Thesis for the degree of Licentiate of Philosophy

# $Q$-fractional Brownian motion and Lévy-driven SPDEs on the sphere: analysis and simulation

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Göteborg, Sweden 2026

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Acknowledgements, dedications, and similar personal statements in this thesis, reflect the author's own views.



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## Abstract

Many real-world phenomena can be modeled by stochastic processes in time, space or space-time. In important application fields like climate modeling and cosmology, the underlying spatial domain is a sphere, which represents the Earth. This thesis is concerned with spatio-temporal stochastic processes and stochastic partial differential equations on the unit sphere and is based on two papers. In the first paper, isotropic $Q$-fractional Brownian motion is discussed, which models isotropic random fields that evolve over time according to a fractional Brownian motion. This $Q$-fractional Brownian motion has applications, for example, in modeling the cosmic microwave background. The spatio-temporal Hölder regularity of $Q$-fractional Brownian motion is analyzed and efficient numerical methods for its simulation are investigated. The second paper focuses on the long-time behavior of stochastic partial differential equations driven by $Q$-Lévy noise on the sphere. For the linear stochastic wave, Schrödinger and Maxwell's equations, physical quantities such as, for example, energy, that are conserved in the deterministic case, are considered. It is proved that under additive noise, the expectations of these quantities grow linearly following trace formulas. Numerical discretization schemes are analyzed and it is proved that exponential Euler-type schemes reproduce the trace formulas while forward and backward Euler–Maruyama methods fail to do so. Extensive numerical experiments illustrate these results.

**Keywords:** Fractional Brownian motion, Lévy processes, Gaussian random fields, Stochastic partial differential equations, spectral discretization, backward Euler–Maruyama scheme, stochastic exponential Euler scheme, geometric numerical integration.

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# List of Publications

The following papers and preprints are included in this thesis:

**Paper I:** Annika Lang, Björn Müller: Isotropic $Q$-fractional Brownian motion on the sphere: regularity and fast simulation, *Philosophical Transactions of the Royal Society A Mathematics Physics Engineering Science* (2025) 383 (2298): 20240238.

**Paper II:** David Cohen, Björn Müller, Andrea Papini: Long-time behavior of some stochastic evolution equations on the sphere and their numerical discretizations, unpublished manuscript.

Author contribution:

**Paper I:** B.M. performed the computations and derived the proofs in discussion with A.L., B.M. implemented the numerical experiments and visualizations, and wrote the first draft of the paper, which was edited and completed together with A.L.

**Paper II:** B.M. contributed significantly to the proofs for the energy of the stochastic wave equation, derived the proofs for energy and mass of the stochastic Schrödinger equation in discussion with A.P. and D.C., implemented the numerical experiments and wrote, in collaboration with A.P. and D.C., the current version of the manuscript.

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*Göteborg, 2026*

# Chapter 1: Introduction

Many phenomena in the natural sciences and related disciplines involve uncertainty and randomness. In order to study such phenomena and make predictions, it is necessary to model this randomness. For this reason, the theory of stochastic processes and stochastic differential equations (SDEs) was developed, see e. g., Kloeden and Platen [1992, Chapter 7]. While a variety of processes can be considered as time series, for example, the movement of particles [Brown, 1828] or stock prices (e. g., Samuelson and Merton [1969], Black and Scholes [1973]), it is often necessary to include a spatial component. Systems with a spatial component are modeled using random fields, spatio-temporal stochastic processes or stochastic partial differential equations (SPDEs). Such models can be found, for example, in material science [Feinauer et al., 2015, Barman and Bolin, 2018, Röding et al., 2022], climate modeling [Guillot, 1999, Fioravanti et al., 2023, Antoniadou et al., 2025] and turbulence [Kraichnan, 1964]. The review by Porcu et al. [2021] provides an overview of various applications of spatio-temporal stochastic models.

In many applications, the spatial component lies in spherical domains, such as when studying phenomena on Earth on a global scale. Therefore, a prominent area of research which uses spatio-temporal stochastic models on spherical domains is climate modeling, see, e. g., Castruccio and Stein [2013], Porcu et al. [2018] and Alegría et al. [2019]. Beyond this, cosmological observations are made from around the globe, so spherical models play an important role here as well, for example for the study of the cosmic microwave background [Marinucci and Peccati, 2011, Anh et al., 2018].

In this thesis, we are concerned with two different types of stochastic spatio-temporal models on the unit sphere: a stochastic process, called $Q$-fractional Brownian motion, as well as linear SPDEs driven by additive $Q$-Lévy noise.

Paper I focuses on $Q$-fractional Brownian motion on the sphere. We first analyze the spatial and temporal Hölder regularity properties of $Q$-fractional Brownian motion, which are relevant for potential applications in modelling. Furthermore, we evaluate different simulation methods of $Q$-fractional Brownian motion in order to determine the most efficient way of

simulating it.

Paper II focuses on the preservation of long-time behavior of physical quantities of interest in linear SPDEs. The study of structure-preserving numerical schemes has a long history in numerical analysis, giving rise to the field of geometric numerical integration, see e. g., Hairer et al. [2006]. This field has recently also been extended to SDEs [Milstein et al., 2002, Hong et al., 2007, D'Ambrosio et al., 2018, Chen et al., 2020, Cohen and Vilmart, 2022]. In Paper II, we consider the long-time behavior of some physical quantities, like the energy of a system, in SPDEs driven by $Q$-Lévy noise on the sphere. Furthermore, we analyze the ability of three numerical schemes to reproduce this long-time behavior.

The goal of this introduction is to provide an intuitive exposition of the background and techniques used in Papers I and II. In order to keep the main arguments and structures in view, we do not present all derivations in full detail. Wherever it is appropriate, we refer to the relevant literature for the detailed and rigorous analyses. A further focus of this thesis is simulation, which is a crucial step in making models applicable to the sciences. Hence, we discuss simulation methods at every step.

The remaining introductory chapters of this thesis are organized as follows: In Chapter 2, we present some finite-dimensional stochastic processes, SDEs and their numerical approximations. Chapter 3 discusses random fields and spatio-temporal stochastic processes on the unit sphere $\mathbb{S}^2$. Lastly, in Chapter 4, we introduce linear SPDEs on the sphere, discuss their numerical treatment and the long-time behavior of physical quantities of interest.

# Chapter 2: Stochastic processes in finite dimensions

We start by considering finite-dimensional stochastic processes and SDEs as they will serve as fundamental building blocks and toy examples for the processes discussed in later chapters. We discuss some theoretical properties as well as their numerical simulations.

Let first $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ be a filtered probability space satisfying the usual conditions. We denote by $\mathbb{T} = [0, T]$ for some $0 < T < \infty$ or $\mathbb{T} = [0, \infty)$ a compact or semi-infinite time interval. A real-valued *stochastic process* is defined as a function $X : \Omega \times \mathbb{T} \to \mathbb{R}$, such that for all $t \in \mathbb{T}$, $X(\cdot, t)$ is a random variable (see, e. g., [Kallenberg, 2021, Ch. 4]).

Under appropriate assumptions on the regularity of each sample path $X(\omega, \cdot)$, we can also view stochastic processes as random variables taking values in corresponding function spaces [Kallenberg, 2021]. If, for example, on a compact time interval $\mathbb{T} = [0, T]$, for $\alpha \geq 0$, $X(\omega, \cdot) \in C^\alpha(\mathbb{T})$ for $\mathbb{P}$-almost all $\omega \in \Omega$, i. e., $X(\omega, \cdot)$ is Hölder continuous with exponent $\alpha$, we might interpret $X$ as a random variable $X : \Omega \to C^\alpha(\mathbb{T})$. Analogously, if $X(\omega, \cdot)$ satisfies a given Sobolev regularity $\eta \geq 0$, we have a random variable $X : \Omega \to H^\eta(\mathbb{T})$.

## 2.1 Brownian motion

An important class of stochastic processes are the *Gaussian processes*, which play a central role in Paper I. A stochastic process $X$ is called Gaussian if for all $n \geq 1$, $t_1, \ldots, t_n \in \mathbb{T}$, and $a_1, \ldots, a_n \in \mathbb{R}$, $\sum_{i=1}^n a_i X(t_i)$ is a Gaussian random variable [Kallenberg, 2021, Chapter 14].

One example of a Gaussian process is *Brownian motion*. It was described in 1827 by Robert Brown, a Scottish botanist, who investigated the microscopic movements of pollen immersed in water [Brown, 1828]. Mathematical models involving Brownian motion were first developed by Bachelier [1900] and Einstein [1905, 1906]. Wiener [1923] provided a rigorous mathematical analysis of Brownian motion as a stochastic process and proved its existence. The mathematical formalization of Brownian motion as a stochastic process is

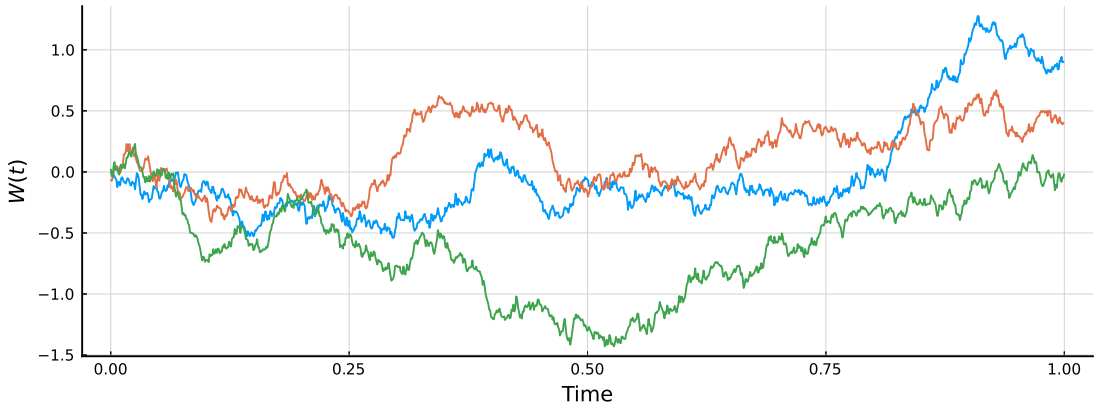



Figure 2.1: Sample paths of a Wiener process.

therefore also called a *Wiener process*. In this thesis, we use both words interchangeably.

We start by defining Brownian motion via its distributional properties as follows:

**Definition 1.** A stochastic process $W$ is called a *Wiener process* or *Brownian motion* if

1. $W(0) = 0$ $\mathbb{P}$-almost surely,
2. $W$ has independent increments, i. e., for all $0 \leq s < t \leq u < v$, $W(t) - W(s)$ is independent of $W(v) - W(u)$,
3. $W(t) - W(s) \sim \mathcal{N}(0, t - s)$, i. e., the increment $W(t) - W(s)$ follows a normal distribution with mean zero and variance $t - s$ for all $0 \leq s \leq t$.

From the definition, we can compute the *covariance function* of Brownian motion as

$$\gamma_W(t, s) = \mathbb{E}[W(s)W(t)] = s \wedge t = \min(s, t). \tag{2.1}$$

As a Gaussian process is uniquely defined by its first and second moments, i. e., its mean and covariance function, we can define Brownian motion equivalently as a Gaussian process with mean zero and covariance $\gamma_W$. We further assume in this thesis that $W$ is a Wiener process with respect to the filtration $(\mathcal{F}_t, t \geq 0)$, i. e., $W$ is adapted to $(\mathcal{F}_t, t \geq 0)$ and $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $t \geq s \geq 0$ (see, e. g., [Kallenberg, 2021, Chapter 14]).

If we want to simulate Brownian motion, we choose an equidistant time grid on $\mathbb{T} = [0, T]$, $0 < T < \infty$. That is, we choose a number of steps $N > 0$ and compute the step size $h = T/N$. Then, $t_n = nh$ for $n = 0, \ldots, N$. This partitions $[0, T]$ into $N$ subintervals of length $h$. All increments $\Delta W_n = W(t_{n+1}) - W(t_n)$, $0 \leq n \leq N - 1$, are now $\mathcal{N}(0, h)$-distributed and independent. Drawing $N$ independent standard Gaussian random numbers,

$\xi_n, 0 \leq n \leq N-1$, we obtain $\Delta W_n = \sqrt{h}\xi_n$ for all $0 \leq n \leq N-1$. The path of $W$ is now a cumulative sum of its increments, i. e., $W(t_n) = \sum_{i=0}^{n-1} \Delta W_i$ for all $1 \leq n \leq N$.

Figure 2.1 illustrates what sample paths of Brownian motion look like over time. Note that these paths follow the correct distribution only in the $N$ discretization points, and are linearly interpolated on the intervals $(t_n, t_{n+1}), 0 \leq n \leq N-1$.

Since Brownian motion was originally described as a model for the movement of pollen particles, it is reasonable to expect that its sample paths are continuous. From the distributional properties in Definition 1, it is, however, not immediately clear. To characterize continuity, let us first consider a *modification* of a stochastic process. A stochastic process $\widetilde{X}$ is called a modification of a process $X$ if $\mathbb{P}(X(t) = \widetilde{X}(t)) = 1$ for all $t \geq 0$ [Øksendal, 2003]. Since distributions are not affected by changes on $\mathbb{P}$-null sets, any modification of a Wiener process $W$ is still a Wiener process according to Definition 1.

The central result on continuity of stochastic processes is the *Kolmogorov continuity theorem* or *Kolmogorov–Chentsov theorem* [Kallenberg, 2021, Theorem 4.23]. This theorem relates distributional properties, namely bounds on moments, to sample path continuity. Let $X$ be a stochastic process and $a, b > 0$. If we can bound

$$\mathbb{E}\left[|X(t) - X(s)|^a\right] \leq C|t-s|^{b+1}$$

for all $s, t \geq 0$, there is a modification of $X$ which has sample paths that are locally Hölder continuous with exponent $\alpha \in (0, b/a)$.

