\pi, \mathrm{Tr}_2 \otimes \tau_{\mathcal{N}}}$. For simplicity of notation identify now $\mathcal{A}$ and $M_2 \otimes \mathcal{N}$. As $\{\pi(e_{x,a,a'})\}_{a,a'}$, $x \in [4]$, and $\{\epsilon_{a,a'} \otimes 1_{\mathcal{N}}\}_{a,a'}$ are systems of matrix units in $M_2 \otimes \mathcal{N}$, by [26, Lemma 2.1], there exists a unitary $V_x \in M_2 \otimes \mathcal{N}$, $x \in [4]$, such that

$$\pi(e_{x,a,a'}) = V_x^* (\epsilon_{a,a'} \otimes 1_{\mathcal{N}}) V_x.$$

A direct calculation shows that

$$\mathrm{Tr}_2(U_1^* \epsilon_{a,a'} U_1 U_3^* \epsilon_{b',b} U_3) = \begin{cases} 1 & \text{if } a = b = a' = b', \\ -1 & \text{if } a = b \neq a' = b', \\ 0 & \text{otherwise.} \end{cases}$$

By (46),

$$(\mathrm{Tr}_2 \otimes \mathrm{id}_{\mathcal{N}})(V_x V_y^*) = 0, \quad x \in [4], a, a' \in [2]. \tag{49}$$

Since $\Gamma^{\pi,\tau} = \Gamma_{\mathcal{K}_4}$, we have that

$$(\mathrm{Tr}_2 \otimes \tau_{\mathcal{N}})((\epsilon_{a,a} \otimes I_n) V_1 V_3^* (\epsilon_{a,a} \otimes I_n) V_3 V_1^*) = 1, \quad a \in [2], \tag{50}$$

and

$$(\mathrm{Tr}_2 \otimes \tau_{\mathcal{N}})((\epsilon_{a,a'} \otimes I_n) V_1 V_3^* (\epsilon_{a',a} \otimes I_n) V_3 V_1^*) = -1, \; a, a' \in [2], a \neq a'.$$

Writing $V_3V_1^*$ as a 2×2-block matrix

$$V_3V_1^* = \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix},$$

with A_3, B_3, C_3 and D_3 in $\mathcal{N}$, by (50) we have that

$$\tau_{\mathcal{N}}(A_3^*A_3) = \tau_{\mathcal{N}}(D_3^*D_3) = 1.$$

As $V_3V_1^*$ is unitary, $A_3^*A_3 + C_3^*C_3 = 1$, giving $\tau_{\mathcal{N}}(C_3^*C_3) = 0$ and hence, since $\tau_{\mathcal{N}}$ is faithful, $C_3 = 0$. Similarly, $B_3 = 0$. From (49) we get $A_3 + D_3 = 0$, that is $V_3V_1^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes u$, where u is unitary in $\mathcal{N}$. Next, writing $V_xV_1^*$, $x = 2, 4$, in a block form

$$V_xV_1^* = \begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix},$$

applying (49) to $V_xV_1^*$ and $V_xV_3^* = V_xV_1^*(V_3V_1^*)^*$, we get $A_x+D_x = 0$ and $A_xu^*-D_xu^* = 0$ and hence $A_x = D_x = 0$ for $x = 2, 4$.

Next we observe that

$$\mathrm{Tr}_2(U_1^*\epsilon_{a,a'}U_1U_2^*\epsilon_{b',b}U_2) = \begin{cases} 1, & a = a' \neq b = b', \\ 1, & a = b' \neq a' = b \\ 0, & \text{otherwise,} \end{cases}$$

implying

$$\tau_{\mathcal{N}}(C_2^*B_2) = \tau_{\mathcal{N}}(B_2^*C_2) = 1$$

and

$$\tau_{\mathcal{N}}(B_2^*B_2) = \tau_{\mathcal{N}}(C_2^*C_2) = 1.$$

By Cauchy-Schwartz, we obtain that $C_2 = \theta B_2$ for a unimodular constant θ; as $\tau_{\mathcal{N}}(C_2^*B_2) = e^{-i\varphi}\tau_{\mathcal{N}}(B_2^*B_2)$, we have that $\theta = 1$ and

$$V_2V_1^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes v$$

for a unitary $v \in \mathcal{N}$. Condition (49) applied to $V_2V_4^* = V_2V_1^*(V_4V_1^*)^*$ gives $vB_4^*+vC_4^* = 0$ and hence $B_4 = -C_4$. Therefore,

$$V_4V_1^* = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes w$$

for a unitary $w \in \mathcal{N}$. With this at hand, we conclude that, if $V = V_1$, then

$$V\pi(e_{x,a,a'})V^* = (V_x V_1^*)^*(\epsilon_{a,a'} \otimes 1_{\mathcal{N}})V_x V_1^* = U_x^* \epsilon_{a,a'} U_x \otimes 1_{\mathcal{N}},$$

$x \in [4]$, $a, a' \in [2]$, completing the proof. □

It follows from [53, Lemma 9.2] that there exists a *-isomorphism $\partial : \mathcal{B}_{4,2} \to \mathcal{B}_{4,2}^{\mathrm{op}}$, such that $\partial(e_{x,a,a'}) = e_{x,a',a}^{\mathrm{op}}$, $x \in [4]$, $a, a' \in [2]$. For $w = e_{x_1,a_1,a_1'} \cdots e_{x_k,a_k,a_k'}$, for some $x_i \in [4]$, $a_i, a_i' \in [2]$, $i = 1, \ldots, k$, set

$$\bar{w} := \partial^{-1}(w^{\mathrm{op}}) = e_{x_k,a_k',a_k} \cdots e_{x_1,a_1',a_1}.$$

Recalling the definition (45), let $\mathcal{S}$ be the set of states of $C_u^*(\mathcal{R}_{4,2}) \otimes_{\max} C_u^*(\mathcal{R}_{4,2})$ that factor through $\mathrm{Hom}(\mathcal{K}_4, \mathcal{Q}_2) \otimes_{\max} \mathrm{Hom}(\mathcal{K}_4, \mathcal{Q}_2)$. Let $\tilde{s}_{\mathcal{K}_4} \in \mathcal{S}$ be the state given by

$$\tilde{s}_{\mathcal{K}_4}(u \otimes v) = \mathrm{tr}_2(\pi_{\mathcal{K}_4}(q_{\mathcal{K}_4}(q_u(u)\overline{q_u(v)}))), \quad u, v \in C_u^*(\mathcal{R}_{4,2}),$$

where $q_u : C_u^*(\mathcal{R}_{4,2}) \to \mathcal{B}_{4,2}$ and $q_{\mathcal{K}_4} : \mathcal{B}_{4,2} \to \mathrm{Hom}(\mathcal{K}_4, \mathcal{Q}_2)$ are the quotient maps, and let $s_{\mathcal{K}_4}$ be the restriction of $\tilde{s}_{\mathcal{K}_4}$ to $\mathcal{R}_{4,2} \otimes_{\mathrm{c}} \mathcal{R}_{4,2}$.

