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RESEARCH ARTICLE

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The Global Glimm Property for C^*-algebras of topological dimension zero

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Abstract
We show that a C^*-algebra with topological dimension zero has the Global Glimm Property (every hereditary subalgebra contains an almost full nilpotent element) if and only if it is nowhere scattered (no hereditary subalgebra admits a finite-dimensional representation). This solves the Global Glimm Problem in this setting. It follows that nowhere scattered C^*-algebras with finite nuclear dimension and topological dimension zero are pure.

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1 | INTRODUCTION

The Global Glimm Problem asks if two relevant regularity properties —known as *nowhere scatteredness* and the *Global Glimm Property*— agree. The problem can be traced back to the pioneering study of purely infinite C^*-algebras by Kirchberg and Rørdam [12], and plays a central role in the study of nonsimple algebras; see, for example, [4, 5, 7, 17].

Concretely, a C^*-algebra A is said to be *nowhere scattered* if no hereditary sub-C^*-algebra of A admits a finite-dimensional, irreducible representation [20], while A has the *Global Glimm Property* if every hereditary sub-C^*-algebra contains an almost full square-zero element (in symbols, for every $a \in A_+$ and $\varepsilon > 0$, there exists $r \in \overline{aAa}$ such that $r^2 = 0$ and $(a-\varepsilon)_+$ is in the ideal generated by r). The name traces back to Glimm, who implicitly established this property for all simple nonelementary C^*-algebras in the course of proving [11, Lemma 4].

The Global Glimm Property always implies nowhere scatteredness, and the Global Glimm Problem asks if the converse is also true. This is known to hold under the additional assumption of real rank zero [7], or stable rank one [2], or Hausdorff finite-dimensional primitive ideal space [5], or finite decomposition rank ([9, Theorem 2.3]).

Recently, a general framework to approach this problem was given in [19], employing Cuntz semigroup techniques. Making use of the tools developed in [19], we solve the Global Glimm Problem for a large class of C^*-algebras:

Theorem A (2.3). *Let A be a C^*-algebra with topological dimension zero. Then A has the Global Glimm Property if and only if A is nowhere scattered.*

A C^*-algebra is said to have *topological dimension zero* if its primitive ideal space has a basis consisting of compact, open sets. This notion was introduced by Brown and Pedersen [6, Remark 2.5(vi)] as a generalization of real rank zero. More generally, it is known that every C^*-algebra with real rank zero has the *ideal property* (that is, its projections separate its closed ideals), and that every C^*-algebra with the ideal property has topological dimension zero.

Theorem A both recovers and extends previous solutions of the Global Glimm Problem. First, since real rank zero implies topological dimension zero, our result subsumes the real rank zero case established in [7]. Second, because Hausdorff zero-dimensional primitive ideal space implies topological dimension zero, we partially recover the Hausdorff finite-dimensional primitive ideal space case treated in [5]. Beyond these cases, Theorem A solves the Global Glimm Problem for classes of C^*-algebras not covered by previous results in the literature. For example, Zhang [22] showed that multiplier algebras of σ-unital, real rank zero C^*-algebras have topological dimension zero, and thus fall within the scope of Theorem A. However, such multiplier algebras are not encompassed by earlier results, since they need not have real rank zero, Hausdorff primitive ideal space, or stable rank one (see Remark 2.9).

As a corollary of Theorem A, we obtain that nowhere scattered C^*-algebras with finite nuclear dimension and topological dimension zero are pure (Corollary 2.4). Further, using results by Zhang [22] and the third-named author [21], we prove that σ-unital, purely infinite C^*-algebras of real rank zero have purely infinite multiplier algebras (Corollary 2.8), thereby partially resolving a problem from [12]. We also recover the result of Elliott and Rouzbehani [8] that a weakly purely infinite C^*-algebra with topological dimension zero is purely infinite (Corollary 2.5).

2 | THE PROOFS

Our results heavily rely on Cuntz semigroup techniques. The reader is referred to [10] for an introduction to the subject.

