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(article starts on next page)

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Baker–Akhiezer Specialisation of Joint Eigenfunctions for Hyperbolic Relativistic Calogero–Moser Hamiltonians

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Abstract
In earlier joint work with Ruijsenaars, we constructed and studied symmetric joint eigenfunctions J_N for quantum Hamiltonians of the hyperbolic relativistic N-particle Calogero–Moser system. For generic coupling values, they are non-elementary functions that in the $N = 2$ case essentially amount to a 'relativistic' generalisation of the conical function specialisation of the Gauss hypergeometric function ${}_2F_1$. In this paper, we consider a discrete set of coupling values for which the solution to the joint eigenvalue problem is known to be given by functions ψ_N of Baker–Akhiezer type, which are elementary, but highly nontrivial, functions. Specifically, we show that J_N essentially amounts to the antisymmetrisation of ψ_N and, as a byproduct, we obtain a recursive construction of ψ_N in terms of an iterated residue formula.

Keywords Baker-Akhiezer function · Calogero-Moser systems

Mathematics Subject Classification 33D52 · 81R12

1 Introduction

Relativistic generalizations of N-particle Calogero–Moser systems were originally conceived by Ruijsenaars to provide integrable quantum mechanical descriptions of relativistic quantum field theories in 1+1 spacetime dimensions, such as the quantum sine-Gordon theory, restricted to a N-particle sector [27, 28]. For this purpose, the relativistic system of hyperbolic type, given by the formally self-adjoint and pairwise commutating analytic difference operators (AΔOs)

$$S_r(g;x) \equiv \sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=r}} \prod_{\substack{j\in I\\ k\notin I}} f_-(g;x_j-x_k) \prod_{l\in I} \exp(-i\hbar\beta\partial_{x_l}) \prod_{\substack{j\in I\\ k\notin I}} f_+(g;x_j-x_k), \quad (1.1)$$

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where

$$f_{\pm}(g;z)=\left(\frac{\sinh(\mu(z\pm i\beta g)/2)}{\sinh(\mu z/2)}\right)^{1/2}, \tag{1.2}$$

is of particular importance. Here, $r=1,\dots,N$, it is natural to view $\beta>0$ as $1/mc$, with m the particle mass and c the speed of light, and in the nonrelativistic limit $c\to\infty$ pairwise commuting Hamiltonians of the nonrelativistic hyperbolic Calogero–Moser system are recovered; see e.g. the surveys [14, 26] and references therein.

In a series of joint papers with Ruijsenaars [15, 16, 18], we developed a recursive scheme producing explicit symmetric joint eigenfunctions of the AΔOs (1.1). In addition to various analyticity and invariance properties, we deduced an explicit formula for dominant asymptotics deep in a Weyl chamber and thereby proved that particles in the hyperbolic relativistic Calogero–Moser system exhibit soliton scattering (i.e. conservation of momenta and factorization of the scattering matrix).

Further fundamental properties of the joint eigenfunctions were obtained by Belousov et al. in a number of recent papers [1–4]. Their results include integral equations, a reflection symmetry of the coupling constant, self-duality under interchange of geometric and spectral variables as well as orthogonality and completeness relations.

On a formal level, Ruijsenaars' hyperbolic AΔOs (1.1) are closely related to Macdonald's q-difference operators

$$D_N^r(z;q,t)\equiv t^{r(r-1)/2}\sum_{\substack{I\subset\{1,\dots,N\}\\|I|=r}}\prod_{\substack{j\in I\\k\notin I}}\frac{tx_j-x_k}{x_j-x_k}\prod_{l\in I}T_{q,x_l}, \tag{1.3}$$

where $(T_{q,x_l}f)(x_1,\dots,x_k,\dots,x_N)=f(x_1,\dots,qx_l,\dots,x_N)$. (The precise relationship can be gleaned from (2.6) and (3.9).) They act on the space of symmetric polynomials in N variables, on which they are simultaneously diagonalised by the symmetric GL_N type Macdonald polynomials; see e.g. [21, 23, 31]. For parameter values of the form $t=q^m$, $m\in\mathbb{Z}$, Etingof and Styrkas [10] and Chalykh [7] constructed and studied non-symmetric joint eigenfunctions of Baker–Akhiezer (BA) type and obtained, in particular, a generalized Weyl character formula for the Macdonald polynomials, first conjectured by Felder and Varchenko [11], where the BA-function replaces the exponential function; see also Chalykh and Etingof [8] for further related results and references.

In this paper, we restrict attention to the discrete set of coupling values $g=m\hbar$, $m\in\mathbb{Z}_+$, and show in Thm. 3.1 that the joint eigenfunctions from [15] of Ruijsenaars' AΔOs $S_r(m\hbar;x)$ are obtained by antisymmetrization of the (self-dual) BA-function associated with the Macdonald operators $D_N^r(z;q^2,q^{-2m})$. As a by-product of our proof, we obtain in Prop. 3.2 a recursive construction of the BA-function by an iterated residue formula.

For the coupling values $g=m\hbar$ under consideration, our main result provides an expansion of symmetric joint S_r-eigenfunctions in terms of eigenfunctions that are 'asymptotically free' deep inside Weyl chambers $x_{\sigma(1)}>x_{\sigma(2)}>\cdots>x_{\sigma(N)}$,

$\sigma \in S_N$. In the case of the Macdonald operators and generic parameter values, such asymptotically free solutions, often referred to as (q-)Harisch–Chandra series, have been studied in detail by Letzter and Stokman [20], van Meer and Stokman [22] as well as Noumi and Shiraishi [24]; and Stokman [30] derived a corresponding (c-function) expansion of Cherednik's basic hypergeometric function associated to root systems. We should also note that Bullimore et al. [5] showed that the partition function of the superconformal field theory $T[U(N)]$, as introduced by Gaiotto and Witten [13], displays the very same recursive structure as the joint S_r-eigenfunctions from [15]; and, identifying integration with summing residues at all (infinitely many) poles above the real axis, they obtained an expression in terms of series of an asympototically free type. Furthermore, in the $N = 2$ case, di Francesco et al. [9] recently obtained a number of interesting results on the S_r-eigenfunctions, including a similar residue computation where the relevant series are identified as (q-)Harish–Chandra series. To establish the precise connection between these expansion results and the recursive construction in [15] is, I believe, an interesting problem to which I plan to return to elsewhere. In the BC_1 case, the analogous connection between integral representations and series expansions was established by van de Bult et al. [6].

The remainder of the paper is structured as follows: In Section 2, we briefly recall definitions and key properties pertaining to joint eigenfunctions of the Ruijsenaars operators (1.1) and BA-(eigen)functions for the Macdonald operators (1.3); and, in Section 3, we give the precise formulations and proofs of our results.

2 Preliminaries

2.1 Joint Eigenfunctions

To begin with, we reparametrize the two length scales of the AΔOs (1.1) as

$$a_+ \equiv 2\pi/\mu > 0, \quad \text{(imaginary period/interaction length)} \tag{2.1}$$

and

$$a_- \equiv \hbar\beta > 0, \quad \text{(shift step size/Compton wavelength)} \tag{2.2}$$

and replace the coupling g by the parameter

$$b \equiv \beta g. \tag{2.3}$$

Rewriting (1.1) in terms of the parameters $a_\pm$, the coefficients become manifestly ia_+-periodic. It follows that the AΔOs obtained after the interchange $a_+ \leftrightarrow a_-$ commute with the given ones. In this way, we obtain $2N$ pairwise commuting AΔOs $H_{r,\delta}(b; x)$, with $r = 1, \ldots, N$ and $\delta = +, -$, and where $H_{r,+}(b; x) = S_r(g; x)$ with b and g related as above; cf. Eq. (1.7) in [15].