Corollary 5.15. *The state $s_{\mathcal{K}_4}$ is an abstract self-test for the family $\mathcal{S}$.*

Proof. It suffices to show that $s_{\mathcal{K}_4}$ has a unique extension to a state $\phi : \mathcal{B}_{4,2} \otimes_{\max} \mathcal{B}_{4,2} \to \mathbb{C}$. Fix such an extension ϕ and note that the canonical correlation $\Gamma_\phi : \mathcal{D}_4 \otimes \mathcal{D}_4 \to M_2 \otimes M_2$ associated with ϕ coincides with $\Gamma_{s_{\mathcal{K}_4}}$. By [8, Theorem 3.2], there exists a tracial von Neumann algebra $(\mathcal{A}, \tau)$ and a *-representation $\pi : \mathcal{B}_{4,2} \to \mathcal{A}$, such that

$$\phi(u \otimes v) = \tau(\pi(u\bar{v})), \quad u, v \in \mathcal{B}_{4,2}.$$

Further, π canonically descends to a *-representation $\tilde{\pi} : \mathrm{Hom}(\mathcal{K}_4, \mathcal{Q}_2) \to \mathcal{A}$, such that, if $\tilde{\phi} : \mathrm{Hom}(\mathcal{K}_4, \mathcal{Q}_2) \otimes_{\max} \mathrm{Hom}(\mathcal{K}_4, \mathcal{Q}_2) \to \mathbb{C}$ is the canonical functional, arising from ϕ in view of the definition of the class $\mathcal{S}$, we have that

$$\tilde{\phi}(u \otimes v) = \tau(\tilde{\pi}(u\bar{v})), \quad u, v \in \mathrm{Hom}(\mathcal{K}_4, \mathcal{Q}_2).$$

Let $\rho : \mathcal{A} \to M_2 \otimes \mathcal{N}$ be the *-homomorphism, arising from Theorem 5.14 (here $(\mathcal{N}, \tau_{\mathcal{N}})$ is a tracial von Neumann algebra). Then, for $w \in \mathrm{Hom}(\mathcal{K}_4, \mathcal{Q}_2)$, we have that

$$\tilde{\phi}(w) = \tau(\tilde{\pi}(w)) = (\tau_2 \otimes \tau_{\mathcal{N}})(\pi_{\mathcal{K}_4}(w) \otimes 1_{\mathcal{N}}) = \tau_2(\pi_{\mathcal{K}_4}(w)).$$

The proof is complete. □

Let $\mathcal{C}$ be the class of quantum models $(H_A \otimes H_B, \varphi_A, \varphi_B, \xi)$ of CQNS correlations, where H_A and H_B are finite-dimensional Hilbert spaces, $H_A \otimes H_B = {}_{\mathcal{B}(H_A)}H_{\mathcal{B}(H_B)^o}$ is a bipartite system defined by the canonical bimodule structure of $H_A \otimes H_B$, and φ_A and φ_B extend to representations π_A and π_B of $\mathcal{A} := \mathcal{B}_{4,2}$ and $\mathcal{B} := \mathcal{B}_{4,2}$, respectively.

Proposition 5.16. *The state $s_{\mathcal{K}_4}$ is a self-test for $\mathcal{C}$.*

Proof. We will apply Theorem 4.3, see also [43, Theorem 4.12]. First we observe that $s_{\mathcal{K}_4}$ is an extreme point in the dual $(\mathcal{R}_{4,2} \otimes_{\min} \mathcal{R}_{4,2})^{\mathrm{d}}$ of $\mathcal{R}_{4,2} \otimes_{\min} \mathcal{R}_{4,2}$. In fact, if $s_{\mathcal{K}_4} = \sum_{i=1}^n \alpha_i s_i$, where $\alpha_i \geq 0$, $\sum_{i=1}^k \alpha_i = 1$, then the corresponding correlations Γ_{s_i} are perfect strategies for quantum colouring of $\mathcal{K}_4$. Indeed, note that, if $J = \Omega_2\Omega_2^*$ is the maximally entangled state in $\mathbb{C}^2 \otimes \mathbb{C}^2$, then

$$0 = \operatorname{Tr}(\Gamma_{s_{\mathcal{K}_4}}(\epsilon_{x,x} \otimes \epsilon_{x,x})J^{\perp}) = \sum_{i=1}^n \alpha_i \operatorname{Tr}(\Gamma_{s_i}(\epsilon_{x,x} \otimes \epsilon_{x,x})J^{\perp})$$

and hence $\operatorname{Tr}(\Gamma_{s_i}(\epsilon_{x,x} \otimes \epsilon_{x,x})J^{\perp})$, being non-negative, must be zero. Similar arguments show that $\operatorname{Tr}(\Gamma_{s_i}(\epsilon_{x,x} \otimes \epsilon_{y,y})J) = 0$ for all i and all $x \neq y$.

Therefore $s(e_{x,a,a'} \otimes e_{y,b,b'}) = \sum_{i=1}^n \alpha_i \tau_i(\pi_i(e_{x,a,a'})\pi_i(e_{y,b',b}))$, where $\pi_i : \operatorname{Hom}(\mathcal{K}_4, \mathcal{Q}_2) \to \mathcal{A}_i$ are $*$-homomorphisms into finite dimensional algebras $\mathcal{A}_i$ with trace τ_i. By Proposition 5.14,

$$\tau_i(\pi_i(e_{x,a,a'})\pi_i(e_{y,b',b})) = \operatorname{tr}_2(U_x^*\epsilon_{a,a'}U_xU_y^*\epsilon_{b',b}U_y)$$

for all i and hence s is extreme. The state $s_{\mathcal{K}_4}$ has also a unique extension to $\mathcal{B}_{4,2} \otimes \mathcal{B}_{4,2}$ and hence by Theorem 4.3 it is a self-test. $\square$

5.5. Schur products

In this subsection, whose main result is Theorem 5.17, we consider classes of no-signalling correlations from group representations. Let G be a finite group, let $\pi : G \to \mathcal{U}(H_\pi)$ and $\rho : G \to \mathcal{U}(H_\rho)$ be irreducible representations of G, and let $\psi \in H_\pi \otimes H_\rho$ be a unit vector. Then

$$u(s,t) := \langle(\pi(s) \otimes \rho(t))\psi, \psi\rangle, \quad s,t \in G,$$

is a normalised positive definite function on $G \times G$ and the associated *Schur multiplier* $\Theta(u) : \mathcal{B}(\ell^2(G)) \otimes \mathcal{B}(\ell^2(G)) \to \mathcal{B}(\ell^2(G)) \otimes \mathcal{B}(\ell^2(G))$ is the unital quantum channel satisfying

$$\Theta(u)(e_{s,s'} \otimes e_{t,t'}) = u(s^{-1}s', t^{-1}t')e_{s,s'} \otimes e_{t,t'}; \tag{51}$$

if further clarity is needed, we write $\Theta(u) = \Theta_{\pi,\rho,\psi}$. Letting $\widetilde{E}_{s,s',g,g'} = \delta_{s,g}\delta_{s',g'}\pi(s^{-1}s')$ and $\widetilde{F}_{t,t',h,h'} = \delta_{t,h}\delta_{t',h'}\rho(t^{-1}t')$, it is straightforward to verify that $\widetilde{E} := (\widetilde{E}_{s,s',g,g'})_{s,s',g,g'}$

and $\widetilde{F} := (\widetilde{F}_{t,t',h,h'})_{t,t',h,h'}$ are unitaries and hence the quadruple $S_{\pi,\rho,\psi} = (H_\pi \otimes H_\rho, \widetilde{E}, \widetilde{F}, \psi)$ is a $\mathcal{Q}_{\mathrm{q}}$-model for $\Theta(u)$.

