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Homogenization of an indefinite spectral problem arising in population genetics

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Abstract

We study an indefinite spectral problem for a second-order self-adjoint elliptic operator in an asymptotically thin cylinder. The operator coefficients and the spectral density function are assumed to be locally periodic in the axial direction of the cylinder. The key assumption is that the spectral density function changes sign, which leads to infinitely many both positive and negative eigenvalues. The asymptotic behavior of the spectrum, as the thickness of the rod tends to zero, depends essentially on the sign of the average of the density function. We study the positive part of the spectrum in a specific case when the local average is negative. We derive a one-dimensional effective spectral problem that is a harmonic oscillator on the real line, and prove the convergence of the spectrum.

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1. Introduction

This paper deals with the homogenization of an indefinite spectral problem for a second-order elliptic operator defined in a thin cylindrical domain. The key assumption is the presence of a sign-changing weight function in front of the spectral parameter and the local periodicity of the operator coefficients. The homogeneous Neumann boundary condition is assumed on the lateral boundary of the cylinder, whereas the homogeneous Dirichlet condition is imposed on its bases. The asymptotic behavior of the spectrum is analyzed as the thickness of the cylindrical domain vanishes. Namely, for a small parameter $\varepsilon > 0$, in the cylinder Ω_ε with thickness of order ε and length of order 1, we consider the following spectral problem:

$$\begin{cases} -\operatorname{div}\left(a^\varepsilon(x)\nabla u^\varepsilon(x)\right) = \lambda^\varepsilon \rho^\varepsilon(x) u^\varepsilon(x), & x \in \Omega_\varepsilon, \\ a^\varepsilon(x) \nabla u^\varepsilon(x) \cdot n = 0, & x \in \Sigma_\varepsilon, \\ u^\varepsilon(-1, x') = u^\varepsilon(1, x') = 0, & x' \in \varepsilon Q. \end{cases} \tag{1}$$

The assumptions on the domain and the coefficients are given in Section 2. If $\rho(x_1, y)$ changes sign, there exist two sequences of eigenvalues. In [22], the authors derived the asymptotics for the positive eigenvalues in the case of a positive local average of the spectral weight, and both positive and negative when the average changes sign. One case was not studied: the asymptotics of the negative part of the spectrum for the positive local average. This is the focus of the present paper. We will show (see Theorem 2.2) that the positive eigenvalues grow as $1/\varepsilon^2$, and the corresponding eigenfunctions localize under the assumption that the principal positive eigenvalue of an auxiliary cell problem (18) attains a unique global minimum in the domain. The existence of a positive principal eigenvalue of a cell spectral problem is of interest in its own and is stated in Lemma 3.3. Moreover, we study the regularity of eigenvalues and eigenfunctions of the cell spectral problem with respect to the slow variable, which plays a role of a parameter (see Lemma 3.4).

Spectral problems with indefinite weight arise when modeling population genetics [9,18,20]. Consider, for example, a model with two types of alleles A_1 and A_2 , and let $p(t, x)$ denote the frequency of the first allele A_1 at time t and point x in some bounded habitat \mathcal{O} . Then $p = p(t, x)$ is assumed to satisfy a nonlinear evolution equation:

$$\begin{aligned} \frac{\partial p}{\partial t} &= m \Delta p + \lambda s(x) p(1 - p) \text{ in } (0, \infty) \times \mathcal{O}, \\ \nabla p \cdot \nu &= 0 \text{ on } (0, \infty) \times \partial \mathcal{O}. \end{aligned}$$

It has been proved in [9] that if $\lambda > 0$ is a bifurcation point of $p = 0$, then λ is an eigenvalue of the linearized problem

$$\begin{aligned} -m \Delta u(x) &= \lambda s(x) u(x) \text{ in } \mathcal{O}, \\ \nabla u \cdot \nu &= 0 \text{ on } \partial \mathcal{O}, \end{aligned} \tag{2}$$

which is a spectral problem with a sign-changing weight. The stability of the trivial equilibria is determined by the sign of the average of the selection coefficient $\int_{\mathcal{O}} s(x) dx$ and by the value of the smallest positive eigenvalue λ_1 of (2).

It is natural to extend the above model to the case when the diffusion m and selection criteria s are spatially heterogeneous and to allow for oscillations in x , which motivates us to consider an indefinite spectral problem with rapidly varying coefficients, with a small period of oscillations denoted by $\varepsilon > 0$. In particular, $s(x)$ might favor allele A_1 at location x , so A_1 has a selective advantage at some regions of \mathcal{O} , and might become a disadvantage in other regions, making $s(x)$ negative, which implies that $s(x)$ changes sign.

We consider a cylindrical domain Ω_ε whose thickness is assumed to be $O(\varepsilon)$, of the same order as the period of oscillations. Modeling population genetics in random media is technically challenging, and periodic approximations are commonly employed. In this work, however, we adopt a more general framework by assuming local periodicity of the coefficients, rather than the classical global periodicity (see problem (3) and hypotheses (H1)–(H5)). From the perspective of population genetics, the asymptotic behavior of the first positive eigenvalue of (3) plays a crucial role, as it determines a critical threshold beyond which instability—and consequently, the emergence of a non-trivial equilibrium—occurs.

The key components of the present work are: the presence of an indefinite weight, locally periodic coefficients, and, as a consequence, the localization of eigenfunctions corresponding to positive eigenvalues in the case of a negative local average of the spectral weight. In addition, the thin domain leads to the dimension reduction problem.

Homogenization of spectral problems has been extensively studied starting from 70s. Kozlov in [17] studied the averaging phenomena of random operators. Homogenization of the elliptic spectral problem with Dirichlet data was studied by Kesavan [14,15]. In the classical case, the convergence of spectra follows from the convergence of the solutions to boundary value problems. Singularly perturbed spectral problems show, however, a different asymptotic behavior. Below, we describe some closely related results.

In [7], a periodic spectral problem for a second-order elliptic operator with a large drift term was studied. The spectral problem with neutronic multigroup diffusion was studied in [1], and a drift in linear transport was studied by Bal in [3]. In these works, under periodicity assumptions, a factorization with an eigenfunction of an auxiliary spectral problem is used to describe the oscillations. In [2] the authors consider the homogenization of the spectral problem for a singularly perturbed diffusion equation in a periodic setting.

In [11], the authors have considered a Dirichlet spectral problem in a bounded domain with a smooth boundary surrounded by a thin band. It is shown that a localization phenomenon occurs near the extrema of the curvature, see [13,21,12] for maxima and for minima [5,10]. In the problems for banded domains, as in [11], they occur for both maxima and minima.

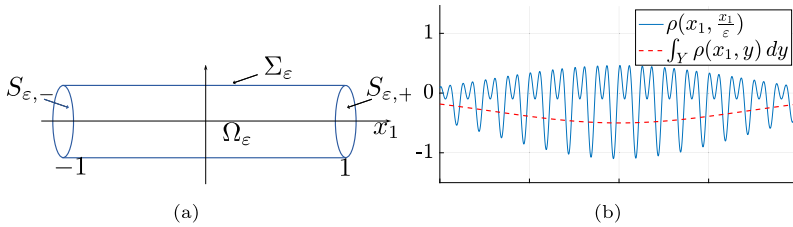


Fig. 1. (a) Thin cylinder $\Omega_\varepsilon = [-1, 1] \times (\varepsilon Q)$; (b) Example of a locally periodic function $\rho(x_1, y) = 0.5 e^{-x_1^2} (\sin(2\pi y) + \cos(4\pi y) - 0.1)$ with $\varepsilon = 1/8$, and its local average $\int_Y \rho(x_1, y) dy = -0.05 e^{-x_1^2} < 0$.

It turns out that introducing, in addition to singular perturbation, local periodicity in the coefficients or in the geometry of the domain leads not only to oscillating eigenfunctions, but also to a localization of eigenfunctions. The spectral problem of a singularly perturbed operator in a locally periodic setting where a localization of eigenfunctions occurs has been considered in [8]. Similar singularly perturbed spectral problems where the localization of eigenfunctions takes place are considered in [23], [26], and [27].

The Dirichlet spectral problem for an elliptic operator with indefinite weight in a bounded domain with periodic coefficients has been considered in [22]. The authors have studied the asymptotics for all cases of the sign of the average.

The rest of this paper is organized as follows. In Section 2, we formulate the problem and state the main result, Theorem 2.2. In Section 3, we introduce an auxiliary spectral problem on a periodicity cell and prove the existence of a principal positive eigenvalue in Lemma 3.1. The regularity of the principal eigenvalue and the eigenfunction is discussed in Lemma 3.4. Section 4 is devoted to the proof of the main theorem. After factorizing the original eigenfunctions with the positive cell eigenfunction of (18) in Section 4.1, we rescale the factorized problem and derive a priori estimates in Section 4.2. In Section 4.3, we pass to the limit using the two-scale convergence in spaces with singular measures. The definition of the two-scale convergence in spaces with singular measures is given in Appendix A, and a mean-value property for the oscillating functions is given in Appendix B.

2. Problem setup

Let Q be a bounded $C^{2,\alpha}$ domain in \mathbb{R}^{d-1} , $d \geq 2$, with a boundary ∂Q . The points in \mathbb{R}^d are denoted $x = (x_1, x')$, where $x' = x_2, \dots, x_d$. For a small parameter $\varepsilon > 0$, denote by $\Omega_\varepsilon = [-1, 1] \times (\varepsilon Q)$ a thin rod with the lateral boundary $\Sigma_\varepsilon = (-1, 1) \times \partial(\varepsilon Q)$ and the bases $S_{\varepsilon,\pm} = \{\pm 1\} \times (\varepsilon Q)$ (see Fig. 1).

In the cylinder Ω_ε , we consider the following spectral problem:

$$\begin{cases} -\operatorname{div}(a^\varepsilon(x) \nabla u^\varepsilon(x)) = \lambda^\varepsilon \rho^\varepsilon(x) u^\varepsilon(x), & x \in \Omega_\varepsilon, \\ a^\varepsilon(x) \nabla u^\varepsilon(x) \cdot n = 0, & x \in \Sigma_\varepsilon, \\ u^\varepsilon(-1, x') = u^\varepsilon(1, x') = 0, & x' \in \varepsilon Q \end{cases} \tag{3}$$

with the $d \times d$ matrix a^ε and the scalar weight ρ^ε :

$$a^\varepsilon(x) = a\left(x_1, \frac{x}{\varepsilon}\right), \quad \rho^\varepsilon(x) = \rho\left(x_1, \frac{x}{\varepsilon}\right). \tag{4}$$

We assume the following conditions hold:

- (H1) $a_{ij}(x_1, y), \rho(x_1, y) \in C^{1,\alpha}([-1, 1]; C^{0,\alpha}(\bar{Y}))$ for some $\alpha > 0$; where $Y = (0, 1] \times Q$ is the periodicity cell with the lateral boundary $\Sigma = (0, 1) \times \partial Q$.
- (H2) The functions $a_{ij}(x_1, y)$ and $\rho(x_1, y)$ are 1-periodic in y_1 .
- (H3) The matrix $a(x_1, y)$ is symmetric and satisfies the uniform ellipticity condition, that is, for any $x_1 \in [-1, 1]$ and $y \in Y$, and for some $\Lambda > 0$ it holds

$$\sum_{i,j=1}^d a_{ij}(x_1, y)\xi_i\xi_j \geq \Lambda|\xi|^2, \quad \xi \in \mathbb{R}^d.$$

- (H4) The weight function $\rho(x_1, y)$ changes sign, that is for any $x_1 \in [-1, 1]$ the sets $\{y \in Y : \rho(x_1, y) < 0\}$ and $\{y \in Y : \rho(x_1, y) > 0\}$ have positive Lebesgue measures.
- (H5) $\int_Y \rho(x_1, y) dy < 0$ for all $x_1 \in [-1, 1]$.

An example of ρ satisfying assumptions (H1)–(H5) is shown in Fig. 1. The quite restrictive regularity assumptions are needed because of the presence of the slow variable x_1 . In the case of purely periodic coefficients, the regularity can be reduced to $L^\infty(Y)$.

The weak formulation of problem (3) reads: Find $\lambda^\varepsilon \in \mathbb{C}$ (eigenvalues) and $u^\varepsilon \in H^1(\Omega_\varepsilon) \setminus \{0\}$ (eigenfunctions) such that $u^\varepsilon(\pm 1, x') = 0$ and

$$(a^\varepsilon \nabla u^\varepsilon, \nabla v)_{L^2(\Omega_\varepsilon)} = \lambda^\varepsilon (\rho^\varepsilon u^\varepsilon, v)_{L^2(\Omega_\varepsilon)}, \tag{5}$$

where $(\cdot, \cdot)_{L^2(\Omega_\varepsilon)}$ denotes the standard scalar product in $L^2(\Omega_\varepsilon)$.

The next lemma characterizes the spectrum of problem (3). For a proof, we refer to [28] or Lemma 1 in [22].

Lemma 2.1. *Under the assumptions (H1) – (H4), the spectral problem (3) has a real and discrete spectrum that consists of two infinite sequences*

$$\begin{aligned} 0 < \lambda_1^{\varepsilon,+} \leq \lambda_2^{\varepsilon,+} \leq \dots \leq \lambda_j^{\varepsilon,+} \leq \dots \rightarrow +\infty, \\ 0 > \lambda_1^{\varepsilon,-} \geq \lambda_2^{\varepsilon,-} \geq \dots \geq \lambda_j^{\varepsilon,-} \geq \dots \rightarrow -\infty. \end{aligned}$$

The corresponding eigenfunctions $u_j^{\varepsilon,\pm}$ may be chosen to satisfy the orthogonality and normalization condition

$$(u_i^{\varepsilon,\pm}, u_j^{\varepsilon,\pm})_{L^2(\Omega_\varepsilon)} = \varepsilon^{1/2} \varepsilon^{d-1} |Q| \delta_{ij}, \tag{6}$$

where $|Q|$ is the Lebesgue measure of Q and δ_{ij} is the Kronecker delta.

Remark 1. The reason for choosing this particular normalization (6) is to get the eigenfunctions of the rescaled problem (42) and the limit problem (8) normalized in a standard way (without the small parameter ε).

