THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Rigidity and embedding phenomena in operator algebras

Jan Gundelach

UNIVERSITY OF GOTHENBURG

Division of Analysis and Probability Theory
Department of Mathematical Sciences
Chalmers University of Technology and University of Gothenburg
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Division of Analysis and Probability Theory
Department of Mathematical Sciences
Chalmers University of Technology and University of Gothenburg
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Sweden

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Sammanfattning

Denna avhandling presenterar inledande kapitel om matematisk bakgrund samt tre artiklar inom forskningsområdena operatoralgebror respektive topologisk dynamik.

Den första artikeln studerar inbäddningar av L^p-operatoralgebror som härrör från tvinnade étale gruppoider med särskild tonvikt på rigiditetsfenomen för $p \neq 2$, vilket gör det möjligt att beskriva inbäddningar mellan reducerade L^p-gruppoidalgebror helt i termer av de underliggande gruppoiderna. Tillämpningar inkluderar AF-inbäddningsförmåga och icke-existensen av unitala kontraktiva homomorfismer från $\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p$ till $\mathcal{O}_2^p$ för $p \neq 2$.

Den andra artikeln handlar om en inbäddningsversion av Rubins sats. Rubins sats säger att om $\Gamma \curvearrowright X$ och $\Delta \curvearrowright Y$ är Rubin-gruppverkningar, så inducerar varje gruppisomorfism $\Gamma \cong \Delta$ en ekvivariant homeomorfism $Y \cong X$. Vi belyser gruppinbäddningar som inducerar en rumslig ekvivariant funktion av en viss form, inklusive exempel på sådana inbäddningar mellan generaliserade Brin-Thompson-grupper.

Den tredje artikeln presenterar två dualitetsresultat för Rokhlin-dimensionen av en partiell gruppverkning. Vi visar att Rokhlin-dimensionen av en partiell verkning av en ändlig abelsk grupp överensstämmer med den duala representerbarhetsdimensionen av den globala duala verkningen av den duala gruppen på den partiella korsprodukten. Vi visar vidare att representerbarhetsdimensionen av en partiell verkningen av en ändlig abelsk grupp överensstämmer med Rokhlin-dimensionen av dess duala verkningen.

Abstract

Starting with introductory chapters of mathematical background, we present three papers in the research areas of operator algebras and topological dynamics, respectively.

The first paper studies embeddings of L^p-operator algebras arising from twisted étale groupoids with particular emphasis on rigidity phenomena for $p \neq 2$ allowing to describe embeddings between reduced L^p-groupoid algebras entirely in terms of the underlying groupoids. Applications include AF-embeddability and the non-existence of unital contractive homomorphisms from $\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p$ into $\mathcal{O}_2^p$ for $p \neq 2$.

Keywords: Rigidity, isometric embedding, twisted étale groupoid, L^p-operator algebra, spatial normalizer, groupoid actor, topological full group.

The second paper provides an embedding version of Rubin's theorem. Rubin's theorem asserts that if $\Gamma \curvearrowright X$ and $\Delta \curvearrowright Y$ are Rubin actions, then any group isomorphism $\Gamma \cong \Delta$ induces an equivariant homeomorphism $Y \cong X$. We highlight group embeddings that induce a spatial equivariant map of a certain form, including instances of such embeddings between generalized Brin-Thompson groups.

Keywords: Rigidity, topological dynamics, locally moving, Rubin embedding, spatial map, regular support, ultrafilter convergence.

The third paper presents two duality results for the Rokhlin dimension of a partial action. We show that the Rokhlin dimension of a partial action by a finite abelian group agrees with the dual representability dimension of the global dual action by the dual group on the partial crossed product. We further show that the representability dimension of a partial action by a finite abelian group agrees with the Rokhlin dimension of its dual action.

Keywords: C*-dynamical system, central sequence algebra, partial action, duality, Rokhlin dimension, representability dimension.

List of publications

This thesis consists of introductory chapters that provide mathematical background for each of the presented papers, and the following appended preprints:

Paper I: E. Gardella and J. Gundelach. Embeddings of L^p-operator algebras. 2026. Preprint, arXiv:2601.15204.

Paper II: J. Gundelach. An embedding version of Rubin's theorem. 2026. Preprint, arXiv:2602.18197.

Paper III: J. Gundelach. Duality of partial Rokhlin dimension. 2026. Preprint, arXiv:2604.09380.

Author contributions

I. Refinement of the author's licentiate thesis from September 2023. Within the scope of the licentiate thesis: Contributed to the literature study, worked out the detailed proofs after discussing the outline with the coauthor, and wrote the entire manuscript. Beyond the scope of the licentiate thesis: Included the treatment of twists and groupoid actors, refined the notion of spatial normalizers, and wrote the entire manuscript except for the introduction.

II. Single author project proofread by Eusebio Gardella.

III. Single author project suggested and proofread by Eusebio Gardella.

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Jan. – Göteborg, 2026.

Contents

1 Introduction

In the middle of the last century, the vast research field of operator algebras emerged as an offspring of functional analysis and topological dynamics. Von Neumann algebras and C*-algebras are well-behaved topological spaces with an algebra structure consisting of bounded operators on a Hilbert space and, to some extent, they describe noncommutative measure spaces and noncommutative topological spaces, respectively. In this sense, operator algebras offer a framework that generalizes the study of topological dynamics and enriches it with both algebraic and analytic tools. A strong motivation to consider the generalization from commutative algebras to potentially noncommutative ones in the first place is that it can be seen as the mathematical background to describe the passage from classical to quantum mechanics in theoretical physics. This connection led to many active research branches within operator algebras that involve the term "quantum", even though, in the last decades, the advances in operator algebras are no longer of direct relevance for physicists.

A C*-algebra is a special Banach algebra in the sense that it is representable as a subalgebra of the bounded operators on a Hilbert space $L^2(\mu)$. More recently, there was the idea to also consider other types of Banach algebras representable as bounded operators on some measure space $L^p(\mu)$ leading to the notion of L^p-operator algebras.

This thesis consists of various background sections that enable the reader to understand the main part consisting of three appended articles that primarily deal with different rigidity phenomena in operator algebras. For dynamical concepts, such as group actions or transformation groupoids, there are associated objects, such as operator algebras or topological full groups, and the articles investigate how much information about these underlying concepts can be recovered from the associated objects. On the one hand, Paper III characterizes certain partial actions in terms of its associated crossed product and its dual action. On the other hand, Paper II characterizes certain group embeddings between topological full groups that remember the underlying spaces they act on. Finally, Paper I, as the most significant part of this thesis, characterizes embeddings between groupoid L^p-operator algebras on the level of the underlying groupoids.

In total, the common theme of the articles is to associate an operator algebra to a groupoid or a partial action and to study the rigidity of this assignment. Rigidity, in this context, means some inflexibility and small range of choices for objects with a certain behavior. For the first rigidity phenomenon, the less symmetric nature of the norm is reflected in a smaller supply of isometric embeddings because such maps automatically have to preserve more structure than their Hilbert counterparts do. Intuitively, for example, compared to the Euclidean norm $\|\cdot\|_2$, the unit ball in the $\|\cdot\|_p$-norm for $p \neq 2$ admits noticeably fewer symmetries. For $p \neq 2$, this forces unital isometric embeddings between L^p-operator algebras to preserve more of the underlying structure than an embedding between C*-algebras typically does. This automatic preservation eventually allows one to describe embeddings on the level of the underlying groupoids.

Another embedding phenomenon studied in this thesis affects Rubin actions. A group which admits a Rubin action heuristically encodes the space it acts on algebraically by its group law already. We consider Rubin embeddings as those group embeddings that allow for a spatial map between the spaces.

Finally, the third rigidity phenomenon studied in this thesis affects partial actions with finite Rokhlin dimension. For finite abelian groups, the Rokhlin dimension is captured by the dual action and suitable forms of representability dimension. This goes both ways. The Rokhlin dimension is the dual representability dimension of its dual, while the Rokhlin dimension of the dual is the representability dimension of the original action.

2 Background

In this chapter, we present an overview over selected topics needed to follow the appended papers conveniently. The aim is to facilitate the access and to comment on concepts that are taken for granted in the papers later on. For a proper treatment including rigorous proofs of the sketched theory, we refer to the literature. That is, to [9] for basics on C*-algebras and to [27], [14], [8] for L^p-operator algebras in the first section; to [32] for groupoid theory in the second section; to [29] for the groupoid model of a Cartan pair, to [24] for groupoid diagrams associated to Cartan maps, and to [33] for basics on groupoid actors in the third section; to [22, Section 3.6] for Rubin's theorem in the fourth section; and to [12] for partial actions in the fifth section.

2.1 Operator algebraic background

For this section on operator algebras, the reader is just assumed to know fundamental functional analysis.

2.1.1 C*-algebras

Starting with their definition in the middle of the last century, C*-algebras are intensely studied objects and since then huge progress has been made at understanding their structure. In this section, we define C*-algebras and state a few classical, but nevertheless nontrivial, results that will have some connection with the appended papers later on. For this introduction, we will closely follow Davidson [9, Chapter 1].

Throughout, all vector spaces and algebras are assumed to be over the field of complex numbers $\mathbb{C}$ and algebras are, unless stated otherwise, not necessarily unital.

Definition 2.1.1. A *Banach algebra* A is an algebra with a submultiplicative norm $\|\cdot\|$, that is,

$$\|ab\| \leq \|a\| \cdot \|b\| \quad \text{for all} \quad a, b \in A,$$

which is complete in the norm topology. It is called a *Banach *-algebra* if there is a conjugate linear involution $*$ on A that reverses the order of multiplication, that is, for all $a, b \in A$ and $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} (a+b)^* &= a^* + b^*, \\ (\lambda a)^* &= \overline{\lambda} a^*, \\ a^{**} &= a, \\ (ab)^* &= b^* a^*. \end{aligned}$$

In this case, $*$ is called the *adjoint*.

Definition 2.1.2. A *C*-algebra* A is a Banach *-algebra whose norm satisfies the *C*-identity*

$$\|a^* a\| = \|a\|^2 \quad \text{for all} \quad a \in A.$$

Note that we do not require our algebras to be unital.

Example 2.1.3. Let $(E, \|\cdot\|)$ be a *Banach space*, that is, a complete normed vector space, and let $\mathcal{B}(E)$ denote the vector space of bounded endomorphisms $E \to E$. Then $\mathcal{B}(E)$ is a Banach algebra under composition as multiplication and the operator norm

$$\|F\| := \sup_{\|\xi\|=1} \|F(\xi)\| \quad \text{for all} \quad F \in \mathcal{B}(E).$$

If $E = \mathcal{H}$ is a *Hilbert space*, that is, if the norm is induced by an inner product $\langle \cdot, \cdot \rangle$, then $\mathcal{B}(\mathcal{H})$ is even a C*-algebra under the usual adjoint operation since, for all $F \in \mathcal{B}(\mathcal{H})$, we have

$$\|F^* F\| = \sup_{\|\xi\|=\|\eta\|=1} |\langle F^* F(\xi), \eta \rangle| = \sup_{\|\xi\|=\|\eta\|=1} |\langle F(\xi), F(\eta) \rangle| = \|F\|^2.$$

Consequently, all norm closed self-adjoint subalgebras of $\mathcal{B}(\mathcal{H})$ are C*-algebras, too. We refer to them as *concrete* C*-algebras.

In fact, [9, Theorem I.9.12] asserts that every abstract C*-algebra is of this concrete form up to C*-isomorphism, namely isometric *-isomorphism.

Theorem 2.1.4. (Gelfand-Naimark). Every C*-algebra A is C*-isomorphic to a concrete C*-algebra. That is, there are a Hilbert space $\mathcal{H}$ and an isometric *-homomorphism $\varphi\colon A \hookrightarrow \mathcal{B}(\mathcal{H})$.

This theorem justifies calling a C*-algebra an *operator algebra* because its elements can be regarded as certain bounded linear operators on a Hilbert space. Moreover, it makes sense to apply the notions for special operators, such as projections, isometries or unitaries to C*-algebras.

Definition 2.1.5. Let A be a C*-algebra. An element $a \in A$ is called *hermitian* if $a = a^*$ and *positive* if there is an element $b \in A$ such that $a = b^*b$. A hermitian element $p \in A$ such that $p^2 = p$ is called a *projection*. An element $v \in A$ is called a *partial isometry* if it satisfies the relations $vv^*v = v$ and $v^*vv^* = v^*$, and in this case v^*v is called the *domain projection* and vv^* the *range projection*.

If A is unital, then $v \in A$ is called an *isometry* if $v^*v = 1$ and a *unitary* if it is an invertible isometry, that is, $v^* = v^{-1}$.

Example 2.1.6. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Then $L^\infty(\mu)$ with the essential supremum norm is a commutative C*-algebra under pointwise operations. We can view $L^\infty(\mu)$ as a concrete C*-algebra since it acts isometrically on $\mathcal{B}(L^2(\mu))$ by multiplication operators. Up to measure zero, the hermitian elements are precisely those function classes represented by $\mathbb{R}$-valued functions, positive ones by $\mathbb{R}_+$-valued functions, projections by $\{0,1\}$-valued functions, and all isometries are necessarily unitaries and representable by $\mathbb{T}$-valued functions.

Even though we theoretically could regard every C*-algebra as an operator algebra, the concrete picture is in practice not necessarily the most convenient way to think about them. There are also C*-algebras of a completely different flavor. We now present another class commutative C*-algebras.

Example 2.1.7. Let X be a topological space that is locally compact and Hausdorff. A continuous function $f\colon X \to \mathbb{C}$ is called a *C_0-function* if for all $\varepsilon > 0$ there is a compact subset $K \subseteq X$ such that $\|f|_{X \setminus K}\|_\infty < \epsilon$. The set of C_0-functions $C_0(X)$ is a Banach algebra under pointwise multiplication and with the supremum norm $\|\cdot\|_\infty$. With the adjoint operation of complex conjugation, $C_0(X)$ is an example of a commutative C*-algebra, that is, for all $f, g \in C_0(X)$, we have $f \cdot g = g \cdot f$.

In fact, combining [9, Theorem I.2.7, Theorem I.3.1] shows that, up to C*-isomorphism, all commutative C*-algebras are of this form.

Theorem 2.1.8. Let A be a commutative Banach algebra. Then there are a locally compact Hausdorff topological space X_A, called the *Gelfand spectrum*, and a contractive algebra homomorphism $\widehat{(\cdot)}\colon A \to C_0(X_A)$, called the *Gelfand transform*. If A is a C*-algebra, then the Gelfand transform is a C*-isomorphism.

Even more, $*$-homomorphisms between commutative C*-algebras are characterized by continuous maps between the Gelfand spectra already. That is, commutative C*-algebraic theory is fully explained in terms of locally compact Hausdorff topological spaces. Interpreted from another angle, one could say that not necessarily commutative C*-algebras generalize classical topological spaces in some sense. This is why research on C*-algebras is sometimes also referred to as *noncommutative topology.*

Theorem 2.1.9. (Gelfand's equivalence). The assignment $C_0(-)$ is a contravariant equivalence between the category of locally compact Hausdorff topological spaces with proper continuous maps as morphisms and the category of commutative C*-algebras with non-degenerate $*$-homomorphisms. Restricting to compact Hausdorff spaces amounts to restricting to unital commutative C*-algebras with unital $*$-homomorphisms on the other end.

Remark 2.1.10. In particular, if X and Y are compact Hausdorff spaces, any unital $*$-homomorphism $\psi\colon C(X) \to C(Y)$ is spatially induced by some continuous map $\rho\colon Y \to X$. That is, for all $f \in C(X)$, we have $\psi(f) = f \circ \rho$. Furthermore, ψ is isometric if and only if it is injective and this is equivalent to the spatial map ρ being surjective.

We now turn to the structure of finite dimensional C*-algebras. Here, the concrete picture of Hilbert space operator algebras helps to find examples. For a finite dimensional Hilbert space $\mathbb{C}^n$, the bounded linear operators are given by the algebra of complex valued $(n \times n)$-matrices M_n with the $\|\cdot\|_2$-operator norm. Its $*$-subalgebras are block-diagonal matrix algebras. Such block-diagonal matrix algebras classify all finite dimensional C*-algebras; see [9, Section III.1].

Theorem 2.1.11. Up to C*-isomorphism, every finite dimensional C*-algebra A is a direct sum of matrix algebras, that is, there are finitely many block sizes $m_1, \ldots, m_k \in \mathbb{N}$ such that $A \cong \bigoplus_{i=1}^k M_{m_i}$.

Out of finite dimensional C*-algebras, there is a straightforward way to construct infinite dimensional C*-algebras with a direct limit procedure; see [9, Section III.2].

Definition 2.1.12. A C*-algebra A is called *approximately finite dimensional* or *AF* if it is the closure of an increasing union of finite dimensional subalgebras A_N for $N \in \mathbb{N}$, that is, $A = \overline{\bigcup_{N\in\mathbb{N}} A_N}$. If A is unital, we use the convention to refer to the scalar multiples of 1 as A_0.

Knowing the form of finite dimensional C*-algebras, an AF-algebra can alternatively be described in terms of a sequence of direct sums of matrix algebras and the way the current one, say $A_N \cong \bigoplus_{j=1}^l M_{n_j}$, should be embedded into its successor, say $A_{N+1} \cong \bigoplus_{i=1}^{k} M_{m_i}$. This is just a matter of bookkeeping how often a specific M_{n_j}-block is repeated along the diagonal in the matrix algebra M_{m_i} and leads to the multiplicity $a_{ij} \in \mathbb{N}_0$. In this way, we can represent the information encoded in the choice of the finite dimensional algebras and the embeddings $A_N \subseteq A_{N+1}$ graphically.