As shown in [14, Corollary 4.3], a separable C^*-algebra has topological dimension zero if and only if $A \otimes \mathcal{O}_2$ has real rank zero. This was used in [19, Proposition 4.18] to prove that a separable

C^*-algebra A has topological dimension zero if and only if $\mathrm{Cu}(A)\otimes\{0,\infty\}$ is algebraic, where we recall that a Cuntz semigroup is *algebraic* if the set of elements x satisfying $x \ll x$ is sup-dense in $\mathrm{Cu}(A)$. Here, we use the tensor product of abstract Cuntz semigroups developed in [3], and we note that $\mathrm{Cu}(A\otimes\mathcal{O}_2)\cong\mathrm{Cu}(A)\otimes\{0,\infty\}$ by [3, Corollary 7.2.15].

In the next result we remove the separability assumption.

Proposition 2.1. *Let A be a C^*-algebra. Then, A has topological dimension zero if and only if* $\mathrm{Cu}(A)\otimes\{0,\infty\}$ *is algebraic.*

Proof. Assume first that $\mathrm{Cu}(A)\otimes\{0,\infty\}$ is algebraic. It follows from [19, Proposition 4.18] that A has topological dimension zero. (We note that the assumption of separability in [19, Proposition 4.18] is not needed for this implication.)

Conversely, assume that A has topological dimension zero. Without loss of generality, we may also assume that A is stable. We will show that for every $x', x \in \mathrm{Cu}(A)$ with $x' \ll x$ there exist $y', y \in \mathrm{Cu}(A)$ such that $x' \ll y' \ll y \ll x$ and $y \ll \infty y'$. It then follows from [19, Lemma 4.16] that $\mathrm{Cu}(A)\otimes\{0,\infty\}$ is algebraic.

So let $x', x \in \mathrm{Cu}(A)$ satisfy $x' \ll x$. Choose $a \in A_+$ and $\varepsilon > 0$ such that

$$x' \ll [(a-\varepsilon)_+], \quad \text{and} \quad x = [a].$$

By [17, Lemma 7.12], for every $\varepsilon' > 0$, the set of elements $c \in A_+$ such that $(c-\varepsilon')_+$ is pseudocompact is dense in A_+, where an element $d \in A_+$ is said to be *pseudocompact* if there exist $\delta > 0$ and $n \in \mathbb{N}$ such that $[d] \leqslant n[(d-\delta)_+]$ in $\mathrm{Cu}(A)$. Applying this, we obtain $c \in A_+$ such that

$$\|a - c\| < \frac{\varepsilon}{2}$$

and such that $(c-\frac{\varepsilon}{2})_+$ is pseudocompact. Using that $\|a-c\| < \frac{\varepsilon}{2}$, it follows from standard Cuntz semigroup techniques that

$$(a-\varepsilon)_+ \precsim (c-\tfrac{\varepsilon}{2})_+ \precsim a.$$

Thus, the pseudocompact element $d := (c-\frac{\varepsilon}{2})_+$ satisfies $x' \ll [d] \leqslant x$. (We have shown that elements in $\mathrm{Cu}(A)$ with a pseudocompact representative are sup-dense.)

Pick $\delta > 0$ and $n \in \mathbb{N}$ such that $[d] \leqslant n[(d-\delta)_+]$. Using that $x' \ll [(a-\varepsilon)_+] \leqslant [d]$, we may assume that δ is so small that $x' \ll [(d-\delta)_+]$. Set $y := [(d-\frac{\delta}{2})_+]$ and $y' := [(d-\delta)_+]$. Then

$$x' \ll y' \ll y \ll [d] = [(c-\tfrac{\varepsilon}{2})_+] \leqslant [a] = x.$$

Further, we have

$$y \ll [d] \leqslant n[(d-\delta)_+] = ny' \leqslant \infty y',$$

which shows that y' and y have the desired properties. □

As defined in [16, Definition 5.1], a Cu-semigroup S is *weakly* $(2,\omega)$*-divisible* if, for every $x', x \in S$ satisfying $x' \ll x$, there exist $n \in \mathbb{N}$ and $d_1, \dots, d_n \in S$ such that

$$2d_1, \dots, 2d_n \leqslant x, \quad \text{and} \quad x' \leqslant d_1 + \dots + d_n.$$