It is often convenient to work with similarity transforms of the AΔOs $H_{r,\delta}$ by the weight function

$$W_N(b;x) \equiv \prod_{1\le j<k\le N}\prod_{\delta=+,-}\frac{G(\delta(x_j-x_k)+i(a_++a_-)/2)}{G(\delta(x_j-x_k)+i(a_++a_-)/2-ib)}, \tag{2.4}$$

where $G(z)\equiv G(a_+,a_-;z)$ is the hyperbolic Gamma function from [25], which is $a_+\leftrightarrow a_-$ invariant, meromorphic in z and satisfies the analytic difference equations

$$\frac{G(z+ia_\delta/2)}{G(z-ia_\delta/2)}=2\cosh(\pi z/a_{-\delta}),\quad \delta=+,-; \tag{2.5}$$

see loc. cit. Prop. III.1 and Prop. III.2. (Unless needed, we suppress dependence of G and functions constructed from G on the parameters $a_\pm$.) Indeed, using these difference equations, it is readily verified that

$$\begin{aligned} A_{r,\delta}(b;x) &\equiv W_N(b;x)^{-1/2}H_{r,\delta}(b;x)W_N(b;x)^{1/2}\\ &=\sum_{\substack{I\subset\{1,\dots,N\}\\ |I|=r}}\prod_{\substack{j\in I\\ k\notin I}}\frac{\sinh(\pi(x_j-x_k-ib)/a_\delta)}{\sinh(\pi(x_j-x_k)/a_\delta)}\prod_{l\in I}\exp(-ia_{-\delta}\partial_{x_l}), \end{aligned} \tag{2.6}$$

which, in contrast to $H_{r,\delta}$, act on the space of meromorphic functions. (We recall that, for $(a_+,a_-,b)\in(0,\infty)^3$ such that $b<a_++a_-$ and $x\in\mathbb{R}^N$, the weight function is regular and positive.)

In [15] functions $J_N(b;x,y)$ having the joint eigenfunction property

$$A_{r,\delta}(b;x)J_N(b;x,y)=J_N(b;x,y)\sum_{1\le j_1<\cdots<j_r\le N}e^{2\pi(y_{j_1}+\cdots+y_{j_r})/a_\delta}, \tag{2.7}$$

are constructed recursively by an explicit integral formula, with integrand built from J_{N-1}, the weight function W_{N-1} as well as the kernel function

$$\mathcal{S}^\sharp_N(b;x,y)\equiv\prod_{j=1}^N\prod_{k=1}^{N-1}\frac{G(x_j-y_k-ib/2)}{G(x_j-y_k+ib/2)}. \tag{2.8}$$

More precisely, with $J_1(x,y)\equiv\exp(2\pi ixy/a_+a_-)$ as the starting point for the recursion, J_N, $N>1$, is given by

$$\begin{aligned} J_N(b;x,y)&=\frac{\exp\left(\frac{2\pi i}{a_+a_-}y_N\sum_{j=1}^N x_j\right)}{(N-1)!}\\ &\cdot\int_{\mathbb{R}^{N-1}}dz\,W_{N-1}(b;z)\mathcal{S}^\sharp_N(b;x,z)J_{N-1}(b;z,(y_1-y_N,\dots,y_{N-1}-y_N)). \end{aligned} \tag{2.9}$$

From Prop. III.2 in [25], we recall that the hyperbolic gamma function is scale invariant, in the sense that

$$G(\lambda a_+, \lambda a_-; \lambda z) = G(a_+, a_-; z), \quad \lambda \in (0, \infty). \tag{2.10}$$

Since J_N is constructed almost entirely from $G(z)$, it is easily seen to have the invariance properties

$$J_N(a_+, a_-, b; x, y) = J_N(a_-, a_+, b; x, y), \tag{2.11}$$
$$J_N(\lambda a_+, \lambda a_-, \lambda b; \lambda x, \lambda y) = \lambda^{N(N-1)/2} J_N(a_+, a_-, b; x, y). \tag{2.12}$$

Hence, we may and shall restrict attention to

$$a_+ = 1, \quad a_- \equiv a \geq 1, \tag{2.13}$$

without any loss of generality. Moreover, to facilitate the comparison with the pertinent BA-function, it is expedient to renormalize J_N and introduce the function

$$\Phi_N(a, b; x, y) \equiv \left(\frac{G(1, a; ib - i(1+a)/2)}{\sqrt{a}}\right)^{N-1} J_N(1, a, b; x, y), \tag{2.14}$$

which, in particular, satisfies the simple duality relation

$$\Phi_N(a, b; x, y) = \Phi_N(a, 1 + a - b; y, x), \tag{2.15}$$

as conjectured in [15], verified for $N \leq 3$ in [16] and proven for arbitrary N in [2] (see Thm. 5 and Eqs. (1.53)–(1.54)).

2.2 Baker–Akhiezer Functions

Take $q \in \mathbb{C}^\times \equiv \mathbb{C} \setminus \{0\}$, $m \in \mathbb{Z}_+$ and let ψ_N be a function of $x, y \in \mathbb{C}^N$ that is of the form

$$\psi_N(x, y) = q^{2(x,y)} \sum_\nu \psi_{N,\nu}(x) q^{2(\nu,y)}, \tag{2.16}$$

where the sum extends over weight vectors

$$\nu = \sum_{1 \leq j < k \leq N} \left(\frac{m}{2} - l_{jk}\right)(e_j - e_k), \quad l_{jk} = 0, \ldots, m, \tag{2.17}$$

with $\{e_1, \ldots, e_N\}$ the standard ON-basis in $\mathbb{R}^N$. Suppose, in addition, that ψ_N satisfies the vanishing condition

$$\psi_N(x, y + s(e_j - e_k)/2) - \psi_N(x, y - s(e_j - e_k)/2) = 0 \text{ when } q^{2(y_j - y_k)} = 1, \tag{2.18}$$

for all $1 \leq j < k \leq N$ and $s = 1, \ldots, m$. Then it is called a Baker–Akhiezer (BA) function associated with the root system A_{N-1} and (positive) integer parameter m. Existence is established by different constructions in Sect. 5 of [10] and Sect. 3.2 of [7]. We note that the iterated residue formula obtained in this paper yields an additional (constructive) existence proof.

Assuming q is not a root of unity, we know from [7, 10] that a function ψ_N with the above properties is unique up to a choice of normalisation; and that, imposing the normalisation condition

$$\psi_{N,\rho_N}(x) = \prod_{1 \leq j < k \leq N} \prod_{n=1}^{m} [n + x_k - x_j], \quad [z] \equiv q^z - q^{-z}, \tag{2.19}$$

yields a self-dual BA function, in the sense that $\psi_N(x, y) = \psi_N(y, x)$. Here, we have used the standard notation

$$\rho_N \equiv \rho_N(m) = \frac{m}{2} \sum_{1 \leq j < k \leq N} (e_j - e_k) = \frac{m}{2} \sum_{j=1}^{N} (N - 2j + 1) e_j, \tag{2.20}$$

obtained by setting all $l_{ij} = 0$ in (2.17). The self-dual BA-function satisfies the bispectral system of q-difference equations

$$D_N^r(q^{2y}; q^2, q^{-2m}) \psi(x, y) = \psi(x, y) \sum_{1 \leq j_1 < \cdots < j_r \leq N} q^{2(x_{j_1} + \cdots + x_{j_r})}, \tag{2.21}$$

$$D_N^r(q^{2x}; q^2, q^{-2m}) \psi(x, y) = \psi(x, y) \sum_{1 \leq j_1 < \cdots < j_r \leq N} q^{2(y_{j_1} + \cdots + y_{j_r})}, \tag{2.22}$$

with $r = 1, \ldots, N$, $q^{2x} \equiv (q^{2x_1}, \ldots, q^{2x_N})$ and $q^{2y} \equiv (q^{2y_1}, \ldots, q^{2y_N})$.