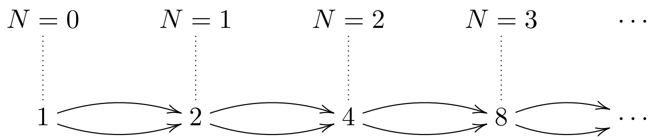
Definition 2.1.13. Let $A = \overline{\bigcup_{N\in\mathbb{N}} A_N}$ be an AF-algebra. The *Bratteli diagram* of the increasing sequence $(A_N)_{N\in\mathbb{N}}$ is a directed graph that is constructed inductively. For $N \in \mathbb{N}$, let $A_N \cong \bigoplus_{j=1}^{l} M_{n_j}$, $A_{N+1} \cong \bigoplus_{i=1}^{k} M_{m_i}$, and let $(a_{ij})_{i=1,\cdots,k,\, j=1,\cdots,l}$ denote the multiplicities of the embedding. The N-th layer of the diagram now consists of l vertices labeled by the block sizes $n_1, \cdots, n_l$ and, for each $j = 1, \cdots, l$, we draw a_{ij} many arrows to the i-th vertex of the successive layer. If A is unital, we start this construction at $A_0 = \mathbb{C}\cdot 1$ already.

Note that an AF-algebra does not admit a unique Bratteli diagram since, for example, any choice for a representing sequence of finite dimensional subalgebras can be thinned out. Correspondingly, the layers of the associated Bratteli diagrams telescope.

Example 2.1.14. If we take $A_N := M_{2^N}$ with the canonical embeddings

$$A_N \hookrightarrow A_{N+1}, \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

of multiplicity two, then the Bratteli diagram looks like



The resulting AF-algebra is called *CAR-algebra* and we denote it by M_{2^∞}. It is a prominent example of a unital infinite dimensional C*-algebra.

Example 2.1.15. If we take $A_N := M_N$ with the non-unital corner embeddings

$$A_N \hookrightarrow A_{N+1}, \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},$$

then the Bratteli diagram looks like $1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$ and describes the AF-algebra of *compact operators* $\mathcal{K}(\ell^2(\mathbb{N}))$. It is a prominent example of a non-unital infinite dimensional C*-algebra.

2.1.2 L^p-operator algebras

In the 1970s, Herz studied group operator algebras for a locally compact group G that act on the family of Banach spaces $L^p(G)$ for $p \in [1, \infty)$ instead of just on the Hilbert space $L^2(G)$. Approximately one decade ago, Phillips [27] introduced L^p-operator algebras in a more general framework and, to a large extent, their study was driven by comparison with C*-algebraic theory since then. In this section, we define L^p-operator algebras and study some guiding examples. We compare them to C*-algebras and end the section with the Banach-Lamperti theorem stating that, for $p \neq 2$, all MP-partial isometries in $\mathcal{B}(L^p(\mu))$ are of a special spatial form.

Recall from Example 2.1.3 that the bounded operators on a Banach space E form a unital Banach algebra $\mathcal{B}(E)$. We take a closer look at those Banach algebras that can be seen as norm closed subalgebras of $\mathcal{B}(E)$ where E is an L^p-space.

Definition 2.1.16. Let $p \in [1, \infty)$. A Banach algebra A is called an *L^p-operator algebra* if there are a measure space $(\Omega, \mathcal{A}, \mu)$ and an isometric embedding of A into the Banach algebra of bounded linear operators $\mathcal{B}(L^p(\mu))$.

Remark 2.1.17. Recall that all Hilbert spaces are of the form $L^2(\mu)$. By Theorem 2.1.4, we have that all C*-algebras are examples of L^2-operator algebras. However, L^2-operator algebras are not necessarily $*$-closed. The upper triangular $(n \times n)$-matrices with the $\|\cdot\|_2$-operator norm provide an example of a unital L^2-operator algebra that is not a C*-algebra.

For $p \neq 2$, there is no canonical $*$-involution to work with. For analogues of the definitions in Definition 2.1.5, equivalent conditions that do not involve the adjoint have to be found; see [8].

Definition 2.1.18. Let A be a unital Banach algebra and let $a \in A$. Since $(\sum_{k=0}^{n} \frac{a^k}{k!})_{n\in\mathbb{N}}$ is Cauchy, the limit exists in A and is denoted by e^a. An element $h \in A$ is called *hermitian* if, for all $t \in \mathbb{R}$, we have that $\|e^{ith}\| = 1$. We denote the set of hermitian elements by A_{h} and define the *core* as the vector space

$$\mathrm{core}(A) := A_{\mathrm{h}} + iA_{\mathrm{h}}.$$

Furthermore, we define the group of *invertible isometries* as

$$\mathcal{U}(A) := \{u \in A\colon u \text{ invertible with } \|u\| = \|u^{-1}\| = 1\}.$$

Remark 2.1.19. These definitions generalize the notions of hermitian elements and unitaries for unital C*-algebras. For general Banach algebras, hermitian elements are not closed under multiplication. However, for L^p-operator algebras, Choi, Gardella and Thiel showed in [8, Theorem 2.9] that the core is not only a vector space, but a C*-subalgebra under complex conjugation.

For a C*-algebra itself, the core is the whole algebra since any element can be written as $a = \frac{a+a^*}{2} + i \cdot \frac{a-a^*}{2i}$. For L^p-operator algebras with $p \neq 2$, however, the core is a much more interesting object. In this case, it is the maximal C*-subalgebra and automatically commutative.

Example 2.1.20. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $p \in [1, \infty)$. Generalizing Example 2.1.6, it is easy to see that multiplication operators by real-valued bounded functions $L^\infty(\mu, \mathbb{R})$ are hermitian operators in $\mathcal{B}(L^p(\mu))$. For $p \neq 2$, however, [8, Proposition 2.7] shows that there are no more of them and thus $\operatorname{core}(\mathcal{B}(L^p(\mu))) \cong L^\infty(\mu)$; see also [8, Example 2.11].

Example 2.1.21. For $n \in \mathbb{N}$, we equip $\{1, \cdots, n\}$ with the counting measure. For fixed $p \in [1, \infty)$, we write M_n^p for the L^p-operator algebra of $(n \times n)$-matrices under the Banach algebraic identification with $\mathcal{B}(\ell^p(\{1, \cdots, n\}))$. Note that, in order to view the matrix algebra as a Banach algebra, the choice of the norm matters, even though algebraically it is the same for all p.

For example, the matrices that rotate by an angle $\theta \in [0, 2\pi)$ given by

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in M_2^p$$

are clearly all invertible, but whether they are invertible isometries depends on p. They are for $p = 2$, but, in the 1-norm, say, the unit ball of $\ell^1(\{1, 2\}, \mathbb{R}) = \mathbb{R}^2$ is diamond-shaped and only rotationally invariant if θ is a multiple of $\pi/2$. Concretely, the computation $\|R_{\frac{\pi}{4}}(\delta_1)\|_1 = \|\frac{\sqrt{2}}{2}(\delta_1 + \delta_2)\|_1 = \sqrt{2}$ shows that $\|R_{\frac{\pi}{4}}\|_1 > 1$ and thus $R_{\frac{\pi}{4}} \notin \mathcal{U}(M_2^1)$.

The C*-algebra among this family $(M_n^p)_{p \in [1,\infty)}$ of L^p-operator algebras does not only admit significantly more invertible isometries, but also more hermitian elements. By Remark 2.1.19, for C*-algebras, the core is the entire matrix algebra, while, for $p \neq 2$, we compute that $\operatorname{core}(M_n^p) \cong C(\{1, \cdots, n\}) = \mathbb{C}^n$ is the commutative subalgebra of diagonal matrices.

Using these matrix algebras as building blocks, we can extend the AF definition in Definition 2.1.12 to L^p-operator algebras.

Definition 2.1.22. Let $p \in [1, \infty)$. A Banach algebra A is called a *spatial AF L^p-operator algebra* if there is some increasing sequence of finite dimensional subalgebras $(A_N)_N$ that are isometrically isomorphic to direct sums of matrix algebras $A_N \cong \bigoplus_{i=1}^{k_N} M_{m_i}^p$ for all $N \in \mathbb{N}$, and such that $A = \overline{\bigcup_{N \in \mathbb{N}} A_N}$. If A is unital, we use the convention to refer to the scalar multiples of 1 as A_0.

Analogously to the previous section, every choice of a sequence $(A_N)_N$ for a spatial AF L^p-operator algebra can be visualized graphically in terms of a Bratteli diagram as in Definition 2.1.13. The Bratteli diagram in Example 2.1.14

encodes all spatial AF L^p-operator algebras $M^p_{2^\infty}$ uniformly in p.

We continue to construct more examples of L^p-operator algebras apart from AF-algebras. The following definition associates an L^p-operator algebra to a discrete group G. We will later unify both constructions in terms of groupoids in Definition 2.2.21.

Definition 2.1.23. Let G be a discrete group and let $p \in [1, \infty)$. The *left-regular representation* $\lambda\colon G \to \mathcal{U}(\mathcal{B}(l^p(G)))$ is given by the left-convolution operators

$$\lambda_g(\xi)(h) := (\delta_g * \xi)(h) = \xi(g^{-1}h) \text{ for all } g, h \in G,\ \xi \in l^p(G).$$

The *reduced group L^p-operator algebra* is $F^p_\lambda(G) := \overline{\operatorname{span}(\lambda(G))} \subseteq \mathcal{B}(l^p(G))$. For $p = 2$, the map λ is a unitary representation. Thus $F^2_\lambda(G)$ is a C*-algebra and more commonly denoted by $C^*_\lambda(G)$.

Remark 2.1.24. Reduced group L^p-operator algebras are unital with unit δ_e. However, for a discrete group G and $p \in [1, \infty) \setminus \{2\}$, the reduced group L^p-operator algebra does not admit any hermitian elements beyond scalar multiples of the unit. That is, $\operatorname{core}(F^p_\lambda(G)) = \mathbb{C}\delta_e$ is trivial.

Example 2.1.25. For a discrete abelian group G, the algebra $F^p_\lambda(G)$ is commutative, too. In this case, the Gelfand transform in Theorem 2.1.8 is applicable and coincides with the Fourier transform $\mathcal{F}\colon F^p_\lambda(G) \to C(\widehat{G})$ given by

$$\mathcal{F}(\lambda_g)(\chi) := \overline{\chi(g)} \text{ for all } g \in G,\ \chi \in \widehat{G}.$$

For $p = 2$, this is a C*-isomorphism $C^*_\lambda(G) \cong C(\widehat{G})$.

Another interesting family of unital L^p-operator algebras is given by the L^p-Cuntz algebras that were introduced by Phillips; see [27, Example 3.1]. They include the omnipresent Cuntz algebras for $p = 2$. In this section, however, we just introduce the L^p-versions for the most prominent Cuntz algebra $\mathcal{O}_2$.

Definition 2.1.26. Fix an integrability parameter $p \in [1, \infty)$. Consider the set of shift operators $\mathcal{S} := \{S_0, T_0, S_1, T_1\} \subseteq \mathcal{B}(l^p(\mathbb{N}_0))$, which for $j = 0, 1$ act on the canonical basis $(\delta_k)_{k \in \mathbb{N}_0}$ as

$$S_j(\delta_k) := \delta_{2k+j} \text{ and } T_j(\delta_k) := \begin{cases} \delta_l, & \text{if } \ k = 2l + j, \\ 0, & \text{otherwise.} \end{cases}$$

The Banach algebra generated by them is called *L^p-Cuntz algebra* and we write $\mathcal{O}^p_2 := \langle \mathcal{S} \rangle \subseteq \mathcal{B}(l^p(\mathbb{N}_0))$. Similarly, the *spatial tensor product* of L^p-Cuntz algebras is generated by the corresponding elementary tensor operators $\mathcal{O}^p_2 \otimes_p \mathcal{O}^p_2 := \langle \mathcal{S} \otimes \mathcal{S} \rangle \subseteq \mathcal{B}(l^p(\mathbb{N}_0 \times \mathbb{N}_0))$. For $p = 2$, we have $T_j = S^*_j$, and $\mathcal{O}^2_2$ is a C*-algebra that we call the *Cuntz algebra* $\mathcal{O}_2$.

Example 2.1.27. The generators of $\mathcal{O}_2^p$ satisfy canonical relations, namely

$$S_0T_0 + S_1T_1 = 1 \text{ and } T_iS_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

If we write elements of the Cantor set $X = \{0,1\}^{\mathbb{N}}$ as infinite binary words without separators, and if $\{0,1\}^*$ denotes the set of finite (possibly empty) initial words, then the topology on X is generated by basic clopen sets of the form μX for $\mu \in \{0,1\}^*$. If $\mu = \mu_1 \cdots \mu_n$ for $\mu_1, \cdots, \mu_n \in \{0,1\}$, by abuse of notation, we write $S_\mu := S_{\mu_1} \cdots S_{\mu_n}$ and $T_\mu := T_{\mu_n} \cdots T_{\mu_1}$ for the corresponding finite words of generators in $\mathcal{O}_2^p$. Note that the operators $\{S_\mu T_\mu \colon \mu \in \{0,1\}^*\}$ are hermitian and commute. They generate a unital abelian subalgebra. It is easy to check that, under the identification of $S_\mu T_\mu$ with the indicator function $\mathbb{1}_{\mu X}$, this subalgebra is isometrically isomorphic to $C(X)$. An analogous argument for tensor operators of these generators shows that $\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p$ admits a unital abelian subalgebra that is isometrically isomorphic to $C(X \times X)$. For $p \neq 2$, we even have that these subalgebras are the respective cores, that is, $\mathrm{core}(\mathcal{O}_2^p) = \langle \{S_\mu T_\mu \colon \mu \in \{0,1\}^*\} \rangle \cong C(X)$ and $\mathrm{core}(\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p) \cong C(X \times X)$. For reasons that will become more apparent in the subsequent background chapters on Cartan subalgebras, these core inclusions can be studied analogously to the inclusion of the diagonal matrices $C(\{1, \cdots, n\}) \subseteq M_n^p$ in Example 2.1.21; see Proposition 2.2.23 and Definition 2.2.30.

Overall, as objects, all of the presented families of spatial AF, group or Cuntz L^p-operator algebras have a lot in common uniformly in the integrability parameter p. There is one striking difference between the cases $p = 2$ and $p \neq 2$, however: Both the core and the group of invertible isometries become noticeably smaller for $p \neq 2$.

The underlying reason for this observation is the Banach-Lamperti theorem that characterizes the analogues of partial isometries in $\mathcal{B}(E)$ for L^p-spaces E and $p \neq 2$. To state it in full generality, we need to introduce more notation. Having found proper Banach algebraic analogues for the C*-algebraic notions of self-adjoint elements and unitaries that do not involve an adjoint operation, the next step is to find an analogue for partial isometries; see also [4, Definition 2.21].

Definition 2.1.28. Let A be a unital Banach algebra. An element $u \in A$ is called an *MP-partial isometry* if there is an element $v \in A$ such that

- $\|u\|, \|v\| \leq 1$,
- $uvu = u$ and $vuv = v$,
- $uv, vu \in A_{\mathrm{h}}$.

We denote the set of MP-partial isometries by $\mathrm{PI}(A)$. If it exists, v is necessarily unique and we refer to v as the *Moore-Penrose inverse* and informally write $u^* := v$. The elements uu^* and u^*u are called *range* and *domain idempotents.*

Remark 2.1.29. Note that $\mathcal{U}(A) \subseteq \mathrm{PI}(A)$. For C*-algebras, MP-partial isometries are partial isometries in the sense of Definition 2.1.5 with the adjoint as Moore-Penrose inverse. This is the justification for our notation.

Further note that $\mathrm{PI}(A)$ is closed under taking Moore-Pensore inverses, but in general not under taking products, unless the corresponding range and domain idempotents commute, because we need

$$u(vv^*u^*u)v = u(u^*uvv^*)v \text{ and } v(uu^*v^*v)u = v(v^*vuu^*)u.$$

This is automatic both for unital L^p-operator algebras with $p \neq 2$ because of the commutative core or for invertible isometries because of $u^* = u^{-1}$. Furthermore, MP-partial isometries are preserved under unital contractive homomorphisms because hermitian elements are and because all other conditions directly follow from contractivity or multiplicativity.