Further, S is said to be $(2,\omega)$-*divisible* if, for every $x', x \in S$ satisfying $x' \ll x$ there exist $n \in \mathbb{N}$ and $d \in S$ such that

$$2d \leqslant x, \quad \text{and} \quad x' \leqslant nd.$$

As shown in [20, Theorem 8.9], a C^*-algebra is nowhere scattered if and only if its Cuntz semigroup is weakly $(2,\omega)$-divisible. Further, by [19, Theorem 3.6], a C^*-algebra has the Global Glimm Property if and only if its Cuntz semigroup is $(2,\omega)$-divisible. In what follows, we use these characterizations to prove Theorem A.

We state Lemma 2.2 below using the language of Cu-semigroups, where properties (O5)–(O8) are always satisfied in the Cuntz semigroup of a C^*-algebra (see [1, 3, 20]). We recall (O5) at the point where it is used in the proof. The remaining properties (O6)–(O8) are required only for the referenced results and are therefore not recalled in this note.

Lemma 2.2. *Let S be a weakly $(2,\omega)$-divisible* Cu*-semigroup with (O5)–(O8), and assume that $S \otimes \{0,\infty\}$ is algebraic. Then S is $(2,\omega)$-divisible.*

Proof. To show that S is $(2,\omega)$-divisible, let $x', x \in S$ with $x' \ll x$. We need to find $y \in S$ such that $2y \leqslant x$ and $x' \ll \infty y$.

Since $S \otimes \{0,\infty\}$ is algebraic, it follows from [19, Lemma 4.17] that S is ideal-filtered in the sense of [19, Definition 4.1]. We can therefore apply [19, Proposition 6.2] for $m = 2$ to obtain $c, d \in S$ such that

$$2c, 2d \leqslant x, \quad \text{and} \quad x' \ll \infty(c+d).$$

Choose $c', d' \in S$ such that $c' \ll c$, $d' \ll d$, and $x' \ll \infty(c'+d')$.

Using again that $S \otimes \{0,\infty\}$ is algebraic, it follows from [19, Lemma 4.16] that elements which generate a compact ideal are dense in S. Specifically, applied for $c' \ll c$, we find $e', e \in S$ such that

$$c' \ll e' \ll e \ll c, \quad \text{and} \quad e \ll \infty e'.$$

The "almost algebraic order" property (O5) means that for all $s', s, t', t, u \in S$ satisfying $s' \ll s$, $t' \ll t$, and $s + t \leqslant u$, there exists $v \in S$ such that $t' \ll v$ and $s' + v \leqslant u \leqslant s + v$. Applying (O5) for $s' = 2e'$, $s = 2e$, $t' = t = 0$ and $u = x$, we obtain $f \in S$ such that

$$2e' + f \leqslant x \leqslant 2e + f.$$

Then

$$2d \leqslant x \leqslant f + 2e \leqslant f + \infty e.$$

Using [20, Proposition 7.8] to "lift" the relation $2d \leqslant f + \infty e$ from the quotient by the ideal generated by e, we obtain $g \in S$ such that

$$2g \leqslant f, \quad \text{and} \quad d' \ll g + \infty e.$$

Set $y := e' + g$. Then

$$2y \leqslant 2e' + 2g \leqslant 2e' + f \leqslant x.$$

Further, we have

$$x' \ll \infty(c' + d') \leqslant \infty(e' + g + e) \leqslant \infty(e' + g) = \infty y,$$

as desired. □

Theorem 2.3. *Let A be a C^*-algebra with topological dimension zero. Then A has the Global Glimm Property if and only if A is nowhere scattered.*

Proof. The forward implication holds in general. For the converse, assume that A is nowhere scattered. Then $\mathrm{Cu}(A)$ is weakly $(2, \omega)$-divisible by [20, Theorem 8.9]. Further, the Cuntz semigroup of every C^*-algebra satisfies (O5)–(O8) (see [3, Proposition 4.6], [15, Proposition 5.1.1], [1, Proposition 2.2] and [20, Theorem 7.4], respectively). Since A has topological dimension zero, $\mathrm{Cu}(A) \otimes \{0, \infty\}$ is algebraic by Proposition 2.1.

