ON QUANTUM SYMMETRIES OF GRAPHS

OLHA OSTROVSKA, VASYL OSTROVSKYI, AND LYUDMILA TUROWSKA

To memory of Yu.M.Berezansky

Abstract. Let G be a simple finite graph, and let $\mathcal{U}_G$ be the related quantum graph. We study the game algebra $C(\mathrm{Qut}(\mathcal{U}_G))$ of quantum automorphism of $\mathcal{U}_G$. We show that for the complete graph K_n, the algebra $C(\mathrm{Qut}(\mathcal{U}_{K_n}))$ is non-commutative already for all $n \geq 3$, in contrast to $C(\mathrm{Qut}(K_n)) = C(S_n^+)$. Moreover, we prove that for any graph G with $|V(G)| \geq 3$, the quantum graph $\mathcal{U}_G$ admits nonlocal symmetry, meaning that there exists a perfect quantum no-signaling correlation for the quantum automorphism game for $\mathcal{U}_G$ which is not local.

Contents

1. Introduction and preliminaries

Let G be a finite simple graph, meaning that it is undirected and has no multiple edges or loops. Write $V(G)$ for its set of vertices and $E(G)$ for the edge set, which is a set of unordered pairs of its vertices. For x, $y \in V(G)$ write $x \sim_G y$ or simply $x \sim y$, if G is clear from the context, if there is an edge between x and y and let $A_G = (a_{x,y})_{x,y\in V(G)}$, be the adjacency matrix of G: we have $a_{x,y} = 1$ if $x \sim_G y$, and $a_{x,y} = 0$ otherwise. Let G and H be two simple graphs. We say that G and H are isomorphic if there exists a bijection $\varphi : V(G) \to V(H)$ such that $\varphi(x) \sim_H \varphi(y)$ if and only if $x \sim_G y$. Clearly, in this case $|V(G)| = |V(H)|$. The map φ is called an isomorphism of graphs. An automorphism φ of a graph G is an isomorphism from G to itself. The automorphisms build a group with respect to composition known as the automorphism group of G.

Graph-based non-local games, and, in particular, isomorphism games have been important in the process of understanding entanglement in quantum mechanics, and first studied in [AMR+19]. In a (2-players) non-local game, a trusted classical verifier interacts with two players, Alice and Bob, who collaborate in attempt to win the game by sending a question/input to each of the two

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players and they must respond with an answer/output. In the case of isomorphism games for simple graphs G and H with $|V(G)| = |V(H)|$ the verifier sends two vertices x and y of the graph G and Alice and Bob must return vertices a and b of the graph H. In order to win, $\mathrm{rel}(x, y) = \mathrm{rel}(a, b)$ must hold where rel is the relationship of two vertices that is the equality of the vertices or adjacency relation or non of the previous two. A strategy for (G, H)-isomorphism is given by joint conditional probability distributions, called also correlations, $p(a, b|x, y)$, where $x, y \in V(G)$ and $a, b \in V(H)$ and where $p(a, b|x, y)$ is thought as the probability for Alice and Bob give the answer (a, b) upon receiving the questions (x, y). We call the strategy is winning or perfect if $p(a, b|x, y) = 0$, whenever $\mathrm{rel}(x, y) \neq \mathrm{rel}(a, b)$. There are several subclasses of strategies that arise from different models of quantum mechanics, related to the way the players compose their individual physical systems. A deterministic classical strategy is given by 01-valued correlations: the Alice's answer is completely determined by the question she received and similar for Bob; in particular, the strategy is given by two functions $\varphi_A : V(G) \to V(H)$, $\varphi_B : V(G) \to V(H)$. For such strategy to be perfect it is easy to see that $\varphi_A = \varphi_B$, where the latter would be the actual isomorphism between G and H. In general, classical (local) strategies, one allows Alice and Bob to pick a deterministic strategy at random. The existence of a perfect local strategy gives the isomorphism of the graphs. A quantum commuting strategy is given by a Hilbert space $\mathcal{H}$, a state ξ, interpreted as a unit vector in $\mathcal{H}$, on which local measurements are jointly performed and mutually commuting families of measurements (POVMs) $E_x = (E_{x,a})_{a\in V(H)}$, $x \in V(G)$, for Alice and $F_y = (F_{y,b})_{b\in V(H)}$, $y \in V(G)$, for Bob, that are used to measure ξ with the resulting outcomes given by

$$p(a, b|x, y) = \langle E_{x,a}F_{y,b}\xi, \xi\rangle.$$

The strategy is called quantum if $\mathcal{H}$ is a tensor product of finite-dimensional Hilbert spaces $\mathcal{H}_A \otimes \mathcal{H}_B$ and the product $E_{x,a}F_{y,b}$ is replaced by the tensor products of the POVMs defined now on the finite-dimensional spaces. Note that a quantum strategy can be obtained as a quantum commuting one by considering the POVMs $E_{x,a} \otimes 1_{\mathcal{H}_B}$ and $1_{\mathcal{H}_A} \otimes F_{y,b}$. Here a POVM on a Hilbert space $\mathcal{H}$ is a family of positive operators on $\mathcal{H}$ which sum up to the identity operator. Local, quantum and quantum commuting strategies are examples of so called no-signalling correlations (NS correlations for short). The isomorphism game is a bisynchronous game, implying that the winning strategy satisfies $p(a, b|x, x) = p(a, a|x, y) = 0$ for all $x \neq y$, $a \neq b$, and by [PR21] the winning strategy of the quantum commuting (qc) type is given by a trace on the associated game algebra. Namely, for the isomorphism game $\mathrm{Iso}(G, H)$, the game algebra is the universal C^*-algebra defined by generators and relations as follows:

$$\begin{aligned}\mathcal{A}(\mathrm{Iso}(G, H)) = C^*\Big\langle e_{x,a}, x \in V(G), a \in V(H) : e_{x,a}^2 = e_{x,a} = e_{x,a}^*,\\ \sum_x e_{x,a} = \sum_a e_{x,a} = 1, \quad (A_G \otimes 1)u = u(A_H \otimes 1)\Big\rangle,\end{aligned}$$

where $u = (e_{x,a})_{x\in V(G),a\in V(H)}$. Any winning qc-strategy is then given by

$$p(a, b|x, y) = \tau(e_{x,a}e_{b,y}),$$

where τ is a tracial state on $\mathcal{A}(\mathrm{Iso}(G, H))$. It is a local (quantum) strategy if τ is factorized through an abelian (finite-dimensional) C^*-algebra $\mathcal{A}$, that is $\tau = \tau_{\mathcal{A}} \circ \pi$, where $\pi : \mathcal{A}(\mathrm{Iso}(G, H)) \to \mathcal{A}$ is a $*$-homomorphism and $\tau_{\mathcal{A}}$ is a state (trace) on $\mathcal{A}$.

In what follows, we will write $C(\mathrm{Qut}(G))$ for $\mathcal{A}(\mathrm{Iso}(G, G))$, the quantum automorphism group; the notion is due to Banica, 2005 [Ban05]. One says that G has quantum symmetry if $C(\mathrm{Qut}(G))$ is not commutative.

If K_N is the complete graph on N vertices, $C(\mathrm{Qut}(K_N)) = C(S_N^+)$, called the free quantum permutation group, introduced by Wang [Wan98] and studied intensively from various perspectives (see [BBC07, LMR20, Web23, Voi17] and references therein). It is the universal C^*-algebra generated by entries of $N \times N$ magic unitary, that is projections $p_{i,j}$, $i, j = 1, \dots N$, such that $(p_{i,j})_{i,j=1}^N$, is unitary, equivalently, $\sum_i p_{i,j} = \sum_j p_{i,j} = 1$. It is known that $C(S_N^+)$ is not commutative iff $N \geq 4$, see [Voi17].

While classically, in the course of a non-local game, Alice and Bob's behaviour is described by a family $p = (p(a, b|x, y))_{(a,b,x,y) \in A \times B \times X \times Y}$ of conditional probability distributions, which can, in a canonical way, be viewed as a noisy information channel $\mathcal{N} : X \times Y \to A \times B$ with well-defined marginal channels, in the quantum setting, one replaces the classical state spaces X, Y, A, B by their quantum analogues (i.e. the Hilbert spaces $\mathbb{C}^{|X|}, \mathbb{C}^{|Y|}$, etc.), and the classical channel $\mathcal{N} : X \times Y \to A \times B$ by a *quantum channel* $\Gamma : M_X \otimes M_Y \to M_A \otimes M_B$, where, for any finite set Z, we have let $M_Z = B(\mathbb{C}^{|Z|})$ be the matrix algebra of linear maps on $\mathbb{C}^{|Z|}$. Quantum no-signalling (QNS) correlations have been introduced by Duan and Winter in [DW16], and their different classes were introduced and studied in [TT24]. An analogue of bisynchronous correlations, called QNS-bicorrelations, was proposed in [BHTT24], as no-signalling QNS correlations satisfying certain concurrency conditions; they are characterised in terms of tracial states on the C^*-algebra $C(\mathbb{P}\mathcal{U}_X^+)$ of functions on *projective free unitary quantum group*. Recall that the C^*-algebra of the free unitary quantum group $C(\mathcal{U}_X^+)$ is the universal unital C^*-algebra generated by the entries $u_{x,a}$ of an $|X| \times |X|$ bi-unitary matrix $U = (u_{x,a})_{x,a}$ which was introduced by Wang, [Wan98], and studied by Banica in [Ban97], and the C^*-algebra $C(\mathbb{P}\mathcal{U}_X^+)$ is the C^*-subalgebra of $C(\mathcal{U}_X^+)$, generated by length two words of the form $u_{x,a}^* u_{x',a'}$, $x, x', a, a' \in X$, which is the quantum automorphism group $\mathrm{Aut}^+(M_X)$ of Wang due to a result by Banica, [Ban99].

