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APPROXIMATION OF CONTINUOUS FUNCTIONS
ON A COMPACT SET
BY SOLUTIONS OF ELLIPTIC EQUATIONS.
QUANTITATIVE RESULTS

BY

GRIGORI ROZENBLUM*

*Department of Mathematics, Chalmers University of Technology
SE-41296 Gathenborg, Sweden*

and

*The Euler International Mathematical Institute, St. Petersburg State University
7-9 Universitetskaya Embankment, St. Petersburg, 199034 Russia
e-mail: grigori@chalmers.se*

AND

NIKOLAI SHIROKOV**

*National Research University Higher School of Economics
Kantemirovskaya Street 3a, St. Petersburg, 194100, Russia*

and

*St. Petersburg State University
7-9 Universitetskaya Embankment, St. Petersburg, 199034 Russia
e-mail: nikolai.shirokov@gmail.com*

ABSTRACT

We establish that a generalized Hölder continuous function on an $(m-2)$ -Ahlfors regular compact set in $\mathbb{R}^m$ can be approximated by solutions of an elliptic equation, with the rate of approximation determined by the continuity modulus of the function.

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1. Introduction

Approximation of functions by some better ones is a classical topic in Analysis. We just mention the Weierstrass theorem on real polynomial approximations, the Lavrentiev-Keldysh-Mergelyan theory of approximation of continuous function of a complex variable by polynomials, and Walsh's results on the rational approximation, to name just a few classical results. Up to now, there are numerous studies of approximation by polynomials, analytic, and harmonic functions, and, further, by solutions of differential equations with constant coefficients. The common feature of these classes of functions is that they possess a lot of algebraic and analytic structures, which enrich possible methods of analysis. (Math. Sci. Net contains several thousand references on this topic.) A detailed presentation, with a number of references, can be found, e.g., in the books [2], [11], [9]. The topic is still under active study, some most recent developments are presented, in particular, in papers by P. Paramonov, P. Gauthier, J. Carmona, M. Mazalov, K. Fedorovskiy and their co-operators (see, e.g., their latest publications [3], [6], [12], [18], [22], [23], and references therein).

Along with this direction, there is an interest in the study of the approximation by means of more special objects, in particular, by solutions of rather general differential equations. The spaces of solutions of equations with variable coefficients possess considerably fewer structures, therefore the approaches developed for the former type of problems fail. A fundamental result was obtained here by F. Browder in [4], where for continuous functions on a compact set of zero Lebesgue measure, there was established the property of approximation by solutions of a very general class of differential equations, including the elliptic ones of order l , having the form $\mathcal{L}u \equiv \mathcal{L}(X, D_X)u = 0$. The conditions imposed on the operator $\mathcal{L}$ were rather permissive, namely, the coefficients of the operator $\mathcal{L}$ as well as the ones of its formal dual $\mathcal{L}'$, should belong to C^1 in some domain Ω , with $\mathcal{L}'$ possessing the property of the local uniqueness in Ω for the Cauchy problem. In this setting, using advanced machinery of Functional Analysis, F. Browder proved that for any compact set $K \subset \Omega$, such that $\Omega \setminus K$ is connected, any continuous function on K can be approximated in $C(K)$ by C^l solutions of the equation $\mathcal{L}u = 0$. This extraordinary result was later extended in the direction of further relaxing the conditions imposed on the structure of the operator and its coefficients. However, certain questions, well understood in the study of polynomial and analytic approximations, remained untouched

upon in [4], and further studies. Namely, these questions concern finding a relation between the quality of the approximated function and the quality (say, the convergence rate) of the approximation. For the traditional approximation methods, there are classical results of this kind. One should mention here, of course, the results by D. Jackson and S. Bernstein connecting the smoothness class of a function on an interval and the rate of its polynomial approximation. Such results have been later called ‘the constructive description of functional classes.’ Further developments by S. Nikolski, V. Dzyadyk, I. Shevchuk, V. Belyi, V. Andrievski, N. Lebedev, and many others dealt with such constructive description of various classes of functions in terms of approximation by algebraic and trigonometrical polynomials, analytical and entire functions, but not by more general ones.

Relatively recently, progress was made in finding such constructive description of Hölder classes in terms of approximation by harmonic functions, which required, as explained above for the qualitative approximation problem, a number of new ideas.

In the present paper, we combine two problems described above. For a compact set $K \subset \mathbb{R}^m$ of zero Lebesgue measure (more exactly, an Ahlfors–David $(m - 2)$ -regular set), we find a necessary and sufficient condition for a function to belong to a prescribed generalized Hölder class in terms of uniform approximation by solutions of a given elliptic second order divergence form equation with Dini continuous coefficients. We use some ideas developed recently in [1] and [24] where the approximation by harmonic functions was considered. Here, it turns out that the sufficient condition in constructive terms for a function to belong to a Hölder class is proved rather elementarily. On the other hand, the proof of the necessary condition, namely that a function in the class under consideration can, in fact, be approximated by solutions of the equation, requires some tricky calculations. The advantage of our results here, compared, say, with the ones in [4], is that we not only prove the existence of approximating solutions but present these approximations explicitly and give sharp error estimates.

To explain our results in the introductory manner, we recall the general philosophy of approximation of functions. Having a ‘bad’ function f , one approximates it by ‘nice’ functions v_δ belonging to some class. The closer, for $\delta \rightarrow 0$, the approximating function v_δ is to f in some sense, the worse is the approximating function v_δ , in particular, its norm in a proper space may grow while $\delta \rightarrow 0$.

Quantitative approximation results specify the correspondence between δ and this worsening of the approximating function. In our case, the function f is continuous on a compact set $K \subset \mathbb{R}^m$, $m \geq 3$, moreover, it belongs to some generalized Hölder class $C^\omega(K)$ corresponding to a continuity modulus $\omega(t)$. The worsening of the approximating function v_δ is measured by the smallness of the δ -neighborhood K_δ of K , where v_δ is 'nice' in the proper sense, and by the size of the gradient of v_δ on K_δ . In this setting, the main results of the paper are as follows (the detailed formulations can be found further on in the text).

THE SUFFICIENT CONDITION. If for a given function $f \in C(K)$ for any sufficiently small $\delta > 0$, there exists a function v_δ on K_δ such that

$$(1.1) \quad |f(X) - v_\delta(X)| \leq C\omega(\delta) \text{ on } K \quad \text{and} \quad |\nabla(v_\delta)(X)| \leq C' \frac{\omega(\delta)}{\delta} \quad \text{on } K_\delta,$$

then $f \in C^\omega(K)$.

THE NECESSARY CONDITION. If K is sufficiently regular, for example, it is a Lipschitz surface of codimension 2 (or, more generally, is Ahlfors $(m-2)$ -regular) and $f \in C^\omega(K)$, then for any small δ , there exists a function v_δ on K_δ satisfying (1.1) and being, additionally, the weak solution in K_δ of a fixed second order elliptic equation $\mathcal{L}u = 0$ in divergence form, with Dini continuous coefficients.

Being put together, these two results can be understood as an example of a 'self-improving' property. Namely, if a function f can be approximated in the sense of (1.1) by some functions v_δ , it can be, in fact, approximated, in the same sense and with the same quality, by essentially more special functions, solutions of the prescribed elliptic equation.

THE STRUCTURE OF THE PAPER. In the second section we present some known basic facts we need, concerning the Hausdorff measure and Ahlfors–David regular sets. Although the results of the paper are valid for a more general class of compact sets, the Ahlfors–David regular ones admit a more explicit and visual description. We make also a certain improvement in the existing cover lemma. We describe also here some key facts about the Green function for elliptic equations with moderately regular coefficients. Then, in Section 3, we establish the sufficient approximation condition. The essential part of our further considerations is based on the construction of an extension of a continuous function from a compact set K to its neighborhood, with control over the derivatives of the

extension; this construction is described in Section 4. After deriving in Section 5 some important estimates of integrals involving the continuity modulus, in the subsequent sections we justify the integral representation for the extended function using the Green function, and finally present the construction of the approximating solutions of the elliptic equation and establish the estimates for the solution, which justifies the necessity part of the approximation theorem.

Our reasoning concerns approximation of an individual Hölder continuous function. It is interesting to extend our considerations in order to study the metric properties of the approximating operator. While we consider the functions on a compact set which is Ahlfors $(m-2)$ -regular (since it directly extends the previously studied case of a curve in $\mathbb{R}^3$, see [1], [24]), certain modifications of our approach enable one to handle θ -regular sets of any dimension $\theta < m-1$. We plan to treat these and other related topics in further publications.