In the present work, we study the asymptotics of the positive eigenvalues $\lambda_j^{\varepsilon,+}$, as $\varepsilon \rightarrow 0$, under the assumption (H5) that the local average of the weight function is negative $\int_Y \rho(x_1, y) dy < 0$. In this case, as will be shown below, the positive eigenvalues grow as $\varepsilon \rightarrow 0$, and the asymptotics can be obtained by using a special factorization with a positive eigenfunction of the auxiliary spectral problem stated on the periodicity cell Y . Namely, we will use a positive principal eigenvalue $\mu(x_1)$ of the auxiliary spectral cell problem with sign-changing weight, for each $x_1 \in [-1, 1]$:

$$\begin{cases} -\operatorname{div}_y(a(x_1, y)\nabla_y\psi(x_1, y)) = \mu(x_1)\rho(x_1, y)\psi(x_1, y), & y \in Y, \\ a(x_1, y)\nabla_y\psi(x_1, y) \cdot \nu = 0, & y \in \Sigma, \\ y_1 \mapsto \psi(x_1, y_1, y') & \text{is } 1 - \text{periodic.} \end{cases} \tag{7}$$

We say that μ is a principal eigenvalue of (7) if it possesses a unique strictly positive eigenfunction. Obviously, $\mu = 0$ is one such principal eigenvalue with a constant eigenfunction. For our purpose, we are interested in a strictly positive principal eigenvalue μ and a non-constant eigenfunction Ψ . We will prove that in the case $\int_Y \rho(x_1, y) dy < 0$ such an eigenvalue exists (see Lemma 3.3).

In what follows we assume that the principal positive eigenvalue $\mu(x_1)$ of (7) satisfies the following assumption.

(H6) The principal positive eigenvalue $\mu(x_1)$ of problem (18) has a unique minimum point at $x_1 = 0$ and $\mu''(0) > 0$.

The main result of the paper is contained in the following theorem.

Theorem 2.2. *Let the hypotheses (H1)–(H6) hold and denote (μ, Ψ) the principal eigenpair of (7). Then, for any j , we have the following convergence result:*

$$\begin{aligned} \lambda_j^{\varepsilon,+} &= \frac{\mu(0)}{\varepsilon^2} + \frac{v_j}{\varepsilon} + o(\varepsilon^{-1}), \quad \varepsilon \rightarrow 0, \\ \varepsilon^{-\frac{d-1}{2}} \|u_j^{\varepsilon,+} - \Psi(0, \frac{x}{\varepsilon}) v_j(\frac{x_1}{\sqrt{\varepsilon}})\|_{L^2(\Omega_\varepsilon)} &\rightarrow 0, \quad \varepsilon \rightarrow 0, \end{aligned}$$

where (v_j, v_j) is the j th eigenpair of the harmonic oscillator

$$-(a^{\text{eff}}v')' + (c^{\text{eff}} + \frac{1}{2}\mu''(0)x_1^2)v = \nu v, \quad x_1 \in \mathbb{R}. \tag{8}$$

The effective coefficients in (8) are defined by

$$a^{\text{eff}} = \frac{1}{|Y|} \int_Y a(0, \zeta)\nabla(\zeta_1 + N_1(\zeta)) \cdot \nabla(\zeta_1 + N_1(\zeta)) d\zeta, \tag{9}$$

$$c^{\text{eff}} = \frac{1}{|Y|} \int_Y \left((a\nabla_x\Psi)(0, \zeta) \cdot \nabla_\zeta\Psi(0, \zeta) - \operatorname{div}_x(a\nabla_\zeta\Psi)(0, \zeta)\Psi(0, \zeta) \right) d\zeta, \tag{10}$$

where N_1 satisfies the cell problem

$$\begin{cases} \operatorname{div}((a \Psi^2)(0, \zeta) \nabla(N_1(\zeta) + \zeta_1)) = 0, & \zeta \in Y, \\ (a \Psi^2)(0, \zeta) \nabla(N_1 + \zeta_1) = 0, & \zeta \in \Sigma, \\ N_1(\cdot, \zeta') \text{ is } 1\text{-periodic.} \end{cases}$$

The proof of this theorem is given in Section 4.

Remark 2. The result can be generalized to the case of a locally periodic varying cross-section and/or locally periodic perforation, as in [23], [25].

Remark 3. The existence of a principal eigenvalue satisfying the assumption **(H6)** is shown for the Laplacian defined on a locally periodic domain in [27]. By a suitable transformation, we can modify the problems defined on a locally periodic domain to a problem on a periodic domain with locally periodic coefficients.

In [24], problem (3) has been studied under different assumptions on the average of the local weight. For the reader’s convenience, we summarize the results about the spectral asymptotics in the cases studied in [24]. We denote

$$\begin{aligned} \mathcal{A}_y u &= -\operatorname{div}_y(a(x_1, y) \nabla_y u), & \mathcal{B}_y u &= a(x_1, y) \nabla_y u \cdot n, \\ \langle \rho(x_1, \cdot) \rangle &= \int_Y \rho(x_1, y) dy. \end{aligned}$$

Case 1: If $\langle \rho(x_1, \cdot) \rangle > 0$ for all $x_1 \in [-1, 1]$, then for any j ,

$$\lambda_j^{\varepsilon,+} \rightarrow \lambda_j, \quad \varepsilon^{-\frac{d-1}{2}} \|u_j^{\varepsilon,+} - u_j\|_{L^2(\Omega_\varepsilon)} \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

where (λ_j, u_j) are solutions of the limit problem

$$\begin{cases} -\left(a^{\text{eff}}(x_1)u'(x_1)\right)' = \lambda^0 \langle \rho(x_1, \cdot) \rangle u^0(x_1), & x_1 \in (-1, 1), \\ u^0(\pm 1) = 0. \end{cases}$$

Here

$$a^{\text{eff}}(x_1) = \int_Y a_{1j}(x_1, y) (\delta_{1j} + \partial_{y_j} N^{1,1}(x_1, y)) dy, \tag{11}$$

and $N^{1,1}$ solves, for each $x_1 \in [-1, 1]$, the cell problem

$$\begin{cases} \mathcal{A}_y N^{1,1}(x_1, y) = \operatorname{div}_y a_{,1}(x_1, y), & y \in Y, \\ \mathcal{B}_y N^{1,1}(x_1, y) = -a_{,1}(x_1, y) \cdot n, & y \in \Sigma, \\ y_1 \mapsto N^{1,1}(x_1, y_1, y') \text{ is } 1\text{-periodic.} \end{cases} \tag{12}$$

Thus, the homogenization of the positive part of the spectrum in the case of a positive average of ρ is classical.

Case 2: If $\langle \rho(x_1, \cdot) \rangle = 0$ for all $x_1 \in [-1, 1]$, then, for any j ,

$$\varepsilon \lambda_j^{\varepsilon, \pm} \rightarrow \lambda_j^{\pm}, \quad \varepsilon^{-\frac{d-1}{2}} \|u_j^{\varepsilon, \pm} - v_j^{\pm}\|_{L^2(\Omega_\varepsilon)} \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

where $(v_j^{\pm}, \lambda_j^{\pm})$ are the j th eigenpairs (with positive and negative eigenvalues) of the following quadratic operator pencil:

$$\begin{cases} -\left(a^{\text{eff}}(x_1)v'(x_1)\right)' + v \mathbf{B}(x_1)v(x_1) - v^2 \mathbf{C}(x_1)v(x_1) = 0, & x_1 \in (-1, 1), \\ v(-1) = v(1) = 0. \end{cases} \tag{13}$$

The functions $\mathbf{B}(x_1), \mathbf{C}(x_1) > 0$ are defined by

$$\mathbf{C}(x_1) = \int_Y (a \nabla_y N^{1,0}, \nabla_y N^{1,0}) dy, \quad \mathbf{B}(x_1) = \frac{\partial}{\partial x_1} \int_Y a \nabla_y N^{1,1} \cdot \nabla_y N^{1,0} dy.$$

The function $N^{1,1}$ solves (12) and $N^{1,0}$ is a solution of

$$\begin{cases} \mathcal{A}_y N^{1,0}(x, y) = \rho(x_1, y), & y \in Y, \\ \mathcal{B}_y N^{1,0}(x_1, y) = 0, & y \in \Sigma, \\ N^{1,0}(x_1, y) \text{ is } 1\text{-periodic in } y. \end{cases} \tag{14}$$

Case 3: If $\langle \rho(x_1, \cdot) \rangle$ changes sign then, for any j ,

$$\lambda_j^{\varepsilon, \pm} \rightarrow \lambda_j^{\pm}, \quad \varepsilon^{-\frac{d-1}{2}} \|u_j^{\varepsilon, \pm} - u_j^{\pm}\|_{L^2(\Omega_\varepsilon)} \rightarrow 0, \quad \varepsilon \rightarrow 0,$$

where $(\lambda_j^{\pm}, u_j^{\pm})$ are solutions of the limit problem

$$\begin{cases} -\left(a^{\text{eff}}(x_1)u'(x_1)\right)' = \lambda^0 \langle \rho(x_1, \cdot) \rangle u^0(x_1), & x_1 \in (-1, 1), \\ u^0(\pm 1) = 0, \end{cases} \tag{15}$$

with the effective coefficients defined by (11).

To summarize, one case which was not studied in [24] is the asymptotics of the positive part of the spectrum for the negative average of the weight function. The case of the negative part of the spectrum for a weight function with a positive average is similar.

Remark 4. The standard asymptotic ansatz does not provide any information about the asymptotics of positive eigenvalues in the case $\langle \rho(x_1, \cdot) \rangle < 0$. Indeed, let us look for a solution $(\lambda^\varepsilon, u^\varepsilon)$ of problem (3) in the form

$$\begin{aligned}
 u^\varepsilon(x) &= u^0(x_1) + \varepsilon u^1(x_1, y) + \varepsilon^2 u^2(x_1, y) + \dots, \quad y = \frac{x}{\varepsilon}, \\
 \lambda^\varepsilon &= \lambda^0 + \varepsilon \lambda^1 + \dots,
 \end{aligned}
 \tag{16}$$

where the unknown functions $u^k(x_1, y)$ are 1-periodic in y_1 . Substituting ansatz (16) into (3), applying the chain rule, and collecting power-like with respect to ε terms, we obtain a cascade of problems for u^k .

In particular, the right-hand side in the problem for u^1 suggests to choose $u^1(x_1, y) = N^{1,1}(x_1, y) \frac{du^0}{dx_1}(x_1)$, where $N^{1,1}$ solves (12) for each $x_1 \in [-1, 1]$. The compatibility condition for the problem for u^2 gives an equation for u^0 :

$$\begin{cases} -\frac{d}{dx_1} \left(a^{\text{eff}}(x_1) \frac{du^0(x_1)}{dx_1} \right) = \lambda^0 \langle \rho(x_1, \cdot) \rangle u^0(x_1), & x_1 \in (-1, 1), \\ u^0(\pm 1) = 0. \end{cases}
 \tag{17}$$

Here a^{eff} is defined by (11). Since $\langle \rho(x_1, \cdot) \rangle < 0$, (17) possesses only negative eigenvalues and thus provides no information about the positive eigenvalues λ_j^ε of the original problem (3).

In this case, when $\langle \rho(x_1, \cdot) \rangle < 0$, to find the asymptotics of the positive eigenvalues and the corresponding eigenfunctions, we will use the factorization technique. Under periodicity assumptions, a factorization with an eigenfunction of an auxiliary spectral problem is used to describe the oscillations in [1–3].

In the following sections, we will show that the positive eigenvalues of (3) tend to infinity, as $\varepsilon \rightarrow 0$, and derive the correct limit spectral problem.

3. Auxiliary spectral cell problem

In order to estimate the positive eigenvalues $\lambda_j^{\varepsilon,+}$ of (3), we will use a positive principal eigenvalue $\mu(x_1)$ of the auxiliary spectral cell problem with sign-changing weight, $x_1 \in [-1, 1]$:

$$\begin{cases} -\text{div}_y(a(x_1, y)\nabla_y \psi(x_1, y)) = \mu(x_1)\rho(x_1, y)\psi(x_1, y), & y \in Y, \\ a(x_1, y)\nabla_y \psi(x_1, y) \cdot \nu = 0, & y \in \Sigma, \\ y_1 \mapsto \psi(x_1, y_1, y') \text{ is } 1\text{-periodic.} \end{cases}
 \tag{18}$$

We say that μ is a principal eigenvalue of (18) if it possesses a strictly positive eigenfunction $\psi(x_1, \cdot) \in H^1(Y)$. Obviously, $\mu = 0$ is one such principal eigenvalue with eigenfunction $\Psi = 1$. We will prove in Lemma 3.3 that there exists a positive principal eigenvalue.

The Neumann problem for the Laplace operator has been studied, for example, in [6], [19]. In particular, in [6] it has been shown that the Neumann problem in a bounded domain \mathcal{O}

$$\begin{cases} -\Delta \psi = \mu \rho \psi, & y \in \mathcal{O}, \\ \nabla \psi \cdot n = 0, & y \in \partial \mathcal{O}, \end{cases}$$

has a positive principal eigenvalue if and only if the measure of the set $\{y : \rho(y) > 0\}$ is positive and the average of the weight is negative $\int_{\mathcal{O}} \rho \, dy < 0$. In this section, we will prove the corresponding result for (18).

Consider the operator $\mathcal{A}_y v = -\operatorname{div}_y(a(x_1, y)\nabla_y v)$ with the domain

$$D(\mathcal{A}_y) = \{v \in H^2(Y) : a\nabla_y v \cdot \nu|_{\Sigma} = 0, y_1 \mapsto v(x_1, y_1, y') \text{ is 1-periodic}\}.$$

We define

$$Q_\mu(v) = (\mathcal{A}_y v, v)_{L^2(Y)} - \mu(\rho v, v)_{L^2(Y)}.$$

Lemma 3.1. *If there is a positive eigenfunction corresponding to an eigenvalue $\mu(x_1)$ of (18), then $Q_\mu(v) \geq 0$ for all $v \in D(\mathcal{A}_y)$.*

Proof. If $u > 0$ is an eigenfunction corresponding to μ of (18), then u is also an eigenfunction corresponding to the eigenvalue $r = k$ of the operator

$$\begin{aligned} T_k \psi &= T_0 \psi + k \psi := \mathcal{A}_y \psi - \mu \rho \psi + k \psi = r \psi \quad \text{in } Y, \\ \mathcal{B}_y \psi &= a \nabla_y \psi \cdot \nu = 0 \quad \text{on } \Sigma, \\ y_1 &\mapsto \psi(y_1, y') \text{ is 1-periodic.} \end{aligned} \tag{19}$$

The spectrum of (19) is discrete for sufficiently large k and consists of a countable number of eigenvalues. The first eigenvalue r_1 is simple, and the corresponding eigenfunction ψ_1 can be chosen positive. Clearly, (r_i, ψ_i) is an eigenpair of T_k (19) if and only if $(r_i - k, \psi_i)$ is an eigenpair of T_0 . Because the eigenfunction u is positive,

$$(u, \psi_1)_{L^2(Y)} = \int_Y u \psi_1 \, dx > 0.$$

As u is an eigenfunction for some eigenvalue in the spectrum of T_k , u must be an eigenfunction corresponding to r_1 due to the fact that the eigenfunctions corresponding to distinct eigenvalues are orthogonal. Then $u \in \operatorname{Span}(\psi_1)$, as r_1 is simple. As $T_k u = k u$ we have $r_1 = k$. Furthermore, $(r_1 - k) = 0$ is a simple eigenvalue for T_0 , and the corresponding eigenfunction ψ_1 does not change sign.