In [BHTT24] bisynchronous bicorrelations were used to study quantum graph isomorphism games, where by a quantum graph we mean a symmetric skew subspace $\mathcal{U} \subset \mathbb{C}^X \times \mathbb{C}^X$ (see [BHTT24, Definition 7.1]). To a classical graph G one can associate naturally the quantum graph $\mathcal{U}_G = \mathrm{span}\{e_x \otimes e_y : x \sim y\}$, where $\{e_x\}_{x \in X}$ is the natural orthogonal basis in $\mathbb{C}^X$, "embedding" in this way classical graphs into the class of quantum graphs. Similar to the classical case one defines quantum isomorphisms between quantum graphs in terms of perfect QNS strategies for a suitable quantum graph isomorphism game. We refer the reader to [BHTT24] for the details. The game algebra whose tracial states encode such perfect strategies is a quotient of $C(\mathbb{P}\mathcal{U}_X^+)$. For the classical graphs G, H, $\mathcal{A}(\mathrm{Iso}(\mathcal{U}_G, \mathcal{U}_H))$ is the C^*-subalgebra generated by $u_{a,x}^* u_{b,y}$, $a, b, x, y \in X = V(G) = V(H)$ of the universal C^*-algebra with generators $u_{a,x}$, $a, x \in X$, and relations such that $U = (u_{x,a})_{x,a \in X}$ is a bi-unitary and $u_{a,x} u_{b,y}^* = 0$ whenever $x \sim y$ and $a \not\sim b$ or $x \not\sim y$ and $a \sim b$. Note, that by some abuse of terminology, in the paper we will always refer to this algebra as the universal C^*-algebra generated by the words $u_{a,x}^* u_{b,y}$, $a, b, x, y \in X$, where $U = (u_{x,a})_{x,a \in X}$ is a bi-unitary, whose entries satisfy the given additional relations. It was proved in [BHTT24] that there exists a local perfect QNS strategy for the quantum $(\mathcal{U}_G, \mathcal{U}_H)$-isomorphism game if and only if G and H are isomorphic, and hence there is a local strategy for (G, H)-isomorphism game. However, it is unknown whether genuinely quantum (quantum/quantum commuting QNS) strategies will have an advantage over quantum/quantum commuting NS correlations to play the (G, H)-isomorphism game[1]. Treating classical graphs as quantum, one may ask whether they have a more rich quantum symmetries and this is the goal of our paper. Namely, in this work we study the algebra $C(\mathrm{Qut}(\mathcal{U}_G)) := \mathcal{A}(\mathrm{Iso}(\mathcal{U}_G, \mathcal{U}_G))$ of the quantum automorphism game of $\mathcal{U}_G$. We show that for the complete graph K_n, the algebra $C(\mathrm{Qut}(\mathcal{U}_{K_n}))$ is non-commutative already for all $n \geq 3$, in contrast to $C(\mathrm{Qut}(K_n)) = C(S_n^+)$. Moreover, there is a surjective homomorphism from $C(\mathrm{Qut}(\mathcal{U}_{K_n}))$ to $C^*(\mathbb{F}_{n-1})$, the C^*-algebra of the free group on $n-1$ generators. In section 3.3 we prove that for any graph G with $|V(G)| \geq 3$,

[1] After the paper was completed, D. Roberson informed the third author, in private communication, that there exist classical graphs G and H such that $\mathcal{U}_G$ and $\mathcal{U}_H$ are quantum isomorphic yet there is no winning quantum no-signalling correlation for the isomorphism game $\mathrm{Iso}(G, H)$.

the corresponding quantum graph $\mathcal{U}_G$ admits nonlocal symmetry, meaning that there exists a perfect QNS correlation for the quantum automorphism game for $\mathcal{U}_G$ which is not local. The similar question for classical graphs was studied by Roberson and Schmidt in [RS21], where they proved among other results, that the complete graph K_n exhibit nonlocal symmetry if and only if $n \geq 5$. In section 3.2 we also present a partition procedure that allows to refine the description of quantum isomorphisms of quantum graphs of type $\mathcal{U}_G$.

For a Hilbert space H we write $B(H)$ for the space of all bounded operators. As usual, $M_n(\mathbb{C})$ or simply M_n is the space of $n \times n$ complex matrices. If A is a C^*-algebra $M_n(A)$ is the C^*-algebra of matrices with entries in A. We write $\mathbb{N}_n$ for the set $\{1, \ldots, n\}$.

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2. Game C^*-algebra for complete graphs

2.1. **Structure of the game algebra $C(\mathrm{Qut}(\mathcal{U}_{K_n}))$.** Recall that if $\mathcal{A}$ is a C^*-algebra, a block matrix $U = (u_{i,j})_{i,j=1}^n \in M_n(\mathcal{A})$ is called a bi-unitary if U and $U^t = (u_{j,i})_{i,j=1}^n$ are unitary.

Let K_n be the complete graph on n vertices, $n \geq 1$. By definition, the algebra $C(\mathrm{Qut}(\mathcal{U}_{K_n}))$ is the universal C^*-algebra generated by $u_{i,j}^* u_{k,l}$, $i, j, k, l = 1, \ldots, n$, where $U = (u_{i,j})_{i,j=1}^n$ is a bi-unitary, such that $u_{i,k} u_{i,l}^* = 0$ for $k \neq l$ and $u_{i,k} u_{j,k}^* = 0$, when $i \neq j$. The next theorem gives an embedding of $C(\mathrm{Qut}(K_n))$ into $C(\mathrm{Qut}(\mathcal{U}_{K_n}))$.

Theorem 1. *For each $n \in \mathbb{N}$, $C(\mathrm{Qut}(\mathcal{U}_{K_n}))$ is the universal C^*-algebra generated by $u_{i,j}^* u_{k,l}$, $i, j, k, l \in \mathbb{N}_n$ where each $u_{i,j}$ is a partial isometry and $U = (u_{i,j})_{i,j=1}^n$ is a bi-unitary. The map $\varphi : C(S_n^+) \to C(\mathrm{Qut}(\mathcal{U}_{K_n}))$, $p_{i,k} \to u_{i,k}^* u_{i,k}$, $i, k \in \mathbb{N}_n$, is an injective $*$-homomorphism.*

Proof. If $u_{k,l}^* u_{i,j}$, $i, j, k, , l \in \mathbb{N}_n$, are the generators of $C(\mathrm{Qut}(\mathcal{U}_{K_n}))$, then $u_{i,j} u_{i,k}^* = 0$ if $j \neq k$. Therefore, for all i, k,

$$u_{i,k}^* = \sum_j u_{i,j}^* u_{i,j} u_{i,k}^* = \sum_{j \neq k} u_{i,j}^* u_{i,j} u_{i,k}^* + u_{i,k}^* u_{i,k} u_{i,k}^* = u_{i,k}^* u_{i,k} u_{i,k}^*,$$

i.e. each $u_{i,k}$ is a partial isometry. Conversely, if $U = (u_{i,j})_{i,j=1}^n$ is a bi-unitary such that each $u_{i,j}$ is a partial isometry we have $u_{i,k} u_{j,k}^* = 0$ and $u_{k,i} u_{k,j}^* = 0$ if $i \neq j$. In fact, in this case each $u_{i,k}^* u_{i,k}$ is a projection and $\sum_i u_{i,k}^* u_{i,k} = 1$ showing that the projections are orthogonal and

$$u_{i,k} u_{j,k}^* = u_{i,k} u_{i,k}^* u_{i,k} u_{j,k}^* u_{j,k} u_{j,k}^* = 0, \quad i \neq j.$$

Similarly, $u_{k,i} u_{k,j}^* = 0$ if $i \neq j$. Therefore we obtain the first statement. Remark also that as $\sum_k u_{i,k}^* u_{j,k} = 0$ for $i \neq j$,

$$u_{i,m} u_{i,m}^* u_{j,m} = u_{i,m} \sum_k u_{i,k}^* u_{j,k} = 0,$$

showing that also $u_{i,m}^* u_{j,m} = 0$ if $i \neq j$, and similarly, $u_{m,i}^* u_{m,j} = 0$.

Clearly, the map φ is a $*$-homomorphism. It is injective. In fact, letting $\mathcal{U}_{n,p.i.}^+$ be the universal C^*-algebra generated by partial isometries $u_{i,j}$ such that $U = (u_{i,j})_{i,j=1}^n$ is a bi-unitary and considering $\rho : \mathcal{U}_{n,p.i.}^+ \to C(S_n^+)$, $u_{i,j} \mapsto p_{i,j}$, we obtain $\rho(\varphi(a)) = a$, $a \in C(S_n^+)$. $\square$

2.2. **Properties of $C(\mathrm{Qut}(\mathcal{U}_{K_2}))$ and $C(\mathrm{Qut}(\mathcal{U}_{K_3}))$.** The C^*-algebra $C(S_2^+)$ is generated by projections $p_{1,1},\ p_{1,2},\ p_{2,1},\ p_{2,2}$ such that $p_{1,1}+p_{1,2}=1,\ p_{2,1}+p_{2,2}=1,\ p_{1,1}+p_{2,1}=1,$ $p_{1,2}+p_{2,2}=1$ showing that $p_{1,2}=p_{2,1}$, $p_{1,1}=p_{2,2}$ and hence $C(S_2^+)$ is the universal C^*-algebra generated by one projection.