For a Brownian motion $W$, we obtain from Definition 1 with $a = 2$

$$\mathbb{E}\left[(W(t) - W(s))^2\right] = |t-s|.$$

Since $W(t) - W(s)$ is a Gaussian random variable, we have further

$$\mathbb{E}\left[(W(t) - W(s))^{2n}\right] \leq C_n \mathbb{E}\left[(W(t) - W(s))^2\right]^n = C_n|t-s|^n$$

for all $n \geq 1, s, t \geq 0$ [Bauer and Burckel, 1995, Lemma 40.2]. The Kolmogorov–Chentsov theorem then states that there exists a modification $\widetilde{W}$ of $W$ that is continuous and, in particular, Hölder-$\alpha$ continuous for $\alpha \in (0, (n-1)/(2n))$ for all $n$. Letting $n$ tend to infinity, we have $\alpha \in (0, 1/2)$. We then write that $\widetilde{W}$ is Hölder continuous with exponent $1/2-$. When restricting to a compact time interval $\mathbb{T}$, we also write $\widetilde{W} \in C^{1/2-}(\mathbb{T})$, meaning that $\widetilde{W}$ is Hölder-$\alpha$ continuous on $\mathbb{T}$ for all $0 < \alpha < 1/2$.

Furthermore, if $\widetilde{W}$ and $\widehat{W}$ are both continuous modifications of $W$, they are *indistinguishable*, i. e., $\mathbb{P}(\widetilde{W}(t) = \widehat{W}(t) \text{ for all } t \geq 0) = 1$. Hence, a continuous modification is almost surely unique. From now on, whenever we consider a Wiener process $W$, we assume it to be this continuous modification.

## 2.2 Fractional Brownian motion

*Fractional Brownian motion* is a generalization of Brownian motion that allows for a wider range of temporal regularities. It was first introduced by Mandelbrot and Van Ness [1968]. It takes a parameter, the *Hurst parameter*, which not only affects statistical properties but also the Hölder regularity of sample paths. We define here fractional Brownian motion similarly to Mishura [2008].

**Definition 2.** Let $0 < H < 1$. A Gaussian process $\beta^H$ is called a *fractional Brownian motion* with *Hurst parameter* $H$, if

1. $\beta^H(0) = 0$ $\mathbb{P}$-almost surely,
2. $\mathbb{E}[\beta^H(t)] = 0$ for all $t \geq 0$,
3. $\gamma_H(s,t) = \mathbb{E}[\beta^H(s)\beta^H(t)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$ for all $s, t \geq 0$.

Setting $H = \frac{1}{2}$, we recover standard Brownian motion as defined in Definition 1.

As for Brownian motion, we are interested in the continuity and Hölder exponent of fractional Brownian motion. Direct computation shows that for all $s, t \geq 0$,

$$\mathbb{E}\left[(\beta^H(t) - \beta^H(s))^2\right] = |t-s|^{2H}$$

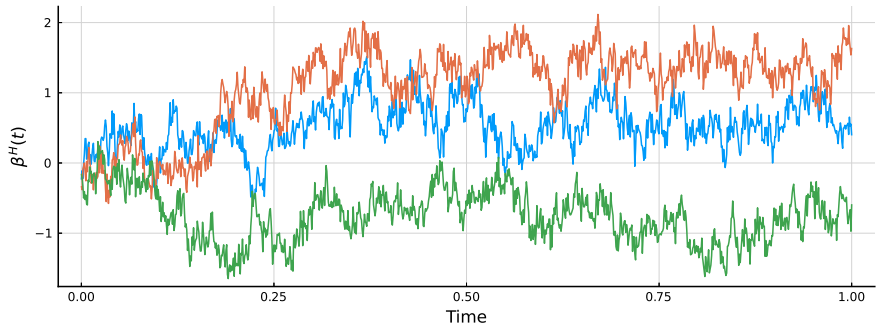
and therefore, using the Gaussianity

$$\mathbb{E}\left[(\beta^H(t) - \beta^H(s))^{2n}\right] = C_n|t-s|^{2nH}$$

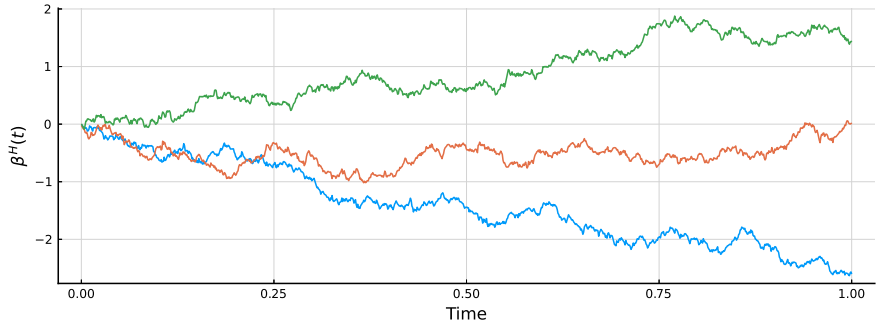
for all $n \geq 1$. The Kolmogorov–Chentsov theorem [Kallenberg, 2021, Theorem 4.23] then yields a ($\mathbb{P}$-almost surely unique) continuous modification of $\beta^H$ with a Hölder-$\alpha$ continuity for $\alpha \in (0, (2nH-1)/(2n))$. Taking $n$ to infinity again, we obtain a Hölder exponent of $H-$. We can see that the Hurst parameter describes the Hölder regularity of our fractional Brownian motion. For the remainder of this thesis, we assume $\beta^H$ to be the continuous modification of fractional Brownian motion.

Figure 2.2 shows sample paths of fractional Brownian motion for different Hurst parameters $H = 0.25, 0.5, 0.75$. It can be seen that for smaller $H$, the sample paths are "rougher", while they appear "smoother" for larger $H$, highlighting the different Hölder regularities described by $H$. Thus, by choosing $H$ correspondingly, we can influence the regularity of the sample paths of fractional Brownian motion.

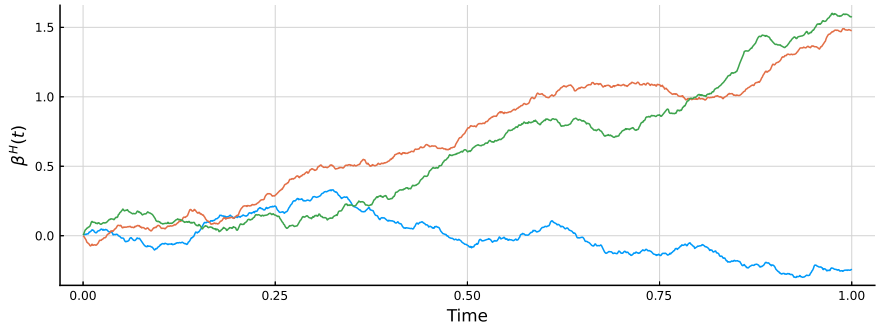
Simulation of Brownian motion as described in Section 2.1 relied heavily on the independence of increments. This property is no longer given for fractional Brownian motion, making simulation less straight-forward. Many




(a) $H = 0.25$




(b) $H = 0.5$




(c) $H = 0.75$

Figure 2.2: Sample paths of fractional Brownian motions with different Hurst parameters $H$.

methods have been explored to efficiently and accurately sample fractional Brownian motion, see Dieker [2004] for a review. These methods include Conditionalized Random Midpoint Displacement [Norros et al., 1999] and Circulant Embedding [Perrin et al., 2002] which we consider in Paper I, and the use of neural networks [Allouche et al., 2022].

One of the conceptually simpler ways of simulating fractional Brownian motion is the Cholesky factorization method (see, e. g. Asmussen and Glynn [2007, Section XI.2]). This method uses the fact that, on the time grid $(t_0, \dots, t_N)$, we aim to simulate a Gaussian random vector with a covariance matrix that we can compute. We denote by $Z_{(N)} = (\beta^H(t_1) - \beta^H(t_0), \dots, \beta^H(t_N) - \beta^H(t_{N-1}))^T$ the vector of increments. The mean of $Z_{(N)}$ is 0 and its covariance matrix is given by $\Sigma = \mathbb{E}\left[Z_{(N)} Z_{(N)}^T\right]$. Since $\Sigma$ is a symmetric and positive definite matrix, we know from linear algebra that there exists a matrix square root $\Sigma^{1/2}$ such that $\Sigma = \Sigma^{1/2}(\Sigma^{1/2})^T$. We then generate an $N$-dimensional standard normal vector $U \sim \mathcal{N}(0, \mathrm{I}_N)$, where $I_N \in \mathbb{R}^{n \times n}$ is an identity matrix, and multiply it from the left by $\Sigma^{1/2}$. The resulting vector $\Sigma^{1/2}U$ has mean zero and covariance matrix

$$\mathbb{E}\left[\Sigma^{1/2}U(\Sigma^{1/2}U)^T\right] = \Sigma^{1/2}\,\mathbb{E}\left[UU^T\right](\Sigma^{1/2})^T = \Sigma^{1/2}(\Sigma^{1/2})^T = \Sigma.$$

Therefore, we can simulate $Z_{(N)}$ as $Z_{(N)} = \Sigma^{1/2}U$. This representation is useful for implementation since it can rely on well-established routines from numerical linear algebra. Its computational complexity is, however, $\mathcal{O}(N^3)$, which is significantly slower than the complexity $\mathcal{O}(N)$ for the simulation of Brownian motion [Asmussen and Glynn, 2007, Section XI.2].

One of the simulation methods for fractional Brownian motion that we discuss in Paper I, the Circulant Embedding method, is based on this principle. However, it uses the fact that the increments of fractional Brownian motion are stationary to compute the required matrix square root more efficiently.

## 2.3 Lévy processes

Stochastic processes need not have continuous sample paths, however. A class of processes that allows for jumps is the class of *Lévy processes.* They generalize Brownian motion by relaxing the assumption of Gaussianity of the increments. Through this, we lose the continuous sample paths that (fractional) Brownian motion has. Instead, we add a weaker, stochastic continuity condition to the definition, see, e. g., Peszat and Zabczyk [2007].

**Definition 3.** A stochastic process $L$ is called a *Lévy process* if

1. $L(0) = 0$ $\mathbb{P}$-almost surely,

2. $L$ has independent increments,

3. $L$ has stationary increments, i. e., for all $t \geq s \geq 0$, the distribution of $L(t) - L(s)$ depends only on $t - s$,

4. $L$ is stochastically continuous, i. e., for all $t_0 \geq 0$, $L(t) \to_p L(t_0)$ for $t \to t_0$, where $\to_p$ denotes a limit in probability.

As for the Wiener process, we assume that any Lévy process $L$ is a Lévy process with respect to the filtration $(\mathcal{F}_t, t \geq 0)$. Additionally, as in Lang et al. [2025] and Paper II, we make the assumption that the Lévy processes we consider are square-integrable, i. e., $L(t) \in L^2(\Omega)$ for all $t \geq 0$, where $L^2(\Omega)$ is the space of square-integrable random variables on $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$. In this case, one can show that the mean satisfies $\mathbb{E}[L(t)] = mt$ for some constant $m \in \mathbb{R}$, and the covariance function is given, for some constant $c \geq 0$, by

$$\gamma_L(t, s) = \mathbb{E}[L(t)L(s)] = c(s \wedge t).$$

For ease of presentation, we assume that $m = 0$ and $c = 1$.

Let us now consider the problem of simulation of Lévy processes. For the sake of this thesis, we consider a subclass of square-integrable, mean-zero Lévy processes that has finitely many jumps in every compact time interval. As shown in [Peszat and Zabczyk, 2007, Section 4.5], such a Lévy process $L$ can be decomposed as

$$L(t) = aW(t) + bP(t), \tag{2.2}$$

where $W$ is a Wiener process, $P$ a *compensated compound Poisson process* and $a, b \in \mathbb{R}$ are constants. A compensated compound Poisson process $P$ with rate $\theta$ and jump size distribution $F$ is given by

$$P(t) = \sum_{i=1}^{N(t)} Z_i - \theta t \, \mathbb{E}[Z], \tag{2.3}$$

where $Z, Z_i, i \geq 1$ are independent and identically distributed (iid) random variables with distribution function $F$ independent of the Poisson process $N$ with rate $\theta$ [Ross, 1996]. A *Poisson process* with rate $\theta$ is a Lévy process $N$ with increments following a Poisson distribution: $N(t) - N(s) \sim \text{Pois}(\theta(t - s))$ for all $t \geq s \geq 0$. The process $P$ can be interpreted as the Poisson process $N$ with jump heights $Z_i$. The term $-\theta t \, \mathbb{E}[Z]$ compensates for the expected height of jumps to ensure that $\mathbb{E}[P(t)] = 0$. By the Blackwell–Girshick equation (see, e. g., [Bauer and Burckel, 1995, Theorem 17.7]), our assumption that $L$ is square-integrable is satisfied provided that $\mathsf{Var}(Z) < \infty$.

A simulation algorithm for Lévy processes can be constructed from the decomposition (2.2). The Wiener process $W$ is simulated as in Section 2.1

[Asmussen and Glynn, 2007, Chapter XII]. For $P$, we recall our time grid $(t_0, t_1 = h, \ldots, t_N = Nh)$ for some $N > 0$ and consider the increments

$$P(t_{n+1}) - P(t_n) = \sum_{i=1}^{N_n} Z_i^n - \theta h \mu,$$

where all $N_n, 0 \leq n \leq N-1$, are Poisson random variables with parameter $\theta h$ and $Z_i^n, 1 \leq i \leq N_n, 0 \leq n \leq N-1$, are iid random variables following any distribution with finite variance and expectation $\mu$. Furthermore, all $Z_i^n, 1 \leq i \leq N_n, 0 \leq n \leq N-1$, are independent of all $N_n, 0 \leq n \leq N-1$. Cumulatively summing increments yields a sample path, as for the Wiener process in Section 2.1. For the simulation and approximation of Lévy processes with infinitely many jumps, we refer the reader to Asmussen and Glynn [2007].

Figure 2.3 illustrates three types of Lévy processes: Wiener processes, compensated compound Poisson processes and the corresponding sums of these processes. Therefore, Figure 2.3a and Figure 2.3b show the continuous and jump parts of the processes in Figure 2.3c, respectively. The rate of the compensated compound Poisson process $P$ is $\theta = 5$ and the jumps follow a uniform distribution on $[-\sqrt{3/5}, \sqrt{3/5}]$ in order to normalize such that $Var(P(1)) = 1$.

## 2.4 Stochastic differential equations

Many real-world phenomena have additional structure that cannot be modelled solely by the stochastic processes described in Sections 2.1, 2.2 and 2.3. For example, there could be an underlying exponential growth, a drift (trend in a certain direction) or mean-reverting behavior. Such structures can be achieved by combining stochastic processes with differential equations, giving rise to *stochastic differential equations (SDEs)*. We refer to Kloeden and Platen [1992, Chapter 7] for examples of applications. Here, we give a short overview of SDEs driven by the Wiener process, since this is the most common way to consider them. The presentation is based on Øksendal [2003].