Since $T_k v = T_0 v + k v$, then for any test function v , by the spectral theorem,

$$(T_0 v, v)_{L^2(Y)} + k \|v\|_{L^2(Y)}^2 = (T_k v, v)_{L^2(Y)} \geq k \|v\|_{L^2(Y)}^2.$$

Since $Q_\mu(v) = (T_0 v, v)_{L^2(Y)}$, we have $Q_\mu(v) \geq 0$ for all $v \in D(\mathcal{A}_y)$. \square

The proof of the existence of a positive principal eigenvalue with a non-constant eigenfunction relies on the following statement (see Lemma 3.9 in [6]).

Lemma 3.2. *Let the hypotheses (H1)-(H5) hold. If $\langle \rho(x_1, \cdot) \rangle = \int_Y \rho \, dy < 0$, there exist $\delta > 0, \gamma > 0$ such that for all $\psi \in D(\mathcal{A}_y)$ satisfying*

$$\int_Y \rho \psi^2 \, dy > -\gamma \int_Y \psi^2 \, dy, \tag{20}$$

we have

$$\int_Y \psi^2 \, dy \leq \frac{1}{\delta} \int_Y |\nabla_y \psi|^2 \, dy. \tag{21}$$

It is clear that for a constant non-zero ψ , the Friedrichs inequality (21) is not satisfied. However, condition (20) does not allow a constant ψ because of the negative local average of ρ . The existence of a nontrivial principal eigenvalue is ensured by the following lemma.

Lemma 3.3. *Let hypotheses (H1)–(H4) hold.*

- (a) *If $\int_Y \rho \, dy \geq 0$, then $\mu = 0$ is the only nonnegative eigenvalue for which the corresponding eigenfunction does not vanish.*
- (b) *If $\int_Y \rho \, dy < 0$, there exists a unique positive principal eigenvalue for which the corresponding eigenfunction does not vanish in Y , and*

$$\mu_1(x_1) = \inf_{\substack{v \in D(\mathcal{A}_y) \\ \langle \rho v, v \rangle_{L^2(Y)} > 0}} \frac{\int_Y a(x_1, y) \nabla_y \psi \cdot \nabla_y \psi \, dy}{\int_Y \rho(x_1, y) \psi^2 \, dy}, \tag{22}$$

where

$$D(\mathcal{A}_y) = \{v \in H^2(Y) : a \nabla_y v \cdot \nu|_{\Sigma} = 0, y_1 \mapsto v(x_1, y_1, y') \text{ is 1-periodic}\}.$$

The eigenfunction Ψ corresponding to μ_1 can be chosen positive and normalized by $\langle \rho \Psi, \Psi \rangle_{L^2(Y)} = 1$.

Proof. In the current proof, x_1 is a fixed parameter, and we omit it for brevity.

- (a) Let μ_1 be defined by (22). Clearly, $\mu_1 \geq 0$. Let us prove that $\mu_1 = 0$ by showing that for any $\mu > 0$ there exists $v \in D(\mathcal{A}_y)$ such that $Q_\mu(v) < 0$, when $\int_Y \rho \, dy \geq 0$. This will contradict Lemma 3.1.

Suppose first that $\int_Y \rho \, dy > 0$. Then we can take $v = 1$ to get

$$Q_\mu(v) = -\mu \int_Y \rho \, dy < 0.$$

If $\int_Y \rho \, dy = 0$, we choose any $w \in D(\mathcal{A}_y)$ such that $\int_Y \rho w \, dy > 0$. For a sufficiently small $s > 0$, we have

$$Q_\mu(1 + sw) = s^2 Q_\mu(w) - 2s\mu \int_Y \rho w \, dy < 0.$$

So μ_1 cannot be positive, and therefore, $\mu_1 = 0$ is the only principal eigenvalue in the case when $\int_Y \rho(x_1, y)dy \geq 0$.

(b) μ_1 given by (22) is positive. Indeed, by Lemma 3.2 and (H3), we have

$$\mu_1 = \inf_{\substack{\psi \in D(\mathcal{A}_y) \\ (\rho\psi, \psi)_{L^2(Y)} > 0}} \frac{\int_Y a(x_1, y) \nabla_y \psi \cdot \nabla_y \psi \, dy}{\int_Y \rho \psi^2 \, dy} \geq \frac{\delta \Lambda}{\|\rho\|_{L^\infty(Y)}} > 0.$$

Next, we prove that μ_1 is a positive eigenvalue of (18). Consider the shifted eigenvalue problem

$$\begin{cases} T_0 \psi = \mathcal{A}_y \psi - \mu_1 \rho \psi = \lambda \psi & \text{in } Y, \\ \mathcal{B}_y \psi = 0 & \text{on } \Sigma, \\ y_1 \mapsto \psi \text{ 1-periodic.} \end{cases} \tag{23}$$

It is clear from (23) that μ_1 is an eigenvalue of (18) if and only if $\lambda = 0$ is an eigenvalue of T_0 . By the definition of μ_1 , $(T_0 v, v)_{L^2(Y)} \geq 0$ for all $v \in D(\mathcal{A}_y)$. The least eigenvalue of T_0 is given by

$$\begin{aligned} \alpha_1 &= \inf\{(\mathcal{A}_y v, v)_{L^2(Y)} - \mu_1(\rho v, v)_{L^2(Y)} : v \in D(\mathcal{A}_y), (\rho v, v)_{L^2(Y)} = 1\} \\ &= \inf\{Q_{\mu_1}(v) : v \in D(\mathcal{A}_y), (\rho v, v)_{L^2(Y)} = 1\}. \end{aligned}$$

Then $\alpha_1 \geq 0$ by the definition of μ_1 . Moreover, there exists a sequence $\{v_n\} \subset D(\mathcal{A}_y)$ such that $\int_Y \rho v_n^2 \, dy = 1$ and

$$\lim_{n \rightarrow \infty} \frac{(\mathcal{A}_y v_n, v_n)_{L^2(Y)}}{(\rho v_n, v_n)_{L^2(Y)}} = \lim_{n \rightarrow \infty} (\mathcal{A}_y v_n, v_n)_{L^2(Y)} = \mu_1.$$

Then $\lim_{n \rightarrow \infty} Q_{\mu_1}(v_n) = \mu_1 - \mu_1 = 0$, and

$$\alpha_1 = \inf_{\substack{v \in D(\mathcal{A}_y) \\ (\rho v, v)_{L^2(Y)} > 0}} Q_{\mu_1}(v) \leq \liminf_{n \rightarrow \infty} Q_{\mu_1}(v_n) = 0.$$

This yields that the smallest eigenvalue of T_0 is $\alpha_1 = 0$. By the Krein-Rutman theorem, it is simple and the corresponding eigenfunction can be chosen positive in Y . This shows that $\mu_1 > 0$ is an eigenvalue of (18) that is simple, and the corresponding eigenfunction can be chosen positive. It is left to show that there are no other principal eigenvalues.

Take $\mu > 0$ such that $\mu \neq \mu_1$. Let us show that μ is not an eigenvalue with a non-negative eigenfunction.

Suppose first that $0 < \mu < \mu_1$. We will show that there exists $\alpha > 0$, depending on μ , such that $Q_\mu(v) \geq \alpha \|v\|_{L^2(Y)}^2$ for all $v \in D(\mathcal{A}_y)$. For $\mu = (1 - s)\mu_1$, $0 < s < 1$, we have

$$\begin{aligned} Q_\mu(v) &= (\mathcal{A}_y v, v)_{L^2(Y)} - \mu(\rho v, v)_{L^2(Y)} \\ &= \frac{\mu}{\mu_1} Q_{\mu_1}(v) + \left(1 - \frac{\mu}{\mu_1}\right) (\mathcal{A}_y v, v)_{L^2(Y)} \geq s (\mathcal{A}_y v, v)_{L^2(Y)}. \end{aligned}$$

Let $\delta, \gamma > 0$ be such that in Lemma 3.2. Then for any $v \in D(\mathcal{A}_y)$ such that $\int_Y \rho v^2 dy > -\gamma \int_Y v^2 dy$, we have

$$Q_\mu(v) \geq s\delta\Lambda \|v\|_{L^2(Y)}^2.$$

If $\int_Y \rho v^2 dy \leq -\gamma \int_Y v^2 dy$, then $Q_\mu(v)$ is estimated in the following way:

$$\begin{aligned} Q_\mu(v) &\geq \Lambda \|\nabla_y v\|_{L^2(Y)}^2 - \mu(\rho v, v)_{L^2(Y)} \\ &\geq -\mu(\rho v, v)_{L^2(Y)} \geq \mu\gamma \|v\|_{L^2(Y)}^2. \end{aligned}$$

Therefore, there exists no non-trivial $v \in D(\mathcal{A}_y)$ for $\mu < \mu_1$ such that $Q_\mu(v) = 0$. So μ is not an eigenvalue of (18).

Next, suppose that μ is such that $0 < \mu_1 < \mu$. Then $Q_\mu(v) \geq 0$ for all $v \in D(\mathcal{A}_y)$. Indeed, there exists v such that

$$\frac{(\mathcal{A}_y v, v)_{L^2(Y)}}{(\rho v, v)_{L^2(Y)}} < \mu \Rightarrow Q_\mu(v) = \int_Y a \nabla_y v \cdot \nabla_y v dy - \mu \int_Y \rho v^2 dy < 0.$$

So μ cannot be an eigenvalue with a positive eigenfunction due to Lemma 3.1. \square

The next lemma characterizes the regularity of $(\mu(x_1), \Psi(x_1, y))$. One of the key arguments we use is the positivity of the eigenfunction, and, thus, the proof is valid only for the principal eigenvalue.

Lemma 3.4. *Let $\mu = \mu(x_1)$ be the principal non-zero eigenvalue of (18) with the corresponding eigenfunction $\Psi = \Psi(x_1, y)$. Under assumptions (H1)–(H4), we have*

$$\mu \in C^{1,\alpha}([-1, 1]), \quad \Psi \in C^{1,\alpha}([-1, 1]; C^{1,\alpha}(\bar{Y}) \cap H^1(Y)). \tag{24}$$

Proof. The classical elliptic regularity of the solutions to elliptic equations ensures that $\Psi(x_1, \cdot) \in H^1(Y) \cap C^{1,\alpha}(\bar{Y})$, for each $x_1 \in [-1, 1]$ (see Corollary 1.4 in [29]). Under the coercivity and boundedness conditions on the coefficient $a(x_1, y)$, the bilinear forms $a_{x_1}(v, v) = (\mathcal{A}_y v, v)_{L^2(Y)}$ and $b_{x_1}(v, \varphi) = (\rho(x_1, \cdot)v, \varphi)_{L^2(Y)}$ are Fréchet differentiable with respect to x_1 (see Theorem 2.4.1 in [16]), and the corresponding differentials are

$$\begin{aligned} a'_{\delta x_1}(v, \varphi) &= \delta x_1 \int_Y \partial_{x_1} a(x_1, y) \nabla v \cdot \nabla \varphi dy, \\ b'_{\delta x_1}(v, \varphi) &= \delta x_1 \int_Y \partial_{x_1} \rho(x_1, y) v \varphi dy. \end{aligned}$$

From the coercivity and differentiability of the bilinear forms it follows that the operator \mathcal{A}_y (given by its bilinear form) with the domain $D(\mathcal{A}_y)$, which is dense in $L^2(Y)$, possesses an inverse \mathcal{A}_y^{-1} that is Fréchet differentiable (Theorem 2.4.2 in [16]). In addition, the operator B_{x_1} defined by the bilinear form $(B_{x_1}v, \varphi)_{L^2(Y)} = (\rho(x_1, \cdot)v, \varphi)_{L^2(Y)}$ is bounded from $L^2(Y)$ into itself, and, thus, is Fréchet differentiable with respect to x_1 . Thus, $\mathcal{A}_y^{-1}B_{x_1}$ is also Fréchet differentiable. The eigenvalue problem (18) can be equivalently rewritten as

$$\mathcal{A}_y^{-1}B_{x_1}\Psi = \frac{1}{\mu(x_1)}\Psi. \tag{25}$$

Then $1/\mu(x_1)$ is continuous with respect to x_1 , and, furthermore, by Theorem 2.5.2 in [16], since the principal eigenvalue is simple, it is differentiable, and the derivative is given by

$$\begin{aligned} \mu'(x_1) &= \int_Y \partial_{x_1}a(x_1, y)\nabla_y\Psi(x_1, y) \cdot \nabla_y\Psi(x_1, y) dy \\ &\quad - \mu(x_1) \int_Y \partial_{x_1}\rho(x_1, y)\Psi(x_1, y)^2 dy, \end{aligned} \tag{26}$$

where Ψ is the corresponding eigenfunction normalized by $\int_Y \rho\Psi^2 dy = 1$.

Having proved that the eigenvalue $\mu(x_1)$ is differentiable with respect to x_1 , we conclude that the corresponding eigenfunction is differentiable in x_1 , as a solution to a boundary value problem with a given $\mu(x_1)$ (see Theorem 2.4.3 in [16]).

