APPROXIMATION OF THE LÉVY-DRIVEN STOCHASTIC HEAT EQUATION ON THE SPHERE

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ABSTRACT. The stochastic heat equation on the sphere driven by additive Lévy random field is approximated by a spectral method in space and forward and backward Euler–Maruyama schemes in time, in analogy to the Wiener case. New regularity results are proven for the stochastic heat equation. The spectral approximation is based on a truncation of the series expansion with respect to the spherical harmonic functions. To do so, we restrict to square-integrable random field and optimal strong convergence rates for a given regularity of the initial condition and two different settings of regularity for the driving noise are derived for the Euler–Maruyama methods. Besides strong convergence, convergence of the expectation and second moment is shown. Weak rates for the spectral approximation are discussed. Numerical simulations confirm the theoretical results.

1. INTRODUCTION

Stochastic partial differential equations (SPDEs) play a central role in modeling systems influenced by random effects across space and time. Traditionally, much of the foundational work in this area has focused on SPDEs, mostly on Euclidean spaces, driven by Wiener noise, leveraging its mathematical tractability and the rich theory developed around Gaussian processes [9, 28, 30]. Numerical methods for such SPDEs have been developed and analyzed for more than thirty years by now, with references for example in the books [18, 26].

However, many real-world phenomena, ranging from turbulent flows on Earth and stellar fluids to biological and financial systems [8, 10, 12, 14, 34], exhibit abrupt, non-Gaussian behavior that cannot be adequately captured by just a Gaussian noise. More so, applications motivate to extend the theory to surfaces and curved spaces. These observations prompt the need to consider more general settings; to our knowledge, literature on surfaces is still lacking with first results in the Gaussian setting on the sphere given in [2, 7, 25, 17, 35, 20, 22, 21].

In this work, we take a step beyond the classical setting by studying SPDEs driven by square-integrable Lévy processes, which naturally can incorporate jumps [11, 13, 30]. We consider the stochastic heat equation on the sphere $\mathbb{S}^2$, on a complete filtered probability

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space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ and a finite time interval $[0,T]$, $T<\infty$, i.e.,

$$\mathrm{d}X(t) = \Delta_{\mathbb{S}^2} X(t)\,\mathrm{d}t + \mathrm{d}L(t), \tag{1}$$

with $\mathcal{F}_0$-measurable initial condition $X(0) = X^0 \in L^2(\Omega, L^2(\mathbb{S}^2))$ driven by an infinite-dimensional Lévy process L, which we will introduce in more detail in Section 2.1.

These processes offer a more realistic [6, 15, 16, 29, 31] and flexible framework for modeling complex stochastic dynamics. The use of Lévy noise introduces a range of analytical and numerical challenges, but also opens new pathways for understanding the behavior of solutions to SPDEs under more general sources of randomness.

Our contributions follow two main directions. First, we establish new regularity results and weak convergence results for spectral approximations of SPDEs, which are new even in the classical Wiener-driven case. These results enhance the theoretical understanding of spectral methods and extend their applicability to broader classes of noise structures. Second, we derive strong convergence estimates for spectral approximations in space and Euler–Maruyama time stepping with non-isotropic noise characterized by its Sobolev regularity, which generalize the results of [22, 20].

The paper is structured as follows: In Section 2, we describe the functional setting and the precise driving noise of (1), showing the properties of the solution to (1) such as the computation of the first and second moments and new regularity results. In Section 3, we prove strong and weak convergence of the spectral approximation where the rates depend on the chosen Lévy noise. In Section 4, the Euler–Maruyama scheme is analyzed, showing optimality of the rate and the convergence to the continuous solution. Finally, in Section 5, numerical simulations confirm the theoretical results for Wiener and Poisson noise. The code that was used to generate the samples, the videos and the convergence rates' plots for the numerical examples is available at [23].

2. Setting

In this section we first introduce the necessary notation, then we define the Lévy noise and derive the solution to (1). We denote by $\mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$, i.e., $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$, where $\|\cdot\|$ denotes the Euclidean norm. We equip $\mathbb{S}^2$ with the geodesic metric given for all $x, y \in \mathbb{R}^3$ by $d(x,y) = \arccos\langle x, y\rangle_{\mathbb{R}^3}$ and couple the Cartesian coordinates $y \in \mathbb{S}^2$ to the polar coordinates $(\vartheta, \varphi) \in [0,\pi] \times [0, 2\pi)$ by the transformation $y = (\sin(\vartheta)\cos(\varphi), \sin(\vartheta)\sin(\varphi), \cos(\vartheta))$. Let σ be the Lebesgue measure on the sphere, which has the representation

$$\mathrm{d}\sigma(y) = \sin(\vartheta)\,\mathrm{d}\vartheta\,\mathrm{d}\varphi$$

and denote by $\mathcal{B}(\mathbb{S}^2)$ the Borel σ-algebra of $\mathbb{S}^2$. Note that $(\mathbb{S}^2, \mathcal{B}(\mathbb{S}^2), \sigma)$ is a measure space and $L^2(\mathbb{S}^2)$ is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_{L^2(\mathbb{S}^2)}$, which is defined for all $f, g \in L^2(\mathbb{S}^2)$ by

$$\langle f, g\rangle_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} \langle f(y), g(y)\rangle_{\mathbb{S}^2}\,\mathrm{d}\sigma(y).$$

We represent the real-valued spherical harmonic functions by $\widetilde{\mathcal{Y}} = (\widetilde{Y}_{\ell,m})_{\ell\in\mathbb{N}_0, m\in\{-\ell,\ldots,\ell\}}$, which consist of $\widetilde{Y}_{\ell,m}\colon [0,\pi]\times[0,2\pi)\to\mathbb{R}$ given by

$$\widetilde{Y}_{\ell,m}(\vartheta,\varphi) = \begin{cases} (|m|\varphi), & m<0 \\ \sqrt{\frac{2\ell+1}{4\pi}}P_{\ell,m}(\cos(\vartheta)), & m=0 \\ \sqrt{2}(-1)^m\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_{\ell,m}(\cos(\vartheta))\sin(m\varphi), & m>0 \end{cases}.$$

Inhere $(P_{\ell,m})_{\ell\in\mathbb{N}_0, m\in\{0,\ldots,\ell\}}$ are the associated Legendre polynomials. The real-valued spherical harmonics form an orthonormal basis of $L^2(\mathbb{S}^2)$.

The spherical Laplacian or Laplace–Beltrami operator is defined by

$$\Delta_{\mathbb{S}^2} = (\sin(\vartheta))^{-1}\frac{\partial}{\partial\vartheta}\Big(\sin(\vartheta)\frac{\partial}{\partial\vartheta}\Big) + (\sin(\vartheta))^{-2}\frac{\partial^2}{\partial\varphi^2},$$

see [5, Chapter 6]. For all $\ell\in\mathbb{N}_0$, $m\in\{-\ell,\ldots,\ell\}$ we have

$$\Delta_{\mathbb{S}^2}\widetilde{Y}_{\ell,m} = -\ell(\ell+1)\widetilde{Y}_{\ell,m}. \tag{2}$$

This is a consequence of the fact that the real-valued spherical harmonics are a linear combination of the complex-valued spherical harmonics.

Next, we define the Sobolev space $H^\eta(\mathbb{S}^2)$ on the unit sphere with smoothness index $\eta\in\mathbb{R}$. For a positive index $\eta\in\mathbb{R}_+$, we use Bessel potentials defining the space by

$$H^\eta(\mathbb{S}^2) = (\mathrm{Id}-\Delta_{\mathbb{S}^2})^{-\eta/2}L^2(\mathbb{S}^2).$$

Note that $x\in L^2(\mathbb{S}^2)$ can be rewritten as $x=\sum_{\ell=0}^\infty\sum_{m=-\ell}^{\ell} x_{\ell,m}\widetilde{Y}_{\ell,m}$ with real-valued coefficients $x_{\ell,m}$. Hence, for $x\in H^\eta(\mathbb{S}^2)$,

$$(\mathrm{Id}-\Delta_{\mathbb{S}^2})^{\eta/2}x = \sum_{\ell=0}^\infty\sum_{m=-\ell}^{\ell}(1+\ell(\ell+1))^{\eta/2}x_{\ell,m}\widetilde{Y}_{\ell,m}\in L^2(\mathbb{S}^2).$$

The inner product in $H^\eta(\mathbb{S}^2)$ is given for all $f,g\in H^\eta(\mathbb{S}^2)$ by

$$\langle f,g\rangle_{H^\eta(\mathbb{S}^2)} = \langle (I-\Delta_{\mathbb{S}^2})^{\eta/2}f, (I-\Delta_{\mathbb{S}^2})^{\eta/2}g\rangle_{L^2(\mathbb{S}^2)}.$$

For $\eta\in\mathbb{R}_-$, we define the negative Sobolev space as the space of distributions generated by

$$H^\eta(\mathbb{S}^2) = \left\{u=(I-\Delta_{\mathbb{S}^2})^\kappa v,\ v\in H^{2\kappa+\eta}(\mathbb{S}^2)\right\},$$

where $\kappa\in\mathbb{N}$ is the smallest integer such that $2\kappa+\eta>0$. In this case, we define the norm to be

$$\|u\|_{H^\eta(\mathbb{S}^2)} = \|v\|_{H^{2\kappa+\eta}(\mathbb{S}^2)}.$$

Finally, we set $H^0(\mathbb{S}^2) = L^2(\mathbb{S}^2)$. As the subsequent sections involve the (formal) series expansion of the noise in terms of real-valued spherical harmonic functions with coefficients in $L^2(\Omega,\mathbb{R})$, we interpret the previously defined Hilbert spaces within the framework of a Gelfand triple, i.e. we consider for $H=L^2(\mathbb{S}^2)$ and $\eta\geq 0$ the following embeddings

$$V=H^\eta\hookrightarrow H\simeq H^*\hookrightarrow V^*\simeq H^{-\eta}.$$

For useful details on the introduced spaces and the Bessel potentials, we refer, for instance, to [33]. In this article we use the Lebesgue–Bochner spaces $L^p(\Omega;H^\eta(\mathbb{S}^2))$ for $p\geq 1,\eta\in\mathbb{R}$ with the norm

$$\|X\|_{L^p(\Omega;H^\eta(\mathbb{S}^2))} := \mathbb{E}[\|X\|^p_{H^\eta(\mathbb{S}^2)}]^{1/p},$$

similar to [22]. In the following, we use a generic constant $C \in (0, \infty)$ to simplify notations. This constant is allowed to change from line to line. In the convergence rate proofs, it is always independent of the investigated quantities. The last thing to introduce from (1) before being able to solve it is the driving noise.