Example 2.1.30. As a continuation of Example 2.1.27, consider the operators $S_\mu \in \mathcal{O}_2^p$ for $\mu \in \{0,1\}^*$. From the relations in Example 2.1.27, it is straightforward to show that they are MP-partial isometries with Moore-Penrose inverses T_μ. Even more, for all $j = 0, 1$ and $\nu \in \{0,1\}^*$, we compute

$$S_j \mathbb{1}_{\nu X} T_j = \mathbb{1}_{j\nu X}, \;\; T_j \mathbb{1}_{j\nu X} S_j = \mathbb{1}_{\nu X}, \;\text{ and }\; T_j \mathbb{1}_{(1-j)\nu X} S_j = 0.$$

Iterating this computation shows that, for all $\mu \in \{0,1\}^*$, the MP-partial isometries S_μ normalize the subalgebra $C(X) \subseteq \mathcal{O}_2^p$ in the sense that

$$S_\mu C(X) T_\mu \cup T_\mu C(X) S_\mu \subseteq C(X).$$

The conjugation operations on $C(X)$ precompose with the spatial maps

$$\underline{\alpha}_{(\emptyset,\mu)} \colon \mu X \to X \text{ and } \underline{\alpha}_{(\mu,\emptyset)} \colon X \to \mu X$$

that delete or append the initial word μ, respectively. More generally, for $\mu, \nu \in \{0,1\}^*$, let $\underline{\alpha}_{(\nu,\mu)} \colon \mu X \to \nu X$ be the map that replaces the initial word μ by ν, and observe that $S_\mu T_\nu f\, S_\nu T_\mu = (f \circ \underline{\alpha}_{(\nu,\mu)})\mathbb{1}_{\mu X}$ for all $f \in C(X)$. Denote the set of $C(X)$-normalizing MP-partial isometries by $\mathrm{PI}_X(\mathcal{O}_2^p)$ and the set of $C(X)$-normalizing invertible isometries by $\mathcal{U}_X(\mathcal{O}_2^p)$. It is not hard to show that partial isometries in $C(X)$ itself are $\mathbb{T} \cup \{0\}$-valued and that every $C(X)$-normalizing MP-partial isometry is of the form nf with $n = \sum_{j=1}^k S_{\mu_j} T_{\nu_j}$ for disjoint initial words $\mu_j \in \{0,1\}^*$, disjoint initial words $\nu_j \in \{0,1\}^*$, and $f \in C(X, \mathbb{T} \cup \{0\})$ supported on $\bigsqcup_{j=1}^k \nu_j X$. Regardless of f, conjugation by

nf implements the initial word replacement map $\underline{\alpha}_{(nf)^*} = \underline{\alpha}_{n^*} : \bigsqcup_{j=1}^{k} \mu_j X \to \bigsqcup_{j=1}^{k} \nu_j X$. We have that nf is invertible, that is, $nf \in \mathcal{U}_X(\mathcal{O}_2^p)$ if and only if $\bigsqcup_{j=1}^{k} \mu_j X = X = \bigsqcup_{j=1}^{k} \nu_j X$. These observations within the L^p-Cuntz algebras provide a concrete example for the rather technical notions of *spatial normalizers* with respect to a given unital abelian C*-subalgebra and their *spatial normalizer action* in [15, Definitions 3.20 and 3.25]. If we denote the group of initial word replacement homeomorphisms associated to $C(X)$-normalizing invertible isometries by $V_2 := \{\underline{\alpha}_U : U \in \mathcal{U}_X(\mathcal{O}_2^p)\} \subseteq \mathrm{Homeo}(X)$, then analogous computations for the tensor product yield a similar word replacement group $2V_2 := \{\underline{\alpha}_U : U \in \mathcal{U}_{X\times X}(\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p)\} \subseteq \mathrm{Homeo}(X \times X)$. Both groups are instances of Brin-Thompson groups and of particular importance for embedding rigidity phenomena in both Paper I and II.

For any $p \in [1, \infty)$, an L^p-operator algebra can be embedded into the bounded operators $\mathcal{B}(L^p(\mu))$ on the L^p-space associated to a measure space $(\Omega, \mathcal{A}, \mu)$. We make use of the fact that, without loss of generality, this measure space $(\Omega, \mathcal{A}, \mu)$ can be chosen to be localizable. That is, any class of infinite measure in the Boolean algebra $\mathcal{A}/\mathcal{N}$ modulo μ-nullsets admits a subclass of finite measure. Localizable measure spaces are the canonical setup to introduce Radon-Nikodym derivatives.

Definition 2.1.31. Let $(\Omega, \mathcal{A}, \mu)$ be a localizable measure space. For $D \in \mathcal{A}$, set $\mathcal{A}_D := \{C \cap D : C \in \mathcal{A}\}$ and $\mu|_D(C) := \mu(C \cap D)$. A *partial set automorphism* on Ω consists of a choice of subsets $D_\Theta, D_{\Theta^*} \in \mathcal{A}$ and a map $\Theta : \mathcal{A}_{D_{\Theta^*}} \to \mathcal{A}_{D_\Theta}$ with $\Theta(\mathcal{N}_{D_{\Theta^*}}) \subseteq \mathcal{N}_{D_\Theta}$ that induces an automorphism of the associated Boolean algebras up to nullsets. Given a partial set automorphism Θ, we denote the inverse map by Θ^* and for a measurable set $C \in \mathcal{A}$, the assignment $\mathbb{1}_C \mapsto \mathbb{1}_{\Theta(C \cap D_{\Theta^*})}$ defines an operator $U_\Theta : L^0(\mu) \to L^0(\mu|_{D_\Theta})$. The measures $\mu|_{D_{\Theta^*}} \circ \Theta^*$ and $\mu|_{D_\Theta}$ on $\mathcal{A}_{D_\Theta}$ are equivalent, since they have the same nullsets by design, and we denote the associated Radon-Nikodym derivative $\frac{\mathrm{d}(\mu|_{D_{\Theta^*}} \circ \Theta^*)}{\mathrm{d}\mu|_{D_\Theta}}$ by $\mathrm{d}_{\mu,\Theta}$.

Definition 2.1.32. Let $(\Omega, \mathcal{A}, \mu)$ be a localizable measure space and let $p \in [1, \infty)$. An operator $s \in \mathcal{B}(L^p(\mu))$ is said to be a *spatial partial isometry* if there exist a partial set automorphism Θ on Ω and a measurable function $w : D_\Theta \to \mathbb{T}$ such that $s = w \cdot \mathrm{d}_{\mu,\Theta}^{\frac{1}{p}} \cdot U_\Theta$.

Spatial partial isometries were introduced by Phillips in [27, Definition 6.4] as a special family of MP-partial isometries. Implicitly, they show up in Example 2.1.21 already when we studied invertible isometries in $M_2^p = \mathcal{B}(\ell^p(\{1, 2\}))$. Now we can put the findings of Example 2.1.21 into a larger context. Indeed, the following theorem [4, Theorem 2.28] states that, in general, there is a lack of symmetries within localizable L^p-spaces for $p \neq 2$ that causes all MP-partial isometries to be spatial.

Theorem 2.1.33. (Banach-Lamperti). Let $(\Omega, \mathcal{A}, \mu)$ be a localizable measure space and let $p \in [1, \infty) \setminus \{2\}$. Then an operator in $\mathcal{B}(L^p(\mu))$ is an MP-partial isometry if and only if it is a spatial partial isometry.

Example 2.1.34. For the counting measure on $\mathbb{N}_0$, the set of spatial partial isometries in $\mathcal{B}(\ell^p(\mathbb{N}_0))$ simplifies. Since Boolean algebra automorphisms preserve atoms, partial set automorphisms on $\mathbb{N}_0$ map singletons to singletons, are fully described by this assignment, and have an associated Radon-Nikodym derivative equal to 1. Up to a multiplication operator by a $\mathbb{T}$-valued function, a spatial partial isometry is described by a bijection $\theta\colon D \to \mathrm{im}(\theta)$ between subsets of $\mathbb{N}_0$, with induced partial set automorphism $\Theta\colon \mathcal{P}(D) \to \mathcal{P}(\mathrm{im}(\theta))$ sending $\{d\}$ to $\{\theta(d)\}$ for all $d \in D$. For example, for the enumeration maps of even and odd numbers with $D_j := \mathbb{N}_0$ and $\theta_j(n) := 2n + j$ for $j = 0, 1$, we obtain the generators of the L^p-Cuntz algebra in Definition 2.1.26 via $S_j = U_{\Theta_j}$ and $T_j = U_{\Theta_j^*}$.

Since unital contractive homomorphisms preserve MP-partial isometries for simple algebraic reasons by Remark 2.1.29, the Banach-Lamperti theorem is the key result for the following non-trivial observation in [15, Section 3] of Paper I:

For $p \neq 2$, unital contractive homomorphisms between L^p-operator algebras preserve both the core and core-normalizing spatial partial isometries. This preservation is exploited heavily to obtain various embedding rigidity results. For L^p-Cuntz algebras, it is already enough to exploit the preservation of core-normalizing invertible isometries. We outline the proof idea of [15, Corollary D] in Paper I.

Example 2.1.35. Let $p \neq 2$ and assume that there exists a unital injective and contractive homomorphism $\varphi\colon \mathcal{O}_2^p \otimes_p \mathcal{O}_2^p \to \mathcal{O}_2^p$. Then φ restricts to an injective $*$-homomorphism between the cores $C(X \times X) \to C(X)$ as in Example 2.1.27. That is, by Remark 2.1.10, there is a surjective continuous spatial map $\rho\colon X \to X \times X$ such that $\varphi(f) = f \circ \rho$ for all $f \in C(X \times X)$. Moreover, φ preserves the set of core-normalizing spatial partial isometries in Example 2.1.30 in the sense that $\varphi(\mathrm{PI}_{X\times X}(\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p)) \subseteq \mathrm{PI}_X(\mathcal{O}_2^p)$. These two preservation results are the keys to show that, with the notation from Example 2.1.30, the map $\underline{\alpha}_U \mapsto \underline{\alpha}_{\varphi(U)}$ for all $U \in \mathcal{U}_{X\times X}(\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p))$ defines an injective group homomorphism $2V_2 \hookrightarrow V_2$. Such a group homomorphism, however, does not exist by group theoretical reasoning.

Theorem 2.1.36. ([15, Corollary D]). By Elliott's C*-isomorphism, we have $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, but, for $p \neq 2$, there is not even a unital injective and contractive homomorphism from $\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p$ into $\mathcal{O}_2^p$.

This result is a first indication that core-preservation arguments help to show that unital isometric homomorphisms between certain classes of L^p-operator algebras and $p \neq 2$ are much more rigid than for C*-algebras. To make the class of L^p-operator algebras with a well-behaved core-inclusion, such as $C(\{0,1\}^{\mathbb{N}_0}) \subseteq \mathcal{O}_2^p$, more precise, we need to introduce some groupoid language.

2.2 Groupoid related background

2.2.1 Topological groupoids

Both groups and equivalence relations are well-known algebraic structures in mathematics. We present the basics of groupoid theory that includes these two structures as special cases. Because of its generality and flexibility, groupoid language is well-suited to unify and treat seemingly disjoint areas in mathematics under one theory.

Roughly speaking, a groupoid $\mathcal{G}$ behaves like a group, but its multiplication is only partially defined, namely on its composable pairs $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$. With this intuition in mind, the following definition [32, Definition 2.1.1] becomes more understandable.

Definition 2.2.1. A *groupoid* $\mathcal{G}$ is a small category such that all morphisms are invertible. We identify $\mathcal{G}$ with the set of its morphisms by identifying the set of objects $\mathcal{G}^{(0)}$ with their identity morphisms.

Unpacking the definitions, a groupoid is a set with an inversion map $(\cdot)^{-1} \colon \mathcal{G} \to \mathcal{G}$, a distinguished set of composable pairs $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$, and a multiplication map $(-) \cdot (-) \colon \mathcal{G}^{(2)} \to \mathcal{G}$ subject to the following conditions:

(I) Involution: $(\gamma^{-1})^{-1} = \gamma$ for all $\gamma \in \mathcal{G}$.

(A) Associativity: For all (η, γ), $(\gamma, \tau) \in \mathcal{G}^{(2)}$, we have $(\eta \cdot \gamma, \tau)$, $(\eta, \gamma \cdot \tau) \in \mathcal{G}^{(2)}$ and their product evaluations $(\eta \cdot \gamma) \cdot \tau$ and $\eta \cdot (\gamma \cdot \tau)$ agree.

(C) Cancellation: It holds $(\gamma, \gamma^{-1}) \in \mathcal{G}^{(2)}$ for all $\gamma \in \mathcal{G}$ and, for all $(\eta, \gamma) \in \mathcal{G}^{(2)}$, we have $(\eta^{-1} \cdot \eta) \cdot \gamma = \gamma$ and $\eta \cdot (\gamma \cdot \gamma^{-1}) = \eta$.

Notation 2.2.2. The associativity condition (A) ensures that iterated multiplication is well-defined and does not depend on the order of pairwise evaluations. From now on, we will omit both parenthesis for iterated multiplications and $\cdot$ symbols. For products like in (A), we therefore just write $\eta\gamma\tau = (\eta \cdot \gamma) \cdot \tau = \eta \cdot (\gamma \cdot \tau)$.

Definition 2.2.3. We call the set $\mathcal{G}^{(0)} := \{\gamma\gamma^{-1} \colon \gamma \in \mathcal{G}\}$ the *unit space* and by (I) it agrees with $\{\gamma^{-1}\gamma \colon \gamma \in \mathcal{G}\}$. We can therefore assign two units to every element via the *range* and *domain* maps $\mathsf{r}, \mathsf{d}\colon \mathcal{G} \to \mathcal{G}^{(0)}$ given by $\mathsf{r}(\gamma) := \gamma\gamma^{-1}$ and $\mathsf{d}(\gamma) := \gamma^{-1}\gamma$ for all $\gamma \in \mathcal{G}$. For $x \in \mathcal{G}^{(0)}$, we denote the range and domain *fibers* by $x\mathcal{G} := \mathsf{r}^{-1}(\{x\})$ and $\mathcal{G}x := \mathsf{d}^{-1}(\{x\})$. The *isotropy bundle* is defined as the set of all elements with common range and domain, which then, in this notation, becomes

$$\mathrm{Iso}(\mathcal{G}) := \bigcup_{x \in \mathcal{G}^{(0)}} x\mathcal{G}x.$$

Remark 2.2.4. Using (A) and (C), we can phrase composability in terms of r and d as

$$\mathcal{G}^{(2)} = \{(\gamma, \eta) \in \mathcal{G} \times \mathcal{G}\colon\ \mathsf{d}(\gamma) = \mathsf{r}(\eta)\}.$$

This is why groupoids can alternatively be defined by first specifying an inverse, a unit space, d, r, and then a multiplication operation.

Example 2.2.5. Every group G is a groupoid under group inversion and multiplication on $G \times G$. It has a singleton unit space $G^{(0)} = \{e\}$ and trivial range and domain maps.

Example 2.2.6. Every equivalence relation $R \subseteq X \times X$ on a set X is a groupoid under the following structure maps for $(x, y), (y, z) \in R$:

$$\begin{aligned} (x,y)^{-1} &:= (y,x), & \mathsf{d}(x,y) &:= y, \\ \mathsf{r}(x,y) &:= x, & (x,y)(y,z) &:= (x,z). \end{aligned}$$

The unit space is given by the diagonal elements $\{(x, x)\colon x \in X\}$ and is often identified with X.

Remark 2.2.7. While groups are the groupoids with trivial unit spaces and therefore solely consist of one isotropy fiber $eGe = G$, equivalence relations are on the opposite end in this regard. They are the groupoids with trivial isotropy bundle $x\mathcal{G}x = \{x\}$ for $x \in \mathcal{G}^{(0)}$ and fully characterized by $(\mathsf{r} \times \mathsf{d})$.

In the same way as we can speak of a topological group or of the subspace topology for an equivalence relation on a topological space, there is a notion of a topological groupoid. For simplicity, we assume them to be locally compact Hausdorff throughout.

Definition 2.2.8. Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. A map $\phi\colon \mathcal{G} \to \mathcal{H}$ is a *groupoid homomorphism* if, whenever $(\gamma, \eta) \in \mathcal{G}^{(2)}$, we have $(\phi(\gamma), \phi(\eta)) \in \mathcal{H}^{(2)}$, and in this case $\phi(\gamma\eta) = \phi(\gamma)\phi(\eta)$. A *topological groupoid* $\mathcal{G}$ is a groupoid equipped with a topology such that $\mathcal{G}$ is locally compact Hausdorff, and multiplication and inversion are continuous. Here we equip $\mathcal{G}^{(2)}$ with the subspace topology coming from the product topology on $\mathcal{G} \times \mathcal{G}$. We say that two topological groupoids $\mathcal{G}$ and $\mathcal{H}$ are *isomorphic* and write $\mathcal{G} \cong \mathcal{H}$ if there is a homeomorphism $\phi\colon \mathcal{G} \to \mathcal{H}$ that is a groupoid homomorphism.

Remark 2.2.9. Note that, for a topological groupoid, r and d are automatically continuous since they are compositions of inversion and multiplication maps. It is easy to see that the Hausdorff property corresponds to the unit space being Hausdorff and relatively closed.

Definition 2.2.10. A topological groupoid $\mathcal{G}$ is called *étale* if the range map $\mathsf{r}\colon \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism. An étale groupoid is called *ample* if $\mathcal{G}^{(0)}$ is totally disconnected.

Remark 2.2.11. Note that, for an étale groupoid, $\mathsf{d} = \mathsf{r} \circ (\cdot)^{-1}$ is a local homeomorphism, too. Since we can cover $\mathcal{G}^{(0)}$ by open r-images of neighborhoods in $\mathcal{G}$, the unit space is necessarily open. Furthermore, fibers, as preimages of a singleton, are discrete.

Example 2.2.12. As a special case of the previous observation, note that a group G is étale if and only if every singleton $\{g\} \subseteq G$ is open, that is, if and only if G is discrete.

We can consider a topological version of the observation we made for equivalence relations. We no longer require the isotropy bundle, but its interior to be trivial.

