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A Study of Distributional Complexity Measures for Boolean Functions

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A number of complexity measures for Boolean functions have previously been introduced. These include (1) sensitivity, (2) block sensitivity, (3) witness complexity, (4) subcube partition complexity and (5) algorithmic complexity. Each of these is concerned with “worst-case” inputs. It has been shown that there is “asymptotic separation” between these complexity measures and very recently, due to the work of Huang, it has been established that they are all “polynomially related”. In this paper, we study the notion of distributional complexity where the input bits are independent and one considers all of the above notions in expectation. We obtain a number of results concerning distributional complexity measures, among others addressing the above concepts of “asymptotic separation” and being “polynomially related” in this context. We introduce a new distributional complexity measure, local witness complexity, which only makes sense in the distributional context and we also study a new version of algorithmic complexity which involves partial information. Many interesting examples are presented including some related to percolation. The latter connects a number of the recent developments in percolation theory over the last two decades with the study of complexity measures in theoretical computer science.

CCS Concepts: • **Theory of computation** → **Computational complexity and cryptography**; *Randomness, geometry and discrete structures*; • **Mathematics of computing** → *Probability and statistics*.

Additional Key Words and Phrases: Distributional Complexity, Boolean Functions, Asymptotic Separation, Percolation Theory

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1 Introduction

A Boolean function is an arbitrary function

$$f : \{0, 1\}^n \rightarrow \{0, 1\}.$$

Boolean functions arise in a number of areas of mathematics including theoretical computer science and its role in the latter will be the main topic of this paper. A number of concepts have been previously introduced which are designed to measure the “complexity” of a Boolean function; see Subsection 2.1 for five of the standard complexity measures which have been intensively studied (see, e.g., [1, 5]). These are (1) sensitivity, (2) block sensitivity, (3) witness complexity, (4) subcube partition complexity and (5) algorithmic complexity. In all of these cases, the definition is given in terms of “worst case” over the possible input strings from $\{0, 1\}^n$. There is an ordering of these complexity notions which holds for all Boolean functions; see Proposition 2.1. We will often use the term *deterministic complexity measures* in order to distinguish these from the *distributional complexity measures* introduced further down. Two particular directions

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which have been successfully investigated within this field are (1) asymptotic separation and (2) polynomial relations. Asymptotic separation means that for any two of the complexity measures (1)–(5), there exists a sequence of Boolean functions $(f_n)_{n \geq 1}$ so that the ratio of these two complexity measures applied to f_n approaches ∞ (or zero); see Theorem 5.1. The term “polynomial relations” means that all of these complexity measures are related by universal polynomial factors; more precisely, there is a constant C so that each complexity measure is at most any other complexity measure raised to the power C ; see Theorem 6.1. Recently in 2019, Huang obtained the final step of this result by proving the so-called sensitivity conjecture [12].

Rather than considering “worst-case” input strings, which is done in the above notions, one can measure complexity in an average sense. One of the main books on Boolean functions when one considers “average case” notions is [17]. Here one assumes that the input string $\{0, 1\}^n$ is chosen randomly; usually one assumes that the bits are chosen independently, 1 or 0, with probabilities p and $1 - p$ for a given p . Then one considers the various notions mentioned earlier ((1) sensitivity, (2) block sensitivity, (3) witness complexity, (4) subcube partition complexity and (5) algorithmic complexity) on expectation rather than on a “worst-case” input string. We use the term *distributional complexity* in this context; see Subsection 2.2 for the precise definitions. One of the various goals of this paper is to study the notions of “asymptotic separation” and “polynomial relations” in the context of distributional complexity; see Theorem 5.3 for the former and Subsection 6.2 for the latter. Some other related references are [4] and [23].

In Subsection 2.2, we also introduce a new complexity measure, called *local witness complexity*, which does not have an analogue in the deterministic (worst-case) setting. This notion of local witness complexity is very natural from a probabilistic perspective, seemingly more natural than the closely aligned notion of subcube partition. In addition, it often captures the minimal structure needed for certain arguments to work; see Theorem 5.2 for an example of this. An analogous notion of local sets also arises in the study of the so-called Gaussian Free Field.

As in the deterministic case, there is an ordering of these distributional complexity measures, including local witness complexity, which holds for all Boolean functions; see Proposition 2.5. We also give examples demonstrating that this new notion of local witness complexity is distinct from all of the other complexity measures.

We proceed to give a brief summary of different parts of the paper. First, many interesting examples of Boolean functions together with either their deterministic complexity measures or their distributional complexity measures (or both) are given throughout the text. A particularly interesting class of Boolean functions is the class of *percolation functions*; see Subsection 4.2 where these are studied. Within these, the sequence corresponding to crossings of a square is especially interesting (see Example 4.11) since many deep results in percolation theory lead to fairly detailed information concerning the various distributional complexity measures. Second, there is a natural notion of composing Boolean functions, and this naturally gives rise, by iteration, to various sequences of Boolean functions. How these different complexity measures, both in the deterministic and in the distributional sense, behave under composition is recalled (for deterministic) and studied (for distributional) in Section 3. In Subsection 2.3, we examine the distinction between algorithmic and subcube partition complexity using a geometric perspective. In particular, we will see that for subcube partition complexity, one can consider the two parts of the cube where f is 1, respectively 0, separately, but this cannot be done for algorithms. In Subsection 2.4, we introduce a unifying perspective on both subcube partitions and local sets. This is done via the concept of a *stopping set* which is analogous to the concept of a stopping time in probability theory. In Section 7, we introduce a new notion of distributional algorithmic complexity with partial information which appears interesting and has certain similarities with the notion of fractional query algorithms introduced in [11]. In addition, it is potentially relevant for studying asymptotic separation of two of the notions of distributional complexity, namely algorithmic and subcube partition; see end of Section 5.

Results from theoretical computer science have in recent years led to new results (or simplifications of older results) in percolation theory and statistical mechanics, such as so-called sharp thresholds for various models. We therefore hope that this paper will motivate the percolation community to look more systematically at distributional complexity measures and at the same time to motivate the theoretical computer science community to study distributional complexity measures (not only worst-case) even more systematically given that these are very relevant from a probabilistic perspective. In addition, a systematic study of distributional complexities for non-product measures would also be an interesting direction to pursue in future research. For example, an important inequality, called the OSSS inequality, has recently been extended in this direction, leading to the resolution of important conjectures in statistical mechanics [7].

After submitting our paper to arXiv, we learned about the paper [15] which appeared on arXiv three months before ours did and which discusses potential examples for super-polynomial separation in the distributional setting. As such, there is some overlap with our paper. In addition, this paper provides a very nice example of a sequence of balanced monotone functions which have full deterministic algorithmic complexity but bounded distributional algorithmic complexity.

2 Definition and hierarchy of complexity measures

2.1 Deterministic complexity measures

In this subsection, we review deterministic complexity measures for Boolean functions. We refer to the coordinates of the n -dimensional hypercube $\{0, 1\}^n$ as bits. An *algorithm* A is an order of querying the bits (i.e. asking for the values) one by one, where the next query is allowed to depend on the outcome of all previous queries. Note that an algorithm will have queried all bits eventually. For a Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ and a realization of the bits $x \in \{0, 1\}^n$, we write $c_f(A, x)$ for the number of queries made by A on x until f is determined. The *deterministic algorithmic complexity* of f (also known as query complexity or decision tree complexity, denoted by $D(f)$) is defined as

$$a_D(f) = \min_A \max_x c_f(A, x). \quad (1)$$

A *subcube partition* C is a partition of the hypercube $\{0, 1\}^n$ into subcubes. It will be convenient to identify a subcube $C \subseteq \{0, 1\}^n$ with a sequence $(c_1, \dots, c_n) \in \{0, 1, \star\}^n$, where $c_i \in \{0, 1\}$ means that bit i is fixed to be 0 respectively 1, and $c_i = \star$ means that bit i is not fixed. We write I_C for the set of bits fixed by C . Given a subcube partition C , we denote the subcube containing x by $C(x) \in C$ and its co-dimension by

$$c(C, x) = n - \log_2(|C(x)|).$$

In other words, $c(C, x)$ equals $|I_{C(x)}|$, the number of bits fixed by $C(x)$, and $\log_2(|C(x)|)$ is the number of bits not fixed by $C(x)$. We say that a subcube partition C determines f , denoted by $C \sim f$, if it is constant on each subcube $C \in C$. The *deterministic subcube partition complexity* of f (also known as unambiguous certificate complexity, denoted by $UC(f)$) is defined as

$$sc_D(f) = \min_{C \sim f} \max_x c(C, x). \quad (2)$$

A set $W \subseteq [n]$ is called a *witness set* for f and x if $x|_W$ determines f . Given x , we refer to the minimum of $|W|$ over all witness sets W for f and x as the *minimum witness size* and denote it by $w_f(x)$. The *deterministic witness complexity* of f (also known as certificate complexity, denoted by $C(f)$) is defined as

$$w_D(f) = \max_x w_f(x).$$

We denote by $b_f(x)$ the maximum number m of disjoint blocks $B_1, \dots, B_m \subseteq [n]$ such that

$$f(x^{B_i}) \neq f(x), \quad \forall 1 \leq i \leq m, \quad (3)$$

where x^{B_i} denotes the realization of the bits obtained from x by flipping all bits in B_i . The *deterministic block sensitivity* of f is defined as

$$b_D(f) = \max_x b_f(x).$$

Finally, we denote by $s_f(x)$ the number of bits i such that $f(x^i) \neq f(x)$, where $x^i := x^{\{i\}}$. These bits are called *pivotal* for f at x . The *deterministic sensitivity* of f is defined as

$$s_D(f) = \max_x s_f(x).$$

Although well-known, we give the proof of the following proposition for the sake of the reader.

PROPOSITION 2.1. *For every Boolean function f , the deterministic complexities satisfy*

$$a_D(f) \geq sc_D(f) \geq w_D(f) \geq b_D(f) \geq s_D(f).$$

PROOF. We prove the four inequalities one by one. First, consider an algorithm A achieving the minimum in (1). When stopped at the moment of determining f , it induces a subcube partition $C(A) \sim f$ with $c(C(A), x) = c_f(A, x)$ for every x . Taking the maximum over x , this implies $a_D(f) \geq sc_D(f)$.

Second, consider a subcube partition $C \sim f$ achieving the minimum (2). For every x , the fixed coordinates of $C(x)$ are a witness set $W_x(C)$ for f and x . Since $c(C, x) = |W_x(C)| \geq w_f(x)$, taking the maximum over x yields $sc_D(f) \geq w_D(f)$.

Third, fix any x and consider a maximal collection of blocks B_1, \dots, B_m as in the definition of $b_f(x)$. We note that any witness set W for f and x must intersect B_i for every $1 \leq i \leq m$. This implies $w_f(x) \geq m = b_f(x)$ and hence, $w_D(f) \geq b_D(f)$.

Finally, for every x , the pivotal bits provide $s_f(x)$ disjoint blocks $B_1, \dots, B_{s_f(x)}$ of size 1 satisfying (3). This implies $s_f(x) \leq b_f(x)$, and hence $s_D(f) \leq b_D(f)$. \square

The following examples show that the five complexity measures are all distinct.

Example 2.2. For n being a multiple of 4, define the Boolean function g on n bits by

$$g(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } \sum_{i=1}^n x_i \in \{\frac{n}{2}, \frac{n}{2} + 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

This function appeared in [16] to show that $w_D(g) > b_D(g) > s_D(g)$. It has deterministic complexities

$$s_D(g) = \frac{n}{2} + 2, \quad b_D(g) = \max\left\{\frac{3n}{4}, \frac{n}{2} + 2\right\}, \quad \text{and } w_D(g) = n - 1.$$

To see this, note that $s_g(x) = \frac{n}{2} + 2$ if x has exactly $\frac{n}{2} + 2$ bits with value 1. Moreover, $b_g(x) = \frac{3n}{4}$ and $w_g(x) = n - 1$ if x has exactly $\frac{n}{2}$ bits with value 1.

We note that the following small proposition also implies $a_D(g) = n$, which could otherwise be seen using an adversary argument.

PROPOSITION 2.3. *For Example 2.2, $sc_D(g) = n$.*

PROOF. Denote by S_i the set containing all x having exactly i bits with value 1. Towards a contradiction, assume that there exists a subcube partition C containing only subcubes fixing at most $n - 1$ bits. Note that for any $x \in S_{n/2}$, $C(x)$

must fix all $n/2$ bits with value 1 and exactly $n/2 - 1$ bits with value 0 since g would otherwise not be constant on $C(x)$. Therefore, $C(x) \neq C(y)$ for all $x, y \in S_{n/2}$ with $x \neq y$. However, note that every such cube $C(x)$ contains an element $x' \in S_{n/2+1}$. Since $|S_{n/2+1}| < |S_{n/2}|$, the cubes cannot be disjoint, a contradiction. \square

Example 2.4. Savický's function on $n = 4$ bits, which was introduced in [24, 25] and which we call MAJ_4 according to [14], is defined by

$$\text{MAJ}_4(x_1, x_2, x_3, x_4) = \begin{cases} 1 & \text{if } 2x_1 + \sum_{i=2}^4 x_i \geq 3, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this is a majority function with the first bit being decisive in the case of a tie. It is easy to see that the deterministic complexities are as follows:

$$\begin{aligned} s_D(\text{MAJ}_4) &= b_D(\text{MAJ}_4) = w_D(\text{MAJ}_4) = sc_D(\text{MAJ}_4) = 3, \\ a_D(\text{MAJ}_4) &= 4, \end{aligned}$$

once one observes that

$$C := \{(\star, 1, 1, 1), (\star, 0, 0, 0), (1, 1, 0, \star), (0, 1, 0, \star), (1, \star, 1, 0), (0, \star, 1, 0), (1, 0, \star, 1), (0, 0, \star, 1)\}$$

is a subcube partition.

2.2 Distributional complexity measures

In this subsection, we define *distributional* complexity measures. We denote the product measure on $\{0, 1\}^n$ with marginal $p \in [0, 1]$ by π_p . We begin with introducing the distributional analogues of the five deterministic complexity measures encountered in the previous subsection. The π_p -*distributional algorithmic complexity* is defined as

$$a(f, \pi_p) = \min_A \mathbb{E}_{x \sim \pi_p} [c_f(A, x)],$$

the π_p -*distributional subcube partition complexity* as

$$sc(f, \pi_p) = \min_{C \sim f} \mathbb{E}_{x \sim \pi_p} [c(C, x)],$$

the π_p -*distributional witness complexity* as

$$w(f, \pi_p) = \mathbb{E}_{x \sim \pi_p} [w_f(x)],$$

the π_p -*distributional block sensitivity* as

$$b(f, \pi_p) = \mathbb{E}_{x \sim \pi_p} [b_f(x)],$$

and the π_p -*distributional sensitivity* (also known as the total influence) as

$$s(f, \pi_p) = \mathbb{E}_{x \sim \pi_p} [s_f(x)].$$

The bits queried by an algorithm A until f is determined and the bits fixed by a subcube partition naturally form random subsets of $[n]$ which are measurable with respect to x . In the following, we also consider random sets $\mathcal{I} \subseteq [n]$ which are not necessarily measurable with respect to x . However, any random set is implicitly assumed to be defined on the *same* probability space as x , and we continue to write $\mathbb{E}_{x \sim \pi_p}$ for the corresponding expectation. A random set $\mathcal{I} \subseteq [n]$ is called a *witness set* for f , denoted $\mathcal{I} \sim f$, if $x|_{\mathcal{I}}$ almost surely determines f . Clearly, we have $w(f, \pi_p) = \min_{\mathcal{I} \sim f} \mathbb{E}_{x \sim \pi_p} [|\mathcal{I}|]$.

