# Jensen's inequality for partial traces in von Neumann algebras

Downloaded from: https://research.chalmers.se, 2026-10-07 19:06 UTC

Citation for the original published paper (version of record):
Rahaman, M., Turowska, L. (2026). Jensen's inequality for partial traces in von Neumann algebras. Bulletin of the London Mathematical Society, 58(4). http://dx.doi.org/10.1112/blms.70347

N.B. When citing this work, cite the original published paper.

research.chalmers.se offers the possibility of retrieving research publications produced at Chalmers University of Technology. It covers all kind of research output: articles, dissertations, conference papers, reports etc. since 2004. research.chalmers.se is administrated and maintained by Chalmers Library

(article starts on next page)

Check for updates

Received: 19 August 2025 | Revised: 29 January 2026 | Accepted: 7 March 2026

DOI: 10.1112/blms.70347

RESEARCH ARTICLE

Bulletin of the London Mathematical Society

# Jensen's inequality for partial traces in von Neumann algebras

**Mizanur Rahaman**[1,2] | **Lyudmila Turowska**[2]

[1]Wallenberg Centre for Quantum Technology, Chalmers University of Technology, Gothenburg, Sweden

[2]Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, Gothenburg, Sweden

**Correspondence**
Mizanur Rahaman, Department of Mathematical Sciences, Chalmers University of Technology, Chalmers Tvärgata 3, Göteborg, SE-SE-412 96, Gothenburg, Sweden.
Email: mizanurr@chalmers.se

**Funding information**
Wallenberg Center for Quantum Technology (WACQT); Swedish Research Council, Grant/Award Number: 2023-04555

**Abstract**

Motivated by a recent result on finite-dimensional Hilbert spaces, we prove Jensen's inequality for partial traces in semifinite von Neumann algebras. We also prove a similar inequality in the framework of general (non-tracial) von Neumann algebras.

**MSC 2020**
46L10, 46N10, 47C15 (primary)

## 1 | INTRODUCTION

Jensen's inequality is a very useful inequality in convex analysis and probability theory. It provides a relationship between convex functions and expectations (or integrals). Extending this connection, Davis [7], and later Choi [6] proved Jensen's inequality for positive maps and operator convex functions. Also, Hansen and Pedersen showed an operator inequality [10] which is a powerful and elegant generalization of Jensen's inequality to the setting of operator algebras, particularly involving operator convex functions. These results play a crucial role in functional analysis and have wide applications in probability and quantum information theory.

In a recent article, Carlen et al. [5] established a Jensen's inequality for partial traces on finite-dimensional Hilbert spaces. More precisely, they showed the following theorem.

© 2026 The Author(s). *Bulletin of the London Mathematical Society* is copyright © London Mathematical Society. This is an open access article under the terms of the Creative Commons Attribution License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.

**Theorem 1** (Carlen–Frank–Larson, 2025). *For two finite-dimensional Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and a self-adjoint operator $H$ acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ whose spectrum lies within an interval $I$, we have*

$$\mathrm{Tr}_2\, f(\mathrm{Tr}_1(\rho \otimes \mathbb{1})^{\frac{1}{2}} H (\rho \otimes \mathbb{1})^{\frac{1}{2}}) \leqslant \mathrm{Tr}_1(\rho^{\frac{1}{2}}\, \mathrm{Tr}_2\, f(H) \rho^{\frac{1}{2}}),$$

*for every convex function $f$ defined on $I$ and every density matrix $\rho$ (a positive semi-definite matrix with trace 1) on $\mathcal{H}_1$.*

This inequality helped to establish some results about eigenvalue asymptotics of operators with homogeneous potentials.

Owing to the general interest in Jensen's inequality, the aim of this article is to extend this new result in the framework of von Neumann algebras. Some notable works have already been done in generalizing tracial version of Jensen's inequality in operator algebras: pioneered by Brown and Kosaki [3] who proved a trace Jensen's inequality in semifinite von Neumann algebras. Later, Petz showed similar inequality involving contractive positive maps [14]. Harada and Kosaki [12] further improved both the previous results involving semi-bounded self-adjoint operators. We also note that there are many variations of Jensen's inequality in different contexts, and for the reader's convenience, we give some references here [1, 9, 11]. It is noteworthy that such an inequality for partial traces in operator algebraic contexts has not been established.

We state our main theorem below.

**Theorem 2** (Jensen's inequality for partial traces in von Neumann algebras). *Let $(\mathcal{M}_1, \tau_1)$ and $(\mathcal{M}_2, \tau_2)$ be two von Neumann algebras with the corresponding normal semifinite traces. Let $H \in \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$ be a self-adjoint element, and $f : \mathbb{R} \to \mathbb{R}$ a continuous convex function. For $a \in L^2(\mathcal{M}_1, \tau_1)$, with $\tau_1(a^*a) = 1$, we have*

$$\tau_2 f(\tau_1 \otimes id)[(a^* \otimes 1) H (a \otimes 1)] \leqslant \tau_1[a^*(id \otimes \tau_2) f(H) a], \tag{1}$$

*whenever both sides are defined.*

*Moreover, if $\tau_1(a^*a) \leqslant 1$, and $f(0) = 0$, then the stated inequality 1 still holds whenever both sides are defined.*

See Remark 1 for what we mean by both sides of the inequality being defined.

We remark that when $\mathcal{M}_1 = B(\mathcal{H}_1)$ and $\mathcal{M}_2 = B(\mathcal{H}_2)$, where $\mathcal{H}_1$, and $\mathcal{H}_2$ are finite-dimensional Hilbert spaces, the inequality above is slightly stronger than the statement in Theorem 1, the authors need the square root of a density operator in the conjugation. For the finite-dimensional case, any operator is trace-class, and hence the extra assumption on the element $a$ in our theorem is not needed. We would like to further point out that we proved a similar inequality in Theorem 8 for general (non-tracial) von Neumann algebras that involve normal states but here we require an operator convex function for the inequality.