Next, we prove the Lipschitz regularity of $\mu(x_1)$. Let us estimate $|\mu(\xi) - \mu(\eta)|$. For brevity, for two values $\xi, \eta \in [-1, 1]$, we write

$$\begin{aligned} L_\xi\Psi_\xi &:= -\operatorname{div}_y(a(\xi, y)\nabla_y\Psi(\xi, y)) = \mu(\xi)\rho(\xi, y)\Psi(\xi, y) =: \mu_\xi\rho_\xi\Psi_\xi, \\ L_\eta\Psi_\eta &:= -\operatorname{div}_y(a(\eta, y)\nabla_y\Psi(\eta, y)) = \mu(\eta)\rho(\eta, y)\Psi(\eta, y) =: \mu_\eta\rho_\eta\Psi_\eta. \end{aligned} \tag{27}$$

Then

$$L_\xi\Psi_\eta = (L_\xi - L_\eta)\Psi_\eta + \mu_\eta\rho_\eta\Psi_\eta. \tag{28}$$

Taking the scalar product of (28) with Ψ_ξ in $L^2(Y)$, integrating by parts and using (27), we obtain

$$\mu_\xi(\rho_\eta\Psi_\eta, \Psi_\xi)_{L^2(Y)} - \mu_\eta(\rho_\xi\Psi_\eta, \Psi_\xi)_{L^2(Y)} = ((L_\xi - L_\eta)\Psi_\eta, \Psi_\xi)_{L^2(Y)}. \tag{29}$$

Using the linear approximations for a_η, ρ_η and integrating by parts in (29), we get

$$\begin{aligned} (\mu_\xi - \mu_\eta)(\rho_\xi\Psi_\eta, \Psi_\xi)_{L^2(Y)} &= (\partial_{x_1}a(\zeta, \cdot)(\eta - \xi)\nabla\Psi_\eta, \nabla\Psi_\xi)_{L^2(Y)} \\ &\quad + (\partial_{x_1}a(\zeta, \cdot)(\eta - \xi)\nabla\Psi_\eta \cdot n, \Psi_\xi)_{L^2(\Sigma)} \\ &\quad - \mu_\eta(\partial_{x_1}\rho(\zeta, \cdot)(\eta - \xi)\Psi_\eta, \Psi_\xi)_{L^2(Y)}. \end{aligned} \tag{30}$$

Since $a_{ij}, \rho \in C^{1,\alpha}([-1, 1]; C^{0,\alpha}(\bar{Y}))$, the partial derivatives $\partial_{x_1}a_{ij}, \partial_{x_1}\rho(x_1, y)$ are Hölder continuous on $[-1, 1]$ and, thus, uniformly bounded. Moreover, by the differentiability of $\mu(x_1)$ and

$\Psi(x_1, y)$ in x_1 , the normalization condition $\int_Y \rho \Psi^2 dy = 1$ implies then that for $|\xi - \eta|$ small enough, $\int_Y \rho_\xi \Psi_\xi \Psi_\eta dy \geq 1/2$. Thus, μ is Lipschitz continuous:

$$|\mu(\xi) - \mu(\eta)| \leq C|\xi - \eta|. \tag{31}$$

Differentiating (18) with respect to x_1 , we obtain:

$$\begin{aligned} & -\operatorname{div}_y(a(x_1, y)\nabla_y(\partial_{x_1}\Psi(x_1, y))) - \mu(x_1)\rho(x_1, y)(\partial_{x_1}\Psi(x_1, y)) \\ & = \operatorname{div}_y(\partial_{x_1}a(x_1, y)\nabla_y\Psi(x_1, y)) + \mu'(x_1)\rho(x_1, y)\Psi(x_1, y) \\ & \quad + \mu(x_1)\partial_{x_1}\rho(x_1, y)\Psi(x_1, y), \quad y \in Y, \tag{32} \\ & a(x_1, y)\nabla_y(\partial_{x_1}\Psi(x_1, y)) \cdot n = -\partial_{x_1}a(x_1, y)\nabla\Psi(x_1, y) \cdot n, \quad y \in \Sigma, \\ & y_1 \mapsto \Psi(x_1, y_1, y') \text{ is 1-periodic.} \end{aligned}$$

The eigenfunction Ψ is normalized by $\int_Y \rho \Psi^2 dy = 1$ which implies

$$\begin{aligned} & \partial_{x_1} \int_Y \rho(x_1, y)\Psi(x_1, y)^2 dy \\ & = \int_Y \partial_{x_1}\rho(x_1, y)\Psi(x_1, y)^2 dy + 2 \int_Y \rho(x_1, y)\partial_{x_1}\Psi(x_1, y)\Psi(x_1, y) dy = 0. \end{aligned}$$

Further, the compatibility condition for (32) is satisfied thanks to (26). Due to the elliptic regularity, $\partial_{x_1}\Psi(x_1, \cdot) \in C^{0,\alpha}(\bar{Y}) \cap H^1(Y)$. To prove the Hölder continuity in x_1 , we estimate the H^1 -norm of the difference $\Psi(\xi, \cdot) - \Psi(\eta, \cdot)$.

$$\begin{aligned} L_\xi(\Psi_\xi - \Psi_\eta) &= (L_\eta - L_\xi)\Psi_\eta + \mu_\xi \rho_\xi \Psi_\xi - \mu_\eta \rho_\eta \Psi_\eta, \\ (L_\xi(\Psi_\xi - \Psi_\eta), (\Psi_\xi - \Psi_\eta))_{L^2(Y)} &= ((L_\eta - L_\xi)\Psi_\eta, (\Psi_\xi - \Psi_\eta))_{L^2(Y)} \\ & \quad + \mu_\xi(\rho_\xi \Psi_\xi, (\Psi_\xi - \Psi_\eta))_{L^2(Y)} \\ & \quad - \mu_\eta(\rho_\eta \Psi_\eta, (\Psi_\xi - \Psi_\eta))_{L^2(Y)}. \end{aligned}$$

By the hypotheses (H1)–(H3), the differentiability of the eigenpair proved above, and Lemma 3.2, we obtain

$$\|\Psi_\xi - \Psi_\eta\|_{H^1(Y)} \leq C|\xi - \eta|^\alpha. \tag{33}$$

Combining (26) and the above estimate yields that $\mu'(x_1)$ is Hölder continuous. For a given $\mu(x_1)$, $\partial_{x_1}\Psi(x_1, y)$ solves (32) and thus, by elliptic regularity, belongs to $H^1(Y) \cup C^{0,\alpha}(\bar{Y})$. We conclude that $\Psi \in C^{1,\alpha}([-1, 1]; C^{0,\alpha}(\bar{Y}) \cap H^1(Y))$. \square

4. Proof of Theorem 2.2

In this section, we prove the main result of the paper, Theorem 2.2. First, we perform the factorization of (3) with the positive eigenfunction defined in Lemma 3.3. After that, we rescale the obtained problem in order to eliminate the singularity. Then the two-scale convergence technique in spaces with singular measure is used to derive the limit problem (8).

4.1. Factorization

As before, we assume that $\langle \rho(x_1, \cdot) \rangle = \int_Y \rho(x_1, y) dy < 0$. Using the factorization $u^\varepsilon(x) = \Psi(x_1, \frac{x}{\varepsilon})v^\varepsilon(x)$, multiplying (3) by $\Psi(x_1, \frac{x}{\varepsilon})$, we obtain the following spectral problem for the new unknowns $(v^\varepsilon, v^\varepsilon)$:

$$\begin{cases} -\operatorname{div}(a_\Psi^\varepsilon \nabla v^\varepsilon) + \frac{1}{\varepsilon} C^\varepsilon v^\varepsilon + \frac{\mu(x_1) - \mu(0)}{\varepsilon^2} \rho_\Psi^\varepsilon v^\varepsilon = v^\varepsilon \rho_\Psi^\varepsilon v^\varepsilon, & x \in \Omega_\varepsilon, \\ a_\Psi^\varepsilon \nabla v^\varepsilon \cdot \nu + (a^\varepsilon \Psi^\varepsilon \nabla_x \Psi(x_1, y)|_{y=x/\varepsilon} \cdot n) v^\varepsilon = 0, & x \in \Sigma_\varepsilon, \\ v^\varepsilon(-1, x') = v^\varepsilon(1, x') = 0, & x' \in \varepsilon Q, \end{cases} \tag{34}$$

where we have denoted

$$v^\varepsilon = \lambda^\varepsilon - \frac{\mu(0)}{\varepsilon^2}, \tag{35}$$

$$a_\Psi^\varepsilon(x) = a(x_1, y) \Psi(x_1, y)^2|_{y=\frac{x}{\varepsilon}}, \quad \rho_\Psi^\varepsilon = \rho(x_1, y) \Psi(x_1, y)^2|_{y=\frac{x}{\varepsilon}}, \tag{36}$$

$$\begin{aligned} C^\varepsilon(x) = & -\varepsilon \Psi(x_1, y) \operatorname{div}(a(x_1, y) \nabla_x \Psi(x_1, y))|_{y=\frac{x}{\varepsilon}} \\ & - \Psi(x_1, y) \operatorname{div}_x(a(x_1, y) \nabla_y \Psi(x_1, y))|_{y=\frac{x}{\varepsilon}}. \end{aligned} \tag{37}$$

Note that the gradient $\nabla_x \Psi$ has only one non-trivial component, the first one $\partial_{x_1} \Psi$. If $a(x_1, y)$ is a scalar function, the second term in the lateral boundary condition in (34) vanishes. The weak form of (34) reads

$$\begin{aligned} A_\Psi(v^\varepsilon, \varphi) \equiv & \int_{\Omega_\varepsilon} a_\Psi^\varepsilon \nabla v^\varepsilon \cdot \nabla \varphi \, dx \\ & + \int_{\Omega_\varepsilon} a(x_1, y) \nabla_x \Psi(x_1, y)|_{y=x/\varepsilon} \cdot \nabla(\Psi^\varepsilon v^\varepsilon \varphi) \, dx \\ & - \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} \operatorname{div}_x(a(x_1, y) \nabla_y \Psi(x_1, y)) \Psi(x_1, y)|_{y=x/\varepsilon} v^\varepsilon \varphi(x) \, dx \\ & + \int_{\Omega_\varepsilon} \frac{\mu(x_1) - \mu(0)}{\varepsilon^2} \rho_\Psi^\varepsilon v^\varepsilon \varphi \, dx = v^\varepsilon \int_{\Omega_\varepsilon} \rho_\Psi^\varepsilon v^\varepsilon \varphi \, dx, \end{aligned} \tag{38}$$

for any $\varphi \in H^1(\Omega_\varepsilon)$, $\varphi(\pm 1, x') = 0$.

By the minmax principle, the first positive eigenvalue of (34) is given by

$$v_1^{\varepsilon,+} = \min_{(\rho_{\Psi}^{\varepsilon} v, v)=1} A_{\Psi}(v, v), \tag{39}$$

where the bilinear form $A_{\Psi}(v, \varphi)$ is defined in (38), and the minimum is taken over functions $v \in H^1(\Omega_{\varepsilon})$ such that $v(\pm 1, x') = 0$. To minimize $A_{\Psi}(v, v)$, one can see that v should be localized, that is v may be of the form $v(\frac{x_1}{\varepsilon^{\gamma}})$ for $\gamma > 0$, as shown in the proof below.

Lemma 4.1. *Under the assumptions (H1)-(H6), the eigenvalues v_j^{ε} of (38) satisfy the following estimate:*

$$|v_j^{\varepsilon}| = |\lambda_j^{\varepsilon,+} - \frac{\mu(0)}{\varepsilon^2}| \leq \frac{C}{\varepsilon}, \quad j \in \mathbf{N}. \tag{40}$$

Proof. We begin by estimating the first positive eigenvalue $v_1^{\varepsilon,+}$. For any $v \in H^1(\Omega_{\varepsilon})$, $(\rho_{\Psi}^{\varepsilon} v, v)_{L^2(\Omega_{\varepsilon})} > 0$ we have:

$$\begin{aligned} v_1^{\varepsilon,+} &\leq \frac{1}{(\rho_{\Psi}^{\varepsilon} v, v)_{L^2(\Omega_{\varepsilon})}} \left(\int_{\Omega_{\varepsilon}} a_{\Psi}^{\varepsilon} \nabla v \cdot \nabla v \, dx \right. \\ &\quad + \int_{\Omega_{\varepsilon}} a(x_1, y) \nabla_x \Psi(x_1, y) \Big|_{y=x/\varepsilon} \cdot \nabla(\Psi^{\varepsilon} v^2) \, dx \\ &\quad - \frac{1}{\varepsilon} \int_{\Omega_{\varepsilon}} \operatorname{div}_x(a(x_1, y) \nabla_y \Psi(x_1, y)) \Psi(x_1, y) \Big|_{y=x/\varepsilon} v^2 \, dx \\ &\quad \left. + \int_{\Omega_{\varepsilon}} \frac{\mu(x_1) - \mu(0)}{\varepsilon^2} \rho_{\Psi}^{\varepsilon} v^2 \, dx \right). \end{aligned} \tag{41}$$

Let us choose a test function $\varphi_{\varepsilon} = \varphi(\frac{x_1}{\varepsilon^{\gamma}})$, for some $0 < \gamma < 1$ and $\varphi \in C_c^{\infty}(\mathbb{R})$ with $\|\varphi\|_{L^2(\mathbb{R})} = 1$. Thus, $\|\varphi_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})}^2 = O(\varepsilon^{(d-1)\varepsilon^{\gamma}})$ and $\|\nabla \varphi_{\varepsilon}\|_{L^2(\Omega_{\varepsilon})}^2 = O(\varepsilon^{(d-1)\varepsilon^{-\gamma}})$. With the help of (H1)-(H3), (H5), and by the regularity properties of Ψ , we have for small enough ε :

$$\begin{aligned} \int_{\Omega_{\varepsilon}} a_{\Psi}^{\varepsilon}(x_1, \frac{x}{\varepsilon}) \nabla \varphi(\frac{x_1}{\varepsilon^{\gamma}}) \cdot \nabla \varphi(\frac{x_1}{\varepsilon^{\gamma}}) \, dx &= O(\varepsilon^{d-1-\gamma}), \\ \int_{\Omega_{\varepsilon}} a(x_1, y) \nabla_x \Psi(x_1, y) \Big|_{y=x/\varepsilon} \cdot \nabla(\Psi^{\varepsilon} \varphi(\frac{x_1}{\varepsilon^{\gamma}})^2) \, dx &= O(\varepsilon^{\gamma-1+d-1}), \\ - \frac{1}{\varepsilon} \int_{\Omega_{\varepsilon}} \operatorname{div}_x(a(x_1, y) \nabla_y \Psi(x_1, y)) \Psi(x_1, y) \Big|_{y=x/\varepsilon} \varphi(\frac{x_1}{\varepsilon^{\gamma}})^2 \, dx &= O(\varepsilon^{\gamma-1+d-1}), \\ \int_{\Omega_{\varepsilon}} \frac{\mu(x_1) - \mu(0)}{\varepsilon^2} \rho_{\Psi}^{\varepsilon} \varphi(\frac{x_1}{\varepsilon^{\gamma}})^2 \, dx &= O(\varepsilon^{d-1+3\gamma-2}). \end{aligned}$$