We call a random set $\mathcal{I} \subseteq [n]$ *local* if for every fixed $J \subseteq [n]$,

$$\sigma(\{\mathcal{I} = J\}, (x_i : i \in J)) \text{ is independent of } \sigma(x_i : i \notin J).$$

In words, conditional on the random set \mathcal{I} being equal to J and on the values of x on J , the bits outside of J are still independent $\text{Ber}(p)$ -distributed. We are now in position to define a sixth distributional complexity measure, which has no deterministic analogue. The π_p -*distributional local witness complexity* is defined as

$$\ell(f, \pi_p) = \min_{\mathcal{I} \sim f, \mathcal{I} \text{ local}} \mathbb{E}_{x \sim \pi_p} [|\mathcal{I}|].$$

PROPOSITION 2.5. *For every Boolean function f and for every $p \in [0, 1]$, the distributional complexities satisfy*

$$a(f, \pi_p) \geq sc(f, \pi_p) \geq \ell(f, \pi_p) \geq w(f, \pi_p) \geq b(f, \pi_p) \geq s(f, \pi_p).$$

PROOF. The inequalities $a(f, \pi_p) \geq sc(f, \pi_p)$ and $w(f, \pi_p) \geq b(f, \pi_p) \geq s(f, \pi_p)$ can be obtained analogously to the deterministic case in Proposition 2.1, by simply replacing taking the maximum over x by expectations with respect to π_p . The inequality $\ell(f, \pi_p) \geq w(f, \pi_p)$ is immediate from the definition. Finally, we establish the inequality $sc(f, \pi_p) \geq \ell(f, \pi_p)$. Given a subcube partition \mathcal{C} , we write $\mathcal{I}_{\mathcal{C}}$ for the random set defined by $\mathcal{I}_{\mathcal{C}}(x) := I_{\mathcal{C}}(x)$. It is straightforward to check that $\mathcal{I}_{\mathcal{C}}$ is local. As $\mathcal{C} \sim f$ implies $\mathcal{I}_{\mathcal{C}} \sim f$, the desired inequality follows. \square

Examples 2.6, 2.8 and 2.9 show that the six complexity measures are all distinct. In Section 5, we will study the stronger notion of asymptotic separation.

Example 2.6. The MAJORITY function on $n = 3$ bits, defined by

$$\text{MAJ}_3(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } \sum_{i=1}^3 x_i \geq 2, \\ 0 & \text{if } \sum_{i=1}^3 x_i \leq 1, \end{cases}$$

has distributional complexities

$$\begin{aligned} s(\text{MAJ}_3, \pi_p) &= 2 - 2p^3 - 2(1-p)^3, \\ b(\text{MAJ}_3, \pi_p) &= 2 - p^3 - (1-p)^3, \\ w(\text{MAJ}_3, \pi_p) &= 2, \\ sc(\text{MAJ}_3, \pi_p) &= a(\text{MAJ}_3, \pi_p) = 2 + 2p(1-p). \end{aligned}$$

Since the notion of local witness complexity involves an infinite number of choices for the joint distribution of (\mathcal{I}, x) , its computation seems to be more difficult than the other distributional complexities. However, in some cases, this complexity can be computed explicitly as the following proposition demonstrates.

PROPOSITION 2.7. $\ell(\text{MAJ}_3, \pi_p) = 2 + 2p(1-p)$.

PROOF. The upper bound follows directly from Proposition 2.5 and the value of $sc(\text{MAJ}_3, \pi_p)$ given above. For the lower bound, let \mathcal{I} be a local witness set for MAJ_3 . We analyze the constraints on the joint distribution of (\mathcal{I}, x) imposed by being a witness set and by locality. First, since \mathcal{I} is a witness set for MAJ_3 , we must have $\{i, j\} \subseteq \mathcal{I}$ for some $i \neq j \in [3]$ satisfying $x_i = x_j$. Thus, there are two possibilities for \mathcal{I} if the bits are not all equal (e.g., $\mathcal{I} = \{1, 2\}$ or $\mathcal{I} = \{1, 2, 3\}$ if $x = (1, 1, 0)$), and four possibilities for \mathcal{I} if the bits are all equal. Second, since \mathcal{I} is local, we have for

every permutation (i, j, k) of $(1, 2, 3)$ and every $z \in \{0, 1\}$,

$$\mathbb{P}_{x \sim \pi_p} [\mathcal{I} = \{i, j\}, x_i = x_j = z, x_k = 1] = \frac{p}{1-p} \cdot \mathbb{P}_{x \sim \pi_p} [\mathcal{I} = \{i, j\}, x_i = x_j = z, x_k = 0]. \quad (4)$$

For abbreviation, we define

$$\beta_{\{i,j\}}^z := \mathbb{P}_{x \sim \pi_p} [\mathcal{I} = \{i, j\}, x_i = x_j = z, x_k = 1 - z] \in [0, p^{1+z}(1-p)^{2-z}],$$

where (i, j, k) is a permutation of $(1, 2, 3)$ and $z \in \{0, 1\}$, and $\beta^z := \beta_{\{1,2\}}^z + \beta_{\{2,3\}}^z + \beta_{\{1,3\}}^z$. The main observation is now that (4) implies

$$\frac{p}{1-p} \beta^1 = \mathbb{P}_{x \sim \pi_p} [|\mathcal{I}| = 2, x = (1, 1, 1)] \leq p^3 \implies \beta^1 \leq p^2(1-p),$$

and analogously, $\beta^0 \leq p(1-p)^2$.

Finally, we express the expected size of \mathcal{I} in terms of β^1 and β^0 .

$$\begin{aligned} \mathbb{E}_{x \sim \pi_p} [|\mathcal{I}|] &= 2\beta^1 + 3 \left(3p^2(1-p) - \beta^1 \right) + 2 \frac{p}{1-p} \beta^1 + 3 \left(p^3 - \frac{p}{1-p} \beta^1 \right) \\ &\quad + 2\beta^0 + 3 \left(3p(1-p)^2 - \beta^0 \right) + 2 \frac{1-p}{p} \beta^0 + 3 \left((1-p)^3 - \frac{1-p}{p} \beta^0 \right) \\ &= 3 - \left(\beta^1 + \frac{p}{1-p} \beta^1 \right) - \left(\beta^0 + \frac{1-p}{p} \beta^0 \right) = 3 - \frac{\beta^1}{1-p} - \frac{\beta^0}{p}. \end{aligned}$$

Using the previously obtained upper bounds on β^1 and β^0 , we conclude that

$$\mathbb{E}_{x \sim \pi_p} [|\mathcal{I}|] \geq 3 - p^2 - (1-p)^2 = 2 + 2p(1-p),$$

and the lower bound on $\ell(\text{MAJ}_3, \pi_p)$ follows. \square

It remains to separate $sc(\cdot, \pi_p)$ and $a(\cdot, \pi_p)$ as well as $\ell(\cdot, \pi_p)$ and $sc(\cdot, \pi_p)$, which we will do in the following two examples.

Example 2.8. The ALL-EQUAL function on $n = 3$ bits, defined by

$$\text{A-EQ}_3(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } x_1 = x_2 = x_3, \\ 0 & \text{otherwise,} \end{cases}$$

has distributional complexities

$$\begin{aligned} s(\text{A-EQ}_3, \pi_p) &= 1 + 2p^3 + 2(1-p)^3, \\ b(\text{A-EQ}_3, \pi_p) &= w(\text{A-EQ}_3, \pi_p) = \ell(\text{A-EQ}_3, \pi_p) = sc(\text{A-EQ}_3, \pi_p) = 2 + p^3 + (1-p)^3, \\ a(\text{A-EQ}_3, \pi_p) &= 2 + p^2 + (1-p)^2. \end{aligned}$$

An optimal subcube partition is given by

$$C = \{(0, 0, 0), (1, 0, \star), (\star, 1, 0), (0, \star, 1), (1, 1, 1)\}.$$

Example 2.9. The Boolean function g on $n = 4$ bits, defined by

$$g(x_1, \dots, x_4) = \begin{cases} 1 & \text{if } (x_1, \dots, x_4) \in \{(1, 0, 0, 1), (0, 0, 0, 1), (0, 1, 0, 1), (0, 1, 1, 0)\}, \\ 0 & \text{otherwise,} \end{cases}$$

has distributional complexities

$$s(g, \pi_{1/2}) = 3/2, \quad b(g, \pi_{1/2}) = 9/4, \quad \text{and } w(g, \pi_{1/2}) = 37/16.$$

PROPOSITION 2.10. *For Example 2.9, $\ell(g, \pi_{1/2}) \leq 21/8$ and $a(g, \pi_{1/2}) = sc(g, \pi_{1/2}) = 11/4$.*

PROOF. To see this, we first construct a local witness set of expected size $21/8$. Define

$$\mathcal{I}_0(x_1, \dots, x_4) = \begin{cases} \{1, 2\} & \text{if } x_1 = x_2 = 1, \\ \{2, 3\} & \text{if } x_2 = 0, x_3 = 1, \\ \{1, 2, 3, 4\} & \text{if } x_1 = 0, x_2 = 1, \\ \{2, 3, 4\} & \text{if } x_2 = x_3 = 0, x_4 = 1, \\ \{2, 4\} & \text{if } x_2 = x_3 = x_4 = 0, \end{cases}$$

and

$$\mathcal{I}_1(x_1, \dots, x_4) = \begin{cases} \{3, 4\} & \text{if } x_3 = x_4, \\ \{1, 2, 3, 4\} & \text{if } x_2 = 1, x_3 \neq x_4, \\ \{2, 3, 4\} & \text{if } x_2 = x_3 = 0, x_4 = 1, \\ \{2, 4\} & \text{if } x_2 = x_4 = 0, x_3 = 1. \end{cases}$$

It is easy to see that \mathcal{I}_0 and \mathcal{I}_1 are witness sets of size $\mathbb{E}_{x \sim \pi_{1/2}}[|\mathcal{I}_0|] = \mathbb{E}_{x \sim \pi_{1/2}}[|\mathcal{I}_1|] = 21/8$, but not local. However, for a $\text{Ber}(1/2)$ -distributed random variable G that is independent of x , one can check that the random set \mathcal{I}_G is a local witness set.

Second, we describe an algorithm whose expected number of queries is $11/4$. The algorithm begins with querying x_3 and x_4 . Note that $f = 0$ if $x_3 = x_4$. The algorithm then queries x_1 . Note that $f = 0$ (resp. $f = 1$) if $(x_1, x_3, x_4) = (1, 1, 0)$ (resp. $(0, 0, 1)$). Lastly, the algorithm queries x_2 .

Third, we need to argue that every subcube partition determining f fixes on average at least $11/4$ bits. We study the subcube partitions of $\{x : f(x) = 1\}$ and $\{x : f(x) = 0\}$ separately. It is easy to see that $\{(\star, 0, 0, 1), (0, 1, 0, 1), (0, 1, 1, 0)\}$ is an optimal subcube partition of $\{x : f(x) = 1\}$, and it fixes on average $7/2$ bits (conditional on $f = 1$). Since $11/4 = (1/4) \cdot (7/2) + (3/4) \cdot (5/2)$, we are thus left with showing that an optimal subcube partition of $\{x : f(x) = 0\}$ fixes on average at least $5/2$ bits (conditional on $f = 0$). We first observe that every subcube in $\{x : f(x) = 0\}$ must fix at least 2 bits, i.e. has size at most 4. Next, if a subcube partition of $\{x : f(x) = 0\}$ contains only one (resp. no) subcube of size 4, it fixes on average at least $8/3$ (resp. 3) bits, which is larger than $5/2$. Otherwise, the subcube partition of $\{x : f(x) = 0\}$ contains at least two subcubes of size 4. A case-by-case analysis allows to verify that for all possible combinations of two subcubes of size 4, the remaining 4 configurations can neither be combined into one subcube of size 4 nor into two subcubes of size 2. This establishes that any subcube partition of $\{x : f(x) = 0\}$ fixes on average at least $(2 \cdot 8 + 3 \cdot 2 + 4 \cdot 2)/12 = 5/2$ bits, and thereby completes the argument. \square

We note that for any Boolean function f , the separation $s_D(f) < b_D(f) < w_D(f)$ directly implies the separation $s(f, \pi_p) < b(f, \pi_p) < w(f, \pi_p)$ for $p \in (0, 1)$ since $s_f(x) \leq b_f(x) \leq w_f(x)$ for every x . However, the converse is generally not true which can be seen, for example, by considering the majority function on three bits satisfying $s(\text{MAJ}_3, \pi_p) < b(\text{MAJ}_3, \pi_p) < w(\text{MAJ}_3, \pi_p)$ but $s_D(\text{MAJ}_3) = b_D(\text{MAJ}_3) = w_D(\text{MAJ}_3) = 2$.

We also note that the separation $sc_D(\text{MAJ}_4) < a_D(\text{MAJ}_4)$ in Example 2.4 does not imply the corresponding separation in the distributional case, and it can be checked that $sc(\text{MAJ}_4, \pi_{1/2}) = a(\text{MAJ}_4, \pi_{1/2})$.

We will see in Proposition 4.9 that distributional witness complexity and distributional block sensitivity are equal for a large class of functions. However, asking for equality of distributional sensitivity and distributional block sensitivity severely restricts which Boolean function we have and we will prove below that this equality implies that the function is a parity function (or negative parity) on some collection of the bits. Note that $s(f, \pi_p) = b(f, \pi_p)$ implies that the sensitivity and block sensitivity agree on every bit string which in term implies that every bit string has a pivotal bit (unless f is constant). The last thing certainly does not imply we are a parity function on some subset of the bits since we can take any Boolean function and XOR it with one bit, guaranteeing at least one pivotal bit.

PROPOSITION 2.11. *Let f be a Boolean function and $p \in (0, 1)$. The following are equivalent:*

- (i) f or $1 - f$ is the PARITY function on some subset of the bits,
- (ii) $b(f, \pi_p) = s(f, \pi_p)$.

In particular, $b(f, \pi_p) = s(f, \pi_p)$ implies that all distributional complexity measures for f are equal. It would be interesting to obtain approximate versions of this result. However, note that for the MAJORITY function on n bits, the distributional sensitivity and block sensitivity differ only by a multiplicative factor $\log(n)$ whereas the distributional sensitivity and witness complexity differ by a multiplicative factor \sqrt{n} (see Example 4.5).

PROOF. The implication (i) \implies (ii) being trivial, we only need to prove (ii) \implies (i). Without loss of generality, fix a Boolean function f that depends on every bit and let n denote the number of bits. First, since $b_f(x) \geq s_f(x)$ for every $x \in \{0, 1\}^n$, the equality in (ii) implies that for every $x \in \{0, 1\}^n$,

$$b_f(x) = s_f(x).$$

Given f , we can consider the hypercube $\{0, 1\}^n$ as a graph, where x and y are connected by an edge if x and y differ in exactly one location and $f(x) = f(y)$. We then denote by $D(x)$ the connected component of x and by $S_f(x)$ the set of pivotal bits for f at x . Now, let us argue that for every $x \in \{0, 1\}^n$,

$$D(x) = \{y \in \{0, 1\}^n : y_i = x_i, \forall i \in S_f(x)\} =: \tilde{D}(x). \quad (5)$$

To this end, we fix any $x \in \{0, 1\}^n$. Since $s_f(x) = b_f(x)$, we must have $f(x^B) = f(x)$ for every $B \subseteq [n] \setminus S_f(x)$, and so it follows that $\tilde{D}(x) \subseteq D(x)$. Note that this implies $S_f(y) \subseteq S_f(x)$ for every $y \in \tilde{D}(x)$. Towards a contradiction, assume that $j \in S_f(x) \setminus S_f(y)$. Then the block

$$B := \{j\} \cup \{i \in [n] \setminus S_f(x) : x_i \neq y_i\}$$

satisfies $f(y) = f(x) \neq f(x^j) = f(y^B)$ and $B \cap S_f(y) = \emptyset$, which contradicts $b_f(y) = s_f(y)$. Thus, $S_f(y) = S_f(x)$ for every $y \in \tilde{D}(x)$ and so (5) follows. In other words, we have established that every connected component is a subcube.