2.1. **The driving Lévy process.** Let us introduce the driving Lévy process $L = (L(t))_{t\in[0,\infty)}$ with its properties in what follows. In the framework of [30, Definition 4.1], we assume that L belongs to the class of square-integrable Lévy processes with values in $H^\eta(\mathbb{S}^2)$, $\eta \in (-1, \infty)$, i.e., $L(t) \in L^2(\Omega, H^\eta(\mathbb{S}^2))$. More specifically, L has stationary and independent increments, is stochastically continuous, and satisfies that $L(0) = 0$. By [30, Theorem 4.3], L has a càdlàg modification and by [30, Theorem 4.44], the mean $\mathbb{E}[L(t)] = t\widetilde{m} \in H^\eta(\mathbb{S}^2)$ and covariance operator $Q \in L_1^+(H^\eta(\mathbb{S}^2))$ exist, where $L_1^+(H^\eta(\mathbb{S}^2))$ denotes the space of all linear, trace class, symmetric, positive semi definite operators from $H^\eta(\mathbb{S}^2)$ into itself.

Since $H^\eta(\mathbb{S}^2) \subset L^2(\mathbb{S}^2)$ for $\eta \geq 0$, we can expand the process in the $L^2(\mathbb{S}^2)$ orthonormal basis of spherical harmonics $\widetilde{\mathcal{Y}}$ to obtain

$$L(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} L_{\ell,m}(t)\widetilde{Y}_{\ell,m} \tag{3}$$

with real-valued càdlàg Lévy processes $L_{\ell,m} = \langle L, \widetilde{Y}_{\ell,m}\rangle_{L^2(\mathbb{S}^2)}$. This converges in $L^2(\Omega, L^2(\mathbb{S}^2))$ with

$$t \operatorname{tr} Q = \mathbb{E}[\|L(t) - \mathbb{E}[L(t)]\|^2_{L^2(\mathbb{S}^2)}] = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathbb{E}[(L_{\ell,m}(t) - \mathbb{E}[L_{\ell,m}(t)])^2] = t \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} v_{\ell,m},$$

see [30, Section 4.8] for details. We set

$$\tilde{v} := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{v_{\ell,m}}\widetilde{Y}_{\ell,m},$$

which is in $H^\eta(\mathbb{S}^2)$ using the above definition of $\operatorname{tr} Q$. Furthermore, the mean is given by

$$t\,\widetilde{m} = \mathbb{E}[L(t)] = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathbb{E}[L_{\ell,m}(t)]\widetilde{Y}_{\ell,m} = t \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \widetilde{m}_{\ell,m}\widetilde{Y}_{\ell,m}$$

For $\eta \in [-1, 0)$, we assume the formal series expansion with respect to $\widetilde{\mathcal{Y}}$, which converges in $L^2(\Omega, H^\eta(S^2))$ and Q is not of trace class in $L^2(\mathbb{S}^2)$ but just in the larger Sobolev space. We gather all considerations in the following assumption:

Assumption 2.1. The Lévy process L is in $L^2(\Omega, H^\eta(\mathbb{S}^2))$ for some $\eta \in [-1, \infty)$ with series expansion (3) converging in $L^2(\Omega, H^\eta(\mathbb{S}^2))$, i.e.,

$$\mathbb{E}[\|L(t)\|^2_{H^\eta(\mathbb{S}^2)}] = t \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (1 + \ell(\ell+1))^\eta (v_{\ell,m} + \widetilde{m}^2_{\ell,m}) = t(\|\tilde{v}\|^2_{H^\eta(\mathbb{S}^2)} + \|\widetilde{m}\|^2_{H^\eta(\mathbb{S}^2)}) < \infty.$$

To connect our results to the isotropic setting considered in [20, 22], we take the following more specific assumption, with angular power spectrum $A_\ell = a_\ell^2$ in the earlier works.

Assumption 2.2. The Lévy process L given by the expansion (3) satisfies for some $\alpha > 0$

$$L_{\ell,m} = a_\ell \hat{L}_{\ell,m},$$

where $a_\ell \leq C\ell^{-\alpha/2}$ for all $\ell \in \mathbb{N}$, a_0 bounded, and $(\hat{L}_{\ell,m})_{\ell\in\mathbb{N}_0, m\in\{-\ell,\ldots,\ell\}}$ is a sequence of identically distributed real-valued Lévy processes with mean $\mathbb{E}[\hat{L}_{\ell,m}(t)] = \hat{m}$ and variance $\mathbb{E}[(\hat{L}_{\ell,m}(t) - \mathbb{E}[\hat{L}_{\ell,m}(t)])^2] = \hat{v}$.

We observe that Assumption 2.2 implies that $L \in L^2(\Omega, H^\eta(\mathbb{S}^2))$ for all $\eta < \alpha/2 - 1$, since

$$\mathbb{E}[\|L(t)\|^2_{H^\eta(\mathbb{S}^2)}] \leq Ct \sum_{\ell=1}^{\infty} (1 + \ell(\ell+1))^\eta \ell^{-\alpha}(1 + 2\ell) < \infty$$

if $2\eta - \alpha + 1 < -1$. Note that in these frameworks, we are not explicitly using the decomposition of the Lévy process L with the eigenfunctions of the covariance operator Q, hence we do not have any information on the independence or uncorrelation of the real–valued processes $L_{\ell,m}$ in contrast with [30, Section 4.8.2].

For the weak convergence analysis, one important setting we consider, is when L admits the Lévy–Khinchin decomposition [30, Theorem 4.23]

$$L(t) = W(t) + \int_\chi \xi(t,\zeta)\, \tilde{N}(t, \mathrm{d}\zeta), \tag{4}$$

where W is a Q-Wiener process, and $\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta) = N(\mathrm{d}t, \mathrm{d}\zeta) - \nu(\mathrm{d}\zeta)\,\mathrm{d}t$ is a compensated Poisson random measure. The function $\xi : [0,T] \times \chi \to L^2(\mathbb{S}^2)$ is the intensity and ν the Lévy measure associated with L [30, Definition 4.14] satisfying with $\chi = L^2(\mathbb{S}^2) \setminus \{0\}$

$$\int_\chi \min(1, \|\zeta\|^2_{L^2(\mathbb{S}^2)})\, \nu(\mathrm{d}\zeta) < \infty.$$

Under this decomposition, L is a square-integrable martingale with zero mean [30, Theorem 4.49].

In this setting, convergence rates will depend on $L^p(\Omega)$ bounds, which require the following assumptions:

Assumption 2.3. Let $\eta \in (-1, \infty)$ and $p > 2$. Assume that $L \in L^p(\Omega, H^\eta(\mathbb{S}^2))$ is a martingale with decomposition (4) satisfying Assumption 2.1 and

$$\sup_{s\leq T} \left(\int_\chi \|\xi(s,\zeta)\|^p_{H^\eta(\mathbb{S}^2)} \nu(\mathrm{d}\zeta) + \left(\int_\chi \|\xi(s,\zeta)\|^2_{H^\eta(\mathbb{S}^2)} \nu(\mathrm{d}\zeta) \right)^{p/2} \right) < +\infty.$$

2.2. **Solution of the SHE.** In the next step, we solve the stochastic heat equation (1). For this, we use the spherical harmonic functions $\widetilde{\mathcal{Y}}$ as ansatz, set $X_{\ell,m}(t) = \langle X(t), \widetilde{Y}_{\ell,m} \rangle_{L^2(\mathbb{S}^2)}$ for all $\ell \in \mathbb{N}_0$, $m \in \{-\ell, \ldots, \ell\}$, and obtain the expansion

$$\sum_{\ell=0}^\infty \sum_{m=-\ell}^{\ell} X_{\ell,m}(t)\widetilde{Y}_{\ell,m} = \sum_{\ell=0}^\infty \sum_{m=-\ell}^{\ell} \left(X_{\ell,m}(0)\widetilde{Y}_{\ell,m} + \int_0^t X_{\ell,m}(s)\Delta_{\mathbb{S}^2}\widetilde{Y}_{\ell,m}\,\mathrm{d}s + L_{\ell,m}(t)\widetilde{Y}_{\ell,m} \right).$$

Since the spherical harmonics are eigenfunctions of the spherical Laplacian, see (2), we get

$$\sum_{\ell=0}^\infty \sum_{m=-\ell}^{\ell} X_{\ell,m}(t)\widetilde{Y}_{\ell,m} = \sum_{\ell=0}^\infty \sum_{m=-\ell}^{\ell} \left(X_{\ell,m}(0) - \ell(\ell+1)\int_0^t X_{\ell,m}(s)\,\mathrm{d}s + L_{\ell,m}(t) \right)\widetilde{Y}_{\ell,m}.$$

Hence the solution of (1) is given by the solutions $(X_{\ell,m})_{\ell\in\mathbb{N}_0,m\in\{-\ell,\dots,\ell\}}$ to the stochastic differential equations

$$X_{\ell,m}(t) = X_{\ell,m}(0) - \ell(\ell+1)\int_0^t X_{\ell,m}(s)\,\mathrm{d}s + L_{\ell,m}(t). \tag{5}$$

By the variation of constants formula, we obtain the solutions

$$X_{\ell,m}(t) = e^{-\ell(\ell+1)t}X_{\ell,m}(0) + \int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}L_{\ell,m}(s),$$

which yields the solution to (1)

$$X(t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Big(e^{-\ell(\ell+1)t}X_{\ell,m}(0) + \int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}L_{\ell,m}(s)\Big)\widetilde{Y}_{\ell,m}. \tag{6}$$

The proof of existence, uniqueness and L^2-regularity of the solution is classical and the reader is directed to [30, 3, 1, 13].

In the following lemma we compute the moments of the solution to the stochastic heat equation.

Lemma 2.4. *Under Assumption 2.1, the stochastic heat equation* (1) *satisfies that*

$$\mathbb{E}[X(t)] = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Big(e^{-\ell(\ell+1)t}\mathbb{E}[X^0_{\ell,m}] + \int_0^t e^{-\ell(\ell+1)(t-s)}\widetilde{m}_{\ell,m}\,\mathrm{d}s\Big)\widetilde{Y}_{\ell,m}$$

and

$$\begin{aligned}&\mathbb{E}\big[\|X(t)\|^2_{L^2(\mathbb{S}^2)}\big]\\ &\quad= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\mathbb{E}\Big[\Big(X^0_{\ell,m}e^{-\ell(\ell+1)t} + \widetilde{m}_{\ell,m}\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}s\Big)^2\Big] + v_{\ell,m}\int_0^t e^{-2\ell(\ell+1)(t-s)}\,\mathrm{d}s.\end{aligned}$$

Proof. For the expectation we obtain

$$\mathbb{E}[X(t)] = \mathbb{E}[X^0] + \int_0^t \Delta_{\mathbb{S}^2}\mathbb{E}[X(s)]\,\mathrm{d}s + \mathbb{E}[L(t)].$$

Since $\mathbb{E}[L(t)] = \widetilde{m}\,t$ by [30, Theorem 4.44] and using $u(t) = \mathbb{E}[X(t)]$ and $u_0 = \mathbb{E}[X^0]$, we need to solve the partial differential equation

$$\partial_t u = \Delta_{\mathbb{S}^2}u + \widetilde{m}, \quad u(0) = u_0.$$

We solve this equation similar to [20, Appendix A], where the homogeneous equation is considered and obtain that

$$\mathbb{E}[X(t)] = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Big(e^{-\ell(\ell+1)t}\mathbb{E}[\langle X(0),\widetilde{Y}_{\ell,m}\rangle_{L^2(\mathbb{S}^2)}] + \int_0^t e^{-\ell(\ell+1)(t-s)}\widetilde{m}_{\ell,m}\,\mathrm{d}s\Big)\widetilde{Y}_{\ell,m}$$

For the second moment of the solution, we calculate using (6),

$$\mathbb{E}[\|X(t)\|^2_{L^2(\mathbb{S}^2)}] = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Big(e^{-2\ell(\ell+1)t}\mathbb{E}[|X^0_{\ell,m}|^2] + \mathbb{E}\Big[\Big|\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}L_{\ell,m}(s)\Big|^2\Big]\Big)$$
$$+ 2\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t}\,\mathbb{E}[X^0_{\ell,m}]\,\mathbb{E}\Big[\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}L_{\ell,m}(s)\Big],$$

where we used that $\widetilde{\mathcal{Y}}$ is an orthonormal basis of $L^2(\mathbb{S}^2)$ and X^0 is independent of L.