3 Baker–Akhiezer Specialisation

When $b = m \in \mathbb{Z}_+$, the weight function W_N (2.4) and kernel function $\mathcal{S}_N^{\sharp}$ (2.8) reduce to elementary functions. More precisely, from the difference equation (2.5) satisfied by $G(z)$, we see that

$$W_N(1, a, m; x) = \prod_{1 \leq j < k \leq N} \sinh\left(\frac{\pi}{a}(x_k - x_j)\right) \prod_{n=-m+1}^{m-1} 2 \sinh\left(\frac{\pi}{a}(x_k - x_j + in)\right) \tag{3.1}$$

and

$$\mathcal{S}_N^\sharp(1,a,m;x,y)=\prod_{j=1}^{N}\prod_{k=1}^{N-1}\prod_{n=0}^{m-1}\frac{1}{2\cosh\left(\frac{\pi}{a}(x_j-y_k+i(m-2n-1)/2)\right)}. \tag{3.2}$$

Up to an overall numerical factor, $W_N(1,a,m;x)$ equals the specialisation $\Delta(q^{2ix};q^2,q^{2m})$ of Macdonald's weight function Δ, while $\mathcal{S}_N^\sharp(1,a,m;x,y)$ is obtained by a similar specialisation of his product function Π after removing a factor depending only on $\sum_{j=1}^N(x_j-y_j)$; cf. Eqs. (9.2) and (2.5), respectively, in Section VI of [21]. Furthermore, (2.5) and the formula $G(1,a;i(1-a)/2)=1/\sqrt{a}$ (see Eq. (3.38) in [25]) entail that

$$\frac{G(1,a;im-i(1+a)/2)}{\sqrt{a}}=\frac{\prod_{n=1}^{m-1}2\sin(\pi n/a)}{a}. \tag{3.3}$$

Consequently, the $b=m$ specialisation of Φ_N is given by recursively by

$$\begin{aligned}\Phi_N(m;x,y)=&\frac{\prod_{n=1}^{m-1}2\sin(\pi n/a)}{a(N-1)!}e^{\frac{2\pi i}{a}y_N(x_1+\cdots+x_N)}\\&\int_{\mathbb{R}^{N-1}}dz\prod_{1\le j<k\le N-1}\sinh\left(\frac{\pi}{a}(z_k-z_j)\right)\prod_{n=-m+1}^{m-1}2\sinh\left(\frac{\pi}{a}(z_k-z_j+in)\right)\\&\cdot\frac{\Phi_{N-1}(m;z,(y_1-y_N,\ldots,y_{N-1}-y_N))}{\prod_{j=1}^{N}\prod_{k=1}^{N-1}\prod_{n=0}^{m-1}2\cosh\left(\frac{\pi}{a}(x_j-z_k+i(m-2n-1)/2)\right)}\end{aligned} \tag{3.4}$$

with $\Phi_1(x,y)=\exp\left(\frac{2\pi}{a}xy\right)$. We note that this is effectively a basic hypergeometric integral, since the weight and kernel functions can be expressed in terms of the product

$$\prod_{n=0}^{m-1}(q^{n+z}-q^{-n-z})=(-1)^m q^{-m(m-1)/2-mz}\frac{(q^{2z};q^2)_\infty}{(q^{2(m+z)};q^2)_\infty}, \tag{3.5}$$

with

$$q=e^{-i\pi/a},\quad a\in\mathbb{R}. \tag{3.6}$$

To be precise, while the (modified) q-Gamma function $(q^{2z};q^2)_\infty$ is only well-defined for $|q|<1$, the above ratio clearly extends to $|q|=1$. For accounts of (multivariable) basic hypergeometric integrals, see e.g. the overview by Koornwinder and Stokman [19] and the survey by Schlosser [29].

Upon setting $(a_+,a_-,b)=(1,a,m)$, the ia-antiperiodicity of the functions involved entails that the eigenvalue equations (2.7), which are are also satisfied by

Φ_N, become trivial for $\delta = +$. As we now show, the remaining equations, corresponding to $\delta = -$, are closely related to the Macdonald eigenvalue equations (2.22), of which the BA-functions are solutions.

Letting

$$\delta_N(x;m) = \prod_{1\le j<k\le N}\prod_{n=-m}^{m}[x_j - x_k + n], \tag{3.7}$$

we recall the well-known similarity transform

$$\delta_N(x;m-1)^{-1} D_N^r(q^{2x};q^2,q^{-2(m-1)})\delta_N(x;m-1) = D_N^r(q^{2x};q^2,q^{2m}), \tag{3.8}$$

which is easily verified by a direct computation; and with the parametrisation (3.6) in effect it is readily seen that

$$q^{mr(1-N)} D_N^r(q^{2ix};q^2,q^{2m}) = \sum_{\substack{I\subset\{1,\dots,N\}\\ |I|=r}}\prod_{\substack{j\in I\\ k\notin I}}\frac{\sinh\left(\frac{\pi}{a}(x_j - x_k im)\ big\right)}{\sinh\left(\frac{\pi}{a}(x_j-x_k)\right)}\prod_{l\in I}\exp(-i\partial_{x_l}). \tag{3.9}$$

Note that the right-hand side coincides with the AΔO $A_{r,-}(1,a,m;x)$ (2.6).

In what follows, it will be important to keep track of dependence on the parameters a or q as well as m. Therefore, we write $\psi_N(q,m;x,y)$ for the self-dual BA-function characterised by (2.16)–(2.19). By the joint eigenfunction properties reviewed in the previous section, it is natural to expect that $\Phi_N(a,m;x,y)$ is proportional to the symmetrisation of $\delta_N(ix;m-1)^{-1}\psi_N(e^{-i\pi/a},m-1;ix,iy)$. Indeed, since both functions satisfy the same set of joint eigenvalue equations with respect to the S_N-invariant AΔOs (3.9), it is plausible that $\Phi_N(a,m;x,y)$ can be expressed as a linear combination of the functions $\delta_N(ix;m-1)^{-1}\psi_N(e^{-i\pi/a},m-1;i\sigma(x),iy)$, $\sigma\in S_N$, where we have used the fact that

$$\delta_N(i\sigma(x);m-1) = (-)^\sigma\delta_N(ix;m-1),\quad \sigma\in S_N, \tag{3.10}$$

which is clear from (3.7). Moreover, given that $\Phi_N(a,m;x,y)$ is manifestly symmetric in x (cf. (2.9) and (2.14)), the coefficients in the linear combination should be of the form $c(y)(-)^\sigma$, with c independent of σ. In this section, we substantiate this expectation by proving the following theorem.

Theorem 3.1 *Let $m\in\mathbb{Z}_+$ and let $a > m-1$ be such that $\exp(-i\pi/a)$ is not a root of unity. Then, we have*

$$\Phi_N(a,m;x,y) = C_N(a,m)\frac{\sum_{\sigma\in S_N}(-)^\sigma\psi_N(e^{-i\pi/a},m-1;i\sigma(x),iy)}{\prod_{1\le j<k\le N}2\sinh\left(\frac{\pi}{a}(y_k-y_j)\right)\prod_{n=-m+1}^{m-1}2\sinh\left(\frac{\pi}{a}(x_k-x_j+in)\right)}$$

(3.11)

with the constant

$$C_N(a,m) = (i^m)^{N(N-1)/2}\left(\frac{a}{\prod_{n=1}^{m-1} 2\sin(\pi n/a)}\right)^{(N-1)(N-2)/2}. \tag{3.12}$$

The first main observation in our proof of Thm. 3.1, as detailed below, is that the product function $\Phi_N(a,m;x,y)\prod_{1\le j<k\le N} 2\sinh(\pi(y_j-y_k))$ can be rewritten as the symmetrisation of a function $\varphi_N(a,m;x,y)$ that is given recursively by an iterated residue formula.