To conclude from here, we assume without loss of generality that $f(0) = 0$ and define for every $I \subseteq S_f(0)$,

$$D_I := \{y \in \{0, 1\}^n : y_i = 0, \forall i \in S_f(0) \setminus I, \text{ and } y_i = 1, \forall i \in I\}.$$

Note that $D_\emptyset = D(0)$ and that the graph distance between D_I and D_J is equal to $|I \Delta J|$. Using (5), it is now straightforward to check by induction on $|I|$ that

$$f|_{D_I} \equiv |I| \pmod{2}, \quad \forall I \subseteq S_f(0). \quad (6)$$

Hence, f does not depend on the bits in $[n] \setminus S_f(0)$, which implies that $S_f(0) = [n]$. Combining this with (6), we conclude that f is the PARITY function. \square

For monotone Boolean functions, the next proposition shows that asking for equality of distributional witness complexity and subcube partition complexity implies that the function is a dictator function on one of its bits. It would be interesting to know if equality of distributional witness complexity and local witness complexity is sufficient.

PROPOSITION 2.12. *Let f be a monotone, non-constant Boolean function and let $p \in (0, 1)$. The following are equivalent:*

- (i) f is a DICTATOR function (i.e. $f(x) = x_i$ for some $i \in [n]$),
- (ii) $sc(f, \pi_p) = w(f, \pi_p)$.

PROOF. The implication (i) \implies (ii) being trivial, we proceed with proving (ii) \implies (i). Assume that f is not a DICTATOR function. Fix an arbitrary subcube partition $C \sim f$, and denote by C_0 (resp. C_1) the subcube containing $(0, \dots, 0)$ (resp. $(1, \dots, 1)$). Since f is monotone and non-constant, we have $f|_{C_0} \equiv 0$ and $f|_{C_1} \equiv 1$. Since f is not a DICTATOR function, it is easy to see that $C \neq \{C_0, C_1\}$. Now, let $C' \in C$ be a subcube different from C_0 and C_1 , and assume without loss of generality that $f|_{C'} \equiv 0$. Since $C' \cap C_0 = \emptyset$, the set $I_{C'}$ of bits fixed by C' must contain some bit i' that is fixed to be 1. However, by monotonicity of f , the set $I_{C'} \setminus \{i'\}$ is a witness set for f and $x \in C'$. This establishes that $sc(f, \pi_p) > w(f, \pi_p)$. \square

QUESTION 2.1. *Does $a(f, \pi_p) = sc(f, \pi_p)$ hold for every monotone Boolean function f and for every $p \in (0, 1)$?*

The following standard example shows why monotonicity is needed in the above proposition.

Example 2.13. The ADDRESS function on $n = m + 2^m$ bits, defined by

$$\text{ADDRESS}_m(x_1, \dots, x_m, y_0, \dots, y_{2^m-1}) = y_{\sum_{i=0}^{m-1} x_i 2^i},$$

has distributional complexities

$$w(\text{ADDRESS}_m, \pi_p) = a(\text{ADDRESS}_m, \pi_p) = m + 1.$$

2.3 A geometric perspective on subcube partitions and algorithms

In this subsection, we follow a more geometric approach. One of the first things we want to understand is when a particular subcube partition arises from an algorithm. We make the following observation: If a subcube partition arises from an algorithm, then it must be the case that (1) there is some hyperplane orthogonal to one of the coordinate directions that splits the cube in half and crosses no edge with endpoints belonging to the same subcube and (2) each of the two subcube partitions induced by this splitting arises from an algorithm. Therefore, Figure 1a illustrates a subcube partition that does not arise from an algorithm.

It follows that in order to show that $sc(f, \pi_p) < a(f, \pi_p)$ for a given Boolean function f , one needs to show that no optimal subcube partition with respect to π_p determining f arises from an algorithm. Since it is easy to verify that there are only two optimal subcube partitions for the ALL-EQUAL function on $n = 3$ bits from Example 2.8 (see Figure 1a for a representation of the optimal subcube partition $\{(0, 0, 0), (1, 0, \star), (\star, 1, 0), (0, \star, 1), (1, 1, 1)\}$; the second one being essentially the same), we see that $sc(\text{A-EQ}_3, \pi_p) < a(\text{A-EQ}_3, \pi_p)$.

We now point out an interesting difference between subcube partitions and algorithms for Boolean functions. Any subcube partition of $\{0, 1\}^n$ which determines a Boolean function f induces in a natural way a subcube partition of the

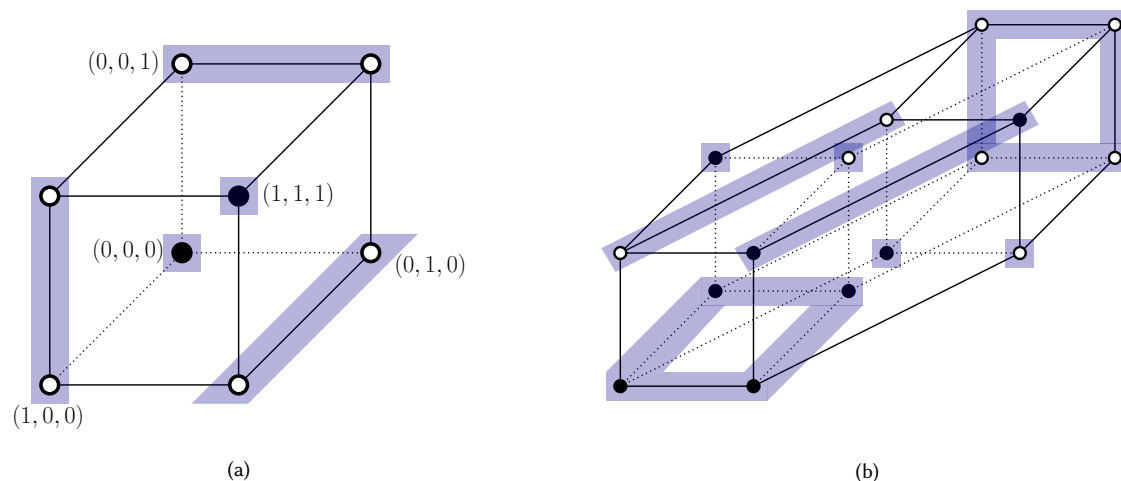


Fig. 1. Illustrations of (a) the ALL-EQUAL function on $n = 3$ bits (see Example 2.8) and (b) the function h on $n = 4$ bits (see Equation (7)) by marking inputs with function value 1 (resp. 0) using a disk with black (resp. white) filling. In (b), the first three directions are as in (a) and additionally the three-dimensional cube in the front (resp. back) contains the inputs with $x_4 = 1$ (resp. 0). The illustrations also show optimal subcube partitions for the respective functions with respect to $\pi_{1/2}$.

subset $f^{-1}(1)$ and a subcube partition of the subset $f^{-1}(0)$. Moreover, it is clear that when trying to find an optimal subcube partition for a Boolean function f (either deterministically in worst case or distributionally with respect to some π_p), one can work separately on $f^{-1}(1)$ and $f^{-1}(0)$, finding optimal solutions for the two parts separately and then just combining them. When we move to algorithms, while it does make sense to talk about optimal algorithms on $f^{-1}(1)$ and $f^{-1}(0)$ separately, one cannot *combine* such algorithms into an algorithm for the whole cube in the same way. (For distributional complexity, optimal algorithms on $f^{-1}(1)$ and $f^{-1}(0)$ means precisely finding algorithms which minimize the expected number of queries made conditional on $f^{-1}(1)$ and $f^{-1}(0)$ respectively.) The following interesting example on 4 bits illustrates such an example.

Define the Boolean function h on $n = 4$ bits by

$$h(x_1, \dots, x_4) = \begin{cases} 1 & \text{if } (x_1, \dots, x_4) \in \{(\star, \star, 0, 1), (1, 1, 1, \star), (1, 0, 0, 0), (0, 0, 1, 1)\}, \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

We refer the reader to Figure 1b for an illustration of h and the unique optimal subcube partition with respect to $\pi_{1/2}$. We leave it to the reader to check that $sc(h, \pi_{1/2}) = 11/4$ and that $a(h, \pi_{1/2}) = 3 > 11/4$, the latter being obtained by a case-by-case analysis. However, if we condition on $h^{-1}(1)$ or $h^{-1}(0)$, one can check that the conditional subcube partition complexities are still $11/4$ but more interestingly, the optimal algorithms on the two pieces also achieve $11/4$. On $h^{-1}(1)$, one sees this by taking the algorithm which first queries x_3 , second x_1 (resp. x_4) if $x_3 = 1$ (resp. 0) and then continues in the obvious way. On $h^{-1}(0)$, one sees this by taking the same algorithm but with the roles of x_1 and x_3 being switched. However, there is no meaningful way to combine these algorithms for the whole cube. Note that the optimal algorithm on $h^{-1}(1)$ above yields a (non-optimal) expected number of queries of $13/4$ on $h^{-1}(0)$ and vice versa. In contrast, for the ALL-EQUAL function, on $A-EQ_3^{-1}(0)$, the optimal subcube partition and algorithmic costs differ (but are equal on $A-EQ_3^{-1}(1)$).

2.4 A stopping set perspective on subcube partitions and local sets

In the proof of Proposition 2.5, we have already seen that any subcube partition naturally induces a local set, and this allowed us to establish the comparison $sc(f, \pi_p) \geq \ell(f, \pi_p)$. Here, we develop a unifying perspective on subcube partitions and local sets.

For the following discussion, we fix a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which the bits $x \sim \pi_p$ are defined. A collection $\mathbb{F} := (\mathcal{F}_J)_{J \subseteq [n]}$ of sub- σ -algebras of \mathcal{F} is a *filtration* if $\mathcal{F}_{J_1} \subseteq \mathcal{F}_{J_2}$ for all $J_1 \subseteq J_2$, and a random set \mathcal{I} is a *stopping set with respect to \mathbb{F}* if $\{\mathcal{I} = J\} \in \mathcal{F}_J$ for every fixed $J \subseteq [n]$. In addition, we call x an *independent field with respect to \mathbb{F}* if for every fixed $J \subseteq [n]$,

$$\sigma(x_i : i \in J) \subseteq \mathcal{F}_J, \text{ and } \sigma(x_i : i \notin J) \text{ is independent of } \mathcal{F}_J.$$

PROPOSITION 2.14. *Fix $(\Omega, \mathcal{F}, \mathbb{P})$ on which the bits $x \sim \pi_p$ are defined. For a random set $\mathcal{I} \subseteq [n]$, the following are equivalent:*

- (i) *There exists a filtration \mathbb{F} such that \mathcal{I} is a stopping set with respect to \mathbb{F} and x is an independent field with respect to \mathbb{F} ,*
- (ii) *\mathcal{I} is local.*

PROOF. The implication (i) \implies (ii) is straightforward from the definitions. To establish the converse, we define the filtration $\mathbb{F} := (\mathcal{F}_J)_{J \subseteq [n]}$ by

$$\mathcal{F}_J := \sigma(\{\mathcal{I} = J'\} : J' \subseteq J, (x_i : i \in J)).$$

\mathcal{I} is clearly a stopping set with respect to \mathbb{F} , and we are left with showing that x is an independent field with respect to \mathbb{F} . First, $\sigma(x_i : i \in J) \subseteq \mathcal{F}_J$ follows from the definition. Second, we fix any $A \in \mathcal{F}_J$ and $B \in \sigma(x_i : i \notin J)$ and, using locality, aim to show $\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]$. It is possible to choose, for $J' \subseteq J$, sets $\mathcal{Y}(J', A) \subseteq \{0, 1\}^J$ and $\mathcal{Y}^*(A) \subseteq \{0, 1\}^J$ such that

$$A = \left(\bigsqcup_{J' \subseteq J} \bigsqcup_{y|_J \in \mathcal{Y}(J', A)} \{\mathcal{I} = J'\} \cap \{x|_J = y|_J\} \right) \sqcup \left(\bigsqcup_{y|_J \in \mathcal{Y}^*(A)} \{\mathcal{I} \not\subseteq J\} \cap \{x|_J = y|_J\} \right),$$

and $\mathcal{Y}(B) \subseteq \{0, 1\}^{J^c}$ such that

$$B = \bigsqcup_{y|_{J^c} \in \mathcal{Y}(B)} \{x|_{J^c} = y|_{J^c}\}.$$

Using these partitions of A and B , we get

$$\begin{aligned} \mathbb{P}[A \cap B] &= \sum_{J' : J' \subseteq J} \sum_{y|_J \in \mathcal{Y}(J', A)} \sum_{y|_{J^c} \in \mathcal{Y}(B)} \mathbb{P}[\{\mathcal{I} = J'\} \cap \{x = y\}] \\ &\quad + \sum_{y|_J \in \mathcal{Y}^*(A)} \sum_{y|_{J^c} \in \mathcal{Y}(B)} \mathbb{P}[\{\mathcal{I} \not\subseteq J\} \cap \{x = y\}]. \end{aligned} \tag{8}$$

Now, for every $J' \subseteq J$ and $y \in \{0, 1\}^n$,

$$\begin{aligned} \mathbb{P}[\{\mathcal{I} = J'\} \cap \{x = y\}] &= \mathbb{P}[\{\mathcal{I} = J'\} \cap \{x|_{J'} = y|_{J'}\}] \cdot \mathbb{P}[x|_{(J')^c} = y|_{(J')^c}] \\ &= \mathbb{P}[\{\mathcal{I} = J'\} \cap \{x|_{J'} = y|_{J'}\}] \cdot \mathbb{P}[x|_{J \setminus J'} = y|_{J \setminus J'}] \cdot \mathbb{P}[x|_{J^c} = y|_{J^c}] \\ &= \mathbb{P}[\{\mathcal{I} = J'\} \cap \{x|_J = y|_J\}] \cdot \mathbb{P}[x|_{J^c} = y|_{J^c}], \end{aligned}$$

where we used in the first and third equality that \mathcal{I} is local. Using the above, one can also show that $\mathbb{P}[\{\mathcal{I} \not\subseteq J\} \cap \{x = y\}] = \mathbb{P}[\{\mathcal{I} \not\subseteq J\} \cap \{x|_J = y|_J\}] \cdot \mathbb{P}[x|_{J^c} = y|_{J^c}]$. Plugging these back into (8), one deduces that $\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]$ and this concludes the proof. \square

The following simple proposition is left to the reader.

PROPOSITION 2.15. *There is a natural bijection between subcube partitions and stopping sets with respect to the canonical filtration $(\sigma(x_i : i \in J))_{J \subseteq [n]}$.*

Note that x is an independent field with respect to the canonical filtration since the bits are independent. The two previous propositions shed light on the relation between subcube partitions and local sets by viewing the two concepts from the unifying perspective of stopping sets and independent fields.