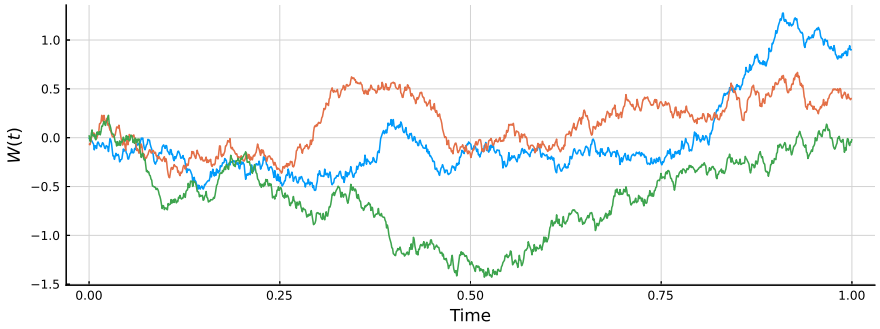
A first-order scalar SDE can be written, in analogy to an ordinary differential equation, as

$$\dot{u}(t) = b(t, u(t)) + \sigma(t, u(t))\dot{W}(t), \quad u(0) = u_0,$$

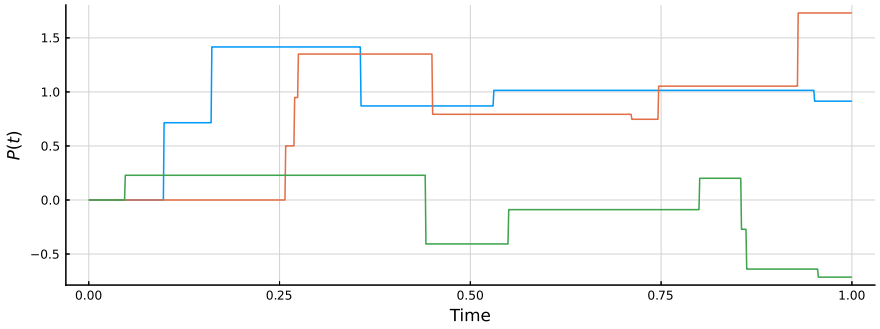
where $u_0 \in \mathbb{R}$, $b, \sigma : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ are the *drift* and *diffusion coefficients* and $\dot{W}(t)$ is *temporal white noise*, the formal time derivative of a Wiener process [Øksendal, 2003].

We can formally rewrite this SDE as

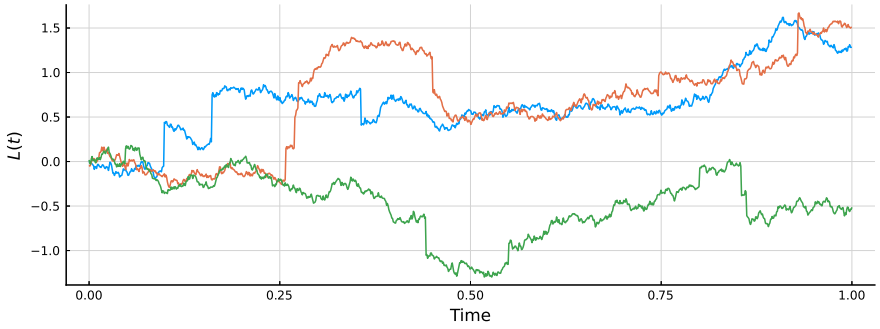
$$\mathrm{d}u(t) = b(t, u(t))\,\mathrm{d}t + \sigma(t, u(t))\,\mathrm{d}W(t), \quad u(0) = u_0, \tag{2.4}$$




(a) Wiener process $W$




(b) Compound Poisson process $P$




(c) Lévy process $L = (W + P)/\sqrt{2}$

Figure 2.3: Sample paths of Lévy processes.

which has the meaning of the integral equation

$$u(t) = u_0 + \int_0^t b(s, u(s))\,\mathrm{d}s + \int_0^t \sigma(s, u(s))\,\mathrm{d}W(s). \tag{2.5}$$

This can be generalized to $n$-dimensional systems of SDEs with $u(t) \in \mathbb{R}^n$. For this, we let $W = (W^{(1)}, \ldots, W^{(m)})^T$ be a vector of $m$ independent Brownian motions and consider drift and diffusion coefficients $b : \mathbb{T} \times R^n \to \mathbb{R}$ and $\sigma : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$.

The integral with respect to Brownian motion on the right-hand side of Equation 2.5 was mathematically first defined by Itô [1944] and is, therefore, called the Itô integral. Its construction can, for example, be found in Øksendal [2003, Chapter 3]. A different formalization of a stochastic integral is provided

by Stratonovich [1966]. Arising from its construction, the Itô integral has two properties that we rely on in this thesis. Firstly, its expectation vanishes, i. e., if $X$ is a square-integrable stochastic process adapted to $(\mathcal{F}_t, t \geq 0)$,

$$\mathbb{E}\left[\int_0^t X(s)\,\mathrm{d}W(s)\right] = 0. \tag{2.6}$$

The second property is the *Itô isometry* which allows to compute second moments of the integral as well. In particular,

$$\mathbb{E}\left[\left(\int_0^t X(s)\,\mathrm{d}W(s)\right)^2\right] = \mathbb{E}\left[\int_0^t X^2(s)\,\mathrm{d}s\right]. \tag{2.7}$$

For a detailed construction of the Itô integral and an analysis of existence and uniqueness of solutions of SDEs, we refer to Øksendal [2003, Chapters 3, 5] and Karatzas and Shreve [1998, Chapter 5].

One of the first examples of an SDE that one usually encounters is the *Ornstein–Uhlenbeck process*

$$\mathrm{d}u(t) = -\lambda u(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \quad u(0) = u_0,$$

where $\lambda, \sigma > 0$ and $u_0 \in \mathbb{R}$. The solution to the Ornstein–Uhlenbeck process is given via a variation-of-constants formula as

$$u(t) = e^{-\lambda t}u_0 + \int_0^t e^{-\lambda(t-s)}\sigma\,\mathrm{d}W(s).$$

The Ornstein–Uhlenbeck process exhibits a mean-reverting behavior, i. e., it tends to move back towards 0. This can be seen in Figure 2.4a which illustrates sample paths of an Ornstein–Uhlenbeck process driven by the Wiener processes shown in Figure 2.1. The parameters chosen for Figure 2.4a are $\lambda = 3$ and $\sigma = 0.5$. The Ornstein–Uhlenbeck process models the random movement of a particle in the presence of friction [Uhlenbeck and Ornstein, 1930, Mao, 2011].

Another example of an SDE is that of the stochastic harmonic oscillator, which is based on a second-order differential equation and is a motivation for Paper II. The stochastic harmonic oscillator with additive noise is given by

$$\ddot{u}(t) = -\lambda u(t) + \sigma\dot{W}(t), \quad u(0) = u_0, \dot{u}(0) = v_0, \tag{2.8}$$

for some $\lambda, \sigma > 0, u_0, v_0 \in \mathbb{R}$. We consider the oscillator (2.8) in terms of positions and velocities $(u, v) = (u, \dot{u})$ and obtain a system of first-order SDEs

$$\begin{aligned}\dot{u}(t) &= v(t),\\ \dot{v}(t) &= -\lambda u(t) + \sigma\dot{W},\end{aligned}$$

which can be rewritten in Itô form as

$$\begin{aligned} \mathrm{d}u(t) &= v(t)\,\mathrm{d}t, \\ \mathrm{d}v(t) &= -\lambda u(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t). \end{aligned} \tag{2.9}$$

The solution of the stochastic harmonic oscillator (2.8), also obtained using a variation-of-constants formula, is [Melbø and Higham, 2004]

$$\begin{aligned} u(t) &= \cos(\sqrt{\lambda}t)u_0 + \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}t)v_0 + \sigma\int_0^t \sin(\sqrt{\lambda}(t-s))\,\mathrm{d}W(s), \\ v(t) &= -\sqrt{\lambda}\sin(\sqrt{\lambda}t)u_0 + \cos(\sqrt{\lambda}t)v_0 + \sigma\int_0^t \cos(\sqrt{\lambda}(t-s))\,\mathrm{d}W(s). \end{aligned} \tag{2.10}$$

Sample paths of both the position and velocity components are illustrated in Figures 2.5a and 2.5b for a stochastic harmonic oscillator with parameters $\lambda = 100, \sigma = 6$ and initial conditions $u_0 = 1, v_0 = 0$. It can be observed that the noise that is clearly visible in the velocity component in Figure 2.5b is smoothed out in the position component in Figure 2.5a.

One can also use Lévy noise as the driving noise for an SDE. For the existence and uniqueness of solutions in this case, we refer the reader to Applebaum [2004]. They also give the corresponding version of Itô's isometry which is relevant in Paper II. Similarly, one can consider SDEs driven by fractional Brownian motion, for which we refer to Mishura [2008].

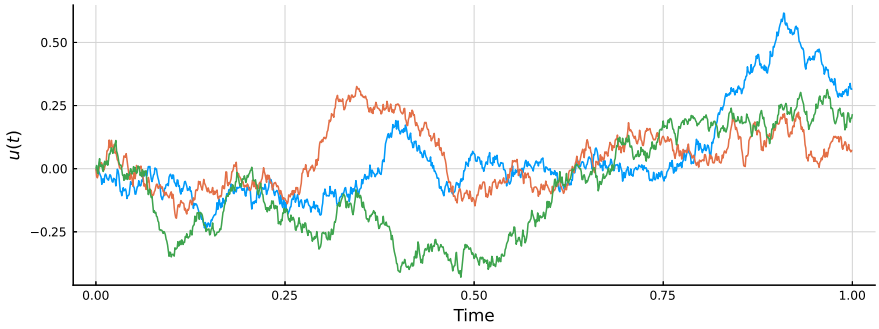
Figures 2.4b, 2.5c and 2.5d show sample paths of the Ornstein–Uhlenbeck process and stochastic harmonic oscillator driven by the Lévy noise shown in Figure 2.3c. The jumps introduced by the Lévy noise can be seen in the Ornstein–Uhlenbeck process in Figure 2.4b and the velocity component of the stochastic harmonic oscillator in Figure 2.5d. In the position component of the oscillator shown in Figure 2.5c, the jumps no longer present as jumps but as kinks.

## 2.5 Numerical solution of SDEs

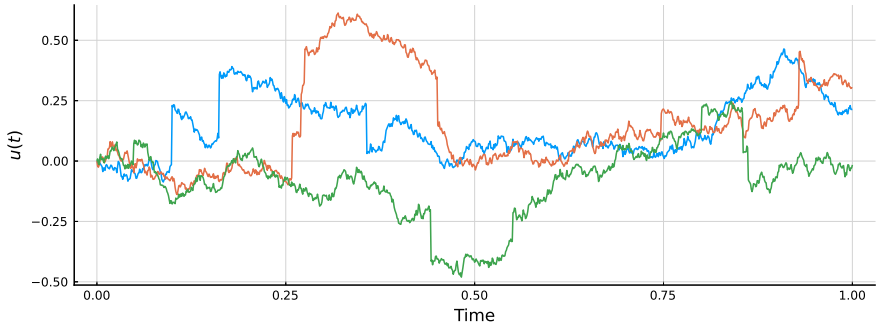
While in the linear cases of the Ornstein–Uhlenbeck process and the stochastic harmonic oscillator, the solution is known explicitly and can be simulated exactly in distribution [Mao, 2011, Chapter 3], this does not hold for more complicated SDEs. For this, numerical methods have been developed to approximate the solution instead.

We consider, like in Section 2.1, an equidistant time grid $t_n = nh$ such that $t_N = Nh = T$, on which we approximate the solution to the SDE (2.4). We denote by $u_n$ the approximation of $u(t_n)$ obtained by means of a numerical method.

The most fundamental numerical method for SDEs is the *Euler–Maruyama (EM)* method, which generalizes the explicit Euler method for ODEs. It




(a) Ornstein–Uhlenbeck process driven by a Wiener process




(b) Ornstein–Uhlenbeck process driven by a Lévy process

Figure 2.4: Sample paths of Ornstein–Uhlenbeck processes with parameters $\lambda = 3$ and $\sigma = 0.5$ and initial condition $u_0 = 0$ driven by Wiener and Lévy processes.

considers the stochastic integral equation (2.5) over the time intervals $[t_n, t_{n+1}]$, $0 \leq n \leq N-1$, as follows

$$u(t_{n+1}) = u(t_n) + \int_{t_n}^{t_{n+1}} b(s, u(s))\,\mathrm{d}s + \int_{t_n}^{t_{n+1}} \sigma(s, u(s))\,\mathrm{d}W(s), \tag{2.11}$$

and freezes the integrands at the left end point $t_n$. So we obtain, within a time step,

$$u(t_{n+1}) \approx u(t_n) + \int_{t_n}^{t_{n+1}} b(t_n, u(t_n))\,\mathrm{d}s + \int_{t_n}^{t_{n+1}} \sigma(t_n, u(t_n))\,\mathrm{d}W(s) \tag{2.12}$$

$$= u(t_{n+1}) + b(t_n, u(t_n))h + \sigma(t_n, u(t_n))\Delta W_n, \tag{2.13}$$

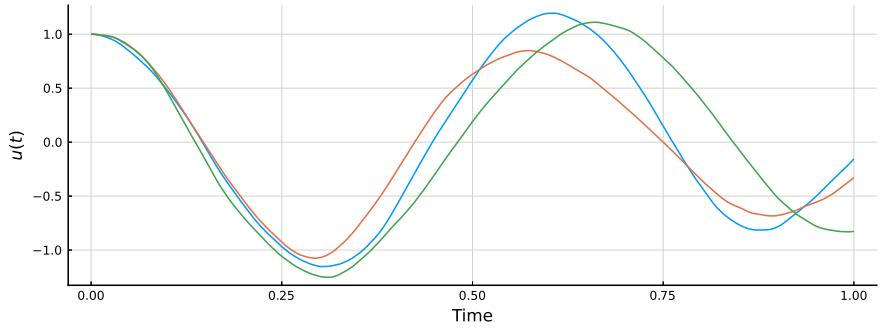
where $\Delta W_n = W(t_{n+1}) - W(t_n)$ is an increment of the Wiener process. We, therefore, define the Euler–Maruyama approximation as

$$u_{n+1} = u_n + b(t_n, u_n)h + \sigma(t_n, u_n)\Delta W_n, \tag{2.14}$$

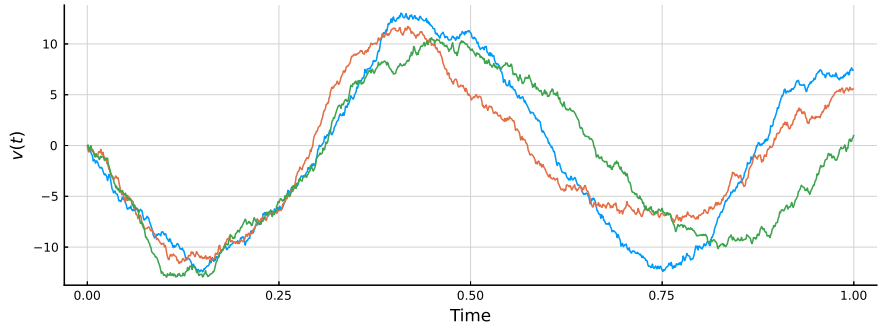
for $0 \leq n \leq N-1$.