By [4, Example 15.12] we obtain for the second term that

(7)
$$\mathbb{E}\Big[\Big|\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}L_{\ell,m}(s)\Big|^2\Big] = v_{\ell,m}\int_0^t e^{-2\ell(\ell+1)(t-s)}\,\mathrm{d}s + \Big(\widetilde{m}_{\ell,m}\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}s\Big)^2;$$
$$\mathbb{E}\Big[\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}L_{\ell,m}(s)\Big] = \widetilde{m}_{\ell,m}\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}s.$$

So in conclusion, we obtain

$$\mathbb{E}\big[\|X(t)\|^2_{L^2(\mathbb{S}^2)}\big]$$
$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\mathbb{E}\Big[\Big(X^0_{\ell,m}e^{-\ell(\ell+1)t} + \widetilde{m}_{\ell,m}\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}s\Big)^2\Big] + v_{\ell,m}\int_0^t e^{-2\ell(\ell+1)(t-s)}\,\mathrm{d}s$$

which finishes the proof. □

In what follows, we establish higher-order regularity of the solution to (1), a key ingredient in deriving weak convergence rates for a broad class of test functions in the next section.

Proposition 2.5. *Let $\rho \in \mathbb{R}$ and let X be the mild solution given in (6) with initial value $X^0 \in L^p(\Omega, H^\nu(\mathbb{S}^2))$. If one of the following assumptions holds:*

(i) *$L = W$ is a Q-Wiener process with $\rho - \eta \le 1$;*
(ii) *$p \ge 2$ and $L \in L^p(\Omega, H^\eta(\mathbb{S}^2))$ satisfies Assumption 2.1, where the processes $(L_{\ell,m})_{\ell,m}$ are assumed to be independent if $p > 2$, and $\rho - \eta \le 1$;*
(iii) *$p > 2$ and L satisfies Assumption 2.3 with $\rho - \eta < 2/p$,*

then

$$\|X(t)\|_{L^p(\Omega,H^\rho(\mathbb{S}^2))} \le C(1 + t^{-\max\{\rho-\nu,0\}/2}\|X^0\|_{L^p(\Omega,H^\nu(\mathbb{S}^2))}) < \infty.$$

Proof. We consider the mild solution X in (6) and denote the stochastic convolution by

$$\tilde{L}(t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}L_{\ell,m}(s)\,\widetilde{Y}_{\ell,m}.$$

Using the triangle inequality, we split the norm

$$\|X(t)\|_{L^p(\Omega,H^\rho(\mathbb{S}^2))} \le \Big\|\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t}X_{\ell,m}(0)\widetilde{Y}_{\ell,m}\Big\|_{L^p(\Omega,H^\rho(\mathbb{S}^2))} + \|\tilde{L}(t)\|_{L^p(\Omega,H^\rho(\mathbb{S}^2))}.$$

The first term satisfies that

$$\Big\|\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t}X_{\ell,m}(0)\widetilde{Y}_{\ell,m}\Big\|^p_{L^p(\Omega,H^\rho(\mathbb{S}^2))} = \mathbb{E}\Big[\Big(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(1+\ell(\ell+1))^\rho e^{-2\ell(\ell+1)t}(X_{\ell,m}(0))^2\Big)^{p/2}\Big],$$

and for $\rho\le\nu$, it is bounded by $\|X^0\|^p_{L^p(\Omega,H^\nu(\mathbb{S}^2))}$. For $\rho>\nu$, we observe with $x^{\rho-\nu}e^{-x}\le C$ for $\ell\neq 0$ that

$$(1+\ell(\ell+1))^\rho e^{-2\ell(\ell+1)t}\le\left(\frac{3}{2}\right)^{\rho-\nu} C\,(1+\ell(\ell+1))^\nu(2t)^{-(\rho-\nu)}, \tag{8}$$

which yields overall

$$\Big\|\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t}X_{\ell,m}(0)\widetilde{Y}_{\ell,m}\Big\|_{L^p(\Omega,H^\rho(\mathbb{S}^2))}\le Ct^{-\max\{\rho-\nu,0\}/2}\|X^0\|_{L^p(\Omega,H^\nu(\mathbb{S}^2))}.$$

For the stochastic convolution $\tilde{L}$, (7) yields for $p=2$ that

$$\|\tilde{L}(t)\|^2_{L^2(\Omega,H^\rho(\mathbb{S}^2))} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(1+\ell(\ell+1))^\rho\left(v_{\ell,m}\int_0^t e^{-2\ell(\ell+1)(t-s)}\,\mathrm{d}s+\Big(\widetilde{m}_{\ell,m}\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}s\Big)^2\right),$$

and since

$$\begin{aligned}(1+\ell(\ell+1))^\rho&\left(v_{\ell,m}\int_0^t e^{-2\ell(\ell+1)(t-s)}\,\mathrm{d}s+\Big(\widetilde{m}_{\ell,m}\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}s\Big)^2\right)\\ &\le\frac{3}{2}(1+\ell(\ell+1))^\eta(v_{\ell,m}+\widetilde{m}^2_{\ell,m})\end{aligned} \tag{9}$$

for $\rho-\eta\le 1$, $\|\hat{L}\|^2_{L^2(\Omega,H^\rho(\mathbb{S}^2))}$ is finite in this range and the claim for $p=2$ follows.

Let us now assume (ii) with $p>2$, i.e., that the Lévy processes $(L_{\ell,m})_{\ell,m}$ are independent. Using the series decomposition and the properties of the norm yields

$$\|\tilde{L}(t)\|^p_{L^p(\Omega,H^\rho(\mathbb{S}^2))}=\mathbb{E}\left[\left(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(1+\ell(\ell+1))^\rho\Big(\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}L_{\ell,m}(s)\Big)^2\right)^{p/2}\right]$$

Since all summands $Z_{\ell,m}(t)=(1+\ell(\ell+1))^\rho(\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}L_{\ell,m}(s))^2$ in the series expansion are independent and non-negative, [24, Corollary 3] as a generalization of the classical Rosenthal inequality (see [32, Theorem 1]) implies

$$\|\tilde{L}(t)\|^p_{L^p(\Omega,H^\rho(\mathbb{S}^2))}\le C\left(\frac{p/2}{\ln p/2}\right)^{p/2}\max\Big\{\Big(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\mathbb{E}[Z_{\ell,m}(t)]\Big)^{p/2},\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\mathbb{E}[Z_{\ell,m}(t)^{p/2}]\Big\}.$$

The first term in the max is bounded as before for $\rho-\eta\le 1$. For the second term, we compute as follows the pth-moment of the integral obtaining

$$\mathbb{E}[Z_{\ell,m}(t)^{p/2}]=(1+\ell(\ell+1))^{\rho\,p/2}\,\mathbb{E}\left[\Big|\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}L_{\ell,m}(s)\Big|^p\right]$$

By [30, Theorem 4.49], $L_{\ell,m}(s) - s\widetilde{m}_{\ell,m}$ is a martingale, so that we can split

$$\begin{aligned}&\mathbb{E}\left[\left|\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}L_{\ell,m}(s)\right|^p\right]\\&\quad\leq 2^{p-1}\left(\mathbb{E}\left[\left|\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}(L_{\ell,m}(s) - s\widetilde{m}_{\ell,m})\right|^p\right] + |\widetilde{m}_{\ell,m}|^p\left(\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}s\right)^p\right),\end{aligned}$$

where the first summand is a martingale by [30, Corollary 8.17] with quadratic variation $\int_0^t v_{\ell,m}e^{-2\ell(\ell+1)(t-s)}\,\mathrm{d}s$. Therefore, we bound

$$\mathbb{E}\left[\left|\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}(L_{\ell,m}(s) - s\widetilde{m}_{\ell,m})\right|^p\right] \leq C\left(\int_0^t v_{\ell,m}e^{-2\ell(\ell+1)(t-s)}\,\mathrm{d}s\right)^{p/2}$$

by the Burkholder–Davis–Gundy inequality [30, Theorem 3.50] and obtain with similar computation as in (9) for $\rho - \eta \leq 1$

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\mathbb{E}[Z_{\ell,m}(t)^{p/2}] \leq C\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(1+\ell(\ell+1))^{\eta\,p/2}(v_{\ell,m}^{p/2} + |\widetilde{m}_{\ell,m}|^p),$$

which is finite since $L \in L^p(\Omega, H^\eta(\mathbb{S}^2))$ with

$$\begin{aligned}\infty > \mathbb{E}[\|L(1)\|^p_{H^\eta(\mathbb{S}^2)}] &= \mathbb{E}\left[\Big(\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(1+\ell(\ell+1))^\eta L_{\ell,m}(1)^2\Big)^{p/2}\right]\\&\geq \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(1+\ell(\ell+1))^{\eta\,p/2}\mathbb{E}[|L_{\ell,m}(1)|^p],\end{aligned}$$

where the martingale $L_{\ell,m}(t) - t\widetilde{m}_{\ell,m}$ has quadratic variation $tv_{\ell,m}$ by [30, Theorem 4.49] and the lower bound of the Burkholder–Davis–Gundy inequality [30, Theorem 3.50] implies

$$\mathbb{E}[|L_{\ell,m}(1)|^p] \geq \mathbb{E}[|L_{\ell,m}(1) - \widetilde{m}_{\ell,m}|^p] + |\widetilde{m}_{\ell,m}|^p \geq C(v_{\ell,m}^{p/2} + |\widetilde{m}_{\ell,m}|^p).$$

Having shown (ii) for $p > 2$, we continue with the proof of (iii), which we prove on a more abstract level.