We conclude this subsection with a brief discussion of randomized subcube partitions. We recall for a subcube partition \mathcal{C} , the random set $\mathcal{I}_{\mathcal{C}}$, defined by $\mathcal{I}_{\mathcal{C}}(x) = I_{\mathcal{C}(x)}$, is a stopping set with respect to the canonical filtration. A randomized subcube partition is a probability measure μ on the space of subcube partitions. If $G \sim \mu$ is independent of x , we observe that $\mathcal{I}_{\mathcal{C}}$ is a stopping set with respect to the filtration

$$(\sigma((x_i : i \in J), G))_{J \subseteq [n]},$$

and x is an independent field with respect to this filtration. Thus, $\mathcal{I}_{\mathcal{C}}$ is a local set by Proposition 2.14. This particular local set has a ‘‘product form’’ and Example 2.9 shows that *not* every local set can be constructed in this way since, in this case, the expected size of the local set is strictly smaller than the distributional subcube partition complexity.

Going the other way around, let \mathcal{G} be a σ -algebra that is independent of $\sigma(x_i : i \in [n])$. If we are now given a stopping set \mathcal{I} with respect to the filtration

$$(\sigma((x_i : i \in J), \mathcal{G}))_{J \subseteq [n]},$$

then we have that x is an independent field with respect to this filtration and \mathcal{I} naturally gives rise to a randomized subcube partition.

3 Boolean function composition

In this section, we analyze the behavior of complexity measures under compositions of Boolean functions. This will be very useful to bound or even compute complexities of more complicated Boolean functions which are constructed using compositions. For two Boolean functions f, g on n respectively m bits, the *composition* $f \circ g$ is the Boolean function on $n \cdot m$ bits defined by

$$f \circ g(x_1^1, \dots, x_m^1, \dots, x_1^n, \dots, x_m^n) = f(g(x_1^1, \dots, x_m^1), \dots, g(x_1^n, \dots, x_m^n)).$$

3.1 Review of the deterministic case

Boolean function composition is well-understood with respect to deterministic complexity measures.

PROPOSITION 3.1. *For all Boolean functions f and g ,*

- (i) $s_D(f \circ g) \leq s_D(f) \cdot s_D(g)$,
- (ii) $w_D(f \circ g) \leq w_D(f) \cdot w_D(g)$,
- (iii) $sc_D(f \circ g) \leq sc_D(f) \cdot sc_D(g)$,

$$(iv) \ a_D(f \circ g) = a_D(f) \cdot a_D(g).$$

We refer to [27, Lemma 3.1] for a proof of (i), (ii) and (iv), and to [14, Prop. 3] for a proof of (iii). The reader will immediately notice that block sensitivity is missing in the above proposition. While one might guess that $b_D(f \circ g) \leq b_D(f) \cdot b_D(g)$, it was observed in [27, Example 5.8] that this inequality is false. The following simple example shows that the first two inequalities can be strict. In the case of subcube partition complexity, we are not aware of an example showing that the inequality in (iii) can be strict.

Example 3.2. The OR-function on n bits is defined by

$$\text{OR}_n(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } x_i = 1 \text{ for some } i \in [n], \\ 0 & \text{otherwise.} \end{cases}$$

The AND-function on n bits is defined by

$$\text{AND}_n(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } x_i = 1 \text{ for all } i \in [n], \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that all deterministic complexity measures of OR_n and of AND_n are equal to n . Now, let f be the OR-function on ℓ bits and let g be the AND-function on m bits. Then $f \circ g$ is the well-known TRIBES-function with ℓ tribes of size m on $n = \ell m$ bits. The deterministic complexity measures are given by

$$a_D(f \circ g) = sc_D(f \circ g) = n \text{ and } s_D(f \circ g) = b_D(f \circ g) = w_D(f \circ g) = \max\{\ell, m\}.$$

These are easily justified, except perhaps $sc_D(f \circ g)$ which can be argued as follows: By [21, Lemma 1], for any Boolean function h , if $\{x : h(x) = 1\}$ is odd, then the deterministic subcube partition complexity equals the number of bits. For the TRIBES function $f \circ g$ as above, it is easy to see that

$$|\{x : f \circ g(x) = 1\}| = \sum_{k=1}^{\ell} (-1)^{k+1} \binom{\ell}{k} (2^m)^{\ell-k},$$

which is odd since the summands corresponding to $k = 1, \dots, \ell - 1$ are even and the summand corresponding to $k = \ell$ is odd.

3.2 Composition results for distributional complexities

The behavior of distributional complexity measures under Boolean function composition differs considerably from the deterministic case as the next result summarizes.

PROPOSITION 3.3. *Set $g(p) := \mathbb{P}_{x \sim \pi_p} [g(x) = 1]$. Then for all Boolean functions f and g ,*

- (i) $s(f \circ g, \pi_p) = s(f, \pi_{g(p)}) \cdot s(g, \pi_p)$,
- (ii) $\ell(f \circ g, \pi_p) \leq \ell(f, \pi_{g(p)}) \cdot \ell(g, \pi_p)$,
- (iii) $sc(f \circ g, \pi_p) \leq sc(f, \pi_{g(p)}) \cdot sc(g, \pi_p)$,
- (iv) $a(f \circ g, \pi_p) \leq a(f, \pi_{g(p)}) \cdot a(g, \pi_p)$.

These results are proven at the end of this subsection. We will often make use of the following immediate corollary.

COROLLARY 3.4. *If a Boolean function f satisfies $\mathbb{P}_{x \sim \pi_p} [f(x) = 1] = p$, then for every $k \geq 1$,*

- (i) $s(f^k, \pi_p) = s(f, \pi_p)^k$,

- (ii) $\ell(f^k, \pi_p) \leq \ell(f, \pi_p)^k$,
- (iii) $sc(f^k, \pi_p) \leq sc(f, \pi_p)^k$,
- (iv) $a(f^k, \pi_p) \leq a(f, \pi_p)^k$,

The previous proposition makes no statement about the distributional block sensitivity or witness complexity of composed Boolean functions. The next example in fact shows that submultiplicativity does not necessarily hold for these two complexities.

Example 3.5. Choose $f = \text{OR}_2$ and $g = \text{AND}_2$ so that $f \circ g$ is the TRIBES-function with $\ell = m = 2$ (see also Example 3.2). On the one hand, it immediately follows from the formulas in Example 4.1 (for the OR function, the complexities are the same as for the AND function with p replaced by $1 - p$) and $g(p) = p^2$ that

$$b(f, \pi_{g(p)}) \cdot b(g, \pi_p) = w(f, \pi_{g(p)}) \cdot w(g, \pi_p) = (1 + (1 - p^2)^2) \cdot (1 + p^2).$$

On the other hand, it is easy to see that $b_{f \circ g}(x) = w_{f \circ g}(x) = 2$ for every $x \in \{0, 1\}^4$ and thus,

$$b(f \circ g, \pi_p) = w(f \circ g, \pi_p) = 2.$$

Since $2 > (1 + (1 - p^2)^2) \cdot (1 + p^2)$ for every $p \in (0, 1)$, this example shows that neither the distributional block sensitivity nor the witness complexity of $f \circ g$ can be bounded from above by the product of the respective complexities of f and g .

Not surprisingly, a general lower bound in terms of the product of the respective complexities cannot hold either for distributional block sensitivity and witness complexity. This can easily be checked by considering $f = g = \text{AND}_2$.

Remark 3.6. It is no coincidence that the distributional block sensitivity and distributional witness complexity are equal in the previous example. We will get back to this in Subsection 4.2.

Finally, let us mention that the inequalities in parts (ii)–(iv) of Proposition 3.3 can be strict. This can be seen by taking $f = g = \text{MAJ}_3$, for which we have seen in Example 2.6 that $\ell(\text{MAJ}_3, \pi_{1/2}) = sc(\text{MAJ}_3, \pi_{1/2}) = a(\text{MAJ}_3, \pi_{1/2}) = 5/2$. It is well-known and can easily be checked that $a(f \circ g, \pi_{1/2}) < (5/2)^2$.

PROOF OF PROPOSITION 3.3. Let f be a Boolean function on n bits, g on m bits.

We begin with part (i) which is a consequence of the standard fact that being pivotal is independent of the bit's value. Note that

$$s_{f \circ g}(x_1^1, \dots, x_m^1, \dots, x_1^n, \dots, x_m^n) = \sum_{i=1}^n \mathbf{1}_{i \text{ pivotal for } f} \cdot s_g(x_1^i, \dots, x_m^i).$$

Taking expectations and using the fact that the event $\{i \text{ pivotal for } f\}$ is measurable with respect to the bits x_j^i with $j \in [m]$ and $i' \in [n] \setminus \{i\}$, we get

$$\begin{aligned} s(f \circ g, \pi_p) &= \sum_{i=1}^n \mathbb{E}_{x \sim \pi_p} \left[\mathbf{1}_{i \text{ pivotal for } f} \cdot s_g(x_1^i, \dots, x_m^i) \right] \\ &= \sum_{i=1}^n \mathbb{P}_{x \sim \pi_p} [i \text{ pivotal for } f] \cdot s(g, \pi_p) = \underbrace{\mathbb{E}_{x \sim \pi_p} \left[\sum_{i=1}^n \mathbf{1}_{i \text{ pivotal for } f} \right]}_{=s(f, \pi_{g(p)})} \cdot s(g, \pi_p). \end{aligned}$$

For the rest of the proof, let us write $x = (x_{B_1}, \dots, x_{B_n})$ with $x_{B_i} := (x_1^i, \dots, x_m^i)$ and $y = y(x) := (g(x_{B_1}), \dots, g(x_{B_n}))$. To prove part (ii), we proceed in three steps.

Step 1: Construction of a natural witness set $\mathcal{I}_f \circ \mathcal{I}_g \sim f \circ g$ from two witnesses sets $\mathcal{I}_f \sim f$ and $\mathcal{I}_g \sim g$

Given a witness set \mathcal{I}_g for g and $x \sim \pi_p$, we denote by μ_g the joint distribution of (\mathcal{I}_g, x) . Moreover, given a witness set \mathcal{I}_f for f and $y \sim \pi_{g(p)}$, we denote the conditional distribution of \mathcal{I}_f given y by μ_f^y . We now construct $\mathcal{I}_f \circ \mathcal{I}_g$. Let $\mathcal{I}_g^1, \dots, \mathcal{I}_g^n$ be random sets such that \mathcal{I}_g^i is a witness set for g on x_{B_i} for every $i \in [n]$, and the pairs $(\mathcal{I}_g^i, x_{B_i})_{i=1}^n$ are i.i.d. with distribution μ_g . Conditional on y , we now choose $\mathcal{I}_f^\star \subseteq [n]$ with conditional law μ_f^y . Note that by construction, \mathcal{I}_f^\star is conditionally independent of x and $\mathcal{I}_g^1, \dots, \mathcal{I}_g^n$ given y . We now define

$$\mathcal{I}_f \circ \mathcal{I}_g := \{(i-1)m + j : i \in \mathcal{I}_f^\star, j \in \mathcal{I}_g^i\}.$$

Note that $\mathcal{I}_f \circ \mathcal{I}_g$ is a witness set for $f \circ g$.

Step 2: If \mathcal{I}_f is local for $y \sim \pi_{g(p)}$ and \mathcal{I}_g is local for $x \in \pi_p$, then $\mathcal{I}_f \circ \mathcal{I}_g$ is local for $x \sim \pi_p$.

We need to show that for every $J \subseteq [nm]$, conditional on $\mathcal{I}_f \circ \mathcal{I}_g = J$ and on $x|_J$, we have $x|_{J^c} \sim \pi_p$. To this end, we partition J^c into J_1 and J_2 defined by

$$J_1 := \{(i-1)m + j : i \notin \mathcal{I}_f^\star, j \in [n]\}, \quad J_2 := \{(i-1)m + j : i \in \mathcal{I}_f^\star, j \notin \mathcal{I}_g^i\}.$$

First, by the construction of $\mathcal{I}_f \circ \mathcal{I}_g$ and the locality of \mathcal{I}_f , it follows that $x|_{J_1} \sim \pi_p$ given $\mathcal{I}_f \circ \mathcal{I}_g = J$ and $x|_J$. Second, again by the construction of $\mathcal{I}_f \circ \mathcal{I}_g$ and by the locality of \mathcal{I}_g , it follows that $x|_{J_2} \sim \pi_p$ given $\mathcal{I}_f \circ \mathcal{I}_g = J$, $x|_J$, and $x|_{J_1}$.

Step 3: $\mathbb{E}_{x \sim \pi_p} [|\mathcal{I}_f \circ \mathcal{I}_g|] = \mathbb{E}_{y \sim \pi_{g(p)}} [|\mathcal{I}_f|] \cdot \mathbb{E}_{x \sim \pi_p} [|\mathcal{I}_g|]$.

It follows immediately from the construction of $\mathcal{I}_f \circ \mathcal{I}_g$ that

$$|\mathcal{I}_f \circ \mathcal{I}_g| = \sum_{i=1}^n \mathbf{1}_{i \in \mathcal{I}_f^\star} \cdot |\mathcal{I}_g^i|. \quad (9)$$

We first note that for every $i \in [n]$,

$$\mathbb{E}_{x \sim \pi_p} \left[\mathbf{1}_{i \in \mathcal{I}_f^\star} \cdot |\mathcal{I}_g^i| \right] = \mathbb{E}_{x \sim \pi_p} \left[\underbrace{\mathbb{E}_{x \sim \pi_p} \left[\mathbf{1}_{i \in \mathcal{I}_f^\star} \mid x_{B_i}, \mathcal{I}_g^i \right]}_{= \mathbb{P}_{y \sim \pi_{g(p)}} [i \in \mathcal{I}_f^\star \mid y_i]} \cdot |\mathcal{I}_g^i| \right] = \mathbb{P}_{y \sim \pi_{g(p)}} [i \in \mathcal{I}_f^\star] \cdot \mathbb{E}_{x \sim \pi_p} [|\mathcal{I}_g^i|],$$

where in the last equality we used that y_i is independent of $\mathbf{1}_{i \notin \mathcal{I}_f^\star}$ by locality. Summing over $i \in [n]$ and using (9), we get

$$\mathbb{E}_{x \sim \pi_p} [|\mathcal{I}_f \circ \mathcal{I}_g|] = \sum_{i=1}^n \mathbb{P}_{y \sim \pi_{g(p)}} [i \in \mathcal{I}_f^\star] \cdot \mathbb{E}_{x \sim \pi_p} [|\mathcal{I}_g^i|] = \mathbb{E}_{y \sim \pi_{g(p)}} [|\mathcal{I}_f|] \cdot \mathbb{E}_{x \sim \pi_p} [|\mathcal{I}_g|],$$

yielding step 3. To conclude the proof of (ii), we take optimal local witness sets $\mathcal{I}_f \sim f$ for $y \sim \pi_{g(p)}$ and $\mathcal{I}_g \sim g$ for $x \sim \pi_p$, and deduce from step 3 that

$$\ell(f \circ g, \pi_p) \leq \mathbb{E}_{x \sim \pi_p} [|\mathcal{I}_f \circ \mathcal{I}_g|] = \mathbb{E}_{y \sim \pi_{g(p)}} [|\mathcal{I}_f|] \cdot \mathbb{E}_{x \sim \pi_p} [|\mathcal{I}_g|] = \ell(f, \pi_{g(p)}) \cdot \ell(g, \pi_p).$$

To prove part (iii), we follow a similar strategy in two steps.