The Euler–Maruyama scheme has, under suitable assumptions, a *mean-square convergence rate* of $1/2$, i. e.,

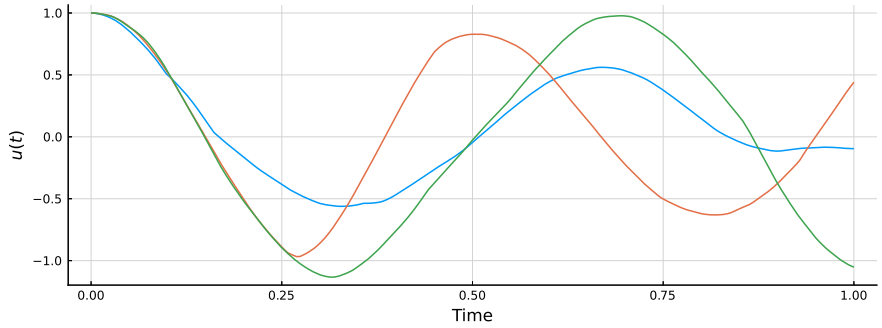
$$\mathbb{E}\left[|u_N - u(t_N)|^2\right]^{\frac{1}{2}} \leq Ch^{\frac{1}{2}},$$



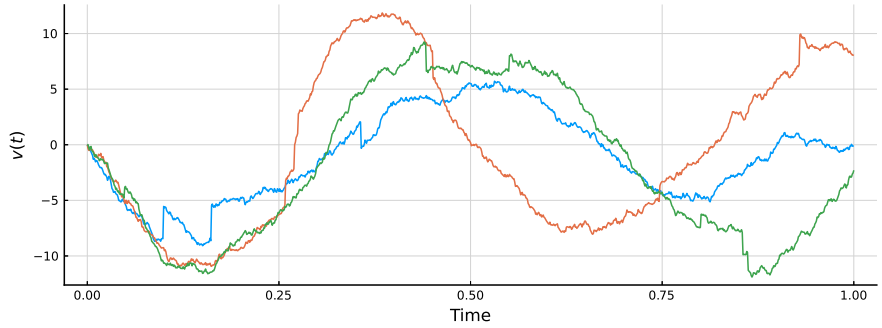

(a) Positions $u(t)$ of stochastic harmonic oscillator driven by Wiener process




(b) Velocities $v(t)$ of stochastic harmonic oscillator driven by Wiener process




(c) Positions $u(t)$ of stochastic harmonic oscillator driven by Lévy process




(d) Velocities $v(t)$ of stochastic harmonic oscillator driven by Lévy process

Figure 2.5: Sample paths of positions and velocities of stochastic harmonic oscillators with parameters $\lambda = 100$, $\sigma = 6$ and initial conditions $u_0 = 1, v_0 = 0$ driven by Wiener and Lévy processes.

for some constant $C > 0$. Furthermore, in the case of linear SDEs with additive noise, as in the Ornstein–Uhlenbeck process and the stochastic harmonic oscillator, a mean-square convergence order of up to 1 can be shown for the Euler–Maruyama scheme [Milstein and Tretyakov, 2004, Section 1.1]. A variety of other numerical schemes exist, with different properties and higher convergence orders, for example, the Milstein scheme [Milstein, 1975], which has a strong convergence rate of 1 also for SDEs with non-linear coefficients. For details on the derivation and convergence analysis for these and other numerical schemes, we refer the reader to the works Kloeden and Platen [1992], Milstein and Tretyakov [2004].

In the case of a driving Lévy noise, similar numerical schemes can be constructed. It is shown in Platen and Bruti-Liberati [2010] that the Euler–Maruyama and backward Euler–Maruyama schemes when applied to SDEs driven by Lévy noise have a mean-square convergence rate of $1/2$. We refer the reader to Platen and Bruti-Liberati [2010] and references therein for detailed analysis.

In Paper II, we consider the Euler–Maruyama scheme, the backward Euler–Maruyama scheme, also known as the drift-implicit Euler–Maruyama scheme [Milstein and Tretyakov, 2004], and an exponential Euler scheme. Exponential Euler schemes for S(P)DEs are analyzed, for example, in Biscay et al. [1996], Jentzen and Kloeden [2009], Cohen [2012], Lord et al. [2014].

# Chapter 3: Random noises on the sphere

We have now gained an intuition of stochastic processes and SDEs. However, in this thesis, we are rather concerned with stochastic processes indexed by space and time, and SPDEs on the unit sphere $\mathbb{S}^2$. These can be seen as infinite-dimensional extensions of the finite-dimensional stochastic processes discussed in Chapter 2.

## 3.1 The unit sphere $\mathbb{S}^2$

We start by introducing the unit sphere and the function spaces we use throughout the rest of the thesis. The unit sphere is given by $\mathbb{S}^2 = \{x \in \mathbb{R}^3 \mid \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2} = 1\}$, embedded in $\mathbb{R}^3$. In other words, we consider the surface of a ball of radius 1 centered at the origin. Through this embedding, we can describe points on the sphere $x \in \mathbb{S}^2$ in Cartesian coordinates $(x_1, x_2, x_3)$, with the restriction that $x_1^2 + x_2^2 + x_3^2 = 1$. Another parametrization is in terms of spherical coordinates, describing each point on the sphere by two angles, the polar angle $\vartheta \in [0, \pi]$ and the azimuthal angle $\varphi \in [0, 2\pi)$ [Marinucci and Peccati, 2011, Section 3.4.1]. We identify here the different notations and write $x = (\vartheta, \varphi)$.

We define the *geodesic distance* for on the sphere as

$$d_{\mathbb{S}^2}(x, y) = \arccos(\langle x, y \rangle_{\mathbb{R}^3}),$$

for all $x, y \in \mathbb{S}^2$ i. e., the smallest angle between the vectors $x$ and $y$ as interpreted in the surrounding Euclidean space $\mathbb{R}^3$. The *surface gradient* is defined as the projection of the gradient in $\mathbb{R}^3$ onto the tangent plane, i. e.,

$$\nabla_{\mathbb{S}^2} f(x) = (\mathrm{I}_3 - xx^T)\nabla f(x), \tag{3.1}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ and $x \in \mathbb{S}^2$. Note that $xx^T \nabla f(x)$ is the radial component of the gradient of $f$ in $\mathbb{R}^3$. Similarly, we carry over the Laplace operator to the sphere. This is done by a coordinate transform to spherical coordinates

and restricting to $r = 1$. The Laplace–Beltrami operator is given as (see, e. g., Marinucci and Peccati [2011, Section 3.4.3])

$$\Delta_{\mathbb{S}^2} = \frac{1}{\sin(\vartheta)} \frac{\partial}{\partial \vartheta} \left( \sin(\vartheta) \frac{\partial}{\partial \vartheta} \right) + \frac{1}{\sin^2(\vartheta)} \frac{\partial^2}{\partial \varphi^2}. \tag{3.2}$$

Lastly, we denote by $\mathrm{d}\sigma(x) = \sin(\vartheta)\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$ the surface measure on the sphere (see, e. g., Marinucci and Peccati [2011], Lang and Schwab [2015]). For an intrinsic treatment of differentiation and integration on curved surfaces in the context of Riemannian geometry, we refer the interested reader to Lee [2018].

The sphere has a variety of favorable properties that make working on it easier than considering general manifolds. The most important such property is that there is an explicitly known basis for the space $L^2(\mathbb{S}^2)$ of square-integrable functions $f : \mathbb{S}^2 \to \mathbb{R}$. The basis functions are the real-valued *spherical harmonic functions*, from now on called spherical harmonics. They are defined in spherical coordinates for $\ell \geq 0, -\ell \leq m \leq \ell$ as

$$Y_{\ell,m}(\vartheta, \varphi) = \begin{cases} (-1)^m \sqrt{2} \sqrt{\frac{(2\ell+1)(\ell-|m|)!}{4\pi(\ell+|m|)!}} P_\ell^{|m|}(\cos\vartheta) \sin(|m|\varphi), & m < 0, \\ \sqrt{\frac{(2\ell+1)}{4\pi}} P_\ell^m(\cos\vartheta), & m = 0, \\ (-1)^m \sqrt{2} \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}} P_\ell^m(\cos\vartheta) \cos(m\varphi), & m > 0, \end{cases} \tag{3.3}$$

where $P_\ell^m, \ell \geq 0, -\ell \leq m \leq \ell$, are the associated Legendre polynomials. For details on the spherical harmonics, we refer to Marinucci and Peccati [2011]. They form an orthonormal basis of $L^2(\mathbb{S}^2)$. In particular,

$$\langle Y_{\ell,m}, Y_{\ell',m'} \rangle_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} Y_{\ell,m}(x) Y_{\ell',m'}(x)\,\mathrm{d}\sigma(x) = \delta_{\ell,\ell'}\delta_{m,m'},$$

where $\delta_{i,j}$ denotes the Kronecker delta of $i$ and $j$. Furthermore, they are eigenfunctions of the Laplace–Beltrami operator (see, e. g., Marinucci and Peccati [2011, Section 3.4.3]) with eigenvalues $-\ell(\ell+1)$ for $\ell \geq 0, -\ell \leq m \leq \ell$, i. e.,

$$\Delta_{\mathbb{S}^2} Y_{\ell,m} = -\ell(\ell+1) Y_{\ell,m}.$$

We note here that the spherical harmonics used in Marinucci and Peccati [2011] and Lang and Schwab [2015], both of which we rely on in this thesis, are the complex-valued spherical harmonics $\widetilde{Y}_{\ell,m}, \ell \geq 0, -\ell \leq m \leq \ell$. With complex-valued coefficients, both $\{Y_{\ell,m}, \ell \geq 0, -\ell \leq m \leq \ell\}$ and $\{\widetilde{Y}_{\ell,m}, \ell \geq 0, -\ell \leq m \leq \ell\}$ span the same space of square-integrable complex-valued functions, $L^2(\mathbb{S}^2; \mathbb{C})$ and a basis transformation allows us to go back and forth between them:

$$\widetilde{Y}_{\ell,m}(\vartheta, \varphi) = \begin{cases} \frac{1}{\sqrt{2}} \left( Y_{\ell,-m} - i Y_{\ell,m} \right), & m < 0, \\ Y_{\ell,0}, & m = 0, \\ \frac{(-1)^m}{\sqrt{2}} \left( Y_{\ell,m} + i Y_{\ell,-m} \right), & m > 0. \end{cases}$$

This expression shows that the complex-valued spherical harmonics are eigenfunctions to the Laplace–Beltrami operator with the same eigenvalues as the real-valued ones, i. e.,

$$\Delta_{\mathbb{S}^2} \widetilde{Y}_{\ell,m} = -\ell(\ell+1)\widetilde{Y}_{\ell,m}.$$

Therefore, the eigenspaces are preserved under this basis transformation. A proof can, for example, be found in Jansson et al. [2024, Appendix A]. Since we focus primarily on real-valued functions, it is more natural to use the real-valued spherical harmonics. For more detailed computations on the conditions necessary for using the complex-valued spherical harmonics as a basis for the space $L^2(\mathbb{S}^2)$ of real-valued functions, we refer the reader to Lang and Schwab [2015].

Any function $f \in L^2(\mathbb{S}^2)$ can be expanded into a series with the spherical harmonics as a basis:

$$f(\vartheta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell,m} Y_{\ell,m}(\vartheta,\varphi), \tag{3.4}$$

with $f_{\ell,m} = \langle f, Y_{\ell,m} \rangle_{L^2(\mathbb{S}^2)}$. This allows us to define fractional powers of the Laplace–Beltrami operator for $\eta \geq 0$ as

$$(-\Delta_{\mathbb{S}^2})^{\eta} f = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell(\ell+1))^{\eta} f_{\ell,m} Y_{\ell,m}, \tag{3.5}$$

and the Helmholtz operator $(\mathrm{I} - \Delta_{\mathbb{S}^2})$ for any $\eta \in \mathbb{R}$ as

$$(\mathrm{I} - \Delta_{\mathbb{S}^2})^{\eta} f = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (1 + \ell(\ell+1))^{\eta} f_{\ell,m} Y_{\ell,m}. \tag{3.6}$$

We define the *Bessel potential spaces* for $\eta \in \mathbb{R}, \eta \geq 0$,

$$H^{\eta}(\mathbb{S}^2) = \{ f \in L^2(\mathbb{S}^2) \mid (\mathrm{I} - \Delta_{\mathbb{S}^2})^{\eta/2} f \in L^2(\mathbb{S}^2) \}, \tag{3.7}$$

equipped with the inner product defined for $f, g \in H^{\eta}(\mathbb{S}^2)$ as

$$\langle f, g \rangle_{H^{\eta}(\mathbb{S}^2)} = \langle (\mathrm{I} - \Delta_{\mathbb{S}^2})^{\eta/2} f, (\mathrm{I} - \Delta_{\mathbb{S}^2})^{\eta/2} g \rangle_{L^2(\mathbb{S}^2)}.$$

The Bessel potential spaces are norm-equivalent to the standard *Sobolev* spaces on $\mathbb{S}^2$ with smoothness index $\eta$. For more details on defining Sobolev spaces via Bessel potentials, we refer the reader to Strichartz [1983], Herrmann et al. [2018] and references therein. We denote $H^0(\mathbb{S}^2) = L^2(\mathbb{S}^2)$. The Sobolev spaces form a scale such that for $0 \leq \xi \leq \eta$, $H^{\eta}(\mathbb{S}^2) \subseteq H^{\xi}(\mathbb{S}^2)$ and for $\eta > 1$, we can use a Sobolev embedding theorem and show that functions $f \in H^{\eta}(\mathbb{S}^2)$ are continuous [Aubin, 1998, Theorem 2.20].