## 2 | PRELIMINARY ON SEMIFINITE VON NEUMANN ALGEBRAS

We refer to [15, Chapter 1] and [16, Chapter V.2] for some preliminary context on von Neumann algebras and we give a brief outline of semifinite traces below. Let $\mathcal{M}$ be a von Neumann algebra

14692120, 2026, 4, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.70347 by Statens Beredning, Wiley Online Library on [27/04/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

acting on $\mathcal{H}$, and write $\mathcal{M}_+$ for the cone of positive elements of $\mathcal{M}$. If $\mathcal{M}$ is equipped with a normal faithful semifinite trace, we call $\mathcal{M}$ tracial. Recall that a trace on $\mathcal{M}$ is $\tau : \mathcal{M}_+ \to [0, \infty]$ satisfying

1. $\tau(x + y) = \tau(x) + \tau(y)$, $x, y \in \mathcal{M}_+$;
2. $\tau(\lambda x) = \lambda\tau(x)$, $\lambda \geqslant 0$, $x \in \mathcal{M}_+$;
3. $\tau(xx^*) = \tau(x^*x)$, $x \in \mathcal{M}$.

$\tau$ is said to be faithful if $\tau(x) = 0$ implies $x = 0$, and $\tau$ is semifinite if for any $x \in \mathcal{M}_+$ there exists a non-zero $y \in \mathcal{M}_+$ such that $y \leqslant x$ and $\tau(y) < \infty$.

Given any tracial von Neumann algebra, let $S_+$ be the set of all $x \in \mathcal{M}_+$ such that $\tau(\mathrm{supp}x) < \infty$, where $\mathrm{supp}x$ denotes the support of $x$, that is the least projection $p$ such that $xp = x$. Let $S$ be the linear span of $S_+$. Then for any $x \in S$ one has $\tau(|x|) < \infty$. The completion of $S$ with respect to $\|x\|_1 := \tau(|x|)$ is the non-commutative $L^1$-space $L^1(\mathcal{M}, \tau)$ associated with $(\mathcal{M}, \tau)$. The space $\mathcal{M} \cap L^1(\mathcal{M}, \tau)$ is dense in $L^1(\mathcal{M}, \tau)$; the trace $\tau$ is finite on $\mathcal{M}_+ \cap L^1(\mathcal{M}, \tau)$ and extends to a functional on $L^1(\mathcal{M}, \tau)$.

The elements of $L^1(\mathcal{M}, \tau)$ can be considered as closed, densely defined operators on $\mathcal{H}$ affiliated with $\mathcal{M}$. An affiliated element is called measurable with respect to $\tau$ if $\tau(e_\lambda(|x|)) < \infty$ for some $\lambda > 0$, where $e_\lambda$ is the characteristic function of the set $(\lambda, \infty)$. Let $L^0(\mathcal{M}, \tau)$ denote the space of measurable elements. It is known that $L^0(\mathcal{M}, \tau)$ is a $*$-algebra, and $\tau$ can be extended to the positive part of $L^0(\mathcal{M}, \tau)$. Then $L^1(\mathcal{M}, \tau) = \{x \in L^0(\mathcal{M}, \tau) : \tau(|x|) < \infty\}$ is a Banach space with the norm $\|x\|_1 = \tau(|x|)$. For $x \in L^0(\mathcal{M}, \tau)$ recall the generalized singular numbers

$$\mu_t(x) = \inf\{\lambda > 0 : \tau(e_\lambda(|x|)) \leqslant t\}, \quad t > 0.$$

It is known that $\tau$ can be extended to $L^0(\mathcal{M}, \tau)_+$, by letting $\tau(x) = \int_0^\infty \mu_t(x)dt$, $x \in L^0(\mathcal{M}, \tau)_+$. Similar to $L^1(\mathcal{M}, \tau)$ one defines the non-commutative $L^2$-space, $L^2(\mathcal{M}, \tau)$, as all elements $x \in L^0(\mathcal{M}, \tau)$ such that $\tau(x^*x) < \infty$. One has that for any $x$, $y \in L^2(\mathcal{M}, \tau)$, $xy^* \in L^1(\mathcal{M}, \tau)$ and $\langle x, y\rangle = \tau(xy^*)$ is an inner product making $L^2(\mathcal{M}, \tau)$ a Hilbert space. Writing $\|x\|_2$ for the corresponding norm, we have $\|xa\|_2^2 = \tau(a^*x^*xa) \leqslant \|x\|^2\|a\|_2^2$ for any $x \in \mathcal{M}$, $a \in L^2(\mathcal{M}, \tau)$.

## 3 | TRACE JENSEN'S INEQUALITY FOR POSITIVE LINEAR MAPS

Recall that a function $f$ defined on an interval $I$ is convex if $f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda)f(y)$ whenever $\lambda \in [0, 1]$ and $x, y \in I$.

We start this section by stating Petz's result (see Theorems A and B in [14]).

**Theorem 3** (Petz, 1987). *Let $\mathcal{M}, \mathcal{N}$ be von Neumann algebras and $f : \mathbb{R}^+ \to \mathbb{R}$ be a continuous convex function. Assume that $\tau$ is a normal semifinite trace on $\mathcal{M}$ and $\Phi : \mathcal{N} \to \mathcal{M}$ is a unital positive map. Then for any $a \in \mathcal{N}_+$ one has*

$$\tau(f(\Phi(a))) \leqslant \tau(\Phi(f(a))),$$

*provided both sides exist.*

*Furthermore, if we assume that $f(0) = 0$, then the above inequality remains valid for contractive positive $\Phi$.*

14692120, 2026, 4, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.70347 by Statens Beredning, Wiley Online Library on [27/04/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Note that a positive unital linear mapping on a C*-algebra is necessarily a contraction [13]. In this section, we show that the assertion of Petz's result holds even when we replace the positive element in the statement of Theorem 3 by a self-adjoint element. This is a crucial step and the main technical ingredient in proving our main result given in the next section.