2. Preliminaries

2.1. HAUSDORFF MEASURE. We recall the definition of the Hausdorff measure (see, e.g., [10], [7]). We denote by $\text{diam}(U)$ the diameter of the set $U \in \mathbb{R}^m$. Let $\mathcal{E} \subset \mathbb{R}^m$ be a bounded set. A finite or countable collection of subsets $U_k \subset \mathbb{R}^m$ is called δ -cover of $\mathcal{E}$, $\delta > 0$, if $\text{diam}(U_k) \leq \delta$ and $\mathcal{E} \subset \bigcup U_k$. For a fixed $\vartheta \geq 0$ and $\delta > 0$, we denote by $\mathcal{H}_\delta^\vartheta(\mathcal{E})$ the quantity

$$\mathcal{H}_\delta^\vartheta(\mathcal{E}) = \inf \left\{ \sum_k \text{diam } U_k^\vartheta : \{U_k\} \text{ is a } \delta\text{-cover of } \mathcal{E} \right\}.$$

It is not excluded that $\mathcal{H}_\delta^\vartheta(\mathcal{E})$ equals zero. As $\delta \searrow 0$, this infimum increases and tends to a limit, which is allowed to be zero or infinity. This limit,

$$\mathcal{H}^\vartheta(\mathcal{E}) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^\vartheta(\mathcal{E}),$$

is called the ϑ -dimensional Hausdorff measure of the set $\mathcal{E}$. It is known (see, e.g., [10]) that $\mathcal{H}^\vartheta$ is a regular Borel measure in $\mathbb{R}^m$.

It follows from the definition that for any set $\mathcal{E}$, there exists a unique $\theta \geq 0$ such that $\mathcal{H}^\vartheta(\mathcal{E}) = \infty$ for $\vartheta < \theta$ and $\mathcal{H}^\vartheta(\mathcal{E}) = 0$ for $\vartheta > \theta$. This number θ is called the Hausdorff dimension of $\mathcal{E}$, $\theta := \dim_{\mathcal{H}}(\mathcal{E})$. Only sets of Hausdorff dimension θ (but not all of them by far) may have finite positive Hausdorff measure $\mathcal{H}^\theta$. Discrete sets (but not only they) have zero Hausdorff dimension.

The Hausdorff measure is a convenient generalization of the Lebesgue measure. For $\theta = m$, it coincides with the m -dimensional Lebesgue measure, up to a constant factor, determined by the dimension. For subsets in a compact Lipschitz surface Σ of dimension θ , $1 \leq \theta < m$, the Hausdorff measure, again, up to a constant factor, coincides with the surface measure on Σ , generated in the standard way by the Lebesgue measure in $\mathbb{R}^m$. The Hausdorff measure is invariant with respect to shifts and rotations; it is also homogeneous under a dilation $\mathcal{D}_r$ with ratio r :

$$\mathcal{D}_r : x \mapsto rx, r > 0 :$$

$\mathcal{H}^\theta(\mathcal{D}_r(\mathcal{E})) = r^\theta \mathcal{H}^\theta(\mathcal{E})$. More generally, under a bi-Lipschitz mapping the Hausdorff measure changes in a controllable way: if $c|x-y| \leq |F(x)-F(y)| \leq C|x-y|$ then

$$c^\theta \mathcal{H}^\theta(\mathcal{E}) \leq \mathcal{H}^\theta(F(\mathcal{E})) \leq C^\theta \mathcal{H}^\theta(\mathcal{E}).$$

2.2. AHLFORS REGULAR SETS. The union of a finite collection of sets having Hausdorff dimension θ is, again, a set of this Hausdorff dimension. On the other hand, the union of finitely many sets of different Hausdorff dimension has Hausdorff dimension coinciding with the largest one of these sets. So, generally, a set may have pieces of different Hausdorff dimension. The following definition describes sets which have one and the same Hausdorff dimension everywhere.

Definition 2.1: Let, for a point $X \in \mathbb{R}^m$, $B_r(X)$ denote the open ball with center X and radius r . The set $\mathcal{E} \subset \mathbb{R}^m$ is called ‘Ahlfors regular of dimension θ ’ (called sometimes ‘Ahlfors–David regular’, or, in brief, ‘ θ -regular’) if there exist positive constants $C_\pm = C_\pm(\mathcal{E}, \theta)$ such that for any $X \in \mathcal{E}$,

$$(2.1) \quad C_- r^\theta \leq \mathcal{H}^\theta(\mathcal{E} \cap B_r(X)) \leq C_+ r^\theta, \quad 0 < r \leq \text{diam}(\mathcal{E}).$$

This property can be considered as a possible convenient generalization of the notion, for $\theta = 1$, of chord-arc curves; see, e.g., [1]. Ahlfors regular sets arise in many topics of Analysis. One can find a discussion of the application of this notion, e.g., in [7], [16], etc.

2.3. EXAMPLES. Obviously, the property to be a θ -regular set is preserved under bi-Lipschitz mappings. Therefore, for an integer θ , Lipschitz surfaces of dimension θ , as well as relatively open sets in them, are θ -regular. A wide family of θ -regular sets is provided by self-similar sets, see [14]. An affine mapping S in $\mathbb{R}^m$ is called **similitude** with coefficient $r < 1$ if it is the composition of a

translation, a rotation and a homothety with coefficient r . Let $S_1, \dots, S_k$ be a collection of similitudes with coefficients r_j . They satisfy **the open set condition** if there exists a nonempty open set $\mathcal{O} \subset \mathbb{R}^m$ such that $S_j(\mathcal{O}) \cap S_{j'}(\mathcal{O}) = \emptyset$, $j \neq j'$ and $\bigcup S_j(\mathcal{O}) \subset \mathcal{O}$. It is established in [14] that if a system S_j satisfies the open set condition, there exists a unique nonempty compact K invariant with respect to each of S_j , moreover $K = \bigcup_j S_j(K)$ (a constructive description of this set K is given in [14]). Such a **self-similar** set K is proved to be θ -regular, for θ being the only positive solution of the equation $\sum_j r_j^\theta = 1$. This construction of self-similar sets produces, in particular, various Cantor-like sets, von Koch snowflakes and many more. Further examples of θ -regular sets can be obtained using bi-Lipschitz maps.

On the other hand, a disjoint union of sets of different Hausdorff dimension can never be regular. In particular, a θ -regular set with $\theta < m$ may never have interior points.

2.4. A COVER LEMMA. Cover lemmas play traditionally an important role in Analysis. We will use the following cover lemma for Ahlfors regular sets, which can be found in [17], Lemma 2.1.

LEMMA 2.2: *Let $\mathcal{E} \subset \mathbb{R}^m$ be a compact θ -regular set. Then for some positive constant $C(\mathcal{E})$, any $r < R$, $r, R \in (0, \text{diam}(\mathcal{E}))$, and any point $X \in \mathcal{E}$, there exists a finite collection of points $\mathcal{Z} = \mathcal{Z}(X, r, R) = \{Z_l, l = 1, \dots, \kappa\}$, $\kappa = \kappa(X, r, R)$ such that*

$$(2.2) \quad (5^\theta C(\mathcal{E}))^{-1} (Rr^{-1})^\theta \leq \kappa(X, r, R) \leq 2^\theta C(\mathcal{E}) (Rr^{-1})^\theta,$$

all points Z_l are contained in $\mathcal{E} \cap B_R(X)$, the balls $B_r(Z_l)$ are disjoint and the balls $B_{5r}(Z_l)$ cover $\mathcal{E} \cap B_R(X)$.

The statement of this Lemma is not sufficient for our needs since the system of points $\mathcal{Z}$ depends, generally, on the starting point X and on the radii r, R . The corollary to follow shows that this system $\mathcal{Z}$ can be constructed depending on r only, so that it satisfies the upper estimate in (2.2) for all points X in the set $\mathcal{E}$ and for all $R > r$ simultaneously.

COROLLARY 2.3: *Under the conditions of Lemma 2.2, let $\mathcal{Z}$ be the system of points Z_l constructed for some $X = X_0$, some fixed $r > 0$ and for*

$$R = R_0 = \text{diam}(\mathcal{E}).$$

Then the inequality

$$(2.3) \quad \kappa(X, R) \leq C'(\mathcal{E})(Rr^{-1})^\theta,$$

holds for any $R > r$ where $\kappa(X, R)$ is the number of points $Z_l \in \mathcal{Z}$ in $B_R(X)$.

Proof. Since $\mathcal{E}$ is θ -regular, for a given R , the Hausdorff measure of $\mathcal{B}_{R+6r}(X) \cap \mathcal{E}$ satisfies

$$(2.4) \quad \mathcal{H}^\theta(B_{R+6r}(X) \cap \mathcal{E}) \leq C_+(R+6r)^\theta \leq 7^\theta C_+ R^\theta.$$

For the points $Z_l \in \mathcal{Z}(R) \equiv \mathcal{Z} \cap B_{R+5r}(X)$, each set $E_r(Z_l) = B_r(Z_l) \cap \mathcal{E}$ lies in $B_{R+6r}(X) \cap \mathcal{E}$, these sets are disjoint, and by the lower estimate in the definition of θ -regular sets (2.1),

$$\mathcal{H}^\theta(E_r(Z_l)) \geq C_- r^\theta.$$

Therefore,

$$(2.5) \quad \mathcal{H}^\theta\left(\bigcup_{Z_l \in \mathcal{Z}(R)} E_r(Z_l)\right) = \sum_{Z_l \in \mathcal{Z}(R)} \mathcal{H}^\theta(E_r(Z_l)) \geq \kappa(X, R) C_- r^\theta.$$

We compare (2.4) and (2.5), which gives us the required upper estimate for $\kappa(X, R)$. It remains to notice that, by the initial construction of the points Z_l , the balls with radius $5r$ centered at the points Z_l in $B_{R+5r}(X)$ cover $B_R(X) \cap \mathcal{E}$ since all other balls centered at Z_l are disjoint with $B_R(X) \cap \mathcal{E}$. ■

It follows also that the balls $B_{6r}(Z_l)$ form a covering of the r -neighborhood $\mathcal{E}_r = \cup_{X \in \mathcal{E}} B_r(X)$ of the set $\mathcal{E}$.