Proposition 1. *$C(\mathrm{Qut}(\mathcal{U}_{K_2}))$ is commutative and $C(\mathrm{Qut}(\mathcal{U}_{K_2}))\simeq C(\mathbb{T})\oplus C(\mathbb{T})$.*

Proof. Let $u_{i,j},\ i,j=1,2$, be partial isometries such that $u_{i,j}^*u_{k,l},\ i,j,k,l=1,2$, generate $C(\mathrm{Qut}(\mathcal{U}_{K_2}))$. The only possibly non-zero products are $u_{1,1}^*u_{1,1}=u_{2,2}^*u_{2,2}$, $u_{1,1}^*u_{2,2}$, $u_{2,2}^*u_{1,1}$, $u_{1,2}^*u_{2,1}$, $u_{2,1}^*u_{1,2}$, $u_{1,2}^*u_{1,2}=1-u_{1,1}^*u_{1,1}=u_{2,1}^*u_{2,1}$. It is a routine to check that they commute.

Let $\varphi: C(\mathcal{U}_2^+)\to C(\mathbb{T})\oplus C(\mathbb{T})$,

$$\varphi(u_{1,1})=(1,0),\quad \varphi(u_{2,2})=(z,0),\quad \varphi(u_{1,2})=(0,1),\quad \varphi(u_{2,1})=(0,z).$$

It is easy to see that φ determines a surjective $*$-homomorphism of of $C(\mathrm{Qut}(\mathcal{U}_{K_2}))$ onto $C(\mathbb{T})\oplus C(\mathbb{T})$. As any one-dimensional representation of $C(\mathrm{Qut}(\mathcal{U}_{K_2}))$ is given by $\pi_z(u_{1,1}^*u_{1,1})=1$, $\pi_z(u_{1,1}^*u_{2,2})=z$, and $\pi_z(u_{1,2}^*u_{1,2})=\pi_z(u_{2,1}^*u_{2,1})=0$ or $\rho_z(u_{1,2}^*u_{1,2})=1$, $\rho_z(u_{1,2}^*u_{2,1})=z$, and $\rho_z(u_{1,1}^*u_{1,1})=\rho_z(u_{2,2}^*u_{2,2})=0$, $z\in\mathbb{T}$, we obtain that the $*$-homomorphism is a $*$-isomorphism. □

As $C(S_n^+)$ is non-commutative for $n\geq 4$ we have already that $C(\mathrm{Qut}(\mathcal{U}_{K_n}))$ is non-commutative for the same values of n, as $C(S_n^+)$ sits in the latter algebra.

We will see that $C(\mathrm{Qut}(\mathcal{U}_{K_3}))$ is not commutative. But first we will establish some structural properties of the algebra $C(\mathrm{Qut}(\mathcal{U}_{K_n}))$.

Lemma 1. *Let $U=(u_{a,x})_{a,x=1}^n$ be a bi-unitary matrix which determines a representation of $C(\mathrm{Qut}(\mathcal{U}_{K_n}))$. There exists a block-diagonal unitary matrix $V=\mathrm{diag}(v_1,\dots,v_n)$ such that VU is a projection permutation matrix, i.e., $(v_au_{a,x})_{a,x=1}^n$ is a family of projections which forms a representation of $C(S_n^+)$.*

Proof. Put

$$v_a=\sum_{x=1}^n u_{a,x}^*,\quad a=1,\dots,n.$$

We claim that v_a, $a\in\mathbb{N}_n$, form the desired family. Indeed, as shown in the proof of Theorem 1, $u_{a,x}u_{a,y}^*=u_{a,x}^*u_{a,y}=0$, $x\not\sim y$, then we have

$$v_a^*v_a=\sum_{x,y=1}^n u_{a,x}u_{a,y}^*=\sum_{x=1}^n u_{a,x}u_{a,x}^*=I,$$
$$v_av_a^*=\sum_{x,y=1}^n u_{a,x}^*u_{a,y}=\sum_{x=1}^n u_{a,x}^*u_{a,x}=I,$$

therefore, v_a is unitary. Further, we have

$$v_au_{a,x}=\sum_{y=1}^n u_{a,y}^*u_{a,x}=u_{a,x}^*u_{a,x}=:p_{a,x},$$

which, as shown above, is a projection, and moreover, these projections form a representation of $C(S_n^+)$. □

Theorem 2. *The algebra $C(\mathrm{Qut}(\mathcal{U}_{K_n}))$ is isomorphic to the universal C^*-algebra $\mathcal{A}$ generated by projections $p_{a,x}$, $a,x\in\mathbb{N}_n$, and unitaries u_i, $i\in\mathbb{N}_{n-1}$, such that $\sum_{a=1}^n p_{a,x}=\sum_{x=1}^n p_{a,x}=1$ and $p_{b,x}u_b\dots u_{a-1}p_{a,x}=0$ for $b<a$.*

Proof. Let $v_a = \sum_{x=1}^{n} u_{a,x}^*$, $a \in \mathbb{N}_n$ and set $\tilde{p}_{a,x} = v_a u_{a,x}$, $\tilde{u}_a = v_a v_{a+1}^*$, $a \in \mathbb{N}_{n-1}$. Then $\tilde{p}_{a,x}$ and $\tilde{u}_a \in C(\mathrm{Qut}(\mathcal{U}_{K_n}))$. We have $u_{a,x}^* u_{b,y} = \tilde{p}_{a,x} v_a v_b^* \tilde{p}_{b,y}$, $v_a v_b^* = \tilde{u}_a \tilde{u}_{a+1} \dots \tilde{u}_{b-1}$, if $b > a$, and $v_a v_b^* = (v_b v_a^*)^*$, otherwise, showing that $\tilde{p}_{a,x}$ and $\tilde{u}_a$ generate $C(\mathrm{Qut}(\mathcal{U}_{K_n}))$. Moreover, from the proof of the previous lemma we get that $\tilde{u}_a$'s are unitary, $\tilde{p} = (\tilde{p}_{a,x})_{a,x=1}^n$ is a projection permutation matrix and

$$\tilde{p}_{b,x} \tilde{u}_b \dots \tilde{u}_{a-1} \tilde{p}_{a,x} = u_{b,x}^* u_{a,x} = 0$$

for $b < a$, as follows from the proof of Theorem 1. Therefore, we get a $*$-homomorphism φ from $C(\mathrm{Qut}(\mathcal{U}_{K_n}))$ to $\mathcal{A}$, given by $\varphi(\tilde{p}_{a,x}) = p_{a,x}$, $\varphi(\tilde{u}_a) = u_a$.

For the generators u_a, $a \in \mathbb{N}_{n-1}$, of $\mathcal{A}$, let $v_1 = 1$, $v_a = u_{a-1}^* \dots u_1^*$, $a > 1$. Clearly, v_a and $p_{a,x}$ generate $\mathcal{A}$. Let $V = \mathrm{diag}(v_1, \dots, v_a)$. Then V is a unitary diagonal block matrix such that $V^* P = (v_a^* p_{a,x})_{a,x=1}^n$ is a bi-unitary block matrix:

$$\begin{aligned}
&\sum_{a=1}^{n} p_{a,y} v_a v_a^* p_{a,x} = \sum_{a=1}^{n} p_{a,y} p_{a,x} = \delta_{x,y} I \\
&\sum_{x=1}^{n} v_a^* p_{a,x} p_{b,x} v_b = v_a^* \Big(\sum_{x=1}^{n} p_{a,x} p_{b,x} \Big) v_b = \delta_{a,b} I \\
&\sum_{x=1}^{n} p_{b,x} v_b v_a^* p_{a,x} = \sum_{x=1}^{n} p_{b,x} u_b \dots u_{a-1} p_{a,x} = \delta_{a,b} I, \quad b < a, \\
&\sum_{a=1}^{n} v_a^* p_{a,x} p_{a,y} v_a = \delta_{x,y} I.
\end{aligned}$$

Moreover, using $\sum_{y=1}^{n} p_{b,y} = 1$ for all $b \in \mathbb{N}_n$, it is easy to see that the elements $p_{b,y} u_b \dots u_{a-1} p_{a,x}$, $b \le a$ (where in the case $b = a$ we think of the latter to be equal to $p_{a,y} p_{a,x}$) generate $\mathcal{A}$. Therefore, we obtain a $*$-homomorphism $\psi : \mathcal{A} \to C(\mathrm{Qut}(\mathcal{U}_{K_n}))$, $\psi(p_{b,y} u_b \dots u_{a-1} p_{a,x}) = u_{b,y}^* u_{a,x}$.

We have

$$\psi(\varphi(\tilde{u}_a)) = \psi(u_a) = \psi\Big(\sum_{x,y=1}^{n} p_{a,y} u_a p_{a+1,x} \Big) = \sum_{x,y=1}^{n} u_{a,y}^* u_{a+1,x} = v_a v_{a+1}^* = \tilde{u}_a,$$

and

$$\psi(\varphi(\tilde{p}_{a,x})) = \psi(p_{a,x}) = u_{a,x}^* u_{a,x} = \tilde{p}_{a,x},$$

i.e. $\psi \circ \varphi = \mathrm{id}$. Similarly, we obtain $\varphi \circ \psi = \mathrm{id}$. ☐

Let $C^*(\mathbb{F}_n)$ be the C^*-algebra of the free group $\mathbb{F}_n$ with n generators. Write w_i, $1 \le i \le n$, for the canonical generators of $C^*(\mathbb{F}_n)$.