Definition 2.2.13. An étale groupoid $\mathcal{G}$ is called *effective* if the interior of the isotropy bundle is trivial, that is, $\mathrm{Iso}(\mathcal{G})^o = \mathcal{G}^{(0)}$. Furthermore, $\mathcal{G}$ is called *minimal* if, for every unit $x \in \mathcal{G}^{(0)}$, the orbit $\mathsf{r}(\mathcal{G}x)$ is dense in $\mathcal{G}^{(0)}$.

The following construction of a transformation groupoid provides the guiding example for an étale groupoid and unifies many of the previous examples.

Example 2.2.14. Let G be a discrete group acting on a locally compact Hausdorff space X by homeomorphisms. The associated *transformation groupoid* is given by the Cartesian product $G \ltimes X := G \times X$ equipped with the following structure maps for $(g, x), (h, gx) \in G \ltimes X$:

$$(g,x)^{-1} := (g^{-1}, gx), \qquad \mathsf{d}(g,x) := x,$$
$$\mathsf{r}(g,x) := gx, \qquad (h,gx)(g,x) := (hg,x).$$

Since G is discrete, this turns $G \ltimes X$ into an étale groupoid with unit space $(G \ltimes X)^{(0)} = \{e\} \times X \cong X$. It is easy to verify that $G \ltimes X$ is effective if and only if the action $G \curvearrowright X$ is topologically free. Furthermore, $G \ltimes X$ is minimal if and only if the action $G \curvearrowright X$ is.

Example 2.2.15. Let $\theta \in \mathbb{R}$ and let $r_\theta \in \mathrm{Homeo}(\mathbb{T})$ be the homeomorphism given by rotation by $2\pi\theta$. This defines an action $\mathbb{Z} \curvearrowright \mathbb{T}$ that is free if and only if θ is irrational. In this case, the associated transformation groupoid $\mathbb{Z} \ltimes_{r_\theta} \mathbb{T}$ is not only effective, but *principal*, that is, $\mathrm{Iso}(\mathbb{Z} \ltimes_{r_\theta} \mathbb{T}) = (\mathbb{Z} \ltimes_{r_\theta} \mathbb{T})^{(0)} \cong \mathbb{T}$.

Using transformation groupoids, we can also define AF constructions as in

Definition 2.1.12 on the level of groupoids.

Definition 2.2.16. Let $M \in \mathbb{N}$. For each $i = 1, \cdots, M$, let $N_i \in \mathbb{N}$ and let X_i be compact Hausdorff spaces, viewed as a *trivial groupoids* with $X_i^{(0)} = X_i$. We equip the Cartesian product $\{1, \cdots, N_i\}^2$ with the discrete topology, and refer to the groupoid structure on it given by $(a, b)(b, c) := (a, c)$ and $(a, b)^{-1} := (b, a)$ for $(a, b), (b, c) \in \{1, \cdots, N_i\}^2$ as the *full equivalence relation* on $\{1, \cdots, N_i\}$. We freely identify $(\{1, \cdots, N_i\}^2)^{(0)}$ with $\{1, \cdots, N_i\}$. Groupoids of the form $\bigsqcup_{i=1}^M X_i \times \{1, \cdots, N_i\}^2$ are called *elementary*. A topological groupoid $\mathcal{F}$ is called *approximately finite dimensional* or *AF* if there is an increasing sequence $(\mathcal{F}_j)_{j\in\mathbb{N}_0}$of elementary groupoids, say $\mathcal{F}_j = \bigsqcup_{i=1}^{M^{(j)}} X_i^{(j)} \times \{1, \cdots, N_i^{(j)}\}^2$, with common unit space

$$\mathcal{F}^{(0)} \cong \mathcal{F}_j^{(0)} \cong \bigcup_j \bigsqcup_{i=1}^{M^{(j)}} X_i^{(j)} \times \{1, \cdots, N_i^{(j)}\},$$

such that $\mathcal{F}$ is isomorphic to the topological direct limit groupoid

$$\varinjlim \mathcal{F}_j = \bigcup_j \bigsqcup_{i=1}^{M^{(j)}} X_i^{(j)} \times \{1, \cdots, N_i^{(j)}\}^2.$$

See further [17, Section 7.2].

Remark 2.2.17. An alternative characterization of an AF-groupoid is given by the tail equivalence relation on the set of rooted infinite paths X_B in a Bratteli diagram B, which has a root vertex at level zero. To see this, fix a level $j \in \mathbb{N}_0$. If we now choose one of the $M^{(j)}$ many vertices i and an infinite path continuing from there, then all of the $N_i^{(j)} := \#\{* \to i\}$ many rooted initial paths to i will be equivalent:

$$\mathcal{F}_B = \bigcup_j \bigsqcup_{i\in B_j^{(0)}} \{1, \cdots, N_i^{(j)}\}^2 \times (X_B)_{i\to\infty}.$$

In particular, AF groupoids are examples of principal ample groupoids.

The idea of infinite path dynamics leads to the final groupoid construction of this subsection, namely groupoids associated to a graph. For our purposes, it suffices to consider the special case of this construction where the graph consists of a single vertex with finitely many loops. In this case, an infinite path is an infinite word with letters the set of loops.

Definition 2.2.18. Let $n \in \mathbb{N}_{\geq 2}$ and consider the Cantor set $X_n := \{0, \ldots, n-1\}^{\mathbb{N}}$. We adapt the notation from Example 2.1.27 and we write elements in

X_n as infinite words without separators. We denote the set of finite (possibly empty) initial words as $\{0,\ldots,n-1\}^*$ and denote the length of $\gamma = (\gamma_1 \cdots \gamma_k) \in \{0,\ldots,n-1\}^*$ by $|\gamma| := k$. We define the *shift map* $\sigma\colon X_n \to X_n$ by $\sigma(x_1x_2\cdots) := (x_2x_3\cdots)$ for all $(x_1x_2x_3\cdots) \in X_n$. The *full shift in the alphabet* $\{0,\ldots,n-1\}$, also known as the *shift of finite type* (SFT), is the groupoid given by

$$\mathcal{G}_n := \left\{(x,k,y) \in X_n \times \mathbb{Z} \times X_n \colon \begin{array}{l} \text{there are } k_+, k_- \in \mathbb{N} \text{ such that} \\ k = k_+ - k_- \text{ and } \sigma^{k_+}(x) = \sigma^{k_-}(y) \end{array}\right\}.$$

The range and domain maps are the projections onto the first and third coordinate, respectively, and multiplication and inversion for $(x,k,y),(y,l,z) \in \mathcal{G}_n$ are defined as $(x,k,y)(y,l,z) := (x,k+l,z)$ and $(x,k,y)^{-1} := (y,-k,x)$. We equip $\mathcal{G}_n$ with the topology generated by the clopen sets

$$Z(\gamma,\delta) := \{(\gamma x, |\gamma| - |\delta|, \delta x)\colon x \in X_n\}$$

for given finite words γ, $\delta \in \{0,\ldots,n-1\}^*$.

Example 2.2.19. Since any two initial finite words can be replaced in any orbit, SFTs are minimal. An inspection of the isotropy bundle of an SFT further shows that non-trivial isotropy element can only arise over rational unit fibers, that is, infinite words that are eventually periodic. Since rational points have empty interior in the totally disconnected Cantor space, it follows that SFTs are examples of minimal effective ample groupoids.

2.2.2 Reduced groupoid operator algebras

In this section, we introduce reduced groupoid L^p-operator algebras. Such algebras generalize Definition 2.1.23, Definition 2.1.22 and Definition 2.1.26, and are, modulo the more technical layer of twists, the main objects of study in Paper I. We end this section with enough background to state [15, Theorem A] of Paper I in an understandable way.

Turning from étale groupoids to operator algebras, we first consider the vector space of compactly supported functions on the groupoid and equip it with a convolution product.

Definition 2.2.20. Let $\mathcal{G}$ be an étale groupoid. For compactly supported functions $f_1, f_2 \in C_c(\mathcal{G})$, we define a convolution product $f_1 * f_2 \in C_c(\mathcal{G})$. For all $\gamma \in \mathcal{G}$, we put

$$(f_1 * f_2)(\gamma) := \sum_{\tau \in \mathcal{G}\mathsf{d}(\gamma)} f_1(\gamma\tau^{-1}) f_2(\tau).$$

This is a well-defined, associative, bilinear operation. Further note that the formula simplifies for a factor $h \in C_c(\mathcal{G}^{(0)})$. We obtain

$$(f_1 * h)(\gamma) = f_1(\gamma)h(\mathsf{d}(\gamma)) \quad \text{and} \quad (h * f_2)(\gamma) = h(\mathsf{r}(\gamma))f_2(\gamma).$$

In particular, for functions supported on the unit space, convolution simplifies to pointwise multiplication and is therefore commutative.

However, in general, the convolution algebra $C_c(\mathcal{G})$ is not a Banach algebra since it is not complete in any of the topologies induced by canonical norm candidates. Dependent on the norm, completing $C_c(\mathcal{G})$ leads to a variety of groupoid Banach algebras; see also [4]. We focus on merely generalizing the concept of a reduced L^p-group C*-algebra in Definition 2.1.23 to groupoids by representing $C_c(\mathcal{G})$ on the ℓ^p-fiber spaces by left convolution operators.

Definition 2.2.21. Let $\mathcal{G}$ be an étale groupoid and let $x \in \mathcal{G}^{(0)}$ be a unit. The convolution algebra $(C_c(\mathcal{G}), *)$ can be represented by left convolution operators on the fiber space $\ell^p(\mathcal{G}x)$ for $p \in [1, \infty)$ by setting

$$\begin{aligned} \lambda_x \colon C_c(\mathcal{G}) &\to \mathcal{B}(\ell^p(\mathcal{G}x)) \\ \lambda_x(f)(h) &:= f * h. \end{aligned}$$

We define $\|f\|_\lambda := \sup\{\|\lambda_x(f)\| \colon x \in \mathcal{G}^{(0)}\}$ for $f \in C_c(\mathcal{G})$. The completion of $C_c(\mathcal{G})$ in this norm is called the *reduced groupoid L^p-operator algebra* and is denoted by $F^p_\lambda(\mathcal{G})$.

Note that the F^p_λ-construction generalizes the one for discrete groups in Definition 2.1.23; see also Example 2.2.12. It is not hard to show that a reduced groupoid L^p-operator algebra is unital if and only if the unit space is compact, in which case the indicator function $1 = \mathbb{1}_{\mathcal{G}^{(0)}}$ serves as a multiplicative unit for the convolution product. Despite of the somewhat intransparent completion we took, we can still view elements in $F^p_\lambda(\mathcal{G})$ as special C_0-functions on the groupoid. Even more, there is a faithful conditional expectation onto $C_0(\mathcal{G}^{(0)})$.

Definition 2.2.22. Let $p \in [1, \infty)$ and let $\mathcal{G}$ be an étale groupoid. For $x \in \mathcal{G}^{(0)}$, denote the pairing of $\ell^p(\mathcal{G}x)$ with the Hölder dual $\ell^q(\mathcal{G}x)$ such that $p^{-1} + q^{-1} = 1$ by $\langle \cdot, \cdot \rangle_x$. As in [8, Section 4], we denote the linear injection induced by the identity map $(C_c(\mathcal{G}), \|\cdot\|_\lambda) \to (C_c(\mathcal{G}), \|\cdot\|_\infty)$ by $j \colon F^p_\lambda(\mathcal{G}) \to C_0(\mathcal{G})$. Concretely, j is given by

$$j_a(\gamma) = \langle \lambda_x(a)(\delta_x), \delta_\gamma \rangle_x$$

for $a \in F^p_\lambda(\mathcal{G})$, $\gamma \in \mathcal{G}x$. This map j allows one to speak of the *open support* of a, and we write

$$\operatorname{supp}'(a) := \{\gamma \in \mathcal{G} \colon j_a(\gamma) \neq 0\}.$$

The injection j further induces a faithful *conditional expectation* $E_{\mathcal{G}}$ onto $C_0(\mathcal{G}^{(0)})$ given by

$$E_{\mathcal{G}} \colon F^p_\lambda(\mathcal{G}) \to C_0(\mathcal{G}^{(0)})$$
$$a \mapsto j_a|_{\mathcal{G}^{(0)}}.$$

That is, $E_{\mathcal{G}}$ is contractive linear and injective on positive hermitian elements, and for all $f, h \in C_0(\mathcal{G}^{(0)})$ and $a \in F^p_\lambda(\mathcal{G})$, we have

$$E_{\mathcal{G}}(f * a * h) = f E_{\mathcal{G}}(a) h.$$

The following proposition is [8, Proposition 5.1] and computes the core for unital reduced groupoid L^p-operator algebras.

Proposition 2.2.23. Let $\mathcal{G}$ be an étale groupoid with compact unit space. Let $p \in [1, \infty) \setminus \{2\}$. Then $\mathrm{core}(F^p_\lambda(\mathcal{G})) = C(\mathcal{G}^{(0)})$.

We continue to show that the examples in Section 2.1 all admit their respective groupoid models and that our core computations there are in line with Proposition 2.2.23.

Example 2.2.24. For $n \in \mathbb{N}$ and $p \in [1, \infty)$, the spatial matrix algebra M^p_n as in Example 2.1.21 is the reduced groupoid L^p-operator algebra of the full equivalence relation $\{1, \cdots, n\}^2$. Indeed, sending the matrix unit $e_{a,b}$ for $(a, b) \in \{1, \cdots, n\}^2$ to $\delta_{(a,b)}$ defines a canonical isometric isomorphism $M^p_n \cong F^p_\lambda(\{1, \cdots, n\}^2)$. Under this isomorphism, the abelian C*-subalgebra of diagonal matrices is identified with continuous functions supported the diagonal $\{(a, a) \colon a \in \{1, \cdots, n\}\}$.

Example 2.2.25. Building up on Example 2.2.24, let $p \in [1, \infty)$, and let A be a spatial AF L^p-operator algebra as in Definition 2.1.22. Then there is an AF groupoid $\mathcal{F}$ such that A is isometrically isomorphic to $F^p_\lambda(\mathcal{F})$. Indeed, A can be realized as the direct limit of spatial matrix algebras $(\bigoplus_{i=1}^{k_N} M^p_{m_i})_N$. Analogously to the C*-algebraic case, there is a one-to-one correspondence between directed systems of this form and Bratteli diagrams as in Definition 2.1.13. If B is the corresponding Bratteli diagram, then the tail equivalence relation $\mathcal{F}_B$ on the set of infinite paths is an AF groupoid by Remark 2.2.17, and we have $A \cong F^p_\lambda(\mathcal{F}_B)$ by design.

Example 2.2.26. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and let $\mathbb{Z} \ltimes_{r_\theta} \mathbb{T}$ be the transformation groupoid of the rotation action by $2\pi\theta$ on the torus as in Example 2.2.15. Note that $\mathbb{Z} \ltimes_{r_\theta} \mathbb{T}$ is not AF since the unit space $\mathbb{T}$ is not totally disconnected. The associated L^p-operator algebras $A^p_\theta := F^p_\lambda(\mathbb{Z} \ltimes_{r_\theta} \mathbb{T})$ are known as the *L^p-noncommutative tori*.

Example 2.2.27. The SFTs in Definition 2.2.18 are groupoid models for the

L^p-Cuntz algebras in Definition 2.1.26. Indeed, for $j = 0, 1$, sending the generators S_j to $\mathbb{1}_{Z(j,\emptyset)}$ and T_j to $\mathbb{1}_{Z(\emptyset,j)}$ defines a canonical isometric isomorphism $\mathcal{O}_2^p \cong F_\lambda^p(\mathcal{G}_2)$.

As the previous examples show, reduced groupoid L^p-operator algebras $F_\lambda^p(\mathcal{G})$ encode a vast variety of L^p-operator algebras. Up to an additional technical layer of twists, reduced groupoid L^p-operator algebras are the main objects of interest in Paper I. A natural question to ask is how rigid the passage $F_\lambda^p(-)$ from étale groupoids to reduced groupoid L^p-operator algebras is, that is, which properties of $\mathcal{G}$ are remembered by the algebra $F_\lambda^p(\mathcal{G})$. To start with the extreme case: For $\mathcal{G}_1$ and $\mathcal{G}_2$ in a certain class of étale groupoids, could it even be true that an isometric isomorphism $F_\lambda^p(\mathcal{G}_1) \cong F_\lambda^p(\mathcal{G}_2)$ on the level of algebras induces an isomorphism $\mathcal{G}_1 \cong \mathcal{G}_2$ on the level of topological groupoids in the sense of Definition 2.2.8?

It is easy to verify that, for $p = 2$ and the class of discrete groups, the answer is no. Already with Example 2.1.25, it becomes apparent that groups $G_1 = \mathbb{Z}/4\mathbb{Z}$ and $G_2 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ admit the same reduced group C*-algebra, namely $C(\widehat{G_1}) \cong \mathbb{C}^4 \cong C(\widehat{G_2})$, despite of not being isomorphic as groups.

It is therefore remarkable that, in contrast, for $p \neq 2$ and the class of discrete groups, the rigidity question has been answered positively by Gardella and Thiel. Together with Palmstrøm, they could improve their result from discrete groups to ample groupoids; see Definition 2.2.10, while, together with Choi, they could also treat the class of effective étale groupoids with compact (not necessarily totally disconnected) unit space. The following results are [18, Theorem B] and [8, Corollary 5.6], respectively.