2.5. THE GREEN FUNCTION AND WEAK SOLUTIONS.

2.5.1. Elliptic equations. In the paper we consider general second order elliptic equations. Unlike the case of analytic or harmonic approximation, the question about the behavior of fundamental solutions at infinity is rather complicated. However for our needs, it suffices to consider the operator in a nice bounded domain Ω ; we accept as Ω the sufficiently large ball and use the Green function instead.

We consider the second order formally self-adjoint divergence form elliptic operator in Ω ,

$$\mathcal{L} = - \sum_{j,k} \partial_j a_{j,k}(X) \partial_k.$$

About the matrix of coefficients

$$\mathbf{a}(X) = (a_{j,k}(X))$$

we suppose that it is real, symmetric, continuous, moreover, it is Dini continuous, namely, for $X, Y \in \Omega$, for a certain continuity modulus $\omega_0(t)$ such that

$$\omega_0(2t) \leq A\omega_0(t); \quad \int_0^1 \frac{\omega_0(t)}{t} dt < \infty,$$

it satisfies

$$(2.6) \quad |a_{j,k}(X) - a_{j,k}(Y)| \leq \omega_0(|X - Y|);$$

finally, it is also uniformly elliptic, $\mathbf{a}(X) \geq \kappa > 0$ in Ω .

With this operator, we associate the **sesquilinear form**

$$(2.7) \quad \mathbf{a}[u, v] = \mathbf{a}_\Omega[u, v] = \int_\Omega \langle \mathbf{a}(X) \nabla u, \nabla \bar{v} \rangle dX,$$

for u, v in the Sobolev space $H^1(\Omega)$, where (and further on) the angle brackets denote the scalar product in $\mathbb{C}^m$, and the symbol dX is used for the integration with respect to the m -dimensional Lebesgue measure.

For a measure μ in Ω with locally finite variation, a function u with locally integrable derivative is called the **weak solution** of the equation $\mathcal{L}u = \mu$ if

$$(2.8) \quad \mathbf{a}[u, \psi] = \int_\Omega \bar{\psi}(X) d\mu(X)$$

for all $\psi \in C_0^\infty(\Omega)$, where $\bar{\psi}$ denotes the complex adjoint of ψ .

Of course, if the coefficients of the operator $\mathcal{L}$ are more regular, say, belong to C^1 , and the measure μ is absolutely continuous with respect to the Lebesgue measure, $d\mu = g dX$, the weak solution is, in fact, the classical strong solution of the equation $\mathcal{L}u = g$ (see, e.g., [21]).

2.5.2. The Green function. An important role in our construction will be played by the Green function. The existence and main properties of the Green function in our setting are established in the classical papers [15], [13], see also [21].

THEOREM 2.4 (Theorem 1.1 in [13]): *Let $\mathcal{L} = -\sum_{j,k} \partial_j a_{j,k} \partial_k$ be an elliptic operator in $\Omega \subset \mathbb{R}^m$, $m \geq 3$, with the matrix of coefficients, $\mathbf{a}(X) = (a_{j,k}(X))$, being bounded and uniformly positive, $\mathbf{a}, \mathbf{a}^{-1} \in L_\infty(\Omega)$. Then there exists the unique Green function $G(X, Y) \geq 0$ such that*

$$(2.9) \quad \mathbf{a}[\phi(\cdot), G(X, \cdot)] = \phi(X), \phi \in C_0^\infty(\Omega),$$

where $\mathbf{a}$ is the sesquilinear form of $\mathcal{L}$ see (2.7); moreover,

$$(2.10) \quad G(X, Y) \leq c_1 |X - Y|^{2-m}, \quad X, Y \in \Omega,$$

and for $|X - Y| < \frac{1}{2} \text{dist}(X, \partial\Omega)$,

$$(2.11) \quad G(X, Y) \geq c_2 |X - Y|^{2-m},$$

where the constants c_1, c_2 depend only on $\|\mathbf{a}\|_{L_\infty}, \|\mathbf{a}^{-1}\|_{L_\infty}$ (and Ω).

Additionally, if the coefficients of $\mathcal{L}$ are Dini continuous, see (2.6), the following estimate holds (see Theorem 3.3 in [13]):

THEOREM 2.5: *Suppose that the boundary of Ω is sufficiently regular (say, satisfies the exterior cone condition). Then*

$$(2.12) \quad |\nabla_X G(X, Y)| \leq C |X - Y|^{1-m}, \quad |\nabla_Y G(X, Y)| \leq C |X - Y|^{1-m},$$

$$(2.13) \quad |\nabla_X \nabla_Y G(X, Y)| \leq C |X - Y|^{-m},$$

for all $X, Y \in \Omega$.

We will use these results for the case when Ω is a (sufficiently large) ball containing the compact set K under consideration, therefore, the regularity conditions are surely satisfied.

The Green function $G(X, Y)$ produces the solution of the Dirichlet problem: for a function $f \in L_2(\Omega)$, the scalar product

$$u(X) = \int_{\Omega} G(X, Y) f(Y) dY \equiv \langle f(\cdot), G(X, \cdot) \rangle$$

defines the (unique) weak solution in $\overset{\circ}{H}^1$ of the equation $\mathcal{L}u = f$. Moreover, by Theorem 6.1 in [15], for any measure μ on Ω , having bounded variation, the integral

$$u(X) = (G(X, \cdot), \mu(\cdot)) \equiv \int_{\Omega} G(X, Y) d\mu(Y)$$

represents a weak solution of the equation $\mathcal{L}u = \mu$, vanishing on $\partial\Omega$.

The Green function will be used in the analysis of the integral representations related to the operator $\mathcal{L}$ and in constructing the required approximation. The standard integral representation is valid for an elliptic equation for a domain $\Omega' \subset \Omega$, (see, e.g., (9.3) in [21]):

$$(2.14) \quad u(X) = \mathbf{a}[u(\cdot), G(X, \cdot)] - \int_{\partial\Omega'} G(X, Y) \partial_{\nu_a(Y)} u(Y) d\sigma(Y).$$

This formula involves the boundary term, usually called the single layer potential, containing the conormal derivative of u on a surface Γ , the boundary of Ω'

$$(2.15) \quad \begin{aligned} \partial_{\nu_a(X)} u(X) &:= \sum_{j,k} a_{j,k}(X) \nu_j(X) \partial_k u(X) \\ &= \langle \mathbf{a}(X) \nabla u(X), \nu(X) \rangle, \quad X \in \Gamma, \end{aligned}$$

where, recall, the angle parentheses denote the standard scalar product in $\mathbb{C}^m$ and $\nu_j(X)$ are components of the unit normal vector $\nu(X)$ of Γ . It is possible that a part of the boundary of Ω' lies on the boundary of Ω ; in this case the integral over the corresponding part of Γ in (2.14) vanishes.

3. The constructive description. Sufficient conditions

Further on, we consider the space $C^\omega(K)$ of continuous functions on the compact set K with continuity modulus, a concave function $\omega(t)$, with decaying $\frac{\omega(t)}{t}$, satisfying

$$(3.1) \quad \int_0^t \frac{\omega(\tau)}{\tau} d\tau + t \int_t^\infty \frac{\omega(\tau)}{\tau^2} d\tau \leq C\omega(t),$$

for some constant C .

THEOREM 3.1: *Let $K \subset \mathbb{R}^m$ be a compact set with zero Lebesgue measure and let $\omega(t)$ be a continuity modulus satisfying (3.1). For $\delta > 0$, denote by K_δ the δ -neighborhood of K in $\mathbb{R}^m$. Suppose that for a function $f(X)$, $X \in K$, for any $\delta > 0$ (equivalently, for a sequence of such δ tending to zero), there exists a function $v_\delta \in C^1(K_\delta)$ such that*

$$(3.2) \quad |f(X) - v_\delta(X)| \leq c' \omega(\delta), \quad X \in K,$$

and

$$(3.3) \quad |\nabla v_\delta(X)| \leq c'' \omega(\delta) \delta^{-1}, \quad X \in K_\delta,$$

with some constants c', c'' (depending, generally, on the function f). Then $f \in C^\omega(K)$.

Proof. Recall that the open ball with radius δ and center X is denoted by $B_\delta(X)$. For a given $\delta > 0$, we take two arbitrary points $X_1, X_2 \in K$ such that $|X_1 - X_2| < \frac{\delta}{2}$. Then both balls $B_\delta(X_j)$ lie inside K_δ and, therefore, the straight segment connecting X_1 and X_2 lies inside K_δ as well. Let $\nu = \frac{X_1 - X_2}{|X_1 - X_2|}$. Then

$$\begin{aligned} |f(X_2) - f(X_1)| &\leq |f(X_2) - v_\delta(X_2)| + |v_\delta(X_2) - v_\delta(X_1)| + |v_\delta(X_1) - f(X_1)| \\ &\leq 2c_1\omega(\delta) + \frac{\delta}{2} \int_0^1 \left| \nabla v_\delta(X_1 + \frac{\delta}{2}t\nu) \right| dt \\ &\leq 2c_1\omega(\delta) + c_2 \frac{\delta \omega(\delta)}{2} \leq C\omega\left(\frac{\delta}{2}\right). \quad \blacksquare \end{aligned}$$

We can see that there are no requirements that the approximating functions v_δ are solutions to some equation or whatever else. The remaining part of the paper is devoted to constructing, for a given function $f \in C^\omega(K)$ and an arbitrary δ such approximating functions, satisfying (3.2), (3.3), which are weak solutions of a given second order elliptic equation.