Corollary 1. *The map* $\varphi\colon C(\mathrm{Qut}(\mathcal{U}_{K_n})) \to C^*(\mathbb{F}_{n-1})$, $p_{a,x} \mapsto \delta_{a,x} 1$, $u_i \mapsto w_i$, $a, x \in \mathbb{N}_n$, $i \in \mathbb{N}_{n-1}$, *is a surjective* $*$*-homomorphism.*

Proof. As $\sum_{x=1}^{n} \varphi(p_{a,x}) = \sum_{a=1}^{n} \varphi(p_{a,x}) = 1$ and

$$\varphi(p_{b,x}) \varphi(u_b \dots u_{a-1}) \varphi(p_{a,x}) = 0$$

for $b \ne a$, the statement now follows from the theorem. ☐

Corollary 2. *The game algebra* $C(\mathrm{Qut}(\mathcal{U}_{K_n}))$ *is non-commutative for all* $n \ge 3$.

Next example presents some other in nature $*$-representations of $C(\mathrm{Qut}(\mathcal{U}_{K_3}))$ showing also that the category of $*$-representations of $C(\mathrm{Qut}(\mathcal{U}_{K_3}))$ is at least as complicated as the category of $*$-representations of the free product of the Cuntz algebras $\mathcal{O}_2$.

Example 1. Let H be a Hilbert space, and let isometries t_1, t_2, s_1, s_2 in $B(H)$ satisfy the Cuntz algebra relations

$$t_1t_1^* + t_2t_2^* = I, \quad s_1s_1^* + s_2s_2^* = I,$$

i.e., determine a representation of the free product $\mathcal{O}_2 * \mathcal{O}_2$, amalgamated over the units. In the space $\mathcal{H} = H \oplus H \oplus H$, consider the operators

$$p_{1,1} = p_{2,3} = p_{3,2} = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$p_{1,2} = p_{2,1} = p_{3,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$p_{1,3} = p_{2,2} = p_{3,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix},$$

and

$$u_1 = \begin{pmatrix} t_1^* & 0 & 0 \\ s_1t_2^* & s_2s_1^* & 0 \\ 0 & t_2s_2^* & t_1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} t_1t_2^* & 0 & t_2t_1^* \\ s_2t_1^* & s_1 & 0 \\ 0 & 0 & t_2^* \end{pmatrix}.$$

It is a straightforward verification that $p_{a,x}$, $a, x = 1, 2, 3$, are projections for which

$$\sum_{b=1}^{3} p_{b,x} = \sum_{y=1}^{3} p_{a,y} = I, \quad a, x = 1, 2, 3,$$

u_1 and u_2 are unitary, and

$$p_{1,k}u_1p_{2,k} = p_{2,k}u_2p_{3,k} = p_{1,k}u_1u_2p_{3,k} = 0, \quad k = 1, 2, 3,$$

and therefore determine a $*$-representation of the category of $*$-representations of $C(\mathrm{Qut}(\mathcal{U}_{K_3}))$, by Theorem 2. Moreover, the representation of $C(\mathrm{Qut}(\mathcal{U}_{K_3}))$ given by u_1, u_2, $p_{a,x}$, $a, x = 1, 2, 3$, is irreducible iff the corresponding representation of $\mathcal{O}_2 * \mathcal{O}_2$ is irreducible, and two such representations of $C(\mathrm{Qut}(\mathcal{U}_{K_3}))$ are unitarily equivalent iff the corresponding representations of $\mathcal{O}_2 * \mathcal{O}_2$ are unitarily equivalent.

Indeed, let $\tilde{t}_1, \tilde{t}_2, \tilde{s}_1, \tilde{s}_2$ be another representation of $\mathcal{O}_2 * \mathcal{O}_2$ in $\tilde{H}$, let $\tilde{p}_{a,x}$, $a, x = 1, 2, 3$ be the corresponding projections in $\tilde{\mathcal{H}} = \tilde{H} \oplus \tilde{H} \oplus \tilde{H}$, and let $\tilde{u}_1, \tilde{u}_2$ be a pair of the corresponding block unitaries. Let C be a bounded operator from $\tilde{\mathcal{H}}$ to $\mathcal{H}$ intertwining u_1, u_2, $p_{a,x}$, $a, x = 1, 2, 3$, and $\tilde{u}_1, \tilde{u}_2$, $\tilde{p}_{a,x}$, $a, x = 1, 2, 3$. Then $C = \mathrm{diag}(C_1, C_2, C_3)$, $C_1, C_2, C_3 \colon \tilde{H} \to H$, and relations

$$u_1C = C\tilde{u}_1, \quad u_1^*C = C\tilde{u}_1^*, \quad u_2C = C\tilde{u}_2, \quad u_2^*C = C\tilde{u}_2^*,$$

are equivalent to the following set of relations

$$\begin{gathered}
t_1C_1=C_1\tilde t_1,\quad t_1^*C_1=C_1\tilde t_1^*,\quad t_2C_3=C_3\tilde t_2,\quad t_2^*C_3=C_3\tilde t_2^*,\\
t_1C_3=C_3\tilde t_1,\quad t_1^*C_3=C_3\tilde t_1^*,\quad s_1C_2=C_2\tilde s_1,\quad s_1^*C_2=C_2\tilde s_1^*,\\
t_2t_1^*C_1=C_1\tilde t_2\tilde t_1^*,\quad t_1t_2^*C_1=C_1\tilde t_1\tilde t_2^*,\\
t_1s_2^*C_2=C_1\tilde t_1\tilde s_2^*,\quad s_2t_1^*C_1=C_2\tilde s_2\tilde t_1^*,\\
t_1t_2^*C_1=C_3\tilde t_1\tilde t_2^*,\quad t_2t_1^*C_3=C_1\tilde t_2\tilde t_1^*,\\
s_1t_2^*C_1=C_2\tilde s_1\tilde t_2^*,\quad t_2s_1^*C_2=C_1\tilde t_2\tilde s_1^*,\\
s_2s_1^*C_2=C_2\tilde s_2\tilde s_1^*,\quad s_1s_2^*C_2=C_2\tilde s_1\tilde s_2^*,\\
s_2t_2^*C_3=C_2\tilde s_2\tilde t_2^*,\quad t_2s_2^*C_2=C_3\tilde t_2\tilde s_2^*,
\end{gathered}$$

which imply

$$\begin{gathered}
t_2C_1=C_1\tilde t_2,\quad t_2^*C_1=C_1\tilde t_2^*,\quad t_2^*C_1=C_3\tilde t_2^*,\quad t_2C_3=C_1\tilde t_2,\\
s_2^*C_2=C_1\tilde s_2^*,\quad s_2C_1=C_2\tilde s_2,\quad s_1C_1=C_2\tilde s_1,\quad s_1^*C_2=C_1\tilde s_1^*,\\
s_2C_2=C_2\tilde s_2,\quad s_2^*C_2=C_2\tilde s_2^*,\quad t_2^*C_3=C_2\tilde t_2^*,\quad t_2C_2=C_3\tilde t_2,
\end{gathered}$$

so that $C_1=C_2=C_3$ intertwines t_1,t_2,s_1,s_2 and $\tilde t_1,\tilde t_2,\tilde s_1,\tilde s_2$. This proves the statements about irreducibility and unitary equivalence above.

3. Nonlocal symmetries

3.1. **Nonlocal symmetries for K_3.** We define nonlocal symmetry of a quantum graph in a similar way as it is done for classical graphs. We say a quantum graph $\mathcal{U}$ admits nonlocal symmetry if there is a perfect QNS correlations for the automorphism game $\mathrm{Iso}(\mathcal{U},\mathcal{U})$ of $\mathcal{U}$, which is not local. We refer the reader to [BHTT24] for the definition of local, quantum and quantum commuting QNS strategies. In what follows we will use the following characterization of perfect QNS correlation for isomorphism game, see [BHTT24, Theorem 7.10].

Theorem 3. *Let G, H be simple undirected graphs such that $|V(G)|=|V(H)|=n$. The following are equivalent for a QNS bicorrelation $\Gamma: M_n\otimes M_n\to M_n\otimes M_n$:*

(i) *Γ is a perfect quantum commuting (resp. quantum/local) strategy for the isomorphism game Iso($\mathcal{U}_G$,$\mathcal{U}_H$);*
(ii) *there exists a trace τ (resp. a trace τ that factors through a finite dimensional/abelian *-representation) of $\mathcal{A}$(Iso($\mathcal{U}_G$,$\mathcal{U}_H$)) such that*

$$\Gamma(\epsilon_{x,x'}\otimes\epsilon_{y,y'})=\left(\tau(u_{a,x}^*u_{a',x'}u_{b',y'}^*u_{b,y})\right)_{a,a',b,b'\in\mathbb{N}_n},\quad x,x',y,y'\in\mathbb{N}_n. \tag{1}$$

We will say that the quantum graphs $\mathcal{U}_G,\mathcal{U}_H$ are qc-isomorphic (q-isomorphic, loc-isomorphic) and write $\mathcal{U}_G\simeq_{\rm qc}\mathcal{U}_H$ (resp. $\mathcal{U}_G\simeq_{\rm q}\mathcal{U}_H$, $\mathcal{U}_G\simeq_{\rm loc}\mathcal{U}_H$) if there exists a perfect quantum commuting (resp. quantum, local) strategy for the isomorphism game $\mathrm{Iso}(\mathcal{U}_G,\mathcal{U}_H)$. Note that by [BHTT24, Proposition 7.8], $\mathcal{U}_G$ and $\mathcal{U}_H$ are loc-isomorphic if and only if G and H are isomorphic.