A *unistochastic operator matrix (USOM)* is a stochastic operator matrix $E = (E_{x,x',a,a'}) \in M_X \otimes M_X \otimes \mathcal{B}(H)$ for which there exists a unitary $U : H^X \to H^X$ such that $E_{x,x',a,a'} = U^*_{a,x} U_{a',x'}$. A $\mathcal{Q}_{\mathrm{q}}$-model will be called *unitary* if it is defined via USOM's $(E_{s,s',g,g'})_{s,s',g,g'}$ and $(F_{t,t',h,h'})_{s,s',g,g'}$ (see Section 6.3). In particular, we have that $S := S_{\pi,\rho,\psi}$ is a unitary model. If H and K are Hilbert spaces, we say that a vector $\psi \in H \otimes K$ is *marginally cyclic* if

$$\overline{(\mathcal{B}(H) \otimes 1)\psi} = \mathcal{B}(H \otimes K) = \overline{(1 \otimes \mathcal{B}(K))\psi}.$$

Let

$$\mathfrak{M}(u) = \{S_{\pi'\rho',\psi'} : \text{ full rank unitary model s.t. } \Theta_{\pi'\rho',\psi'} = \Theta(u)\}.$$

Theorem 5.17. *If ψ is marginally cyclic in $H_\pi \otimes H_\rho$ and*

$$\operatorname{span}\{(\pi(s) \otimes \rho(t))\psi\psi^*(\pi(s^{-1}) \otimes \rho(t^{-1})) : s,t \in G\} = \mathcal{B}(H_\pi \otimes H_\rho) \tag{52}$$

then $(H_\pi \otimes H_\rho, \widetilde{E}, \widetilde{F}, \psi)$ is a self-test for $\mathfrak{M}(u)$.

Proof. Suppose that $(H_A \otimes H_B, (E_{s,s',g,g'})_{s,s',g,g'}, (F_{t,t',h,h'})_{t,t',h,h'}, \xi)$ is a full rank unitary model for $\Theta(u)$ in $\mathcal{Q}_{\mathrm{q}}$. This means that $E_{s,s',g,g'} = U^*_{g,s} U_{g',s'}$ and $F_{t,t',h,h'} = V^*_{h,t} V_{h',t'}$ for some block operator unitaries $(U_{g,s})_{g,s} : H_A^G \to H_A^G$ and $(V_{h,t})_{h,s} : H_B^G \to H_B^G$, the reduced densities $(\mathrm{id} \otimes \mathrm{Tr})(\xi\xi^*)$ and $(\mathrm{Tr} \otimes \mathrm{id})(\xi\xi^*)$ have full rank, and

$$\langle (E_{s,s',g,g'} \otimes F_{t,t',h,h'})\xi, \xi\rangle = \delta_{s,g}\delta_{s',g'}\delta_{t,h}\delta_{t',h'}\langle \pi(s^{-1}s') \otimes \rho(t^{-1}t')\psi, \psi\rangle.$$

In particular,

$$\langle (U_{s',s'} \otimes V_{t',t'})\xi, (U_{s,s} \otimes V_{t,t})\xi\rangle = \langle (\pi(s') \otimes \rho(t'))\psi, (\pi(s) \otimes \rho(t))\psi\rangle \tag{53}$$

for all $s, s', t, t' \in G$. We now observe that the unitaries $(U_{g,s})_{g,s}$ and $(V_{h,t})_{h,t}$ are necessarily diagonal. Since (under the trace-duality convention)

$$\Theta(u)(X^{\mathrm{t}})^{\mathrm{t}} = (\mathrm{id} \otimes \mathrm{Tr}_{2,4})((U \otimes V)(X \otimes \xi\xi^*)(U^* \otimes V^*)),$$

setting $\rho_\xi = (\mathrm{id} \otimes \mathrm{Tr})(\xi\xi^*)$, we have

$$\Theta(u)(\rho^{\mathrm{t}} \otimes 1)^{\mathrm{t}} = (\mathrm{id} \otimes \mathrm{Tr})U(\rho \otimes \rho_\xi)U^*.$$

Since the transformation $\rho \mapsto \Theta(u)(\rho^{\mathrm{t}} \otimes 1)^{\mathrm{t}}$ is a $\mathcal{D}_G$-bimodule map, if $\{e_i\}_{i=1}^{m_A}$ is an orthonormal basis for H_A, it follows that the Kraus operator

$$(\mathrm{id} \otimes e_i^*)U((\cdot) \otimes \sqrt{\rho_\xi} e_j) : \ell^2(G) \to \ell^2(G)$$

belongs to $\mathcal{D}'_G = \mathcal{D}_G$ for each $i, j \in [m_A]$. Hence,

$$U(1 \otimes \sqrt{\rho_\xi})(x \otimes 1) = (x \otimes 1)U(1 \otimes \sqrt{\rho_\xi}), \quad x \in \mathcal{D}_G.$$

Since ρ_ξ, and hence $\sqrt{\rho_\xi}$, has full rank, it follows that $U(x \otimes 1) = (x \otimes 1)U$, so that $U \in \mathcal{D}_G \otimes \mathcal{B}(H_A)$. Hence, $U_{s,s'} = \delta_{s,s'} U_{s,s}$ with $U_s := U_{s,s} \in \mathcal{U}(H_A)$. Without loss of generality, we may also assume $U_e = 1$ (indeed, if not, redefine U_s as $U_e^* U_s$, noting that $E_{s,s',g,g'} = U_{g,s}^* U_e U_e^* U_{g',s'}$). An analogous argument shows that $V_{t,t'} = \delta_{t,t'} V_{t,t}$ with $V_t := V_{t,t} \in \mathcal{U}(H_B)$ and we may assume that $V_e = 1$.

Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the (unital) C^*-subalgebra of $\mathcal{B}(H_A)$ (resp. $\mathcal{B}(H_B)$) generated by $\{U_s\}_{s \in G}$ (resp. $\{V_t\}_{t \in G}$). By the finite-dimensionality of $\mathcal{A}$ and $\mathcal{B}$, there exist unitaries $W_A : H_A \to \bigoplus_{i=1}^{n_A} H_A^i \otimes K_A^i$ and $W_B : H_B \to \bigoplus_{j=1}^{n_B} H_B^j \otimes K_B^j$ such that

$$W_A a W_A^* = \bigoplus_{i=1}^{n_A} \sigma_A^i(a) \otimes I_{K_A^i}, \quad a \in \mathcal{A},$$

and

$$W_B b W_B^* = \bigoplus_{j=1}^{n_B} \sigma_B^j(a) \otimes I_{K_B^j}, \quad b \in \mathcal{B},$$

with σ_A^i and σ_B^j irreducible representations of $\mathcal{A}$ and $\mathcal{B}$, respectively. Fix orthonormal bases $\{e_k^i\}_{k=1}^{d_A^i}$ and $\{e_l^j\}_{l=1}^{d_B^j}$ of K_A^i and K_B^j, respectively. It follows from (53) that

$$\begin{aligned}
&\langle (U_{s'} \otimes V_{t'})\xi, (U_s \otimes V_t)\xi \rangle \\
&= \sum_{i=1}^{n_A} \sum_{j=1}^{n_B} \sum_{k=1}^{d_A^i} \sum_{l=1}^{d_B^j} p_{i,j,k,l} \langle (\sigma_A^i(U_{s'}) \otimes \sigma_B^j(V_{t'}))\xi_{i,j,k,l}, (\sigma_A^i(U_s) \otimes \sigma_B^j(V_t))\xi_{i,j,k,l} \rangle,
\end{aligned}$$

where $\xi_{i,j,k,l} \in H_A^i \otimes H_B^j$ is the normalisation of $(\mathrm{id} \otimes e_k^i \otimes \mathrm{id} \otimes e_l^j)^* \xi$, and

$$p_{i,j,k,l} = \|(\mathrm{id} \otimes e_k^i \otimes \mathrm{id} \otimes e_l^j)^* \xi\|^2 \geq 0,$$

and note that $\sum_{i,j,k,l} p_{i,j,k,l} = 1$. Combining (51) and (53), we obtain a convex decomposition of the channel $\Theta(u)$, which is extreme within the set of completely positive trace preserving maps by (52) (see e.g. [35, Theorem 3]). Indeed, setting $\psi_{(s,t)} := (\pi(s) \otimes \varphi(t))\psi$, $s, t \in G$, in the notation and terminology of [35], $\{\psi_{(s,t)}\}_{(s,t) \in G \times G}$ is a full set of vectors by (52) and $C_{(s,t),(s',t')} := u(s^{-1}s', t^{-1}t') = \langle \psi_{(s',t')}, \psi_{(s,t)} \rangle$. Thus, for every $\lambda = (i,j,k,l) \in \Lambda = \{(i,j,k,l) : p_{i,j,k,l} > 0\}$, we have

$$\langle (\pi(s')\otimes\rho(t'))\psi, (\pi(s)\otimes\rho(t))\psi\rangle$$
$$= \langle (\sigma_A^i(U_{s'})\otimes\sigma_B^j(V_{t'}))\xi_{i,j,k,l}, (\sigma_A^i(U_s)\otimes\sigma_B^j(V_t))\xi_{i,j,k,l}\rangle.$$