4. The extension of a continuous function

It is known since long ago, see [20], [26], that a continuous function on a compact set in the Euclidean space, with a given continuity modulus $\omega(t)$, admits an extension to the whole space with the same continuity modulus. The following statement provides us with an extension which, additionally, is infinitely differentiable outside K , with the first order derivative controlled by this continuity modulus and by the distance to the compact. The continuity modulus $\omega(t)$ is fixed, and it is supposed to be a concave function with decaying $\frac{\omega(t)}{t}$, satisfying (3.1)

In the following reasoning (and further on in the paper), the symbol c without subscript denotes constants whose value is of no importance, somehow depending on the function f and the set K , which may vary from one formula to another one and even within one and the same formula. For a point $X \in \mathbb{R}^m \setminus K$ we denote by $d(X)$ the distance of X to K .

LEMMA 4.1: *Let $K \subset \mathbb{R}^m$, $m \geq 3$, be a compact set with connected complement in the open ball Ω . Let f be a continuous function, $f \in C^\omega(K)$ with continuity modulus ω . Then there exists an extension f_0 of f , $f_0 \in C_0(\mathbb{R}^m) \cap C^\infty(\mathbb{R}^m \setminus K)$,*

such that

$$(4.1) \quad |f_0(X_1) - f_0(X_2)| \leq c\omega(|X_1 - X_2|); \quad X_1, X_2 \notin K,$$

$$(4.2) \quad |\nabla f_0(X)| \leq c \frac{\omega(d(X))}{d(X)}, \quad X \notin K.$$

Remark: In fact, for any $k > 0$, the estimate $\nabla^k f_0(X) = O(\frac{\omega(d(X))}{d(X)^k})$ holds.

Proof. The construction of f_0 goes in the following way. Let $f_1 \in C_0(\mathbb{R}^m)$ be the McShane–Whitney continuation of f (see [20], [26]) such that

$$|f_1(X_1) - f_1(X_2)| \leq c\omega(|X_1 - X_2|), \quad X_1, X_2 \in \mathbb{R}^m.$$

There exists (see, e.g., [25]) a smooth approximation

$$d_0(X) > 0, \quad d_0 \in C^\infty(\mathbb{R}^m \setminus K),$$

of the distance function $d(X)$, such that

$$(4.3) \quad c_1 d(X) \leq d_0(X) \leq c_2 d(X), \quad X \in \mathbb{R}^m \setminus K,$$

and also

$$|\operatorname{grad} d_0(X)| \leq c, \quad |\operatorname{grad}^2 d_0(X)| \leq \frac{c}{d(X)},$$

with constant c depending only on the compact K .

We set $\rho(X) = \frac{d_0(X)}{2c_2}$, where c_2 is the constant in (4.3), so that $\rho(X) \leq \frac{d(X)}{2}$, and denote $\phi(X) = \rho(X)^{-2}$, $X \notin K$; obviously,

$$(4.4) \quad |\nabla \phi(X)| \leq c d(X)^{-3}.$$

We also set $\psi(X) = \phi(X)^{m/2} = \rho(X)^{-m}$; for the derivatives of ψ , we have

$$(4.5) \quad |\nabla \psi(X)| \leq c d(X)^{-m-1}.$$

The function whose existence is proclaimed in Lemma 4.1 is constructed as the averaging of the function f_1 with variable radius depending on $\rho(X)$.

We denote by $B(X)$ the ball $B_{\rho(X)}(X)$, centered at the point $X \notin K$ and with radius $\rho(X)$; this ball satisfies $\operatorname{dist}(B(X), K) \geq d(X)/2$. Let the even function $h \geq 0$, $h(r) \in C_0^\infty((-1, 1))$, satisfy

$$\int_{\mathbb{R}^m} h(|X|^2) dX = 1.$$

Now we introduce the mollification kernel $\mathcal{K}(X, Y)$, smooth in both variables, by

$$\mathcal{K}(X, Y) = \psi(X) h(\phi(X)|X - Y|^2),$$

thus

$$\int_{\mathbb{R}^m} \mathcal{K}(X, Y) dY \equiv \int_{B(X)} \mathcal{K}(X, Y) dY = 1.$$

Using this kernel, we introduce the mollified function f_0 by setting

$$f_0(X) = \int_{\mathbb{R}^m} \mathcal{K}(X, Y) f_1(Y) dY \equiv \int_{B(X)} \mathcal{K}(X, Y) f_1(Y) dY,$$

so f_0 is a variable radius mollification of the function f_1 with radius vanishing as the point approaches K (in a somewhat different setting, the idea of using a variable radius mollification was earlier implemented by E. Dyn'kin, [8], and V. Burenkov, [5]). The smoothness of f_1 follows from the smoothness of the kernel $\mathcal{K}$.

For a given point $X \notin K$, let X_0 be (one of) the point(s) in K , closest to X , $d(X) = |X - X_0|$. Then

$$(4.6) \quad f(X) - f(X_0) = \int_{B(X)} \mathcal{K}(X, Y) (f_1(Y) - f(X_0)) dY.$$

Since for $Y \in B(X)$ we have $|Y - X_0| \leq |Y - X| + |X - X_0| \leq cd(X)$, it follows

$$(4.7) \quad |f_1(Y) - f(X_0)| \leq c\omega(d(X));$$

in particular, for $Y = X$, we have

$$(4.8) \quad |f_1(X) - f(X_0)| \leq c\omega(d(X)).$$

This implies that f_0 is, in fact, a continuous extension of the function f to $\mathbb{R}^m$.

Our aim now is to estimate the derivatives of f_0 .

For any constant γ , the derivatives of f_0 and of $f_0^\gamma := f_0 - \gamma$, of course, coincide. Therefore, when estimating derivatives of f_0 at the point $X \in \mathbb{R}^m \setminus K$, we may handle the function f_0^γ instead, using the value $\gamma = f(X_0)$ (where, recall, X_0 is the closest to point X in K). We keep in mind here that, by (4.7) and (4.8),

$$(4.9) \quad |f_0^\gamma(Y)| \leq c\omega(d(X)), \quad \text{for } Y \in B(X).$$

So, we have

$$\nabla f_0^\gamma(X) = \int_{\mathbb{R}^m} \nabla_X \mathcal{K}(X, Y) f_1^\gamma(Y) dY = \int_{B(X)} \nabla_X \mathcal{K}(X, Y) f_1^\gamma(Y) dY.$$

Since $|f_1^\gamma(Y)| = |f_1(Y) - f(X_0)| \leq c\omega(d(X))$ for $Y \in B(X)$,

$$|\nabla f_0(X)| \leq c\omega(d(X)) \int_{B(X)} |\nabla_X \mathcal{K}(X, Y)| dY.$$

For the derivatives of the kernel $\mathcal{K}$, we have

$$(4.10) \quad \begin{aligned} \nabla_X \mathcal{K}(X, Y) \\ = (\nabla \psi(X))h(\phi(X)|X - Y|^2) + \psi(X)\nabla_X h(\phi(X)|X - Y|^2); \end{aligned}$$

where

$$(4.11) \quad \begin{aligned} \nabla_X h(\phi(X)|X - Y|^2) \\ = h'(\phi(X)|X - Y|^2)(\nabla \phi(X)|X - Y|^2 + 2\phi(X)(X - Y)). \end{aligned}$$

Using (4.4), (4.5), and the fact that $|X - Y| \leq \rho(X)$, we obtain from (4.10), (4.11) that

$$|\nabla_X \mathcal{K}(X, Y)| \leq cd(X)^{-m-1},$$

and, therefore,

$$\int_{B(X)} |\nabla_X \mathcal{K}(X, Y)| \leq cd(X)^{-1}.$$

We recall (4.9), which gives us

$$|\nabla f_0(X)| \equiv |\nabla f_0^\gamma(X)| \leq c \frac{\omega(d(X))}{d(X)}. \quad \blacksquare$$

Finally, we can take a smooth function $\chi(X) \in C_0^\infty(\Omega)$, $\chi(X) = 1$ in a neighborhood of K , and consider χf_0 instead of f_0 , to have a continuation of f with compact support in Ω , with all the required properties.

5. Integral estimates for the continuity modulus

In this section we derive important estimates of some integrals involving the continuity modulus $\omega(t)$ and the extended function f_0 constructed in Section 4. Let, as before, Ω be an open ball in $\mathbb{R}^m$, such that $K \subset \Omega$. Suppose that $\mathcal{L}$ is an elliptic operator in Ω with coefficients satisfying the conditions in Section 2.5 and let $G(X, Y)$ be the Green function for $\mathcal{L}$ in Ω .

We suppose that our compact set K satisfies the following condition.