We split $\tilde{L}$ into a Wiener and a Poisson part by

$$\tilde{L}(t) = \tilde{W}(t) + \tilde{N}(t) = \int_0^t e^{\Delta_{\mathbb{S}^2}(t-s)}\,\mathrm{d}W(s) + \int_0^t\int_\chi e^{\Delta_{\mathbb{S}^2}(t-s)}\xi(s,\zeta)\tilde{N}(ds,d\zeta),$$

where $(e^{\Delta_{\mathbb{S}^2}t})_{t\geq 0}$ denotes the semigroup generated by $\Delta_{\mathbb{S}^2}$, which is given by $e^{\Delta_{\mathbb{S}^2}t}\widetilde{Y}_{\ell,m} = e^{-\ell(\ell+1)t}\widetilde{Y}_{\ell,m}$. This yields

$$\mathbb{E}\left[\|\tilde{L}(t)\|^p_{H^\rho(\mathbb{S}^2)}\right] \leq 2^{p-1}\mathbb{E}\left[\|\tilde{W}(t)\|^p_{H^\rho(\mathbb{S}^2)}\right] + 2^{p-1}\mathbb{E}\left[\|\tilde{N}(t)\|^p_{H^\rho(\mathbb{S}^2)}\right],$$

where the first term is bounded for all p by [9, Theorem 4.36] and the above result for $p = 2$, which also proves the claim for (i).

For the second term, [27, Lemma 3.1] and the same computations as in (8) imply for $\rho - \eta < 2/p$

$$
\begin{aligned}
&\mathbb{E}\left[\|\tilde{N}(t)\|^p_{H^\rho(\mathbb{S}^2)}\right] \\
&\quad = \mathbb{E}\left[\| \int_0^t \int_\chi (I - \Delta_{\mathbb{S}^2})^{\rho/2} e^{-(t-s)\Delta_{\mathbb{S}^2}} \xi(s,\zeta) \tilde{N}(\mathrm{d}s, \mathrm{d}\zeta) \|^p_{L^2(\mathbb{S}^2)}\right] \\
&\quad \le C \int_0^t \left(\int_\chi \|e^{(t-s)\Delta_{\mathbb{S}^2}} \xi(s,\zeta)\|^p_{H^\rho(\mathbb{S}^2)} \nu(\mathrm{d}\zeta) + \Big(\int_\chi \|e^{(t-s)\Delta_{\mathbb{S}^2}} \xi(s,\zeta)\|^2_{H^\rho(\mathbb{S}^2)} \nu(\mathrm{d}\zeta)\Big)^{p/2} \right) \mathrm{d}s \\
&\quad \le C \int_0^t (t-s)^{-\max((\rho-\eta),0)\, p/2} \Big(\int_\chi \|\xi(s,\zeta)\|^p_{H^\eta(\mathbb{S}^2)} \nu(\mathrm{d}\zeta) + \big(\int_\chi \|\xi(s,\zeta)\|^2_{H^\eta(\mathbb{S}^2)} \nu(\mathrm{d}\zeta)\big)^{p/2} \Big) \mathrm{d}s \\
&\quad \le C \sup_{s \le T} \left(\int_\chi \|\xi(s,\zeta)\|^p_{H^\eta(\mathbb{S}^2)} \nu(\mathrm{d}\zeta) + \Big(\int_\chi \|\xi(s,\zeta)\|^2_{H^\eta(\mathbb{S}^2)} \nu(\mathrm{d}\zeta)\Big)^{p/2} \right),
\end{aligned}
$$

which is finite by Assumption 2.3 and concludes the proof. □

3. Spectral approximation in space

To compute solutions to (1), we start with a semi-discrete scheme obtained by a spatial truncation similarly to [22, 7, 20]. More specifically, using the series expansion (6), we set for $\kappa \in \mathbb{N}$

$$
X^{(\kappa)}(t) = \sum_{\ell=0}^{\kappa} \sum_{m=-\ell}^{\ell} \Big(e^{-\ell(\ell+1)t} X^0_{\ell,m} + \int_0^t e^{-\ell(\ell+1)(t-s)} \, \mathrm{d}L_{\ell,m}(s) \Big) \widetilde{Y}_{\ell,m}. \tag{10}
$$

With the same computations as in Lemma 2.4, we observe that the expectation satisfies

$$
\mathbb{E}[X^{(\kappa)}(t)] = \sum_{\ell=0}^{\kappa} \sum_{m=-\ell}^{\ell} \Big(\mathbb{E}[X^0_{\ell,m}] e^{-\ell(\ell+1)t} + \widetilde{m}_{\ell,m} \int_0^t e^{-\ell(\ell+1)(t-s)} \, \mathrm{d}s \Big) \widetilde{Y}_{\ell,m} \tag{11}
$$

and similarly the second moment

$$
\begin{aligned}
&\mathbb{E}\big[\|X^{(\kappa)}(t)\|^2_{L^2(\mathbb{S}^2)}\big] \\
&\quad = \sum_{\ell=0}^{\kappa} \sum_{m=-\ell}^{\ell} \Big(\mathbb{E}\Big[\Big(X^0_{\ell,m} e^{-\ell(\ell+1)t} + \widetilde{m}_{\ell,m} \int_0^t e^{-\ell(\ell+1)(t-s)} \, \mathrm{d}s \Big)^2 \Big] + v_{\ell,m} \int_0^t e^{-2\ell(\ell+1)(t-s)} \, \mathrm{d}s \Big).
\end{aligned} \tag{12}
$$

Similarly to [22, Lemma 7.1] and [20, Lemma 3.1], we show strong convergence, where the rates coincide with those in the Wiener setting under Assumption 2.2.

Theorem 3.1. *Let X be given by* (6) *and $X^{(\kappa)}$ by its spectral approximation* (10). *Then,*

$$
\|X(t) - X^{(\kappa)}(t)\|_{L^2(\Omega, L^2(\mathbb{S}^2))} \le e^{-(\kappa+1)^2 t} \|X^0\|_{L^2(\Omega, L^2(\mathbb{S}^2))} + C\kappa^{-\beta},
$$

where $\beta = \eta + 1$ under Assumption 2.1 and $\beta = \alpha/2$ if Assumption 2.2 holds.

Proof. Using the series expansions (6) and (10), we obtain by the triangle inequality

$$\begin{aligned}\|X(t)-X^{(\kappa)}(t)\|_{L^2(\Omega,L^2(\mathbb{S}^2))} \leq \Big\| \sum_{\ell=\kappa+1}^{\infty}\sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t} X^0_{\ell,m}\widetilde{Y}_{\ell,m}\Big\|_{L^2(\Omega,L^2(\mathbb{S}^2))}\\ + \Big\|\sum_{\ell=\kappa+1}^{\infty}\sum_{m=-\ell}^{\ell}\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}L_{\ell,m}(s)\widetilde{Y}_{\ell,m}\Big\|_{L^2(\Omega,L^2(\mathbb{S}^2))}.\end{aligned}$$

The first summand satisfies analogously to [20, Lemma 3.1]

$$\Big\| \sum_{\ell=\kappa+1}^{\infty}\sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t} X^0_{\ell,m}\widetilde{Y}_{\ell,m}\Big\|_{L^2(\Omega,L^2(\mathbb{S}^2))} \leq e^{-(\kappa+1)(\kappa+2)t}\|X^0\|_{L^2(\Omega,L^2(\mathbb{S}^2))},$$

while (7) in the proof of Lemma 2.4, implies for the second term

$$\begin{aligned}&\Big\|\sum_{\ell=\kappa+1}^{\infty}\sum_{m=-\ell}^{\ell}\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}L_{\ell,m}(s)\widetilde{Y}_{\ell,m}\Big\|^2_{L^2(\Omega,L^2(\mathbb{S}^2))}\\ &\quad= \sum_{\ell=\kappa+1}^{\infty}\sum_{m=-\ell}^{\ell}\Big(v_{\ell,m}\int_0^t e^{-2\ell(\ell+1)(t-s)}\,\mathrm{d}s + \Big(\widetilde{m}_{\ell,m}\int_0^t e^{-\ell(\ell+1)(t-s)}\,\mathrm{d}s\Big)^2\Big)\\ &\quad\leq \sum_{\ell=\kappa+1}^{\infty}\sum_{m=-\ell}^{\ell}\Big(v_{\ell,m}\frac{1}{2}\ell^{-2} + \widetilde{m}^2_{\ell,m}\ell^{-4}\Big),\end{aligned}$$

where we computed and bounded the integrals.

If Assumption 2.1 is satisfied for some $\eta > -1$,

$$\begin{aligned}\sum_{\ell=\kappa+1}^{\infty}\sum_{m=-\ell}^{\ell} v_{\ell,m}\ell^{-2} &= \sum_{\ell=\kappa+1}^{\infty}\sum_{m=-\ell}^{\ell} v_{\ell,m}(1+\ell(\ell+1))^{\eta}(1+\ell(\ell+1))^{-\eta}\ell^{-2}\\ &\leq C(\kappa+1)^{-2-2\eta}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} v_{\ell,m}(1+\ell(\ell+1))^{\eta} \leq C\kappa^{-2(1+\eta)}\|\tilde{v}\|^2_{H^\eta(\mathbb{S}^2)},\end{aligned}$$

and with the same arguments,

$$\sum_{\ell=\kappa+1}^{\infty}\sum_{m=-\ell}^{\ell}\widetilde{m}^2_{\ell,m}\ell^{-4} \leq C\kappa^{-2(2+\eta)}\|\widetilde{m}\|^2_{H^\eta(\mathbb{S}^2)}.$$

This proves the claim with $\beta = 1+\eta$ and implies under Assumption 2.2 convergence of order up to $\alpha/2$ but not including $\beta = \alpha/2$. We sharpen this bound by using the properties of the special Lévy process in Assumption 2.2.

$$\begin{aligned}\sum_{\ell=\kappa+1}^{\infty}\sum_{m=-\ell}^{\ell}\Big(v_{\ell,m}\frac{1}{2}\ell^{-2} + \widetilde{m}^2_{\ell,m}\ell^{-4}\Big) &= \sum_{\ell=\kappa+1}^{\infty}\sum_{m=-\ell}^{\ell}\Big(a_\ell^2\hat{v}_{\ell,m}\frac{1}{2}\ell^{-2} + a_\ell^2\hat{m}^2_{\ell,m}\ell^{-4}\Big)\\ &\leq C\hat{v}\sum_{\ell=\kappa+1}^{\infty}(2\ell+1)\ell^{-\alpha-2} + C\hat{m}^2\sum_{\ell=\kappa+1}^{\infty}(2\ell+1)\ell^{-\alpha-4}.\end{aligned}$$

This expression is bounded by $\kappa^{-\alpha}$ as computed, e.g., in the proof of [22, Prop. 5.2], which yields the improved bound $\alpha/2$. □

Having established strong convergence rates, we continue with weak convergence. More precisely, taking an appropriate class of test functions φ, we aim to show that $\mathbb{E}[\varphi(X_n)]$ converge to $\mathbb{E}[\varphi(X)]$ in a suitable sense with X_n, X the solutions of our approximated and initial problem, respectively. The reason for such an effort lies in the fact that for the approximation of stochastic partial differential equations the weak convergence rate is approximately twice the strong convergence rate; see [19] for more details.