Proposition 2. *$\mathcal{U}_{K_3}$ admits nonlocal symmetry.*

Proof. For the generators $p_{x,a}$, $a, x = 1, 2, 3$, u_1, u_2 of $C(\mathrm{Qut}(\mathcal{U}_{K_3}))$ from Theorem 2, define

$$\phi(u_1) = \phi(u_2) = V = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\phi(p_{1,1}) = \phi(p_{2,2}) = \phi(p_{3,3}) = P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\phi(p_{1,3}) = \phi(p_{2,1}) = \phi(p_{3,2}) = P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\phi(p_{1,2}) = \phi(p_{2,3}) = \phi(p_{3,1}) = P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then ϕ can be extended to a homomorphism

$$\phi\colon C(\mathrm{Qut}(\mathcal{U}_{K_3})) \to M_3(\mathbb{C}).$$

Indeed, since P_1, P_2, P_3 are projections and V is unitary, it is sufficient to verify that

$$\sum_{a=1}^{3} \phi(p_{x,a}) = \sum_{x=1}^{3} \phi(p_{x,a}) = I,$$

$$\phi(p_{1,a})V\phi(p_{2,a}) = \phi(p_{2,a})V\phi(p_{3,a}) = \phi(p_{1,a})V^2\phi(p_{3,a}) = 0, \quad a = 1, 2, 3.$$

But it is a straightforward verification that these relations hold. Moreover, since

$$P_jVP_k \neq 0 \iff (j,k) \in \{(1,3),(2,1),(3,2)\},$$
$$P_jV^2P_k \neq 0 \iff (j,k) \in \{(1,2),(2,3),(3,1)\},$$

we see that

$$\begin{aligned}
\phi(u_{1,1}^*u_{2,3}) &= \phi(p_{1,1})\phi(u_1)\phi(p_{2,3}) = P_1VP_3 = E_{13},\\
\phi(u_{1,2}^*u_{2,1}) &= \phi(p_{1,2})\phi(u_1)\phi(p_{2,1}) = P_3VP_2 = E_{32},\\
\phi(u_{1,3}^*u_{2,2}) &= \phi(p_{1,3})\phi(u_1)\phi(p_{2,2}) = P_2VP_1 = E_{21},\\
\phi(u_{2,1}^*u_{3,3}) &= \phi(p_{2,1})\phi(u_2)\phi(p_{3,3}) = P_2VP_1 = E_{21},\\
\phi(u_{2,2}^*u_{3,1}) &= \phi(p_{2,2})\phi(u_2)\phi(p_{3,1}) = P_1VP_3 = E_{13},\\
\phi(u_{2,3}^*u_{3,2}) &= \phi(p_{2,3})\phi(u_2)\phi(p_{3,2}) = P_3VP_2 = E_{32},\\
\phi(u_{1,1}^*u_{3,2}) &= \phi(p_{1,1})\phi(u_1u_2)\phi(p_{3,2}) = P_1V^2P_2 = E_{12},\\
\phi(u_{1,2}^*u_{3,3}) &= \phi(p_{1,2})\phi(u_1u_2)\phi(p_{3,3}) = P_3V^2P_1 = E_{31},\\
\phi(u_{1,3}^*u_{3,1}) &= \phi(p_{1,3})\phi(u_1u_2)\phi(p_{3,1}) = P_2V^2P_3 = E_{23},
\end{aligned}$$

where E_{xa}, $a, x = 1, 2, 3$, are the standard matrix units, and all other elements $u_{a,x}^*u_{b,y}$, $b > a$, are mapped by ϕ to zero.

Define a linear subspace $\mathcal{R} \subset C(\mathrm{Qut}(\mathcal{U}_{K_3}))$ as follows

$$\mathcal{R} = \mathrm{span}\{u_{a,x}^*u_{b,y}u_{a',x'}^*u_{b,'y'}, a, b, a', b', x, y, x', y' = 1, 2, 3\}.$$

Our aim is to show that on $C(\mathrm{Qut}(\mathcal{U}_{K_3}))$ there exists a trace τ which is "nonlocal" in the following sense: there is no commutative C^*-algebra $\mathcal{A}$ such that for all $x \in \mathcal{R}$

$$\tau(x) = \tau_{\mathcal{A}}(\pi(x)),$$

where $\pi\colon C(\mathrm{Qut}(\mathcal{U}_{K_3}))\to\mathcal{A}$ is a homomorphism, and $\tau_{\mathcal{A}}\in\mathcal{A}^*$.

We let $\tau=\mathrm{tr}\circ\phi$, where $\phi\colon C(\mathrm{Qut}(\mathcal{U}_{K_3}))\to M_3(\mathbb{C})$ is as constructed above, and tr is the standard normalized trace on $M_3(\mathbb{C})$. Then we have

$$\begin{aligned}\rho(u_{1,1}^*u_{2,3}u_{3,1}^*u_{2,2}) &= \mathrm{tr}(\phi(u_{1,1}^*u_{2,3})\phi(u_{2,2}^*u_{3,1})^*)\\ &= \mathrm{tr}(E_{13}E_{31})=\mathrm{tr}(E_{11})=\frac{1}{3}.\end{aligned}$$

On the other hand, for any commutative C^*-algebra $\mathcal{A}$, a $*$-homomorphism, $\pi\colon C(\mathrm{Qut}(\mathcal{U}_{K_3}))\to\mathcal{A}$, and $\tau_{\mathcal{A}}\in\mathcal{A}^*$, by the commutativity we have

$$\begin{aligned}\tau_{\mathcal{A}}(\pi(u_{1,1}^*u_{2,3}u_{3,1}^*u_{2,2})) &= \tau_{\mathcal{A}}(\pi(p_{1,1}u_1p_{2,3}p_{3,1}u_2^*p_{2,2}))\\ &= \tau_{\mathcal{A}}(\pi(p_{1,1})\pi(u_1)\pi(p_{2,3})\pi(p_{3,1})\pi(u_2^*)\pi(p_{2,2}))\\ &= \tau_{\mathcal{A}}(\pi(p_{1,1})\pi(p_{3,1})\pi(u_1)\pi(u_2^*)\pi(p_{2,3})\pi(p_{2,2}))=0,\end{aligned}$$

since $p_{1,1}p_{3,1}=p_{2,3}p_{2,2}=0$ in $C(\mathrm{Qut}(\mathcal{U}_{K_3}))$. □

Remark 1. It was proved in [RS21] that the graph K_n does not admit nonlocal symmetry only if $n\geq 5$, meaning that in this case there is no perfect NS qc-strategy that cannot be produced classically, i.e. not in the loc-class. This shows that QNS strategies provide more nonlocal "symmetries".

Remark 2. We note that the representation ϕ from the proof of Proposition 2 comes from the bi-unitary $\tilde{\mathcal{U}}=(\tilde{u}_{a,x})_{a,x=1}^3$ whose entries are 3×3 matrices and given by

$$\begin{aligned}&\tilde{u}_{1,1}=E_{11}, && \tilde{u}_{1,2}=E_{23}, && \tilde{u}_{1,3}=E_{32},\\ &\tilde{u}_{2,1}=E_{22}, && \tilde{u}_{2,2}=E_{31}, && \tilde{u}_{2,3}=E_{13},\\ &\tilde{u}_{3,1}=E_{33}, && \tilde{u}_{3,2}=E_{12}, && \tilde{u}_{3,3}=E_{21},\end{aligned}$$

$\phi(u_{a,x}^*u_{b,y})=\tilde{u}_{a,x}^*\tilde{u}_{b,y}$, $a,b,x,y=1,2,3$. It is easy to see that $\phi(u_{a,x}^*u_{b,y})=\tilde{u}_{a,x}^*\tilde{u}_{b,y}$, $a,b,x,y=1,2,3$. Also notice that for such a choice of $(\tilde{u}_{a,x})_{a,x=1}^3$, the matrix $\tilde{\mathcal{U}}$ is a 9×9 classical permutation matrix.

Remark 3. The bi-unitary matrix $\tilde{\mathcal{U}}$ from the remark above can be obtained as a result of a slight modification of the Latin square construction discussed in [RS21, Section 4.1].

Let A,B,C,D be flat unitary (complex Hadamard) $n\times n$ matrices, that is unitary matrices whose entries all have the same modulus (necessarily equal to $\frac{1}{\sqrt{n}}$ when the matrices are $n\times n$). Define

$$\psi_{jk}=\sqrt{n}\begin{pmatrix}a_{1j}b_{1k}\\ \vdots\\ a_{nj}b_{nk}\end{pmatrix},\quad \phi_{jk}=\sqrt{n}\begin{pmatrix}c_{1j}d_{1k}\\ \vdots\\ c_{nj}d_{nk}\end{pmatrix},\qquad j,k=1,\dots,n,$$

where a_{ij}, b_{ij}, c_{ij}, d_{ij} are the entries of A, B, C, D respectively. Then, since the matrices A and B are flat unitary, their entries have absolute value $1/\sqrt{n}$, and

$$\psi_{jk}^*\psi_{jl}=n\sum_{i=1}^n\bar{b}_{ik}\bar{a}_{ij}a_{ij}b_{il}=\sum_{i=1}^n\bar{b}_{ik}b_{il}=\delta_{k,l},\quad j,k,l=1,\dots,n,$$

and similarly $\psi_{kj}^*\psi_{lj} = \delta_{k,l}$. Also, since $\psi_{jk}\psi_{jk}^* = n(a_{lj}b_{lk}\bar{b}_{mk}\bar{a}_{mj})_{l,m=1}^n$, we have

$$\begin{aligned}
\sum_{j=1}^n \psi_{jk}\psi_{jk}^* &= \Big(n\sum_{j=1}^n a_{lj}b_{lk}\bar{b}_{mk}\bar{a}_{mj}\Big)_{l,m=1}^n \\
&= \Big(nb_{lk}\bar{b}_{mk}\sum_{j=1}^n a_{lj}\bar{a}_{mj}\Big)_{l,m=1}^n = (\delta_{l,m})_{l,m=1}^n = I. \\
\sum_{j=1}^n \psi_{kj}\psi_{kj}^* &= \Big(n\sum_{j=1}^n a_{lk}b_{lj}\bar{b}_{mj}\bar{a}_{mk}\Big)_{l,m=1}^n \\
&= \Big(na_{lk}\bar{a}_{mk}\sum_{j=1}^n b_{lj}\bar{b}_{mj}\Big)_{l,m=1}^n = (\delta_{l,m})_{l,m=1}^n = I.
\end{aligned}$$

The same relations hold obviously for $(\phi_{jk})_{j,k=1}^n$.