Following [3], we make the following definition

**Definition 1.** Let $x$ be a self-adjoint operator in $\mathcal{M}$ with the Jordan decomposition $x = x_+ - x_-$. We say that $\tau(x)$ is defined if $\tau(x_+) < +\infty$ or $\tau(x_-) < +\infty$; in this case, we set $\tau(x) = \tau(x_+) - \tau(x_-) \in [-\infty, +\infty]$.

We have

**Theorem 4.** *Let $\mathcal{M}, \mathcal{N}$ be von Neumann algebras and $f : \mathbb{R} \to \mathbb{R}$ be a continuous convex function. Assume that $\tau$ is a normal semifinite trace on $\mathcal{M}$ and $\Phi : \mathcal{N} \to \mathcal{M}$ is a unital positive map. Then for any self-adjoint $x \in \mathcal{N}$, we have*

$$\tau(f(\Phi(x))) \leqslant \tau(\Phi(f(x))),$$

*provided both sides exist.*

*Furthermore, the above inequality remains valid if we have a contractive positive map $\Phi$ and a continuous convex function with $f(0) = 0$.*

Note that the above statement not only improves Petz's result, but also gives a generalization of one of the main results of [12], as in Theorem 7 in [12] they prove a trace Jensen's inequality for maps of the form $x \to a^* x a$, where $a$ is a contraction.

We need a few lemmas, which are generalizations of Lemmas 4 and 5 given in [12]. Indeed, in [12] the following lemma was stated for the special map $\Phi(x) = a^* x a$, for a contraction $a \in \mathcal{M}$. Here we show that the assertion holds for a positive contractive map. In the following we will use the concept of spectral pre-order.

**Definition 2.** Let $a, b \in \mathcal{M}$ be two self-adjoint elements. Then we say that $a \lesssim b$ if $e_{(s,\infty)}(a)$, is equivalent, in the Murray-von Neumann sense, to a subprojection of $e_{(s,\infty)}(b)$ for every real number $s$, where $e_I(z)$ is the spectral projection of any self-adjoint element $z \in \mathcal{M}$ corresponding to the subset $I \subseteq \mathbb{R}$.

We remark that if $a \lesssim b$ and $\tau$ is a normal semifinite trace on $\mathcal{M}$ then $\tau(a) \leqslant \tau(b)$ (see [3, Lemma 3].

The following Lemma was proved in [3] for maps $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$, $\Phi(x) = a^* x a$, where $a \in B(\mathcal{H})$ is a contraction. It is easily generalized to any contractive positive normal map on a von Neumann algebra $\mathcal{M}$.

**Lemma 5.** *Let $\Phi : \mathcal{M} \to \mathcal{M}$ be a unital positive map and suppose $\mathcal{M} \subseteq B(\mathcal{H})$. Then for any continuous convex function $f : \mathbb{R} \to \mathbb{R}$ and a self-adjoint $x \in \mathcal{M}$, we have*

$$f(\langle \Phi(x)\xi, \xi\rangle) \leqslant \langle \Phi(f(x))\xi, \xi\rangle,$$

*for any unit vector $\xi \in \mathcal{H}$.*

14692120, 2026, 4, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.70347 by Statens Beredning, Wiley Online Library on [27/04/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

*Furthermore, if $\Phi$ is assumed to be a positive contraction, then the above inequality holds for any continuous convex function $f : \mathbb{R} \to \mathbb{R}$ with the assumption $f(0) = 0$.*

*Proof.* Note that a positive unital map on $\mathcal{M}$ is necessarily contractive. Let $\Phi : \mathcal{M} \to \mathcal{M}$ be a positive contraction. As $g \mapsto \langle \Phi(g(x))\xi, \xi\rangle$ is a positive bounded linear functional on $C_0(\mathbb{R})$, by the Riesz theorem there exists a Borel measure $\mu_\Phi$ such that

$$\langle \Phi(g(x))\xi, \xi\rangle = \int_{\mathbb{R}} g(t) d\mu_\Phi(t).$$

As the spectrum $\sigma(x)$ is a compact subset of $\mathbb{R}$ and $g(x) = 0$ if supp $g \cap \sigma(x) = \emptyset$, supp $\mu_\Phi$ is compact. Taking $g \in C_0(\mathbb{R})$, $g = 1$ on a compact containing $\sigma(x)$ and supp $\mu_\Phi$, and using the fact that $\Phi$ is a positive contraction, we obtain $\mu_\Phi(\mathbb{R}) = \langle \Phi(1)\xi, \xi\rangle \leqslant 1$. Let $\nu_\Phi = \mu_\Phi + (1 - \langle \Phi(1)\xi, \xi\rangle)\delta_0$, where $\delta_0$ is the Dirac measure supported in $\{0\}$. Then $\nu_\Phi$ is a probability measure. Now suppose that $f(0) = 0$. As the functions $f(t)$ and $g(t) = t$ vanish at 0, we obtain from the classical Jensen's inequality that

$$\begin{aligned}\langle \Phi(f(x))\xi, \xi\rangle &= \int_{\mathbb{R}} f(t) d\mu_\Phi(t) = \int_{\mathbb{R}} f(t) d\nu_\Phi(t)\\ &\geqslant f\left(\int_{\mathbb{R}} t d\nu_\Phi(t)\right) = f\left(\int_{\mathbb{R}} t d\mu_\Phi(t)\right) = f(\langle \Phi(x)\xi, \xi\rangle).\end{aligned}$$

Note that if $\Phi$ is unital, then $\mu_\Phi$ is a probability measure and the rest of the proof follows for any continuous convex function $f$ without the assumption $f(0) = 0$. □