Based on the strong convergence proof of Theorem 3.1, we obtain weak rates for the expectation and second moment as specially chosen test functions. While the error of the expectation contains an additional term compared to the Wiener case in [20, Lemma 3.2], since the Lévy process is not assumed to be centered, the second moment converges with the same rate under Assumption 2.2.

Corollary 3.2. *Let $\beta = \eta + 1$ under Assumption 2.1 and $\beta = \alpha/2$ with Assumption 2.2. Given the mild solution X in* (6) *and its spectral approximation $X^{(\kappa)}$ in* (10)*, the error of the mean is bounded by*

$$\|\mathbb{E}[X(t)] - \mathbb{E}[X^{(\kappa)}(t)]\|_{L^2(\mathbb{S}^2)} \leq e^{-(\kappa+1)^2 t}\|\mathbb{E}[X^0]\|_{L^2(\mathbb{S}^2)} + C\kappa^{-(\beta+1)}.$$

Furthermore, the error of the second moment is bounded by

$$|\mathbb{E}[\|X(t)\|^2_{L^2(\mathbb{S}^2)} - \|X^{(\kappa)}(t)\|^2_{L^2(\mathbb{S}^2)}]| \leq 2e^{-2(\kappa+1)^2 t}\|X^0\|^2_{L^2(\Omega,L^2(\mathbb{S}^2))} + C\kappa^{-2\beta}.$$

Proof. We plug in the expressions for the two expectations derived in Lemma 2.4 and (11) to obtain

$$\|\mathbb{E}[X(t)] - \mathbb{E}[X^{(\kappa)}(t)]\|_{L^2(\mathbb{S}^2)} \leq e^{-2(\kappa+1)^2 t}\|\mathbb{E}[X^0]\|_{L^2(\mathbb{S}^2)} + \Big(\sum_{\ell=\kappa+1}^{\infty}\sum_{m=-\ell}^{\ell} \ell^{-4}\widetilde{m}^2_{\ell,m}\Big)^{1/2}.$$

The claim follows with the same computations as in Theorem 3.1. To bound the error of the second moment, we observe that the same computations as in [7, Prop. 4] yield

$$|\mathbb{E}[\|X(t)\|^2_{L^2(\mathbb{S}^2)} - \|X^{(\kappa)}(t)\|^2_{L^2(\mathbb{S}^2)}]| = \mathbb{E}[\|X(t) - X^{(\kappa)}(t)\|^2_{L^2(\mathbb{S}^2)}],$$

which implies the claim by Theorem 3.1. □

Concerning the spectral approximation in space, we can derive weak error rates for a broader class of test functions, with rates depending on the regularity of the test function, the solution, the initial condition, and the noise. We first present the assumption on the test functions.

Assumption 3.3. Let φ be a Fréchet differentiable test function that satisfies for fixed $\rho \geq 0$ and all $t \in [0, T]$,

$$\|\int_0^1 \varphi'(\rho X(t) + (1-\rho)X^{(\kappa)}(t))\,\mathrm{d}\rho\|_{L^2(\Omega,H^\rho(\mathbb{S}^2))} \leq C < \infty.$$

As shown in [7], independently of the specific equation, if the solution to (1) satisfies $X(t) \in L^{2m}(\Omega, H^\rho(\mathbb{S}^2))$, this implies that the class of test functions includes those with polynomial growth of the derivative in $H^\rho(\mathbb{S}^2)$, i.e. for all $x \in H^\rho(\mathbb{S}^2)$,

$$\|\varphi'(x)\|_{H^\rho(\mathbb{S}^2)} \leq C\left(1 + \|x\|^m_{H^\rho(\mathbb{S}^2)}\right). \tag{13}$$

Hence the class of test functions satisfying Assumption 3.3 is not trivial as in Proposition 2.5 is shown that $X(t) \in L^{2m}(\Omega, H^\rho(\mathbb{S}^2))$. With this, we are able to state our weak convergence result.

Theorem 3.4. *Assume that X and $X^{(\kappa)}$, given by* (6) *and* (10)*, respectively, and φ satisfy Assumption 3.3 for some $\rho \geq 0$. Then the weak error is bounded by*

$$\left|\mathbb{E}\left[\varphi(X(t)) - \varphi(X^{(\kappa)}(t))\right]\right| \leq C(e^{-(\kappa+1)^2 t}(\kappa+1)^{-\rho}\|X^0\|_{L^2(\Omega,L^2(\mathbb{S}^2))} + C\kappa^{-(\beta+\rho)},$$

where $\beta = \eta + 1$ under Assumption 2.1 and $\beta = \alpha/2$ under Assumption 2.2.

Proof. As in [7], we consider the Gelfand triple $V \subset H \subset V^*$ with $V = H^\rho(\mathbb{S}^2)$, $H = L^2(\mathbb{S}^2)$, $V^* = H^{-\rho}(\mathbb{S}^2)$ and obtain using the mean value theorem and Hölder's inequality

$$\begin{aligned}
&\mathbb{E}\left[\left|\varphi(X(t)) - \varphi(X^{(\kappa)}(t))\right|\right] \\
&= \mathbb{E}\left[\left\langle \int_0^1 \varphi'\left(\rho X(t) + (1-\rho)X^\kappa(t)\right) d\rho, X(t) - X^{(\kappa)}(t)\right\rangle_{V\times V^*}\right] \\
&\leq \|\int_0^1 \varphi'\left(\rho X(t) + (1-\rho)X^\kappa(t)\right) d\rho\|_{L^2(\Omega,V)}\|X(t) - X^{(\kappa)}(t)\|_{L^2(\Omega,V^*)}.
\end{aligned}$$

The first norm is bounded by Assumption 3.3, while the second norm is the strong error computed in the weaker $H^{-\rho}(\mathbb{S}^2)$ norm than in Theorem 3.1. Following the proof of Theorem 3.1 with new weights therefore yields

$$\begin{aligned}
&\|X(t) - X^{(\kappa)}(t)\|^2_{L^2(\Omega,V^*)} \\
&\leq \sum_{\ell=\kappa+1}^{\infty}\sum_{m=-\ell}^{\ell}\left(e^{-2\ell(\ell+1)t}\mathbb{E}\left[|X^0_{\ell,m}|^2\right] + v_{\ell,m}\frac{1}{2}\ell^{-2} + \widetilde{m}^2_{\ell,m}\ell^{-4}\right)(1+\ell(\ell+1))^{-\rho}
\end{aligned}$$

and increased polynomial convergence rates in the initial condition and noise by ρ. This finishes the proof. □

Based on the general weak convergence result and our regularity estimates of the solution in Proposition 2.5, we conclude with the following corollary. We obtain the usual rule of thumb with a weak convergence rate that is twice the strong one in the first two settings of Proposition 2.5. For the more general third setting, the rate is essentially $\eta + 1/m$, i.e., twice the strong if ϕ' grows linearly and else depending on the degree of the polynomial growh condition in (13).

Corollary 3.5. *Assume that X and $X^{(\kappa)}$ are given by* (6) *and* (10)*, respectively, satisfying Assumption 2.1 with $L = W$ being a Q-Wiener prcoess or $L \in L^p(\Omega, H^\eta(\mathbb{S}^2))$ with independent $(L_{\ell,m})_{\ell,m}$. Then, for all test functions φ satisfying* (13)*, $X^{(\kappa)}$ converges weakly to X with error bounded by*

$$\left|\mathbb{E}\left[\varphi(X(t)) - \varphi(X^{(\kappa)}(t))\right]\right| \leq C(e^{-(\kappa+1)^2 t}(\kappa+1)^{-(\eta+1)}\|X^0\|_{L^2(\Omega,L^2(\mathbb{S}^2))} + C\kappa^{-2(\eta+1)}.$$

In the more general setting, where L satisfies Assumption 2.3, for all $\rho < \eta + 1/m$, the error is bounded by

$$\left|\mathbb{E}\left[\varphi(X(t)) - \varphi(X^{(\kappa)}(t))\right]\right| \leq C(e^{-(\kappa+1)^2 t}(\kappa+1)^{-\rho}\|X^0\|_{L^2(\Omega,L^2(\mathbb{S}^2))} + C\kappa^{-(\eta+1+\rho)}.$$

4. Euler–Maruyama approximation in time

In the previous section, we have approximated the solution to (1) in space by spatial truncation in (10). To compute the solution with this scheme, one would have to simulate the stochastic convolution, which is possible in the Wiener case as shown in [22]. Unfortunately, this is no longer the case for more general Lévy noise. Therefore, we need to introduce an additional time discretization. As in [20], we introduce a forward and a backward Euler–Maruyama scheme and prove convergence.

For $n \in \mathbb{N}$ we take an equidistant grid on the interval $[0,T]$ with grid size $h = T/n$ and grid points $t_k = kh$ for $k \in \{0,\dots,n\}$. The forward resp. backward Euler–Maruyama scheme for (5) is given by

$$X^{(h)}_{\ell,m}(t_k) = \xi X^{(h)}_{\ell,m}(t_{k-1}) + \xi^{\delta} \Delta L_{\ell,m}(t_k),$$

where $\Delta L_{\ell,m}(t_k) = L_{\ell,m}(t_k) - L_{\ell,m}(t_{k-1})$, $\xi = (1 - \ell(\ell+1)h)$ and $\delta = 0$ for the forward scheme, while $\xi = (1 + \ell(\ell+1)h)^{-1}$ and $\delta = 1$ for the backward scheme. Recursively, we obtain

$$X^{(h)}_{\ell,m}(t_k) = \xi^k X_{\ell,m}(0) + \sum_{j=1}^{k} \xi^{k-j+\delta} \Delta L_{\ell,m}(t_j),$$

and hence, the fully discrete approximation of (1) is given by

$$X^{(\kappa,h)}(t_k) = \sum_{\ell=0}^{\kappa} \sum_{m=-\ell}^{\ell} X^{(h)}_{\ell,m}(t_k) \widetilde{Y}_{\ell,m}. \tag{14}$$

As discussed in [20], for the forward Euler–Maruyama scheme, stability is only guaranteed if $\ell(\ell+1)h \leq 1$, while the backward Euler–Maruyama scheme is unconditionally stable. Our proofs rely on a coupling between space and time discretization even for the backward scheme, and we assume from here on that all schemes satisfy that $\ell(\ell+1)h \leq C_c$ for some constant C_c with $C_c \leq 1$ for the forward scheme.