Now define the block martix

$$\mathcal{W} = (w_{jk})_{j,k=1}^n, \quad w_{jk} = \psi_{jk}\phi_{jk}^*.$$

We claim that $\mathcal{W}$ is bi-unitary. Indeed,

$$\begin{aligned}
\sum_{i=1}^n w_{ij}^* w_{ik} &= \sum_{i=1}^n \phi_{ij}\psi_{ij}^*\psi_{ik}\phi_{ik}^* = \delta_{jk}\sum_{i=1}^n \phi_{ij}\phi_{ij}^* = \delta_{jk}I, \\
\sum_{i=1}^n w_{ji}^* w_{ki} &= \sum_{i=1}^n \phi_{ji}\psi_{ji}^*\psi_{ki}\phi_{ki}^* = \delta_{jk}\sum_{i=1}^n \phi_{ji}\phi_{ji}^* = \delta_{jk}I.
\end{aligned}$$

To obtain the bi-unitary matrix $\mathcal{U}$ from the previous Remark, take

$$A = B = C = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^2 \\ 1 & \epsilon^2 & \epsilon \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \epsilon^2 & \epsilon \\ 1 & \epsilon & \epsilon^2 \end{pmatrix},$$

and orthonormal basis

$$e_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad e_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \epsilon \\ \epsilon^2 \end{pmatrix}, \quad e_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \epsilon^2 \\ \epsilon \end{pmatrix}.$$

where $\epsilon = e^{2\pi i/3}$. Then

$$\begin{aligned}
&\psi_{11} = \psi_{23} = \psi_{32} = \phi_{11} = \phi_{22} = \phi_{33} = e_1, \\
&\psi_{12} = \psi_{21} = \psi_{33} = \phi_{13} = \phi_{21} = \phi_{32} = e_2, \\
&\psi_{13} = \psi_{22} = \psi_{31} = \phi_{12} = \phi_{23} = \phi_{31} = e_3,
\end{aligned}$$

and therefore,

$$\begin{aligned}
&u_{1,1} = e_1e_1^* = E_{11}, \quad u_{1,2} = e_2e_3^* = E_{23}, \quad u_{1,3} = e_3e_2^* = E_{32}, \\
&u_{2,1} = e_2e_2^* = E_{22}, \quad u_{2,2} = e_3e_1^* = E_{31}, \quad u_{2,3} = e_1e_3^* = E_{13}, \\
&u_{3,1} = e_3e_3^* = E_{33}, \quad u_{3,2} = e_1e_2^* = E_{12}, \quad u_{3,3} = e_2e_1^* = E_{21}.
\end{aligned}$$

3.2. **Decomposition into regular graphs.** Let G_1 and G_2 be simple undirected finite graphs. We assume that G_1 and G_2 have the same number of vertices n. We also write $V_j(G)$ for the set of vertices of G of degree (valence) j.

By Theorem 3, the quantum graphs $\mathcal{U}_{G_1}$ and $\mathcal{U}_{G_2}$ are qc-isomorphic if there exist a tracial C^*-algebra $\mathcal{A}$ and a bi-unitary matrix $U = (u_{a,x})_{a\in V(G_2),x\in V(G_1)}$ which satisfies $u^*_{a,x}u_{b,y} \in \mathcal{A}$, $x, y \in V(G_1)$, $a, b \in V(G_2)$, and the following orthogonality conditions

$$u_{a,x}u^*_{b,y} = 0, \quad \text{if } a \sim b \text{ in } G_2,\ x \not\sim y \text{ in } G_1 \text{ or } a \not\sim b \text{ in } G_2,\ x \sim y \text{ in } G_1, \tag{2}$$

here we assume that $a \not\sim a$, $x \not\sim x$ for any $a \in V(G_2)$, $x \in V(G_1)$.

The following statements are similar to the ones obtained in [Ful06] in the context of quantum automorphism of graphs.

Lemma 2. *Let a bi-unitary block matrix U satisfies the orthogonality conditions (2). If $x \in V_j(G_1)$, $a \in V_k(G_2)$, $j \neq k$, then $u_{a,x} = 0$.*

Proof. Let $n = |V(G_1)| = |V(G_2)|$. Take $x \in V_j(G_1)$, $a \in V_k(G_2)$, $j \neq k$. Since U is unitary, for any $y \in V(G_1)$, we have

$$\sum_{b\in V(G_2)} u^*_{b,y}u_{b,y} = I.$$

Then for any $y \sim x$ in G_1, from $u_{a,x}u^*_{b,y} = 0$, $a \not\sim b$ in G_2, we get

$$u_{a,x} = u_{a,x} \sum_{b\in V(G_2)} u^*_{b,y}u_{b,y} = u_{a,x} \sum_{b\sim a} u^*_{b,y}u_{b,y}.$$

As $x \in V_j(G_1)$, taking sum over all $y \sim x$ we obtain

$$ju_{a,x} = u_{a,x} \sum_{y\sim x}\sum_{b\sim a} u^*_{b,y}u_{b,y}.$$

Similarly, since U is bi-unitary, for any $b \in V(G_2)$, we have

$$\sum_{y\in V(G_1)} u^*_{b,y}u_{b,y} = I,$$

and since $a \in V_k(G_2)$, we get

$$ku_{a,x} = u_{a,x} \sum_{y\sim x}\sum_{b\sim a} u^*_{b,y}u_{b,y},$$

which implies $ju_{a,x} = ku_{a,x}$ and hence $u_{a,x} = 0$. □

Proposition 3. *Assume that there exists a bi-unitary matrix U which satisfies the orthogonality conditions (2). Then $|V_k(G_1)| = |V_k(G_2)|$, $k \in \mathbb{N}_{n-1}$.*

Proof. Let $x \in V_j(G_1)$. Since U is unitary, applying Lemma 2, we obtain

$$I = \sum_{a\in V(G_2)} u^*_{a,x}u_{a,x} = \sum_{a\in V_j(G_2)} u^*_{a,x}u_{a,x},$$

which implies, in particular, that $V_j(G_2)$ is non-empty. Then

$$\sum_{x\in V_j(G_1)}\sum_{a\in V_j(G_2)} u^*_{a,x}u_{a,x} = |V_j(G_1)|I.$$

Similarly, since U is bi-unitary, we see that for $a \in V_j(G_2)$,

$$I = \sum_{x\in V(G_1)} u^*_{a,x}u_{a,x} = \sum_{x\in V_j(G_1)} u^*_{a,x}u_{a,x},$$

which yields

$$\sum_{x\in V_j(G_1)}\sum_{a\in V_j(G_2)} u^*_{a,x}u_{a,x} = |V_j(G_2)|I. \qquad \square$$

Recall that for $S \subset V(G)$ a subgraph of G induced by S is defined as a graph whose set of vertices is S, and two vertices in S are connected by edge iff they are connected by edge in G.

Lemma 3. *Let G_1^k and G_2^k, $k \in \mathbb{N}_{n-1}$, $n = |V(G_1)| = |V(G_2)|$, be subgraphs of G_1, G_2 induced respectively by $V_k(G_1)$ and $V_k(G_2)$. Order the vertices of G_1 and G_2 by their degree. Then a bi-unitary U satisfying (2) has a block diagonal form, $U = \operatorname{diag}(U_1, \dots, U_{n-1})$, $\dim U_k = |V_k(G_1)| = |V_k(G_2)|$, where U_k is a bi-unitary which satisfies (2) for graphs G_1^k, G_2^k (if $|V_k(G_1)| = |V_k(G_2)| = 0$, then the corresponding block is empty).*

Proof. The block diagonal form of U follows from Lemma 2 and Proposition 3. Since U is bi-unitary, each diagonal block is obviously bi-unitary. The orthogonality conditions follow since G_1^k, G_2^k are induced subgraphs. $\square$

Recall that a graph is called regular if each its vertex has the same number of neighbors (degree).

Theorem 4. *Let U be a bi-unitary matrix which satisfies the orthogonality conditions (2).*

Then there exist a finite set Λ, a family $(G_1^\lambda)_{\lambda\in\Lambda}$ of regular induced subgraphs of G_1 and a family $(G_2^\lambda)_{\lambda\in\Lambda}$, of regular induced subgraphs of G_2 such that:

(i) Degrees of vertices in G_1^λ and G_2^λ coincide for all $\lambda \in \Lambda$;

(ii) $|V(G_1^\lambda)| = |V(G_2^\lambda)|$, $\lambda \in \Lambda$;

(iii) after an appropriate ordering of vertices of G_1, G_2 (and the corresponding rows and columns of U), the matrix U takes the form

$$U = \operatorname{diag}(U_\lambda)_{\lambda\in\Lambda},$$

where each U_λ, $\lambda \in \Lambda$ is a bi-unitary block matrix of size $n_\lambda = |V(G_1^\lambda)| = |V(G_2^\lambda)|$ which satisfies the orthogonality conditions imposed by the pair of graphs G_1^λ, G_2^λ.