Using the normalization condition $\int_Y \rho \Psi^2 dy = 1$ and Lemma 3.2, we obtain

$$\begin{aligned} \int_{\Omega_\varepsilon} \rho_\Psi^\varepsilon \varphi_\varepsilon^2 dx &= \frac{1}{|Q|} \int_{\Omega_\varepsilon} \left(\int_Y \rho \Psi^2 dy \right) \varphi_\varepsilon^2 dx + O(\varepsilon \|\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla \varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)}) \\ &= \|\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)}^2 + O(\varepsilon \|\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla \varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)}) = O(\varepsilon^\gamma \varepsilon^{d-1}). \end{aligned}$$

Then the estimate (41) becomes

$$|v_1^{\varepsilon,+}| \leq C \varepsilon^{-\gamma} (\varepsilon^{-\gamma} + \varepsilon^{\gamma-1} + \varepsilon^{3\gamma-2}).$$

Equating the powers of ε on the right-hand side of the last inequality, we can see that the best choice of γ for the considered type of test function is $\gamma = \frac{1}{2}$, and the estimate (40) is proved for $j = 1$. The following eigenvalues $v_j^{\varepsilon,+}$, $j = 2, 3, \dots$ can be estimated in a similar way by choosing a test function that concentrates in a vicinity of $x_1 = 0$, which is the minimum point of $\mu(x_1)$, and is orthogonal to the first $j - 1$ eigenfunctions u_k^ε , $k = 1, \dots, j - 1$. \square

4.2. Rescaling

We change the variables in (34) setting $z = \frac{x}{\sqrt{\varepsilon}}$. With $v^\varepsilon(\sqrt{\varepsilon}z) = w^\varepsilon(z)$, (34) becomes

$$\begin{cases} -\operatorname{div}(a_\Psi^\varepsilon(\sqrt{\varepsilon}z) \nabla w^\varepsilon(z)) + C_\Psi^\varepsilon w^\varepsilon(z) \\ + \frac{\mu(\sqrt{\varepsilon}z_1) - \mu(0)}{\varepsilon} \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z) w^\varepsilon(z) = \varepsilon v^\varepsilon \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z) w^\varepsilon(z), & z \in \varepsilon^{-\frac{1}{2}} \Omega_\varepsilon, \\ a_\Psi^\varepsilon(\sqrt{\varepsilon}z) \nabla w^\varepsilon(z) \cdot \nu + w^\varepsilon(z) a^\varepsilon \Psi^\varepsilon \nabla_z \Psi(\sqrt{\varepsilon}z_1, y)|_{y=z/\sqrt{\varepsilon}} \cdot n = 0, & \varepsilon^{-\frac{1}{2}} \Sigma_\varepsilon, \\ w^\varepsilon(-\varepsilon^{-1/2}, z') = w^\varepsilon(\varepsilon^{-1/2}, z') = 0, & z' \in \sqrt{\varepsilon} Q, \end{cases} \tag{42}$$

where

$$\begin{aligned} a_\Psi^\varepsilon(\sqrt{\varepsilon}z) &= a_\Psi \left(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}} \right), & \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z) &= \rho_\Psi \left(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}} \right), \\ C_\Psi^\varepsilon(\sqrt{\varepsilon}z) &= \left(-\Psi^\varepsilon \operatorname{div}(a \nabla_z \Psi) - \frac{1}{\sqrt{\varepsilon}} \Psi^\varepsilon \operatorname{div}_z(a \nabla_z \Psi) \right) \left(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}} \right). \end{aligned}$$

We will derive a priori estimates for the eigenfunctions of (42) in terms of a singular measure $d\mu_\varepsilon$ charging the rescaled domain $\varepsilon^{-1/2} \Omega_\varepsilon$. Let us define a Radon measure on \mathbb{R}^d by

$$\mu_\varepsilon(B) = \frac{\varepsilon^{-(d-1)/2}}{|Q|} \int_B \chi_{\varepsilon^{-\frac{1}{2}} \Omega_\varepsilon} dx, \tag{43}$$

for all Borel sets B , where $\chi_{\varepsilon^{-\frac{1}{2}} \Omega_\varepsilon}$ is the characteristic function on the rescaled domain $\varepsilon^{-\frac{1}{2}} \Omega_\varepsilon$.

Let us choose the normalization in the original domain Ω_ε : $\int_{\Omega_\varepsilon} v_i^\varepsilon v_j^\varepsilon dx = \varepsilon^{1/2} \varepsilon^{d-1} |Q| \delta_{ij}$. Then, in the rescaled domain, the normalization condition becomes

$$\int_{\varepsilon^{-1/2}\Omega_\varepsilon} v_i^\varepsilon(\sqrt{\varepsilon}z)v_j^\varepsilon(\sqrt{\varepsilon}z) dz = \varepsilon^{(d-1)/2}|Q|\delta_{ij}, \tag{44}$$

and in terms of the singular measure:

$$\int_{\mathbb{R}^d} v_i^\varepsilon v_j^\varepsilon d\mu_\varepsilon = \delta_{ij}. \tag{45}$$

Lemma 4.2. *Suppose that (H1)–(H6) are satisfied. Let $(v^\varepsilon, w^\varepsilon)$ be an eigenpair of (42) with the eigenfunctions normalized by $\int_{\mathbb{R}^d} w_i^\varepsilon w_j^\varepsilon d\mu_\varepsilon = \delta_{ij}$. Then w^ε satisfies the following estimate*

$$\int_{\mathbb{R}^d} |\nabla w_\varepsilon|^2 d\mu_\varepsilon + \int_{\mathbb{R}^d} |z_1 w_\varepsilon|^2 d\mu_\varepsilon \leq C, \tag{46}$$

with a constant C independent of ε .

Proof. Weak formulation of the rescaled problem (42) reads

$$\begin{aligned} & \int_{\varepsilon^{-1/2}\Omega_\varepsilon} a_\Psi^\varepsilon(\sqrt{\varepsilon}z)\nabla w^\varepsilon(z) \cdot \nabla \varphi(z) dz \\ & + \sqrt{\varepsilon} \int_{\varepsilon^{-1/2}\Omega_\varepsilon} a(x_1, \zeta) \nabla_x \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} \cdot \nabla(\Psi^\varepsilon(\sqrt{\varepsilon}z)) w^\varepsilon(z) \varphi(z) dz \\ & + \sqrt{\varepsilon} \int_{\varepsilon^{-1/2}\Omega_\varepsilon} a(x_1, \zeta) \nabla_x \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} \cdot \nabla w^\varepsilon(z) \varphi(z) \Psi^\varepsilon(\sqrt{\varepsilon}z) dz \\ & + \sqrt{\varepsilon} \int_{\varepsilon^{-1/2}\Omega_\varepsilon} a(x_1, \zeta) \nabla_x \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} \cdot w^\varepsilon(z) \nabla \varphi(z) \Psi^\varepsilon(\sqrt{\varepsilon}z) dz \\ & - \int_{\varepsilon^{-1/2}\Omega_\varepsilon} \operatorname{div}_x(a(x_1, \zeta) \nabla_\zeta \Psi(x_1, \zeta)) \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} w^\varepsilon(z) \varphi(z) dz \\ & + \int_{\varepsilon^{-1/2}\Omega_\varepsilon} \frac{\mu(\sqrt{\varepsilon}z_1) - \mu(0)}{\varepsilon} \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z) w^\varepsilon(z) \varphi(z) dz \\ & = \varepsilon v^\varepsilon \int_{\varepsilon^{-1/2}\Omega_\varepsilon} \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z) w^\varepsilon(z) \varphi(z) dz, \end{aligned} \tag{47}$$

for all $\phi \in H^1(\varepsilon^{-1/2}\Omega_\varepsilon)$ such that $\phi(-1/\sqrt{\varepsilon}, z') = \phi(1/\sqrt{\varepsilon}, z') = 0$. We use w^ε as a test function in the weak formulation above and rewrite it in terms of μ_ε :

$$\begin{aligned}
 & \int_{\mathbb{R}^d} a_{\Psi}^{\varepsilon}(\sqrt{\varepsilon}z) \nabla w^{\varepsilon} \cdot \nabla w^{\varepsilon} d\mu_{\varepsilon} \\
 & + \sqrt{\varepsilon} \int_{\mathbb{R}^d} a(x_1, \zeta) \nabla_x \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} \cdot \nabla \Psi(\sqrt{\varepsilon}z) (w^{\varepsilon})^2 d\mu_{\varepsilon} \\
 & + \sqrt{\varepsilon} \int_{\mathbb{R}^d} a(x_1, \zeta) \nabla_x \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} \cdot \nabla w^{\varepsilon} w^{\varepsilon} \Psi(\sqrt{\varepsilon}z) d\mu_{\varepsilon} \\
 & + \sqrt{\varepsilon} \int_{\mathbb{R}^d} a(x_1, \zeta) \nabla_x \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} \cdot w^{\varepsilon}(z) \nabla w^{\varepsilon} \Psi(\sqrt{\varepsilon}z) d\mu_{\varepsilon} \\
 & - \int_{\mathbb{R}^d} \operatorname{div}_x (a(x_1, \zeta) \nabla_{\zeta} \Psi(x_1, \zeta)) \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} (w^{\varepsilon})^2 d\mu_{\varepsilon} \\
 & + \int_{\mathbb{R}^d} \frac{\mu(\sqrt{\varepsilon}z_1) - \mu(0)}{\varepsilon} \rho_{\Psi}^{\varepsilon}(\sqrt{\varepsilon}z) (w^{\varepsilon})^2 d\mu_{\varepsilon} \tag{48} \\
 & = \varepsilon v^{\varepsilon} \int_{\varepsilon^{-1/2}\Omega_{\varepsilon}} \rho_{\Psi}^{\varepsilon}(\sqrt{\varepsilon}z) (w^{\varepsilon})^2 d\mu_{\varepsilon}.
 \end{aligned}$$

The right-hand side of the above identity is estimated using Lemma B.1, the normalization in Lemma 3.3, the estimate for the eigenvalues (40), and the normalization condition for w_i^{ε} :

$$\varepsilon v^{\varepsilon} \left| \int_{\varepsilon^{-1/2}\Omega_{\varepsilon}} \rho_{\Psi}^{\varepsilon}(\sqrt{\varepsilon}z) |w^{\varepsilon}|^2 dz \right| \leq C.$$

Let us estimate the second term in the equality above separately, using the regularity properties of the coefficients and $\Psi(x_1, y)$:

$$\begin{aligned}
 & \left| \sqrt{\varepsilon} \int_{\varepsilon^{-1/2}\Omega_{\varepsilon}} a(x_1, \zeta) \nabla_x \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} \cdot \nabla (\Psi^{\varepsilon}(\sqrt{\varepsilon}z)) (w^{\varepsilon}(z))^2 dz \right| \\
 & \leq \sqrt{\varepsilon} \left| \int_{\varepsilon^{-1/2}\Omega_{\varepsilon}} a(x_1, \zeta) \nabla_x \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} \cdot \nabla_z (\Psi^{\varepsilon}(\sqrt{\varepsilon}z)) (w^{\varepsilon}(z))^2 dz \right| \\
 & + \left| \int_{\varepsilon^{-1/2}\Omega_{\varepsilon}} a(x_1, \zeta) \nabla_x \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} \cdot \nabla_{\zeta} \Psi^{\varepsilon}(\sqrt{\varepsilon}z) (w^{\varepsilon}(z))^2 dz \right| \\
 & \leq C_1 \varepsilon \int_{\varepsilon^{-1/2}\Omega_{\varepsilon}} |w^{\varepsilon}|^2 dz + C_2 \int_{\varepsilon^{-1/2}\Omega_{\varepsilon}} |w^{\varepsilon}|^2 dz \leq C_0 \int_{\varepsilon^{-1/2}\Omega_{\varepsilon}} |w^{\varepsilon}|^2 dz.
 \end{aligned}$$

This shows that this term does not vanish, as $\varepsilon \rightarrow 0$, and, thus, we shift the spectrum by adding $C_0 \|w^\varepsilon\|_{L^2(\varepsilon^{-1/2}\Omega_\varepsilon)}^2$ to both sides of the weak formulation for the rescaled equation.

As $\rho_\Psi^\varepsilon(\sqrt{\varepsilon}z)$ is sign-changing, we cannot estimate the term containing $(\mu(\sqrt{\varepsilon}z_1) - \mu(0))$ directly from below. Then we will use the following mean-value theorem to estimate this term, using Corollary 1

$$\begin{aligned} & \int_{\mathbb{R}^d} \frac{(\mu(\sqrt{\varepsilon}z_1) - \mu(0))}{\varepsilon} \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z) (w^\varepsilon(z))^2 d\mu_\varepsilon \\ & - \int_{\mathbb{R}^d} \frac{(\mu(\sqrt{\varepsilon}z_1) - \mu(0))}{\varepsilon} \left(\int_Y \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z_1, \zeta) d\zeta \right) (w^\varepsilon(z))^2 d\mu_\varepsilon \\ & = O(\sqrt{\varepsilon} \|w^\varepsilon\|_{L^2(\mathbb{R}^d, \mu_\varepsilon)} \|\nabla w^\varepsilon\|_{L^2(\mathbb{R}^d, \mu_\varepsilon)}). \end{aligned}$$

By (H5),

$$\mu(\sqrt{\varepsilon}z_1) - \mu(0) = \frac{1}{2} \mu''(0) (\sqrt{\varepsilon}|z_1|)^2 + o(|\sqrt{\varepsilon}z_1|^2).$$

Since we do not know yet that w^ε is localized, Taylor’s expansion cannot be used to obtain an estimate for the remainder. Instead, we will use a quadratic equivalence, a forward consequence of Taylor’s theorem. We substitute $(\mu(\sqrt{\varepsilon}z_1) - \mu(0)) \left(\int_Y \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z_1, \zeta) d\zeta \right)$ with the equivalent quadratic function $\frac{\mu''(0)}{2} (\sqrt{\varepsilon}|z_1|)^2$ in the weak formulation (48).

Finally, by the coercivity of a and the regularity properties of Ψ , we derive (46). Note that the estimate $\|z_1 w^\varepsilon\|_{L^2(\mathbb{R}^d, \mu_\varepsilon)} \leq C$ implies $\|w^\varepsilon\|_{L^2(\mathbb{R}^d, \mu_\varepsilon)} \leq C$ because of the growing weight $|z_1|$. \square

The proof of the following lemma can be found in [25], [23].