Since all of these spaces are Hilbert spaces, and we will later on be interested in choosing different spaces along this scale, e. g., $L^2(\mathbb{S}^2)$ and $H^1(\mathbb{S}^2)$, we will in the following use the notation $\mathcal{H}$ whenever we mean an arbitrary space along this scale. For the sake of being explicit, all computations will be performed specifically with Bessel potential spaces on the sphere in mind.

A further class of function spaces that play a role in Paper I are Hölder spaces, which we define in the sense of metric spaces like in Krätschmer and Urusov [2023]. A function $f : \mathbb{S}^2 \to \mathbb{R}$ is called Hölder continuous with Hölder exponent $0 < \alpha < 1$ if

$$|f(x) - f(y)| \leq C d_{\mathbb{S}^2}(x, y)^\alpha \tag{3.8}$$

for all $x, y \in \mathbb{S}^2$. We denote the space of Hölder-$\alpha$ continuous functions on the sphere as $C^\alpha(\mathbb{S}^2)$. $C^0(\mathbb{S}^2)$ is chosen to denote the space of all continuous functions. We denote the space of functions that Hölder-$\alpha$ continuous for all $\alpha < \gamma, 0 < \gamma \leq 1$, as $C^{\gamma-}(\mathbb{S}^2)$.

We further consider Hölder spaces with $\alpha > 1$. We set $\alpha = k + \gamma$ for some $k \in \mathbb{N}$ and $0 \leq \gamma < 1$. Then, $C^\alpha(\mathbb{S}^2)$ is the space of functions whose $k$-th derivative is Hölder continuous with exponent $\gamma$. As shown in Taylor [1981, Theorem XI.2.5], $f$ is in $C^\alpha(\mathbb{S}^2)$ with $\alpha$ as above if $(\mathrm{I} - \Delta_{\mathbb{S}^2})^{k/2} f$ is in $C^\gamma(\mathbb{S}^2)$.

## 3.2 Random fields on the sphere

We are now equipped to consider random fields on the sphere, which are random variables in some function space $\mathcal{H}$. The presentation here follows mostly that in Marinucci and Peccati [2011], Lang and Schwab [2015].

Let us start with a pointwise definition of a random field. A *random field on the sphere*, in the following referred to simply as random field, is a stochastic process that is indexed by $\mathbb{S}^2$, a function $Z : \Omega \times \mathbb{S}^2 \to \mathbb{R}$. Therefore, for every point $x \in \mathbb{S}^2$, $Z(x)$ is a random variable. Note that, in the following, we reserve the term stochastic process for time-dependent processes, using only the term random field for purely space-dependent processes. Analogous to a Gaussian process, we call $Z$ a *Gaussian random field (GRF)* if, for all $n \geq 1$, $x_1, \ldots, x_n \in \mathbb{S}^2$ and $a_1, \ldots, a_n \in \mathbb{R}$, $\sum_{i=1}^n a_i Z(x_i)$ is a Gaussian random variable.

We often require that the samples $Z(\omega)$ of our random field are in some function space, such as a Sobolev space $\mathcal{H}$. We can then interpret $Z$ as an $\mathcal{H}$-valued random variable, i. e., $Z : \Omega \to \mathcal{H}$. In this thesis, we assume all random fields to be in $L^2(\Omega; \mathcal{H})$, i. e., $Z \in \mathcal{H}$ $\mathbb{P}$-almost surely and $\mathbb{E}\left[\|Z\|_{\mathcal{H}}^2\right] < \infty$. This means that the covariance of $Z$ is well-defined. For a more extensive discussion on Hilbert space-valued (Gaussian) random variables, we refer to Da Prato and Zabczyk [2014].

Every random field $Z \in L^2(\Omega; \mathcal{H})$ has an expansion in spherical harmonics, called the *Karhunen–Loève expansion* (see, e. g., Marinucci and Peccati [2011, Section 5.2])

$$Z(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \alpha_{\ell,m} Y_{\ell,m}(x), \tag{3.9}$$

where $\alpha_{\ell,m}, \ell \geq 0, -\ell \leq m \leq \ell$, are random variables and $x \in \mathbb{S}^2$. If $Z$ is a Gaussian random field, then $\alpha_{\ell,m}$ is Gaussian for all $\ell \geq 0, -\ell \leq m \leq \ell$ [Marinucci and Peccati, 2011, Section 6.3]. We require that the variances of the random variables $\alpha_{\ell,m}, \ell \geq 0, -\ell \leq m \leq \ell$, decay sufficiently fast as $\ell$ increases for the sum (3.9) to converge in $L^2(\Omega; \mathcal{H})$, i. e.,

$$\mathbb{E}\left[\|Z\|_{\mathcal{H}}^2\right] = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathbb{E}\left[\alpha_{\ell,m}^2\right] < \infty. \tag{3.10}$$

In this case, the mean of $Z$ exists and is given by

$$m(x) = \mathbb{E}\left[Z(x)\right] = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathbb{E}[\alpha_{\ell,m}] Y_{\ell,m}(x). \tag{3.11}$$

Therefore, any random field can be represented as $Z(x) = \widetilde{Z}(x) + m(x)$, where $\mathbb{E}[\widetilde{Z}] = 0$. For ease of presentation, we only consider mean-zero random fields, with the understanding that an arbitrary function in $\mathcal{H}$ can be added as a mean.

We now consider the second moments, i. e., the covariance. Assuming that $Z$ is sufficiently regular to allow for pointwise evaluation and considering it from a pointwise perspective, we can compute the *covariance kernel*

$$\begin{aligned}\mathbb{E}\left[Z(x)Z(y)\right] &= \mathbb{E}\left[\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \alpha_{\ell,m} Y_{\ell,m}(x) \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} \alpha_{\ell',m'} Y_{\ell',m'}(y)\right] \\ &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} \mathbb{E}\left[\alpha_{\ell,m}\alpha_{\ell',m'}\right] Y_{\ell,m}(x) Y_{\ell',m'}(y).\end{aligned}$$

If the $\alpha_{\ell,m}, \ell \geq 0, -\ell \leq m \leq \ell$, are uncorrelated, we obtain

$$\mathbb{E}\left[Z(x)Z(y)\right] = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathbb{E}\left[\alpha_{\ell,m}^2\right] Y_{\ell,m}(x) Y_{\ell,m}(y) = \gamma_Q(x, y).$$

Like the covariance function of a temporal stochastic process, the covariance kernel can be used to obtain the Hölder exponent of a Hölder continuous random field. We discuss this in Section 3.2.1.

Returning to the function space perspective, since $Z$ is in $L^2(\Omega; \mathcal{H})$, $Z$ has a well-defined covariance, expressed through a covariance operator $Q$. The covariance operator $Q$ is defined as the unique linear operator satisfying

$$\mathbb{E}\left[\langle Z, \varphi\rangle_{\mathcal{H}} \langle Z, \psi\rangle_{\mathcal{H}}\right] = \langle Q\varphi, \psi\rangle_{\mathcal{H}}, \tag{3.12}$$

for all $\varphi, \psi \in \mathcal{H}$, see, e. g., Lang et al. [2013] and Peszat and Zabczyk [2007, Section 3.5]. Plugging in spherical harmonics, we see from the Karhunen–Loève expansion that $Q$ then describes the covariance between the coefficients:

$$\mathbb{E}\left[\alpha_{\ell,m}\alpha_{\ell',m'}\right] = \mathbb{E}\left[\langle Z, Y_{\ell,m}\rangle_{\mathcal{H}} \langle Z, Y_{\ell',m'}\rangle_{\mathcal{H}}\right] = \langle QY_{\ell,m}, Y_{\ell',m'}\rangle_{\mathcal{H}}$$

for all $\ell \geq 0, -\ell \leq m \leq \ell$ and $\ell' \geq 0, -\ell' \leq m' \leq \ell'$. Furthermore, $Q$ is a trace-class operator, i. e.,

$$\operatorname{Tr}(Q) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \langle QY_{\ell,m}, Y_{\ell,m}\rangle_{\mathcal{H}} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathbb{E}\left[\alpha_{\ell,m}^2\right] < \infty,$$

if and only if $Z \in L^2(\Omega; \mathcal{H})$ [Peszat and Zabczyk, 2007, Sections 3.5, 4.9]. Furthermore, like any covariance operator, $Q$ is non-negative definite and Hermitian. We denote the space of trace-class covariance operators by $L_1^+(H)$.

### 3.2.1 Isotropy and Hölder regularity

A property that is of relevance for our computations later in this context is that of *isotropy* of random fields on the sphere. Isotropy means rotation invariance, meaning the random field "looks the same" whichever way one rotates the sphere. While there are various definitions of isotropy [Marinucci and Peccati, 2011], we are mainly concerned with 2-weak isotropy. 2-weak isotropy requires the first and second moment (expectation and covariance) to be constant under rotation. Specifically, Marinucci and Peccati [2011, Def. 5.9] define a random field $Z$ as 2-weakly isotropic if $\mathbb{E}\left[Z^2(x)\right] < \infty$, $\mathbb{E}\left[Z(x)\right] = m, m \in \mathbb{R}$, and

$$\mathbb{E}\left[Z(x)Z(y)\right] = \mathbb{E}\left[Z(gx)Z(gy)\right]$$

for all $x, y \in \mathbb{S}^2$ and any rotation matrix $g$. Alternatively, 2-weak isotropy can be described by constant mean and the covariance kernel depending only on $d_{\mathbb{S}^2}(x, y)$, i. e., for some function $f : [0, \pi] \to \mathbb{R}$,

$$\mathbb{E}\left[Z(x)Z(y)\right] = f(d_{\mathbb{S}^2}(x, y)), \tag{3.13}$$

for all $x, y \in \mathbb{S}^2$ [Marinucci and Peccati, 2011, Section 6.4].

It is shown in Marinucci and Peccati [2011, Section 6.3] that a 2-weakly isotropic random field can be written as

$$Z(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{A_\ell} z_{\ell,m} Y_{\ell,m}(x), \tag{3.14}$$

where $(A_\ell, \ell \geq 0)$ is the *angular power spectrum of* $Z$ and $z_{\ell,m}, \ell \geq 0, -\ell \leq m \leq \ell$, are uncorrelated random variables of mean zero and variance 1.

Conversely, a direct computation of the covariance, using an addition formula for spherical harmonics [Marinucci and Peccati, 2011, Equation (3.42)], shows that any random field given by (3.14) is 2-weakly isotropic.

If $Z$ is an isotropic random field, the covariance operator $Q$ is diagonalized by the spherical harmonics, i. e., for all $\ell \geq 0, -\ell \leq m \leq \ell$,

$$QY_{\ell,m} = A_\ell Y_{\ell,m}. \tag{3.15}$$

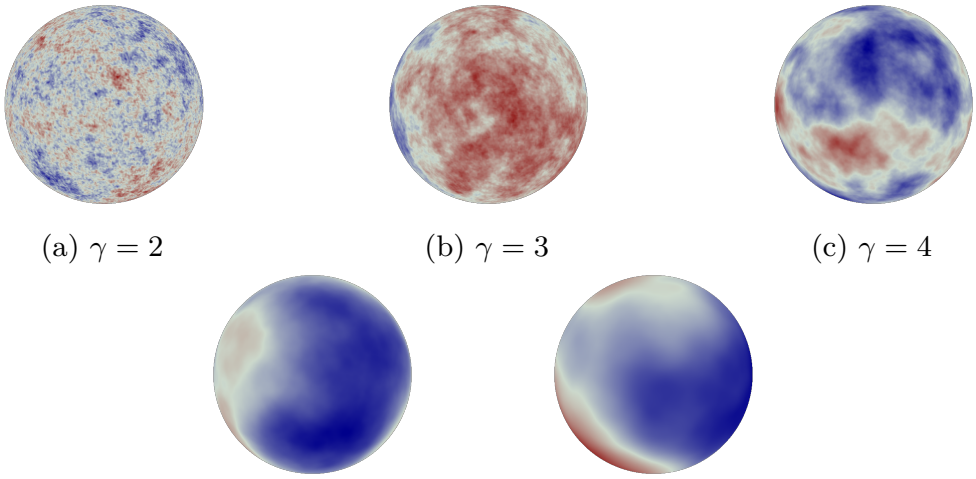
This can be computed directly via the definition of the covariance operator in Equation (3.12) by plugging in $Y_{\ell,m}$ in place of $\varphi$ and $\psi$.

The regularity of an isotropic random field can be characterized by its angular power spectrum: the faster the decay of $(A_\ell, \ell \geq 0)$, the more regular the random field. This has been extensively investigated in Lang and Schwab [2015] for isotropic Gaussian random fields on the sphere. Their main result states that if $Z$ is a Gaussian random field with Karhunen–Loève expansion (3.14) and it holds for some $\beta \in (0, 2]$ that $\sum_{\ell=0}^{\infty} A_\ell \ell^{\beta+1} < \infty$, then there exists a continuous modification of $Z$ that is Hölder continuous with exponent $\alpha$ for all $\alpha < \beta/2$. The proof of this result is based on a Kolmogorov–Chentsov-type theorem on manifolds that was proved in Andreev and Lang [2014] and independently for the sphere in Lang and Schwab [2015]. Beyond this, Lang and Schwab [2015] even show that for $\beta > 2$, the theorem still applies analogously and one obtains modifications of $Z$ that are $C^{\beta/2-}(\mathbb{S}^2)$-valued.

Figure 3.1 illustrates how the regularity of isotropic Gaussian random fields on the sphere is influenced by the decay of the angular power spectrum. We choose an algebraic decay of $A_\ell \sim (\ell + 1)^{-\gamma}$ for $\gamma = 2, \ldots, 6$, yielding Hölder regularities of $\alpha < \gamma/2 - 1$. Note that $\gamma = 2$ in Figure 3.1a is the boundary case of a random field that is not continuous, it merely appears continuous due to the spectral truncation method (see Section 3.2.2) used for its simulation.