Proof. Order the vertices of G_1, G_2 by their degree (at this step, the order of vertices of the same degree can be taken arbitrarily). Then by Lemma 3, construct induced subgraphs $G_1^{k_1}, G_2^{k_1}$, $k_1 \in \mathbb{N}_{n-1}$, and decompose U into a direct sum $U = \operatorname{diag}(U_{k_1})_{k_1=1}^{n-1}$ of bi-unitary matrices corresponding to pairs $G_1^{k_1}, G_2^{k_1}$. If for some k_1 the graph $G_1^{k_1}$ is not regular (and therefore, by Proposition 3 $G_2^{k_1}$ is not regular, too), repeat the same procedure: order vertices inside $G_1^{k_1}$ and $G_2^{k_1}$ by their degree in these subgraphs, consider induced subgraphs $G_1^{k_1k_2}, G_2^{k_1k_2}$, $k_2 \in \mathbb{N}_{n_1-1}$, $n_1 = |V_{k_1}(G_1)| = |V_{k_1}(G_2)|$, and further decompose $U_{k_1} = \operatorname{diag}(U_{k_1,1}, \dots, U_{k_1,n_1-1})$; do it recursively for all non-regular subgraphs. As a result, we obtain a family of regular induced subgraphs G_1^λ, G_2^λ, where λ belongs to some set Λ of multiindices of varied length, and a decomposition $U = \operatorname{diag}(U_\lambda)_{\lambda\in\Lambda}$ such that (i), (iii) holds by the construction, and (ii) follows from Proposition 3. $\square$

Corollary 3. *Let G_1 and G_2 be simple graphs such that $\mathcal{U}_{G_1} \simeq_{\mathrm{t}} \mathcal{U}_{G_2}$, $\mathrm{t} \in \{\mathrm{loc}, \mathrm{q}, \mathrm{qc}\}$. Then for each $\lambda \in \Lambda$ and the regular induced subgraphs G_1^λ and G_2^λ, we have $\mathcal{U}_{G_1^\lambda} \simeq_{\mathrm{t}} \mathcal{U}_{G_2^\lambda}$.*

3.3. **Nonlocal symmetries for higher order graphs.** In this section we show that all quantum graphs $\mathcal{U}_G$ admit nonlocal symmetries if and only if $|V(G)| \geq 3$. This stands in contrast to the situation for nonlocal symmetries of the underlying classical graphs, where the theory is considerably richer, see [RS21]. In particular, it was proved in [RS21] that the only graphs on five vertices that does not admit nonlocal symmetries are K_5 and its complement $\bar{K}_5$.

We first define the class of synchronous correlations and recall a characterization which will be used in the proof of the main theorem.

Let $\mathbb{F}(n,k)$ be the group that is the free product of n copies of the cyclic group of order k and let $C^*(\mathbb{F}(n,k))$ be the C^*-algebra of $\mathbb{F}(n,k)$. It is generated by n unitaries $u_1,\dots,u_n$, each of order k, i.e. $u_i^k=1$, $i\in\mathbb{N}_n$. Decomposing u_i in terms of spectral projections $u_i=\sum_{a=0}^{k-1}\omega^a e_{i,a}$, where $\omega=e^{2\pi i/k}$, we obtain a system of PVMs $\{e_{i,a}:a=0,\dots,k-1\}$, i.e. projections $e_{i,a}$ such that $\sum_{a=0}^{k-1}e_{i,a}=1$ for all i, which is also a universal family of n k-PVM's, meaning that for any family of PVMs $\{E_{x,a}:a=0,\dots,k-1\}\subset\mathcal{B}(H)$, $x\in\mathbb{N}_n$, there exists a $*$-homomorphism of $C^*(\mathbb{F}(n,k))\to\mathcal{B}(H)$ such that $e_{x,a}\mapsto E_{x,a}$.

Let $X=Y$, $A=B$ be finite sets. Recall that a correlation $\{p(a,b|x,y)\}_{x,y\in X,a,b\in A}$ is synchronous if $p(a,b|x,x)=0$ whenever $a=b$. We write $\mathcal{C}^s_{\rm t}(n,k)$ for the t-class of synchronous correlations with $|X|=|Y|=n$, $|A|=|B|=k$, where $\rm t\in\{\rm loc,q,qc\}$.

The following characterization of synchronous correlations in terms of traces is from [PSS+16].

Theorem 5. *We have $p\in\mathcal{C}^s_{\rm qc}(n,k)$ if and only if there is a family of n k-PVM's $\{E_{x,a}:1\le x\le n,0\le a\le k-1\}$ in a C^*-algebra $\mathcal{A}$ with a trace τ such that $p(a,b|x,y)=\tau(E_{x,a}E_{y,b})$. Moreover,*

- *$p\in\mathcal{C}^s_{\rm loc}(n,k)$ if and only if $\mathcal{A}$ can be taken to be abelian,*
- *$p\in\mathcal{C}^s_{\rm q}(n,k)$ if and only if $\mathcal{A}$ can be taken to be finite-dimensional.*

An example showing a separation between local $\mathcal{C}^s_{\rm loc}(3,2)$ and quantum synchronous correlations $\mathcal{C}^s_{\rm q}(3,2)$ is given in [MPTW23, Example 4.4]. In particular, together with Theorem 5, it provides a trace $\tilde\tau$ on $C^*(\mathbb{F}(3,2))$ such that there is no representation π of $C^*(\mathbb{F}(3,2))$ into an abelian C^*-algebra $\mathcal{A}$ and a linear functional $f\in\mathcal{A}^*$ such that $\tilde\tau(e_{x,a}e_{y,b})=f(\pi(e_{x,a}e_{y,b}))$ for all $1\le x,y\le 3$, $a,b\in\{0,1\}$. Write v_x, $1\le x\le 3$, for the generators of $C^*(\mathbb{F}(3,2))$ and let $\mathcal{R}=\operatorname{span}\{1,v_x,v_xv_y,1\le x,y\le 3\}$. As $p_{x,a}=\frac12(1+(-1)^a v_x)$, we have that the restriction $\tilde\tau|_{\mathcal{R}}$ is not equal to the restriction $f\circ\pi|_{\mathcal{R}}$ for any π and f as above.

We are now ready to prove the main theorem of this section.

Theorem 6. *Let G be a graph with $|V(G)|\ge 3$. Then $\mathcal{U}_G$ admits a nonlocal symmetry.*

Proof. Assume first that $|V(G)|=3$. In Proposition 2 we have already proved the statement for the full graph K_3. The three remaining graphs are $\bullet-\bullet-\bullet$, $\bullet\quad\bullet-\bullet=K_1\cup K_2$, and $\bullet\quad\bullet\quad\bullet=K_1\cup K_1\cup K_1$. As in the case of K_3, we prove that for each such graph G, there exists a trace τ_G on $C(\mathrm{Qut}(\mathcal{U}_G))$, which cannot be factored through a commutative C^*-algebra $\mathcal{A}$ in such a way that

$$\tau(x)=\tau_{\mathcal{A}}(\pi(x)),$$

where $\tau_{\mathcal{A}}\in\mathcal{A}^*$, $\pi\colon C(\mathrm{Qut}(\mathcal{U}_G))\to\mathcal{A}$ is a homomorphism, and

$$x\in\mathcal{R}=\operatorname{span}\{u^*_{a,x}u_{b,y}u^*_{a',x'}u_{b,'y'},a,b,a',b',x,y,x',y'=1,2,3\}.$$

Consider the graph $G=\overset{1}{\bullet}\quad\overset{2}{\bullet}-\overset{3}{\bullet}=K_1\cup K_2$. Let $U=(u_{a,x})_{a,x\in V(G)}$ be the bi-unitary such that $u^*_{a,x}u_{b,y}$, $x,y,a,b\in V(G)$ generate $C(\mathrm{Qut}(\mathcal{U}_G))$. By Lemma 2, $u_{1,1}$ is unitary, $u_{1,2}=u_{2,1}=u_{1,3}=u_{3,1}=0$ and $(u_{a,x})^3_{a,x=2}$ is a bi-unitary such that $u^*_{a,x}u_{b,y}$, $x,y,a,b=2,3$, generate a subalgebra isomorphic to $C(\mathrm{Qut}(\mathcal{U}_{K_2}))$. By Theorem 1 and the proof of Proposition 1, $u_{a,x}$, $a,x=2,3$, are partial isometries, so $u^*_{2,2}u_{2,2}=u^*_{3,3}u_{3,3}:=p$, $u^*_{2,3}u_{2,3}=u^*_{3,2}u_{3,2}=1-p$ are projections. Therefore, if $\pi\colon C(\mathrm{Qut}(\mathcal{U}_G))\to\mathcal{A}$ is a $*$-homomorphism to a commutative

C^*-algebra $\mathcal{A}$, then

$$\begin{aligned}\pi(u_{2,2}^*u_{1,1}u_{3,2}^*u_{1,1}) &= \pi(u_{2,2}^*u_{2,2})\pi(u_{2,2}^*u_{1,1})\pi(u_{3,2}^*u_{3,2})\pi(u_{3,2}^*u_{1,1})\\ &= \pi(p)\pi(u_{2,2}^*u_{1,1})\pi(1-p)\pi(u_{3,2}^*u_{1,1})\\ &= \pi(u_{2,2}^*u_{1,1})\pi(p)\pi(1-p)\pi(u_{3,2}^*u_{1,1}) = 0.\end{aligned}$$

On the other hand, for the bi-unitary $U = (u_{a,x})_{a,x=1}^3$ given by

$$u_{1,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad u_{1,2} = u_{1,3} = u_{2,1} = u_{3,1} = 0,$$

$$u_{2,2} = u_{3,3} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad u_{2,3} = u_{3,2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

which can be easily verified to determine a two-dimensional representation of $C(\mathrm{Qut}(\mathcal{U}_G))$, we have that

$$\mathrm{tr}(u_{2,2}^*u_{1,1}u_{3,2}^*u_{1,1}) = \mathrm{tr}(p) = \frac{1}{2},$$

which proves the existence of a nonlocal symmetry for this graph.