Theorem 2.2.28. *Let $\mathcal{G}$ and $\mathcal{H}$ be ample groupoids, and let $p \in [1, \infty) \setminus \{2\}$. Then $F_\lambda^p(\mathcal{G}) \cong F_\lambda^p(\mathcal{H})$ if and only if $\mathcal{G} \cong \mathcal{H}$.*

Theorem 2.2.29. *Let $\mathcal{G}$ and $\mathcal{H}$ be effective étale groupoids with compact unit spaces, and let $p \in [1, \infty) \setminus \{2\}$. Then $F_\lambda^p(\mathcal{G}) \cong F_\lambda^p(\mathcal{H})$ if and only if $\mathcal{G} \cong \mathcal{H}$.*

The proof of the latter Theorem 2.2.29 relies on the automatic core preservation in conjunction with the following C*-algebraic concept.

Definition 2.2.30. Let A be a C*-algebra and let $B \subseteq A$ be a maximal abelian C*-subalgebra. Then (A, B) is called a *Cartan pair* if A is generated by normalizers $\mathcal{N}_B(A) := \{n \in A \colon nBn^* \cup n^*Bn \subseteq B\}$ and if there is a faithful conditional expectation $E_A \colon A \to B$. In this case, B is called a *Cartan subalgebra*. Given two Cartan pairs (A_i, B_i) for $i = 1, 2$, a *Cartan map* $\varphi \colon (A_1, B_1) \to (A_2, B_2)$ is a $*$-homomorphism $\varphi \colon A_1 \to A_2$ that satisfies $\varphi(B_1) \subseteq B_2$, $\varphi(\mathcal{N}_{B_1}(A_1)) \subseteq \mathcal{N}_{B_2}(A_2)$, and $E_{A_2} \circ \varphi = \varphi \circ E_{A_1}$. An injective Cartan map is automatically isometric and called a *Cartan embedding*, while a bijective Cartan map is automatically a C*-isomorphism and called a *Cartan isomorphism*. Equivalently, a Cartan isomorphism is a C*-isomorphism that restricts to a C*-isomorphism between the Cartan subalgebras.

For an effective étale groupoid $\mathcal{G}$, that we call a *Weyl groupoid* in [15, Definition 2.13], $C_0(\mathcal{G}^{(0)}) \subseteq C^*_\lambda(\mathcal{G})$ is an instance of a Cartan subalgebra. There is a neat isomorphism rigidity theory for Cartan pairs developed by Renault in [29] showing that, up to a twist, all Cartan pairs are of this form. Without introducing Weyl twists, a weaker form of [29, Proposition 4.11] can be phrased as follows:

Proposition 2.2.31. Let $\mathcal{G}$ and $\mathcal{H}$ be effective étale groupoids. Then any Cartan isomorphism $\varphi\colon (C^*_\lambda(\mathcal{G}), C_0(\mathcal{G}^{(0)})) \to (C^*_\lambda(\mathcal{H}), C_0(\mathcal{H}^{(0)}))$ induces an isomorphism $\mathcal{G} \cong \mathcal{H}$ on the level of topological groupoids.

The proof of Theorem 2.2.29 is an analogue of Proposition 2.2.31 for the core inclusion $C(\mathcal{G}^{(0)}) \subseteq F^p_\lambda(\mathcal{G})$ and $p \neq 2$, and exploits that any isometric isomorphism automatically restricts to an isomorphism between the cores.

There is a version of Proposition 2.2.31 that characterizes Cartan maps on the level of groupoids. For effective étale groupoid models with compact unit spaces and $p \neq 2$, this motivates to study the embedding rigidity of core inclusions with the theory of Cartan embeddings. Indeed, in Paper I, we work towards [15, Theorem A], which is a generalization of Theorem 2.2.29 from isomorphism to embedding rigidity, and reads as follows.

Theorem 2.2.32. Let $\mathcal{G}$ and $\mathcal{H}$ be effective étale groupoids with compact unit spaces, let $p \in (1,\infty) \setminus \{2\}$ and let $\varphi\colon F^p_\lambda(\mathcal{G}) \to F^p_\lambda(\mathcal{H})$ be a unital contractive homomorphism. Then the following are equivalent:

1. $\varphi|_{C(\mathcal{G}^{(0)})}$ is injective and satisfies $\varphi \circ E_\mathcal{G} = E_\mathcal{H} \circ \varphi$.
2. φ is isometric and satisfies $\varphi \circ E_\mathcal{G} = E_\mathcal{H} \circ \varphi$.
3. φ is induced at the level of the groupoids, in the following sense: There are an effective étale groupoid $\mathcal{K}$ and groupoid homomorphisms

$$\mathcal{G} \xleftarrow{\;\pi\;} \mathcal{K} \xrightarrow{\;\iota\;} \mathcal{H}$$

such that π is surjective and fiberwise bijective and ι has open image, is injective and bijective on units, and $\varphi(f) = \mathbb{1}_{\iota(\mathcal{K})} \cdot (f \circ \pi \circ \iota^{-1})$ for $f \in C_c(\mathcal{G})$.

In a nutshell, unital isometric homomorphisms between reduced L^p-operator algebras associated to effective groupoids as in Theorem 2.2.32 are that rare that they necessarily stem from a basic groupoid diagram. The remainder of the background chapters for Paper I is designed to illustrate the main ideas of the proof.

We use Banach-Lamperti in Theorem 2.1.33 to show that, for $p \neq 2$, the geometry of L^p-spaces has so few symmetries that a unital contractive homomorphism between the $\mathcal{B}$-algebras automatically preserves both the core and core-normalizing MP-partial isometries in the sense of Example 2.1.30. For $p \neq 1$, this hands down to unital contractive homomorphisms between general L^p-operator algebras, and leads to a spatial normalizer action for effective groupoid models. This spatial normalizer action determines the intermediate groupoid via a transformation groupoid construction. Together with the extra assumption of intertwining the conditional expectation, the classification techniques for Cartan maps apply and eventually lead to the reconstruction of the homomorphism from its underlying groupoid diagram as in Theorem 2.2.32.

In many cases, the extra assumption of intertwining the conditional expectation is even automatic, for example, if $\mathcal{G}$ is principal. Then injectivity on the unit space algebra suffices to obtain a groupoid diagram

$$\mathcal{G} \xleftarrow{\pi} \mathcal{K} \xrightarrow{\iota} \mathcal{H}$$

as in Theorem 2.2.32. Given such a diagram, it is not hard to show that $\mathcal{G}$ has to be AF once $\mathcal{H}$ is AF. This is the proof idea for the AF-embeddability result [15, Theorem 5.5] in Paper I. The following is [15, Corollary B] and a direct consequence of these observations because of Example 2.2.15 and Example 2.2.26.

Corollary 2.2.33. Let $p \in (1, \infty) \setminus \{2\}$, let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let $r_\theta \in \mathrm{Homeo}(\mathbb{T})$ be the homeomorphism given by rotation by $2\pi\theta$. Let $A^p_\theta = F^p_\lambda(\mathbb{T} \rtimes_{r_\theta} \mathbb{Z})$ be the associated L^p-noncommutative torus, and let B be a unital spatial AF L^p-operator algebra. Then there is no unital contractive homomorphism from A^p_θ into B.

To get a better understanding of the outlined proof strategy and the meaning of the groupoid diagram in Theorem 2.2.32, it is beneficial to provide more background on inverse semigroups and groupoid actors between transformation groupoids. This is the goal of the following subsections.

2.3 Further background for Paper I

2.3.1 Inverse semigroup actions

For étale groupoids, their open bisections play a pivotal role in their study. In this subsection, we show that they form an inverse semigroup acting on the unit space by partial homeomorphisms. This action recovers the étale groupoid in the sense that it is canonically isomorphic to the associated transformation groupoid.

Definition 2.3.1. A *semigroup* $\mathcal{S}$ is a set with an associative binary operation $\cdot\colon \mathcal{S}\times\mathcal{S}\to\mathcal{S}$. It is called *unital* if it admits a multiplicative unit. A semigroup S is an *inverse semigroup* if for every $s\in\mathcal{S}$ there is a unique element $s^*\in\mathcal{S}$ such that $ss^*s=s$ and $s^*ss^*=s^*$. A map between two inverse semigroups $\mathcal{S}\to\mathcal{T}$ is called an *inverse semigroup homomorphism* if it preserves both multiplication and $*$.

Remark 2.3.2. Intuitively, groupoids and inverse semigroups are two different ways to generalize groups. While, in groupoids, there is a partially defined multiplication, in inverse semigroups, there is a partial inverse operation that implicitly defines range and domain elements.

Example 2.3.3. Although the notation might suggest it, for a C*-algebra A, the set of partial isometries $\mathrm{PI}(A)$ is in general not an inverse semigroup since it might not be closed under taking products. As discussed in Remark 2.1.29, however, if A is a unital L^p-operator algebra for some $p\in[1,\infty)\setminus\{2\}$, then the set of MP-partial isometries $\mathrm{PI}(A)$ is a unital inverse semigroup.

Example 2.3.4. Let X be a topological space. A *partial homeomorphism* on X is a homeomorphism $\alpha\colon U\to V$ between open subsets $U,V\subseteq X$, and we write $\mathrm{dom}(\alpha)=U$ and $\mathrm{ran}(\alpha)=V$. The set of partial homeomorphisms $\mathrm{Homeo}_{\mathrm{par}}(X)$ forms a unital inverse semigroup under composition and inversion with unit $\mathrm{id}_X\in\mathrm{Homeo}_{\mathrm{par}}(X)$. Concretely, for partial homeomorphisms $\alpha\colon\mathrm{dom}(\alpha)\to\mathrm{ran}(\alpha)$ and $\beta\colon\mathrm{dom}(\beta)\to\mathrm{ran}(\beta)$ on X, the composition is meant to be

$$\beta\circ\alpha\colon\mathrm{dom}(\alpha)\cap\alpha^{-1}(\mathrm{dom}(\beta))\to\beta(\mathrm{ran}(\alpha)\cap\mathrm{dom}(\beta)).$$

An inverse semigroup homomorphism $\mathcal{S}\to\mathrm{Homeo}_{\mathrm{par}}(X)$ is also called an *inverse semigroup action* $S\curvearrowright X$.

Definition 2.3.5. Let $\mathcal{G}$ be an étale groupoid. An *open bisection* is an open subset $S\subseteq\mathcal{G}$ such that $\mathsf{r}|_S$ and $\mathsf{d}|_S$ are injective. We denote the set of open bisections by $\mathcal{B}(\mathcal{G})$. It is a unital inverse semigroup under setwise operations with unit $\mathcal{G}^{(0)}\in\mathcal{B}(\mathcal{G})$, that is, for $S,T\in\mathcal{B}(\mathcal{G})$, we have

$$ST:=\{\gamma\tau\colon(\gamma,\tau)\in\mathcal{G}^{(2)}\cap S\times T\}\ \text{ and }\ S^*=S^{-1}=\{\gamma^{-1}\colon\gamma\in S\}.$$

Every open bisection $S\in\mathcal{B}(\mathcal{G})$ induces a partial homeomorphism on the unit space

$$\alpha_S:=\mathsf{r}|_S\circ\mathsf{d}|_S^{-1}\colon\mathsf{d}(S)\to\mathsf{r}(S).$$

This assignment $\alpha\colon\mathcal{B}(\mathcal{G})\to\mathrm{Homeo}_{\mathrm{par}}(\mathcal{G}^{(0)})$ is an inverse semigroup homomorphism and called the *bisection action* $\alpha\colon\mathcal{B}(\mathcal{G})\curvearrowright\mathcal{G}^{(0)}$. Restricting the bisection action to open bisections with domain and range equal to the whole unit space yields a group of homeomorphisms called the *topological full group*

$$[\![\mathcal{G}]\!] := \{\alpha_S \in \mathrm{Homeo}(\mathcal{G}^{(0)}) \colon S \in \mathcal{B}(\mathcal{G}),\ \mathsf{d}(S) = \mathsf{r}(S) = \mathcal{G}^{(0)}\}.$$

Remark 2.3.6. One way to visualize an étale groupoid $\mathcal{G}$ is as a collection of arrows on the clopen subspace $\mathcal{G}^{(0)}$ from the domain to the range units. For a fixed unit $x \in \mathcal{G}^{(0)}$, the fibers are discrete and countable by Remark 2.2.11, and the topology on $\mathcal{G}$ is generated by open bisections. A fixed $S \in \mathcal{B}(\mathcal{G})$ can then by seen as one major arrow that associates to each unit $x \in \mathsf{d}(S) = S^*S$ a unique unit $\alpha_S(x) \in \mathsf{r}(S) = SS^*$. The bisection action describes this intrinsic dynamic on the unit space.

The following is [29, Proposition 3.3] and addresses when the bisection action is faithful.

Proposition 2.3.7. Let $\mathcal{G}$ be an étale groupoid. Then the inverse semigroup homomorphism α in Definition 2.3.5 is injective if and only if $\mathcal{G}$ is effective.

Example 2.3.8. For the principal groupoid $\{1, \cdots, N\}^2$, an open bisection $S \in \mathcal{B}(\{1, \cdots, N\}^2)$ is the graph of a partial set automorphism $\alpha_S \colon \mathsf{d}(S) \to \mathsf{r}(S)$ on the unit space $\{1, \cdots, N\}$. Thus, elements of the topological full group are permutations on $\{1, \cdots, N\}$ and we can identify $[\![\{1, \cdots, N\}^2]\!]$ with the symmetric group Σ_N.

Similarly, the SFT $\mathcal{G}_2$ is effective and any open bisection is compact and a finite disjoint union of compact open bisections of the form $Z(\nu, \mu)$ for $\mu, \nu \in \{0, 1\}^*$. The partial homeomorphism $\alpha_{Z(\nu,\mu)} \colon \mu\{0, 1\}^{\mathbb{N}} \to \nu\{0, 1\}^{\mathbb{N}}$ replaces the initial word μ by ν. The computations in Example 2.1.30 show that the groups of local word replacement homeomorphisms V_2 and $2V_2$ are in fact the topological full groups of $\mathcal{G}_2$ and $\mathcal{G}_2 \times \mathcal{G}_2$, respectively.

The concept of a transformation groupoid in Example 2.2.14 generalizes from group to inverse semigroup actions.

Definition 2.3.9. Let X be a locally compact Hausdorff space and let $\mathcal{S}$ be an inverse semigroup. Let $\beta \colon \mathcal{S} \to \mathrm{Homeo}_{\mathrm{par}}(X)$ be an inverse semigroup homomorphism, which we abbreviate to $\beta \colon \mathcal{S} \curvearrowright X$. On the set $\{(s, x) \colon s \in \mathcal{S},\ x \in \mathrm{dom}(\beta_s)\}$ we define an equivalence relation by

$$(s, x) \sim (s', y) \quad \text{if and only if} \quad \begin{cases} x = y \text{ and there is an idempotent } e \in \mathcal{S} \\ \text{such that } se = s'e \text{ and } x \in \mathrm{dom}(\beta_e). \end{cases}$$

The equivalence class of (s, x) is called the *germ* of (s, x), and is denoted by $[s, x]$. The set $\mathcal{S} \ltimes_\beta X := \{[s, x] \colon s \in \mathcal{S},\ x \in \mathrm{dom}(\beta_s)\}$ admits a groupoid structure via

$$[s, \beta_t(y)] \cdot [t, y] := [st, y] \text{ and } [s, x]^{-1} := [s^*, \beta_s(x)].$$

We call $\mathcal{S} \ltimes_\beta X$ the *transformation groupoid* of $\beta\colon \mathcal{S} \curvearrowright X$. Unless stated otherwise, we equip the transformation groupoid with the topology generated by basic open subsets of the form $\{[s, x]\colon x \in U\}$ for $s \in \mathcal{S}$ and $U \subseteq \mathrm{dom}(\beta_s)$ open. If this topology is Hausdorff, then $\mathcal{S} \ltimes_\beta X$ is an étale groupoid with unit space homeomorphic to X. If β is injective, then $\mathcal{S}$ is isomorphic to its image under β and $\mathcal{S} \ltimes_\beta X \cong \beta(\mathcal{S}) \ltimes X$ is effective by construction. The latter is called a *germ groupoid*.

Definition 2.3.10. Let $\mathcal{G}$ be an étale groupoid and let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{G})$ be an inverse subsemigroup. We say that $\mathcal{S}$ is *wide* if $\mathcal{S}$ covers $\mathcal{G}$ and if, for every $S, T \in \mathcal{S}$, the intersection $S \cap T$ is a union of bisections in $\mathcal{S}$. We abbreviate the corresponding restricted bisection action to $\alpha\colon \mathcal{S} \curvearrowright \mathcal{G}^{(0)}$.

The following is [11, Proposition 5.4] and asserts that all étale groupoids are transformation groupoids of their respective bisection inverse semigroup actions.

Proposition 2.3.11. Let $\mathcal{G}$ be an étale groupoid. Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{G})$ be a wide inverse subsemigroup and let $\alpha\colon \mathcal{S} \curvearrowright \mathcal{G}^{(0)}$ be the associated bisection action. Then the map $\Psi\colon \mathcal{S} \ltimes_\alpha \mathcal{G}^{(0)} \to \mathcal{G}$ given by $\Psi([S, x]) := Sx$ for all $[S, x] \in \mathcal{S} \ltimes_\alpha \mathcal{G}^{(0)}$ is an isomorphism of topological groupoids.