We can therefore apply Lemma 2.2 to deduce that $\mathrm{Cu}(A)$ is $(2, \omega)$-divisible. Then, by [19, Theorem 3.6], A has the Global Glimm Property. □

In the rest of the paper we give some applications of Theorem 2.3.

Corollary 2.4. *Let A be a nowhere scattered C^*-algebra with finite nuclear dimension and topological dimension zero. Then A is pure.*

Proof. The C^*-algebra A has the Global Glimm Property by Theorem 2.3. Thus, the result follows from [4, Theorem 6.5]. □

In [12, Proposition 4.15], Kirchberg and Rørdam show that a C^*-algebra is purely infinite if and only if it is weakly purely infinite and has the Global Glimm Property, and they asked if every weakly purely infinite C^*-algebra must in fact be purely infinite ([12, Question 9.5]). This question was settled affirmatively by Elliott and Rouzbehani [8] for C^*-algebras with topological dimension zero, a class that includes both the simple and the real rank zero cases already treated in [12]. We recover their result.

Corollary 2.5 (Elliott and Rouzbehani [8]). *Let A be a C^*-algebra with topological dimension zero. Then A is purely infinite if and only if A is weakly purely infinite.*

Proof. The forward implication holds in general. For the converse, assume that A is weakly purely infinite. Then A is nowhere scattered ([20, Example 3.3]), and therefore has the Global Glimm Property by Theorem 2.3. Now it follows from [12, Proposition 4.15] that A is purely infinite. □

We now turn to applications of Theorem 2.3 to multiplier algebras, which rest on the following result of Zhang.

Theorem 2.6 (Zhang). *Let A be a σ-unital C^*-algebra of real rank zero. Then its multiplier algebra $\mathcal{M}(A)$ has topological dimension zero.*

Proof. By [22, Theorem 2.2], every closed ideal in $\mathcal{M}(A)$ is the closed linear span of its projections. Consequently, projections in $\mathcal{M}(A)$ separate closed ideals, which means that $\mathcal{M}(A)$ has the ideal property. This implies that $\mathcal{M}(A)$ has topological dimension zero; see, for example, [13, Theorem 2.8]. □

Corollary 2.7. *Let A be a σ-unital, nowhere scattered C^*-algebra of real rank zero. Then its multiplier algebra $\mathcal{M}(A)$ has the Global Glimm Property.*

Proof. By Theorem 2.6, $\mathcal{M}(A)$ has topological dimension zero. Further, we know from [21, Theorem 5.12] that $\mathcal{M}(A)$ is nowhere scattered. Using Theorem 2.3, we get that $\mathcal{M}(A)$ satisfies the Global Glimm Property. □

In [12, Proposition 4.11], Kirchberg and Rørdam show that if A is a σ-unital, purely infinite C^*-algebra, then its multiplier algebra $\mathcal{M}(A)$ is weakly purely infinite, and they raise the question of whether $\mathcal{M}(A)$ is purely infinite. We resolve this problem in the case that A has real rank zero.

Corollary 2.8. *Let A be a σ-unital, purely infinite C^*-algebra of real rank zero. Then $\mathcal{M}(A)$ is purely infinite.*

Proof. By Theorem 2.6, $\mathcal{M}(A)$ has topological dimension zero. Further, $\mathcal{M}(A)$ is weakly purely infinite by [12, Proposition 4.11]. Now it follows from Corollary 2.5 that $\mathcal{M}(A)$ is purely infinite. □

Remark 2.9. Let A be a σ-unital, purely infinite C^*-algebra of real rank zero. Then $\mathcal{M}(A)$ is purely infinite and has topological dimension zero by Theorem 2.6 and Corollary 2.8. However, $\mathcal{M}(A)$ need not have real rank zero. For example, if Q denotes the Calkin algebra, then the stabilization $Q \otimes \mathcal{K}$ is a σ-unital, purely infinite C^*-algebra of real rank zero, but $\mathcal{M}(Q \otimes \mathcal{K})$ has real rank one by [18, Example 3.4].

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