For the bi-unitary matrix $U = (u_{a,x})_{a,x\in V(G)}$ related to the graph $G = \overset{2}{\bullet} - \overset{1}{\bullet} - \overset{3}{\bullet}$ we have by Lemma 2, as above, that u_{11} is unitary, $u_{1,2} = u_{2,1} = u_{1,3} = u_{3,1} = 0$, and u_{ax}, $a, x \in \{2, 3\}$ form a bi-unitary matrix, but unlike above, there are no non-trivial orthogonality conditions. Consider C^*-algebra $C^*(\mathbb{F}(3,2))/\mathcal{J}$, where $\mathcal{J}$ is a C^*-ideal generated by $u_1 + u_2 + u_3$, where u_i, $i = 1, 2, 3$, are the canonical selfadjoint unitary generators of $C^*(\mathbb{F}(3,2))$. This algebra has a unique, up to unitary equivalence, irreducible representation π,

$$\pi(u_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pi(u_2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}, \quad \pi(u_3) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}, \tag{3}$$

and thus is simple. Using directly verified relation $u_1u_2 = u_2u_3$, it is a routine verification that

$$\mathcal{U} = (\pi(u_{a,x}))_{a,x=1}^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}I & 0 & \\ 0 & \pi(u_1) & \pi(u_2) \\ 0 & \pi(u_2) & -\pi(u_3) \end{pmatrix}$$

is a bi-unitary matrix.

For the corresponding representation π of $C(\mathrm{Qut}(\mathcal{U}_G))$, we have, in particular,

$$\pi(u_{1,1}^*u_{2,2}) = \pi(u_1), \quad \pi(u_{1,1}^*u_{2,3}) = \pi(u_2), \quad \pi(u_{1,1}^*u_{3,3}) = -\pi(u_3).$$

By Theorem 3, the trace $\tau = \mathrm{tr} \circ \pi$ gives rise to a perfect quantum strategy on $C(\mathrm{Qut}(\mathcal{U}_G))$ by

$$\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = \left(\tau(u_{a,x}^*u_{a',x'}u_{b',y'}^*u_{b,y})\right)_{a,a',b,b'}, \quad x, x', y, y', a, a', b, b' \in \mathbb{N}_3.$$

Assuming this strategy being local, there exists a commutative C^*-algebra $\mathcal{A}$, a state $\gamma \in \mathcal{A}^*$, and a homomorphism $\rho\colon C(\mathrm{Qut}(\mathcal{U}_G)) \to \mathcal{A}$, such that for each

$$s \in \mathcal{S} = \mathrm{span}\{u_{a,x}^*u_{a',x'}u_{b,y}^*u_{b',y'} \mid x, x', y, y', a, a', b, b' \in \mathbb{N}_n\}$$

we have $\tau(s) = \gamma(\rho(s))$. Here we may assume that γ is faithful. Then we have

$$\begin{aligned}&\gamma(\rho((u_{1,1}^*u_{2,2} - u_{2,2}^*u_{1,1})^*(u_{1,1}^*u_{2,2} - u_{2,2}^*u_{1,1}))\\ &\quad= \mathrm{tr}\,\pi((u_{1,1}^*u_{2,2} - u_{2,2}^*u_{1,1})^*(u_{1,1}^*u_{2,2} - u_{2,2}^*u_{1,1}))\\ &\quad= \mathrm{tr}(\pi(u_1) - \pi(u_1^*))^*(\pi(u_1) - \pi(u_1^*)) = 0,\end{aligned}$$

so that $\rho(u_{1,1}^*u_{2,2})$ is self-adjoint. Here we used the fact that

$$(u_{1,1}^*u_{2,2} - u_{2,2}^*u_{1,1})^*(u_{1,1}^*u_{2,2} - u_{2,2}^*u_{1,1}) \in \mathcal{S}.$$

Notice that $1 = (u_{1,1}^* u_{1,1})^2 \in \mathcal{S}$, then, as $\|u_{1,1}^* u_{2,2}\| \le 1$, we have $0 \le 1 - u_{1,1}^* u_{2,2} u_{2,2}^* u_{1,1} \in \mathcal{S}$, and

$$\gamma(\rho(1 - u_{1,1}^* u_{2,2} u_{2,2}^* u_{1,1})) = \operatorname{tr}\pi(1 - u_{1,1}^* u_{2,2} u_{2,2}^* u_{1,1})) = \operatorname{tr}\pi(1 - u_1 u_1) = 0,$$
$$\gamma(\rho(1 - u_{2,2}^* u_{1,1} u_{1,1}^* u_{2,2})) = \operatorname{tr}\pi(1 - u_{2,2}^* u_{1,1} u_{1,1}^* u_{2,2})) = \operatorname{tr}\pi(1 - u_1 u_1) = 0,$$

therefore, $\rho(u_{1,1}^* u_{2,2})$ is also unitary. The same way we see that $\rho(u_{1,1}^* u_{2,3})$ and $\rho(u_{1,1}^* u_{3,3})$ are self-adjoint unitaries. Similarly, from

$$\begin{aligned}\gamma(\rho((u_{1,1}^* u_{2,2} + u_{1,1}^* u_{2,3} - u_{1,1}^* u_{3,3})^*(u_{1,1}^* u_{2,2} + u_{1,1}^* u_{2,3} - u_{1,1}^* u_{3,3}))) \\ = \operatorname{tr}\pi((u_1 + u_2 + u_3)^*(u_1 + u_2 + u_3)) = 0\end{aligned}$$

we conclude that $\rho(u_{1,1}^* u_{2,2} + u_{1,1}^* u_{2,3} - u_{1,1}^* u_{3,3}) = 0$.

This way, the mapping $u_1 \mapsto \rho(u_{1,1}^* u_{2,2}), u_2 \mapsto \rho(u_{1,1}^* u_{2,3}), u_3 \mapsto -\rho(u_{1,1}^* u_{3,3})$ gives rise to a non-trivial $*$-homomorphism of $C^*(\mathbb{F}(3,2))/\mathcal{J} \to \mathcal{A}$, which is impossible as $C^*(\mathbb{F}(3,2))/\mathcal{J} \simeq M_2(\mathbb{C})$, and $\mathcal{A}$ is commutative.

For the graph $G = K_1 \cup K_1 \cup K_1$, the set of orthogonality conditions (2) is empty, therefore, any biunitary 3×3 matrix determines a representation of $C(\mathrm{Qut}(\mathcal{U}_G))$. Then the arguments used above for the graph $\bullet - \bullet - \bullet$ give an example of nonlocal symmetry.

Assume now $n := |V(G)| \ge 4$. Let v_i, $i \in \mathbb{N}$, be unitary operators on a Hilbert space H. Then $U = \operatorname{diag}(v_1, \ldots, v_n) \in M_n(B(H))$ is a bi-unitary operator which generates a representation of $C(\mathrm{Qut}(\mathcal{U}_G))$ by letting

$$\rho(u_{a,x}^* u_{a',x'}) = \delta_{a,x}\delta_{a',x'} v_a^* v_{a'}. \tag{4}$$

Indeed,

$$\rho(u_{a,x}^* u_{a',x'} u_{b,y}^* u_{b',y'}) = \delta_{a,x}\delta_{a',x'} v_a^* v_{a'} \delta_{b,y}\delta_{b',y'} v_b^* v_{b'}$$

is zero for $a' \sim b, x' \not\sim y$ or $a' \not\sim b, x' \sim y$ since for nonzero cases we have $a' = x', b = y$. In particular, we have $\rho(u_{a,x}^* u_{a,x}) = \delta_{a,x} I$.

In (4), take $H = \mathbb{C}^2$ and π as defined in (3) and let

$$v_2 = v_1\pi(u_1), \quad v_3 = v_2\pi(u_2) = v_1\pi(u_1u_2), \quad v_4 = v_3\pi(u_3) = v_1\pi(u_1u_2u_3),$$

so that $v_1^* v_2 = \pi(u_1)$, $v_2^* v_3 = \pi(u_2)$, $v_3^* v_4 = \pi(u_3)$. If we assume that the corresponding winning strategy for $C(\mathrm{Qut}(\mathcal{U}_G))$ is local, then the arguments similar to the above show that the mapping $s\colon u_a \mapsto \pi'(u_{a,a}^* u_{a+1,a+1})$, $a = 1, 2, 3$, gives rise to a $*$-homomorphism of $C^*(\mathbb{F}(3,2))/\mathcal{J}$ into a commutative $\mathcal{A}$ which leads to a contradiction. □

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National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute"
Email address: olyushka.ostrovska@gmail.com

Institute of Mathematics, NAS of Ukraine
Email address: vo@imath.kiev.ua

Chalmers University of Technology, Department of Mathematics
Email address: turowska@chalmers.se