**Lemma 6.** *Let $\Phi : \mathcal{M} \to \mathcal{M}$ be a unital positive map. For any continuous convex function $f$ and a self-adjoint $x \in \mathcal{M}$, let $I \subset \mathbb{R}$ be an interval on which $f(t)$ is monotone, that is, either increasing or decreasing. And set $p = e_I(\Phi(x))$ to be the spectral projection of $\Phi(x)$ on the interval $I$.*

(a) *If $f(t) \geqslant 0$ on the interval $I$, then we have $p\Phi(f(x))p \geqslant 0$ and moreover*

$$p\Phi(f(x))p \gtrsim pf(\Phi(x))p.$$

(b) *If $f(t) \leqslant 0$ on the interval $I$, then the negative part $(p\Phi(f(x))p)_-$ of the Jordan decomposition satisfies*

$$-pf(\Phi(x))p \gtrsim (p\Phi(f(x))p)_-.$$

*Furthermore, if $\Phi$ is assumed to be a positive contraction, then the assertions of $(a)$ and $(b)$ follow for any continuous convex functions $f$ with $f(0) = 0$.*

*Proof.*

(a) To see how to prove $p\Phi(f(x))p \geqslant 0$, we note that if we choose $\xi \in \text{Range}(p)$, then using Lemma 5 yields $\langle \Phi(f(x))\xi, \xi\rangle \geqslant f(\langle \Phi(x)\xi, \xi\rangle) \geqslant 0$. The last inequality follows because $f$ was assumed to be positive on $I$.

14692120, 2026, 4, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.70347 by Statens Beredning, Wiley Online Library on [27/04/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

We have to prove that for any $s > 0$ there exists a projection $q_s$ such that

$$e_{(s,+\infty)}(pf(\Phi(x))p) \sim q_s \leqslant e_{(s,+\infty)}(p\Phi(f(x))p).$$

For $s > 0$ we write $[f > s] \cap I$ for $\{t \in I : f(t) > s\}$. We observe first that $e_{(s,+\infty)}(pf(\Phi(x))p) = e_{[f>s]\cap I}(\Phi(x))$, as $\chi_I(t)f(t) > s$ if and only if $t \in I$ and $f(t) > s$. Assuming that $e_{[f>s]\cap I}(\Phi(x))$ is non-zero, in particular, $[f > s] \cap I \neq \emptyset$, and taking $\xi$ a unit vector in Range$(e_{[f>s]\cap I}(\Phi(x))$, we obtain using monotonicity of $f$ on $I$, that $\langle \Phi(x)\xi, \xi\rangle \in \overline{[f > s] \cap I}$. In fact, letting $q = e_{[f>s]\cap I}(\Phi(x))$, the spectrum $\sigma(q\Phi(x)q)$ is in the closure of $[f > s] \cap I$ which is a closed interval, say $[\alpha, \beta] \subset \bar{I}$, giving $\alpha \leqslant \langle \Phi(x)\xi, \xi\rangle \leqslant \beta$. Assume that $f$ is increasing on $I$ (the case when $f$ is decreasing is treated similarly). Then either $f(\alpha) = s$ or $f(\alpha) > s$; in the latter case $f(\langle \Phi(x)\xi, \xi\rangle) \geqslant f(\alpha) > s$ and hence

$$s < f(\langle \Phi(x)\xi, \xi\rangle) = f(\langle p\Phi(x)p\xi, \xi\rangle);$$

and in the former case if $f(\langle \Phi(x)\xi, \xi\rangle) = f(\alpha) = s$, then $\langle q\Phi(x)q\xi, \xi\rangle = \langle \Phi(x)\xi, \xi\rangle \leqslant \alpha$, as $f(t) > s$ for all $t > \alpha$, $t \in I$; on the other hand, $q\Phi(x)q \geqslant \alpha$, giving $\Phi(x)\xi = q\Phi(x)q\xi = \alpha\xi$ and $\xi = e_{[f=s]}(\Phi(x))\xi$, a contradiction. Therefore,

$$f(\langle p\Phi(x)p\xi, \xi\rangle) = f(\langle \Phi(x)\xi, \xi\rangle) > f(\alpha) = s.$$

Applying Lemma 5, we get

$$\langle p\Phi(f(x))p\xi, \xi\rangle > s$$

giving $e_{(-\infty,s]}(p\Phi(f(x))p)\xi = 0$ and hence

$$e_{(s,+\infty)}(pf(\Phi(x))p) \wedge e_{(-\infty,s]}(p\Phi(f(x))p) = 0.$$

If $e_{(s,+\infty)}(pf(\Phi(x))p) = e_{[f>s]\cap I}(\Phi(x)) = 0$, the latter holds trivially.

Now using Kaplanski's formula $p_1 \vee p_2 - p_1 \sim p_2 - p_1 \wedge p_2$ for any pair of projections $(p_1, p_2)$, we obtain

$$\begin{aligned} &e_{(s,+\infty)}(pf(\Phi(x))p) = e_{(s,+\infty)}(pf(\Phi(x))p) - e_{(s,+\infty)}(pf(\Phi(x))p) \wedge e_{(-\infty,s]}(p\Phi(f(x))p) \\ &\sim e_{(s,+\infty)}(pf(\Phi(x))p) \vee e_{(-\infty,s]}(p(\Phi(f(x)))p) - e_{(-\infty,s]}(p\Phi(f(x))p) \\ &\leqslant I - e_{(-\infty,s]}(p\Phi(f(x))p) = e_{(s,+\infty)}(p\Phi(f(x))p). \end{aligned}$$

(b) is obtained using similar arguments. □

*Proof of Theorem* 4. We follow the same line of argument as given in [12]. Without loss of generality, we consider $f(t)$ to be decreasing at the origin. We assume that $f(t)$ is

1. positive and decreasing on $I_1 = (-\infty, 0)$,
2. negative and decreasing on $I_2 = [0, t_1)$,
3. negative and increasing on $I_3 = [t_1, t_2)$,
4. positive and increasing on $I_4 = [t_2, \infty)$.