Step 1: Construction of a natural subcube partition $C_f \circ C_g \sim f \circ g$ from two subcube partitions $C_f \sim f$ and $C_g \sim g$

Let $C(y) \in C_f$ be the subcube containing y . For every $x \in \{0, 1\}^{n-m}$, we now define $C(x) := \prod_{i=1}^n C_i(x)$, where

$$C_i(x) := \begin{cases} C(x_{B_i}) \in C_g & \text{if } i \text{ is fixed by } C(y), \\ \{0, 1\}^m & \text{if } i \text{ is not fixed by } C(y). \end{cases}$$

It now remains to show that $\{C(x)\}_{x \in \{0,1\}^{n \cdot m}}$ is a subcube partition for $f \circ g$, which we will denote by $C_f \circ C_g$. First, note that $C(y) \cap C(y') = \emptyset$ implies $C(x) \cap C(x') = \emptyset$. Hence, if $C(x) \cap C(x') \neq \emptyset$, then $C(y) = C(y')$. Hence, $C(x_{B_i}) = C(x'_{B_i})$ for every i that is fixed by $C(y)$, and so $C(x) = C(x')$. Finally, we observe that the constructed subcube partition satisfies $C_f \circ C_g \sim f \circ g$.

Step 2: The equality $sc(C_f \circ C_g, \pi_p) = sc(C_f, \pi_{g(p)}) \cdot sc(C_g, \pi_p)$ holds, where we used the notation $sc(C, \pi_p) = \mathbb{E}_{x \sim \pi_p} [c(C, x)]$.

It follows immediately from the construction of $C_f \circ C_g$ that

$$c(C_f \circ C_g, x) = \sum_{i=1}^n \mathbf{1}_{i \text{ is fixed by } C(y)} \cdot c(C_g, x_{B_i}) \quad (10)$$

We first note that for every $i \in [n]$,

$$\begin{aligned} & \mathbb{E}_{x \sim \pi_p} [\mathbf{1}_{i \text{ is fixed by } C(y)} \cdot c(C_g, x_{B_i})] \\ &= \mathbb{E}_{x \sim \pi_p} \left[\underbrace{\mathbb{E}_{x \sim \pi_p} [\mathbf{1}_{i \text{ is fixed by } C(y)} \mid x_{B_i}]}_{= \mathbb{P}_{y \sim \pi_{g(p)}} [i \text{ is fixed by } C(y) \mid y_i]} \cdot c(C_g, x_{B_i}) \right] \\ &= \mathbb{P}_{y \sim \pi_{g(p)}} [i \text{ is fixed by } C(y)] \cdot sc(C_g, \pi_p), \end{aligned}$$

where in the last equality we used that y_i is independent of $\{i \text{ is not fixed by } C(y)\}$. Now, taking $\mathbb{E}_{x \sim \pi_p}$ in (10), we get

$$sc(C_f \circ C_g, \pi_p) = \sum_{i=1}^n \mathbb{P}_{y \sim \pi_{g(p)}} [i \text{ is fixed by } C(y)] \cdot sc(C_g, \pi_p) = sc(C_f, \pi_{g(p)}) \cdot sc(C_g, \pi_p),$$

yielding step 2. To conclude the proof of (iii), we take optimal subcube partitions $C_f \sim f$ and $C_g \sim g$ and deduce from step 2 that

$$sc(f \circ g, \pi_p) \leq sc(C_f \circ C_g, \pi_p) = sc(C_f, \pi_{g(p)}) \cdot sc(C_g, \pi_p) = sc(f, \pi_{g(p)}) \cdot sc(g, \pi_p).$$

Finally, we show part (iv). We choose an optimal algorithm A_f for f and an optimal algorithm A_g for g , and define an algorithm for $f \circ g$, denoted by $A_f \circ A_g$, as follows: Whenever the algorithm A_f queries a bit $i \in [n]$, the algorithm $A_f \circ A_g$ determines the value of y_i by using the algorithm A_g on the bits corresponding to (x_1^i, \dots, x_m^i) , and as soon as $y_i = g(x_{B_i})$ is determined, $A_f \circ A_g$ proceeds analogously with the next query of A_f . Clearly, this algorithm determines $f \circ g$ and from here, we conclude as in part (ii). \square

Remark 3.7. Analogously to part (i) of the last proof, one can show that for all Boolean functions f and g ,

$$b(f \circ g, \pi_p) \geq s(f, \pi_{g(p)}) \cdot b(g, \pi_p).$$

Remark 3.8. Under the additional assumption that

$$\mathbb{E}_{x \sim \pi_p} [w_g(x) \mid g(x) = 1] = \mathbb{E}_{x \sim \pi_p} [w_g(x) \mid g(x) = 0],$$

one can prove that $w(f \circ g, \pi_p) \leq w(f, \pi_{g(p)}) \cdot w(g, \pi_p)$.

4 Examples

In the first subsection here, we consider classical examples of Boolean functions and study their distributional complexities. In the second subsection, we consider examples related to percolation theory.

4.1 Classical examples

Example 4.1. Concerning the AND-function on n bits (see Example 3.2), it is easy to see that some of its distributional complexities are given by

$$\begin{aligned} s(\text{AND}_n, \pi_p) &= n \cdot p^{n-1}, \\ b(\text{AND}_n, \pi_p) &= w(\text{AND}_n, \pi_p) = n \cdot p^n + (1 - p^n), \\ a(\text{AND}_n, \pi_p) &= \frac{1 - p^n}{1 - p}. \end{aligned}$$

For the distributional subcube partition complexity, we have the following partial result.

PROPOSITION 4.2. *For $p \leq 1/2$, we have $sc(\text{AND}_n, \pi_p) = a(\text{AND}_n, \pi_p)$.*

PROOF. We prove by induction on n that $sc(\text{AND}_n, \pi_p) = \frac{1-p^n}{1-p}$ with $n = 1$ being obvious. Let C be a subcube partition achieving the minimum $sc(\text{AND}_{n+1}, \pi_p)$. Since $p \leq 1/2$, we have $a(\text{AND}_n, \pi_p) < 2$ for every $n \geq 1$, and hence C must contain a subcube with only 1 fixed bit. Without loss of generality, C contains the subcube $(0, \star, \dots, \star)$. We are thus left with choosing an optimal subcube partition of $(1, \star, \dots, \star)$, which is equivalent to choosing an optimal subcube partition for AND_n . Using the induction hypothesis,

$$\mathbb{E}_{x \sim \pi_p} [c(C, x)] = (1-p) \cdot 1 + p \cdot \left(\frac{1-p^n}{1-p} + 1 \right) = \frac{1-p^{n+1}}{1-p}.$$

□

Remark 4.3. We believe that the equality $sc(\text{AND}_n, \pi_p) = a(\text{AND}_n, \pi_p)$ holds for all $p \in [0, 1]$, but the above argument does not extend to $p > 1/2$.

Example 4.4. Iterated 3-MAJORITY is the Boolean function f^k on $n = 3^k$ bits defined by $f := \text{MAJ}_3$ (see Example 2.6) and iteratively by $f^k := f \circ f^{k-1}$ for $k \geq 2$. Fix $p = 1/2$ and recall that the distributional complexities of $f = \text{MAJ}_3$ are given by

$$\begin{aligned} s(\text{MAJ}_3, \pi_{1/2}) &= 3/2, \quad b(\text{MAJ}_3, \pi_{1/2}) = 7/4, \quad w(\text{MAJ}_3, \pi_{1/2}) = 2, \\ \text{and } \ell(\text{MAJ}_3, \pi_{1/2}) &= sc(\text{MAJ}_3, \pi_{1/2}) = a(\text{MAJ}_3, \pi_{1/2}) = 5/2. \end{aligned}$$

We note that $f = \text{MAJ}_3$ is balanced, i.e. $\mathbb{P}_{x \sim \pi_{1/2}} [f(x) = 1] = 1/2$. Hence, by part (i) of Corollary 3.4,

$$s(f^k, \pi_{1/2}) = (3/2)^k.$$

Moreover, we note that $w_{f^k}(x) = 2^k$ for all $x \in \{0, 1\}^{3^k}$ and so

$$w(f^k, \pi_{1/2}) = 2^k.$$

From the inequality of O’Donnell-Servedio (see Theorem 5.2) and part (iii) of Corollary 3.4, we get

$$(5/2)^k \geq a(f^k, \pi_{1/2}) \geq sc(f^k, \pi_{1/2}) \geq \ell(f^k, \pi_{1/2}) \stackrel{\text{(OS)}}{\geq} \left(s(f^k, \pi_{1/2}) \right)^2 = (9/4)^k.$$

Example 4.5. The MAJORITY function on n bits (with n odd) is defined by

$$\text{MAJ}_n(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } \sum_{i=1}^n x_i \geq \frac{n+1}{2}, \\ 0 & \text{if } \sum_{i=1}^n x_i \leq \frac{n-1}{2}. \end{cases}$$

For $p = 1/2$, it is balanced and its distributional sensitivity is given by

$$s(\text{MAJ}_n, \pi_{1/2}) = \frac{n+1}{2} \cdot \mathbb{P}_{x \sim \pi_{1/2}} \left[\sum_{i=1}^n x_i \in \left\{ \frac{n-1}{2}, \frac{n+1}{2} \right\} \right] \sim \sqrt{2/\pi} \cdot \sqrt{n}.$$

Next, we compute its distributional block sensitivity. Let $2\ell + 1$ with $\ell \in \{0, \dots, \frac{n-1}{2}\}$ be the difference between the number of 1's and 0's in x with $\text{MAJ}_n(x) = 1$. Then

$$b_{\text{MAJ}_n}(x) = \lfloor \frac{\max\{\# \text{ of } 1\text{'s in } x, \# \text{ of } 0\text{'s in } x\}}{\ell + 1} \rfloor = \lfloor \frac{n + 2\ell + 1}{2\ell + 2} \rfloor$$

and so

$$b(\text{MAJ}_n, \pi_{1/2}) = 2 \sum_{\ell=0}^{\frac{n-1}{2}} \binom{n}{\frac{n+2\ell+1}{2}} \cdot 2^{-n} \cdot \lfloor \frac{n + 2\ell + 1}{2\ell + 2} \rfloor \asymp \log(n) \cdot \sqrt{n}.$$

Since $w_{\text{MAJ}_n}(x) = \frac{n+1}{2}$ for every x , the distributional witness complexity is

$$w(\text{MAJ}_n, \pi_{1/2}) = \frac{n+1}{2}.$$

Finally, we note that the distributional algorithmic complexity is

$$a(\text{MAJ}_n, \pi_{1/2}) = n - o(n).$$

Example 4.6. Recalling from Example 3.2, TRIBES with ℓ tribes of size m is the Boolean function on $n = \ell \cdot m$ bits defined by

$$\text{TRIBES}_{\ell, m} = \text{OR}_{\ell} \circ \text{AND}_m.$$

We fix $p = 1/2$ and $\ell = 2^m$, and note that for these choices

$$\mathbb{P}_{x \sim \pi_{1/2}} [\text{TRIBES}_{2^m, m}(x) = 1] = 1 - (1 - \frac{1}{2^m})^{2^m} \rightarrow 1 - e^{-1} \text{ as } m \rightarrow \infty.$$

It is easy to see that the distributional complexities are given by

$$\begin{aligned} s(\text{TRIBES}_{2^m, m}, \pi_{1/2}) &= m \cdot 2^m \cdot \frac{1}{2^{m-1}} (1 - \frac{1}{2^m})^{2^m-1} \sim 2m \cdot e^{-1} \sim \frac{2}{e} \log_2(n), \\ b(\text{TRIBES}_{2^m, m}, \pi_{1/2}) &= w(\text{TRIBES}_{2^m, m}, \pi_{1/2}) \sim 2^m \cdot e^{-1} + m \cdot (1 - e^{-1}) \asymp \frac{n}{\log_2(n)}, \\ a(\text{TRIBES}_{2^m, m}, \pi_{1/2}) &\asymp \frac{n}{\log_2(n)}. \end{aligned}$$

Thus, we also have $sc(\text{TRIBES}_{2^m, m}, \pi_{1/2}) \asymp \frac{n}{\log_2(n)}$ and $\ell(\text{TRIBES}_{2^m, m}, \pi_{1/2}) \asymp \frac{n}{\log_2(n)}$.

4.2 Percolation functions

Let $G = (V, E)$ be a finite multigraph. A (bond) percolation configuration $\omega \in \{0, 1\}^E$ declares edges *open* if $\omega(e) = 1$ or *closed* if $\omega(e) = 0$. We identify ω with the subgraph of open edges $G_{\omega} := (V, E_{\omega})$, where $E_{\omega} := \{e \in E : \omega(e) = 1\}$, and introduce some notation to describe its connectivity properties. A *path* $\gamma = (\gamma_i)_{i=0}^n \subseteq V$ of length n is a sequence of distinct vertices with $\{\gamma_{i-1}, \gamma_i\} \in E$ for all $1 \leq i \leq n$, and the path is called *open* if $\omega(\{\gamma_{i-1}, \gamma_i\}) = 1$ for all $1 \leq i \leq n$. We say that two subsets $A, B \subseteq V$ are *connected* if there exists an open path that starts in A and ends in B , and denote this event by $\{A \leftrightarrow B\}$.

Definition 4.7. A Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is called a *percolation function* if it is of the form

$$f(\omega) = \begin{cases} 1 & \text{if } \omega \in \{A \leftrightarrow B\}, \\ 0 & \text{otherwise,} \end{cases}$$

for some multigraph $G = (V, E)$ with n edges and some $A, B \subseteq V$.

Remark 4.8. We note that any percolation function is monotone. We also note that the function $\text{AND}_2 \circ \text{OR}_2$ is a percolation function according to the previous definition, but it cannot be represented by a simple graph.

AND-functions, OR-functions and TRIBES-functions are percolation functions. However, MAJORITY-functions are not percolation functions. We recall from Example 4.1 that for every $p \in (0, 1)$ and $n \geq 1$,

$$a(\text{AND}_n, \pi_p) > w(\text{AND}_n, \pi_p) = b(\text{AND}_n, \pi_p) > s(\text{AND}_n, \pi_p).$$

In fact, Corollary 4.10 will show that the above comparisons are general properties of percolation functions.

PROPOSITION 4.9. *For every percolation function f and for every $\omega \in \{0, 1\}^n$,*

$$w_f(\omega) = b_f(\omega).$$

In particular, for every $p \in [0, 1]$, we have $w(f, \pi_p) = b(f, \pi_p)$.

PROOF. Let ω be such that $f(\omega) = 1$. Since f is monotone, it suffices to consider witnesses $W \subseteq E_\omega$ and blocks that are subsets of E_ω . The witness sets of minimum size are now the shortest open paths from A to B and so $w_f(\omega)$ is the length of a shortest path from A to B in the multigraph G_ω . The maximum number $b_f(\omega)$ of disjoint blocks that flip the outcome of f is thus equal to the maximum number of disjoint (A, B) -edge-cutsets for G_ω . While the inequality $b_f(\omega) \leq w_f(\omega)$ holds for every Boolean function f and every ω , the inequality $b_f(\omega) \geq w_f(\omega)$ follows by considering the disjoint (A, B) -edge-cutsets for G_ω defined by

$$\{e \in E_\omega : d_{G_\omega}(e, A) = i\}$$

for $0 \leq i \leq w_f(\omega) - 1$.