The expectation of $X^{(\kappa,h)}$ satisfies

$$\mathbb{E}[X^{(\kappa,h)}(t_k)] = \sum_{\ell=0}^{\kappa} \sum_{m=-\ell}^{\ell} \Big(\xi^k \mathbb{E}[X^0_{\ell,m}] + \sum_{j=1}^{k} \xi^{k-j+\delta} h \, \widetilde{m}_{\ell,m} \Big) \widetilde{Y}_{\ell,m}, \tag{15}$$

which follows with the linearity of the expectation.

To show that the second moment satisfies

$$\mathbb{E}[\|X^{(\kappa,h)}(t_k)\|^2_{L^2(\mathbb{S}^2)}] = \sum_{\ell=0}^{\kappa} \sum_{m=-\ell}^{\ell} \mathbb{E}\Big[\Big(X^0_{\ell,m} \xi^k + \widetilde{m}_{\ell,m} h \sum_{j=1}^{k} \xi^{k-j+\delta} \Big)^2 \Big] + v_{\ell,m} h \sum_{j=1}^{k} \xi^{2(k-j+\delta)} \Big), \tag{16}$$

we plug in (14) and compute the square obtaining

$$\begin{aligned}\mathbb{E}[\|X^{(\kappa,h)}(t_k)\|^2_{L^2(\mathbb{S}^2)}] &= \mathbb{E}\Big[\Big\|\sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell}\xi^k X_{\ell,m}(0)\widetilde{Y}_{\ell,m}\Big\|^2_{L^2(\mathbb{S}^2)}\Big] \\ &\quad + 2\mathbb{E}\Big[\Big\langle\sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell}\xi^k X^0_{\ell,m}\widetilde{Y}_{\ell,m},\sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell}\sum_{j=1}^{k}\xi^{k-j+\delta}\Delta L_{\ell,m}(t_j)\widetilde{Y}_{\ell,m}\Big\rangle_{L^2(\mathbb{S}^2)}\Big] \\ &\quad + \mathbb{E}\Big[\Big\|\sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell}\sum_{j=1}^{k}\xi^{k-j+\delta}\Delta L_{\ell,m}(t_j)\widetilde{Y}_{\ell,m}\Big\|^2_{L^2(\mathbb{S}^2)}\Big].\end{aligned}$$

While the first and third term are computed as in [20], the additional second term satisfies with the independence of the Lévy process and the initial condition that

$$\begin{aligned}&\mathbb{E}\Big[\Big\langle\sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell}\xi^k X^0_{\ell,m}\widetilde{Y}_{\ell,m},\sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell}\sum_{j=1}^{k}\xi^{k-j+\delta}\Delta L_{\ell,m}(t_j)\widetilde{Y}_{\ell,m}\Big\rangle_{L^2(\mathbb{S}^2)}\Big] \\ &\quad = \mathbb{E}\Big[\sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell}\xi^k X^0_{\ell,m}\sum_{j=1}^{k}\xi^{k-j+\delta}\Delta L_{\ell,m}(t_j)\Big] = \sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell}\xi^k h\sum_{j=1}^{k}\xi^{k-j+\delta}\,\mathbb{E}[X^0_{\ell,m}]\widetilde{m}_{\ell,m}.\end{aligned}$$

The following proposition will be needed in the convergence proofs and summarizes the results on the approximation of the exponential function from [20], which are obtained by a combination of Propositions 4.1–4.3 and parts of the proofs of Theorems 4.4 and 4.5 in [20].

Proposition 4.1. *For all $\mu \in (0,1]$, there exists a constant $C_{\mu,\widetilde{c}} \in (0,\infty)$ such that for all $\ell, k \in \mathbb{N}$ and $h \in (0,\infty)$ with $\ell(\ell+1)h \leq C_c$ and $a \in \{1,2\}$*

$$|e^{-\ell(\ell+1)h\cdot k} - \xi^k| \leq C_{\mu,\widetilde{c}}(\ell(\ell+1))^{1+\mu}h^{1+\mu}ke^{-\ell(\ell+1)h\cdot(k-1)} \leq C_{\mu,\widetilde{c}}(\ell(\ell+1))^{\mu}h^{\mu}, \tag{17}$$

$$\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\big(e^{-\ell(\ell+1)(t_k-s)} - \xi^{k-j+\delta}\big)^2\,\mathrm{d}s \leq C_{\mu,\widetilde{c}}(\ell(\ell+1))^{2\mu-1}h^{2\mu}, \tag{18}$$

$$\Big|\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j} e^{-a\ell(\ell+1)(t_k-s)} - \xi^{a(k-j+\delta)}\,\mathrm{d}s\Big| \leq C_{\mu,\widetilde{c}}(\ell(\ell+1))^{\mu-1}h^{\mu}. \tag{19}$$

With all prerequisites at hand, we are now ready to prove first strong convergence in Theorem 4.2 and then convergence of the first and second moments in Theorem 4.3.

Theorem 4.2. *Let $X^0 \in L^2(\Omega, H^\nu(\mathbb{S}^2))$ for some $\nu \geq 0$ and let $X^{(\kappa)}$ be given by (10) with Euler–Maruyama approximation $X^{(\kappa,h)}$ defined by (14). Then the strong error in the time discretisation is bounded by*

$$\|X^{(\kappa)}(t_k) - X^{(\kappa,h)}(t_k)\|_{L^2(\Omega,L^2(\mathbb{S}^2))} \leq C\big(h^\mu t_k^{\min\{1,\nu/2\}-\mu}\|X^0\|_{L^2(\Omega,H^{\min\{2,\nu\}}(\mathbb{S}^2))} + h^{\min\{\beta/2,1\}}\big)$$

for $\mu \in (0,1]$, where $\beta = \eta + 1$ under Assumption 2.1 and $\beta < \alpha/2$ with Assumption 2.2.

Proof. Plugging in the definitions of the schemes and computing the norm yields

$$\begin{aligned}&\|X^{(\kappa)}(t_k) - X^{(\kappa,h)}(t_k)\|^2_{L^2(\Omega,L^2(\mathbb{S}^2))}\\ &\leq 2\sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell}\Big((e^{-\ell(\ell+1)t_k} - \xi^k)^2\mathbb{E}[|X^0_{\ell,m}|^2]\\ &\qquad + \mathbb{E}\big[\big(\int_0^{t_k} e^{-\ell(\ell+1)(t_k-s)}\,\mathrm{d}L_{\ell,m}(s) - \sum_{j=1}^{k}\xi^{k-j+\delta}\Delta L_{\ell,m}(t_j)\big)^2\big]\Big).\end{aligned}$$

By the first inequality in (17) in Proposition 4.1, the first summand satisfies for $\mu \in (0,1]$

$$(e^{-\ell(\ell+1)t_k} - \xi^k)^2 \leq C(\ell(\ell+1))^{2\mu+2}h^{2\mu}t_k^2 e^{-2\ell(\ell+1)t_{k-1}} \leq Ch^{2\mu}t_k^{\min\{2,\nu\}-2\mu}(\ell(\ell+1))^{\min\{2,\nu\}},$$

where we used that $\exp(2\ell(\ell+1)h)$ and $x^{2\mu-\min\{2,\nu\}+2}\exp(-x)$ are bounded. This implies the claimed bound for the first term using the definition of the Sobolev norm.

To write the second term as a stochastic integral, we denote for $s \in [0,T]$ the projection to the previous grid point by $\underline{s} = \max\{t_k : k \in \{0,\dots,n\}, t_k \leq s\}$. Applying [4, Example 15.12] as earlier and Jensen's inequality, we obtain

$$\begin{aligned}&\mathbb{E}\big[\big(\int_0^{t_k} e^{-\ell(\ell+1)(t_k-s)}\,\mathrm{d}L_{\ell,m}(s) - \sum_{j=1}^{k}\xi^{k-j+\delta}\Delta L_{\ell,m}(t_j)\big)^2\big]\\ &= \mathbb{E}\Big[\Big|\int_0^{t_k} e^{-\ell(\ell+1)(t_k-s)} - \xi^{k-(\underline{s}+1)h^{-1}+\delta}\,\mathrm{d}L_{\ell,m}(s)\Big|^2\Big]\\ &\leq \big(v_{\ell,m} + \widetilde{m}^2_{\ell,m}T\big)\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\big(e^{-\ell(\ell+1)(t_k-s)} - \xi^{k-j+\delta}\big)^2\,\mathrm{d}s\end{aligned}$$

Setting $\mu = \min\{(\eta+1)/2, 1\}$ in (18) bounds the last expression by

$$\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}\big(e^{-\ell(\ell+1)(t_k-s)} - \xi^{k-j+\delta}\big)^2\,\mathrm{d}s \leq C(\ell(\ell+1))^{\min\{\eta,1\}}h^{2\min\{(\eta+1)/2,1\}}$$

and yields for as bound for the second term using the definition of the Sobolev norms

$$\begin{aligned}&\mathbb{E}\big[\big(\int_0^{t} e^{-\ell(\ell+1)(t_k-s)}\,\mathrm{d}L_{\ell,m}(s) - \sum_{j=1}^{k}\xi^{k-j+\delta}\Delta L_{\ell,m}(t_j)\big)^2\big]\\ &\qquad\leq Ch^{2\min\{(\eta+1)/2,1\}}\big(\|\tilde{v}\|^2_{H^\eta(\mathbb{S}^2)} + T\|\widetilde{m}\|^2_{H^\eta(\mathbb{S}^2)}\big).\end{aligned}$$

This finishes the proof under Assumption 2.1. The claim for Assumption 2.2 follows since L in $L^2(\Omega, H^\eta(\mathbb{S}^2))$ for all $\eta < \alpha/2 - 1$ by (2.1). □

Theorem 4.3. *Let $\beta = \eta + 1$ under Assumption 2.1 and $\beta < \alpha/2$ with Assumption 2.2. Given the spectral approximation $X^{(\kappa)}$ in* (10) *and the fully discrete approximation $X^{(\kappa,h)}$ in* (14) *with $X^0 \in L^2(\Omega, H^\nu(\mathbb{S}^2))$ for some $\nu \geq 0$, the error of the mean is bounded by*

$$\|\mathbb{E}[X^{(\kappa)}(t_k) - X^{(\kappa,h)}(t_k)]\|_{L^2(\mathbb{S}^2)} \leq Ch^\mu(t_k^{\min\{1,\nu/2\}-\mu}\|\mathbb{E}[X^0]\|_{H^{\min\{2,\nu\}}(\mathbb{S}^2)} + t_k^{\min\{1,(\beta+1)/2\}-\mu}).$$

Furthermore, the error of the second moment is bounded by

$$|\mathbb{E}[\|X^{(\kappa)}(t)\|^2_{L^2(\mathbb{S}^2)} - \|X^{(\kappa,h)}(t)\|^2_{L^2(\mathbb{S}^2)}]| \leq C(h^\mu t_k^{\min\{1,\nu\}-\mu}\|X^0\|^2_{L^2(\Omega,H^{\min\{1,\nu\}}(\mathbb{S}^2))} + h^{\min\{\beta,1\}}).$$