14692120, 2026, 4, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.70347 by Statens Beredning, Wiley Online Library on [27/04/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Some of these intervals may be empty, for example, if $f(t) \geqslant 0$, then $I_2 = I_3 = \varnothing$. Now, for the self-adjoint element $x \in \mathcal{M}$ we set $p_i = e_{I_i}(\Phi(x))$ and let $E(y) = \sum_{i=1}^{4} p_i y p_i$. We can write $f(\Phi(x)) = E(f(\Phi(x))) = \sum_i p_i f(\Phi(x)) p_i$ in the matrix form:

$$f(\Phi(x)) = \begin{bmatrix} p_1 f(\Phi(x))p_1 & 0 & 0 & 0 \\ 0 & p_2 f(\Phi(x))p_2 & 0 & 0 \\ 0 & 0 & p_3 f(\Phi(x))p_3 & 0 \\ 0 & 0 & 0 & p_4 f(\Phi(x))p_4 \end{bmatrix}.$$

Note that for $i = 1, 4$, $p_i f(\Phi(x)) p_i \geqslant 0$ and for $i = 2, 3$, we have $p_i f(\Phi(x)) p_i \leqslant 0$.

However, $\Phi(f(x))$ is not diagonal, and we have

$$E(\Phi(f(x))) = \begin{bmatrix} p_1 \Phi(f(x))p_1 & 0 & 0 & 0 \\ 0 & p_2 \Phi(f(x))p_2 & 0 & 0 \\ 0 & 0 & p_3 \Phi(f(x))p_3 & 0 \\ 0 & 0 & 0 & p_4 \Phi(f(x))p_4 \end{bmatrix}.$$

Note that using part $(a)$ of the Lemma 6 we have $p_1 \Phi(f(x)) p_1 \geqslant 0$ and $p_4 \Phi(f(x)) p_4 \geqslant 0$.

The Jordan decomposition of $f(\Phi(x)) = f(\Phi(x))_+ - f(\Phi(x))_-$, where $f(\Phi(x))_+ \geqslant 0$ and $f(\Phi(x))_- \geqslant 0$ have orthogonal supports, is now given by

$$f(\Phi(x))_+ = \begin{bmatrix} p_1 f(\Phi(x))p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_4 f(\Phi(x))p_4 \end{bmatrix},$$

and

$$f(\Phi(x))_- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -p_2 f(\Phi(x))p_2 & 0 & 0 \\ 0 & 0 & -p_3 f(\Phi(x))p_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

However, the Jordan decomposition of $\Phi(f(x))$ is difficult to describe, but $E(\Phi(f(x)))$ is given by

$$(E(\Phi(f(x))))_+ = \begin{bmatrix} p_1 \Phi(f(x))p_1 & 0 & 0 & 0 \\ 0 & (p_2 \Phi(f(x))p_2)_+ & 0 & 0 \\ 0 & 0 & (p_3 \Phi(f(x))p_3)_+ & 0 \\ 0 & 0 & 0 & p_4 \Phi(f(x))p_4 \end{bmatrix},$$

$$(E(\Phi(f(x))))_- = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (p_2 \Phi(f(x))p_2)_- & 0 & 0 \\ 0 & 0 & (p_3 \Phi(f(x))p_3)_- & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now using Lemma 6 we can compare the diagonal blocks of $f(\Phi(x))_\pm$ and $(E(\Phi(f(x))))_\pm$:

14692120, 2026, 4, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.70347 by Statens Beredning, Wiley Online Library on [27/04/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$p_i\Phi(f(x))p_i \gtrsim p_i f(\Phi(x))p_i, \text{ for } i = 1, 4$$

$$-p_i f(\Phi(x))p_i \gtrsim (p_i\Phi(f(x))p_i)_- \text{ for } i = 2, 3.$$

Taking direct sum, we get

$$(E(\Phi(f(x)))_+ \geqslant \begin{bmatrix} p_1\Phi(f(x))p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_4\Phi(f(x))p_4 \end{bmatrix}$$

$$\gtrsim \begin{bmatrix} p_1 f(\Phi(x))p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_4 f(\Phi(x))p_4 \end{bmatrix} = f(\Phi(x))_+.$$

Thus we get

$$(E(\Phi(f(x)))_+ \gtrsim f(\Phi(x))_+. \tag{2}$$

And similarly one obtains

$$f(\Phi(x))_- \gtrsim (E(\Phi(f(x)))_-. \tag{3}$$

Thus, Equation (2) yields $\tau((E(\Phi(f(x)))_+) \geqslant \tau(f(\Phi(x))_+)$ and Equation (3) yields $\tau(f(\Phi(x))_-) \geqslant \tau((E(\Phi(f(x)))_-)$. Now assume that both the quantities $\tau(\Phi(f(x)))$ and $\tau(f(\Phi(x)))$ are well defined. Using the fact that $E$ is $\tau$-preserving, we get that $\tau(E(\Phi(f(x))))$ is well defined. Indeed, from the minimality of the Jordan decomposition (see [12, Section 2.4]) we see that

$$\tau(E(\Phi(f(x)))_\pm) \leqslant \tau(E(\Phi(f(x))_\pm)) = \tau(\Phi(f(x))_\pm).$$

Hence we obtain

$$\begin{aligned} \tau(E(\Phi(f(x))) &= \tau((E(\Phi(f(x)))_+) - \tau((E(\Phi(f(x)))_-) \\ &\geqslant \tau(f(\Phi(x))_+) - \tau(f(\Phi(x))_-) \\ &= \tau(f(\Phi(x))). \end{aligned}$$