More specifically, the formula involves denominator factors $\sinh\big(\frac{\pi}{a}(x_j - z_k + i(m-2n-1)/2)\big)$, producing poles at

$$z_k = x_j + i(m-2n-1)/2 + i\ell a, \quad n = 0,\dots,m-1, \quad \ell\in\mathbb{Z}, \tag{3.13}$$

where $j = 1,\dots,N$ and $k = 1,\dots,N-1$. Assuming that $\mathrm{Re}\,(x_j - x_k)\neq 0$ for all $1\le j<k\le N$ and $a > m-1$, we can find contours γ_k that only encircle the poles corresponding to a fixed value of $k = 1,\dots,N-1$ and $\ell = 0$; see the discussions preceding Props. 3.3 and 3.6 for explicit examples. Up to normalisation, the recursive formula defining φ_N can then be obtained from (3.4) by the substitutions $\Phi\to\varphi$, $\mathbb{R}^{N-1}\to\gamma_1\times\cdots\times\gamma_{N-1}$ and $\cosh\to\sinh$.

The second main observation, which more or less completes the proof of Thm. 3.1, is that φ_N essentially amounts to a renormalisation of ψ_N and, as a byproduct, we thus find the following recursive construction of the self-dual BA-function ψ_N.

Proposition 3.2 *Under the above assumptions on m and a, the BA-function ψ_N is obtained from ψ_{N-1} by the iterated residue formula*

$$\begin{aligned}&\psi_N(e^{-i\pi/a},m;ix,iy)\\ &=\left(\frac{\prod_{n=1}^m\sin(\pi n/a)}{a(-i)^{m+1}}\right)^{N-1}\frac{W_N(1,a,m+1;z)}{\prod_{1\le j<k\le N}2\sinh\left(\frac{\pi}{a}(x_kx_j)\right)}e^{\frac{2\pi i}{a}y_N\sum_{j=1}^N x_j}\\ &\cdot\int_{\underline{\gamma}}dz\,\frac{\psi_{N-1}(e^{-i\pi/a},m;iz,i(y_1-y_N,\dots,y_{N-1}-y_N))\prod_{1\le j<k\le N-1}2\sinh\left(\frac{\pi}{a}(z_k-z_j)\right)}{\prod_{j=1}^N\prod_{k=1}^{N-1}\prod_{n=0}^m 2\sinh\left(\frac{\pi}{a}\left(x_j-z_k+\frac{i}{2}(m-2n)\right)\right)},\end{aligned} \tag{3.14}$$

where $\underline{\gamma}\equiv\gamma_1\times\cdots\times\gamma_{N-1}$.

This result provides a natural generalisation of the iterated residue formulae for BA-functions for rational and trigonometric Calogero–Moser–Sutherland operators obtained by Felder and Veselov [12].

In the two subsections below, we provide our proofs of Thm. 3.1 and Prop. 3.2. To begin with, we establish the $N = 2$ cases of the results. These we then use as the basis for proofs by induction on N. To shorten some of the formulae involved, we typically suppress dependence on the parameters a and m.

3.1 The $N = 2$ Case

Setting $N = 2$ in (3.4), we deduce

$$\begin{aligned}
&\frac{2a\sinh\left(\frac{\pi}{a}(y_1-y_2)\right)}{\prod_{n=1}^{m-1}2\sin(\pi n/a)}\Phi_2(x,y)\\
&=\int_{\mathbb{R}}dz\frac{e^{\frac{2\pi i}{a}(y_1-y_2)(z-ia/2)}-e^{\frac{2\pi i}{a}(y_1-y_2)(z+ia/2)}}{\prod_{j=1}^{2}\prod_{n=0}^{m-1}2\cosh\left(\frac{\pi}{a}(x_j-z+i(m-2n-1)/2)\right)}\\
&=(-1)^m\left(\int_{\mathbb{R}-ia/2}dz-\int_{\mathbb{R}+ia/2}dz\right)\\
&\frac{e^{\frac{2\pi i}{a}(y_1-y_2)z}}{\prod_{j=1}^{2}\prod_{n=0}^{m-1}2\sinh\left(\frac{\pi}{a}(x_j-z+i(m-2n-1)/2)\right)}.
\end{aligned}\tag{3.15}$$

We note that $a > m-1$ and $x, y \in \mathbb{R}^2$ ensures that Φ_2 is well-defined and that the only poles of the integrand located within the strip $|\mathrm{Im}\, z| < a/2$ are

$$z = x_j + i(m-2n-1)/2, \quad j = 1,2, \quad n = 0,\dots,m-1. \tag{3.16}$$

At first, we assume $x = (x_1, x_2) \in \mathbb{R}^2$ is such that $x_1 \neq x_2$, which ensures that we can introduce a contour γ_1 encircling only the poles (3.16) with $j = 1$ counterclockwise. For example, with $\delta \equiv |x_1 - x_2| > 0$, we can take the contour consisting of line segments $x_1 + u \pm ia/2$, $|u| \le \delta/2$, and $x_1 \pm \delta/2 + iv$, $|v| \le a/2$. Introducing the function

$$\varphi_2(x,y) \equiv e^{\frac{2\pi i}{a}y_2(x_1+x_2)}\int_{\gamma_1}dz\frac{e^{\frac{2\pi i}{a}(y_1-y_2)z}}{\prod_{j=1}^{2}\prod_{n=0}^{m-1}2\sinh\left(\frac{\pi}{a}(x_j-z+i(m-2n-1)/2)\right)}, \tag{3.17}$$

the following result is an easy consequence of (3.15) and Cauchy's residue theorem.

Proposition 3.3 *For $m \in \mathbb{Z}_+$ and $a > m-1$, we have*

$$\Phi_2(x,y) = (-1)^m\frac{\prod_{n=1}^{m-1}2\sin(\pi n/a)}{a}\frac{\sum_{\sigma\in S_2}\varphi_2(\sigma(x),y)}{2\sinh\left(\frac{\pi}{a}(y_1-y_2)\right)}. \tag{3.18}$$

We proceed to establish the precise connection between φ_2 and the self-dual BA-function ψ_2. As detailed in the following proposition, when applying Cauchy's residue thm. to (3.17) we obtain a series expansion for φ_2 analogous to the expansion of ψ_2 given by the $N = 2$ instance of (2.16).

Proposition 3.4 *Assuming that $m \in \mathbb{Z}_+$ and $a > m-1$, we have*

$$\varphi_2(x,y) = e^{\frac{2\pi i}{a}(x,y)}\sum_{l=0}^{m-1}\varphi_{2,l}(x)e^{\frac{2\pi}{a}\left(\frac{m-1}{2}-l\right)(y_1-y_2)} \tag{3.19}$$

with

$$\varphi_{2,l}(x) = \frac{a(-i)^m}{\prod_{n\neq l} 2\sin\left(\frac{\pi}{a}(n-l)\right)} \frac{1}{\prod_{n=0}^{m-1} 2\sinh\left(\frac{\pi}{a}(x_2 - x_1 + i(n-l))\right)}. \tag{3.20}$$

We note that the above series essentially amounts to a special (terminating) case of the basic hypergeometric series

$${}_2\phi_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\middle| q, z\right) = \sum_{l=0}^{\infty} \frac{(a:q)_l (b:q)_l}{(c;q)_l (q;q)_l} z^l, \tag{3.21}$$

where the q-shifted factorial is given by $(a;q)_0 = 1$ and $(a;q)_l = (1-a)\cdots(1-aq^{l-1})$ when $l > 0$. To verify this claims, it suffices to observe that

$$\begin{aligned} \frac{1}{\prod_{n\neq l} 2\sin\left(\frac{\pi}{a}(n-l)\right)} &= \frac{1}{\prod_{n=1}^{m-1} 2\sin\left(\frac{\pi}{a}n\right)} \prod_{n=0}^{l-1} \frac{2\sin\left(\frac{\pi}{a}(m-1-n)\right)}{2\sin\left(\frac{\pi}{a}(-n-1)\right)} \\ &= \frac{t_m^l}{\prod_{n=1}^{m-1} 2\sin\left(\frac{\pi}{a}n\right)} \frac{(q^2/t_m^2;q^2)_l}{(q^2;q^2)_l} \end{aligned} \tag{3.22}$$

and similarly that

$$\begin{aligned} &\frac{1}{\prod_{n=0}^{m-1} 2\sinh\left(\frac{\pi}{a}(x_2 - x_1 + i(n-l))\right)} \\ &= t_m^l \frac{1}{\prod_{n=0}^{m-1} 2\sinh\left(\frac{\pi}{a}(x_2 - x_1 + in)\right)} \frac{(q^2 X_2/t_m^2 X_1;q^2)_l}{(q^2 X_2/X_1;q^2)_l}. \end{aligned} \tag{3.23}$$