### 3.2.2 Simulation of random fields on the sphere

One approach to simulating Gaussian random fields, as we saw above for Gaussian processes, is to select discrete points and apply the Cholesky factorization



(a) $\gamma = 2$ (b) $\gamma = 3$ (c) $\gamma = 4$

(d) $\gamma = 5$ (e) $\gamma = 6$

Figure 3.1: Independent realizations of isotropic Gaussian random fields on the sphere with Karhunen–Loève expansion (3.14) and angular power spectrum decaying algebraically as $A_\ell \sim (\ell + 1)^{-\gamma}$.

method, which allows us to reproduce the exact covariances. Beyond the high computational cost of this method, constructing the covariance functions needed to apply this method is not straight-forward on the sphere. Lindgren et al. [2011] report, for example, that the Matérn covariance, a widely used model, is no longer a valid covariance function if used with the geodesic distance on the sphere. Instead, various approximation algorithms have been proposed and analyzed in the literature. These can be found, for example, in Lindgren et al. [2011], Jansson et al. [2022, 2024], Bachmayr and Djurdjevac [2023] and the review by Porcu et al. [2018].

One simulation method for isotropic Gaussian random fields is the spectral truncation method [Lang and Schwab, 2015]. This method takes the Karhunen–Loève expansion (3.14) with a specified angular power spectrum $A(_\ell, \ell \geq 0)$ and instead of letting $\ell$ run to infinity, truncates at $\kappa \geq 0$, obtaining

$$Z^\kappa = \sum_{\ell=0}^{\kappa} \sum_{m=-\ell}^{\ell} \sqrt{A_\ell} z_{\ell,m} Y_{\ell,m}. \tag{3.16}$$

Strong and almost sure convergence rates that depend on the decay of $(A_\ell, \ell \geq 0)$ are shown in Lang and Schwab [2015].

It is of interest here to briefly mention an efficient way of implementing the evaluation of $Z^\kappa$ at points on the sphere, for example for the purpose of visualizing the random field. Just like there are pre-implemented efficient routines for evaluating one-dimensional Fourier series on discrete points called *Fast Fourier Transforms (FFTs)* [Cooley and Tukey, 1965], there exists the *Spherical Harmonics Transform (SHT)*. The SHT decomposes the series (3.16)

into classical Fourier series that are then evaluated through highly optimized FFT routines [Slevinsky, 2019]. The library used for this thesis and Paper I is the `Julia` library `FastSphericalHarmonics`[1].

## 3.3 Space-time stochastic processes

We now combine the temporal stochastic processes from Chapter 2 with the random fields of Section 3.2 to obtain spatio-temporal stochastic processes on the sphere. We saw in Section 3.2 that we can expand random fields into the Karhunen–Loève expansion (3.9). We now replace the random variables $\alpha_{\ell,m}$ with stochastic processes $a_{\ell,m}X_{\ell,m}(t)$, where we assume that the variance $\mathsf{Var}(X_{\ell,m}(1)) = 1$ and $a_{\ell,m} \in \mathbb{R}$ are rescaling constants for all $\ell \geq 0, -\ell \leq m \leq \ell$. The Karhunen–Loève expansion of a spatio-temporal process $X(t)$ then reads

$$X(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell,m} X_{\ell,m}(t) Y_{\ell,m}, \tag{3.17}$$

where we again assume that the sequence $(a_{\ell,m}, \ell \geq 0, -\ell \leq m \leq \ell)$ decays sufficiently fast.

Analogous to the random field setting, we can interpret $X$ as a stochastic process indexed by space and time, $X : \Omega \times \mathbb{T} \times \mathbb{S}^2 \to \mathbb{R}$, or as a stochastic process taking values in a function space $\mathcal{H}$, $X : \Omega \times \mathbb{T} \to \mathcal{H}$. In this thesis, we require that $X(t)$ almost surely lies in the space $\mathcal{H}$ for all $t \in \mathbb{T}$ and that $X(t)$ is in $L^2(\Omega; \mathcal{H})$. Then, $X(t)$ is a random field with a well-defined covariance operator $tQ$ that is trace-class according to our assumptions above.

We call a stochastic process $X$ isotropic if $X(t)$ is isotropic for all $t \in \mathbb{T}$. Like in Section 3.2.1, this is the case if we can write the Karhunen–Loéve expansion as

$$X(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{A_\ell} X_{\ell,m}(t) Y_{\ell,m}, \tag{3.18}$$

where all $X_{\ell,m}, \ell \geq 0, -\ell \leq m \leq \ell$, are iid processes.

Before we move to some explicit examples, let us consider the simulation of such a spatio-temporal process. In the isotropic case, we can use the spectral truncation method of Lang and Schwab [2015] that we described in Section 3.2.2. We simulate

$$X^\kappa(t) = \sum_{\ell=0}^{\kappa} \sum_{m=-\ell}^{\ell} \sqrt{A_\ell} X_{\ell,m}(t) Y_{\ell,m} \tag{3.19}$$

[1] https://github.com/eschnett/FastSphericalHarmonics.jl

on a discrete time grid by simulating each $X_{\ell,m}, 0 \leq \ell \leq \kappa, -\ell \leq m \leq \ell$, for example as described in Chapter 2. The convergence results in Lang and Schwab [2015] can be applied almost directly in this case, with an added dependence on $t$. We prove this in Paper I for the case where $X_{\ell,m}, \ell \geq 0, -\ell \leq m \leq \ell$, are fractional Brownian motions.

### 3.3.1 $Q$-Wiener process on the sphere

If we choose Wiener processes $W_{\ell,m}$ as our coefficient processes $X_{\ell,m}$ for all $\ell \geq 0, -\ell \leq m \leq \ell$, we obtain the *$Q$-Wiener process*

$$W^Q(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell,m} W_{\ell,m}(t) Y_{\ell,m}, \tag{3.20}$$

where $Q$ is the covariance operator of the random field $W^Q(1)$. The $Q$-Wiener process satisfies the same properties as a one-dimensional Brownian motion, except that we now have to phrase them in the language of Hilbert spaces. We define the $Q$-Wiener process similarly to Da Prato and Zabczyk [2014].

**Definition 4.** An $\mathcal{H}$-valued stochastic process $W^Q(t)$ is called a *$Q$-Wiener process* if

1. $W^Q(0) = 0$ $\mathbb{P}$-almost surely,
2. $W^Q$ has independent increments,
3. $W^Q(t) - W^Q(s) \sim \mathcal{N}(0, (t-s)Q)$ for all $0 \leq s \leq t$.

An application of a Kolmogorov–Chentsov theorem (e. g., Kallenberg [2021, Theorem 4.23]) yields a unique continuous modification that is locally Hölder-$\alpha$ continuous for $\alpha < 1/2$ with respect to the norm of $\mathcal{H}$. We consider in this thesis $W^Q$ to be this continuous modification.

For $Q$-Wiener processes, there is a simulation technique that allows us to use any of the methods we described for random fields in Section 3.2.2. It can be extracted from Definition 4 above: Observe that an increment $\Delta W_n^Q = W^Q(t_{n+1}) - W^Q(t_n)$ follows an $\mathcal{N}(0, hQ)$ distribution, i. e., $\Delta W_n^Q = \sqrt{h} Z_n$, where $Z_n \sim \mathcal{N}(0, Q)$ and all $Z_n$ are independent. We see that every increment of $W^Q$ can be viewed as a (rescaled) Gaussian random field with covariance operator $Q$. So by simulating $Z_n, n \geq 0$, using any method referenced in Section 3.2.2 and cumulatively summing up, we obtain

$$W^Q(t_n) = \sum_{i=0}^{n-1} Z_i.$$

### 3.3.2 $Q$-fractional Brownian motion on the sphere

Similarly, plugging in fractional Brownian motions $\beta_{\ell,m}^H$ with Hurst parameter $0 < H < 1$ in the Karhunen–Loève expansion, we obtain *$Q$-fractional Brownian motion*

$$B_Q^H(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell,m} \beta_{\ell,m}^H(t) Y_{\ell,m}, \tag{3.21}$$

$Q$-fractional Brownian motion was first introduced by Grecksch and Anh [1999] in an abstract Hilbert-space setting. Here, we define an $\mathcal{H}$-valued $Q$-fractional Brownian motion as follows, see, e. g., Grecksch et al. [2009].

**Definition 5.** Let $0 < H < 1$. An $\mathcal{H}$-valued Gaussian process $B_Q^H$ is called a *$Q$-fractional Brownian motion* with Hurst parameter $H$ if

1. $\mathbb{E}\left[B_Q^H(t)\right] = 0$ for all $t \geq 0$,
2. $\mathbb{E}\left[\langle B_Q^H(t), \varphi\rangle_{\mathcal{H}} \langle B_Q^H(s)\psi\rangle_{\mathcal{H}}\right] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})\langle Q\varphi, \psi\rangle_{\mathcal{H}}$ for all $s, t \geq 0$.

We choose, again, the continuous modification of fractional Brownian motion which has, by the Kolmogorov–Chentsov theorem, local Hölder continuity with exponent $H-$ with respect to the norm of $\mathcal{H}$.

Computing the space-time covariance kernel for an isotropic $Q$-fractional Brownian motion $B_Q^H$,

$$\begin{aligned}\mathbb{E}\left[B_Q^H(s,x)B_Q^H(t,y)\right] &= \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \\ &\quad \cdot \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell Y_{\ell,m}(x) Y_{\ell,m}(y) \\ &= \gamma_H(s,t)\gamma_Q(x,y),\end{aligned}$$

we see that it separates into a spatial and temporal component. We call such a covariance a separable covariance [Porcu et al., 2021]. For a fixed point in time, $B_Q^H(t,\cdot)$ is a random field with covariance operator $tQ$ and for a fixed spatial point, $B_Q^H(\cdot, x)$ is a (rescaled) fractional Brownian motion with Hurst parameter $H$. Thus, we can use $(A_\ell, \ell \geq 0)$ and $H$ to vary spatial and temporal regularity, respectively.

### 3.3.3 $Q$-Lévy processes on the sphere

Lastly, to allow for jumps, we can also choose $X_{\ell,m}$ in Equation (3.17) to be Lévy processes $L_{\ell,m}$ as defined in Section 2.3 and obtain an infinite-dimensional

Lévy process

$$L(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell,m} L_{\ell,m}(t) Y_{\ell,m}. \tag{3.22}$$

We define it here abstractly as in Peszat and Zabczyk [2007].

**Definition 6.** An $\mathcal{H}$-valued stochastic process $L$ is called a Lévy process if

1. $L(0) = 0$ $\mathbb{P}$-almost surely,
2. $L$ has independent increments,
3. $L$ has stationary increments, i. e., for all $t \geq s \geq 0$, the distribution of $L(t) - L(s)$ depends only on $t - s$,
4. $L$ is stochastically continuous, i. e., for all $t_0 \geq 0$, $L(t) \to_p L(t_0)$ for $t \to t_0$.

Analogously to Section 2.3, we assume that $L(t)$ is in $L^2(\Omega; \mathcal{H})$ for all $t \geq 0$. As given in Peszat and Zabczyk [2007, Theorem 4.44], if $L$ is a square-integrable Lévy process, its mean $\mathbb{E}[L(t)] = mt$ as well as its covariance operator $Q$ are well-defined. For ease of presentation, we assume $m = 0$. Then, we can define a $Q$-Lévy process as follows [Peszat and Zabczyk, 2007].

**Definition 7.** Let $L^Q$ be a square-integrable Lévy process on $\mathcal{H}$ and $Q \in L_1^+(\mathcal{H})$. Then, $L^Q$ is called a $Q$-Lévy process if its covariance is given by

$$\mathbb{E}\left[\langle L^Q(t), \varphi\rangle_{\mathcal{H}} \langle L^Q(s), \psi\rangle_H\right] = (s \wedge t)\langle Q\varphi, \psi\rangle_{\mathcal{H}}.$$

Note that for every fixed point in time, $L^Q(t)$ is a random field with covariance operator $tQ$. It need not be a Gaussian random field since $Q$-Lévy processes allow for more general distributions.

## 3.4 Summary of Paper I

In Paper I, we consider the spatio-temporal Hölder regularity of sample paths of $Q$-fractional Brownian motion on the sphere. That is, we interpret $B_Q^H$ as a random variable taking values in a Hölder space on the compact manifold $M = [0, T] \times \mathbb{S}^2$. Using a variant of the Kolmogorov–Chentsov theorem, developed by Krätschmer and Urusov [2023], we obtain a Hölder continuous modification of $B_Q^H$ on $M$ with Hölder exponent $\min(H, \eta/2)-$, where $\eta/2-$ is the spatial Hölder regularity of $B_Q^H(t)$ on $\mathbb{S}^2$. As the spatial and temporal regularities ($\eta/2$ and $H$) differ, one Hölder exponent is not sufficient to fully describe the regularity. We, therefore, consider a Hölder criterion allowing for different regularities in space and time and, using a continuity argument,

prove that $B_Q^H$ satisfies this sharper regularity. We further establish a similar result on $(d-1)$-dimensional hyperspheres $\mathbb{S}^{d-1}$ embedded into $\mathbb{R}^d$.

Beyond this, we consider efficient simulation methods for $Q$-fractional Brownian motion on the sphere. We apply a spectral truncation as considered in Section 3.2.2, and derive a convergence result depending on the time horizon $T$. For the simulation of the one-dimensional fractional Brownian motions making up the coefficients of the truncated series expansion, we investigate the Conditionalized Random Midpoint Displacement method and the Circulant Embedding method. While the Circulant Embedding method produces a distributionally exact result, the Conditionalized Random Midpoint Displacement method is approximate. We perform simulations to estimate the error of the latter method, obtaining an error decay rate of approximately 1.

# Chapter 4: Linear SPDEs driven by additive $Q$-Lévy noise on the sphere

Having seen the background on spatio-temporal stochastic processes, we now impose a differential equation structure on these processes. Through this, we obtain *stochastic partial differential equations (SPDEs).*

At the heart of the modern-day theory of SPDEs lies the $Q$-Wiener process described in Section 3.3.1. Detailed discussions of SPDEs driven by $Q$-Wiener processes can be found in standard works such as Da Prato and Zabczyk [2014], Prévôt and Röckner [2007], Liu and Röckner [2015], Lord et al. [2014].