Now note that

$$E(\Phi(f(x)))_+ - E(\Phi(f(x)))_- = E(\Phi(f(x))) = E(\Phi(f(x))_+) - E(\Phi(f(x))_-)$$

and hence

$$\tau(E(\Phi(f(x)))_+) + \tau(E(\Phi(f(x))_-)) = \tau(E(\Phi(f(x))_+)) + \tau(E(\Phi(f(x)))_-),$$

and we get using the trace preservation of $E$

$$\begin{aligned} \tau(\Phi(f(x))) &= \tau(\Phi(f(x))_+) - \tau(\Phi(f(x))_-) \\ &= \tau(E(\Phi(f(x))_+)) - \tau(E(\Phi(f(x))_-)) \end{aligned}$$

14692120, 2026, 4, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.70347 by Statens Beredning, Wiley Online Library on [27/04/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$= \tau((E(\Phi(f(x)))_+) - \tau((E(\Phi(f(x)))_-)$$
$$= \tau(E(\Phi(f(x))) \geqslant \tau(f(\Phi(x))).$$ □

## 4 | THE PROOF OF THE MAIN THEOREM

Let $(\mathcal{M}_1, \tau_1), (\mathcal{M}_2, \tau_2)$ be two tracial von Neumann algebras. For $\omega \in (\mathcal{M}_2)_* \simeq L^1(\mathcal{M}_2, \tau_2)$ let $R_\omega : \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2 \to \mathcal{M}_1$ be the right slice operator defined on elementary tensors as $R_\omega(a \otimes b) = a\omega(b)$. Similarly, we define a left slice map $L_w : \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2 \to \mathcal{M}_2$, $w \in (\mathcal{M}_1)_*$. The maps are completely bounded normal maps such that $\|L_w\|_{\text{cb}} = \|w\|$, $\|R_\omega\|_{cb} = \|\omega\|$. Moreover, if $\omega$ is a positive linear functional (a state), then $R_\omega$ is normal completely positive (and unital).

The same holds for the left slice map. Then for any $X \in \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$, $a, b \in \mathcal{M}_1$, we have $R_\omega((a \otimes 1)X(b \otimes 1)) = aR_\omega(X)b$ (checked on elementary tensor products). Assuming $\tau_1(a^*a) < \infty$, that is, $a \in L^2(\mathcal{M}_1, \tau_1)$, we obtain $a^*R_\omega(X)a$ is in $L^1(\mathcal{M}_1, \tau_1)$, with

$$|\tau_1(a^*R_\omega(X))a| \leqslant \|a\|_2\|R_\omega(X)\| \leqslant \|a\|_2\|\omega\|\|X\|.$$

Therefore we can define an element $(\tau_1 \otimes \text{id})((a^* \otimes 1)X(a \otimes 1))$ in $\mathcal{M}_2$ by letting

$$\langle(\tau_1 \otimes \text{id})((a^* \otimes 1)X(a \otimes 1)), \omega\rangle = \tau_1(a^*R_\omega(X)a).$$

We have also $\|(\tau_1 \otimes \text{id})((a^* \otimes 1)X(a \otimes 1))\| \leqslant \|a\|_2\|X\|$. Note that letting $\omega_{\tau_1,a}(Y) = \tau_1(a^*Ya)$, $Y \in \mathcal{M}_1$, which is a bounded linear functional on $\mathcal{M}_1$, we have $(\tau_1 \otimes \text{id})((a^* \otimes 1)X(a \otimes 1)) = L_{\omega_{\tau_1,a}}(X)$.

*Remark* 1. For $X \in \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$, we say that $(\text{id} \otimes \tau_2)(X)$ is **defined** whenever $L_\omega(X) \in L^1(\mathcal{M}_2, \tau_2)$ for any $\omega \in (\mathcal{M}_1)_*$, and let $\langle(\text{id} \otimes \tau_2)(X), \omega\rangle := \tau_2(L_\omega(X))$. By the Uniform Boundedness Principle, $(\text{id} \otimes \tau_2)(X)$ is indeed a bounded linear functional on $(\mathcal{M}_1)_*$ and hence is an element of $\mathcal{M}_1$.

In Theorem 2, by (1) being defined we mean that $(\text{id} \otimes \tau_2)(f(H))$ is defined and show that in this case the left-hand side is defined in the sense of Definition 1.

**Lemma 7.** *Let $a \in L^2(\mathcal{M}_1, \tau_1)$ and $X \in \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$. If $(\text{id} \otimes \tau_2)(X)$ is defined, then $(\tau_1 \otimes \text{id})((a^* \otimes 1)X(a \otimes 1)) \in L^1(\mathcal{M}_2, \tau_2)$, and*

$$\tau_2((\tau_1 \otimes \text{id})((a^* \otimes 1)X(a \otimes 1))) = \tau_1(a^*(\text{id} \otimes \tau_2)(X)a).$$

*Proof.* Consider $\omega \in (\mathcal{M}_1)_*$, given by $\omega(M) = \tau_1(a^*Ma)$, $M \in \mathcal{M}_1$. Then for any $w \in (\mathcal{M}_2)_*$, as

$$\langle L_\omega(X), w\rangle = \langle X, \omega \otimes w\rangle = \omega(R_w(X))$$
$$= \tau_1(a^*R_w(X)a) = \langle(\tau_1 \otimes \text{id})(a^* \otimes 1)X(a \otimes 1)), w\rangle,$$

we obtain $L_\omega(X) = (\tau_1 \otimes \text{id})((a^* \otimes 1)X(a \otimes 1)) \in L^1(\mathcal{M}_2, \tau_2)$.