Now, let ω be such that $f(\omega) = 0$. Since f is monotone, it suffices to consider witnesses $W \subseteq E \setminus E_\omega$ and blocks that are subsets of $E \setminus E_\omega$. By collapsing each connected component of G_ω into a single vertex, we define a new multigraph $\hat{G} = (\hat{V}, \hat{E})$ whose vertices \hat{V} correspond to the connected components of G_ω and whose edges \hat{E} correspond to the closed edges $E \setminus E_\omega$ in the original multigraph G . $w_f(\omega)$ is now equal to the size of a minimum (A, B) -edge-cutset for \hat{G} and $b_f(\omega)$ is equal to the maximal number of edge-disjoint paths from A to B in \hat{G} . The equality $w_f(\omega) = b_f(\omega)$ follows from Menger's Theorem. \square

The following corollary is a direct consequence of Propositions 2.11, 2.12, and 4.9.

COROLLARY 4.10. *For every percolation function f which depends on at least 2 bits, we have that*

$$sc(f, \pi_p) > w(f, \pi_p) = b(f, \pi_p) > s(f, \pi_p).$$

We are not aware of any monotone Boolean function f satisfying $a(f, \pi_p) > sc(f, \pi_p)$ (see Question 2.1) or $sc(f, \pi_p) > \ell(f, \pi_p)$, and thus also not of a percolation function satisfying this.

For the last upper bound in (ii), we make use of a standard algorithm that determines f_m by exploring the connected component of each of the points on the left side. We note that an edge $e \in E_m$ is queried by this algorithm before determining f_m only if there is an open path in G_m from one of the endpoints of e to the left side of the square. Therefore,

$$\begin{aligned} a(f_m, \pi_{1/2}) &\leq \sum_{e \in E_m} \mathbb{P}_{x \sim \pi_{1/2}} [e \xrightarrow{G_m} \{0\} \times [0, m]] \\ &\leq (2m+1) \cdot \sum_{k=0}^m \mathbb{P}_{x \sim \pi_{1/2}} \left[(0, 0) \xrightarrow{\mathbb{Z}^2} \partial([k, k]^2) \right], \end{aligned}$$

where the k -th summand accounts for the edges with minimal x -coordinate k . By Theorem 11.89 in [10], there are constants $c, C \in (0, \infty)$ such that for every $k \geq 1$,

$$\mathbb{P}_{x \sim \pi_{1/2}} \left[(0, 0) \xrightarrow{\mathbb{Z}^2} \partial([-k, k]^2) \right] \leq Ck^{-c}.$$

From this, the last upper bound in (ii) follows.

The equality $b(f_m, \pi_{1/2}) = w(f_m, \pi_{1/2})$ in (ii) follows from Proposition 4.9, and all the other inequalities, except the first, follow from the hierarchy (see Proposition 2.5). Finally, the fact that $w(f_m, \pi_{1/2}) \gtrsim m^{1+c}$ is stated in (1.1) in [6]. \square

Critical percolation on the square lattice is conjectured to be conformally invariant, which would yield exact values for the so-called arm exponents. Using these conjectured values, we arrive at the following conjecture. We note that in the case of critical site percolation on the triangular lattice, conformal invariance is known, and hence so are the arm exponents [26].

CONJECTURE 4.14.

- (i) $s(f_m, \pi_{1/2}) = m^{3/4+o(1)}$,
- (ii) $w(f_m, \pi_{1/2}) \leq m^{4/3+o(1)}$
- (iii) $\ell(f_m, \pi_{1/2}) \geq m^{3/2+o(1)}$,
- (iv) $a(f_m, \pi_{1/2}) \leq m^{7/4+o(1)}$.

We now comment on this conjecture: For (i), the probability of being pivotal is essentially $m^{-\alpha_4}$ where the four-arm exponent α_4 is conjectured to be $5/4$. Using the inequality of O’Donnell-Servedio (see Theorem 5.2), (iii) follows immediately from (i). For (ii), one first notes that $w_{f_m}(\omega)$ is at most the length of the lowest crossing (when $f_m(\omega) = 1$). Since an edge being on the lowest crossing entails a three-arm event starting at that edge and since the three-arm exponent is conjectured to be $2/3$, the result would follow. For (iv), one makes use of an algorithm that “follows the interface”. Being queried then corresponds to the two-arm exponent which is conjectured to be $1/4$ (see [8, Theorem 8.4]).

Non-rigorous arguments suggest that $w(f_m, \pi_{1/2})$ behaves like $m^{1.130\dots}$ and $a(f_m, \pi_{1/2})$ behaves like m^α with $\alpha \in [1.5, 1.6]$ (see [20]). Finding the exact exponents in these two cases are well-known open problems.

5 Asymptotic separations

In Section 2, we have seen examples showing that the five complexity measures are distinct, both in the deterministic and distributional case. In this section, we study asymptotic separations.

5.1 Review of the deterministic case

Asymptotic separations are well-studied in the deterministic case and we refer the reader to the recent paper [1] (see, in particular, Table 1) for an excellent overview. We summarize a small fraction of the known results in the following theorem.

THEOREM 5.1. *There exist sequences of Boolean functions $(f_n^{(i)})_{n \geq 1}$, $1 \leq i \leq 4$, and $\epsilon > 0$ such that asymptotically as $n \rightarrow \infty$,*

$$\frac{a_D(f_n^{(1)})}{sc_D(f_n^{(1)})^{1+\epsilon}} \rightarrow \infty, \quad \frac{sc_D(f_n^{(2)})}{w_D(f_n^{(2)})^{1+\epsilon}} \rightarrow \infty, \quad \frac{w_D(f_n^{(3)})}{b_D(f_n^{(3)})^{1+\epsilon}} \rightarrow \infty, \quad \text{and} \quad \frac{b_D(f_n^{(4)})}{s_D(f_n^{(4)})^{1+\epsilon}} \rightarrow \infty.$$

We refer to [2, 14] for the first, to [9] for the third, and to [22] for the fourth separation. For the second, the TRIBES function (see Example 3.2) with ℓ tribes of size m provides such an example assuming $\min\{\ell, m\} \rightarrow \infty$ since $w_D(\text{TRIBES}_{\ell, m}) = \max\{\ell, m\}$ and $sc_D(\text{TRIBES}_{\ell, m}) = n = \ell \cdot m$.

5.2 Asymptotic separation for distributional complexities

The following theorem yields a first general asymptotic separation [19].

THEOREM 5.2 (O'DONNELL-SERVEDIO). *For every monotone Boolean function f and for every $p \in [0, 1]$,*

$$sc(f, \pi_p) \geq 4p(1-p) \cdot (s(f, \pi_p))^2.$$

In fact, the same inequality holds with the left-hand side replaced by $\ell(f, \pi_p)$. The easiest way to see this is by adapting the proof given in [8, Thm. 8.8].

Based on the examples studied in Subsection 4.1, we obtain the following result.

THEOREM 5.3. *There exist sequences of Boolean functions $(f_n^{(i)})_{n \geq 1}$, $1 \leq i \leq 3$, and $\epsilon > 0$ such that asymptotically as $n \rightarrow \infty$,*

$$\frac{\ell(f_n^{(1)}, \pi_{1/2})}{w(f_n^{(1)}, \pi_{1/2})^{1+\epsilon}} \rightarrow \infty, \quad \frac{w(f_n^{(2)}, \pi_{1/2})}{b(f_n^{(2)}, \pi_{1/2})^{1+\epsilon}} \rightarrow \infty, \quad \text{and} \quad \frac{b(f_n^{(3)}, \pi_{1/2})}{s(f_n^{(3)}, \pi_{1/2})^{1+\epsilon}} \rightarrow \infty.$$

The reader will immediately notice that asymptotic separations between distributional algorithmic complexity, subcube partition complexity and local witness complexity are missing. Before discussing the difficulties in asymptotically separating distributional algorithmic complexity and subcube partition complexity, we prove the three asymptotic separations.

PROOF. (1) Consider iterated 3-MAJORITY f^k on $n = 3^k$ bits as defined in Example 4.4. Then we have

$$\frac{\ell(f^k, \pi_{1/2})}{w(f^k, \pi_{1/2})} \geq (9/8)^k = n^{\log_3(9/8)},$$

which yields the result since $w(f^k, \pi_{1/2}) \leq n$.

(2) Consider MAJ $_n$ on n bits (with n odd) as defined in Example 4.5. Then we have

$$\frac{w(\text{MAJ}_n, \pi_{1/2})}{b(\text{MAJ}_n, \pi_{1/2})} \asymp \frac{\sqrt{n}}{\log(n)},$$

which yields the result as in (1).

(3) Consider $\text{TRIBES}_{2^m, m}$ with 2^m tribes of size m as studied in Example 4.6. Then we have

$$\frac{b\left(\text{TRIBES}_{2^m, m, \pi_{1/2}}\right)}{s\left(\text{TRIBES}_{2^m, m, \pi_{1/2}}\right)} \asymp \frac{n}{\log(n)^2},$$

which yields the result as in (1). □

We end this section with a proposed example for asymptotically separating distributional algorithmic complexity and distributional subcube partition complexity. Let f be the ALL-EQUAL function on $n = 3$ bits (see Example 2.8). We want to iterate f but since f is not balanced, it makes things simpler to XOR f with one bit so that it is balanced and hence all its iterates are balanced. In fact, we believe that in order for this example to have a chance of working, we need to take f and XOR it with a large number of bits. Denote by $f \oplus \text{PAR}_k$ the function on $3 + k$ bits which is f XORed with k bits. Clearly, when $p = 1/2$, the distributional algorithmic complexity and distributional subcube partition complexity are $5/2 + k$ and $9/4 + k$ respectively. We now consider the sequence of Boolean functions $(f \oplus \text{PAR}_k)^\ell$ with k fixed and indexed by ℓ . By Corollary 3.4, we know that the distributional subcube partition complexity of $(f \oplus \text{PAR}_k)^\ell$ is at most $(9/4 + k)^\ell$. The hope is that for k large, the distributional algorithmic complexity of $(f \oplus \text{PAR}_k)^\ell$ is precisely $(5/2 + k)^\ell$ for every $\ell \geq 1$. If this were the case, we would have the desired separation. The above would correspond to the optimal algorithm for $(f \oplus \text{PAR}_k)^\ell$ just being the optimal algorithm for $(f \oplus \text{PAR}_k)$ iterated ℓ times. The reason this might be believable is that whenever one deviates from the latter, one has an initial cost of k corresponding to looking at the k added XOR bits which one always needs to do first. If k is very large, this cost is too much and it is not worth deviating from the simple "iterative" algorithm.

This approach will naturally lead to Section 7 where we introduce the notion of partial information revelation. The connection arises by viewing the querying of the k parity bits of $f \oplus \text{PAR}_k$ as providing partial information on the value of $f \oplus \text{PAR}_k$. We will revisit this approach in Section 8.

6 Polynomial relations

In 2019, Huang proved the long-standing sensitivity conjecture [12], which, given previous results, implies that for some C , $a_D(f) \leq s_D(f)^C$ for every Boolean function f . In other words, all deterministic complexity measures are polynomially related to each other. We first review the polynomial relations in the deterministic case and then move to the distributional setting.

6.1 Review of the deterministic case

The five deterministic complexity measures discussed so far are actually part of a larger family studied in computational complexity theory. Let us introduce one additional complexity measure which was particularly important in the proof of the sensitivity conjecture. We denote by $\deg(f)$ the *degree* of the multilinear polynomial p that represents f , i.e. $p(x) = f(x)$ for all $x \in \{0, 1\}^n$.

THEOREM 6.1. *For every Boolean function f ,*

$$a_D(f) \stackrel{(1)}{\leq} w_D(f) \cdot b_D(f), \quad w_D(f) \stackrel{(2)}{\leq} b_D(f) \cdot s_D(f), \quad b_D(f) \stackrel{(3)}{\leq} \deg(f)^2 \quad \text{and} \quad \deg(f) \stackrel{(4)}{\leq} s_D(f)^2.$$

In particular, all these deterministic complexity measures are polynomially related.

Inequality (1) was proven in [3], (2) in [16], (3) in [16] with an extra factor 2 and then improved to the above in [27], and (4) in [12]. We refer to the survey [5] and to [1] for more background.

Moreover, Nisan showed equality of witness complexity and sensitivity for monotone Boolean functions [16, Proposition 2.2].

PROPOSITION 6.2. *For every monotone Boolean function f ,*

$$w_D(f) = b_D(f) = s_D(f).$$

6.2 Polynomial relations for distributional complexities

We begin by considering the AND-function on n bits from Example 4.1 whose distributional sensitivity and block sensitivity are given by

$$s(\text{AND}_n, \pi_p) = n \cdot p^{n-1}, \quad \text{and} \quad b(\text{AND}_n, \pi_p) = 1 + (n-1) \cdot p^n \geq 1.$$

Hence, for any $p \in (0, 1)$, we do not have a polynomial relation between $s(\text{AND}_n, \pi_p)$ and $b(\text{AND}_n, \pi_p)$. However, note that for any $p \in (0, 1)$, $\text{Var}_p(\text{AND}_n) \rightarrow 0$ as $n \rightarrow \infty$.

Our goal is thus to obtain polynomial relations for Boolean functions with variances bounded away from 0.

QUESTION 6.1. *Fix $c > 0$. Are there any polynomial relations between*

$$a(f, \pi_p), \quad sc(f, \pi_p), \quad \ell(f, \pi_p), \quad w(f, \pi_p), \quad b(f, \pi_p), \quad \text{and} \quad s(f, \pi_p)$$

for general Boolean functions f satisfying $\text{Var}_p(f) \geq c$?

We are also interested in such relations for restricted classes of Boolean functions, for example when restricting to monotone, transitive, or symmetric Boolean functions (transitive means that the set of permutations in S_n leaving the Boolean function invariant, acts transitively on $[n]$).

A first negative answer to this question is given by the function $\text{TRIBES}_{2^m, m}$ with 2^m tribes of size m (see Example 4.6). It is transitive, monotone, and for $p = 1/2$,

$$\text{Var}_{1/2}(\text{TRIBES}_{2^m, m}) \rightarrow e^{-1}(1 - e^{-1}) \text{ as } m \rightarrow \infty.$$

However, its distributional sensitivity and block sensitivity are *not* polynomially related since

$$s(\text{TRIBES}_{2^m, m}, \pi_{1/2}) \asymp \log_2(n), \quad \text{but} \quad b(\text{TRIBES}_{2^m, m}, \pi_{1/2}) \asymp \frac{n}{\log_2(n)}.$$

A first positive answer under certain conditions follows from the so-called OSSS inequality [18].

THEOREM 6.3 (O'DONNELL-SAKS-SCHRAMM-SERVEDIO). *Let f be a Boolean function on n bits. For every algorithm A and for every $p \in (0, 1)$,*

$$\text{Var}_p(f) \leq 4p(1-p) \sum_{i=1}^n \mathbb{P}_{x \sim \pi_p}[i \in A_T(f)] \cdot \mathbb{P}_{x \sim \pi_p}[i \text{ pivotal for } f],$$

where $A_T(f)$ denotes the (random) set of bits queried by A before determining f .

In particular, for every transitive f ,

$$n \cdot \text{Var}_p(f) \leq 4p(1-p) \cdot a(f, \pi_p) \cdot s(f, \pi_p).$$

We deduce the following immediate corollary.