CONDITION A. For any $\delta > 0$ there exists a set $\mathcal{Z}(\delta)$ of points $Z_l \in K$, $l = 1, \dots, M(\delta)$, such that the distance between any two points in $\mathcal{Z}(\delta)$ is not less than δ , additionally, for some constant c_0 , the set K_δ , the δ -neighborhood of K is covered by the collection of closed balls $B_{c_0\delta}(Z_l)$,

$$(5.1) \quad K_\delta \subset \bigcup_{Z_l \in \mathcal{Z}(\delta)} B_{c_0\delta}(Z_l),$$

and, finally, for any $R \geq \delta$, some constant c_1 and for any point $X \in \mathbb{R}^m$, the quantity $M(X, R, \delta)$ of the points of $\mathcal{Z}(\delta)$ in the ball $\overline{B_R(X)}$ is not greater than $c_1(\frac{R}{\delta})^{m-2}$, in particular, the number $M(\delta)$ of balls in $\mathcal{Z}(\delta)$ satisfies

$$(5.2) \quad M(\delta) \leq c_1 \left(\frac{\text{diam}(K)}{\delta} \right)^{m-2}.$$

As follows from Lemma 2.2 and Corollary 2.3, if the set K is $(m-2)$ -regular, condition **A** is satisfied. We recall that in Section 2.2 above, some basic examples of construction of regular sets are described, the most visual being Lipschitz surfaces of dimension $m-2$, as well as some examples are given of sets which are not regular.

Having the set $\mathcal{Z}(\delta)$, for a fixed $c > 0$, we denote by $\mathcal{B}_c(\delta)$ the union of balls with radius $c\delta$ centered at the points $Z_l \in \mathcal{Z}(\delta)$,

$$(5.3) \quad \mathcal{B}_c(\delta) = \bigcup_{Z_l \in \mathcal{Z}(\delta)} B_{c\delta}(Z_l).$$

The value of c in (5.3) will be fixed so that $c > \max(2c_0, c_1)$, where c_0, c_1 are the constants in Condition **A**, and we will omit it in the notation, writing simply $\mathcal{B}(\delta) := \mathcal{B}_c(\delta)$. With this choice, the set $\mathcal{B}(\delta)$ covers K_δ .

Let $f(X)$ be a function on K with the continuity modulus ω , i.e., $f \in C^\omega(K)$, and let f_0 be its extension to Ω as constructed in Section 4, in particular, supported in Ω .

For a fixed point $X \in (\Omega \setminus K)$, we denote the distance from X to K by $\delta = d(X)$ and let δ_1 satisfy $2c_0\delta_1 < \delta$, where c_0 is the constant in (5.1).

We introduce the set $\Theta_{\delta_1} \subset \Omega$,

$$(5.4) \quad \Theta_{\delta_1} = \Omega \setminus \overline{\mathcal{B}(\delta_1)}.$$

It follows that the initial point X belongs to Θ_{δ_1} ; this means, it lies outside $\overline{\mathcal{B}(\delta_1)}$. The distance of the points in Θ_{δ_1} to K can be estimated from below,

$$(5.5) \quad \text{dist}(\Theta_{\delta_1}, K) \geq c_0\delta_1.$$

In fact, to justify (5.5), we suppose that $\text{dist}(K, \Theta_{\delta_1}) < c_0\delta_1$ and consider a point Y in the joint boundary of Θ_{δ_1} and $\mathcal{B}(\delta_1)$, where (5.5) breaks down. Let Y' be its closest point in K , so that $\text{dist}(Y, K) \equiv d(Y) = |Y - Y'|$. The point Y' must belong to one of the balls $B_{c_0\delta_1}(Z_l)$, $Z_l \in \mathcal{Z}(\delta_1)$. Therefore, for this particular l ,

$$|Z_l - Y| \leq |Z_l - Y'| + |Y' - Y| < c_0\delta_1 + c_0\delta_1 = 2c_0\delta_1.$$

This would mean that Y is an interior point in $\mathcal{B}(\delta_1)$ and not a boundary point, contrary to what was assumed.

The boundary $\Sigma(\delta_1) := \partial(\mathcal{B}(\delta_1))$ of $\mathcal{B}(\delta_1)$ is a piecewise smooth surface; it consists of a finite set of domains on spheres in $\mathbb{R}^m$, having piecewise smooth $(m-2)$ -dimensional boundaries. We denote by σ the $(m-1)$ -dimensional surface measure on $\Sigma(\delta_1)$, generated by the Lebesgue measure on $\mathbb{R}^m$. Since the surface measure of a sphere equals $\sigma(\partial(B_{c\delta_1}(Z))) = c_m \delta_1^{m-1}$ and the quantity $M(\delta_1)$ of balls satisfies (5.2), we have

$$(5.6) \quad \sigma(\partial\mathcal{B}(\delta_1)) \leq cM(\delta_1)\delta_1^{m-1} \leq c \operatorname{diam}(K)^{m-2}\delta_1^{2-m} \times \delta_1^{m-1} = c\delta_1.$$

Further on, when estimating the expressions in the integral representations to follow, we will use the following lemma.

LEMMA 5.1: *For the continuity modulus ω as above, the following estimate holds:*

$$(5.7) \quad \int_{\mathcal{B}(\delta_1)} \frac{\omega(d(Y))}{d(Y)} dY < \infty, \quad \delta_1 > 0.$$

Proof. First of all, since the set K has Lebesgue measure zero, we may replace the integration domain in (5.7) by $\mathcal{B}(\delta_1) \setminus K$. We introduce the notation $\delta_n = 2^{1-n}\delta_1$, $n = 1, 2, \dots$; for each n , consider the covering of K by the balls with radius $c_1\delta_n$, in accordance with Condition **A** (in particular, for an $(m-2)$ -regular set K , using Lemma 2.2), and construct the corresponding neighborhoods $\mathcal{B}(\delta_n)$ of the compact set K . Accordingly, the domain $\mathcal{B}(\delta_1) \setminus K$ admits the representation

$$\mathcal{B}(\delta_1) \setminus K = \bigcup_{n=1}^{\infty} \mathcal{G}_n$$

where $\mathcal{G}_n$ denotes $\mathcal{B}(\delta_n) \setminus \mathcal{B}(\delta_{n+1})$. Therefore,

$$(5.8) \quad \int_{\mathcal{B}(\delta_1) \setminus K} \frac{\omega(d(Y))}{d(Y)} dY = \sum_{n=1}^{\infty} \int_{\mathcal{G}_n} \frac{\omega(d(Y))}{d(Y)} dY.$$

On the set $\mathcal{G}_n$, we have $d(Y) \geq c\delta_n$ and $\frac{\omega(d(Y))}{d(Y)} \leq C \frac{\omega(\delta_{n+1})}{\delta_{n+1}}$. The quantity $M(\delta_n)$ of balls composing $\mathcal{B}(\delta_n)$, by Condition **A**, is majorated by $(\frac{\operatorname{diam}(K)}{\delta_n})^{m-2}$, therefore,

$$\operatorname{meas}_m(\mathcal{B}(\delta_n)) \leq c\delta_n^m \left(\frac{\operatorname{diam}(K)}{\delta_n} \right)^{m-2} \leq c\delta_n^2,$$

which implies

$$\int_{\mathcal{G}_n} \frac{\omega(d(Y))}{d(Y)} dY \leq c \frac{\omega(\delta_{n+1})}{\delta_{n+1}} \text{meas}_m(\mathcal{G}_n) \leq c\omega(\delta_{n+1})\delta_n^2.$$

We substitute the last estimate in (5.8) and obtain

$$\int_{\mathcal{B}(\delta)} \frac{\omega(d(Y))}{d(Y)} dY \leq \sum_n \frac{\delta_1}{2^n} \omega\left(\frac{\delta_1}{2^n}\right) < \infty. \quad \blacksquare$$

Lemma 5.1 implies the following important property.

COROLLARY 5.2: *As $\delta_1 \rightarrow +0$, the integral in (5.7) tends to zero.*

We will also need the localized estimate.