Lemma 4.3. *Under conditions in Lemma 4.2, there exists $w \in H^1(\mathbb{R}^d, \mu^*)$ such that*

$$\begin{aligned} w^\varepsilon & \xrightarrow{2} w(z_1, 0) \quad \text{in } L^2(\mathbb{R}^d, \mu_\varepsilon), \\ \nabla w^\varepsilon & \xrightarrow{2} \nabla^{\mu^*} w(z_1, 0) + \nabla_\zeta w^1(z_1, \zeta) \quad \text{in } L^2(\mathbb{R}^d, \mu_\varepsilon), \end{aligned}$$

where $w^1(z_1, \zeta) \in L^2(\mathbb{R}; H^1(Y))$ is 1-periodic in ζ_1 .

4.3. Passage to the limit in (47)

Recall the weak formulation of the rescaled problem (42):

$$\int_{\mathbb{R}^d} a_\Psi^\varepsilon(\sqrt{\varepsilon}z) \nabla w^\varepsilon(z) \cdot \nabla \varphi(z) d\mu_\varepsilon$$

$$\begin{aligned}
 & + \sqrt{\varepsilon} \int_{\mathbb{R}^d} a(x_1, \zeta) \nabla_x \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} \cdot \nabla \Psi^\varepsilon(\sqrt{\varepsilon}z) w^\varepsilon(z) \varphi(z) d\mu_\varepsilon \\
 & + \sqrt{\varepsilon} \int_{\mathbb{R}^d} a(x_1, \zeta) \nabla_x \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} \cdot \nabla w^\varepsilon(z) \varphi(z) \Psi^\varepsilon(\sqrt{\varepsilon}z) d\mu_\varepsilon \\
 & + \sqrt{\varepsilon} \int_{\mathbb{R}^d} a(x_1, \zeta) \nabla_x \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} \cdot w^\varepsilon(z) \nabla \varphi(z) \Psi^\varepsilon(\sqrt{\varepsilon}z) d\mu_\varepsilon \\
 & - \int_{\mathbb{R}^d} \operatorname{div}_x (a(x_1, \zeta) \nabla_\zeta \Psi(x_1, \zeta)) \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} w^\varepsilon(z) \varphi(z) d\mu_\varepsilon \tag{49} \\
 & + \int_{\mathbb{R}^d} \frac{\mu(\sqrt{\varepsilon}z_1) - \mu(0)}{\varepsilon} \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z) w^\varepsilon(z) \varphi(z) d\mu_\varepsilon \\
 & = \varepsilon v^\varepsilon \int_{\mathbb{R}^d} \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z) w^\varepsilon(z) \varphi(z) d\mu_\varepsilon,
 \end{aligned}$$

for all $\phi \in H^1(\varepsilon^{-1/2}\Omega_\varepsilon)$ such that $\phi(-1/\sqrt{\varepsilon}, z') = \phi(1/\sqrt{\varepsilon}, z') = 0$.

Step 1.

Choose a test function $\Phi_\varepsilon = \varepsilon^{1/2} \phi(z) \psi(\frac{z}{\varepsilon^{1/2}})$ in (49), where $\phi \in C^\infty(\mathbb{R}^d)$, $\psi \in C^\infty(\bar{Y})$. The gradient of the test function is

$$\nabla \Phi_\varepsilon = \varepsilon^{1/2} \psi(\frac{z}{\varepsilon^{1/2}}) \nabla \phi(z) + \phi(z) \nabla_\zeta \psi(\zeta) \Big|_{\zeta=z/\varepsilon^{1/2}}.$$

The limit of the first term in (47) is

$$\begin{aligned}
 & \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} a_\Psi^\varepsilon(\sqrt{\varepsilon}z) \nabla w^\varepsilon(z) \cdot \nabla \Phi^\varepsilon(z) d\mu_\varepsilon \\
 & = \frac{1}{|Y|} \int_{\mathbb{R}^d} \left(\int_Y a_\Psi(0, \zeta) \left(\nabla^{\mu^*} w(z_1, 0) + \nabla_\zeta w^1(z_1, \zeta) \right) \cdot \nabla_\zeta \psi(\zeta) d\zeta \right) \phi(z_1, 0) d\mu^*.
 \end{aligned}$$

The limit of the next four terms is zero due to the regularity properties of Ψ and the small factor $\sqrt{\varepsilon}$ in the test function. The sixth term can be proved to go to zero by using Corollary 1 in [23]. Indeed,

$$\begin{aligned}
 & \int_{\mathbb{R}^d} \frac{\mu(\sqrt{\varepsilon}z_1) - \mu(0)}{\varepsilon} \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z) w^\varepsilon \Phi^\varepsilon(z) d\mu_\varepsilon \\
 & = \frac{\sqrt{\varepsilon}}{|Y|} \int_{\mathbb{R}^d} \frac{\mu(\sqrt{\varepsilon}z_1) - \mu(0)}{\varepsilon} \left(\int_Y \rho_\Psi(\sqrt{\varepsilon}z_1, \zeta) \psi(\zeta) d\zeta \right) w^\varepsilon \phi(z) d\mu_\varepsilon + O(\sqrt{\varepsilon})
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\sqrt{\varepsilon}\mu''(0)}{2|Y|} \int_{\mathbb{R}^d} |z_1|^2 \left(\int_Y \rho_\Psi(\sqrt{\varepsilon}z_1, \zeta)\psi(\zeta) d\zeta \right) w^\varepsilon \phi(z) d\mu_\varepsilon \\
 &\quad + \frac{\sqrt{\varepsilon}}{|Y|} \int_{\mathbb{R}^d} \frac{o(\varepsilon|z_1|^2)}{2\varepsilon} \left(\int_Y \rho_\Psi(\sqrt{\varepsilon}z_1, \zeta)\psi(\zeta) d\zeta \right) w^\varepsilon \phi(z) d\mu_\varepsilon + O(\sqrt{\varepsilon}).
 \end{aligned}$$

Since ϕ has a compact support in z_1 , $o(\varepsilon|z_1|^2) = o(\varepsilon)$ as $\varepsilon \rightarrow 0$. Therefore

$$\begin{aligned}
 &\left| \int_{\mathbb{R}^d} \frac{\mu(\sqrt{\varepsilon}z_1) - \mu(0)}{\varepsilon} \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z) w^\varepsilon \Phi^\varepsilon(z) d\mu_\varepsilon \right| \\
 &\leq \left| \frac{\sqrt{\varepsilon}\mu''(0)}{2|Y|} \int_{\mathbb{R}^d} |z_1|^2 \left(\int_Y \rho_\Psi(\sqrt{\varepsilon}z_1, \zeta)\psi(\zeta) d\zeta \right) w^\varepsilon \phi(z) d\mu_\varepsilon + O(\sqrt{\varepsilon}) \right| \leq C\sqrt{\varepsilon}.
 \end{aligned}$$

As for the right-hand side of the weak formulation (49), it is estimated by using the bound for the eigenvalues (40):

$$\left| \varepsilon v^\varepsilon \int_{\mathbb{R}^d} \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z) w^\varepsilon \Phi^\varepsilon(z) d\mu_\varepsilon \right| \leq C\sqrt{\varepsilon}.$$

Passing limit as $\varepsilon \rightarrow 0$ in (49), we obtain a problem for w^1 :

$$\int_{\mathbb{R}^d} \left(\int_Y a_\Psi(0, \zeta) \left(\nabla^{\mu^*} w(z_1, 0) + \nabla_\zeta w^1(z_1, \zeta) \right) \cdot \nabla_\zeta \psi(\zeta) d\zeta \right) \phi(z_1, 0) d\mu^* = 0. \tag{50}$$

We are looking for the solution in the form $w^1(z_1, \zeta) = N(\zeta) \cdot \nabla^{\mu^*} w(z_1, 0)$, where $N : Y \rightarrow \mathbb{R}^d$ are periodic in ζ_1 solving

$$\int_Y a_\Psi(0, \zeta) \nabla_\zeta N_j(\zeta) \cdot \nabla_\zeta \psi(\zeta) d\zeta = - \int_Y (a_\Psi)_{kj} \partial_{\zeta_k} \psi(\zeta) d\zeta.$$

The last integral identity yields the problem for N_j in its strong form:

$$\begin{cases} -\operatorname{div}(a_\Psi(0, \zeta) \nabla_\zeta N_j(\zeta)) = \partial_{\zeta_k} (a_\Psi)_{kj}(0, \zeta) & \text{in } Y \\ a_\Psi(0, \zeta) \nabla_\zeta N_j(\zeta) \cdot n = -(a_\Psi)_{kj}(0, \zeta) n_k & \text{on } \Sigma, \\ N_j(\cdot, \zeta') \text{ is } 1\text{-periodic,} & j = 1, 2, \dots, d. \end{cases} \tag{51}$$

In this way, we have the following convergence

$$\nabla w^\varepsilon(z) \xrightarrow{2} (I + \nabla N(\zeta)) \nabla^{\mu^*} w(z_1, 0) \text{ as } \varepsilon \rightarrow 0.$$

We proceed with deriving the effective problem for the limit function w .

Step 2: Let us take a test function $\varphi \in C_0^\infty(\mathbb{R}; C^\infty(\varepsilon^{1/2}Q))$ in (49)

$$\begin{aligned} & \int_{\mathbb{R}^d} a_{\Psi}^\varepsilon(\sqrt{\varepsilon}z) \nabla w^\varepsilon \cdot \nabla \varphi \, d\mu_\varepsilon + \int_{\mathbb{R}^d} (a^\varepsilon \nabla_z \Psi^\varepsilon \cdot \nabla(\Psi^\varepsilon w^\varepsilon \varphi)) \, d\mu_\varepsilon \\ & - \int_{\mathbb{R}^d} \Psi^\varepsilon \operatorname{div}_z (a^\varepsilon \nabla_\zeta \Psi^\varepsilon) w^\varepsilon \varphi(z) \, d\mu_\varepsilon \\ & + \int_{\mathbb{R}^d} \frac{\mu(\sqrt{\varepsilon}z_1) - \mu(0)}{\varepsilon} \rho_{\Psi}^\varepsilon(\sqrt{\varepsilon}z) w^\varepsilon \varphi(z) \, d\mu_\varepsilon \\ & = \varepsilon \nu^\varepsilon \int_{\mathbb{R}^d} \rho_{\Psi}^\varepsilon(\sqrt{\varepsilon}z) w^\varepsilon \varphi(z) \, d\mu_\varepsilon. \end{aligned} \tag{52}$$

The first term in (52), as $\varepsilon \rightarrow 0$, yields

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} a_{\Psi}^\varepsilon(\sqrt{\varepsilon}z) \nabla w^\varepsilon(z) \cdot \nabla \varphi(z) \, d\mu_\varepsilon \\ & = \frac{1}{|Y|} \int_{\mathbb{R}^d} \left(\int_Y a_{\Psi}(0, \zeta) (I + \nabla N(\zeta)) \, d\zeta \right) \nabla^{\mu^*} w(z_1, 0) \cdot \nabla \varphi(z_1, 0) \, d\mu^* \\ & = \int_{\mathbb{R}^d} A^{\text{eff}} \nabla^{\mu^*} w(z_1, 0) \cdot \nabla \varphi(z_1, 0) \, d\mu^*, \end{aligned}$$

where we denote

$$A_{ij}^{\text{eff}} = \frac{1}{|Y|} \int_Y (a_{\Psi})_{ik}(0, \zeta) (\delta_{kj} + \partial_{\zeta_k} N_j(\zeta)) \, d\zeta.$$

The fourth term in (52):

$$\begin{aligned} & \int_{\mathbb{R}^d} \frac{\mu(\sqrt{\varepsilon}z_1) - \mu(0)}{\varepsilon} \rho_{\Psi}^\varepsilon(\sqrt{\varepsilon}z) w^\varepsilon \varphi(z) \, d\mu_\varepsilon \\ & = \frac{1}{|Y|} \int_{\mathbb{R}^d} \frac{\mu(\sqrt{\varepsilon}z_1) - \mu(0)}{\varepsilon} \left(\int_Y \rho_{\Psi}(0, \zeta) \, d\zeta \right) w^\varepsilon \varphi(z) \, d\mu_\varepsilon \\ & + O(\sqrt{\varepsilon} \|w^\varepsilon\|_{L^2(\mathbb{R}^d, \mu_\varepsilon)} \|\nabla w^\varepsilon\|_{L^2(\mathbb{R}^d, \mu_\varepsilon)}) \end{aligned}$$

$$= \frac{1}{|Y|} \int_{\mathbb{R}^d} \frac{\mu(\sqrt{\varepsilon}z_1) - \mu(0)}{\varepsilon} \left(\int_Y \rho_\Psi(\sqrt{\varepsilon}z_1, \zeta) d\zeta \right) w^\varepsilon \varphi(z) d\mu_\varepsilon + O(\sqrt{\varepsilon}).$$

Writing the Taylor expansions for $\mu(\sqrt{\varepsilon}z_1)$ and $\langle \rho_\Psi(\sqrt{\varepsilon}z_1, \cdot) \rangle$ we obtain, as $\varepsilon|z_1|^2 \rightarrow 0$:

$$\begin{aligned} \mu(\sqrt{\varepsilon}z_1) &= \mu(0) + \varepsilon z_1^2 \mu''(0) + o(\varepsilon|z_1|^2), \\ \langle \rho_\Psi(\sqrt{\varepsilon}z_1, \cdot) \rangle &= \langle \rho_\Psi(0, \cdot) \rangle + \sqrt{\varepsilon}z_1 \langle \partial_{z_1} \rho_\Psi(0, \cdot) \rangle + \frac{\varepsilon|z_1|^2}{2} \langle \partial_{z_1}^2 \rho_\Psi(0, \cdot) \rangle + o(\varepsilon|z_1|^2). \end{aligned}$$

By the Lebesgue dominated convergence theorem, since the test function has compact support,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} \frac{\mu(\sqrt{\varepsilon}z_1) - \mu(0)}{\varepsilon} \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z) w^\varepsilon \varphi(z) d\mu_\varepsilon \\ = \frac{\mu''(0)}{2} \langle \rho_\Psi(0, \cdot) \rangle \int_{\mathbb{R}^d} |z_1|^2 w(z_1, 0) \varphi(z_1, 0) d\mu^*. \end{aligned}$$

As for the terms 2-4 in (49) coming from C^ε , we see that the only one that contributes to the limit is the one containing $\nabla_\zeta \Psi$ (in the second term).