Proof. To compute the error of the expection, we plug in the representations (11) and (15) and obtain

$$\begin{aligned}
&\|\mathbb{E}[X^{(\kappa)}(t_k) - X^{(\kappa,h)}(t_k)]\|^2_{L^2(\mathbb{S}^2)}\\
&\quad\leq 2\sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell}\Big(\big(e^{-\ell(\ell+1)t_k} - \xi^k\big)^2\,\mathbb{E}[X^0_{\ell,m}]^2 + \Big(\int_0^{t_k} e^{-\ell(\ell+1)(t_k-s)}\,\mathrm{d}s - \sum_{j=1}^{k}\xi^{k-j+\delta}h\Big)^2\widetilde{m}^2_{\ell,m}\Big).
\end{aligned}$$

The bound for the first summand follows as in Theorem 4.2. For the second summand, we observe computing the integral and the geometric series that

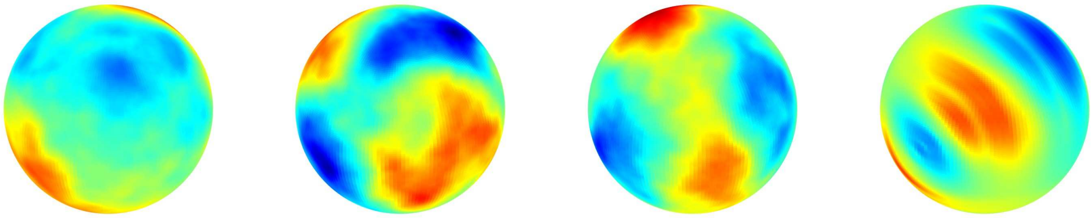
$$\Big(\int_0^{t_k} e^{-\ell(\ell+1)(t_k-s)}\,\mathrm{d}s - \sum_{j=1}^{k}\xi^{k-j+\delta}h\Big)^2 = (\ell(\ell+1))^{-2}(e^{-\ell(\ell+1)t_k} - \xi^k)^2. \tag{20}$$

Therefore the second term is equal to the first with coefficients $(\widetilde{m}_{\ell,m}/(\ell(\ell+1)))^2$, which yields the claim when ν is set to $\eta+2$. As in the previous theorem, the result for Assumption 2.2 follows since $L \in L^2(\Omega, H^\eta(\mathbb{S}^2))$ for all $\eta < \alpha/2 - 1$ by (2.1). For the second moment we plug in (12) and (16) to obtain using for the non-variance term that $(a^2 - b^2) = (a-b)(a+b)$,

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\|X^{(\kappa)}(t_k)\|^2_{L^2(\mathbb{S}^2)} - \|X^{(\kappa,h)}(t_k)\|^2_{L^2(\mathbb{S}^2)}\Big]\Big|\\
&\quad\leq \sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell} v_{\ell,m}\Big|\int_0^{t_k} e^{-2\ell(\ell+1)(t_k-s)}\,\mathrm{d}s - h\sum_{j=1}^{k}\xi^{2(k-j+\delta)}\Big|\\
&\qquad\qquad + \Big|\mathbb{E}\Big[\Big(X^0_{\ell,m}\big(e^{-\ell(\ell+1)t_k} - \xi^k\big) + \widetilde{m}_{\ell,m}\Big(\int_0^{t_k} e^{-\ell(\ell+1)(t_k-s)}\,\mathrm{d}s - h\sum_{j=1}^{k}\xi^{k-j+\delta}\Big)\Big)\\
&\qquad\qquad\qquad \Big(X^0_{\ell,m}\big(e^{-\ell(\ell+1)t_k} + \xi^k\big) + \widetilde{m}_{\ell,m}\Big(\int_0^{t_k} e^{-\ell(\ell+1)(t_k-s)}\,\mathrm{d}s + h\sum_{j=1}^{k}\xi^{k-j+\delta}\Big)\Big)\Big]\Big|
\end{aligned}$$

Note that $e^{-\ell(\ell+1)} \leq 1$ and $\xi \leq 1$, hence we use this estimates whenever possible and obtain with $ab \leq a^2 + b^2$ that

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\|X^{(\kappa)}(t_k)\|^2_{L^2(\mathbb{S}^2)} - \|X^{(\kappa,h)}(t_k)\|^2_{L^2(\mathbb{S}^2)}\Big]\Big|\\
&\quad\leq \sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell} v_{\ell,m}\Big|\int_0^{t_k} e^{-2\ell(\ell+1)(t_k-s)}\,\mathrm{d}s - h\sum_{j=1}^{k}\xi^{2(k-j+\delta)}\Big|\\
&\qquad + C\sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell}\mathbb{E}[|X^0_{\ell,m}|^2]\big|e^{-\ell(\ell+1)t_k} - \xi^k\big|\\
&\qquad + C\sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell}\Big(\mathbb{E}[|X^0_{\ell,m}|]^2 + \widetilde{m}^2_{\ell,m}\Big)\big|e^{-\ell(\ell+1)t_k} - \xi^k\big|\Big|\int_0^{t_k} e^{-\ell(\ell+1)(t_k-s)}\,\mathrm{d}s + h\sum_{j=1}^{k}\xi^{k-j+\delta}\Big|\\
&\qquad + C\sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell}\Big(\mathbb{E}[|X^0_{\ell,m}|]^2 + \widetilde{m}^2_{\ell,m}\Big)\Big|\int_0^{t_k} e^{-\ell(\ell+1)(t_k-s)}\,\mathrm{d}s - h\sum_{j=1}^{k}\xi^{k-j+\delta}\Big|
\end{aligned}$$



(1) t=1 (2) t=1 (3) t=1 (4) t=1

FIGURE 1. Realization of different Lévy processes on the sphere: (1) Wiener process, (2) Poisson process, (3) sum of a Wiener and a Poisson process, (4) same one–dimensional Poisson process on all components of the decomposition.

To resort and simplify this expression, we compute with the definition of ξ and δ and the bounds on $\ell(\ell+1)h$ that

$$\begin{aligned}\Big|\int_0^{t_k} e^{-\ell(\ell+1)(t_k-s)}\,\mathrm{d}s + h\sum_{j=1}^k \xi^{k-j+\delta}\Big| \\ \le \Big|\int_0^{t_k} e^{-\ell(\ell+1)(t_k-s)}\,\mathrm{d}s\Big| + \Big|h\sum_{j=1}^k \xi^{k-j+\delta}\Big| \le C(\ell(\ell+1))^{-1}\end{aligned}$$

and use (20). This yields

$$\begin{aligned}&\Big|\mathbb{E}\Big[\|X^{(\kappa)}(t_k)\|^2_{L^2(\mathbb{S}^2)} - \|X^{(\kappa,h)}(t_k)\|^2_{L^2(\mathbb{S}^2)}\Big]\Big| \\ &\le \sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell} v_{\ell,m}\Big|\int_0^{t_k} e^{-2\ell(\ell+1)(t_k-s)}\,\mathrm{d}s - h\sum_{j=1}^k \xi^{2(k-j+\delta)}\Big| \\ &\quad + C\sum_{\ell=1}^{\kappa}\sum_{m=-\ell}^{\ell}\Big(\mathbb{E}[|X^0_{\ell,m}|^2] + (\ell(\ell+1))^{-1}(\mathbb{E}[|X^0_{\ell,m}|]^2 + \widetilde{m}^2_{\ell,m})\Big)\big|e^{-\ell(\ell+1)t_k} - \xi^k\big|.\end{aligned}$$

Since $\mathbb{E}[|X^0_{\ell,m}|]^2$ and $\widetilde{m}^2_{\ell,m}$ are smoothed with $(\ell(\ell+1))^{-1}$, convergence is dominated by the remaining two terms. While the claim for the first one follows with Proposition 4.1(19), the second one is bounded similarly to the proof of Theorem 4.2 by

$$\sum_{\ell=0}^{\kappa}\sum_{m=-\ell}^{\ell}\big|e^{-\ell(\ell+1)t_k} - \xi^k\big|\mathbb{E}[|X^0_{\ell,m}|^2] \le Ch^{\mu} t_k^{\min\{1,\nu\}-\mu}\|X^0\|^2_{L^2(\Omega,H^{\min\{1,\nu\}}(\mathbb{S}^2))}. \qquad \square$$

5. Numerical simulation

We are now ready to support the theoretical conclusions from Sections 3 and 4 with numerical experiments. Specifically, we will compare the convergence rates of various errors for the spectral approximation, as well as for the forward and backward Euler–Maruyama schemes.

Although our theorems do not require assuming any particular structure in the decomposition components $L_{\ell,m}$ of our Lévy random field L, for the purpose of numerical simulation we