There has also been research on SPDEs driven by $Q$-fractional Brownian motion, starting with Grecksch and Anh [1999]. Further theory on existence and uniqueness was built, among others, by Grecksch et al. [2009], Duncan et al. [2002, 2009], Tindel et al. [2003], Maslowski and Nualart [2003], Röckner and Wang [2009].

In this thesis, we will focus on the case of SPDEs driven by $Q$-Lévy noises as is considered in Paper II. Note that since the $Q$-Wiener process is a type of $Q$-Lévy process, SPDEs driven by the former are covered by the theory presented here. For results on existence, uniqueness, and regularity of solutions of $Q$-Lévy driven SPDEs, we refer to, for example, Hausenblas [2005], Albeverio et al. [2009], Benth and Krühner [2023]. The presentation here follows Peszat and Zabczyk [2007].

## 4.1 Linear SPDEs driven by additive Lévy noise

A first-order linear SPDE driven by additive $Q$-Lévy noise $L^Q$ can formally be written as an SDE in the Hilbert space $\mathcal{H}$

$$\mathrm{d}u(t) = Au(t)\,\mathrm{d}t + \mathrm{d}L^Q(t), \quad u(0) = u_0. \tag{4.1}$$

Equivalently, we can write the integral equation

$$u(t) = u_0 + \int_0^t Au(s)\,\mathrm{d}s + \int_0^t \mathrm{d}L^Q(s). \tag{4.2}$$

For ease of presentation, we assume here that the initial condition $u_0 \in \mathcal{H}$ is non-random. The differential operator $A$ is a linear unbounded operator mapping from its domain $D(A) = \{x \in \mathcal{H} \mid Ax \in \mathcal{H}\}$ into $\mathcal{H}$. Most commonly, $A$ is a second-order differential operator, such as the Laplace–Beltrami operator $\Delta_{\mathbb{S}^2}$, or a function thereof. We require that $A$ generates a *strongly continuous semigroup* $S(t) = e^{At}$ and assume further that $A$ is diagonalized by the spherical harmonics with eigenvalues $-\lambda_{\ell,m}$,

$$AY_{\ell,m} = -\lambda_{\ell,m} Y_{\ell,m}, \tag{4.3}$$

where $\lambda_{\ell,m} \geq 0$ for all $\ell \geq 0, -\ell \leq m \leq \ell$. The semigroup generated by $A$ can then be defined via a spectral expansion, analogously to Equations (3.5) and (3.6):

$$e^{At}x = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} e^{-\lambda_\ell t} x_{\ell,m} Y_{\ell,m},$$

where $x_{\ell,m} = \langle x, Y_{\ell,m}\rangle_{\mathcal{H}}$ for all $\ell \geq 0, -\ell \leq m \leq \ell, x \in \mathcal{H}$ [Lord et al., 2014, Section 3.2]. For the theory of operator semigroups in the more general case, we refer to Pazy [1983].

We can formally write down a variation-of-constants formula as a solution to (4.1) as

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)}\,\mathrm{d}L^Q(s). \tag{4.4}$$

This is called a *mild solution* and is well-defined under the conditions given in [Peszat and Zabczyk, 2007, Sections 9.2, 9.4].

This decomposition allows us to spectrally decompose the SPDE and its solution, and obtain an infinite-dimensional system of SDEs. We start out from the integral equation (4.2) and spectrally expand

$$u(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell,m}(t) Y_{\ell,m}, \quad u_0 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_0^{\ell,m} Y_{\ell,m}$$

and $L^Q$ according to Equation (3.22) to obtain, using Equation (4.3),

$$\begin{aligned}\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell,m}(t) Y_{\ell,m} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_0^{\ell,m} Y_{\ell,m} - \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^t \lambda_{\ell,m} u_{\ell,m}(s)\,\mathrm{d}s Y_{\ell,m} \\ + \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^t a_{\ell,m}\,\mathrm{d}L_{\ell,m}(s) Y_{\ell,m}.\end{aligned} \tag{4.5}$$

Using the orthogonality of the spherical harmonics $Y_{\ell,m}, \ell \geq 0, -\ell \leq m \leq \ell$, Equation (4.5) decomposes into a system

$$u_{\ell,m}(t) = u_0^{\ell,m} - \int_0^t \lambda_{\ell,m} u_{\ell,m}(s)\,\mathrm{d}s + \int_0^t a_{\ell,m}\,\mathrm{d}L_{\ell,m}(s),$$

or, equivalently,

$$\mathrm{d}u_{\ell,m}(t) = -\lambda_{\ell,m} u_{\ell,m}(t)\,\mathrm{d}t + a_{\ell,m}\,\mathrm{d}L_{\ell,m}(t), \quad u_{\ell,m}(0) = u_0^{\ell,m}, \tag{4.6}$$

for $\ell \geq 0, -\ell \leq m \leq \ell$. Note that we are not assuming that the $L_{\ell,m}$, $\ell \geq 0, -\ell \leq m \leq \ell$, are independent. Hence, the system is coupled via the noise.

## 4.2 Numerical discretization

For numerical approximation of the solution $u$ to Equation (4.1), we discretize by truncating the expansion (4.5) at $\kappa \geq 0$, denoting the *spatially semi-discrete approximation* of $u$ by

$$u^\kappa(t) = \sum_{\ell=0}^{\kappa} \sum_{m=-\ell}^{\ell} u_{\ell,m}(t) Y_{\ell,m}. \tag{4.7}$$

Assuming that $u_{\ell,m}, 0 \leq \ell \leq \kappa, -\ell \leq m \leq \ell$, are given by Equation (4.5), this implicitly carries with it a discretization $A^\kappa$ of the differential operator $A$. We can define $A^\kappa : \mathcal{H}^\kappa \to \mathcal{H}^\kappa$ as an orthogonal projection of $A$ onto the finite-dimensional subspace $\mathcal{H}^\kappa$ spanned by $\{Y_{\ell,m}, 0 \leq \ell \leq \kappa, -\ell \leq m \leq \ell\}$. Specifically, we require

$$\langle A^\kappa u, v\rangle_{\mathcal{H}} = \langle Au, v\rangle_{\mathcal{H}}$$

for all $u, v \in \mathcal{H}^\kappa$. It can be computed from this definition that $A^\kappa Y_{\ell,m} = -\lambda_{\ell,m} Y_{\ell,m} = AY_{\ell,m}$ for all $0 \leq \ell \leq \kappa, -\ell \leq m \leq \ell$. So, $u^\kappa$ satisfies the system of SDEs

$$\mathrm{d}u^\kappa(t) = A^\kappa u^\kappa(t)\,\mathrm{d}t + \mathrm{d}L^\kappa(t), \quad u(0) = u_0, \tag{4.8}$$

where $L^\kappa(t) = \sum_{\ell=0}^{\kappa} \sum_{m=-\ell}^{\ell} a_{\ell,m} L_{\ell,m}(t) Y_{\ell,m}$ is a spectral truncation of the $Q$-Lévy noise $L^Q$. The spectral truncation method is discussed, for example, in Larsson and Thomée [2003, Section 14.2].

Applying a time discretization to the semi-discrete approximation (4.7), we obtain a *fully discrete approximation* of $u$. Here, we give the Euler–Maruyama scheme for Equation (4.8):

$$u_{n+1}^\kappa = u_n^\kappa + hA^\kappa u_n^\kappa + \Delta L_n^\kappa,$$

where $\Delta L_n^\kappa = L^\kappa(t_{n+1}) - L^\kappa(t_n)$ for all $0 \leq n \leq N-1$. Since $A^\kappa$ is diagonalized by the spherical harmonics $Y_{\ell,m}$, the system decomposes and we can apply the time discretization scheme to Equation (4.6) separately for every coefficient $u_{\ell,m}, \ell \geq 0, -\ell \leq m \leq \ell$. Note that, however, the noise terms $L_{\ell,m}, \ell \geq 0, -\ell \leq m \leq \ell$, remain correlated.

In the spherical setting, convergence analysis for numerical schemes for some SPDEs driven by $Q$-Wiener noise has been performed in Lang and Schwab [2015], Lang and Motschan-Armen [2024], Cohen and Lang [2022] and in Lang et al. [2025] for the $Q$-Lévy setting. For general references on the numerical discretization of SPDEs, we refer to Lord et al. [2014], Kruse [2014], Larsson and Thomée [2003], Thomée [2006], Jentzen and Kloeden [2011] in the $Q$-Wiener case and Barth and Lang [2012] in the $Q$-Lévy case.

## 4.3 Some SPDEs and their physical quantities

Let us now consider the space $\mathcal{H} = L^2(\mathbb{S}^2)$ and denote, for the remainder of this chapter, by $\|\cdot\|$ the norm $\|\cdot\|_{L^2(\mathbb{S}^2)}$. We are interested in three different SPDEs or systems of SPDEs in $\mathcal{H}$, namely

- the *stochastic wave equation*

$$u_{tt}(t) = \Delta_{\mathbb{S}^2} u(t) + \dot{L}^Q(t), \tag{4.9}$$

- the *stochastic Schrödinger equation*

$$iu_t(t) = \Delta_{\mathbb{S}^2} u(t) + \dot{L}^Q(t), \tag{4.10}$$

- the *stochastic Maxwell's equations*

$$\partial_t \begin{pmatrix} E(t) \\ H(t) \end{pmatrix} = \begin{pmatrix} \nabla_{\mathbb{S}^2} \times H(t) \\ \nabla_{\mathbb{S}^2} \times E(t) \end{pmatrix} + \begin{pmatrix} \dot{L}_E^Q(t) \\ \dot{L}_H^Q(t) \end{pmatrix}, \tag{4.11}$$

where $\dot{L}^Q, \dot{L}_E^Q, \dot{L}_H^Q$ denote the formal temporal derivatives of $Q$-Lévy processes. We give a short introduction to all three equations before focusing on the stochastic wave equation in order to illustrate the research questions answered and techniques used in Paper II. Note that most research on these equations has been done in Euclidean settings and we carry them over to the sphere. Details for this can be found in Paper II.

The stochastic wave equation (4.9) is an SPDE that is second-order in time. Analogously to the stochastic harmonic oscillator, we can rewrite Equation (4.9) into a system of SPDEs that match Equation (4.1). We denote by $(u(t), v(t)) = (u(t), u_t(t))$ and obtain, in Itô form,

$$\begin{aligned} \mathrm{d}u(t) &= v(t)\,\mathrm{d}t, \\ \mathrm{d}v(t) &= \Delta_{\mathbb{S}^2} u(t)\,\mathrm{d}t + \mathrm{d}L^Q(t). \end{aligned} \tag{4.12}$$

As the wave equation can be used to describe physical waves, it is reasonable to expect that the system satisfies a conservation of energy. The energy for the wave equation is defined in analogy to Evans [2010, Section 2.4.3] as

$$\mathcal{E}(t) = \frac{1}{2}\|(-\Delta_{\mathbb{S}^2})^{1/2}u(t)\|^2 + \frac{1}{2}\|u_t(t)\|^2. \tag{4.13}$$

It is a physical quantity that is *conserved* in the deterministic wave equation, i. e., it is constant over time and satisfies $\mathcal{E}(t) = \mathcal{E}(0)$ [Evans, 2010, Section 2.4.3]. Adding random noise, however, adds energy to the system over time. Thus, the energy is no longer a conserved quantity for the stochastic wave equation. This was previously considered for the Euclidean case in, e. g., Schurz [2008], Cohen et al. [2013].

The deterministic linear Schrödinger equation, called the *free Schrödinger equation* describes the evolution of the wave function of a quantum particle. In particular, the probability density function for the position of the particle at time $t$ is given by

$$f(t,x) = |u(t,x)|^2,$$

where $u(t)$ is a solution to Equation (4.10) [Lubich, 2008]. Note that the values of the Schrödinger equation are complex, hence $u(t)$ lives in the space $L^2(\mathbb{S}^2;\mathbb{C})$ of square-integrable complex-valued functions on the sphere. We equip this space with the norm $\|\cdot\|_{L^2(\mathbb{S}^2;\mathbb{C})} = \int_{\mathbb{S}^2}|u(t,x)|^2\,\mathrm{d}\sigma(x)$. We further denote by $\bar{u}$ the complex conjugate of $u$.

The free Schrödinger equation has multiple conserved quantities, called the *mass*

$$\mathcal{M}(t) = \|u(t)\|^2_{L^2(\mathbb{S}^2;\mathbb{C})} = \int_{\mathbb{S}^2} f(t,x)\,\mathrm{d}\sigma(x),$$

*energy*

$$\mathcal{E}(t) = \|\nabla_{\mathbb{S}^2}u(t)\|^2_{L^2(\mathbb{S}^2;\mathbb{C}^3)},$$

and *momentum*

$$p(t) = \int_{\mathbb{S}^2} u(t)\nabla_{\mathbb{S}^2}\bar{u}(t) - \bar{u}(t)\nabla_{\mathbb{S}^2}u(t)\,\mathrm{d}x.$$

See, for example, Carles [2008, Section 4.4] for the definition and conservation properties of these quantities in the Euclidean setting.

Lastly, we consider the stochastic Maxwell's equations on the sphere (4.11). These equations are given as a system of equations in $L^2(\mathbb{S}^2)$, where $L^Q_E$ and $L^Q_H$ are both $Q$-Lévy processes. Maxwell's equations originate from electrodynamics, where $E$ and $H$ describe the electric and magnetic field, respectively. A conserved quantity in the deterministic Maxwell's equations is the energy

$$\mathcal{E}(t) = \|E(t)\|^2_{L^2(\mathbb{S}^2)} + \|H(t)\|^2_{L^2(\mathbb{S}^2)}.$$

This is shown, e. g., in Cohen et al. [2020] for the Euclidean setting.

In Paper II, we are concerned with the long-time behavior of the quantities defined above in the presence of additive noise. Inspired by works in the Euclidean setting with $Q$-Wiener noise [Cohen et al., 2013, Anton and Cohen, 2018, Cohen et al., 2020], we prove the linear growth of the quantities of interest in the exact solution and investigate which numerical schemes reproduce this linear growth. To explain the main concepts behind Paper II, we now give example computations for the stochastic wave equation.

## 4.4 Energy of the stochastic wave equation

As we hinted at in Section 4.3, we are interested in the long-time behavior of physical quantities in the three SPDEs (4.9),(4.10) and (4.11) and in their numerical discretizations. Here, we show the ideas underlying these questions and their answers, focusing on the energy of the stochastic wave equation. We consider an unbounded time interval $\mathbb{T} = [0, \infty)$ and show computations for any $t \in \mathbb{T}$.