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \sqrt{\varepsilon} \int_{\mathbb{R}^d} a(x_1, \zeta) \nabla_x \Psi(x_1, \zeta) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} \cdot \frac{1}{\sqrt{\varepsilon}} (\nabla_\zeta \Psi)(\sqrt{\varepsilon}z) w^\varepsilon(z) \varphi(z) d\mu_\varepsilon \\ = \frac{1}{|Y|} \int_{\mathbb{R}^d \times Y} (a \nabla_x \Psi)(0, \zeta) \cdot \nabla_\zeta \Psi(0, \zeta) w(z_1) \varphi(z_1, 0) d\zeta d\mu^*; \end{aligned}$$

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} \Psi(x_1, \zeta) \operatorname{div}_x (a(x_1, \zeta) \nabla_\zeta \Psi(x_1, \zeta)) \Big|_{x_1=\sqrt{\varepsilon}z_1, \zeta=z/\sqrt{\varepsilon}} w^\varepsilon(z) \varphi(z) d\mu_\varepsilon \\ = \frac{1}{|Y|} \int_{\mathbb{R}^d \times Y} \Psi(0, \zeta) \operatorname{div}_x (a \nabla_\zeta \Psi)(0, \zeta) w(z_1) \varphi(z_1, 0) d\zeta d\mu^*. \end{aligned}$$

Denote

$$c^{\text{eff}} = \frac{1}{|Y|} \int_Y \left((a \nabla_x \Psi)(0, \zeta) \cdot \nabla_\zeta \Psi(0, \zeta) - \operatorname{div}_x (a \nabla_\zeta \Psi)(0, \zeta) \Psi(0, \zeta) \right) d\zeta.$$

The limit of the right-hand side of the weak formulation (49) is

$$\lim_{\varepsilon \rightarrow 0} \varepsilon v^\varepsilon \int_{\mathbb{R}^d} \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z) w^\varepsilon(z) \varphi(z) d\mu_\varepsilon = v \langle \rho_\Psi(0, \cdot) \rangle \int_{\mathbb{R}^d} w(z_1, 0) \varphi(z_1, 0) d\mu^*.$$

Finally, passing to the limit of the weak formulation (49), we have

$$\begin{aligned} & \int_{\mathbb{R}^d} A^{\text{eff}} \nabla^{\mu^*} w(z_1, 0) \cdot \nabla \phi(z_1, 0) d\mu^* \\ & + \frac{\mu''(0)}{2} \langle \rho_\Psi(0, \cdot) \rangle \int_{\mathbb{R}^d} |z_1|^2 w(z_1, 0) \phi(z_1, 0) d\mu^* \\ & + \int_{\mathbb{R}^d} c^{\text{eff}} w(z_1, 0) \phi(z_1, 0) d\mu^* = v \langle \rho_\Psi(0, \cdot) \rangle \int_{\mathbb{R}^d} w(z_1, 0) \phi(z_1, 0) d\mu^*. \end{aligned} \tag{53}$$

Take any test function with zero trace $\varphi(z_1, 0, \dots, 0) = 0$ and a non-zero μ^* -gradient, e.g. $\varphi(z) = \sum_{j \neq 1} z_j \psi(z_1)$, with arbitrary $\psi \in C_c^\infty(\mathbb{R}) \setminus \{0\}$. Then

$$\begin{aligned} \varphi(z) &= z \cdot (0, \psi_2(z_1), \dots, \psi_d(z_1)), \quad \varphi(z_1, 0, \dots, 0) = 0, \\ \nabla \varphi(z) &= \left(\sum_{j \neq 1} z_j \psi'_j(z_1), \psi_2(z_1), \dots, \psi_d(z_1) \right), \\ \nabla^{\mu^*} \varphi(z) &= \nabla \varphi(z_1, 0, \dots, 0) = (0, \psi_2(z_1), \dots, \psi_d(z_1)). \end{aligned}$$

By the density of $C_c^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$, we can take $\psi_j \in L^2(\mathbb{R}^d)$. Taking this test function in (53) gives

$$\int_{\mathbb{R}^d} A^{\text{eff}} \nabla^{\mu^*} w(z_1, 0) \cdot (0, \psi_2(z_1), \dots, \psi_d(z_1)) d\mu^* = 0$$

which implies

$$A^{\text{eff}} \nabla^{\mu^*} w(z_1, 0) = \left(\sum_{j=1}^d A_{1j}^{\text{eff}} \partial_{z_j}^{\mu^*} w(z_1, 0), 0, \dots, 0 \right).$$

Let us reformulate the equation for N_k in the following form:

$$\begin{aligned} -\text{div}(a_\Psi(0, \zeta) \nabla(N_k(\zeta) + \zeta_k)) &= 0, \quad y \in Y \\ a_\Psi(0, \zeta) \nabla(N_k + \zeta_k) &= 0, \quad y \in \Sigma, \\ N_k(\cdot, \zeta') &\text{ is } 1 - \text{periodic.} \end{aligned} \tag{54}$$

Multiplying (54) by ζ_m for $m \neq 1$ (ζ_m is then 1-periodic in ζ_1 and thus it can be used as a test function), and integrating by parts over Y , we obtain

$$\int_Y a_\Psi(0, \zeta) \nabla(N_k(\zeta) + \zeta_k) \cdot \nabla \zeta_m \, d\zeta = 0,$$

which yields

$$0 = \int_Y \sum_{j=1}^d (a_\Psi)_{mj}(0, \zeta) (\partial_{\zeta_j} N_k + \delta_{jk}) \, d\zeta = A_{mk}^{\text{eff}}, \quad m \neq 1.$$

Then the weak formulation of the limit problem (53) becomes

$$\begin{aligned} & \int_{\mathbb{R}^d} A_{11}^{\text{eff}} \partial_{z_1}^{\mu^*} w(z_1, 0) \partial_{z_1} \phi(z_1, 0) \, d\mu^* + \frac{\mu''(0)}{2} \langle \rho_\Psi(0, \cdot) \rangle \int_{\mathbb{R}^d} |z_1|^2 w(z_1, 0) \phi(z_1, 0) \, d\mu^* \\ & + \int_{\mathbb{R}^d} c^{\text{eff}} w(z_1, 0) \phi(z_1, 0) \, d\mu^* = \nu \langle \rho_\Psi(0, \cdot) \rangle \int_{\mathbb{R}^d} w(z_1, 0) \phi(z_1, 0) \, d\mu^*. \end{aligned}$$

Choosing N_i as a test function in the cell problem of N_k we have

$$\int_Y a_\Psi(0, \zeta) \nabla_\zeta(N_k + \zeta_k) \cdot \nabla_\zeta N_i \, d\zeta = 0,$$

then A_{ik}^{eff} can be written as

$$A_{ik}^{\text{eff}} = \int_Y a_\Psi(0, \zeta) \nabla_\zeta(N_k + \zeta_k) \cdot \nabla_\zeta(N_i + \zeta_i) \, d\zeta$$

which gives

$$\begin{aligned} A_{11}^{\text{eff}} &= A^{\text{eff}} e_1 \cdot e_1 = \frac{1}{|Y|} \int_Y a(0, \zeta) \nabla(\zeta_1 + N_1(\zeta)) \cdot \nabla(\zeta_1 + N_1(\zeta)) \, d\zeta \\ &\geq \frac{\Lambda}{|Y|} \int_Y |\nabla(\zeta_1 + N_1(\zeta))|^2 \, d\zeta. \end{aligned}$$

Assuming that $\partial_{\zeta_i}(\zeta_1 + N_1(\zeta)) = 0$, for all i leads to the contradiction since N_1 is periodic in ζ_1 . Thus, the effective coefficient is strictly positive.

Denoting $a^{\text{eff}} := A_{11}^{\text{eff}}$, $w(z_1) := w(z_1, 0)$, the last integral identity is the weak formulation of the harmonic oscillator equation on \mathbb{R} :

$$-a^{\text{eff}} w'' + \left(\frac{\mu''(0)}{2} \langle \rho_\Psi(0, \cdot) \rangle |z_1|^2 + c^{\text{eff}} \right) w = \nu \langle \rho_\Psi(0, \cdot) \rangle w, \quad w \in L^2(\mathbb{R}). \tag{55}$$

Due to the normalization condition and the strong convergence of w^ε in $L^2(\mathbb{R}^d, \mu_\varepsilon)$ (see Lemma A.5), the limit function $w(z_1) \neq 0$. Thus, $(\nu, w(z_1))$ is an eigenpair of the effective spectral problem (55).

Remark 5. The eigenpairs (v_j, w_j) of the Sturm-Liouville problem

$$-a^{\text{eff}} w'' + \left(\frac{\mu''(0)}{2} \langle \rho_\Psi(0, \cdot) \rangle |z_1|^2 + c^{\text{eff}} \right) w = \nu \langle \rho_\Psi(0, \cdot) \rangle w, \quad w \in L^2(\mathbb{R})$$

admit explicit representation:

$$v_j = (c^{\text{eff}} + (2j - 1) \sqrt{\frac{a^{\text{eff}} \mu''(0)}{2}}) / \langle \rho_\Psi(0, \cdot) \rangle,$$

$$w_j(z_1) = H_j(\theta^{1/4} z_1) e^{-\sqrt{\theta} z_1^2 / 2}, \quad j = 1, 2, \dots,$$

where $\theta = \frac{\mu''(0)}{2a^{\text{eff}}}$ and $H_j(x) = e^{x^2} \frac{d^{j-1}}{dx^{j-1}} e^{-x^2}$ are the Hermite polynomials. Note that the eigenvalues ν_j are simple [4].

5. Convergence of spectra

In this section, we will show that the j th eigenvalue of the rescaled problem (42) converges to the j th eigenvalue of the homogenized problem (55), as well as the convergence of the corresponding eigenfunctions.

Lemma 5.1. *For sufficiently small ε , along a subsequence, the eigenvalues $\varepsilon \nu_j^\varepsilon$ of problem (42) are simple.*

Proof. Suppose that some eigenvalue $\varepsilon \nu^\varepsilon$ of (42) has multiplicity at least two for all $\varepsilon > 0$, i.e. there exist two linearly independent eigenfunctions $v_1^\varepsilon, v_2^\varepsilon$ corresponding to $\varepsilon \nu^\varepsilon$. Let us normalize the eigenfunctions by $\int_{\mathbb{R}^d} v_1^\varepsilon v_2^\varepsilon d\mu_\varepsilon = \delta_{ij}$. By the strong L^2 -compactness, v_1^ε and v_2^ε , extended by zero to $\mathbb{R} \times \varepsilon^{1/2} Q$, converge strongly in $L^2(\mathbb{R}^d, \mu_\varepsilon)$ to the eigenfunctions v_1, v_2 of the limit problem corresponding to some eigenvalue ν^* . Passing to the limit in the normalization condition yields $\int_{\mathbb{R}} v_1 v_2 dx = 0$. Since the eigenvalues of the one-dimensional limit problem are simple, v_1, v_2 should be linearly dependent, which leads to a contradiction. \square

The next lemma shows that the convergence of eigenvalues, as $\varepsilon \rightarrow 0$, preserves the order.

Lemma 5.2. *For any j , the j th eigenvalue ν_j^ε of problem (42) converges to the j th eigenvalue ν_j of (55), and the corresponding eigenfunction w_j^ε converges, along a subsequence, to the eigenfunction w_j of (55).*

Proof. By the a priori estimates and the compactness, we have proved that all the eigenvalues of (42) converge to some eigenvalues of (55). It is left to prove that all eigenvalues of the effective problem (55) are limits of some eigenvalues of (42). We follow the ideas of the proof of Lemma 3.12 in [23] and will prove this by contradiction. Assume that the first eigenvalue ν_1^ε of (42), converges to the second eigenvalue ν_2 of (55), and not to ν_1 . The first eigenvalue ν_1^ε is simple and the corresponding eigenfunction v_1^ε converges to the eigenfunction v_2 . By the variational principle,

$$v_1^\varepsilon = \inf_w \frac{\mathcal{F}_\varepsilon(w)}{\int_{\mathbb{R}^d} \rho_\Psi^\varepsilon w^2 d\mu_\varepsilon},$$

where the infimum is taken over $H^1(\mathbb{R}^d, \mu_\varepsilon) \setminus \{0\}$ such that $w|_{z_1=\varepsilon^{-1/2}\Gamma_\varepsilon^\pm} = 0$, and

$$\begin{aligned} \mathcal{F}_\varepsilon(w) := & \int_{\mathbb{R}^d} a_\Psi^\varepsilon(\sqrt{\varepsilon}z) \nabla w \cdot \nabla w d\mu_\varepsilon + \int_{\mathbb{R}^d} a^\varepsilon \nabla_z \Psi^\varepsilon \cdot \nabla(\Psi^\varepsilon w^2) d\mu_\varepsilon \\ & - \int_{\mathbb{R}^d} \Psi^\varepsilon \operatorname{div}_z(a^\varepsilon \nabla_z \Psi^\varepsilon) w^2 d\mu_\varepsilon \\ & + \int_{\mathbb{R}^d} \frac{\mu(\sqrt{\varepsilon}z_1) - \mu(0)}{\varepsilon} \rho_\Psi^\varepsilon(\sqrt{\varepsilon}z) w^2 d\mu_\varepsilon. \end{aligned}$$

The minimum is attained on the first eigenfunction w_1^ε . Since $v_1^\varepsilon \rightarrow v_2$, as $\varepsilon \rightarrow 0$, we can write $v_1^\varepsilon = v_2 + o(1)$, as $\varepsilon \rightarrow 0$.

Let $w_1(z_1)$ be the first eigenfunction of (55) and N be the normalized solution of the auxiliary cell problem satisfied by N_1 . Denote

$$W_\varepsilon = \left(w_1(z_1) + \varepsilon^{1/2} N\left(\frac{z}{\sqrt{\varepsilon}}\right) w_1'(z_1) \right) \phi_\varepsilon(z_1),$$

where

$$\phi_\varepsilon(z_1) = \begin{cases} 1, & z_1 \in [-\frac{\varepsilon^{-1/2}}{6}, \frac{\varepsilon^{-1/2}}{6}] \\ 0, & z_1 \in \mathbb{R} \setminus [-\frac{\varepsilon^{-1/2}}{3}, \frac{\varepsilon^{-1/2}}{3}] \end{cases}$$

and such that $0 \leq \phi_\varepsilon \leq 1, |\phi_\varepsilon'(z_1)| \leq C\varepsilon^{1/2}$.