On the other hand, $\langle(\text{id} \otimes \tau_2)(X), \omega\rangle = \tau_1(a^*(\text{id} \otimes \tau_2)(X)a)$, giving

$$\tau_2(L_\omega(X)) = \tau_2((\tau_1 \otimes \text{id})(a^* \otimes 1)X(a \otimes 1)) = \tau_1(a^*(\text{id} \otimes \tau_2)(X)a).$$ □

14692120, 2026, 4, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.70347 by Statens Beredning, Wiley Online Library on [27/04/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

*Proof of Theorem* 2. Define a map $\Phi : \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2 \to \mathcal{M}_2$ by

$$\Phi(X) = (\tau_1 \otimes \mathrm{id})((a^* \otimes 1)X(a \otimes 1)).$$

Note that the condition $\tau_1(a^*a) = 1$ makes the map well-defined, completely positive, and unital as the slice map is $L_{\omega_{\tau_1,a}}$ with $\omega_{\tau_1,a}(y) = \tau_1(a^*ya)$, $y \in \mathcal{M}_1$; see the above arguments.

Using Theorem 4, we get for any self-adjoint $H$

$$\tau_2(f(\Phi(H))) \leqslant \tau_2(\Phi(f(H))),$$

whenever both sides are defined.

As $\Phi(f(H)) \in L^1(\mathcal{M}_2, \tau_2)$, the right-hand side is defined and finite. Moreover, if $E$ is as in the proof of Theorem 4, $E(\Phi(f(H))) \in L^1(\mathcal{M}_2, \tau_2)$ and $\tau_2(\Phi(f(H))) = \tau_2(E(\Phi(f(H))))$ and $\tau_2((E(\Phi(f(H)))_+)$, $\tau_2(E(\Phi(f(H)))_-)$ are finite. It now follows from the proof of Theorem 4 that $\tau_2(f(\Phi(H))_+) \leqslant \tau_2(E(\Phi(f(H)))_+) < \infty$ and hence the left-hand side is defined, but can take the value $-\infty$. This yields

$$\tau_2 f(\tau_1 \otimes \mathrm{id})[(a^* \otimes 1)H(a \otimes 1)] \leqslant \tau_2(\tau_1 \otimes \mathrm{id})[(a^* \otimes 1)f(H)(a \otimes 1)].$$

Now using Lemma 7 to the RHS of the above inequality we get

$$\tau_2 f(\tau_1 \otimes \mathrm{id})[(a^* \otimes 1)H(a \otimes 1)] \leqslant \tau_1(a^*(\mathrm{id} \otimes \tau_2)(f(H))a),$$

which yields the desired result. □

## 5 | JENSEN'S INEQUALITY FOR STATES (NON-TRACIAL CASE)

In this section we show a similar Jensen inequality with general, not necessarily, tracial states. We need the notion of operator convexity which is a stronger notion than that of ordinary convexity. Note that a continuous function $f$ on an interval $I$ is said to be **operator convex** if $f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y)$ holds whenever $\lambda \in [0,1]$ and for any self-adjoint operators $x, y$ with spectra included in $I$. Operator convexity is stronger than ordinary convexity [2].

**Theorem 8.** *Let $\mathcal{M}_1, \mathcal{M}_2$ be two von Neumann algebras and two normal states $\rho_1, \rho_2$ acting on $\mathcal{M}_1, \mathcal{M}_2$ respectively. Suppose that $H$ is a self-adjoint element in $\mathcal{M}_1 \bar{\otimes} M_2$. Then for any contraction $a \in \mathcal{M}_1$ and any operator convex function $f$ defined on the spectrum of $H$, we have*

$$\rho_2 f[(\rho_1 \otimes \mathrm{id})((a^* \otimes 1)H(a \otimes 1))] \leqslant \rho_1(a^*(\mathrm{id} \otimes \rho_2)(f(H))a). \tag{4}$$

*Proof.* We note that for states $\rho_1, \rho_2$, the corresponding left and right slice maps are unital completely positive maps, that is, $L_{\rho_1} : \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2 \to \mathcal{M}_2$ defined by $L_{\rho_1}(x \otimes y) = (\rho_1 \otimes \mathrm{id})(x \otimes y) = \rho_1(x)y$ and similarly $R_{\rho_2} : \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2 \to \mathcal{M}_1$ defined by $R_{\rho_2}(x \otimes y) = (\mathrm{id} \otimes \rho_2)(x \otimes y) = x\rho_2(y)$. Note that these maps are unital because the $\rho_1, \rho_2$ are states, so we have $\rho_1(1) = 1 = \rho_2(1)$, where we abused our notations slightly and denoted the identity element of both the algebras as 1 and the number appears in the middle is the numeric 1.

Note that for any ucp map $\Phi$ and an operator convex function $f$ on an interval $I$, we have $\Phi(f(H)) \geqslant f(\Phi(H))$ for every self-adjoint $H$ (see [6, 7]) such that spectrum of $H$ is inside $I$. Now,

14692120, 2026, 4, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.70347 by Statens Beredning, Wiley Online Library on [27/04/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

calling $X := (a^* \otimes 1)H(a \otimes 1)$, which is a self-adjoint element, we have for the left-hand side of Equation (4)

$$\rho_2 f[(\rho_1 \otimes \mathrm{id})((a^* \otimes 1)H(a \otimes 1))] = \rho_2(f(L_{\rho_1}(X))) \leqslant \rho_2(L_{\rho_1}(f(X))). \tag{5}$$