For effective groupoids, there are other canonical ways to model them as transformation groupoids. We end this section with the reconstruction idea of an effective étale groupoid $\mathcal{G}$ from its reduced L^p-operator algebra $F^p_\lambda(\mathcal{G})$. The following theorem is a consequence of [15, Section 3] in Paper I.

Theorem 2.3.12. Let $\mathcal{G}$ be an effective étale groupoid and let $p \in [1, \infty)$. Then there is a subset of spatial normalizers $\mathcal{SN}^p_\lambda(\mathcal{G}) \subseteq F^p_\lambda(\mathcal{G})$ that induce an inverse semigroup of partial homeomorphisms

$$\mathcal{R}^p_\lambda = \{\underline{\alpha}_n \in \mathrm{Homeo}_{\mathrm{par}}(\mathcal{G}^{(0)})\colon n \in \mathcal{SN}^p_\lambda(\mathcal{G})\}.$$

Furthermore, the set of open supports $\{\mathrm{supp}'(n)\colon n \in \mathcal{SN}^p_\lambda(\mathcal{G})\}$ forms a wide inverse subsemigroup of $\mathcal{B}(\mathcal{G})$ and $\underline{\alpha}_n = \alpha_{\mathrm{supp}'(n)}$ for all $n \in \mathcal{SN}^p_\lambda(\mathcal{G})$. Hence, there is a canonical isomorphism $\mathcal{R}^p_\lambda \ltimes \mathcal{G}^{(0)} \cong \mathcal{G}$ of topological groupoids that reconstructs $\mathcal{G}$ from the algebra $F^p_\lambda(\mathcal{G})$ and the unit space.

To avoid the technical definition of this set of spatial normalizers, we illustrate the idea of Theorem 2.3.12 for the SFT $\mathcal{G}_2$. Here, it suffices to consider the spatial MP-normalizers.

Example 2.3.13. As a continuation of Example 2.1.30, recall that for any

L^p-Cuntz algebra $\mathcal{O}_2^p$, the set of $C(X)$-normalizing MP-partial isometries

$$\mathrm{PI}_X(\mathcal{O}_2^p) = \{u \in \mathrm{PI}(\mathcal{O}_2^p) \colon uC(X)u^* \cup u^*C(X)u \subseteq C(X)\}$$

is closed under products and taking MP-inverses and thus yields an inverse semigroup. Under the isomorphism with $F^p_\lambda(\mathcal{G}_2)$ in Example 2.2.27, any $C(X)$-normalizing MP-partial isometry is of the form $u = nf$ with

$$n = \sum_{j=1}^{k} \mathbb{1}_{Z(\mu_j,\emptyset)} \mathbb{1}_{Z(\emptyset,\nu_j)} = \mathbb{1}_{\bigsqcup_{j=1}^{k} Z(\mu_j,\nu_j)}$$

for disjoint initial words $\mu_j \in \{0,1\}^*$, disjoint initial words $\nu_j \in \{0,1\}^*$, and $f \in \mathrm{PI}(C(X))$ supported on $\bigsqcup_{j=1}^{k} \nu_j X$. Furthermore, regardless of f, conjugation by u^* or n^* implements the same initial word replacement map $\underline{\alpha}_u = \underline{\alpha}_n \colon \bigsqcup_{j=1}^{k} \nu_j X \to \bigsqcup_{j=1}^{k} \mu_j X$. It is straightforward to check that this assignment defines an inverse semigroup action $\underline{\alpha} \colon \mathrm{PI}_X(\mathcal{O}_2^p) \curvearrowright X$ that we call the *spatial normalizer action*. Note that the above computations show that taking open supports yields the same clopen bisection $\mathrm{supp}'(u) = \mathrm{supp}'(n) = \bigsqcup_{j=1}^{k} Z(\mu_j,\nu_j)$ and that the map $\mathrm{supp}' \colon \mathrm{PI}_X(\mathcal{O}_2^p) \to \mathcal{B}(\mathcal{G}_2)$ is a surjective inverse semigroup homomorphism. We observe that $\underline{\alpha}_u \colon \bigsqcup_{j=1}^{k} \nu_j X \to \bigsqcup_{j=1}^{k} \mu_j X$ agrees with the corresponding map of the bisection action

$$\alpha_{\mathrm{supp}'(u)} = \alpha_{\bigsqcup_{j=1}^{k} Z(\mu_j,\nu_j)},$$

that is, $\underline{\alpha} = \alpha \circ \mathrm{supp}'$. In total, combining Proposition 2.3.7 and Proposition 2.3.11, shows that the transformation groupoid $\mathrm{PI}_X(\mathcal{O}_2^p) \ltimes_{\underline{\alpha}} X$ is canonically isomorphic to $\mathcal{G}_2$.

2.3.2 Groupoid actors

In this subsection, we present the concept of groupoid actors. Generalizing Proposition 2.2.31, for C*-algebras, there is a correspondence between free actors and Cartan maps. All Cartan maps are induced by a free actor or, equivalently, a certain groupoid diagram. It is possible to generalize this characterization from Cartan maps between Cartan pairs to Cartan like maps between general pairs of the form $(F^p_\lambda(\mathcal{G}), C_0(\mathcal{G}^{(0)}))$ for an effective étale groupoid $\mathcal{G}$. In the unital case of a compact unit space and for $p \neq 2$, this is the core-inclusion and the Cartan map assumptions of preserving the subalgebra and its core-normalizers are automatic for unital contractive homomorphisms. Only the intertwining assumption of the conditional expectations is not automatic and this explains the proof idea of [15, Theorem A] in Paper I. We begin by defining groupoid left actions; see [26, Definition 4.15].

Notation 2.3.14. Let X, Y and Z be topological spaces and let $f\colon X \to Z$ and $g\colon Y \to Z$ be continuous maps. We denote the topological pullback of f and g by $X \times_{f,\,g} Y$, that is,

$$X \times_{f,\,g} Y = \{(x,y) \in X \times Y \colon f(x) = g(y)\}.$$

We endow $X \times_{f,\,g} Y$ with the topology given by the subspace topology in $X \times Y$.

Definition 2.3.15. Let $\mathcal{G}$ be a topological groupoid and let Y be a topological space. A *groupoid left action* $h\colon \mathcal{G} \curvearrowright Y$ consists of a continuous map $\rho_h\colon Y \to \mathcal{G}^{(0)}$, called the *anchor map*, and a continuous multiplication $\cdot_h\colon \mathcal{G} \times_{\mathsf{d},\,\rho_h} Y \to Y$ such that

1. $\rho_h(\gamma \cdot_h y) = \mathsf{r}(\gamma)$ for all $(\gamma, y) \in \mathcal{G} \times_{\mathsf{d},\,\rho_h} Y$,
2. $\gamma_1 \cdot_h (\gamma_2 \cdot_h y) = (\gamma_1\gamma_2) \cdot_h y$ for all $(\gamma_2, y) \in \mathcal{G} \times_{\mathsf{d},\,\rho_h} Y$ and $\gamma_1 \in \mathcal{G}\mathsf{r}(\gamma_2)$, and
3. $\rho_h(y) \cdot_h y = y$ for all $y \in Y$.

Example 2.3.16. Any topological groupoid $\mathcal{G}$ acts by left multiplication on itself. In this case, the anchor map is the range and the actor multiplication is the standard multiplication on $\mathcal{G}^{(2)}$. Analogously to Definition 2.3.15, one defines groupoid right actions and right multiplication is a canonical example of it with the domain as the anchor map.

Definition 2.3.17. Let $\mathcal{G}$ and $\mathcal{H}$ be topological groupoids. An *actor* $h\colon \mathcal{G} \curvearrowright \mathcal{H}$ is a groupoid left action commuting with the canonical right multiplication on $\mathcal{H}$. That is, for all $(\gamma, \eta_1) \in \mathcal{G} \times_{\mathsf{d},\,\rho_h} \mathcal{H}$ and $\eta_2 \in \mathsf{d}(\eta_1)\mathcal{H}$, we have

1. $\rho_h(\eta_1\eta_2) = \rho_h(\eta_1)$,
2. $\mathsf{d}(\gamma \cdot_h \eta_1) = \mathsf{d}(\eta_1)$, and
3. $\gamma \cdot_h (\eta_1\eta_2) = (\gamma \cdot_h \eta_1)\eta_2$.

Since the action on units determines an actor, we refer to the restriction $\rho := \rho_h|_{\mathcal{H}^{(0)}} \colon \mathcal{H}^{(0)} \to \mathcal{G}^{(0)}$ as the *anchor map* of h. We say that an actor $h\colon \mathcal{G} \curvearrowright \mathcal{H}$ is *free* at a unit $y \in \mathcal{H}^{(0)}$ if $\gamma \cdot_h y = y$ implies $\gamma = \rho(y)$. We say that h is *free* if it is free at all units.

The following construction of actors between transformation groupoids is from [33, Example 3.4].

Example 2.3.18. Let $\beta^X\colon \mathcal{S} \curvearrowright X$ and $\beta^Y\colon \mathcal{T} \curvearrowright Y$ be actions of unital inverse semigroups on locally compact Hausdorff spaces. We claim that any pair (ψ, ρ) consisting of an inverse semigroup homomorphism $\psi\colon \mathcal{S} \to \mathcal{T}$ and a continuous map $\rho\colon Y \to X$ satisfying

$$\operatorname{dom}(\beta^Y_{\psi(S)}) = \rho^{-1}(\operatorname{dom}(\beta^X_S)) \;\text{ and }\; \rho \circ \beta^Y_{\psi(S)} = \beta^X_S \circ \rho \;\text{ for all }\; S \in \mathcal{S}$$

defines an actor $h_{(\psi,\rho)}\colon \mathcal{S} \ltimes_{\beta^X} X \curvearrowright \mathcal{T} \ltimes_{\beta^Y} Y$ between the transformation groupoids as in Definition 2.3.9. Concretely, using the homeomorphisms $X \cong (\mathcal{S} \ltimes_{\beta^X} X)^{(0)}$ and $Y \cong (\mathcal{T} \ltimes_{\beta^Y} Y)^{(0)}$, we take ρ as the anchor map. Hence $[S,x] \in \mathcal{S} \ltimes_{\beta^X} X$ and $[T,y] \in \mathcal{T} \ltimes_{\beta^Y} Y$ are composable if and only if $x = \rho(\beta^Y_T(y))$ and in this case we define

$$[S,x] \cdot_{h_{(\psi,\rho)}} [T,y] := [\psi(S)T, y].$$

To see that this does not depend on the germ representatives, let $(S,x) \sim (S',x)$ and let $(T,y) \sim (T',y)$. That is, there are idempotents $U \in \mathcal{S}$ and $V \in \mathcal{T}$ with $x \in \operatorname{dom}(\beta^X_U)$ and $y \in \operatorname{dom}(\beta^Y_V)$ such that $SU = S'U$ and $TV = T'V$. By assumption, we have $x = \rho(\beta^Y_T(y)) \in \operatorname{dom}(\beta^X_U)$ and thus $\beta^Y_T(y) \in \operatorname{dom}(\beta^Y_{\psi(U)})$. We compute

$$\beta^Y_{\psi(S)T}(y) = \beta^Y_{\psi(S)\psi(U)TV}(y) = \beta^Y_{\psi(S')\psi(U)T'V}(y) = \beta^Y_{\psi(S')T'}(y).$$

Hence, the multiplication $\cdot_{h_{(\psi,\rho)}}$ is well-defined. Since $[S,x] \cdot_{h_{(\psi,\rho)}} [T,y]$ is left multiplication by $[\psi(S), \mathsf{r}([T,y])]$, it trivially commutes with right multiplication in $\mathcal{T} \ltimes_{\beta^Y} Y$. It is left to show that the restriction $\cdot_h\colon (\mathcal{S} \ltimes_{\beta^X} X) \times_{\mathsf{d},\rho} Y \to \mathcal{T} \ltimes_{\beta^Y} Y$ defines a left action. To see that $\cdot_h$ is continuous with respect to the standard topologies on the transformation groupoids, we consider the preimage of a basic open subset $\{([T,y]\colon y \in U\}$ for given $T \in \mathcal{T}$ and open $U \subseteq \operatorname{dom}(\beta^Y_T)$ as in Definition 2.3.9. The preimage is

$$\bigcup_{S \in \psi^{-1}(T)} \{([S,\rho(y)], y)\colon\ y \in U\} = \bigcup_{S \in \psi^{-1}(T)} \{[S,x]\colon\ x \in \operatorname{dom}(\beta^X_S)\} \times_{\mathsf{d},\rho} U$$

and thus open in $(\mathcal{S} \ltimes_{\beta^X} X) \times_{\mathsf{d},\rho} Y$. Now the first criterion in Definition 2.3.15 follows from the equivariance condition on ρ, the second one holds since ψ is a homomorphism, and the third one holds since ψ is unital. This shows that $h_{(\psi,\rho)}$ is an actor.

In fact, between étale groupoids, all actors are of this form; see [33, Lemma 2.5] and [33, Proposition 3.5].

Proposition 2.3.19. Let $\mathcal{G}$ and $\mathcal{H}$ be étale groupoids, and let $h\colon \mathcal{G} \curvearrowright \mathcal{H}$ be an actor. Then $\cdot_h$ is open. Moreover, for every open bisection $S \in \mathcal{B}(\mathcal{G})$, the product $S \cdot_h \mathcal{H}^{(0)}$ is an open bisection satisfying $\rho \circ \alpha_{S \cdot_h \mathcal{H}^{(0)}} = \alpha_S \circ \rho$.

Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{G})$ and $\mathcal{T} \subseteq \mathcal{B}(\mathcal{H})$ be wide inverse subsemigroups and use Proposition 2.3.11 to identify $\mathcal{G}$ and $\mathcal{H}$ with transformation groupoids $\mathcal{S} \ltimes_\alpha \mathcal{G}^{(0)}$ and $\mathcal{T} \ltimes_\alpha \mathcal{H}^{(0)}$, respectively. Then there is a canonical one-to-one correspondence between actors $h\colon \mathcal{G} \curvearrowright \mathcal{H}$ and pairs (ψ,ρ) consisting of an inverse semigroup homomorphism $\psi\colon \mathcal{S} \to \mathcal{T}$ and a continuous map $\rho\colon \mathcal{H}^{(0)} \to \mathcal{G}^{(0)}$ satisfying

$$\mathsf{d}(\psi(S)) = \rho^{-1}(\mathsf{d}(S)) \ \text{ and } \ \rho \circ \alpha_{\psi(S)} = \alpha_S \circ \rho \ \text{ for all } \ S \in \mathcal{S}.$$

Given $h\colon \mathcal{G} \curvearrowright \mathcal{H}$, the associated pair (ψ_h, ρ_h) consists of $\psi_h(S) := S \cdot_h \mathcal{H}^{(0)}$ for $S \in \mathcal{S}$ and the anchor map. The assignment $h \mapsto (\psi_h, \rho_h)$ and the construction $(\psi, \rho) \mapsto h_{(\psi,\rho)}$ in Example 2.3.18 are mutual inverses.

Remark 2.3.20. Building up on Remark 2.3.6, intuitively, an actor $h\colon \mathcal{G} \curvearrowright \mathcal{H}$ between étale groupoids lifts, for any given $y \in \mathcal{H}^{(0)}$ and $\gamma \in \mathcal{G}\rho(y)$, the arrow γ consistently with the groupoid multiplication along ρ to $\gamma \cdot_h y \in \mathcal{H}y$. Equivalently, it lifts an open bisection consistently along ρ.

$$\begin{array}{ccc} \mathsf{r}(\gamma \cdot_h y) & \xleftarrow[\in \mathcal{H}]{\gamma \cdot_h y} & y \\ \big\downarrow \rho & & \big\downarrow \rho \\ \mathsf{r}(\gamma) & \xleftarrow[\in \mathcal{G}]{\gamma} & \mathsf{d}(\gamma) \end{array} \qquad \begin{array}{ccc} \rho^{-1}(\mathsf{r}(S)) & \xleftarrow[\in \mathcal{T}]{\psi(S)} & \rho^{-1}(\mathsf{d}(S)) \\ \big\downarrow \rho & & \big\downarrow \rho \\ \mathsf{r}(S) & \xLeftarrow[\in \mathcal{S}]{S} & \mathsf{d}(S). \end{array}$$

Freeness at a unit $y \in \mathcal{H}^{(0)}$ means that the representation of $\eta = \gamma \cdot_h y \in \mathcal{G} \cdot_h y$ as a product is unique.

Recall from Theorem 2.3.12 that effective étale groupoids can be recovered from their unit space and their reduced groupoid L^p-operator algebra. For $p \neq 2$ and compact unit space, the reduced groupoid L^p-operator algebra suffices since the unit space is implicit as the core spectrum. Furthermore, for $p \neq 1, 2$, the preservation of the involved set of spatial normalizers leads to the following result.

Theorem 2.3.21. Let $\mathcal{G}$ and $\mathcal{H}$ be effective étale groupoids with compact unit spaces X and Y. Let $p \in (1, \infty) \setminus \{2\}$ and let $\varphi\colon F^p_\lambda(\mathcal{G}) \to F^p_\lambda(\mathcal{H})$ be a unital contractive homomorphism. Then φ induces an actor $h_\varphi = h_{(\psi,\rho)}\colon \mathcal{G} \curvearrowright \mathcal{H}$, where the anchor map $\rho\colon Y \to X$ stems from $\varphi|_{C(X)} = \rho^*$ and the inverse semigroup homomorphism $\psi\colon \mathcal{R}^p_\lambda(\mathcal{G}) \to \mathcal{R}^p_\lambda(\mathcal{H})$ is given by $\psi(\underline{\alpha}_n) := \underline{\alpha}_{\varphi(n)}$ for all $n \in \mathcal{SN}^p_\lambda(\mathcal{G})$.