COROLLARY 6.4. *Let $p \in (0, 1)$ and let $(f_n)_{n \geq 1}$ be a sequence of transitive Boolean functions (with f_n on n bits) satisfying for some $c, \epsilon > 0$ and $C < \infty$,*

$$\text{Var}_p(f_n) \geq c \quad \text{and} \quad a(f, \pi_p) \leq C \cdot n^{1-\epsilon}, \quad \forall n \geq 1.$$

Then,

$$s(f_n, \pi_p) \geq \frac{c}{C^{\frac{1}{1-\epsilon}}} \cdot (a(f_n, \pi_p))^{\frac{\epsilon}{1-\epsilon}}$$

and hence $s(f_n, \pi_p)$ and $a(f_n, \pi_p)$ are polynomially related.

Remark 6.5. In the case of critical planar percolation, this corollary can be used to show polynomial relations at criticality (see Example 4.11).

We close this section with a geometric perspective on the ratio of $sc(f, \pi_{1/2})$ and $s(f, \pi_{1/2})$, namely

$$\frac{sc(f, \pi_{1/2})}{s(f, \pi_{1/2})} = \frac{\inf_{C: E(C) \subseteq E(f)} |\Delta_E(C)|}{|\Delta_E(f)|} =: \gamma(f),$$

where E_n denotes the edges of the cube $\{0, 1\}^n$, $E(f) := \{\{x, y\} \in E_n : f(x) \neq f(y)\}$, $E(C) := \{\{x, y\} \in E_n : C(x) \neq C(y)\}$, and the two edge boundaries are defined as $\Delta_E(f) = E_n \setminus E(f)$ and $\Delta_E(C) = E_n \setminus E(C)$. The equality above is relatively straightforward to check and Theorem 5.2 immediately yields $\gamma(f) \geq s(f, \pi_{1/2})$. Establishing upper bounds on $\gamma(f)$ in terms of other distributional complexity measures could then be a way to obtain polynomial relations.

7 Distributional algorithmic complexity with partial information

In this section, we will only be dealing with the product measure $\pi_{1/2}$. A standard algorithm queries the bits one by one and pays a cost of 1 for each query. Our goal is to allow algorithms to ask for partial information concerning the value of a bit at a cost depending on the amount of information asked for.

7.1 Distributional Γ -algorithmic complexity

Consider $x = (x_1, \dots, x_n)$, where each

$$x_i = (x_i(t))_{t=0}^{T_{\{0,1\}}}$$

is an independent Brownian motion started at $1/2$ and stopped when hitting the set $\{0, 1\}$. A *generalized algorithm* A starts in the state $(1/2, \dots, 1/2)$ and initially picks some $i \in [n]$ and $p \in (1/2, 1]$. To obtain the new state, we let x_i evolve up to time $T_{\{1-p, p\}}$, the hitting time of $\{1-p, p\}$. The new state is then

$$\left(1/2, \dots, 1/2, x_i(T_{\{1-p, p\}}), 1/2, \dots, 1/2\right).$$

At any later stage, if the present state is (p_1, \dots, p_n) , A chooses some $j \in [n]$ and $p \in (\max\{1-p_j, p_j\}, 1]$, and goes to state $(p_1, \dots, p_{j-1}, x_j(T_{\{1-p, p\}}), p_{j+1}, \dots, p_n)$. Note that $p_j \rightarrow 1-p$ with probability $\frac{p-p_j}{p-(1-p)}$ and $p_j \rightarrow p$ with probability $\frac{p_j-(1-p)}{p-(1-p)}$. The algorithm continues until reaching a state $(p_1, \dots, p_n) \in \{0, 1\}^n$. We will only consider algorithms that almost surely terminate after finitely many steps. Note that standard algorithms as defined in Section 2 correspond to those algorithms which always pick $p = 1$.

A *cost function* $\Gamma : [1/2, 1] \rightarrow [0, 1]$ is a non-decreasing function satisfying $\Gamma(1/2) = 0$ and $\Gamma(1) = 1$. Given a Boolean function f on n bits and a cost function Γ , we define the Γ -cost of A for determining f as

$$c_\Gamma(A, f) = \mathbb{E}_x \left[\sum_{i=1}^n \Gamma(\max\{1 - \bar{p}_i, \bar{p}_i\}) \right],$$

where \bar{p}_i is the state of bit i at the first moment at which the p_j 's that are 0 or 1 determine f . The motivation for this cost function is to assign a cost $\Gamma(p) - \Gamma(\max\{1 - p_i, p_i\})$ for the transition from p_i or $1 - p_i$ to $\{p, 1 - p\}$. Finally, we define the *distributional Γ -algorithmic complexity* as

$$a_\Gamma(f, \pi_{1/2}) = \inf_A c_\Gamma(A, f).$$

We note that $w(f, \pi_{1/2}) \leq a_\Gamma(f, \pi_{1/2}) \leq a(f, \pi_{1/2})$ for all Γ and for all f . Note that for the trivial choice $\Gamma = \mathbf{1}_{(1/2, 1]}$ (respectively $\Gamma = \mathbf{1}_{\{1\}}$), the right (respectively left) inequality saturates. We recall that for PARITY functions and ADDRESS functions (see Example 2.13), $w(f, \pi_{1/2}) = a(f, \pi_{1/2})$. The next example illustrates that this new complexity measure can have more interesting behavior.

Example 7.1. The AND function on $n = 2$ bits (see Examples 3.2 and 4.1) has distributional complexities

$$a(\text{AND}_2, \pi_{1/2}) = 3/2 \quad \text{and} \quad w(\text{AND}_2, \pi_{1/2}) = 5/4.$$

For some fixed $p_0 \in (1/2, 1]$, we consider the generalized algorithm A_{p_0} which initially asks the $(1 - p_0)/p_0$ question for bit 1 (meaning that it picks bit 1 and goes to the state $(x_1(T_{\{1-p_0, p_0\}}), 1/2)$). If $x_1(T_{\{1-p_0, p_0\}}) = 1 - p_0$, it asks the 0/1 question for bit 1 in the second step, and then the 0/1 question for bit 2. If $x_1(T_{\{1-p_0, p_0\}}) = p_0$, it asks the 0/1 question for bit 2 in the second step, and then the 0/1 question for bit 1. One can check that the Γ -cost of A_{p_0} for determining AND_2 is

$$c_\Gamma(A_{p_0}, \text{AND}_2) = 3/2 + \frac{\Gamma(p_0) - 2p_0 + 1}{4}.$$

We deduce that $a_\Gamma(\text{AND}_2, \pi_{1/2}) < a(\text{AND}_2, \pi_{1/2})$ if $\Gamma(p_0) < 2p_0 - 1$.

We are interested in the following question, which is related to the discussion in Section 8.

QUESTION 7.1. *Given a Boolean function f , for which cost functions Γ do we have*

$$a_\Gamma(f, \pi_{1/2}) = a(f, \pi_{1/2}) ?$$

It is interesting to compare our notion of distributional Γ -algorithmic complexity with the notion of Gross [11]. On the one hand, his concept of “fractional query algorithms” is more general than the “generalized algorithms” defined above since it allows for queries, starting at p_i , that let x_i evolve up to the hitting time $T_{\{q, p\}}$, where $p_i \in (q, p)$. Unlike in our model, the amount of information revealed by the algorithm is therefore not monotone in time. On the other hand, Gross assumes cost to be quadratic which corresponds to the specific choice of $\Gamma(p) = (2p - 1)^2$ in our model. We also refer the reader to [13] where, again considering more general algorithms and quadratic cost, the authors study the special case of the MAJORITY function on $n = 3$ bits.

7.2 Distributional (p, κ) -algorithmic complexity

In this subsection, we study the family of generalized algorithms

$$\mathcal{A}_p = \{\text{algorithms which are only allowed to ask } (1 - p)/p \text{ or } 0/1 \text{ questions}\}$$

for some fixed $p \in (1/2, 1)$. We note that the Γ -cost of an algorithm $A \in \mathcal{A}_p$ depends on Γ only through $\Gamma(p) \in [0, 1]$. Hence, we can represent the cost of a $(1-p)/p$ question by a number $\kappa \in [0, 1]$. We write $c_{p,\kappa}(A, f)$ for $c_\Gamma(A, f)$ and naturally define the *distributional (p, κ) -algorithmic complexity of f* as

$$a_{p,\kappa}(f, \pi_{1/2}) = \min_{A \in \mathcal{A}_p} c_{p,\kappa}(A, f).$$

An algorithm $A \in \mathcal{A}_p$ is called *(p, κ) -optimal for f* if it achieves the above minimum. We ask the analogue of Question 7.1 in this simplified setting.

QUESTION 7.2. *Given a Boolean function f , for which p and κ do we have*

$$a_{p,\kappa}(f, \pi_{1/2}) = a(f, \pi_{1/2}) ?$$

It will be convenient to define for $1 \leq i \leq n$, the random variables

$$Z_i := \mathbf{1}_{x_i(T_{(1-p,p)})=p} \quad \text{and} \quad X_i := x_i(T_{(0,1)}).$$

We note that the answers to the questions that any algorithm $A \in \mathcal{A}_p$ asks only depend on these random variables. Z_i and X_i are $\text{Ber}(1/2)$ -distributed, but not independent as $\mathbb{P}_x[X_i = Z_i] = p$. We now give an alternative description of algorithms in \mathcal{A}_p , which will turn out to be useful.

Definition 7.2. A sequence of random partitions $A = (A_t)_{0 \leq t \leq 2n} = (A_t^0, A_t^1, A_t^2)_{0 \leq t \leq 2n}$ of $[n]$ is an *algorithm* if

(i) $A_0 = ([n], \emptyset, \emptyset)$ and $A_{2n} = (\emptyset, \emptyset, [n])$,

and for every $t \in \{1, \dots, 2n\}$,

(ii) A_t is obtained from A_{t-1} by either moving an element from A_{t-1}^0 to A_{t-1}^1 or from A_{t-1}^1 to A_{t-1}^2 ,

(iii) A_t is measurable with respect to A_{t-1} , $(Z_i)_{i \in A_{t-1}^1}$, and $(X_i)_{i \in A_{t-1}^2}$.

The random partition represents bits for which we have no information (A^0), bits for which we have partial information (A^1), and bits for which we have full information (A^2). Moving bit i from A^0 to A^1 corresponds to querying Z_i or equivalently asking the $(1-p)/p$ question for bit i , thereby providing partial information about X_i . Also moving bit i from A^1 to A^2 corresponds to querying X_i after already having queried Z_i , or equivalently asking the 0/1 question for bit i after already having asked the $(1-p)/p$ question for bit i .

An algorithm A is called *induced* if it always queries X_i immediately after Z_i ; equivalently A_t^1 has at most one element for every t . Clearly, there is a one-to-one correspondence between the subfamily of induced algorithms, denoted by $\mathcal{I} \subseteq \mathcal{A}_p$, and the family of standard algorithms for f . Given a Boolean function f on n bits and an algorithm $A \in \mathcal{A}_p$, we define the (random) first time at which the algorithm determines f as

$$T_A = T_A(f) := \min \left\{ t \geq 0 : f \text{ is determined by } (X_i)_{i \in A_t^2} \right\}.$$

We note that the cost of $A \in \mathcal{A}_p$ to determine f is given by

$$c_{p,\kappa}(A, f) = \mathbb{E}_x \left[\kappa \cdot |A_{T_A}^1| + |A_{T_A}^2| \right].$$

Remark 7.3. For $\kappa = 1$ and any $p \in (1/2, 1)$, we have $a_{p,1}(f, \pi_{1/2}) = a(f, \pi_{1/2})$ since asking a 0/1 question after the corresponding $(1-p)/p$ question is of no disadvantage as it has no additional cost. In other words,

$$\min_{A \in \mathcal{A}_p} \mathbb{E}_x \left[|A_{T_A}^1| + |A_{T_A}^2| \right] = \min_{I \in \mathcal{I}} \mathbb{E}_x \left[|I_T^2| \right].$$

The next easy proposition establishes monotonicity in p and κ with respect to Question 7.2.

PROPOSITION 7.4. Consider $(p, \kappa), (p', \kappa') \in (1/2, 1) \times [0, 1]$. If $p \geq p'$ and $\kappa' \geq \kappa$, then we have

$$a_{p,\kappa}(f, \pi_{1/2}) = a(f, \pi_{1/2}) \implies a_{p',\kappa'}(f, \pi_{1/2}) = a(f, \pi_{1/2})$$

PROOF. First, fix p , let $\kappa' \geq \kappa$ and assume that there exists an induced (p, κ) -optimal algorithm $I \in \mathcal{I}$ for f . We note that the cost of any algorithm $A \in \mathcal{A}_p$ to determine f is non-decreasing in κ , and so also the distributional (p, κ) -algorithmic complexity of f . Moreover, the cost of any induced algorithm to determine f is constant in κ . Therefore,

$$a_{p,\kappa}(f, \pi_{1/2}) \leq a_{p,\kappa'}(f, \pi_{1/2}) \leq c_{p,\kappa'}(I, f) = c_{p,\kappa}(I, f),$$

and we deduce from the (p, κ) -optimality of I that both inequalities are in fact equalities. Hence, $a(f, \pi_{1/2}) = a_{p,\kappa}(f, \pi_{1/2})$ implies $a(f, \pi_{1/2}) = a_{p,\kappa'}(f, \pi_{1/2})$.

Second, fix κ and let $p \geq p'$. Since $a_{p',\kappa}(f, \pi_{1/2}) \leq a(f, \pi_{1/2})$, it suffices to show that $a_{p,\kappa}(f, \pi_{1/2}) \leq a_{p',\kappa}(f, \pi_{1/2})$. To this end, we consider the family $\mathcal{A}_{p',p}$ of generalized algorithms, which are only allowed to ask $(1-p')/p'$, $(1-p)/p$ or 0/1 questions, and we associate cost κ to asking a $(1-p')/p'$ question, cost 0 to asking a $(1-p)/p$ question (after the corresponding $(1-p')/p'$ question), and cost $1-\kappa$ to asking a 0/1 question (after the corresponding $(1-p)/p$ question). Since it is clearly optimal for any algorithm $A \in \mathcal{A}_{p',p}$ to always ask the $(1-p)/p$ question directly after the $(1-p')/p'$ question, we have using obvious notation

$$a_{p',\kappa}(f, \pi_{1/2}) = \min_{A \in \mathcal{A}_{p'}} c_{p',\kappa}(A, f) \geq \min_{A \in \mathcal{A}_{p',p}} c_{p',p,\kappa}(A, f) = \min_{A \in \mathcal{A}_p} c_{p,\kappa}(A, f) = a_{p,\kappa}(f, \pi_{1/2}).$$

□

For $p \in (1/2, 1)$ and $\kappa \in [0, 1]$, we define

$$\kappa_c(f, p) = \inf \{ \kappa : a_{p,\kappa}(f, \pi_{1/2}) = a(f, \pi_{1/2}) \},$$

and

$$p_c(f, \kappa) = \sup \{ p : a_{p,\kappa}(f, \pi_{1/2}) = a(f, \pi_{1/2}) \}.$$

By the previous proposition $p \mapsto \kappa_c(f, p)$ and $\kappa \mapsto p_c(f, \kappa)$ are (weakly) increasing functions. For the AND function on $n = 2$ bits, it can easily be shown based on the computations in Example 7.1 that $\kappa_c(\text{AND}_2, p) = 2p - 1$.

THEOREM 7.5. For every Boolean function f on $n \geq 1$ bits and $p \in (1/2, 1)$,

$$\kappa_c(f, p) \leq \kappa_0(n, p) := \frac{1}{1 + \frac{1}{n} \left(\frac{1-p}{2} \right)^n}.$$

We divide the proof into two lemmas.