Indeed, using the parametrisation (3.6) for q and introducing

$$t_m \equiv e^{-\frac{i\pi}{a}m} = q^m \tag{3.24}$$

as well as

$$X_j \equiv e^{\frac{2\pi}{a}x_j} = q^{i2x_j}, \quad Y_j \equiv e^{\frac{2\pi}{a}y_j} = q^{i2y_j}, \quad j = 1, 2, \tag{3.25}$$

it becomes clear from the above computations that

$$\begin{aligned} \varphi_2(x, y) = &\frac{a(-i)^m \exp\left(\frac{2\pi i}{a}(xi\ rho o_2(m-1), y)\right)}{\prod_{n=1}^{m-1} 2\sin(\pi n/a) \prod_{n=0}^{m-1} 2\sinh\left(\frac{\pi}{a}(x_2 - x_1 + in)\right)} \\ &\cdot {}_2\phi_1\left(\begin{matrix} q^2/t_m^2,\ q^2 X_2/t_m^2 X_1 \\ q^2 X_2/X_1 \end{matrix}\middle| q^2, t_m^2 Y_2/Y_1\right). \end{aligned} \tag{3.26}$$

Starting from the q-difference eigenvalue equation satisfied by ${}_2\phi_1$, it would now be a simple exercise to show that φ_2 is an eigenfunction of the $N = 2$ Macdonald operator D_2^1 for suitable parameter values. One could also invoke the duality relation (2.15) and linear independence of $\varphi_2(\sigma(x), y)$, $\sigma \in S_2$, for generic $x \in \mathbb{R}^2$; see Lemma 3.8 below. Either way, one could then use a uniqueness result such as Prop. 4.1 in [23] for 'asymptotically free' eigenfunctions of D_2^1 to arrive at the connection between φ_2 and ψ_2. We note that such uniqueness results for arbitrary N can be found in the paper of Letzter and Stokman [20] (in a general root system setting) as well as that of Noumi and Shiraishi [24].

Here, we rely instead on the defining properties of ψ_2 and show that φ_2 has vanishing properties similar to those given by (2.18) with $N = 2$.

Proposition 3.5 *For $s = 1, \dots, m-1$, we have*

$$\varphi_2(x, y + is(e_1 - e_2)/2) - \varphi_2(x, y - is(e_1 - e_2)/2) = 0 \ \text{ when } \ e^{\frac{2\pi}{a}(y_1 - y_2)} = 1. \tag{3.27}$$

Proof Up to an overall factor $2\exp(\frac{2\pi i}{a}(y_2(x_1 + x_2)))$, the LHS of (3.27) is given by the integral

$$\int_{\gamma_1} dz \frac{\exp\left(\frac{2\pi i}{a}(y_1 - y_2)z\right)\sinh(\frac{\pi s}{a}(x_1 + x_2 - 2z))}{\prod_{j=1}^2\prod_{n=0}^{m-1} 2\sinh(\frac{\pi}{a}(x_j - z + i(m - 2n - 1)/2))}, \tag{3.28}$$

which, by Cauchy's theorem, equals

$$\begin{aligned}(-i)^m e^{\frac{2\pi i}{a}(y_1 - y_2)x_1} \sum_{l=0}^{m-1} \frac{a}{\prod_{n\neq l} 2\sin(\frac{\pi}{a}(n-l))} \\ \cdot \frac{\sinh(\frac{\pi s}{a}(x_2 - x_1 - i(m - 2l - 1)))}{\prod_{n=0}^{m-1} 2\sinh(\frac{\pi}{a}(x_2 - x_1 + i(n-l)))} e^{\frac{\pi}{a}(m-2l-1)(y_1 - y_2)}.\end{aligned} \tag{3.29}$$

Since $e^{-\frac{\pi}{a}(m-2l-1)(y_1-y_2)} = e^{\frac{\pi}{a}(m-1)(y_1-y_2)}$ when $e^{\frac{2\pi}{a}(y_1-y_2)} = 1$, we only have to show that

$$\sum_{l=0}^{m-1} \frac{a}{\prod_{n\neq l} 2\sin(\frac{\pi}{a}(n-l))} \frac{\sinh(\frac{\pi s}{a}(x_2 - x_1 + i(m - 2l - 1)))}{\prod_{n=0}^{m-1} 2\sinh(\frac{\pi}{a}(x_2 - x_1 + i(n-l)))} = 0. \tag{3.30}$$

To this end, we note that $s < m$ entails that the left-hand side is a meromorphic function in $x_1 - x_2$ that is bounded away from its poles and decays exponentially as $|\mathrm{Re}\,(x_1 - x_2)| \to \infty$. Hence, Liouville's theorem will imply the above identity once we show that the residues at the (simple) poles

$$x_2 - x_1 + i\ell = i\ell' a, \quad \ell = -m + 1, \dots, m - 1, \quad \ell' \in \mathbb{Z}, \tag{3.31}$$

all vanish; and, by ia-(anti)periodicity, we may and shall restrict attention to $\ell' = 0$. For a fixed $\ell = -m+1, \ldots, m-1$, we observe that there exists $n = 0, \ldots, m-1$ such that $n - l = \ell$ if and only if

$$l = \max(-\ell, 0), \ldots, \min(m-1, m-1-\ell). \tag{3.32}$$

It follows that the residue of the left-hand side of (3.30) at $x_2 - x_1 + i\ell = 0$ is proportional to

$$\sum_{l=\max(-\ell,0)}^{\min(m-1,m-1-\ell)} f_l, \quad f_l(a) \equiv \frac{2a}{\prod_{n\neq l} \sin(\frac{\pi}{a}(n-l))} \frac{\sin(\frac{\pi s}{a}(2l+\ell-m+1))}{\prod_{n\neq l+\ell} \sin(\frac{\pi}{a}(n-l-\ell))}. \tag{3.33}$$

On the set of integers (3.32), we have the involution $\sigma : l \mapsto m-1-\ell-l$. Since $f_{\sigma(l)} = -f_l$, as is easily checked, this clearly implies that the residue vanishes. □

Setting $l = 0$ in (3.20), we find that

$$\varphi_{2,0}(x) = \frac{2a(-i)^m}{\prod_{n=1}^{m-1} 2\sin(\pi n/a)} \frac{1}{\prod_{n=0}^{m-1} 2\sinh\left(\frac{\pi}{a}(x_2 - x_1 + in)\right)}. \tag{3.34}$$

Comparing (3.34), (3.27) and (3.26) with the $N = 2$ instances of (2.19), (2.18) and (2.16), respectively, the uniqueness of the BA-function, once a normalisation has been fixed, implies that

$$\varphi_2(x, y) = \frac{2a(-i)^m}{\prod_{n=1}^{m-1} 2\sin(\pi n/a)} \frac{\psi_2(\exp(-i\pi/a), m-1; i\sigma(x), iy)}{\prod_{n=-m+1}^{m-1} 2\sinh\left(\frac{\pi}{a}(x_2 - x_1 + in)\right)}. \tag{3.35}$$

When combined with Prop. 3.3, we obtain the $N = 2$ case of Thm. 3.1; and a comparison with (3.17) yields the $N = 2$ instance of Prop. 3.2.