It follows that

$$W_\lambda : H_\pi\otimes H_\rho \ni (\pi(s)\otimes\rho(t))\psi \to (\sigma_A^i(U_s)\otimes\sigma_B^j(V_t))\xi_{i,j,k,l}\in H_A^i\otimes H_B^j$$

is a well-defined isometry for each $\lambda\in\Lambda$. Recalling that ψ is marginally cyclic and that π and ρ are irreducible (so $\mathrm{span}\{\pi(s) : s\in G\} = \mathcal{B}(H_\pi)$ and $\mathrm{span}\{\rho(t) : t\in G\} = \mathcal{B}(H_\rho)$) for any $\eta\in H_\pi\otimes H_\rho$ and $s\in G$, for suitable scalars c_t, $t\in G$, we have

$$\begin{aligned}
W_\lambda(\pi(s)\otimes 1)\eta &= \sum_{t\in G} c_t W_\lambda(\pi(s)\otimes\rho(t))\psi\\
&= \sum_{t\in G} c_t(\sigma_A^i(U_s)\otimes\sigma_B^j(V_t))\xi_{i,j,k,l}\\
&= \sum_{t\in G} c_t(\sigma_A^i(U_s)\otimes I_{H_B^j})(\sigma_A^i(U_e)\otimes\sigma_B^j(V_t))\xi_{i,j,k,l}\\
&= \sum_{t\in G} c_t(\sigma_A^i(U_s)\otimes I_{H_B^j})W_\lambda(1\otimes\rho(t))\psi\\
&= (\sigma_A^i(U_s)\otimes I_{H_B^j})W_\lambda\eta.
\end{aligned}$$

Thus, $W_\lambda(\pi(s)\otimes 1) = (\sigma_A^i(U_s)\otimes I_{H_B^j})W_\lambda$, $s\in G$. Similarly, $W_\lambda(1\otimes\rho(t)) = (I_{H_A^i}\otimes\sigma_B^j(V_t))W_\lambda$, $t\in G$, so that

$$W_\lambda(\pi(s)\otimes\rho(t)) = (\sigma_A^i(U_s)\otimes\sigma_B^j(V_t))W_\lambda, \quad s,t\in G.$$

Then the unitaries $\sigma_A^i(U_s)\otimes\sigma_B^j(V_t)$ are respectively mapped to the unitaries $\pi(s)\otimes\rho(t)$ under the unital completely positive map $\Phi_\lambda : \mathcal{B}(H_A^i\otimes H_B^j)\to\mathcal{B}(H_\pi\otimes H_\rho)$, given by $\Phi_\lambda(T) = W_\lambda^* T W_\lambda$. Thus, the unitaries $\sigma_A^i(U_s)\otimes\sigma_B^j(V_t)$ belong to the multiplicative domain $\mathcal{M}$ of Φ_λ. Hence, by [12, Proposition 1.5.7], $\mathcal{M}$ contains the C^*-algebra generated by $\sigma_A^i(U_s)\otimes\sigma_B^j(V_t)$, which, by the irreducibility of σ_A^i and σ_B^j (and the definitions of $\mathcal{A}$ and $\mathcal{B}$), is equal to $\mathcal{B}(H_A^i\otimes H_B^j)$. So $\Phi_\lambda : \mathcal{B}(H_A^i\otimes H_B^j)\to\mathcal{B}(H_\pi\otimes H_\rho)$ is a unital $*$-homomorphism, necessarily injective by simplicity of $\mathcal{B}(H_A^i\otimes H_B^j)$ and surjective by irreducibility of π and ρ. Hence, W_λ is unitary.

Moreover, the local intertwining properties above show that the maps $\sigma_A : \mathcal{A}\to\mathcal{B}(H_\pi)\otimes I_{H_\rho}$ and $\sigma_B : \mathcal{B}\to I_{H_\pi}\otimes\mathcal{B}(H_\rho)$, given by

$$\sigma_A(a) = W_\lambda^*(\sigma_A^i(a)\otimes I_{H_B^j})W_\lambda \;\text{ and }\; \sigma_B(b) = W_\lambda^*(I_{H_A^i}\otimes\sigma_B^j(b))W_\lambda$$

are irreducible representations of $\mathcal{A}$ and $\mathcal{B}$, respectively, and are independent of λ. Clearly, $\sigma_A\otimes\sigma_B$ is unitarily equivalent to each $\sigma_A^i\otimes\sigma_B^j$, (via W_λ) so we must have local unitary equivalence: $\sigma_A \cong \sigma_A^i$ and $\sigma_B \cong \sigma_B^j$, say, via $W_A^i : H_A^i \to H_\pi$ and $W_B^j : H_B^j \to H_\varphi$. Then $W_A^i \otimes W_B^j = \alpha_{i,j} W_\lambda^*$ for some phase $\alpha_{i,j} \in \mathbb{T}$.

We now follow ideas from [43, Theorem 4.12]. Let

$$\Lambda_A = \{i : p_{i,j,k,l} > 0 \text{ for some } j, k, l\},$$

and define Λ_B similarly. Fix unit vectors $\eta_A \in H_\pi$ and $\eta_B \in H_\rho$. For $i \notin \Lambda_A$, let $T_A^i : H_A^i \otimes K_A^i \to H_\pi \otimes H_A^i \otimes K_A^i$ be given by $T_A^i(\eta) = \eta_A \otimes \eta$, and define $T_B^j : H_B^j \otimes K_B^j \to H_\rho \otimes H_B^j \otimes K_B^j$, for $j \notin \Lambda_B$, similarly. Finally, set

$$H_A^{\text{aux}} := \Big(\bigoplus_{i\in\Lambda_A} K_A^i\Big) \bigoplus \Big(\bigoplus_{i\notin\Lambda_A} H_i^A \otimes K_A^i\Big),$$

and define the isometry $T_A : H_A \to H_\pi \otimes H_A^{\text{aux}}$ by

$$T_A = \bigg(\Big(\bigoplus_{i\in\Lambda_A} W_A^i \otimes I_{K_A^i}\Big) \oplus \Big(\bigoplus_{i\notin\Lambda_A} T_A^i\Big)\bigg) \circ W_A.$$

Define H_B^{aux} and $T_B : H_B \to H_\rho \otimes H_B^{\text{aux}}$ similarly, and let

$$\xi^{\text{aux}} = \bigoplus_{i=1}^{n_A} \bigoplus_{j=1}^{n_B} \sum_{k=1}^{d_A^i} \sum_{l=1}^{d_B^j} \alpha_{i,j} \sqrt{p_{i,j,k,l}}\, e_k^i \otimes e_l^j \in H_A^{\text{aux}} \otimes H_B^{\text{aux}}.$$

By construction, for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$ we have

$$(T_A \otimes T_B)(a \otimes b)\xi = ((\sigma_A(a) \otimes \sigma_B(b))\psi) \otimes \xi^{\text{aux}}.$$