LEMMA 7.6. Let f be a Boolean function on $n \geq 1$ bits. If $\kappa > \kappa_0(n, p)$, then every algorithm $A \in \mathcal{A}_p$ that is (p, κ) -optimal for f satisfies

$$\mathbb{E}_x \left[|A_{T_A^1}^1(f)| + |A_{T_A^2}^2(f)| \right] = \min_{I \in \mathcal{I}} \mathbb{E}_x \left[|I_{T_I}^2(f)| \right]. \quad (11)$$

Moreover, if (11) holds and if Z_i is a (p, κ) -optimal first query among all algorithms in \mathcal{A}_p , then Z_i is also an optimal first query among all algorithms in \mathcal{I} .

By Remark 7.3, the inequality \geq in (11) holds for all κ . AND₂ provides an easy counterexample showing that equality can fail for small κ . The following example shows that κ large is also a necessary assumption to ensure that an optimal first query among all algorithms in \mathcal{A}_p is also an optimal first query among all algorithms in \mathcal{I} .

Example 7.7. Consider a variant of the ADDRESS function on $n = 7$ bits, defined by

$$f(x_1, \dots, x_7) = \begin{cases} x_1 \oplus x_2 & \text{if } x_5 \oplus x_6 \oplus x_7 = 1, \\ x_3 \oplus x_4 & \text{if } x_5 \oplus x_6 \oplus x_7 = 0. \end{cases}$$

On the one hand, the algorithm I which first queries X_5, X_6, X_7 , and then either X_1, X_2 or X_3, X_4 depending on the value of $X_5 \oplus X_6 \oplus X_7$, has cost 5. In fact, every induced algorithm that is optimal among \mathcal{I} starts with querying X_5, X_6 and X_7 in some order. On the other hand, consider the algorithm A which first queries Z_1, Z_2, Z_3, Z_4 . If $Z_1 \oplus Z_2 = Z_3 \oplus Z_4$, it then queries X_1, X_2, X_3, X_4 and if needed also X_5, X_6, X_7 . Otherwise, it queries X_5, X_6, X_7 , and then either X_1, X_2 or X_3, X_4 depending on the value of $X_5 \oplus X_6 \oplus X_7$. Note that A has expected cost $9/2 + \kappa + 6p(1-p)(p^2 + (1-p)^2)$, which is strictly smaller than 5 if $\kappa < 1/2 - 6p(1-p)(p^2 + (1-p)^2)$. Furthermore, for such κ , one can check that Z_1, Z_2, Z_3, Z_4 are the only optimal first queries among \mathcal{A}_p .

To prove the lemma, it will be convenient to define for $1 \leq i \leq n$,

$$Y_i := \mathbf{1}_{Z_i=X_i}.$$

Note that $Y_i = 0$ means either the Brownian motion x_i terminated at 0 even though it hit $p \in (1/2, 1)$ before $1-p$ or it terminated at 1 even though it hit $1-p \in (0, 1/2)$ before p .

PROOF OF LEMMA 7.6. Fix a Boolean function f on $n \geq 1$ bits. By Remark 7.3,

$$\min_{B \in \mathcal{A}_p} \mathbb{E}_x \left[|B_{T_B}^1| + |B_{T_B}^2| \right] = \min_{J \in \mathcal{I}} \mathbb{E}_x \left[|J_{T_J}^2| \right], \quad (12)$$

and in the same way, we have for every $y_1, \dots, y_n \in \{0, 1\}$,

$$\begin{aligned} \min_{B \in \mathcal{A}_p} \mathbb{E}_x \left[|B_{T_B}^1| + |B_{T_B}^2| \mid Y_1 = y_1, \dots, Y_n = y_n \right] \\ = \min_{J \in \mathcal{I}} \mathbb{E}_x \left[|J_{T_J}^2| \mid Y_1 = y_1, \dots, Y_n = y_n \right]. \end{aligned} \quad (13)$$

We now fix a (p, κ) -optimal algorithm $A \in \mathcal{A}_p$ and an induced algorithm $I \in \mathcal{I}$ achieving the minimum in the right-hand side of (12). Since the cost of an induced algorithm is independent of $(Y_i)_{i=1}^n$, I also achieves the minimum in the right-hand side of (13) for all $y_1, \dots, y_n \in \{0, 1\}$. Thus, for every $y_1, \dots, y_n \in \{0, 1\}$,

$$\begin{aligned} \mathbb{E}_x \left[|A_{T_A}^1| + |A_{T_A}^2| \mid Y_1 = y_1, \dots, Y_n = y_n \right] &\geq \mathbb{E}_x \left[|I_{T_I}^2| \mid Y_1 = y_1, \dots, Y_n = y_n \right] \\ &= \mathbb{E}_x \left[|I_{T_I}^2| \right]. \end{aligned} \quad (14)$$

Towards a contradiction, assume that $\kappa > \kappa_0(n, p)$ and that in (11), the left-hand side is strictly larger than the right-hand side. Then inequality (14) must be strict for some $\hat{y}_1, \dots, \hat{y}_n \in \{0, 1\}$ and so we must have

$$\mathbb{E}_x \left[|A_{T_A}^1| + |A_{T_A}^2| \mid Y_1 = \hat{y}_1, \dots, Y_n = \hat{y}_n \right] \geq 2^{-n} + \mathbb{E}_x \left[|I_{T_I}^2| \mid Y_1 = \hat{y}_1, \dots, Y_n = \hat{y}_n \right] \quad (15)$$

since the conditional expectations are multiples of 2^{-n} . Indeed, they only depend on the randomness of Z_1, \dots, Z_n , which are independent of Y_1, \dots, Y_n . Comparing the (p, κ) -costs of A and I , we obtain

$$\begin{aligned} & c_{p,\kappa}(A, f) - c_{p,\kappa}(I, f) \\ &= (1 - \kappa) \cdot \underbrace{\left(\mathbb{E}_x \left[|A_{T_A}^2| \right] - \mathbb{E}_x \left[|I_{T_I}^2| \right] \right)}_{\geq -n} + \kappa \cdot \underbrace{\left(\mathbb{E}_x \left[|A_{T_A}^1| + |A_{T_A}^2| \right] - \mathbb{E}_x \left[|I_{T_I}^2| \right] \right)}_{\geq 2^{-n}(1-p)^n} \\ &\geq -n + \kappa \cdot (n + 2^{-n}(1-p)^n) > 0, \end{aligned}$$

where we bounded the difference $\mathbb{E}_x[|A_{T_A}^1| + |A_{T_A}^2|] - \mathbb{E}_x[|I_{T_I}^2|]$ from below using (14) and (15) and $\mathbb{P}_x[Y_1 = \hat{y}_1, \dots, Y_n = \hat{y}_n] \geq \min\{1-p, p\}^n = (1-p)^n$. This contradicts the (p, κ) -optimality of A . This concludes the proof of (11). The final statement is straightforward and left to the reader. \square

LEMMA 7.8. *Let f be a Boolean function on $n \geq 1$ bits. For every algorithm $A \in \mathcal{A}_p$,*

$$\mathbb{E}_x \left[|A_{T_A(f)}^1| + |A_{T_A(f)}^2| \right] = \min_{I \in \mathcal{I}} \mathbb{E}_x \left[|I_{T_I(f)}^2| \right] \implies \mathbb{E}_x \left[|A_{T_A(f)}^1| \right] = 0.$$

PROOF. Recall that for $p \in (1/2, 1)$, we have defined the random variables

$$Z_i = \mathbf{1}_{x_i(T_{(1-p,p)})=p}, \quad X_i = x_i(T_{(0,1)}), \quad \text{and} \quad Y_i = \mathbf{1}_{Z_i=X_i}$$

for $1 \leq i \leq n$. We naturally extend these definitions to $p = 1/2$ by choosing $(Z_i)_{1 \leq i \leq n}$ to be $\text{Ber}(1/2)$ -distributed random variables that are independent of each other and of the Brownian motions $(x_i)_{1 \leq i \leq n}$ in this case. For every $p \in [1/2, 1)$, we note that $Y_i \sim \text{Ber}(p)$, Y_i is independent of Z_i , and $X_i = Z_i \cdot Y_i + (1 - Z_i) \cdot (1 - Y_i)$.

Fix a Boolean function f on $n \geq 1$ bits, an algorithm $A \in \mathcal{A}_p$ and an algorithm $I \in \mathcal{I}$ that is optimal among all induced algorithms. As argued at the beginning of the proof of Lemma 7.6 (see (13)), we have

$$\mathbb{E}_x \left[|A_{T_A}^1| + |A_{T_A}^2| \mid Y_1 = y_1, \dots, Y_n = y_n \right] \geq \mathbb{E}_x \left[|I_{T_I}^2| \mid Y_1 = y_1, \dots, Y_n = y_n \right] \quad (16)$$

for every $y_1, \dots, y_n \in \{0, 1\}$. From now on, we assume $\mathbb{E}_x[|A_{T_A}^1| + |A_{T_A}^2|] = \min_{I \in \mathcal{I}} \mathbb{E}_x[|I_{T_I}^2|]$. Equation (16) then implies

$$\mathbb{E}_x \left[|A_{T_A}^1| + |A_{T_A}^2| \mid Y_1 = y_1, \dots, Y_n = y_n \right] = \mathbb{E}_x \left[|I_{T_I}^2| \mid Y_1 = y_1, \dots, Y_n = y_n \right] \quad (17)$$

for every $y_1, \dots, y_n \in \{0, 1\}$. Viewed as a sequence of random partitions (see Definition 7.2), it makes sense to also consider the algorithm A for other values of $p \in [1/2, 1)$. Neither of the two sides in (17) depends on p and we thus deduce that for every $p \in [1/2, 1)$, we have

$$\mathbb{E}_x \left[|A_{T_A}^1| + |A_{T_A}^2| \right] = \mathbb{E}_x \left[|I_{T_I}^2| \right].$$

However, for $p = 1/2$, we actually have no information about the values of the bits in A^1 , meaning that given the Z_i 's with $i \in A^1$, the X_i 's with $i \in A^1$ are still independent and $\text{Ber}(1/2)$ -distributed. Hence, the algorithm A can be viewed as a randomized algorithm where the external randomness is coming from the Z_i bits. It now follows from the optimality of I among all induced algorithms that we must have

$$\mathbb{E}_x \left[|A_{T_A}^2| \right] \geq \mathbb{E}_x \left[|I_{T_I}^2| \right], \quad \text{and thus,} \quad \mathbb{E}_x \left[|A_{T_A}^1| \right] = 0.$$

We deduce that for every $y_1, \dots, y_n \in \{0, 1\}$ and $p = 1/2$,

$$\mathbb{E}_x \left[|A_{T_A}^1| \mid Y_1 = y_1, \dots, Y_n = y_n \right] = 0.$$

This implies $\mathbb{E}_x[|A_{T_A}^1|] = 0$ for every $p \in [1/2, 1)$, thereby concluding the proof. \square

We are now ready to prove the theorem.

PROOF OF THEOREM 7.5. We need to show that for any Boolean function f on $n \geq 1$ bits and $\kappa > \kappa_0(n, p)$, we have

$$a_{p,\kappa}(f, \pi_{1/2}) = a(f, \pi_{1/2}). \quad (18)$$

Now, fix such an f and κ . Combining Lemmas 7.6 and 7.8, we deduce that every (p, κ) -optimal algorithm $A \in \mathcal{A}_p$ for f satisfies

$$\mathbb{E}_x \left[|A_{T_A}^1(f)| \right] = 0.$$

Hence, for every (p, κ) -optimal algorithm $A \in \mathcal{A}_p$ for f , we have

$$c_{p,\kappa}(A, f) = \mathbb{E}_x \left[|A_{T_A}^2(f)| \right] = c_{p,1}(A, f),$$

and so

$$a_{p,\kappa}(f, \pi_{1/2}) = a_{p,1}(f, \pi_{1/2}) = a(f, \pi_{1/2}),$$

where the last equality is due to Remark 7.3. This concludes the proof. \square

Remark 7.9. Equality (18) implies that for $\kappa > \kappa_0(n, p)$, any optimal first query among \mathcal{I} is also an optimal first query among \mathcal{A}_p . However, Example 7.7 shows that this might not be the case for κ small.

Recall that we have seen that $\kappa_c(\text{AND}_2, p) = 2p - 1$. Moreover, one can check that for any permutation invariant Boolean function f on $n = 3$ bits, $\kappa_c(f, p) \in \{0, 2p - 1\}$. This naturally leads to the following questions.

QUESTION 7.3. Let \mathcal{S} denote the set of permutation invariant Boolean functions. Do we have

$$\sup_{f \in \mathcal{S}} \kappa_c(f, p) = 2p - 1 ?$$

QUESTION 7.4. Do we have

$$\sup_f \kappa_c(f, p) = 2p - 1 \quad \text{or} \quad \sup_f \kappa_c(f, p) < 1 ?$$

8 Distributional algorithmic complexity under composition and an outlook on asymptotic separation

While this section does not contain theorems, it provides a natural conjecture together with a connection to asymptotic separation of distributional algorithmic and subcube partition complexity. It also brings us back to the discussion at the end of Sections 5. We believe that the ideas of the proof of Theorem 7.5 could lead to the following result which we therefore state as a conjecture.

CONJECTURE 8.1. For every $n, m \geq 1$, there exists $k_0(n, m) \in \mathbb{N}$, such that for any Boolean functions f and g on $n \geq 1$ respectively $m \geq 1$ bits and for any $k \geq k_0(n, m)$, we have

$$a \left(f \circ (g \oplus \text{PAR}_k), \pi_{1/2} \right) = a(f, \pi_{1/2}) \cdot a(g \oplus \text{PAR}_k, \pi_{1/2}).$$

The above statement easily implies that this also holds for f replaced by $f \oplus \text{PAR}_\ell$ for any ℓ . Therefore, the above conjecture would then imply that for every $n \geq 1$ and for every $L \geq 1$, there exists $k_1(n, L) \in \mathbb{N}$ such that for every Boolean function f on $n \geq 1$ bits and for every $k \geq k_1(n, L)$, we have

$$a \left((f \oplus \text{PAR}_k)^\ell, \pi_{1/2} \right) = \left(a(f \oplus \text{PAR}_k, \pi_{1/2}) \right)^\ell, \quad \forall 1 \leq \ell \leq L.$$

This naturally leads to the next question.

QUESTION 8.1. For which Boolean functions f does there exist $k_1(f) \in \mathbb{N}$ such that for every $k \geq k_1(f)$, we have

$$a\left((f \oplus \text{PAR}_k)^\ell, \pi_{1/2}\right) = \left(a(f \oplus \text{PAR}_k, \pi_{1/2})\right)^\ell, \quad \forall 1 \leq \ell < \infty?$$

In particular, does this hold for $f = A\text{-EQ}_3$ (see Example 2.8)?

While the parity function trivially provides such an example, one approach in achieving asymptotic separation of algorithmic and subcube partition complexity is to find an example of a Boolean function f for which $sc(f, \pi_p) < a(f, \pi_p)$ and which satisfies the above. As explained at the end of Section 5, our hope would be that the second question has a positive answer, implying that the sequence

$$\left((A\text{-EQ}_3 \oplus \text{PAR}_k)^\ell\right)_{\ell \geq 1}$$

asymptotically separates distributional algorithmic and subcube partition complexity for some sufficiently large $k \geq 1$.

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