3.2 The Inductive Step

We can now use the $N = 2$ case as the basis for proofs by induction. More precisely, we consider $N \geq 3$ and assume that Thm. 3.1 along with Props. 3.6–3.7, Lemma 3.8 and Props. 3.9–3.10 hold true when N is replaced by $N-1$. (In the $N = 2$ case, the pertinent results have all been established above.)

To begin with, we require that $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ with $x_j \neq x_k$ for all $1 \leq j < k \leq N$. Letting $\delta = \min\{|x_j - x_k| \mid 1 \leq j < k \leq N\}$, we construct N contours γ_j, $j = 1, \ldots, N$, from the line segments $x_j + u \pm ia/2$, $|u| \leq \delta/2$, and $x_j \pm \delta/2 + iv$, $|v| \leq a/2$; and define the function

$$\varphi_N(x, y) \equiv e^{\frac{2\pi i}{a} y_N \sum_{j=1}^N x_j}$$

$$\cdot \int_{\underline{\gamma}} dz \prod_{1\leq j<k\leq N-1} \sinh\left(\frac{\pi}{a}(z_k - z_j)\right) \prod_{n=-m+1}^{m-1} 2\sinh\left(\frac{\pi}{a}(z_k - z_j + in)\right)$$
$$\cdot \frac{\varphi_{N-1}(z, (y_1 - y_N, \ldots, y_{N-1} - y_N))}{\prod_{j=1}^{N}\prod_{k=1}^{N-1}\prod_{n=0}^{m-1} 2\sinh\left(\frac{\pi}{a}\left(x_j z_k + \frac{i}{2}(m-2n-1)\right)\right)}, \tag{3.36}$$

with $\underline{\gamma} \equiv \gamma_1 \times \cdots \times \gamma_{N-1}$, and where the z_j-contour γ_j only encircles the simple poles

$$z_j = x_j + i(m-2n-1)/2, \quad n = 0, \ldots, m-1. \tag{3.37}$$

The arbitrary-N generalisation of Prop. 3.3 now follows.

Proposition 3.6 *Assuming $m \in \mathbb{Z}_+$ and $a > m-1$, we get*

$$\prod_{1\leq j<k\leq N} 2\sinh(\pi(y_j - y_k)) \cdot \Phi_N(x, y)$$
$$= (-1)^{mN(N-1)/2} \left(\frac{\prod_{n=1}^{m-1} 2\sin(\pi n/a)}{a}\right)^{N-1} \sum_{\sigma\in S_N} \varphi_N(\sigma(x), y). \tag{3.38}$$

Proof To ease the notation, we suppress dependence on the parameters a and m throughout the proof. Substituting (3.4) in the left-hand side of (3.38), using (3.1)–(3.2) to reduce the size of the resulting expression and invoking the above statement after taking $N \to N-1$, we obtain

$$\left(\frac{\prod_{n=1}^{m-1} 2\sin(\pi n/a)}{a}\right)^{N-1} \prod_{j=1}^{N-1} 2\sinh(\pi(y_j - y_N)) \frac{e^{\frac{2\pi i}{a} y_N \sum_{j=1}^{N} x_j}}{(N-1)!}$$
$$\cdot \sum_{\sigma\in S_{N-1}} \int_{\mathbb{R}^{N-1}} dz W_{N-1}(z) \mathcal{S}_N^{\sharp}(x, z) \varphi_{N-1}(\sigma(z), (y_1 - y_N, \ldots, y_{N-1} - y_N)).$$

Thanks to the S_{N-1}-invariance of the domain of integration $\mathbb{R}^{N-1}$ as well as $W_{N-1}(z)$ and $\mathcal{S}_N^{\sharp}(x, z)$, we can replace the sum over $\sigma \in S_{N-1}$ by a factor $(N-1)!$. By also substituting $2\sinh(\pi(y_j - y_N)) = e^{\pi(y_j - y_N)} - e^{-\pi(y_j - y_N)}$, we thus obtain

$$\left(\frac{\prod_{n=1}^{m-1} 2\sin(\pi n/a)}{a}\right)^{N-1} e^{\frac{2\pi i}{a} y_N \sum_{j=1}^{N} x_j} \sum_{\delta\in\{\pm 1\}^{N-1}} \delta_1 \cdots \delta_{N-1}$$
$$\cdot \int_{\mathbb{R}^{N-1}} dz W_{N-1}(z) \mathcal{S}_N^{\sharp}(x, z) e^{\pi \sum_{j=1}^{N-1} \delta_j (y_j - y_N)} \varphi_{N-1}(z, (y_1 - y_N, \ldots, y_{N-1} - y_N)). \tag{3.39}$$

To rewrite the integral further we shall make use of the invariance property

$$e^{\pi\sum_{j=1}^{N-1}\delta_j u_j}\varphi_{N-1}(z,u)=\varphi_{N-1}(z-(ia/2)\delta,u)\prod_{j=1}^{N-1}\delta_j^{m(N-2)}. \tag{3.40}$$

When $N-1=2$ it is a simple consequence of Prop. 3.4 and, using this as the basis for an induction argument, the validity of the formula in the general-N case is readily inferred from (3.49)–(3.50) and the observation that $W_{N-1}(z+(ia/2)\delta)=W_{N-1}(z)$. Substituting (3.40) (with $u_j=y_j-y_N$) in the above integral, taking $z\to z+(ia/2)\delta$, under which $W_{N-1}(z)$ is invariant, and noting that

$$\mathcal{S}_N^\sharp(x,z+(ia/2)\delta)=\frac{(-1)^{mN(N-1)/2}\prod_{j=1}^{N-1}\delta_j^{mN}}{\prod_{j=1}^{N}\prod_{k=1}^{N-1}\prod_{n=0}^{m-1}2\sinh\left(\frac{\pi}{a}(x_j-z_k+i(m-2n-1)/2)\right)}, \tag{3.41}$$

we arrive at

$$\begin{aligned}&(-1)^{mN(N-1)/2}\left(\frac{\prod_{n=1}^{m-1}2\sin(\pi n/a)}{a}\right)^{N-1}e^{\frac{2\pi i}{a}y_N\sum_{j=1}^{N}x_j}\sum_{\delta\in\{\pm1\}^{N-1}}\delta_1\cdots\delta_{N-1}\\ &\cdot\int_{\mathbb{R}-\frac{ia}{2}\delta_1}dz_1\cdots\int_{\mathbb{R}-\frac{ia}{2}\delta_{N-1}}dz_{N-1}\frac{W_{N-1}(z)\varphi_{N-1}(z,(y_1-y_N,\ldots,y_{N-1}-y_N))}{\prod_{j=1}^{N}\prod_{k=1}^{N-1}\prod_{n=0}^{m-1}2\sinh\left(\frac{\pi}{a}\left(x_j-z_k+\frac{i}{2}(m-2n-1)\right)\right)},\end{aligned} \tag{3.42}$$

where the sum of integrals amounts to

$$\int_{\mathcal{C}^{N-1}}dz\frac{W_{N-1}(z)\varphi_{N-1}(z,(y_1-y_N,\ldots,y_{N-1}-y_N))}{\prod_{j=1}^{N}\prod_{k=1}^{N-1}\prod_{n=0}^{m-1}2\sinh\left(\frac{\pi}{a}\left(x_j-z_k+\frac{i}{2}(m-2n-1)\right)\right)} \tag{3.43}$$

with $\mathcal{C}$ the contour consisting of the lines $\mathbb{R}\pm ia/2$, traversed from right/left to left/right. By Cauchy's residue theorem, this one integral equals

$$\begin{aligned}&\sum_{\sigma\in S_N}\int_{\gamma_{\sigma(1)}}dz_1\cdots\int_{\gamma_{\sigma(N-1)}}dz_{N-1}\\ &\cdot\frac{W_{N-1}(z)\varphi_{N-1}(z,(y_1-y_N,\ldots,y_{N-1}-y_N))}{\prod_{j=1}^{N}\prod_{k=1}^{N-1}\prod_{n=0}^{m-1}2\sinh\left(\frac{\pi}{a}\left(x_j-z_k+\frac{i}{2}(m-2n-1)\right)\right)}+R(x,y)\end{aligned}$$

for some remainder term $R(x,y)$, resulting from residues of the integrand at points $z=(z_1,\ldots,z_{N-1})$ given by

$$z_j=x_{i_j}+i(m-2n_j+1)/2,\quad j=1,\ldots,N-1, \tag{3.44}$$

where at least two i_j coincide.