In particular,

$$\begin{aligned}
&(T_A \otimes T_B)(E_{s,s',g,g'} \otimes F_{t,t',h,h'})\xi \\
&= \delta_{s,g}\delta_{s',g'}\delta_{t,h}\delta_{t',h'} T_A \otimes T_B(U_s^* U_{s'} \otimes V_t^* V_{t'})\xi \\
&= \delta_{s,g}\delta_{s',g'}\delta_{t,h}\delta_{t',h'} (\pi(s)^*\pi(s') \otimes \rho(t)^*\rho(t'))\psi \otimes \xi^{\text{aux}} \\
&= \delta_{s,g}\delta_{s',g'}\delta_{t,h}\delta_{t',h'} (\pi(s^{-1}s') \otimes \rho(t^{-1}t'))\psi \otimes \xi^{\text{aux}}.
\end{aligned}$$

Thus, the model $(H_\pi \otimes H_\rho, \{\widetilde{E}_{s,s',g,g'}\}, \{\widetilde{F}_{t,t',h,h'}\}, \psi)$ is a self-test, as claimed. □

We now exhibit a class of examples satisfying the hypotheses of Proposition 5.17.

Example 5.18. Let $G = S_3$, the symmetric group on 3 points. S_3 has a two dimensional irreducible representation $\pi : S_3 \to \mathcal{U}(\mathbb{C}^2)$ given by

$$\pi(e) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \pi(123) = \begin{bmatrix} e^{i2\pi/3} & 0 \\ 0 & e^{-i2\pi/3} \end{bmatrix}, \quad \pi(132) = \begin{bmatrix} e^{-i2\pi/3} & 0 \\ 0 & e^{i2\pi/3} \end{bmatrix},$$
$$\pi(12) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \pi(23) = \begin{bmatrix} 0 & e^{-i2\pi/3} \\ e^{i2\pi/3} & 0 \end{bmatrix}, \quad \pi(13) = \begin{bmatrix} 0 & e^{i2\pi/3} \\ e^{-i2\pi/3} & 0 \end{bmatrix}.$$

In the argument below, we will interpret the above matrices as rotations on the Bloch sphere. To that end, recall that the single qubit rotation operators

$$R_x(\theta) = e^{-i\frac{\theta}{2}\sigma_x}, \quad R_y(\theta) = e^{-i\frac{\theta}{2}\sigma_y}, \quad R_z(\theta) = e^{-i\frac{\theta}{2}\sigma_z},$$

induce rotations of angle $\theta \in \mathbb{R}$ about the x, y, and z axes, respectively, where σ_x, σ_y and σ_z are the 2×2 Pauli matrices. Then

$$\pi(123) = R_z(2\pi/3), \quad \pi(132) = R_z(4\pi/3), \quad \pi(12) = X = -iR_x(\pi).$$

If $\{e_0, e_1\}$ denotes the standard basis of $\mathbb{C}^2$, for $\theta \in (0, \pi/2) \cup (\pi/2, \pi)$, we let $\{e_\theta, f_\theta\}$ denote the following y-rotated basis:

$$e_\theta = R_y(\theta)e_0 = \begin{bmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{bmatrix}, \quad f_\theta = R_y(\theta)e_1 = \begin{bmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{bmatrix}.$$

Let $\psi := \alpha e_\theta \otimes e_\theta + \beta f_\theta \otimes f_\theta \in \mathbb{C}^2 \otimes \mathbb{C}^2$ for fixed $\alpha, \beta \in \mathbb{C}$ satisfying $|\alpha|^2 + |\beta|^2 = 1$ and $|\alpha|^2 \in (0, 1/2)$. Then ψ has full Schmidt rank so is marginally cyclic. We now show that

$$V := \mathrm{span}\{(\pi(s) \otimes \pi(t))\psi\psi^*(\pi(s^{-1}) \otimes \pi(t^{-1})) : s, t \in S_3\} = \mathcal{B}(\mathbb{C}^2 \otimes \mathbb{C}^2),$$

thereby obtaining a self-testing $\mathcal{Q}_{\mathrm{q}}$-model for the channel $\Theta(u)$, where $u(s,t) = \langle(\pi(s) \otimes \pi(t))\psi, \psi\rangle$, by Proposition 5.17.

First, by irreducibility of π and the orthogonality relation ([49, Theorem III.1.1]),

$$\sum_{s \in S_3} \pi(s)(\cdot)\pi(s)^* = \mathrm{Tr}(\cdot)1,$$

so that

$$\pi(s)\rho\pi(s)^* \otimes 1, \; 1 \otimes \pi(t)\rho\pi(t)^* \in V, \quad s, t \in G,$$

where ρ is the reduced density matrix of ψ:

$$\rho = (\mathrm{id} \otimes \mathrm{Tr})(\psi\psi^*) = |\alpha|^2 e_\theta e_\theta^* + |\beta|^2 f_\theta f_\theta^* = (\mathrm{Tr} \otimes \mathrm{id})(\psi\psi^*).$$

Therefore, it suffices to show that

$$\mathrm{span}_{\mathbb{R}}\{\pi(s)\rho\pi(s)^* : s \in S_3\} = M_2(\mathbb{C})_{sa}. \tag{54}$$

Recall that the space of trace-1 self-adjoint 2×2 matrices is affinely isomorphic to $\mathbb{R}^3$ via

$$(x, y, z) \Leftrightarrow \begin{bmatrix} 1/2 + z & x - iy \\ x + iy & 1/2 - z \end{bmatrix},$$

with the convex subset of density operators corresponding to the unit ball. In particular, we can visualize ρ as the vector $r_\rho = (x_\rho, y_\rho, z_\rho) \in \mathbb{R}^3_{\|\cdot\| \leq 1}$, which lies on the straight line between the points $r_{e_\theta e_\theta^*}$ and $r_{f_\theta f_\theta^*}$ on the boundary sphere. Both these latter points lie on the great circle in the xz-plane, and, since $\theta \in (0, \pi/2) \cup (\pi/2, \pi)$, are neither on the z-axis, nor the xy-plane. (Recall that $r_{e_0 e_0^*}$ and $r_{e_1 e_1^*}$ are the North and South poles, respectively.) Thus, the points in the unit ball associated to the rotated states

$$\begin{aligned} \pi(123)\rho\pi(123)^* &= R_z(2\pi/3)\rho R_z(2\pi/3)^*, \\ \pi(132)\rho\pi(132)^* &= R_z(4\pi/3)\rho R_z(4\pi/3)^*, \\ \pi(12)\rho\pi(12)^* &= R_x(\pi)\rho R_x(\pi)^* \end{aligned}$$

are affinely independent in $\mathbb{R}^3$, so their affine hull yields all hermitian matrices of trace 1, and the equality (54) follows.

6. Connections with C^*-envelopes

Recall that, if $\mathcal{S}$ is an operator system, its *C^*-envelope* $C_e^*(\mathcal{S})$ is the unital C*-algebra, uniquely determined up to isomorphism by the following universal property: there is a unital complete order embedding $\iota : \mathcal{S} \to C_e^*(\mathcal{S})$ such that $C^*(\iota(\mathcal{S})) = C_e^*(\mathcal{S})$, and for any C^*-algebra $\mathcal{A}$ and unital complete order embedding $\varphi : \mathcal{S} \to \mathcal{A}$ with $C^*(\varphi(\mathcal{S})) = \mathcal{A}$, there is a surjective $*$-homomorphism $\pi : \mathcal{A} \to C_e^*(\mathcal{S})$ such that $\pi \circ \varphi = \iota$. In this penultimate section, we show that the examples in Sections 5.1 (PVM self-tests), 5.4 (quantum graph colouring) and 5.5 (Schur product channels) are all instances of a single phenomenon: unique state extension across

$$\mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B \subseteq C_e^*(\mathcal{S}_A) \otimes_{\max} C_e^*(\mathcal{S}_B),$$

for pertinent operator systems $\mathcal{S}_A$ and $\mathcal{S}_B$ where the latter inclusion is valid.