LEMMA 5.3: *Let X_0 be an arbitrary point in K and let $r \geq \delta$. Then*

$$(5.9) \quad I(X_0, r) := \int_{B_r(X_0) \cap \mathcal{B}(\delta_1)} \frac{\omega(d(Y))}{d(Y)} dY \leq cr^{m-2} \delta_1 \omega(\delta_1).$$

Proof. We use the notations for $\delta_n, \mathcal{G}_n$ from the previous proof. Then

$$(5.10) \quad I(X_0, r) = \sum_{n=1}^{\infty} \int_{B_r(X_0) \cap \mathcal{G}_n} \frac{\omega(d(Y))}{d(Y)} dY.$$

For $Y \in B_r(X_0) \cap \mathcal{G}_n$, we have

$$d(Y) \geq c_1 \delta_{n+1}, \quad \frac{\omega(d(Y))}{d(Y)} \leq c \frac{\omega(\delta_{n+1})}{\delta_{n+1}}.$$

The ball $B_{r+2c_1\delta_n}(X_0)$ contains no more than

$$\left(\frac{r+2c_1\delta_n}{\delta_n}\right)^{m-2} \leq c \left(\frac{r}{\delta_n}\right)^{m-2}$$

points of the set $\mathcal{Z}(\delta_1)$, therefore,

$$\text{meas}_m(B_r(X_0) \cap \mathcal{B}(\delta_n)) \leq c\delta_n^m \left(\frac{r}{\delta_n}\right)^{m-2} = cr^{m-2} \delta_n^2.$$

The latter inequality enables us to estimate the integral in (5.10):

$$(5.11) \quad \begin{aligned} \sum_n \int_{B_r(X_0) \cap \mathcal{G}_n} \frac{\omega(d(Y))}{d(Y)} dY &\leq c \sum_n \frac{\omega(\delta_{n+1})}{\delta_{n+1}} r^{m-2} \delta_n^2 \\ &\leq c\delta_1 \omega(\delta_1) r^{m-2}. \quad \blacksquare \end{aligned}$$

The reasoning in the proof of Lemma 5.3 produces one more important estimate. For $X_0 \in K$ and an arbitrary $\delta > 0$, we have $B_{c_1\delta}(X_0) \subset \mathcal{B}(\delta)$. Therefore,

$$(5.12) \quad \int_{B_\delta(X_0)} \frac{\omega(d(Y))}{d(Y)} \leq c\delta^{m-1}\omega(\delta).$$

6. The integral representation

For a fixed function f on K with continuity modulus ω , as above, let f_0 be the extension of f to $\mathbb{R}^m$ constructed in accordance with Lemma 4.1. The aim of this section is to justify the integral representation for the function f_0 in Ω . At the moment, we may not apply immediately the general formula (2.14) since the derivatives of f_0 may have rather strong singularities when approaching the set K . Therefore, a limit procedure is needed in order to justify the resulting formula.

6.1. INTEGRAL REPRESENTATION OUTSIDE K . Recall that f_0 vanishes at the boundary of the ball Ω . We fix some point $X \notin K$. For a sufficiently small $\delta_1 > 0$, such that $X \in \Theta_{\delta_1}$ (see the definition in (5.4)), and for the function f_0 which by construction is smooth outside K , we apply the formula for the integral representation in the domain Θ_{δ_1} , which is possible since f_0 is smooth in this domain; we use the Green function $G(X, Y)$ for the ball Ω . The boundary of Θ_{δ_1} consists of two parts. The first one is the boundary of the ball Ω . Since G vanishes on this sphere, this part of the boundary does not contribute to the integral representation. The other part of the boundary of Θ_{δ_1} is the interior part, namely, the joint boundary $\Sigma(\delta_1)$ of Θ_{δ_1} and $\mathcal{B}(\delta_1)$, the latter being the neighborhood of the compact K . Thus, for $X \in \Theta_{\delta_1}$, we have

$$(6.1) \quad \begin{aligned} f_0(X) &= \int_{\Theta_{\delta_1}} \langle \mathbf{a}(Y) \nabla_Y G(X, Y), \nabla_Y f_0(Y) \rangle dY \\ &\quad + \int_{\Sigma(\delta_1)} G(X, Y) \partial_{\nu_{\mathbf{a}}(Y)} f_0(Y) d\sigma(Y). \end{aligned}$$

When δ_1 is sufficiently small, namely when the distance from X to $\Sigma(\delta_1)$ is greater than $\frac{1}{2}d(X)$, the Green function $G(X, Y)$ and its derivative $G(X, Y)$ satisfy for $Y \in \Sigma(\delta_1)$ the estimates

$$G(X, Y) \leq cd(X)^{2-m}, \quad |\nabla_Y G(X, Y)| \leq cd(X)^{1-m},$$

according to (2.10), (2.12).

We are able now to estimate the integrals in (6.1). For the surface integral, by (4.2) and (5.6), we have

$$\begin{aligned}
 (6.2) \quad & \left| \int_{\Sigma(\delta_1)} G(X, Y) \partial_{\nu_a(Y)} f_0(Y) d\sigma(Y) \right| \\
 & \leq cd(X)^{1-m} \int_{\Sigma(\delta_1)} \frac{\omega(d(Y))}{d(Y)} dY \\
 & \leq cd(X)^{1-m} \delta_1 \frac{\omega(\delta_1)}{\delta_1} = cd(X)^{1-m} \omega(\delta_1).
 \end{aligned}$$

Estimate (6.2) shows that the surface integral in (6.1) tends to zero as $\delta_1 \rightarrow 0$.

Now we consider the volume integral in (6.1). With $X \notin K$ still fixed, we use estimate (2.12) for the gradient of the Green function and, again, estimate (4.2) for the gradient of the function f_0 . Together with Lemma 5.1 and Corollary 5.2, this implies that as $\delta_1 \rightarrow 0$, the volume integral tends to

$$\int_{\Omega} \langle \mathbf{a}(Y) \nabla_Y G(X, Y), \nabla f_0(Y) \rangle dY.$$

So, we may pass to the limit as $\delta_1 \rightarrow 0$ in (6.1), use Corollary 5.2, and obtain the integral representation of the function f_0 outside K :

$$(6.3) \quad f_0(X) = \int_{\Omega} \langle \mathbf{a}(Y) \nabla_Y G(X, Y), \nabla f_0(Y) \rangle dY, \quad X \notin K.$$

Our next task is to show that this representation for $f_0 = f$ is valid for all points in K as well. For this, we need the following estimate.

LEMMA 6.1: *Let K satisfy Condition **A** and let $X_0 \in K$. Then for $r > 0$,*

$$(6.4) \quad I(X_0, r) \equiv \int_{B_r(X_0)} \frac{\omega(d(Y))}{d(Y)} |Y - X_0|^{1-m} dY \leq c\omega(r).$$

Proof. We set $r_n = 2^{-n}r$, $n = 0, \dots, \mathcal{V}_n = B_{r_n}(X_0) \setminus B_{r_{n+1}}(X_0)$. Then the integral in (6.4) splits into the sum

$$I(X_0, r) = \sum_{n=0}^{\infty} J_n; \quad J_n \equiv \int_{\mathcal{V}_n} \frac{\omega(d(Y))}{d(Y)} |Y - X_0|^{1-m} dY.$$

When Y belongs to the spherical annulus $\mathcal{V}_n$, we have $|Y - X_0| \geq r_{n+1}$, and, therefore,

$$\begin{aligned}
 J_n & \leq r_{n+1}^{1-m} \int_{\mathcal{V}_n} \frac{\omega(d(Y))}{d(Y)} dY \leq r_{n+1}^{1-m} \int_{B_{r_n}} \frac{\omega(d(Y))}{d(Y)} dY \\
 & \leq cr_{n+1}^{1-m} r_n^{m-1} \omega(r_n) \leq c\omega(2^{-n}r).
 \end{aligned}$$

We take into account the properties of the continuity modulus ω to obtain

$$\begin{aligned} I(X_0, r) &\leq c \sum_{n=0}^{\infty} \omega(2^{-n}r) \leq c \int_0^{\infty} \omega(2^{-t}r) dt \\ &= c \int_1^{\infty} \omega(\tau^{-1}r) \frac{d\tau}{\tau} = c \int_0^{\rho} s^{-1} \omega(s) ds \leq c\omega(r). \quad \blacksquare \end{aligned}$$

Our next aim is to determine the behavior of the quantity

$$(6.5) \quad H(X, \delta_0) = \int_{B_{2\delta_0}(X)} \frac{\omega(d(Y))}{d(Y)} |X - Y|^{1-m} dY$$

as the point $X \notin K$ approaches K , while $\delta_0 = d(X)$. Note that in (6.5), the ball where the integration is performed, although having center X outside K , still may contain a certain portion of K . Therefore, for $Y \in K$, in other words, when $d(Y) = 0$, we set formally $\frac{\omega(d(Y))}{d(Y)} = 0$, so that the integral in (6.5) makes sense.

LEMMA 6.2: For $X \notin K$,

$$(6.6) \quad H(X, \delta_0) \leq c\omega(\delta_0).$$

Proof. We consider two domains in the ball $B_{2\delta_0}(X)$, where separate estimates will be performed.

If Y belongs to the ball $B_{\delta_0/2}(X)$, we have $d(Y) \geq \frac{1}{2}\delta_0$. Therefore, for the integral $I^{(1)}$ over this ball, the estimate holds

$$I^{(1)} \leq \frac{\omega(\delta_0/2)}{\delta_0} \int_{B_{\delta_0/2}(X)} |X - Y|^{1-m} dY \leq c\omega(\delta_0/2) \leq c\omega(\delta_0).$$

For the integral $I^{(2)}$ over the remaining part of the ball $B_{2\delta_0}(X)$, the points X, Y are controllably separated, $|X - Y| \geq \frac{1}{2}\delta_0$. Let $X_0 \in K$ be a point in K such that $|X - X_0| = \delta_0$. Then the ball $B_{2\delta_0}(X)$ is contained in the ball $B_{4\delta_0}(X_0)$, and therefore, by Lemma 6.1,

$$\begin{aligned} I^{(2)} &= \int_{B_{2\delta_0}(X) \setminus B_{\delta_0/2}(X)} \frac{\omega(d(Y))}{d(Y)} |X - Y|^{1-m} dY \\ &\leq c\delta_0^{1-m} \int_{B_{4\delta_0}(X_0)} \frac{\omega(\delta(Y))}{d(Y)} dY \\ &\leq c\delta_0^{1-m} \delta_0^{m-1} \omega(2\delta_0) \leq c\omega(\delta_0). \end{aligned}$$

We sum these estimates for $I^{(1)}$ and $I^{(2)}$ to obtain (6.6). $\blacksquare$

6.2. INTEGRAL REPRESENTATION ON K . We define the function f_0^* on the compact K in the following way: for a point $X_0 \in K$, we set

$$(6.7) \quad f_0^*(X_0) = \mathbf{a}[f_0(\cdot), G(X_0, \cdot)] \equiv \int_{\Omega} \langle \nabla f_0(Y), \mathbf{a}(Y) \nabla_Y G(X_0, Y) \rangle dY.$$

LEMMA 6.3: Let $X \notin K$, $X_0 \in K$, $|X - X_0| = d(X) = \delta$. Then

$$(6.8) \quad |f_0^*(X_0) - f_0(X)| \leq c\omega(\delta).$$

Similarly, for $X_0, X_1 \in K$, $|X_0 - X_1| = \delta$,

$$(6.9) \quad |f_0^*(X_0) - f_0^*(X_1)| \leq c\omega(\delta).$$

Proof. We present the proof of (6.8). The reasoning establishing (6.9) is quite similar.