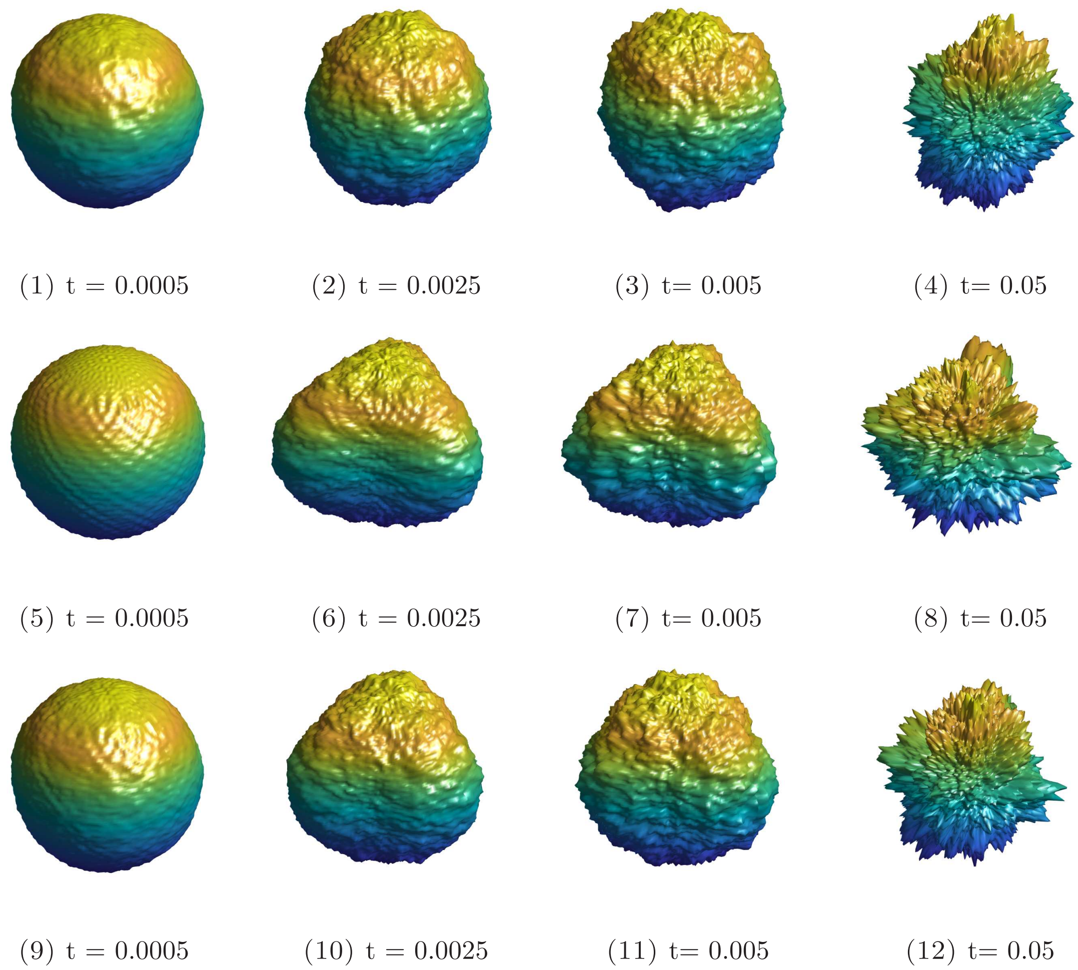
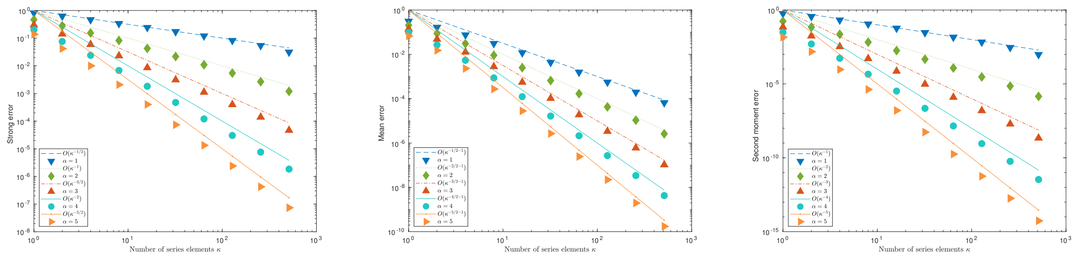


Figure 2. Sample of the noise evolving in time: (1)-(4) Wiener process, (5)-(8) Poisson process, (9)-(12) Sum of a Wiener and a Poisson process.

restrict ourselves to Assumption 2.2. Specifically, we assume that $L_{\ell,m} \sim \ell^{-\alpha} \hat{L}_{\ell,m}$ for a given α, where the processes $\hat{L}_{\ell,m}$ are independent and identically distributed. In Figure 1 we can see realizations of such Lévy processes L_t with Brownian components, Poisson component, a mixture of the two and a special case in which the same one–dimensional Poisson process drives all the components $(L_{\ell,m})_{\ell,m}$. More so, using the exponential transformation $e^{L(t)/\|L(t)\|}$ on the sphere $\mathbb{S}^2$, as in [7, 20, 22], we can see in Figure 2 realization of the evolving stochastic noise through time. For these realizations, we set $\alpha = 3$ and take for the combination of the Wiener and Poisson process the same samples as for the separate consideration.

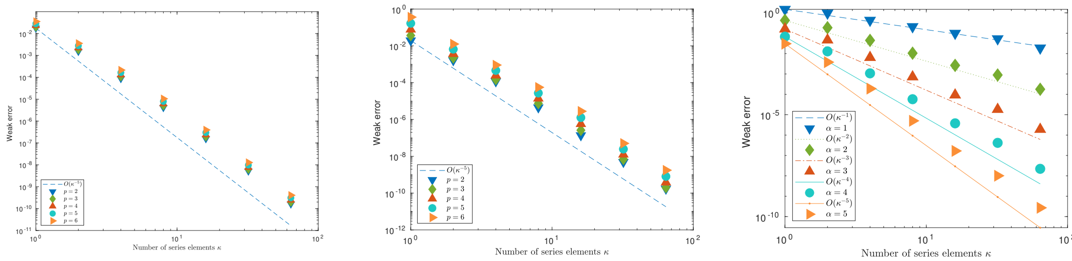
For the spectral approximation, we performed simulations using a reference solution at $\kappa = 2^{10}$ and unit time $T = 1$. The approximations were computed at different truncation levels $\kappa = 2^j$, with $j = 0, ..., 9$.

In Figure 3.(1), we present the strong error computed explicitly with initial condition $X^0 = 0$ and a Poisson random field satisfying Assumption 2.2 with independent Poisson components with intensity $\lambda = 1$. The rates obtained for $\alpha = 1, ..., 5$ confirm the theoretical results in Theorem 3.1. In Figure 3.(2), we observe the behavior of the expectation, which differs from the classical case of Wiener noise. As shown in [20], in that case only the exponential decay of the initial condition is observed, whereas here we see a decay in κ that depends on the regularity of the mean of the driving noise confirming Corollary 3.2. Similarly, as shown in Figure 3.(3), the convergence of the second moment validates the theoretical results of Corollary 3.2.



(1) Strong convergence. (2) Mean. (3) Second moment.

FIGURE 3. Convergence of the spectral method for different α.



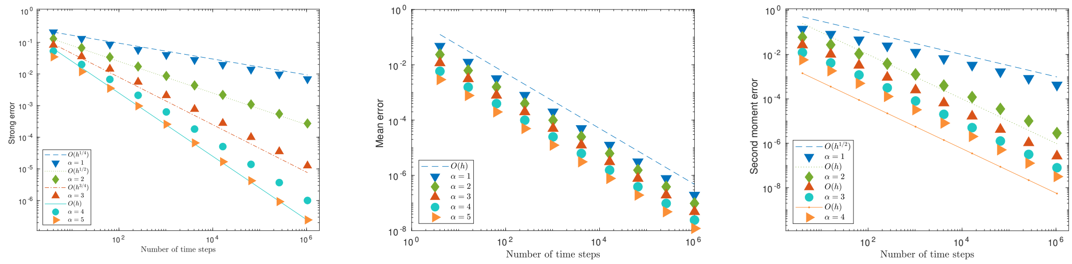
(1) Fixed $\alpha = 5$, Wiener. (2) Fixed $\alpha = 5$, Poisson. (3) Fixed $p = 3$, Poisson.

FIGURE 4. Spectral weak convergence for different processes L, p, and α: (1) Weak error of Wiener process with fixed $\alpha = 5$, (2) Weak error of Poisson process with fixed $\alpha = 5$, (3) Weak error of Poisson process with fixed $p = 3$.

To confirm the weak convergence rates in Theorem 3.4, we consider a reference solution at time $T = 1$ with $\kappa = 2^8$ with initial condition $X^0 = 0$. The test functions are given by $\varphi(X) = \|X\|^p_{L^2(\mathbb{S}^2)}$, where $p \in \{2, 3, 4, 5, 6\}$, and we perform Monte Carlo simulations with 20 samples.

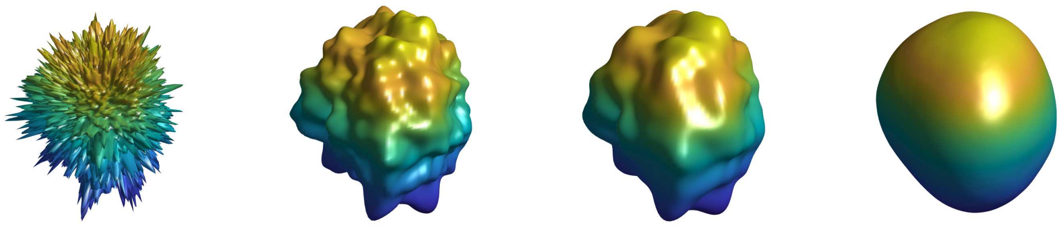
In Figure 4.(1), we show that the weak rates for a Wiener noise with a covariance decay proportional to ℓ^{-5} are independent of p as proven in Theorem 3.4.

In Figures 4.(2), we simulate the weak rates for a Lévy random field whose components are independent Poisson processes with decay rate proportional to $\ell^{-\alpha}$. We observe agreement of the decay with respect to the parameter α and no dependence on p on the rate of convergence as shown in the theorem. Furthermore, Figure 4.(3) illustrates for $p = 3$ the dependence of the decay on the regularity parameter α in agreement with the theoretical predictions. We notice that we have assumed a strong property in the simulation of the processes, i.e. the



(1) Strong error. (2) Mean error. (3) Second moment error.

FIGURE 5. Convergence of the Euler-Maruyama method for different α.



(1) t = 0 (2) t = 0.0025 (3) t= 0.005 (4) t= 0.05

FIGURE 6. Sample of the stochastic heat equation on the sphere evolving in time driven by the sum of a Wiener and a Poisson process.

independence of the components $(L_{\ell,m})$. This limitation, arising from the simplified numerical framework, and how to surpass it, will be addressed in future work.

Having verified the spectral convergence, we now turn to simulating the time discretization using the forward and backward Euler–Maruyama schemes. For this, we focus on the error between X^κ and $X^{(\kappa,h)}$. Simulations are performed on time grids with step size $h = 2^{-2m}$ for $m = 1, ..., 10$, coupled with $\kappa = 2^m$ to ensure stability for the forward Euler–Maruyama scheme and to satisfy the estimates for the backward scheme. As with the spectral approximations, we set $X^0 = 0$ to concentrate on the convergence relative to the noise smoothness parameter α. The Lévy noise is decomposed into independent Poisson processes satisfying Assumption 2.2.

The results for the backward Euler–Maruyama scheme in Figure 5.(1), computed explicitly, with a reference solution using $h = 2^{-14}$ and $\kappa = 2^7$, confirm the expected convergence rate of $h^{\min\{\alpha/4,1/2\}}$ from Theorem 4.2.

Figure 5.(2) shows the simulated convergence of the expectation for $\eta \in \{-1, -1/2, 0, 1, 2\}$. According to Theorem 4.3, we expect a convergence rate of order h in time. To minimize smoothing over time, we used $T = 0.05$, and, as proven in the theory, the simulations show $O(h)$ order of convergence. Finally, in Figure 5.(3) we see the weak convergence of the second moment for the Euler-Maruyama scheme. The setting is the same as the strong error, and it is computed explicitly. The results show the expected convergence rate $h^{\min\{\alpha/2,1\}}$ as in Theorem 4.3.

We conclude the section showing in Figure 6 the evolution of the solution in time $X(t)$, with a mixture of Brownian and Poisson components, projected with the exponential transformation $\exp\left(X(t)/\|X(t)\|\right)$ on the sphere. For this we take the same sample of the sum of a Wiener and a Poisson process as in Figure 2 and start from a rough initial condition, i.e. in $L^2(\Omega, H^-(\mathbb{S}^2))$.

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