By contrast, general abstract self-testing concerns unique state extensions across

$$\mathcal{S}_A \otimes_{\mathrm{c}} \mathcal{S}_B \subseteq C_u^*(\mathcal{S}_A) \otimes_{\max} C_u^*(\mathcal{S}_B),$$

the tensor product of *universal* C*-covers. Similar comparisons can be made when self-testing among models which factor through $C_e^*(\mathcal{S}_A) \otimes_{\max} C_e^*(\mathcal{S}_B)$.

The observations that follow rely mainly on the work [5], but along the way we establish new dilation results for stochastic operator matrices.

6.1. PVM's

As mentioned in Remark 5.5, PVM self-testing fits in our general framework, where the family of states $s : C_u^*(\mathcal{S}_{X,A}) \otimes_{\max} C_u^*(\mathcal{S}_{Y,B}) \to \mathbb{C}$ to be self-tested is restricted to those that factor through the quotient map

$$\pi_A \otimes \pi_B : C_u^*(\mathcal{S}_{X,A}) \otimes_{\max} C_u^*(\mathcal{S}_{Y,B}) \to \mathcal{A}_{X,A} \otimes_{\max} \mathcal{A}_{Y,B}.$$

Note that the set of such states is precisely

$$(\pi_A \otimes \pi_B)^*(\mathcal{S}(\mathcal{A}_{X,A} \otimes_{\max} \mathcal{A}_{Y,B})) \cong \mathcal{S}(\mathcal{A}_{X,A} \otimes_{\max} \mathcal{A}_{Y,B}).$$

It is well-known that

$$\mathcal{A}_{X,A} \cong \underbrace{\mathcal{D}_A *_1 \cdots *_1 \mathcal{D}_A}_{|X| \text{ times}} \cong C_e^*(\mathcal{S}_{X,A}),$$

and similarly for $\mathcal{A}_{Y,B}$, (for the latter isomorphism, see e.g. [5, Corollary 2.9]). Thus, a state $f : \mathcal{S}_{X,A} \otimes_c \mathcal{S}_{Y,B} \to \mathbb{C}$ has a unique extension to

$$(\pi_A \otimes \pi_B)^*(\mathcal{S}(C_e^*(\mathcal{S}_{X,A}) \otimes_{\max} C_e^*(\mathcal{S}_{Y,B}))$$

if and only if it extends uniquely across

$$\mathcal{S}_{X,A} \otimes_c \mathcal{S}_{Y,B} \subseteq C_e^*(\mathcal{S}_{X,A}) \otimes_{\max} C_e^*(\mathcal{S}_{Y,B}),$$

the above inclusion being valid by [5, Lemma 3.10].

6.2. Semi-classical SOM's

Extending the setup of Subsection 5.4, let

$$\mathcal{B}_{X,A} = \underbrace{M_A *_1 \cdots *_1 M_A}_{|X| \text{ times}}.$$

For each $x \in X$, write $\{\epsilon_{x,a,a'} : a, a' \in A\}$ for the canonical matrix unit system of the x-th copy of M_A, and let

$$\mathcal{R}_{X,A} = \operatorname{span}\{\epsilon_{x,a,a'} : x \in X, a, a' \in A\},$$

considered as an operator subsystem of $\mathcal{B}_{X,A}$. By [5, Corollary 2.9], the C^*-algebra $\mathcal{B}_{X,A}$ is universal for unital $*$-homomorphisms $M_A \to \mathcal{D}_X \otimes \mathcal{A}$, where $\mathcal{A}$ is a unital C^*-algebra (see [5, Definition 2.2]), and the pair $(\mathcal{R}_{X,A}, \mathcal{B}_{X,A})$ satisfies the hypotheses of [5, Theorem 3.8]. Hence, $C_e^*(\mathcal{R}_{X,A}) = \mathcal{B}_{X,A}$, and from [5, Lemma 3.10] (or [46, Lemma 2.8]),

$$\mathcal{R}_{X,A} \otimes_{\mathrm{c}} \mathcal{R}_{Y,B} \subseteq C_e^*(\mathcal{R}_{X,A}) \otimes_{\max} C_e^*(\mathcal{R}_{Y,B}).$$

Following [53, §7.1], an SOM $E \in M_X \otimes M_A \otimes \mathcal{B}(H)$ is called *semi-classical* if $E = \sum_{x \in X} \epsilon_{x,x} \otimes E_x$ with $E_x \in (M_A \otimes \mathcal{B}(H))^+$ and $\mathrm{Tr}_A E_x = I_H$, $x \in X$. In [53, Theorem 7.5] it was shown that semi-classical SOMs correspond to unital completely positive maps $\mathcal{R}_{X,A} \to \mathcal{B}(H)$. Moreover, the state space of $\mathcal{R}_{X,A} \otimes_{\mathrm{c}} \mathcal{R}_{Y,A}$ is affinely isomorphic to the set of classical-to-quantum no-signalling correlations $\Gamma : \mathcal{D}_X \otimes \mathcal{D}_Y \to M_A \otimes M_B$ [53, Theorem 7.7].

In light of the previous paragraphs, Corollary 5.15 means that the state $\tilde{s}_{\mathcal{K}_4} : \mathcal{R}_{4,2} \otimes_{\mathrm{c}} \mathcal{R}_{4,2} \to \mathbb{C}$ given by

$$\tilde{s}_{\mathcal{K}_4}(w) = \mathrm{tr}_2((\pi_{\mathcal{K}_4} \cdot \pi_{\mathcal{K}_4})(w)), \quad w \in \mathcal{R}_{4,2} \otimes_{\mathrm{c}} \mathcal{R}_{4,2},$$

has a unique extension to the finite-dimensional states of $C_e^*(\mathcal{R}_{4,2}) \otimes_{\max} C_e^*(\mathcal{R}_{4,2})$. Similarly, Proposition 5.16 means that $\tilde{s}_{\mathcal{K}_4}$ is a self-test for the class of finite-dimensional semi-classical SOM models which factor through $C_e^*(\mathcal{R}_{4,2}) \otimes_{\max} C_e^*(\mathcal{R}_{4,2})$.

6.3. Unistochastic operator matrices

It is known that the entries of any SOM $E = (E_{x,x',a,a'}) \in M_X \otimes M_A \otimes \mathcal{B}(H)$ can be represented as $E_{x,x',a,a'} = V_{a,x}^* V_{a',x'}$ for an isometry $V : H^X \to K^A$, where $V = (V_{a,x})_{a,x}$ [53, Theorem 3.1]. When $X = A$, it is natural to study further the unistochastic SOM's (USOM's), that is, those SOM's for which V can be taken unitary (see Section 5.5). In this subsection, we show that any SOM can be dilated to a USOM, thereby establishing a matricial version of Naimark dilation between POVM's and PVM's. Along the way, we connect these notions to C^*-envelopes of pertinent operator systems, as done in the previously in this section.

Let $\mathcal{B}_X$ denote the universal C^*-algebra generated by the elements $u_{a,x}$ $a, x \in X$ of a unitary matrix $u = (u_{a,x})_{a,x}$, commonly known as the Brown algebra [11]. It was shown in [25, Proposition 2.3] that if S_1^X is the dual operator space $(M_X)^*$, then the map $e_{a,x} \mapsto u_{a,x}$, from S_1^X into $\mathrm{span}\{u_{a,x} : a, x \in X\} \subseteq \mathcal{B}_X$ is a completely isometric isomorphism.