Since the function f_0 is smooth outside K , at the points $X \notin K$ the integral representation (6.1) is valid. The aim of this lemma consists in establishing that this integral representation is valid for all points in K as well. We set $r_n = 2^{n+1}\delta$, $n \geq 1$, and, further, we have

$$\begin{aligned} & f_0^*(X_0) - f_0(X) \\ &= \int_{\Omega} \langle \mathbf{a}(Y) \nabla f_0(Y), \nabla_Y G(X_0, Y) \rangle dY \\ &\quad - \int_{\Omega} \langle \mathbf{a}(Y) \nabla f_0(Y), \nabla_Y G(X, Y) \rangle dY \\ &= \int_{B_{r_0}(X_0)} \langle \mathbf{a}(Y) \nabla f_0(Y), \nabla_Y (G(X_0, Y)) \rangle dY \\ (6.10) \quad &\quad - \int_{B_{r_0}(X_0)} \langle \mathbf{a}(Y) \nabla f_0(Y), \nabla_Y G(X, Y) \rangle dY \\ &\quad + \sum_{n=1}^{\infty} \int_{B_{r_n}(X_0) \setminus B_{r_{n-1}}(X_0)} \langle \mathbf{a}(Y) \nabla f_0(Y), \nabla_Y (G(X_0, Y) - G(X, Y)) \rangle dY \\ &\equiv (I_0(X_0) - I_0(X)) + \sum_{n=1}^{\infty} (I_n(X_0) - I_n(X)). \end{aligned}$$

We recall the estimates (2.12) for the Green function $G(X, Y)$:

$$|\nabla_Y G(X, Y)| \leq c|X - Y|^{1-m}, \quad |\nabla_Y G(X_0, Y)| \leq c|X_0 - Y|^{1-m}.$$

Therefore, for the first term on the last line in (6.10), we have, by Lemma 6.1 and Lemma 6.2,

$$\begin{aligned}
 I_0(X_0) &\leq c \int_{B_{r_0}(X_0)} |Y - X_0|^{1-m} |\nabla f_0(Y)| dY \\
 &\leq c \int_{B_{r_0}(X_0)} |Y - X_0|^{1-m} \frac{\omega(d(Y))}{d(Y)} dY \\
 &\leq c\omega(r_0) \leq c\omega(\delta),
 \end{aligned}
 \tag{6.11}$$

and, similarly,

$$\begin{aligned}
 I_0(X) &\leq c \int_{B_{r_0}(X_0)} |Y - X|^{1-m} |\nabla f_0(Y)| dY \\
 &\leq c \int_{B_{2r_0}(X)} |Y - X|^{1-m} \frac{\omega(d(Y))}{d(Y)} dY \leq c\omega(\delta).
 \end{aligned}$$

To handle the sum on the last line in (6.10), we use the estimate (2.13) for the Green function, which implies

$$|\nabla_Y G(X_0, Y) - \nabla_Y G(X, Y)| \leq c|X_0 - X| \max_{\tau \in [0,1]} |Y - (\tau X + (1 - \tau)X_0)|^{-m}.$$

For $Y \notin B_{r_n}(X_0)$, $n > 1$, we have $|Y - X_0| \geq 2\delta = 2|X - X_0|$, therefore,

$$\begin{aligned}
 |Y - (\tau X + (1 - \tau)X_0)| &\geq |Y - X_0| - (1 - \tau)|X - X_0| \geq |Y - X_0| - \delta \\
 &\geq \frac{1}{2}|Y - X_0|,
 \end{aligned}$$

which implies

$$|\nabla_Y G(X_0, Y) - \nabla_Y G(X, Y)| \leq c\delta|Y - X_0|^{-m}.$$

As a result,

$$\begin{aligned}
 |I_n(X_0) - I_n(X)| &\leq c\delta \int_{B_{r_n}(X_0) \setminus B_{r_{n-1}}(X_0)} |Y - X_0|^{-m} \frac{\omega(d(Y))}{d(Y)} dY \\
 &\leq c\delta(2^n\delta)^{-m} \frac{\omega(2^n\delta)}{2^n\delta} \int_{B_{r_n}} dY \\
 &= c\delta(2^n\delta)^{-m} \frac{\omega(2^n\delta)}{2^n\delta} \times (2^n\delta)^m \\
 &= c2^{-n}\omega(2^n\delta).
 \end{aligned}$$

Therefore, for the sum in (6.10), we have

$$\begin{aligned}
 (6.12) \quad \sum_1^\infty (I_n(X_0) - I_n(X)) &\leq c \sum_1^\infty 2^{-n} \omega(2^n \delta) \leq c \int_0^\infty \frac{\omega(2^t \delta)}{2^t} dt \\
 &= c \int_1^\infty \frac{\omega(\delta \varsigma)}{\varsigma^2} d\varsigma = c\delta \int_\delta^\infty \frac{\omega(v)}{v^2} dv \leq c\omega(\delta),
 \end{aligned}$$

by (3.1).

The substitution of (6.11) and (6.12) into (6.10) gives (6.8) and thus proves the Lemma. ■

LEMMA 6.4: *Let X_0 be a point in K . Let $f_0^*(X_0)$ be constructed as in (6.7). Then*

$$f_0^*(X_0) = f(X_0).$$

Proof. We take an arbitrarily small $\delta > 0$ and find a point $X \notin K$ such that $\frac{\delta}{2} \leq |X - X_0| \leq \delta$ (such a point, of course, exists since K possesses no interior points). Let, further, $X_1 \in K$ be (one of) the closest point(s) in K to X and we set $d(X) = |X_1 - X|$. Then $d(X) \leq \delta$ and

$$|X_0 - X_1| \leq |X_0 - X| + |X_1 - X| \leq 2\delta.$$

By Theorem 6.3,

$$|f_0^*(X_0) - f_0(X)| \leq |f_0^*(X_0) - f_0^*(X_1)| + |f_0^*(X_1) - f_0(X)| \leq c\omega(\delta).$$

We recall now that the function f_0 satisfies $|f(X_0) - f_0(X)| \leq c\omega(\delta)$, according to the way f_0 was constructed. Therefore,

$$|f_0^*(X_0) - f(X_0)| \leq |f_0^*(X_0) - f_0(X)| + |f_0(X) - f(X_0)| \leq c\omega(\delta).$$

Since $\delta > 0$ was chosen arbitrarily, it follows that $f_0^*(X_0) - f(X_0) = 0$. ■

Thus, we have established that the integral representation (6.3) is valid not only for $X \notin K$ but for all $X \in K$ as well, this means on the whole domain Ω .

7. Construction of the approximating function

7.1. FORMULATION OF THE MAIN THEOREM. We are ready now to prove the approximation theorem.

THEOREM 7.1: *Let $K \subset \mathbb{R}^m$, $m \geq 3$, be a compact set satisfying Condition **A**. Let $\mathcal{L}$ be an elliptic operator in the neighborhood of K , satisfying conditions of Section 2.5.2. Suppose that the function $f(X)$, $X \in K$, belongs to $C^\omega(K)$, where ω is a continuity modulus as above. Then for some constants $c_1, c_2, c_3 > 0$, for any $\delta > 0$ there exists a function $v_\delta \in H^1(\Omega)$, which is a weak solution of the equation $\mathcal{L}v_\delta = 0$ in the δ -neighborhood K_δ of K , satisfies the estimate $|\nabla v_\delta| \leq c_2 \frac{\omega(\delta)}{\delta}$ in K_δ , and it approximates f on K :*

$$|f(X) - v_\delta(X)| \leq c_3 \omega(\delta), \quad X \in K.$$

It stands to reason that if the set K is θ -regular for $\theta = m-2$, the Condition **A** is satisfied, therefore, the inference of Theorem 7.1 is valid.

7.2. THE APPROXIMATION. Having δ fixed, we introduce the function

$$(7.1) \quad v_\delta(X) = \int_{\Omega \setminus \mathcal{B}(\delta)} \langle \mathbf{a}(Y) \nabla f_0(Y), \nabla_Y G(X, Y) \rangle dY.$$

The property of main importance for our studies is the following.