First, a spectral decomposition of the stochastic wave equation (4.12) with

$$u(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell,m}(t) Y_{\ell,m}, \quad v(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} v_{\ell,m}(t) Y_{\ell,m}$$

yields a system of stochastic harmonic oscillators for $\ell \geq 0, -\ell \leq m \leq \ell$

$$\begin{aligned} \mathrm{d}u_{\ell,m}(t) &= v_{\ell,m}(t)\,\mathrm{d}t \\ \mathrm{d}v_{\ell,m}(t) &= -\lambda_\ell u_{\ell,m}(t)\,\mathrm{d}t + a_{\ell,m}\,\mathrm{d}L_{\ell,m}(t) \end{aligned} \tag{4.14}$$

with initial conditions $u_{\ell,m}(0) = u_0^{\ell,m}, v_{\ell,m}(0) = v_0^{\ell,m}$ [Cohen and Lang, 2022]. Here we have set $\lambda_\ell = \ell(\ell+1)$.

Using Parseval's identity, we can also decompose the energy (4.13) of the stochastic wave equation (4.9) into

$$\mathcal{E}(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathcal{E}_{\ell,m}(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2}(\dot{u}_{\ell,m}(t))^2 + \frac{1}{2}\lambda_\ell (u_{\ell,m}(t))^2. \tag{4.15}$$

Note that the energy of the harmonic oscillator $u_{\ell,m}$ is

$$\mathcal{E}_{\ell,m}(t) = \frac{1}{2}(\dot{u}_{\ell,m}(t))^2 + \frac{1}{2}\lambda_\ell (u_{\ell,m}(t))^2 \tag{4.16}$$

for all $\ell \geq 0, -\ell \leq m \leq \ell$. Thus, the total energy of the stochastic wave equation is the sum of the contributions of all the harmonic oscillators in the spectral expansion.

We now compute the expected energy $\mathbb{E}[\mathcal{E}_{\ell,m}]$ of a stochastic harmonic oscillator $u_{\ell,m}$ for arbitrary $\ell \geq 0, -\ell \leq m \leq \ell$. This was first done by

Melbø and Higham [2004] in the one-dimensional setting and we follow their computations here. Plugging the variation-of-constants formula (2.10) for the stochastic harmonic oscillator (4.14) into the definition of the energy in Equation (4.16), we obtain

$$\begin{aligned}
2\,\mathbb{E}\left[\mathcal{E}_{\ell,m}(t)\right] &= \mathbb{E}\left[(\dot{u}_{\ell,m}(t))^2 + \lambda_\ell (u_{\ell,m}(t))^2\right] \\
&= \mathbb{E}\left[\left(-\sqrt{\lambda_\ell}\sin(\sqrt{\lambda_\ell}t)u_0^{\ell,m} + \cos(\sqrt{\lambda_\ell}t)v_0^{\ell,m}\right.\right. \\
&\qquad \left.\left. + a_{\ell,m}\int_0^t \cos(\sqrt{\lambda_\ell}(t-s))\,\mathrm{d}L_{\ell,m}(s)\right)^2\right] \\
&\quad + \mathbb{E}\left[\lambda_\ell\left(\cos(\sqrt{\lambda_\ell}t)u_0^{\ell,m} + \frac{1}{\sqrt{\lambda_\ell}}\sin(\sqrt{\lambda_\ell}t)v_0^{\ell,m}\right.\right. \\
&\qquad \left.\left. + a_{\ell,m}\int_0^t \sin(\sqrt{\lambda_\ell}(t-s))\,\mathrm{d}L_{\ell,m}(s)\right)^2\right].
\end{aligned}$$

Multiplying out the square, cancelling appropriately and noting that by Equation (2.6) all cross-terms containing an Itô integral vanish, we obtain

$$\begin{aligned}
2\,\mathbb{E}\left[\mathcal{E}_{\ell,m}(t)\right] &= \lambda_\ell \sin^2(\sqrt{\lambda_\ell}t)(u_0^{\ell,m})^2 + \cos^2(\sqrt{\lambda_\ell}t)(v_0^{\ell,m})^2 \\
&\quad + \mathbb{E}\left[a_{\ell,m}^2\left(\int_0^t \cos(\sqrt{\lambda_\ell}(t-s))\,\mathrm{d}L_{\ell,m}(s)\right)^2\right] \\
&\quad + \lambda_\ell \cos^2(\sqrt{\lambda_\ell}t)(u_0^{\ell,m})^2 + \sin^2(\sqrt{\lambda_\ell}t)(v_0^{\ell,m})^2 \\
&\quad + \mathbb{E}\left[a_{\ell,m}^2\left(\int_0^t \sin(\sqrt{\lambda_\ell}(t-s))\,\mathrm{d}L_{\ell,m}(s)\right)^2\right].
\end{aligned}$$

By Itô's isometry (2.7), which holds by the assumptions that $\mathsf{Var}(L_{\ell,m}(1)) = 1$ and $\mathbb{E}\left[L_{\ell,m}(t)\right] = 0$ (see, e. g., Applebaum [2004, Chapter 4]), we have

$$\begin{aligned}
\mathbb{E}\left[a_{\ell,m}^2\left(\int_0^t \sin(\sqrt{\lambda_\ell}(t-s))\,\mathrm{d}L_{\ell,m}(s)\right)^2\right] &= a_{\ell,m}^2\int_0^t \sin^2(\sqrt{\lambda_\ell}(t-s))\,\mathrm{d}s, \\
\mathbb{E}\left[a_{\ell,m}^2\left(\int_0^t \cos(\sqrt{\lambda_\ell}(t-s))\,\mathrm{d}L_{\ell,m}(s)\right)^2\right] &= a_{\ell,m}^2\int_0^t \cos^2(\sqrt{\lambda_\ell}(t-s))\,\mathrm{d}s.
\end{aligned}$$

Using that $\sin^2(x) + \cos^2(x) = 1$ for all $x \in \mathbb{R}$, we now sum the corresponding terms which yields

$$\mathbb{E}\left[\mathcal{E}_{\ell,m}(t)\right] = \frac{1}{2}\left(\lambda_\ell (u_0^{\ell,m})^2 + (v_0^{\ell,m})^2 + t a_{\ell,m}^2\right) = \mathbb{E}\left[\mathcal{E}_{\ell,m}(0)\right] + \frac{t}{2}a_{\ell,m}^2.$$

Summing up according to Equation (4.15), we obtain that

$$\mathbb{E}\left[\mathcal{E}(t)\right] = \mathbb{E}\left[\mathcal{E}(0)\right] + \frac{t}{2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell,m}^2 = \mathbb{E}\left[\mathcal{E}(0)\right] + \frac{t}{2}\operatorname{Tr}(Q). \tag{4.17}$$

We observe a linear growth in the energy with a slope depending on the trace of the covariance operator. Due to the involvement of the trace, this is called a *trace formula* [Schurz, 2008, Cohen et al., 2013, Anton and Cohen, 2018].

Having proved a trace formula for the exact solution of the stochastic wave equation (4.12), we are also interested in how this is reproduced by numerical discretization methods. We first perform a spatial semi-discretization of $u$ and $v$ through spectral truncation as in Equation (4.7). Then, the energy $\mathcal{E}^\kappa$ is given as $\mathcal{E}^\kappa(t) = \sum_{\ell=0}^{\kappa} \sum_{m=-\ell}^{\ell} \mathcal{E}_{\ell,m}(t)$. It satisfies a trace formula with reduced slope given by the trace of $Q^\kappa$, the covariance operator of $L^\kappa$:

$$\mathbb{E}\left[\mathcal{E}^\kappa(t)\right] = \mathbb{E}\left[\mathcal{E}^\kappa(0)\right] + \frac{t}{2} \sum_{\ell=0}^{\kappa} \sum_{m=-\ell}^{\ell} a_{\ell,m}^2 = \mathbb{E}\left[\mathcal{E}^\kappa(0)\right] + \frac{t}{2} \operatorname{Tr}\left(Q^\kappa\right). \tag{4.18}$$

We are further interested in whether applying time-discretization methods to Equation (4.14) reproduces the trace formula (4.18). Let us consider, as Melbø and Higham [2004], the numerical approximation obtained by the Euler–Maruyama scheme for $0 \le n \le N-1$,

$$\begin{aligned} u_{n+1}^{\ell,m} &= u_n^{\ell,m} + h v_n^{\ell,m} \\ v_{n+1}^{\ell,m} &= v_n^{\ell,m} - h\lambda_\ell u_n^{\ell,m} + a_{\ell,m} \Delta L_{\ell,m}^n. \end{aligned}$$

We can compute the energy $\mathcal{E}_{n+1}^{\ell,m} = (v_{n+1}^{\ell,m})^2/2 + \lambda_\ell (u_{n+1}^{\ell,m})^2/2$ recursively now, based on $(u_n^{\ell,m}, v_n^{\ell,m})^T$:

$$\begin{aligned} 2\,\mathbb{E}\left[\mathcal{E}_{n+1}^{\ell,m}\right] &= \mathbb{E}\left[(v_{n+1}^{\ell,m})^2 + \lambda_\ell (u_{n+1}^{\ell,m})^2\right] \\ &= \mathbb{E}\left[(v_n^{\ell,m} - h\lambda_\ell u_n^{\ell,m} + a_{\ell,m}\Delta L_{\ell,m}^n)^2 + \lambda_\ell (u_n^{\ell,m} + h v_n^{\ell,m})^2\right] \\ &= \mathbb{E}\left[(v_n^{\ell,m})^2 + h^2\lambda_\ell^2 (u_n^{\ell,m})^2 + (a_{\ell,m}\Delta L_{\ell,m}^n)^2 - 2 v_n^{\ell,m} h \lambda_\ell u_n^{\ell,m}\right. \\ &\qquad \left. + \lambda_\ell (u_n^{\ell,m})^2 + \lambda_\ell h^2 (v_n^{\ell,m})^2 + 2\lambda_\ell u_n^{\ell,m} h v_n^{\ell,m}\right], \end{aligned}$$

where we already dropped all cross-terms containing $\Delta L_{\ell,m}^n$, since these have vanishing expectation and are, by construction, independent of $(u_n^{\ell,m}, v_n^{\ell,m})^T$. Cancelling out terms, we see

$$\begin{aligned} 2\,\mathbb{E}\left[\mathcal{E}_{n+1}^{\ell,m}\right] &= 2\,\mathbb{E}\left[\mathcal{E}_n^{\ell,m}\right] + \mathbb{E}\left[h^2\lambda_\ell^2 (u_n^{\ell,m})^2 + (a_{\ell,m}\Delta L_{\ell,m}^n)^2 + \lambda_\ell h^2 (v_n^{\ell,m})^2\right] \\ &= 2(1 + h^2\lambda_\ell)\,\mathbb{E}\left[\mathcal{E}_n\right] + a_{\ell,m}^2 h, \end{aligned}$$

where we used that the variance of $\Delta L_{\ell,m}^n$ is $h$. Unraveling this recursion down to $\mathbb{E}\left[\mathcal{E}_0^{\ell,m}\right]$, we see an exponential growth,

$$\mathbb{E}\left[\mathcal{E}_n^{\ell,m}\right] \ge (1 + h^2\lambda_\ell)^n\,\mathbb{E}\left[\mathcal{E}_0^{\ell,m}\right] \ge e^{\frac{1}{2}\lambda_\ell h t_n}\,\mathbb{E}\left[\mathcal{E}_0^{\ell,m}\right],$$

for $\lambda_\ell h^2 < 1$, where we used the inequality $(1 + x/y)^y > e^{xy/(x+y)}$ with $x = nh^2\lambda_\ell, y = n$. Taking a sum analogously to Equation (4.15), using $\lambda_\ell \geq 1$ for all $\ell \geq 1$, and assuming $\lambda_\ell h^2 < 1$ for all $1 \leq \ell \leq \kappa$, we obtain

$$\mathbb{E}\left[\mathcal{E}_n^\kappa\right] = \sum_{\ell=0}^{\kappa} \sum_{m=-\ell}^{\ell} \mathbb{E}\left[\mathcal{E}_n^{\ell,m}\right] \geq e^{\frac{1}{2}ht_n}\, \mathbb{E}\left[\mathcal{E}_0^\kappa - \mathcal{E}_0^{0,0}\right] + \mathbb{E}\left[\mathcal{E}_0^{0,0}\right].$$

We have thereby shown that the Euler–Maruyama scheme does not reproduce the trace formula. Instead, it produces an exponential growth in the energy. This raises the question: Are there numerical discretization methods that can reproduce the long-time behavior of the energy in the exact solution to the stochastic wave equation? Moreover, can similar results be obtained for the physical quantities of interest for the stochastic Schrödinger and stochastic Maxwell's equations? In Paper II, we investigate these questions in detail, using computations analogous to the ones shown in the infinite-dimensional setting, providing theoretical results as well as numerical illustrations.

## 4.5 Summary of Paper II

In Paper II, we consider the long-time behavior of the expectation of the physical quantities introduced in Section 4.3 for the linear stochastic wave equation (4.9), linear stochastic Schrödinger equation (4.10) and linear stochastic Maxwell's equations (4.11) driven by additive $Q$-Lévy noise on the sphere.

For the stochastic wave equation, we prove that the energy of the exact solution satisfies a trace formula. That is, a linear growth of the expected energy dependent on the trace of $Q$. We then investigate the ability of numerical schemes to preserve this trace formula. First, we show that the energy in a spectrally truncated stochastic wave equation satisfies the same trace formula based on the discretized covariance operator. Investigating time-discretization schemes, we prove that the Euler–Maruyama scheme produces an exponential growth in energy. The backward Euler–Maruyama scheme exhibits a dampening behavior, leading to lower energies than for the exact solution. Lastly, we prove that the exponential Euler scheme satisfies the trace formula for the energy for arbitrary time horizons.

We further investigate the same questions for the mass, energy and momentum in the stochastic Schrödinger equation and the energy of the stochastic Maxwell's equations, showing analogous trace formulas. In all cases, we prove that the exponential Euler scheme is able to preserve the long-time behavior of the expected quantities from the exact solution to the SPDE. For the mass and energies we further prove that the Euler–Maruyama and backward Euler–Maruyama fail to do so. Finally, we perform extensive numerical experiments to illustrate our theoretical results.

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