Recall from Subsection 5.1 the universal TRO of a block operator isometry $v = (v_{a,x})_{a,x \in X}$ [53], hereby denoted simply $\mathcal{V}_X$ as we assume that $X = A$ throughout this subsection; thus, $\mathcal{V}_X$ is universal for the relations

$$\sum_{a \in X} v_{a,x}^* v_{a,x'} = \delta_{x,x'} 1, \quad x, x' \in X, \tag{55}$$

in that every (concrete) block operator isometry $V = (v_{a,x})_{a,x}$, whose entries lie in $\mathcal{B}(H, K)$ for some Hilbert spaces H and K, gives rise to a unique ternary morphism $\theta_V : \mathcal{V}_X \to \mathcal{B}(H)$ such that $\theta(v_{a,x}) = V_{a,x}$, $x, a \in X$. Since the entries of u satisfy the relations (55), there exists a (unique) ternary morphism $\varphi : \mathcal{V}_X \to \mathcal{B}_X$ such that

$\varphi(v_{a,x}) = u_{a,x}$. On the other hand, since $v = (v_{a,x})_{a,x}$ belongs to the unit ball of $M_X(\mathcal{V}_X)$, the canonical completely isometric identification $M_X(\mathcal{V}_X) \cong \mathcal{CB}(S_1^X, \mathcal{V}_X)$ (see e.g. [4, Proposition 1.5.14]), yields a complete contraction $\varphi' : \mathrm{span}\{u_{a,x} : a, x \in X\} \to \mathrm{span}\{v_{a,x} : a, x \in X\}$ satisfying $\varphi'(u_{a,x}) = v_{a,x}$. Since ternary morphisms are completely contractive, it follows that the restriction

$$\kappa : \mathrm{span}\{u_{a,x} : a, x \in X\} \to \mathrm{span}\{v_{a,x} : a, x \in X\}$$

of φ' is a complete isometry (see also [18, Proposition 4.1]). This first order isomorphism leads to the following dilation result.

Proposition 6.1. *Let H and K be Hilbert spaces, for which $(v_{a,x})_{a,x} \in M_X(\mathcal{B}(H,K))$. There exists a Hilbert space L, isometries $w_1 : H \to L$, $w_2 : K \to L$ and a unital *-homomorphism $\pi : \mathcal{B}_X \to \mathcal{B}(L)$ satisfying*

$$v_{a,x} = w_2^* \pi(u_{a,x}) w_1 \quad \text{and} \quad w_2 w_2^* \pi(u_{a,x}) w_1 = \pi(u_{a,x}) w_1 \quad x, a \in X.$$

Proof. By the injectivity of $\mathcal{B}(H,K)$, we can extend κ to a complete contraction $\widetilde{\kappa} : \mathcal{B}_X \to \mathcal{B}(H,K)$. By the Haagerup-Paulsen-Wittstock representation theorem for complete contractions, there exists a Hilbert space L, isometries $w_1 : H \to L$, $w_2 : K \to L$ and a unital *-homomorphism $\pi : \mathcal{B}_X \to \mathcal{B}(L)$ such that $\widetilde{\kappa}(a) = w_2^* \pi(a) w_1$, $a \in \mathcal{B}_X$. (We note that this can be derived by applying the proof of [12, Theorem B.7] together with the rectangular version of Paulsen's off-diagonal trick [4, Lemma 1.3.15] and refer to [24, Theorem 2.1.12] for a complete argument.) In particular,

$$v_{a,x} = \kappa(u_{a,x}) = w_2^* \pi(u_{a,x}) w_1, \quad x, a \in X.$$

Then $v = (w_2^* \otimes I_X)\pi_X(u)(w_1 \otimes I_X)$, where $\pi_X := (\pi \otimes \mathrm{id}_{M_X})$. Since

$$(w_2 \otimes I_X)v = (w_2 w_2^* \otimes I_X)\pi_X(u)(w_1 \otimes I_X)$$

with both $(w_2 \otimes I_X)v$ and $\pi_X(u)(w_1 \otimes I_X)$ isometries, we have that $\pi_X(u)(w_1 \otimes I_X)H \otimes \mathbb{C}^X \subseteq \mathrm{ran}(w_2 \otimes I_X)$. $\square$

Corollary 6.2. *There exists a complete order isomorphism $\varphi : \mathcal{T}_X \to \mathrm{span}\{u_{a,x}^* u_{a',x'} : x, x', a, a' \in X\}$ satisfying*

$$\varphi(v_{a,x}^* v_{a',x'}) = u_{a,x}^* u_{a',x'}, \quad x, x', a, a' \in X. \tag{56}$$

Proof. By the universal property of $v = (v_{a,x})_{a,x}$, there exists a (non-degenerate) TRO morphism $\mathcal{V}_X \to \mathcal{B}_X$ mapping $v_{a,x}$ to $u_{a,x}$, $x, a \in X$. The latter morphism induces a unital $*$-homomorphism $\varphi : \mathcal{C}_X \to \mathcal{B}_X$ mapping $v_{a,x}^* v_{a',x'}$ to $u_{a,x}^* u_{a',x'}$, $x, x', a, a' \in X$, where $\mathcal{C}_X := \mathcal{C}_{X,X}$. Thus $\varphi|_{\mathcal{T}_X}$ is a unital completely positive map satisfying (56).

Represent v faithfully inside $M_X(\mathcal{B}(H,K))$ for some Hilbert spaces H and K. By Proposition 6.1, there exists a Hilbert space L, isometries $w_1 : H \to L$, $w_2 : K \to L$ and a unital *-homomorphism $\pi : \mathcal{B}_X \to \mathcal{B}(L)$ satisfying

$$v_{a,x} = w_2^*\pi(u_{a,x})w_1 \quad \text{and} \quad w_2w_2^*\pi(u_{a,x})w_1 = \pi(u_{a,x})w_1, \quad x, a \in X.$$

But then

$$v_{a,x}^*v_{a',x'} = w_1^*\pi(u_{a,x}^*)w_2w_2^*\pi(u_{a',x'})w_1 = w_1^*\pi(u_{a,x}^*u_{a',x'})w_1,$$

for all $x, x', a, a' \in X$. It follows that $w_1^*\pi(\cdot)w_1$ is a unital completely positive inverse to φ. □

Let E be a unistochastic operator matrix acting on a Hilbert space H. When $H = \mathbb{C}$, and E is diagonal in the sense that $E_{x,x',a,a'} = \delta_{x,x'}\delta_{a,a'}E_{x,x,a,a}$, we recover the usual notion of unistochastic matrices. A simple application of the previous results yields the following dilation property.