LEMMA 7.2: *The function v_δ is a weak solution of the equation $\mathcal{L}v_\delta = 0$ in $\mathcal{B}(\delta)$.*

Proof. By the definition of a weak solution, see (2.8) with $\mu = 0$ in $\mathcal{B}(\delta)$, it suffices to show that for any function $\varpi \in C_0^\infty(\mathcal{B}(\delta))$,

$$\mathbf{a}[v_\delta, \varpi] = 0,$$

where $\mathbf{a}$ is the sesquilinear form (2.7). So, we take an arbitrary function $\varpi \in C_0^\infty(\mathcal{B}(\delta))$ and obtain

$$\begin{aligned} & \mathbf{a}[v_\delta, \varpi] \\ &= \int_{\mathcal{B}(\delta)} \langle \mathbf{a}(X) \nabla_X v_\delta(X), \nabla \varpi(X) \rangle dX \\ (7.2) \quad &= \int_{\mathcal{B}(\delta)} \langle \mathbf{a}(X) \nabla_X \left(\int_{\Omega \setminus \mathcal{B}(\delta)} \langle \mathbf{a}(Y) \nabla_Y f_0(Y), \nabla_Y G(X, Y) \rangle dY \right), \nabla \varpi(X) \rangle dX \\ &= \int_{\mathcal{B}(\delta)} \langle \mathbf{a}(X) \left(\int_{\Omega \setminus \mathcal{B}(\delta)} \langle \mathbf{a}(Y) \nabla_Y f_0(Y), \nabla_Y \nabla_X G(X, Y) \rangle dY \right), \nabla \varpi(X) \rangle dX. \end{aligned}$$

We change the order of integration in (7.2):

$$\mathbf{a}[v_\delta, \varpi] = \int_{\Omega \setminus \mathcal{B}(\delta)} \langle \mathbf{a}(Y) \nabla f_0(Y), \nabla_Y \mathbf{a}[G(\cdot, Y), \varpi(\cdot)] \rangle dY.$$

Due to the main property of the Green function (Theorem 1.1 in [13], see (2.9)), we have

$$\mathbf{a}[G(\cdot, Y), \varpi(\cdot)] = \varpi(Y).$$

Therefore,

$$\mathbf{a}[v_\delta, \varpi] = \int_{\Omega \setminus \mathcal{B}(\delta)} \langle \mathbf{a}(Y) \nabla f_0(Y), \nabla \varpi(Y) \rangle dY,$$

and the last expression equals zero since $\varpi(Y) = 0$ on $\Omega \setminus \mathcal{B}(\delta)$. ■

7.3. THE APPROXIMATION ESTIMATE. Next, for $X_0 \in K$, it follows from (7.1) that

$$\begin{aligned} f(X_0) - v_\delta(X_0) &= f_0^*(X_0) - v_\delta(X_0) \\ (7.3) \qquad &= \int_{\mathcal{B}(\delta)} \langle \mathbf{a}(Y) \nabla f_0(Y), \nabla_Y G(X_0, Y) \rangle dY. \end{aligned}$$

To estimate the quantity in (7.3), we set again $r_n = 2^n \delta$, $n = 0, 1, \dots$, denote $\mathcal{C}_n = \mathcal{B}(\delta) \cap (B_{r_n}(X_0) \setminus B_{r_{n-1}}(X_0))$, and consider the decomposition

$$\begin{aligned} &\int_{\mathcal{B}(\delta)} \langle \mathbf{a}(Y) \nabla f_0(Y), \nabla_Y G(X_0, Y) \rangle dY \\ (7.4) \qquad &= \int_{\mathcal{B}(\delta) \cap B_{r_0}(X_0)} \langle \mathbf{a}(Y) \nabla f_0(Y), \nabla_Y G(X_0, Y) \rangle dY \\ &\quad + \sum_{n=1}^{\infty} \int_{\mathcal{C}_n} \langle \mathbf{a}(Y) \nabla f_0(Y), \nabla_Y G(X_0, Y) \rangle dY. \end{aligned}$$

We estimate separately the terms in (7.4). First, according to estimate (2.12) for the derivative of the Green function,

$$\begin{aligned} &\left| \int_{\mathcal{B}(\delta) \cap B_{r_0}(X_0)} \langle \mathbf{a}(Y) \nabla f_0(Y), \nabla_Y G(X_0, Y) \rangle dY \right| \\ (7.5) \qquad &\leq \int_{B_{r_0}(X_0)} |Y - X_0|^{1-m} \frac{\omega(d(Y))}{d(Y)} dY \leq c\omega(\delta), \end{aligned}$$

by Lemma 6.1. Further, we have

$$\begin{aligned} &\left| \int_{\mathcal{C}_n} \langle \mathbf{a}(Y) \nabla f_0(Y), \nabla_Y G(X, Y) \rangle dY \right| \leq cr_{n-1}^{1-m} \int_{\mathcal{C}_n} \frac{\omega(\delta(Y))}{\delta(Y)} dY \\ (7.6) \qquad &\leq cr_{n-1}^{1-m} \int_{\mathcal{B}(\delta) \cap B_{r_n}(X_0)} \frac{\omega(d(Y))}{d(Y)} dY \\ &\leq cr_n^{1-m} r_n^{m-2} \delta \omega(\delta) \\ &= cr_n^{-1} \delta \omega(\delta) = c2^{-n} \omega(\delta), \end{aligned}$$

the last inequality following from (5.9). As a result, the sum on the last line in (7.4) is bounded by

$$(7.7) \quad c \sum_{n=0}^{\infty} 2^{-n} \omega(\delta) = c\omega(\delta).$$

We, finally, collect (7.4), (7.5), (7.6), (7.7), which gives us

$$(7.8) \quad |f(X_0) - v_\delta(X_0)| \leq c\omega(\delta).$$

7.4. GRADIENT ESTIMATE. For any points X in the δ -neighborhood K_δ of K and $Y \in \partial\mathcal{B}(\delta)$, we have $|Y - X| \geq (c_1 - 1)\delta$, by the triangle inequality.

We set here $r_n = 3c_1 2^n \delta$, $n = 0, 1, \dots$. For $X \in K_\delta$, we estimate derivatives of the function $v_\delta(X)$ (defined in (7.1)), starting with

$$(7.9) \quad \nabla_X v_\delta(X) = \int_{\Omega \setminus \mathcal{B}(\delta)} \mathfrak{A}[X, Y] dY,$$

where the integrand $\mathfrak{A}[X, Y]$ denotes

$$\mathfrak{A}[X, Y] \equiv \nabla_X \langle \mathfrak{a}(Y) \nabla f_0(Y), \nabla_Y G(X, Y) \rangle = \langle \mathfrak{a}(Y) \nabla_Y f_0(Y), \nabla_X \nabla_Y G(X, Y) \rangle.$$

Therefore,

$$(7.10) \quad |\nabla_X v_\delta(X)| \leq \left| \int_{(\Omega \setminus \mathcal{B}(\delta)) \cap B_{r_0}(X_0)} \mathfrak{A}[X, Y] dY \right| + \sum_{n=1}^{\infty} \left| \int_{\mathcal{S}_n} \mathfrak{A}[X, Y] dY \right|,$$

where $X_0 \in K$ is (one of) the point(s) in K lying at distance δ from X and $\mathcal{S}_n$ is shorthand for

$$\mathcal{S}_n \equiv (\Omega \setminus \mathcal{B}(\delta)) \cap (B_{r_n}(X_0) \setminus B_{r_{n-1}}(X_0)).$$

We recall the estimate (2.13) for the double gradient of the Green function. It implies

$$|\nabla_X \nabla_Y G(X, Y)| \leq cr_0^{-m}, \quad \text{for } Y \in (\Omega \setminus \mathcal{B}(\delta)) \cap B_{r_0}(X_0)$$

and

$$|\nabla_X \nabla_Y G(X, Y)| \leq cr_n^m, \quad \text{for } Y \in \mathcal{S}_n.$$

As a result, the inequalities (7.9), (7.10) and the estimate (4.2) for the gradient of the function f_0 give us

$$\begin{aligned} |\nabla v_\delta(X)| &\leq c \left(r_0^{-m} \int_{B_{r_0}(X)} \frac{\omega(d(Y))}{d(Y)} dY + \sum_{n=1}^{\infty} r_n^{-m} \int_{B_{r_n}(X)} \frac{\omega(d(Y))}{d(Y)} dY \right) \\ &\leq c \left(r_0^{-m} \rho_0^{m-1} \omega(r_0) + \sum_{n=1}^{\infty} r_0^{-m} r_n^{m-1} \omega(r_n) \right) \\ &= c \left(\frac{\omega(r_0)}{r_0} + \sum_{n=1}^{\infty} \frac{\omega(r_n)}{r_n} \right) = \frac{c}{\delta} \left(\omega(3c_1\delta) + \sum_{n=1}^{\infty} \frac{\omega(3 \cdot 2^n \delta)}{3 \cdot 2^n c_1} \right) \leq c \frac{\omega(\delta)}{\delta}, \end{aligned}$$

due to the calculation as in the last line in (6.12).

The results of the last two subsections show that the function v_δ does in fact satisfy all requirements claimed; this concludes the proof of our main theorem.

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