Referring back to (3.36), we find that it remains to prove that $R = 0$. To this end, we take $N \to N-1$ in Prop. 3.10 and use the resulting formula to rewrite the integrand as

$$C_{N-1}\frac{\psi_{N-1}(e^{-i\pi/a}, m-1; iz, i(y_1-y_N, \ldots, y_{N-1}-y_N))\prod_{1\le j<k\le N-1}\sinh\left(\frac{\pi}{a}(z_j-z_k)\right)}{\prod_{j=1}^{N}\prod_{k=1}^{N-1}\prod_{n=0}^{m-1}2\sinh\left(\frac{\pi}{a}\left(x_j-z_k+\frac{i}{2}(m-2n-1)\right)\right)}, \tag{3.45}$$

where C_{N-1} is a constant (for fixed m and a), whose value is of no consequence for the arguments below. From Prop. 4.4 in [7] (part (iii)), we know that $\psi_{N-1}(u, v)$ is an entire function. The presence of the factors $\sinh\left(\frac{\pi}{a}(z_j - z_k)\right)$ thus entails that (iterated) residues at points (3.44) such that $i_j = i_k$ and $n_j = n_k$ for some $1 \le j \ne k \le N-1$ all vanish. In the remaining cases we have $n_j \ne n_k$ while $i_j = i_k$. Thanks to the S_{N-1}-invariance of the integrand (cf. Lemma 3.8), we may and shall restrict attention to $j = 1$ and $k = 2$. Letting $\widetilde{\varphi}_{N-1}(z) = \psi_{N-1}(e^{-i\pi/a}, m-1; iz, i(y_1-y_N, \ldots, y_{N-1}-y_N))$, $t = x_{i_1} = x_{i_2}$ and $s = n_1 - n_2$, we observe that

$$\begin{aligned}&\tilde{\varphi}_{N-1}\big((t+i(m-2n_1-1)/2, t+i(m-2n_2-1)/2, \ldots)\big)\\ &\quad=\tilde{\varphi}_{N-1}\big((t+i(m-n_1-n_2-1)/2-is/2, t+i(m-n_1-n_2-1)/2+is/2, \ldots)\big).\end{aligned}$$

Note that the self-duality of ψ_{N-1} entails that $\widetilde{\varphi}_{N-1}(z)$ satisfies the vanishing conditions in Prop. 3.9 with $N \to N-1$ in z. Since each $n_j = 0, \ldots, m-1$, we have $s = -m+1, \ldots m-1$. From the vanishing conditions for $\tilde{\varphi}$, we can thus infer that its specialisation at (3.44) with $i_1 = i_2$ is invariant under the interchange $n_1 \leftrightarrow n_2$. Moreover, we have

$$\sinh(\pi(z_1-z_2)/a) = \sinh(i\pi(n_2-n_1)/a), \tag{3.46}$$

so that the same specialisation of $\varphi(z)W_{N-1}(z)$ is antisymmetric under $n_1 \leftrightarrow n_2$. In fact, more generally, when $i_1 = \cdots = i_M$, $M \ge 2$, coincide, we can, similarly, show that antisymmetry extends to any permutation of the corresponding M distinct integers $n_1, \ldots, n_M$. (Essentially, we need only to decompose such a permutation in terms of (elementary) transpositions and appeal to the above discussion.) Either way, we see that the residues corresponding to a fixed choice of indices i_j, at least two of which coincide, and integers n_j, fixed up to permutations among the ones corresponding to equal i_j, cancel (in pairs) and consequently that $R = 0$. □

We proceed to show that φ_N (3.36) essentially amounts to the BA-function reviewed above. This requires that we establish two things: the relevant series expansion and vanishing properties.

Proposition 3.7 *We have*

$$\varphi_N(x; y) = e^{\frac{2\pi i}{a}(x,y)}\sum_{\nu}\varphi_{N,\nu}(x)e^{\frac{2\pi}{a}(\nu,y)} \tag{3.47}$$

with the sum running over weight vectors of the form (2.17) with $m \to m-1$ and where

$$\varphi_{N,\rho_N(m-1)}(x) = \left(\frac{a(-i)^m}{\prod_{n=1}^{m-1} 2\sin(\pi n/a)}\right)^{N(N-1)/2} \cdot \frac{1}{\prod_{1\le j<k\le N}\prod_{n=0}^{m-1} 2\sinh\left(\frac{\pi}{a}(x_k - x_j + in)\right)}. \tag{3.48}$$

Proof By our induction assumption, we may invoke the proposition after taking $N \to N-1$. Substituting the resulting expansion (3.47) in (3.36) and making use of Cauchy's residue theorem, we deduce

$$\begin{aligned}\varphi_N(x;y) &= e^{\frac{2\pi i}{a}y_N\sum_{j=1}^N x_j}\sum_{\nu'} e^{\frac{2\pi}{a}(\nu',y)} \\ &\quad\cdot\int_{\underline{\gamma}} dz \frac{W_{N-1}(z)\varphi_{N-1,\nu'}(z)e^{\frac{2\pi i}{a}(z,(y_1-y_N,\ldots,y_{N-1}-y_N))}}{\prod_{j=1}^N\prod_{k=1}^{N-1}\prod_{n=0}^{m-1}2\sinh\left(\frac{\pi}{a}(x_j - z_k + i(m-2n-1)/2)\right)} \\ &= e^{\frac{2\pi i}{a}(x,y)}\sum_{\nu'} e^{\frac{2\pi}{a}(\nu',(y_1,\ldots,y_{N-1}))}\sum_{n_1=0}^{m-1} \\ &\quad\cdots\sum_{n_{N-1}=0}^{m-1} C_{\nu';n_1,\ldots,n_{N-1}}(x)e^{\frac{2\pi}{a}\sum_{k=1}^{N-1}(n_k-(m-1)/2)(y_k-y_N)},\end{aligned} \tag{3.49}$$

with ν' running over the set of weight vectors obtained after taking $N \to N-1$ and $m \to m-1$ in (2.17), and where

$$\begin{aligned}&= (-ia)^{N-1}\frac{W_{N-1}(x_1 + i(m-2n_1-1)/2,\ldots,x_{N-1}+i(m-2n_{N-1}-1)/2)}{\prod_{k=1}^{N-1}\prod_{n'\neq n_k}2\sinh\left(\frac{i\pi}{a}(n_k-n')\right)} \\ &\quad\cdot\frac{\varphi_{N-1,\nu'}(x_1 + i(m-2n_1-1)/2,\ldots,x_{N-1}+i(m-2n_{N-1}-1)/2)}{\prod_{k=1}^{N-1}\prod_{j\neq k}\prod_{n'=0}^{m-1}2\sinh\left(\frac{\pi}{a}(x_j-x_k+i(n_k-n'))\right)}\end{aligned} \tag{3.50}$$

and we have used that $(\nu', (y_N,\ldots,y_N)) = 0$, since ν' is a linear combination of vectors $e_j - e_k$ with $1 \le j < k \le N-1$. Setting $l_{j,N-1} = m-1-n_j = 0,\ldots,m-1$, we get the equality

$$\begin{aligned}\sum_{1\le j<k\le N-1}\left(\frac{m-1}{2} - l_{jk}\right)(e_j - e_k) + \sum_{j=1}^{N-1}\left(n_j - \frac{m-1}{2}\right)(e_j - e_N) \\ = \sum_{1\le j<k\le N}\left(\frac{m-1}{2} - l_{jk}\right)(e_j - e_k)\end{aligned} \tag{3.51}$$

and the validity of (3.47) with ν from (2.17) clearly follows.