Corollary 6.3. *Let H be a Hilbert space and $E = (E_{x,x',a,a'}) \in M_X \otimes M_X \otimes \mathcal{B}(H)$ be a SOM. Then there exists a Hilbert space K, an isometry $W : H \to K$ and a block operator unitary $(U_{a,x})_{a,x\in X} \in M_X \otimes \mathcal{B}(K)$, such that $E_{x,x',a,a'} = W^*U_{a,x}^*U_{a',x'}W$.*

Proof. By the universal property of $\mathcal{T}_X$, there is unital completely positive map $\psi : \mathcal{T}_X \to \mathcal{B}(H)$ satisfying $\psi(e_{x,x',a,a'}) = E_{x,x',a,a'}$. Let $\varphi^{-1} : \mathrm{span}\{u_{a,x}^*u_{a',x'} : x, x', a, a' \in X\} \to \mathcal{T}_X$ be the complete order isomorphism from Corollary 6.2. Extending the unital completely positive map $\psi \circ \varphi^{-1}$ to $\mathcal{B}_X \to \mathcal{B}(H)$ (by injectivity of $\mathcal{B}(H)$) and appealing to a Stinespring representation of the extension yields the desired conclusion. □

Using results from [5], we now show that the C^*-envelope of $\mathcal{T}_X$ is the C*-subalgebra $C^*(\mathcal{U})$ of $\mathcal{B}_X$, generated by the operator system

$$\mathcal{U} = \mathrm{span}\{u_{a,x}^*u_{a',x'} : x, x', a, a' \in X\}.$$

Proposition 6.4. *Let X be a finite set. Then $C_e^*(\mathcal{T}_X) = C^*(\mathcal{U})$.*

Proof. Throughout the proof, a unital $*$-homomorphism between unital C^*-algebras will simply be called a morphism. Let $\mathcal{A}$ be the unital C^*-algebra which is universal for morphisms $M_X \mapsto M_X \otimes \mathcal{A}$ in the sense that there exists a morphism $\alpha : M_X \to M_X \otimes \mathcal{A}$ such that, for any unital C^*-algebra $\mathcal{B}$ and morphism $\beta : M_X \to M_X \otimes \mathcal{B}$, there is a unique morphism $\Lambda : \mathcal{A} \to \mathcal{B}$ such that $\beta = (\mathrm{id} \otimes \Lambda)\alpha$ (see [5, Theorem 2.3]). Viewing $\mathcal{A} \subseteq \mathcal{B}(H)$, there exists a unitary $w : \mathbb{C}^X \otimes H \to \mathbb{C}^X \otimes H$ such that $\alpha(T) = w^*(T \otimes I_H)w$, $T \in M_X$. Since α has range in $M_X \otimes \mathcal{A}$, we have that $w_{a,x}^*w_{a',x'} \in \mathcal{A}$ for all $x, x', a, a' \in X$.

Moreover, $\mathcal{A}$ is generated by $\{(\rho \otimes \mathrm{id})\alpha(A) : \rho \in M_X^*,\ A \in M_X\}$ [5, Theorem 2.3], so $\mathcal{A} = C^*(\{w_{a,x}^* w_{a',x'} : x, x', a, a' \in X\})$.

If $u = (u_{a,x})_{a,x}$ is the generating matrix for $\mathcal{B}_X$,

$$M_X \ni T \mapsto u^*(T \otimes 1)u \in M_X \otimes \mathcal{B}_X$$

is a morphism, so the universal property of $\mathcal{A}$ supplies a morphism $\Lambda : \mathcal{A} \to \mathcal{B}_X$ satisfying $\Lambda(w_{a,x}^* w_{a',x'}) = u_{a,x}^* u_{a',x'}$, $x, x', a, a' \in X$. Then $\Lambda(\mathcal{A}) \subseteq C^*(\mathcal{U})$, necessarily.

By the universal property of $\mathcal{B}_X$, there is a morphism $\pi : \mathcal{B}_X \to \mathcal{B}(H)$ such that $\pi(u_{a,x}) = w_{a,x}$, $x, a \in X$. The restriction of π to $C^*(\mathcal{U})$ is an inverse to Λ, so that $C^*(\mathcal{U}) \cong \mathcal{A}$. The desired conclusion then follows from [5, Theorem 3.8]. $\square$

It follows from the proof of Proposition 6.4 that the operator system $\mathcal{T}_X$ satisfies the universal property of [5, Theorem 3.3] for unital completely positive maps $M_X \to M_X \otimes \mathcal{T}_X$, so that [5, Corollary 3.11] implies

$$\mathcal{T}_X \otimes_c \mathcal{T}_Y \subseteq C_e^*(\mathcal{T}_X) \otimes_{\max} C_e^*(\mathcal{T}_Y).$$

Therefore, the self-testing examples from Proposition 5.17 give rise to finite-dimensional states $f : \mathcal{T}_X \otimes_c \mathcal{T}_Y \to \mathbb{C}$ which extend uniquely to finite-dimensional states of $C_e^*(\mathcal{T}_X) \otimes_{\max} C_e^*(\mathcal{T}_Y)$.

7. Questions

The notion of an approximate dilation $\tilde{S}$ of a model S over the pair $(\mathcal{S}_{X,A}, \mathcal{S}_{Y,B})$ was defined in [56] for the case of classes of quantum models, and can be easily extended to classes of quantum commuting models over a finitary context $(\mathcal{S}_A, \mathcal{S}_B)$, where $\mathcal{S}_A = \mathrm{span}\{f_i\}_{i=1}^k$ and $\mathcal{S}_B = \mathrm{span}\{g_j\}_{j=1}^l$: given $\delta > 0$, a quantum commuting model $\tilde{S} = (\tilde{H}, \tilde{\varphi}_A, \tilde{\varphi}_B \tilde{\xi})$ is said to δ-dilate a quantum commuting model $S = (H, \varphi_A, \varphi_B, \xi)$ if there exist an auxiliary system H_{aux} and a unit vector $\xi_{\mathrm{aux}} \in H_{\mathrm{aux}}$, and a local isometry $V : H \to \tilde{H} \otimes H_{\mathrm{aux}}$ such that

$$V\phi_A(f_i)\phi_B(g_j)\xi \sim^\delta \tilde{\phi}_A(f_i)\tilde{\phi}_B(g_j)\tilde{\xi} \otimes \xi_{\mathrm{aux}}, \quad i \in [k], j \in [l].$$

We say that a correlation p of quantum commuting type over $(\mathcal{S}_A, \mathcal{S}_B)$ is a robust quantum commuting self-test if there exists a model $\tilde{S}$ of p such that for every $\epsilon > 0$ there exists $\delta > 0$ with the property that, whenever p_S is a correlation arising from a quantum commuting model S over $(\mathcal{S}_A, \mathcal{S}_B)$ with $\|p_S - p\|_1 < \epsilon$, we have that $\tilde{S}$ is a δ-dilation of S, where

$$\|p_S - p\|_1 := \sum_{i=1}^k \sum_{j=1}^l |p_s(f_i \otimes g_j) - p(f_i \otimes g_j)|$$

We have the following implications for a correlation p:

$$p \text{ is a robust self-test } \Rightarrow \ p \text{ is a self-test } \Rightarrow \ p \text{ is a weak self-test.}$$

These implications show that weak self-testing is a natural concept to study, but suggest the following initial question:

Question 7.1. Is every weak self-test a self-test?

If S is a centrally supported quantum POVM model for $p \in \mathcal{C}_{\mathrm{q}}$ and S_r is the reduced model of S, then $S \preceq S_r$ (see [43, Lemma 4.2]), which reduces the question of self-testing of p to the study of full rank models of p. The latter was important for establishing a number of self-testing results and allowed the application of representations of certain algebraic relations (see e.g. [50] for a self-test of the optimal strategy of CHSH game, and [38] for self-tests of some synchronous correlations). We therefore formulate the following:

Question 7.2. For general model $S = ({}_{\mathcal{A}}H_{\mathcal{B}}, \varphi_A, \varphi_B, \xi)$, is it true that $S \preceq S_r$?

We point out that, currently, we do not see how the construction of local isometries for the dilation $S \preceq S_r$ can be modified to give one in the non-tensor split case.

In the definition of self-testing, an assumption is made on the existence of an auxiliary quantum system that ampliates the ideal model for the self-test in question. What types of auxiliary systems may arise is an interesting question in its own right. As an example, we formulate the following:

Question 7.3. When applying our definition of self-testing to (quantum) no-signalling correlations of quantum type (e.g. for the classes $\mathcal{C}_{\mathrm{q}}$ and $\mathcal{Q}_{\mathrm{q}}$), do the auxiliary bimodules arising from the dilation pre-order automatically tensor factorise into quantum spacial systems?

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