Example 2.3.22. Let $p \in [1, \infty)$. The map $\varphi\colon \mathcal{O}^p_2 \to \mathcal{O}^p_2 \otimes_p \mathcal{O}^p_2$ given by $\varphi(a) := a \otimes 1$ for all $a \in \mathcal{O}^p_2$ is a unital isometric homomorphism. It induces the actor $h_\varphi\colon \mathcal{G}_2 \curvearrowright \mathcal{G}_2 \times \mathcal{G}_2$ given by the first coordinate projection $\rho\colon X \times X \to X$ and the inverse semigroup homomorphism $\psi\colon \alpha(\mathcal{B}(\mathcal{G}_2)) \to \alpha(\mathcal{B}(\mathcal{G}_2 \times \mathcal{G}_2))$ given by $\psi(\alpha_Z) := \alpha_Z \times \mathrm{id}_X = \alpha_{Z \times X}$ for all $Z \in \mathcal{B}(\mathcal{G}_2)$. The actor h_φ is free and characterized by its associated subgroupoid $\mathcal{G}_2 \cdot_{h_\varphi} (X \times X) = \mathcal{G}_2 \times X \subseteq \mathcal{G}_2 \times \mathcal{G}_2$.

On the level of the C_c-algebras of the groupoids, any actor canonically induces a $*$-homomorphism. In this context, [15, Theorem A] of Paper I states that, if h_φ is free, then this induced homomorphism extends to F^p_λ and recovers the original homomorphism φ. The following is a special case of theory developed in [33, Section 4].

Proposition 2.3.23. Let $\mathcal{G}$ and $\mathcal{H}$ be étale groupoids with compact unit spaces, and let $h\colon \mathcal{G} \curvearrowright \mathcal{H}$ be an actor. Then h induces a unital $*$-homomorphism $\varphi_h\colon C_c(\mathcal{G}) \to C_c(\mathcal{H})$ such that the following convolution identity holds

$$[\varphi_h(f) * \xi](\eta) = \sum_{\gamma \in \rho(\mathsf{r}(\eta))\mathcal{G}} f(\gamma)\xi(\gamma^{-1} \cdot_h \eta)$$

for all $f \in C_c(\mathcal{G})$, $\xi \in C_c(\mathcal{H})$, and $\eta \in \mathcal{H}$. That is,

$$\varphi_h(f)(\eta) = \sum_{\gamma \in \mathcal{G}\colon\ \gamma \cdot_h \mathsf{d}(\eta) = \eta} f(\gamma)$$

for all $f \in C_c(\mathcal{G})$ and $\eta \in \mathcal{H}$. In particular, if $\operatorname{supp}'(f) =: S$ is an open bisection in $\mathcal{G}$, then $\operatorname{supp}'(\varphi_h(f)) = S \cdot_h \mathcal{H}^{(0)}$ is an open bisection in $\mathcal{H}$, and we have $\varphi_h(f)(\gamma \cdot_h y) = f(\gamma)$ for all $(\gamma, y) \in S \times_{\mathsf{d},\rho} \mathcal{H}^{(0)}$. Moreover, the actor h is free if and only if $\varphi_h \circ E_{\mathcal{G}} = E_{\mathcal{H}} \circ \varphi_h$. In this case, φ_h is contractive with respect to the λ-norm and extends to a $*$-homomorphism between the reduced C*-algebras by continuity.

Remark 2.3.24. Let $p \in [1, \infty)$ and let $h\colon \mathcal{G} \curvearrowright \mathcal{H}$ be a free actor between étale groupoids with compact unit spaces. Analogously to the final paragraph in Proposition 2.3.23, we have that h induces a unital contractive homomorphism $\varphi_h\colon F^p_\lambda(\mathcal{G}) \to F^p_\lambda(\mathcal{H})$ that preserves bisection supports and satisfies $\varphi_h \circ E_{\mathcal{G}} = E_{\mathcal{H}} \circ \varphi_h$. If $\mathcal{G}$ and $\mathcal{H}$ are effective and if $p = 2$, we therefore obtain that φ_h is a Cartan map as in Definition 2.2.30. The assignment $\mathcal{G} \mapsto (C^*_\lambda(\mathcal{G}), C(\mathcal{G}^{(0)}))$ on effective étale groupoids with compact unit spaces and $h \mapsto \varphi_h$ on actors is functorial. In fact, by [33, Corollary 5.9], actors canonically induce all Cartan maps between such untwisted Cartan pairs $(C^*_\lambda(\mathcal{G}), C(\mathcal{G}^{(0)}))$. The described functor is an equivalence between the category of effective étale groupoids with compact unit spaces as objects and free actors as morphisms, and unital untwisted Cartan pairs with unital Cartan maps as morphisms.

We show next that, for a free actor $h\colon \mathcal{G} \curvearrowright \mathcal{H}$, the associated homomorphism $\varphi_h\colon F^p_\lambda(\mathcal{G}) \to F^p_\lambda(\mathcal{H})$ is induced by a groupoid diagram.

Definition 2.3.25. Let $\mathcal{G}$ and $\mathcal{H}$ be étale groupoids with compact unit spaces X and Y. Let $h\colon \mathcal{G} \curvearrowright \mathcal{H}$ be a free actor. We define the *intermediate groupoid* associated to h to be the image groupoid $\mathcal{G} \cdot_h Y \subseteq \mathcal{H}$, and define the *$h$-diagram* to be

$$\mathcal{G} \xleftarrow{\ \pi_h\ } \mathcal{G} \cdot_h Y \xrightarrow{\ \iota_h\ } \mathcal{H},$$

where π_h and is the canonical fiberwise injective homomorphism given by $\pi_h(\gamma \cdot_h y) := \gamma$ for all $\gamma \cdot_h y \in \mathcal{G} \cdot_h Y$, and ι_β is the unitwise bijective inclusion with open image. For $p \in [1, \infty)$, we define induced maps

$$\pi^*\colon F^p_\lambda(\mathcal{G}) \to F^p_\lambda(\mathcal{G} \cdot_h Y) \text{ and } \iota_*\colon F^p_\lambda(\mathcal{G} \cdot_h Y) \to F^p_\lambda(\mathcal{H})$$

given by the extensions by continuity of

$$\pi^*(f) := f \circ \pi_h \quad \text{and} \quad \iota_*(g) := \mathbb{1}_{\iota_h(\mathcal{G} \cdot_h Y)} \cdot \left(g \circ \iota_h^{-1}\right)$$

for all $f \in C_c(\mathcal{G})$ and for all $g \in C_c(\mathcal{G} \cdot_h Y)$.

We end this section by showcasing the reconstruction route for Theorem A in terms of actor language. The following are [15, Proposition 4.17] and [15, Proposition 4.19] in Paper I.

Proposition 2.3.26. Let $\mathcal{G}$ and $\mathcal{H}$ be étale groupoids with compact unit spaces. Let $h\colon \mathcal{G} \curvearrowright \mathcal{H}$ be a free actor and let $p \in [1, \infty)$. Then the homomorphism $\varphi_h\colon F^p_\lambda(\mathcal{G}) \to F^p_\lambda(\mathcal{G})$ is induced by the h-diagram in the sense that $\varphi_h = \iota_* \circ \pi^*$ as in Definition 2.3.25. Moreover, φ_h is isometric if and only if the anchor ρ is surjective.

Proposition 2.3.27. Let $\mathcal{G}$ and $\mathcal{H}$ be effective étale groupoids with compact unit spaces, and let $p \in (1, \infty) \setminus \{2\}$. Let $\varphi\colon F^p_\lambda(\mathcal{G}) \to F^p_\lambda(\mathcal{H})$ be a unital contractive homomorphism and let $h_\varphi\colon \mathcal{G} \curvearrowright \mathcal{H}$ be its associated actor as in Theorem 2.3.21. If we have $\varphi \circ E_\mathcal{G} = E_\mathcal{H} \circ \varphi$, then the actor homomorphism of h_φ recovers the original one in the sense of $\varphi = \varphi_{h_\varphi} = \iota_* \circ \pi^*$.

Setting $\mathcal{K} := \mathcal{G} \cdot_{h_\varphi} Y = \psi(\mathcal{R}^p_\lambda(\mathcal{G})) \ltimes Y$ as the intermediate groupoid, and $\pi([\underline{\alpha}_{\varphi(n)}, y]) := [\underline{\alpha}_n, \rho(y)]$ for all $[\underline{\alpha}_{\varphi(n)}, y] \in \mathcal{K}$, these two propositions lead to the proof of Theorem 2.2.32.

2.4 Rubin's theorem and Paper II

An effective étale groupoid is the germ groupoid of its bisection action on the unit space by Proposition 2.3.11. This result allows us to study operator algebraic Cartan pairs in terms of its (twisted) groupoid models without loss of information. For SFTs, such as $\mathcal{G}_2$, this connection between operator algebras, groupoids and topological dynamics goes even deeper because of the additional groupoid properties mentioned in Example 2.2.19. In fact, Matui established the following rigidity result in [25, Section 3].

Theorem 2.4.1. Let $\mathcal{G}$ and $\mathcal{H}$ be minimal effective étale groupoids with Cantor unit spaces. If $[\![\mathcal{G}]\!]$ and $[\![\mathcal{H}]\!]$ are isomorphic as discrete groups, then $\mathcal{G}$ and $\mathcal{H}$ are isomorphic as topological groupoids.

By exploiting minimality, any partial homeomorphism of the bisection action locally acts like an element of the topological full group. Hence, any germ can be represented by an element of the topological full group and, by effectivity, its action on the Cantor space suffices to recover the groupoid up to isomorphism. Thus, in essence, Matui's spatial reconstruction of the groupoid $\mathcal{G}$ from the topological full group $[\![\mathcal{G}]\!]$ in Theorem 2.4.1 reduces to the task of reconstructing the action $[\![\mathcal{G}]\!] \curvearrowright \mathcal{G}^{(0)}$ on the unit space up to equivariant homeomorphism. That is, it suffices to show that an isomorphism on the level of the topological full groups induces an equivariant spatial homeomorphism between the unit spaces. From this perspective, Matui's theorem is one ramification of Rubin's spatial reconstruction theory. The action $V_2 \curvearrowright \{0,1\}^{\mathbb{N}}$ is a prominent example of a *Rubin action*; see Definition 2.4.3. Rubin's theorem states that, for two Rubin actions, a group isomorphism between the acting groups induces an equivariant homeomorphism between the target spaces. In this sense, Rubin's theorem is the glue that makes it possible to study operator algebras such as $\mathcal{O}_2^p$ not only in terms of its groupoid model $\mathcal{G}_2$, but in terms of its topological full group V_2 without loss of any information; see also Example 2.1.35.

In this section, we define Rubin actions and discuss the proof ideas for Rubin's theorem. As the title suggests, in [19], we prove an embedding version of this theorem. Instead of group isomorphisms, we specify the class of *Rubin embeddings*, that is, those group embeddings between topological full groups that implicitly encode a spatial equivariant map between the unit spaces. In the context of actors as in Example 2.3.18, this induced map serves as an anchor map, and similarly to Theorem 2.4.1, this will be used to show that any Rubin embedding between topological full groups induces an actor on groupoid level; see Example 2.1.30.

Definition 2.4.2. Let Γ be a group acting on a topological space X. The *open support* of $\gamma \in \Gamma$ is given by $\mathrm{supp}'_X(\gamma) := \{x \in X : \gamma \cdot x \neq x\}$, the *support* by $\mathrm{supp}_X(\gamma) := \overline{\mathrm{supp}'_X(\gamma)}$ and the *regular support* by $\mathrm{rsupp}_X(\gamma) := \mathrm{supp}_X(\gamma)^o$. For an open set $U \subseteq X$, we define the associated *localized subgroup* as

$$\Gamma_U := \{\gamma \in \Gamma : \ \mathrm{supp}'_X(\gamma) \subseteq U\}.$$

Definition 2.4.3. Let Γ be a group and let X be a Hausdorff topological space. A faithful action $\Gamma \curvearrowright X$ is called

1. *locally dense*, if for all open $U \subseteq X$ and all $p \in U$, we have $p \in \overline{(\Gamma_U \cdot p)}^o$;
2. *Rubin*, if it is locally dense and X is locally compact with no isolated points;
3. *locally moving*, if $\Gamma_U \neq \{1\}$ for all nonempty open sets $U \subseteq X$.

Remark 2.4.4. Note that any Rubin action is locally moving because if $\Gamma_U = \{1\}$ for some nonempty open set $U \subseteq X$, then local density would imply that any $p \in U$ would be isolated. However, unlike locally dense actions, locally moving actions may have nowhere dense orbits under localized subgroups.

Example 2.4.5. Thompson's group F consists of all orientation-preserving homeomorphisms of the unit interval $[0,1]$ that are piecewise linear with slopes which are powers of two and whose non-differentiable points are dyadic rationals. The canonical faithful action $F \curvearrowright [0,1]$ is locally moving, but admits global fixed points at 0 and 1. There are two more Thompson groups of a similar kind. Modulo integers, F acts by automorphisms on the torus $\mathbb{T}$ and the group generated by F and the automorphism that adds $\frac{1}{2}$ modulo integers is denoted by T. Finally, the group V is generated by T and the discontinuous map that fixes $[0, \frac{1}{2})$ and permutes the intervals $[\frac{1}{2}, \frac{3}{4})$ and $[\frac{3}{4}, 1)$. Since elements in V permute dyadic intervals, they can be canonically viewed as homeomorphisms of the Cantor space $\{0,1\}^{\mathbb{N}}$.

Under this identification, we have $V = V_2$ as in Example 2.1.30. The faithful actions of Thompson's groups $F \curvearrowright (0,1)$, $T \curvearrowright \mathbb{T}$ and $V \curvearrowright \{0,1\}^{\mathbb{N}}$ are examples of Rubin actions, as we show next. Let $\Gamma \curvearrowright X$ denote any of these actions. Then X is locally compact and has no isolated points. In the first two cases, any $p \in U$ is contained in an open dyadic interval $p \in D \subseteq U$, and by scaling partitions of X to D and leaving $X \setminus \overline{D}$ fixed, it follows that $\Gamma_D \curvearrowright D$ and $\Gamma \curvearrowright X$ are conjugate. Now minimality shows that $\overline{\Gamma_D \cdot p} = \overline{D}$ and hence the actions are locally dense. The third action on the Cantor space is an instance of a bigger family of Rubin actions treated in Example 2.4.7.

Definition 2.4.6. Let $m \in \mathbb{N}$ and let $k_1, \dots k_m \in \mathbb{N}_{\geq 2}$ be alphabet sizes. For each $i = 1, \dots, m$, we set $X_{k_i} := \{0, \dots, k_i - 1\}^{\mathbb{N}}$ and denote its elements as tuples of infinite strings without separators. A *table* for the Cantor set $X = \prod_{i=1}^{m} X_{k_i}$ of size $l \in \mathbb{N}$ is a matrix of the form

$$\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} v^{(1)} & v^{(2)} & \cdots & v^{(l)} \\ u^{(1)} & u^{(2)} & \cdots & u^{(l)} \end{pmatrix}$$

with entries which are tuples of finite words $v^{(j)}, u^{(j)} \in \prod_{i=1}^{m} \{0, \dots, k_i - 1\}^*$ for all $j = 1, \dots, l$ such that both rows describe a partition of

$$X = \bigsqcup_{j=1}^{l} \prod_{i=1}^{m} u_i^{(j)} X_{k_i} = \bigsqcup_{j=1}^{l} \prod_{i=1}^{m} v_i^{(j)} X_{k_i}.$$

That is, for every $y \in X$, there is a unique index j and tail $x \in X$ such that $y = v^{(j)} x$. Every table induces a homeomorphism of X by replacing the v-initial word tuple with the one of u, that is,

$$\overline{\begin{pmatrix} v \\ u \end{pmatrix}} \left(v^{(j)} x \right) := u^{(j)} x.$$

The composition of two such homeomorphisms is induced by a refined table of possibly greater size and we call the thereby formed group of induced homeomorphisms the *generalized Brin-Thompson group* $V_{k_1,\dots,k_m}$:

$$V_{k_1,\dots,k_m} := \overline{\left\{ \begin{pmatrix} v \\ u \end{pmatrix} \in \mathrm{Homeo}(X) \colon \begin{pmatrix} v \\ u \end{pmatrix} \text{ is a table} \right\}}.$$

The name is motivated by the definition of the *Brin-Thompson group* mV_k in the case $k_1 = \cdots = k_m = k$. Brin-Thompson groups generalize V and $2V$ from Example 2.1.30 and Example 2.3.8. Generalized Brin-Thompson groups are the topological full groups of products of SFTs, namely $V_{k_1,\dots,k_m} = [\![\mathcal{G}_{k_1} \times \cdots \times \mathcal{G}_{k_m}]\!]$, by design. The argument in the following example applies to all canonical actions of topological full groups associated to minimal effective étale groupoids with Cantor unit space, but we limit ourselves to the generalized Brin-Thompson groups for clarity. See also [15, Definition 6.8, Proposition 7.7].