Since the only term in (3.49) that contributes to $\varphi_{N,\rho_N(m-1)}(x)$ corresponds to $\nu' = \rho_{N-1}(m-1)$ and $n_1 = \cdots = n_{N-1} = m-1$, we have

$$\varphi_{N,\rho_N(m-1)}(x) = \left(\frac{a(-i)^m}{\prod_{n=1}^{m-1} 2\sin(\pi n/a)}\right)^{N-1} \cdot \frac{W_{N-1}(x)\varphi_{N-1,\rho_{N-1}(m-1)}(x)}{\prod_{k=1}^{N-1}\prod_{j\neq k}\prod_{n'=0}^{m-1} 2\sinh\left(\frac{\pi}{a}(x_j - x_k + i(m-1-n'))\right)}, \tag{3.52}$$

where we have used the fact that both $W_{N-1}(x)$ and $\varphi_{N-1,\rho_{N-1}(m-1)}$ are invariant under translations $x \to x + (t,\ldots,t)$ for any $t \in \mathbb{C}$. Using (3.1) and (3.48) for $N \to N-1$, we easily arrive at the right-hand side of (3.48). □

To establish quasi-invariance, we make use of the fact that φ_N is S_N-invariant in the sense of the following lemma.

Lemma 3.8 *We have*

$$\varphi_N(\sigma(x),\sigma(y)) = (-)^{|\sigma|}\varphi_N(x,y), \quad \sigma \in S_N. \tag{3.53}$$

Proof From (3.36) and the induction assumption, we see immediately that the statement holds true whenever $\sigma \in S_{N-1}$. Hence, we need only to consider the special case $\sigma = \sigma_{N-1}$. By a direct computation, similar to the one in the proof of Proposition 3.7, we obtain

$$\begin{aligned}\varphi_{N,\sigma_{N-1}(\rho_N(m-1))}(x) &= -\left(\frac{a(-i)^m}{\prod_{n=1}^{m-1} 2\sin(\pi n/a)}\right)^{N(N-1)/2} \\ &\quad\cdot \frac{1}{\prod_{k=1}^{N}\prod_{j=1}^{\min(k,N-1)-1}\prod_{n=0}^{m-1} 2\sinh\left(\frac{\pi}{a}(x_k - x_j + in)\right)} \\ &\quad\cdot \frac{1}{\prod_{n=0}^{m-1} 2\sinh\left(\frac{\pi}{a}(x_{N-1} - x_N + in)\right)} \\ &= -\varphi_{N,\rho_N(m-1)}(\sigma_{N-1}(x)).\end{aligned} \tag{3.54}$$

(Note that only the term in (3.49) corresponding to $\nu' = \rho_{N-1}$, $n_1 = \cdots = n_{N-2} = m-1$ and $n_{N-1} = 0$ contributes.) It follows that $\varphi_N(x,y)$ and $-\varphi_N(\sigma_{N-1}(x),\sigma_{N-1}(y))$ are both (finite) series of the form

$$e^{\frac{2\pi i}{a}(x - i\rho_N(m-1),y)} f_N(x,y), \quad f_N(x,y) = \Gamma_0(x) + \sum_{\nu>0}\Gamma_\nu(x)e^{-\frac{2\pi}{a}(\nu,y)}, \tag{3.55}$$

where $\Gamma_0 = \varphi_{N,\rho_N(m-1)}$ and $\nu > 0$ means that $\nu = \sum_{1\le j<k\le N} l_{jk}(e_j - e_k)$ with $l_{jk} \ge 0$ and not all equal to zero. From Proposition 3.7, we get that the functions $\varphi_N(\sigma(x),y)$, $\sigma \in S_N$, form a linearly independent set; and, since the pre-factor $\prod_{j<k} 2\sinh(\pi(y_j -$

$y_k)$) in (3.38) is (anti)periodic under shifts $y_j \to y_j - i$ and the Macdonald operators D_N^r are S_N-invariant, it follows that $\varphi_N(x, y)$ and $-\varphi_N(\sigma_{N-1}(x), \sigma_{N-1}(y))$ satisfy the difference equation

$$D_N(y)\varphi(x, y) = \varphi(x, y)\sum_{j=1}^{N} e^{\frac{2\pi}{a}x_j}, \quad D_N(y) \equiv D_N^1(q^{i2y}; q^2, q^{2(1-m)}). \tag{3.56}$$

For the corresponding series f_N, this translates into the difference equation $L_N(y)f(x, y) = f(x, y)\sum_{j=1}^{N} e^{\frac{2\pi}{a}x_j}$ given by the difference operator

$$\begin{aligned} L_N(y) &\equiv e^{-\frac{2\pi i}{a}(x-i\rho_N(m-1),y)} D_N(y) e^{\frac{2\pi i}{a}(x-i\rho_N(m-1),y)} \\ &= \sum_{j=1}^{N} e^{\frac{2\pi}{a}x_j} \prod_{k<j} \frac{1 - e^{\frac{2\pi}{a}(y_j-y_k+i(m-1))}}{1 - e^{\frac{2\pi}{a}(y_j-y_k)}} \cdot \prod_{k>j} \frac{1 - e^{\frac{2\pi}{a}(y_k-y_j-i(m-1))}}{1 - e^{\frac{2\pi}{a}(y_k-y_j)}} \cdot e^{-i\partial_{y_j}}. \end{aligned} \tag{3.57}$$

By expanding the coefficients in power series in $e^{\frac{2\pi}{a}(y_k-y_j)}$, $1 \le j < k \le N$, this, together with the leading coefficient Γ_0, is readily seen to uniquely determine f_N. This essentially amounts to the uniqueness result in Theorem 2.3 in [24], wherein a detailed proof can be found. □

Proposition 3.9 *For all $1 \le j < k \le N$ and $s = 1, \ldots, m-1$, we have the vanishing property*

$$\varphi_N\left(x; y + \frac{is}{2}(e_j - e_k)\right) = \varphi_N\left(x; y - \frac{is}{2}(e_j - e_k)\right), \quad e^{\frac{2\pi}{a}(y_j-y_k)} = 1. \tag{3.58}$$

Proof Thanks to Lemma 3.8, it suffices to establish the vanishing property when $j = 1$ and $k = 2$; and, in this case, it follows immediately from (3.36) and our induction assumption. □

Comparing Props. 3.7 and 3.9 with the properties (2.16) and (2.18)–(2.19), which uniquely characterises the self-dual BA-function ψ_N, we find the following result.

Proposition 3.10 *Given $m \in \mathbb{Z}_+$ and $a > m - 1$, we have*

$$\begin{aligned} \varphi_N(a, m; x, y) = &\left(\frac{a(-i)^m}{\prod_{n=1}^{m-1} 2\sin(\pi n/a)}\right)^{N(N-1)/2} \\ &\cdot \frac{\psi_N(\exp(-i\pi/a), m-1; ix, iy)}{\prod_{1\le j<k\le N}\prod_{n=-m+1}^{m-1} 2\sinh\left(\frac{\pi}{a}(x_k - x_j + in)\right)}. \end{aligned} \tag{3.59}$$

Finally, by combining Props. 3.6 and 3.10, we arrive at the statement in Thm. 3.1 for arbitrary N; and, by using the above expression to express φ_N and φ_{N-1} in (3.36) in terms of ψ_N and ψ_{N-1}, respectively, we easily verify the claim in Prop. 3.2.

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Declarations

Conflict of Interest The author has no conflicting interest to declare that are relevant to the content of this article.

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