The cut-off function is introduced in order to make the test function W_ε satisfy the Dirichlet boundary condition at the ends of the rod. Using Corollary 1 and taking into account the exponential decay of $w_1(z_1)$, we estimate the norm of W_ε :

$$\begin{aligned} \|W_\varepsilon\|_{L^2(\mathbb{R}^d, \mu_\varepsilon)}^2 &= \int_{\mathbb{R}^d} \left(w_1(z_1) + \varepsilon^{1/2} N\left(\frac{z}{\sqrt{\varepsilon}}\right) w_1'(z_1) \right)^2 |\phi_\varepsilon(z_1)|^2 d\mu_\varepsilon \\ &= \int_{\mathbb{R}^d} |w_1(z_1)|^2 |\phi_\varepsilon(z_1)|^2 d\mu_\varepsilon + \varepsilon \int_{\mathbb{R}^d} \left| N\left(\frac{z}{\sqrt{\varepsilon}}\right) w_1'(z_1) \right|^2 |\phi_\varepsilon(z_1)|^2 d\mu_\varepsilon \\ &\quad + 2\sqrt{\varepsilon} \int_{\mathbb{R}^d} w_1(z_1) N\left(\frac{z}{\sqrt{\varepsilon}}\right) w_1'(z_1) |\phi_\varepsilon(z_1)|^2 d\mu_\varepsilon \\ &= \int_{\mathbb{R}^d} |w_1(z_1)|^2 |\phi_\varepsilon(z_1)|^2 d\mu_\varepsilon + o(\varepsilon) \end{aligned}$$

$$= \int_{\mathbb{R}} |w_1(z_1)|^2 dz_1 + o(\sqrt{\varepsilon}), \text{ as } \varepsilon \rightarrow 0.$$

Let us compute the derivatives of W_ε :

$$\begin{aligned} \partial_{z_i} W_\varepsilon &= \left(\delta_{1i} + \partial_{\zeta_i} N(\zeta) \Big|_{\zeta=z/\sqrt{\varepsilon}} \right) w'_1(z_1) \phi_\varepsilon(z_1), \quad i \neq 1, \\ \partial_{z_1} W_\varepsilon &= \left(\delta_{11} + \partial_{\zeta_1} N(\zeta) \Big|_{\zeta=z/\sqrt{\varepsilon}} \right) w'_1(z_1) \phi_\varepsilon(z_1) + \varepsilon^{1/2} N\left(\frac{z}{\sqrt{\varepsilon}}\right) w''_1(z_1) \phi_\varepsilon(z_1) \\ &\quad + \left(w_1(z_1) + \varepsilon^{1/2} N\left(\frac{z}{\varepsilon}\right) w'_1(z_1) \right) \phi'_\varepsilon(z_1). \end{aligned}$$

Substituting W_ε into the functional \mathcal{F}_ε , we obtain

$$\begin{aligned} \mathcal{F}_\varepsilon(W_\varepsilon) &= \int_{\mathbb{R}^d} (a^\varepsilon_\Psi)_{ij} \left(\delta_{1j} + \partial_{\zeta_j} N\left(\frac{z}{\sqrt{\varepsilon}}\right) \right) \left(\delta_{1i} + \partial_{\zeta_i} N\left(\frac{z}{\sqrt{\varepsilon}}\right) \right) \phi_\varepsilon^2(z_1) (w'_1(z_1))^2 d\mu_\varepsilon \\ &\quad + \int_{\mathbb{R}^d} (a^\varepsilon \nabla_z \Psi^\varepsilon) \cdot \nabla (\Psi^\varepsilon W_\varepsilon^2) d\mu_\varepsilon + \frac{1}{\sqrt{\varepsilon}} \int_{\mathbb{R}^d} \Psi^\varepsilon \operatorname{div}_z (a^\varepsilon \nabla_\zeta \Psi^\varepsilon) W_\varepsilon^2 d\mu_\varepsilon \\ &\quad + \int_{\mathbb{R}^d} \frac{\mu(\sqrt{\varepsilon} z_1) - \mu(0)}{\varepsilon} \rho_\Psi W_\varepsilon^2 d\mu_\varepsilon + o(1), \quad \varepsilon \rightarrow 0. \end{aligned}$$

Taking the limit as $\varepsilon \rightarrow 0$

$$\begin{aligned} \mathcal{F}_\varepsilon(W_\varepsilon) &= \int_{\mathbb{R} \times Y} (a_\Psi(0, \zeta))_{ij} (\delta_{1j} + \partial_{\zeta_j} N(\zeta)) (\delta_{1i} + \partial_{\zeta_i} N(\zeta)) (w'_1(z_1))^2 d\zeta dz_1 \\ &\quad + \int_{\mathbb{R} \times Y} (a \nabla_x \Psi)(0, \zeta) \cdot \nabla_\zeta (\Psi(0, \zeta)) (w_1(z_1))^2 d\zeta dz_1 \\ &\quad - \int_{\mathbb{R} \times Y} \Psi(0, \zeta) \operatorname{div} (a \nabla_\zeta \Psi)(0, \zeta) (w_1(z_1))^2 d\zeta dz_1 \\ &\quad + \frac{\mu''(0)}{2} \langle \rho_\Psi(0, \cdot) \rangle \int_{\mathbb{R}} |z_1|^2 (w_1(z_1))^2 dz_1 + o(1) \\ &= \int_{\mathbb{R}} a^{\text{eff}}(w_1(z_1))^2 dz_1 + \int_{\mathbb{R}} c^{\text{eff}}(w_1(z_1))^2 dz_1 \\ &\quad + \frac{\mu''(0)}{2} \langle \rho_\Psi(0, \cdot) \rangle \int_{\mathbb{R}} |z_1|^2 (w_1(z_1))^2 dz_1 + o(1), \quad \varepsilon \rightarrow 0. \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{\mathcal{F}(W_\varepsilon)}{\int_{\mathbb{R}^d} \rho_\Psi^\varepsilon W_\varepsilon^2 d\mu_\varepsilon} \\ &= \frac{\int_{\mathbb{R}} a^{\text{eff}}(w_1')^2 dz_1 + \int_{\mathbb{R}} c^{\text{eff}}(w_1)^2 dz_1 + \frac{\mu''(0)}{2} \langle \rho_\Psi(0, \cdot) \rangle \int_{\mathbb{R}} |z_1|^2 (w_1)^2 dz_1}{\langle \rho_\Psi(0, \cdot) \rangle \int_{\mathbb{R}} (w_1)^2 dz_1} + o(1) \\ &= \nu_1 + o(1), \quad \varepsilon \rightarrow 0. \end{aligned}$$

Since $\nu_1 < \nu_2$, we have constructed a test function giving a smaller eigenvalue than $\nu_1^\varepsilon = \nu_2 + o(1)$, which is a contradiction. \square

6. Conclusion

The paper addresses a case that was open in previous work [24]: the asymptotics of the positive part of the spectrum when the local average of the sign-changing weight is negative. This is a non-classical regime where standard homogenization techniques do not apply. Such indefinite spectral problems arise in population genetics, where the sign-changing weight models spatially varying selection that can favor or disfavor certain alleles in different regions. For the case where the local average of the weight is negative, the positive eigenvalues of the spectral problem grow as ε^{-2} as the thickness ε of the cylinder tends to zero. Moreover, the eigenfunctions oscillate and are approximated by a product of the principal eigenfunction of an auxiliary spectral cell problem at ε scale and the other function satisfying the harmonic oscillator equation at $\sqrt{\varepsilon}$ scale. The existence of a positive principal eigenvalue of an indefinite auxiliary spectral problem on a periodicity cell is of interest on its own. We prove that the eigenfunctions localize near the minimum of the principal eigenvalue of an auxiliary cell problem. We derive a homogenized spectral problem which is a one-dimensional harmonic oscillator on the real line, with explicit formulas for the effective coefficients. The technique employed combines homogenization methods, including factorization with an auxiliary cell eigenfunction and two-scale convergence in spaces with singular measures, with dimension reduction.

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Appendix A. Two-scale convergence in spaces with measure

Definition A.1. Let μ_ε be the measure defined by (43). A sequence $g^\varepsilon(x) \in L^2(\mathbb{R}^d, \mu_\varepsilon)$ is said to converge weakly in $L^2(\mathbb{R}^d, \mu_\varepsilon)$ to a function $g(x_1) \in L^2(\mathbb{R}^d, \mu^*)$, as $\varepsilon \rightarrow 0$, if

- (i) $\|g^\varepsilon\|_{L^2(\mathbb{R}^d, \mu_\varepsilon)} \leq C$,
- (ii) for any $\phi \in C_c^\infty(\mathbb{R}^d)$ the following relation holds:

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} g^\varepsilon(x) \phi(x) d\mu_\varepsilon = \int_{\mathbb{R}^d} g(x_1) \phi(x) d\mu^*.$$

A sequence g^ε is said to converge strongly to $g(x_1)$ in $L^2(\mathbb{R}^d, \mu_\varepsilon)$, as $\varepsilon \rightarrow 0$, if it converges weakly and

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} g^\varepsilon(x) \psi^\varepsilon(x) d\mu_\varepsilon = \int_{\mathbb{R}^d} g(x_1) \psi(x_1) d\mu^*$$

for any sequence $\{\psi^\varepsilon(x)\}$ weakly converging to $\psi(x_1)$ in $L^2(\mathbb{R}^d, \mu_\varepsilon)$.

The weak compactness results are valid in spaces with measure [30].

In the present context, two-scale convergence is described as follows.

Definition A.2. We say $g^\varepsilon \in L^2(\mathbb{R}^d, \mu_\varepsilon)$ converges two-scale weakly, as $\varepsilon \rightarrow 0$, in $L^2(\mathbb{R}^d, \mu_\varepsilon)$ if

- (i) $\|g^\varepsilon\|_{L^2(\mathbb{R}^d, \mu_\varepsilon)} \leq C, \quad \varepsilon > 0,$
- (ii) there exists a function $g(x_1, y) \in L^2(\mathbb{R}^d \times Y, \mu^* \times d\zeta)$ such that the following limit relation holds:

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} g^\varepsilon(x) \phi(x) \psi\left(\frac{x}{\sqrt{\varepsilon}}\right) d\mu_\varepsilon = \frac{1}{|Y|} \int_{\mathbb{R}^d} \int_Y g(x_1, y) \phi(x_1, 0) \psi(\zeta) d\zeta d\mu^*,$$

for any $\phi \in C_c^\infty(\mathbb{R}^d)$ and $\psi(\zeta) \in C^\infty(\bar{Y})$ periodic in ζ_1 .

We write $g^\varepsilon \xrightarrow{2} g(x_1, y)$ if g^ε converges two-scale weakly to $g(x_1, y)$ in $L^2(\mathbb{R}^d, \mu_\varepsilon)$.

Note that the last definition holds for more general classes of test functions, e.g. $\Phi(x, y) \in C(\mathbb{R}^d; L^\infty(Y))$ or $\Phi(x, y) = \phi(x) \psi(y)$ with $\phi \in C(\mathbb{R}^d), \psi \in L^2(Y)$, [30].

Lemma A.3. Suppose that g^ε satisfies the following estimate

$$\|g^\varepsilon\|_{L^2(\mathbb{R}^d, \mu_\varepsilon)} \leq C.$$

Then g^ε , up to a subsequence, converges two-scale weakly in $L^2(\mathbb{R}^d, \mu_\varepsilon)$ to some function $g(x_1, y) \in L^2(\mathbb{R}^d \times Y, \mu^* \times d\zeta)$.

Definition A.4. A sequence g^ε is said to converge two-scale strongly to a function $g(x_1, y) \in L^2(\mathbb{R}^d \times Y, \mu^* \times d\zeta)$ if

- (i) $g^\varepsilon \xrightarrow{2} g(x_1, y).$
- (ii) The following limit relation holds

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} (g^\varepsilon(x))^2 d\mu_\varepsilon = \frac{1}{|Y|} \int_{\mathbb{R}^d} \int_Y (g(x_1, y))^2 d\zeta d\mu^*$$

We write $g^\varepsilon \xrightarrow{2} g(x_1, y)$ if g^ε converges two-scale strongly to the function $g(x_1, y)$ in $L^2(\mathbb{R}^d, \mu_\varepsilon)$.

Lemma A.5. Assume that v_ε is such that

$$\int_{\mathbb{R}^d} |\nabla v|^2 d\mu_\varepsilon + \int_{\mathbb{R}^d} |x_1 v|^2 d\mu_\varepsilon \leq C.$$

Then v_ε converges strongly in $L^2(\mathbb{R}^d, \mu_\varepsilon)$ to $v \in L^2(\mathbb{R})$.

The proof of Lemma A.5 follows the lines of Lemma 4.4 in [8].

Appendix B. Integral estimates for oscillating functions

The proof of the following integral estimate for oscillating functions can be found in [23].

Lemma B.1. Let $v_\varepsilon \in H_0^1(\Omega_\varepsilon)$, $v_\varepsilon(\pm 1, x') = 0$, and $w(x_1, y) \in C^1(\bar{I}, L^\infty(Y))$. Then as $\varepsilon \rightarrow 0$,

$$\begin{aligned} & \int_{\Omega_\varepsilon} w(x_1, \frac{x}{\varepsilon}) v_\varepsilon^2(x) dx - \frac{1}{|Y|} \int_{\Omega_\varepsilon} \left(\int_Y w(x_1, y) dy \right) v_\varepsilon^2(x) dx \\ &= O(\varepsilon \|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)}). \end{aligned}$$

Corollary 1. Let $w_\varepsilon \in H^1(\mathbb{R}^d, \mu_\varepsilon)$ be such that $w_\varepsilon = 0$ on $\varepsilon^{-1/2}\Gamma_\varepsilon^\pm$, and $c(x_1, y) \in C^1(\mathbb{R}; L^\infty(Y))$. Then as $\varepsilon \rightarrow 0$,

$$\begin{aligned} & \int_{\mathbb{R}^d} c(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}) w_\varepsilon^2(z) d\mu_\varepsilon - \int_{\mathbb{R}^d} \left(\int_Y c(\sqrt{\varepsilon}z_1, \zeta) d\zeta \right) w_\varepsilon^2(z) d\mu_\varepsilon \\ &= O(\sqrt{\varepsilon} \|w_\varepsilon\|_{L^2(\mathbb{R}^d, \mu_\varepsilon)} \|\nabla w_\varepsilon\|_{L^2(\mathbb{R}^d, \mu_\varepsilon)}). \end{aligned}$$

Data availability

No data was used for the research described in the article.

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