Now note that for any $A \in \mathcal{M}_1 \bar{\otimes} M_2$ we have

$$\rho_2(L_{\rho_1}(A)) = (\rho_1 \otimes \rho_2)(A) = \rho_1(R_{\rho_2}(A)).$$

Now using the above commutation relation we see the right-hand side of Equation (5)

$$\rho_2(L_{\rho_1}(f(X))) = \rho_1(R_{\rho_2}(f[(a^* \otimes 1)H(a \otimes 1)])) \leqslant \rho_1(R_{\rho_2}((a^* \otimes 1)f(H)(a \otimes 1))), \tag{6}$$

where we have used Jensen's operator inequality (contractive version) from Hansen and Pederson (see Corollary 2.3 in [10]):

$$f((a^* \otimes 1)H(a \otimes 1)) \leqslant (a^* \otimes 1)f(H)(a \otimes 1),$$

and positivity of the map $\rho_1 R_{\rho_2}$. As for any $a \in \mathcal{M}_1$ and $A \in \mathcal{M}_1 \bar{\otimes} \mathcal{M}_2$,

$$R_{\rho_2}((a^* \otimes 1)A(a \otimes 1)) = a^* R_{\rho_2}(A)a,$$

we get that the right-hand side of Equation (6) is

$$\rho_1 R_{\rho_2}((a^* \otimes 1)f(H)(a \otimes 1))) = \rho_1[a^* R_{\rho_2}(f(H))a].$$

Putting all together we have

$$\rho_2 f[(\rho_1 \otimes \mathrm{id})((a^* \otimes 1)H(a \otimes 1))] \leqslant \rho_1[a^*(\mathrm{id} \otimes \rho_2)(f(H))a]. \qquad \square$$

*Remark* 2. We remark that normal states $\rho_1$, $\rho_2$ in Theorem 8 can be replaced by unital normal completely positive maps to have the following operator inequality:

$$(\mathrm{id} \otimes \rho_2)(f[(\rho_1 \otimes \mathrm{id})((a^* \otimes 1)H(a \otimes 1))]) \leqslant (\rho_1 \otimes \mathrm{id})(a^* \otimes 1(\mathrm{id} \otimes \rho_2)(f(H))a \otimes 1).$$

Working with the scalar case it would be interesting to know whether the statement of Theorem 8 holds with continuous convex function $f$ instead of operator convex one.

## 6 | OUTLOOK

The finite-dimensional techniques used in the proof of Theorem 1 can be extended to the infinite-dimensional setting under certain assumptions on the discreteness of the spectrum, as noted in [5]. Under these conditions, the authors obtained results on eigenvalue asymptotics. We expect that our theorem will allow one to treat more general situations beyond this setting.

Trace inequalities, and operator Jensen's inequality in particular, have played an important role in the study of quantum entropy of quantum systems (see [4, 8]). It would be interesting to investigate whether the results of this paper have implications for problems arising in quantum information theory.

14692120, 2026, 4, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.70347 by Statens Beredning, Wiley Online Library on [27/04/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

## ACKNOWLEDGMENTS

We thank Simon Larson for many interesting discussions during the course of this work. We also thank the anonymous referee for many helpful suggestions that improved the exposition of the article. M.R. acknowledges support from the Wallenberg Center for Quantum Technology (WACQT) and L.T. was supported by the Swedish Research Council project grant 2023-04555.

## JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

## ORCID

*Mizanur Rahaman* https://orcid.org/0000-0002-0301-0496
*Lyudmila Turowska* https://orcid.org/0000-0002-3141-5968

## REFERENCES

1. J. Antezana, P. Massey, and D. Stojanoff, *Jensen's inequality for spectral order and submajorization*, J. Math. Anal. Appl. **331** (2007), no. 1, 297–307.
2. R. Bhatia, *Operator monotone and operator convex functions*, Springer, New York, 1997, pp. 112–151.
3. L. G. Brown and H. Kosaki, *Jensen's inequality in semi-finite von Neumann algebras*, J. Oper. Theory **23** (1990), no. 1, 3–19.
4. E. Carlen, *Inequalities in matrix algebras*, Grad. Stud. Math., vol. 251, Amer. Math. Soc., Providence, RI, 2023, pp. 416.
5. E. A. Carlen, R. L. Frank, and S. Larson, *A Jensen inequality for partial traces and applications to partially semiclassical limits*, Lett. Math. Phys. **115** (2025), no. 3, 52.
6. M. D. Choi, *A Schwarz inequality for positive linear maps on C*-algebras*, Illinois J. Math. **18** (1974), 565–574.
7. C. Davis, *A Schwarz inequality for convex operator functions*, Proc. Amer. Math. Soc. **8** (1957), no. 1, 42–44.
8. E. G. Effros, *A matrix convexity approach to some celebrated quantum inequalities*, Proc. Natl. Acad. Sci. USA **106** (2009), no. 4, 1006–1008.
9. D. R. Farenick and F. Zhou, *Jensen's inequality relative to matrix-valued measures*, J. Math. Anal. Appl. **327** (2007), no. 2, 919–929.
10. F. Hansen and G. K. Pedersen, *Jensen's operator inequality*, Bull. Lond. Math. Soc. **35** (2003), no. 4, 553–564.
11. F. Hansen and G. K. Pedersen, *Jensen's trace inequality in several variables*, Internat. J. Math. **14** (2003), no. 6, 667–681.
12. T. Harada and H. Kosaki, *Trace Jensen inequality and related weak majorization in semi-finite von Neumann algebras*, J. Oper. Theory **63** (2010), no. 1, 129–150.
13. V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Stud. Adv. Math., vol. 78, Cambridge Univ. Press, Cambridge, New York, 2002, pp. xii+300.
14. D. Petz, *Jensen's inequality for positive contractions on operator algebras*, Proc. Amer. Math. Soc. **99** (1987), no. 2, 273–277.
15. G. Pisier and Q. Xu, *Non-commutative $L_p$-spaces*, Handbook of the Geometry of Banach Spaces, vol. 2, North-Holland, Amsterdam, 2003, pp. 1459–1517.
16. M. Takesaki, *Theory of operator algebras. I*, Springer-Verlag, New York–Heidelberg, 1979, pp. x+415.

14692120, 2026, 4, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/blms.70347 by Statens Beredning, Wiley Online Library on [27/04/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License