Example 2.4.7. The canonical faithful actions $V_{k_1,\dots,k_m} \curvearrowright \prod_{i=1}^m X_{k_i}$ of generalized Brin-Thompson groups are Rubin actions. To verify local density at $p \in \prod_{i=1}^m X_{k_i}$, without loss of generality, we can assume that the open neighborhood $p \in U$ is a cylinder set $\mu \prod_{i=1}^m X_{k_i}$ for some $\mu \in \prod_{i=1}^m \{0, \dots, k_i - 1\}^*$. Then the associated localized subgroup action $(V_{k_1,\dots,k_m})_U \curvearrowright U$ is conjugate to $V_{k_1,\dots,k_m} \curvearrowright \prod_{i=1}^m X_{k_i}$ via

$$\iota_U \colon V_{k_1,\dots,k_m} \to (V_{k_1,\dots,k_m})_U, \quad \iota_U(\psi)(y) := \begin{cases} \mu\psi(x), & \text{if } y = \mu x \in \mu \prod_{i=1}^m X_{k_i}; \\ y, & \text{otherwise.} \end{cases}$$

Since $V_{k_1,\dots,k_m} \curvearrowright \prod_{i=1}^m X_{k_i}$ is minimal, we have $U = \overline{((V_{k_1,\dots,k_m})_U \cdot p)}^o$ showing local density.

The following is Rubin's theorem [30, Corollary 3.5].

Theorem 2.4.8. Let $\Gamma \curvearrowright X$ and $\Delta \curvearrowright Y$ be Rubin actions. Then any group isomorphism $\varphi \colon \Gamma \to \Delta$ induces a (Δ, Γ)-equivariant homeomorphism $\rho \colon Y \to X$.

The first out of two key steps in the proof of Rubin's theorem is to use the locally moving property to describe localized subgroups of regular open subsets algebraically and independently from the ambient space. To make this precise, we introduce the notion of algebraic disjointness; see [22, Section 3.6] and [6, Section 2].

Definition 2.4.9. Let Γ be a group, let $f, g \in \Gamma$ and let $C_\Gamma(g) := \{\gamma \in \Gamma \colon \gamma g = g\gamma\}$ denote the centralizer of g. We say that g is *algebraically disjoint* from f if for all $h \in \Gamma \backslash C_\Gamma(f)$ there are $f_1,\ f_2 \in C_\Gamma(g)$ such that $[f_1, [f_2, h]] \in C_\Gamma(g) \backslash \{1\}$. We write $g \lhd_{alg}^{\Gamma} f$.

The relation $\lhd_{alg}^{\Gamma}$ on Γ is neither symmetric, reflexive, nor preserved under group homomorphisms in general. However, despite its bad permanence properties, for locally moving actions, the following proposition justifies viewing the algebraic disjointness relation as a way to describe the lattice of regular localized subgroups in Γ independently from X.

Recall the notion of locally moving in Definition 2.4.3. The following is [22, Theorem 3.6.3] or [6, Proposition 2.1], respectively.

Proposition 2.4.10. Let $\Gamma \curvearrowright X$ be locally moving and let $f \in \Gamma$. Then $\Gamma_{\mathrm{rsupp}_X(f)} = C_\Gamma(\{g^{12} \colon g \lhd_{alg}^{\Gamma} f\})$.

The second key step in the proof of Rubin's theorem is to use local density to recover the space X from the isomorphism class of Γ by expressing points in X algebraically as limits of ultrafilters consisting of regular supports. This information is contained in the poset of localized subgroups, and since these subgroups are preserved under group isomorphisms, this finally allows one to construct the desired spatial equivariant homeomorphism from a group isomorphism between the acting groups of two Rubin actions.

In Paper II, we follow these ideas for sufficiently well-behaved group embeddings instead of group isomorphisms with the goal to construct a spatial equivariant map in this way that is not necessarily a homeomorphism.

Remark 2.4.11. Note that if $\Gamma \curvearrowright X$ is Rubin, if $\Delta \curvearrowright Y$ is a locally moving faithful action, and if $\Phi \colon \Gamma \to \Delta$ is a group embedding, then the following desirable preservation of localized subgroups

$$\Phi(\Gamma_{\mathrm{rsupp}_X(\gamma)}) = \Phi(\Gamma)_{\mathrm{rsupp}_Y(\Phi(\gamma))} \text{ for all } \gamma \in \Gamma$$

is not guaranteed. It is an algebraic condition that does not depend on the ambient actions on X or Y by Proposition 2.4.10 and is therefore met by isomorphisms, but not necessarily by every injective group homomorphism. In fact, for given $\gamma \in \Gamma$, neither containment of the two subgroups $\Phi(\Gamma_{\mathrm{rsupp}_X(\gamma)})$ and $\Phi(\Gamma)_{\mathrm{rsupp}_Y(\Phi(\gamma))}$ is automatic. By Proposition 2.4.10 and injectivity of Φ, the subgroups are

$$C_{\Phi(\Gamma)}(\{\Phi(\tau)^{12} \colon \tau \lhd_{alg}^{\Gamma} \gamma\}) \text{ and } C_{\Phi(\Gamma)}(\{\delta^{12} \colon \delta \lhd_{alg}^{\Delta} \Phi(\gamma)\}),$$

and it is easy to verify that neither of the two group theoretical relations $\tau \lhd_{alg}^{\Gamma} \gamma$ or $\Phi(\tau) \lhd_{alg}^{\Delta} \Phi(\gamma)$ is implied by the other.

Another conceptual complication of the passage from isomorphisms to embeddings is that assuming the image action $\Phi(\Gamma) \curvearrowright Y$ to be locally dense would be too restrictive to cover any proper embeddings. Indeed, in this case, $\Phi(\Gamma) \curvearrowright Y$ would be Rubin and the restricted group isomorphism $\Phi \colon \Gamma \to \Phi(\Gamma)$ would induce a homeomorphism $Y \to X$ by Theorem 2.4.8. That is why, for a more suitable preservation property of the locally dense action $\Gamma \curvearrowright X$ under Φ, we need to introduce a potentially coarser topology on the target space Y.

Definition 2.4.12. Let $\Gamma \curvearrowright Y$ be an action on a topological space. We say that $\Gamma \curvearrowright Y$ is *of full support* if it has no global fixed points. We call the coarser topology on Y generated by

$$\left\{ \bigcap_{j=1}^{n} \mathrm{supp}'_Y(\gamma_j) \subseteq Y : \gamma_1, \ldots, \gamma_n \in \Gamma \text{ and } \bigcap_{j=1}^{n} \mathrm{supp}'_Y(\gamma_j) \neq \emptyset \right\}$$

the *saturated topology.* We further refer to open subsets with respect to the saturated topology as *saturated* open subsets and denote closures and interiors with respect to this topology by an indexed *Sat*.

Remark 2.4.13. If Y is locally compact and if $\Gamma \curvearrowright Y$ is of full support, then Y is also locally compact in the saturated topology. If $\Gamma \curvearrowright Y$ is Rubin, then all open subsets are saturated by [6, Proposition 3.2].

The considerations in Remark 2.4.11 and in Remark 2.4.13 lead to a meaningful generalization from group isomorphisms to the correct class of group embeddings that allow for the following embedding version of Rubin's theorem, which is [19, Theorem A] in Paper II.

Theorem 2.4.14. Let X and Y be locally compact Hausdorff spaces with no isolated points, let $\Gamma \curvearrowright X$ be a Rubin action, let $\Delta \curvearrowright Y$ be a faithful action, and let $\Phi \colon \Gamma \to \Delta$ be an injective group homomorphism. The following are equivalent:

1. $\Phi(\Gamma) \curvearrowright Y$ is of full support, satisfies $\Phi(\Gamma_{\mathrm{rsupp}_X(\gamma)}) = \Phi(\Gamma)_{\mathrm{rsupp}_Y(\Phi(\gamma))}$ for all $\gamma \in \Gamma$, and, for all saturated open $U \subseteq Y$ and all $p \in U$, we have $p \in \overline{(\Phi(\Gamma)_U \cdot p)}^{o}_{Sat}$.
2. There is a unique continuous surjective spatial map $\rho \colon Y \to X$ such that $\rho^{-1}(\mathrm{rsupp}_X(\gamma)) = \mathrm{rsupp}_Y(\Phi(\gamma))$ and $\rho \circ \Phi(\gamma) = \gamma \circ \rho$ for all $\gamma \in \Gamma$.

We call an embedding Φ as in (1) a *Rubin embedding* and the map ρ as in (2) its *anchor map.* The name anchor is justified because, for Rubin embeddings between topological full groups, a Rubin embedding induces an actor with this anchor map. See also [15, Theorem 6.11] and Example 2.3.18.

Example 2.4.15. Applying tables coordinatewise, duplicating them or permuting them provide canonical ways to embed generalized Brin-Thompson groups into others with potentially more alphabet factors. We illustrate this idea with the example of the embedding $\iota \colon V_2 \hookrightarrow 2V_2$ given by $\iota\,(\psi)\,(v_1^{(j)}x, y) :=$

$(u_1^{(j)}x, y)$ for any table $\begin{pmatrix} v_1 \\ u_1 \end{pmatrix}$ of X_2 implementing ψ and $x, y \in X_2$. That is, using empty words for all $v_2^{(j)}$ and $u_2^{(j)}$ constitutes a table of $X_2 \times X_2$ implementing $\iota(\psi)$. We claim that ι is a Rubin embedding with the projection onto the first coordinate $\pi_1 \colon X_2 \times X_2 \to X_2$ as its anchor map; see also Example 2.3.22. To see this, it is immediate that $\pi_1 \colon X_2 \times X_2 \to X_2$ is continuous, surjective and ι-equivariant. Furthermore, the regular support of any homeomorphism

$$\overline{\begin{pmatrix} v \\ u \end{pmatrix}} \in V_{k_1,\ldots,k_m} \quad \text{is} \quad \bigsqcup_{j \colon u^{(j)} \neq v^{(j)}} u^{(j)} \prod_{i=1}^{m} X_{k_i},$$

since the periodic fixed points under nonidentical prefixes are nowhere dense. Hence, we get $\mathrm{rsupp}_{X_2}(\psi) = \bigsqcup_{j \colon u_1^{(j)} \neq v_1^{(j)}} u_1^{(j)} X_2$ and $\mathrm{rsupp}_{X_2 \times X_2}(\iota(\psi)) = \bigsqcup_{j \colon u^{(j)} \neq v^{(j)}} u_1^{(j)} X_2 \times X_2$. This shows that $\pi_1^{-1}(\mathrm{rsupp}_{X_2}(\psi)) = \mathrm{rsupp}_{X_2 \times X_2}(\iota(\psi))$.

2.5 Partial actions and Paper III

Having treated special group actions on topological spaces in Section 2.4, in this section, we introduce partial actions on topological spaces and use theory developed by Exel in the 1990s and 2000s to clarify their connection to inverse semigroup actions as in Example 2.3.4. See also [12] for a proper introduction to partial actions. Finally, limiting ourselves to finite groups, we introduce partial actions on C*-algebras, their associated partial crossed product, and, for abelian groups, the dual action on the partial crossed product. We end this section with the definition of the partial Rokhlin property and the aims of Paper III to, first, characterize the partial Rokhlin property in terms of the dual action and, second, characterize the partial actions whose dual has the Rokhlin property.

Definition 2.5.1. Let G be a discrete group and let X be a topological space. A *partial action* of G on X is a collection $\theta = ((D_g)_{g\in G}, (\theta_g)_{g\in G})$ of open sets $D_g \subseteq X$ and partial homeomorphisms $\theta_g \colon D_{g^{-1}} \to D_g$ for all $g \in G$ with $\theta_e = \mathrm{id}_X$ and such that, for all $g, h \in G$, the map θ_{gh} extends $\theta_g \circ \theta_h$ whenever the composition is well-defined. A partial action $\theta = ((D_g)_{g\in G}, (\theta_g)_{g\in G})$ is *global* if $D_g = X$ for all $g \in G$.

A partial action is global if and only if it is a group action in the classical sense. The set of partial homeomorphisms is an inverse semigroup as in Definition 2.3.1 and features in the definition of an inverse semigroup action Example 2.3.4. To clarify the connection between inverse semigroup actions and partial group actions, we introduce more notation.

Definition 2.5.2. Let S be an inverse semigroup. For $s, t \in S$, we write $s \leq t$

if $ts^*s = s$. This defines the canonical *partial order* relation on S. The partial order is invariant under multiplication from either side and, restricted to the commutative subinverse semigroup of *idempotents* $E(S) := \{e \in S : e^2 = e\}$, we have, for $e, f \in E(S)$, that $e \leq f$ if and only if $ef = e$.

Example 2.5.3. If $\mathcal{G}$ is an étale groupoid, then the idempotents in $\mathcal{B}(\mathcal{G})$ are precisely the open subsets of the unit space, while the canonical partial order on $\mathcal{B}(\mathcal{G})$ agrees with the partial order of set-wise inclusion. Similarly, for a topological space X, the idempotents in $\mathrm{Homeo}_{\mathrm{par}}(X)$ are identity maps on open subsets, while we have $\eta_1 \leq \eta_2$ in $\mathrm{Homeo}_{\mathrm{par}}(X)$ if and only if η_2 is an extension of η_1.

In particular, if $\theta = ((D_g)_{g\in G}, (\theta_g)_{g\in G})$ is a partial action as in Definition 2.5.1, then for all $g, h \in G$, we have $\theta_g\theta_h \leq \theta_{gh}$. Exel showed in [10] that, in the definition of partial actions, this inequality relation is in fact equivalent to the equality relations $\theta_g\theta_h\theta_{h^{-1}} = \theta_{gh}\theta_{h^{-1}}$ and $\theta_{g^{-1}}\theta_g\theta_h = \theta_{g^{-1}}\theta_{gh}$. These relations allow us to formalize partial actions as certain inverse semigroup actions.

Definition 2.5.4. Let G be a discrete group. We define the inverse semigroup $S(G)$ generated by a set of formal generators $\{[g] : g \in G\}$ subject to the relations

$$[g^{-1}][g][h] = [g^{-1}][gh], \;\; [g][h][h^{-1}] = [gh][h^{-1}] \text{ and } [g][e] = [g]$$

for all $g, h \in G$. For an inverse semigroup T, we call a map $\pi : G \to T$ a *partial homomorphism* if the induced map $\widetilde{\pi} : S(G) \to T$ given by $\widetilde{\pi}([g]) := \pi(g)$ for all $g \in G$ is an inverse semigroup homomorphism. If T is an inverse semigroup contained in $\mathrm{PI}(B)$ for some unital C*-algebra B, then we say that π is a *partial representation* on B.

Proposition 2.5.5. Let G be a discrete group, let X be a topological space, and let $\theta : G \to \mathrm{Homeo}_{\mathrm{par}}(X)$ be a map. Then the following are equivalent.

- $((\mathrm{ran}(\theta_g))_{g\in G}, (\theta_g)_{g\in G})$ is a partial action;
- θ is a partial homomorphism;
- θ induces an inverse semigroup action $\widetilde{\theta} : S(G) \curvearrowright X$.

Motivated by Gelfand's equivalence in Theorem 2.1.9 that takes an open subset $U \subseteq X$ of a locally compact Hausdorff space X to an ideal $C_0(U) \lhd C_0(X)$, the concept of partial actions in Definition 2.5.1 generalizes from topological spaces to C*-algebras.

Definition 2.5.6. Let A be a C*-algebra and let G be a group. A *partial action* is a collection $\alpha = ((A_g)_{g\in G}, (\alpha_g)_{g\in G})$ of ideals $A_g \lhd A$ and C*-isomorphisms $\alpha_g : A_{g^{-1}} \to A_g$ for all $g \in G$ with $\alpha_e = \mathrm{id}_A$ and such that, for all $g, h \in G$, the map α_{gh} extends $\alpha_g \circ \alpha_h$ whenever the composition is well-defined. We say that a partial action is *global* if $A_g = A$ for all $g \in G$.

Example 2.5.7. Global actions are group actions by C*-automorphisms in the classical sense. For any global action $\alpha\colon G \curvearrowright A$ and any ideal $I \lhd A$, by restricting to the dynamics inside I, we obtain a partial action $\alpha|_I$ by setting $I_g := I \cap \alpha_g(I)$ and $(\alpha|_I)_g := \alpha_g|_{I_{g^{-1}}}$ for $g \in G$. Partial actions of this form are called *globalizable* since they admit a global enveloping action on A. However, not all partial actions are globalizable. In [13], Ferraro discusses this question and provides a necessary condition for partial actions to be globalizable. For a commutative C*-algebra $I \cong C_0(X)$, for example, any partial action on I is spatially induced by a partial action on the Gelfand spectrum X, and all partial actions on topological spaces are globalizable.

There are several ways to associate a C*-algebra to a partial action. Among others, there are the constructions of full and reduced partial crossed products as completions of the section algebra for an associated Fell bundle over the group; see [12, Section 11]. Since we want to avoid the theory of C*-algebras associated to Fell bundles, we merely say that the idea is analogous to the construction of groupoid algebras as potentially different norm completions of the $*$-algebra $C_c(\mathcal{G})$ associated to an étale groupoid $\mathcal{G}$ in Definition 2.2.20.

Example 2.5.8. For example, for the abelian C*-algebra $C(X)$ over the Cantor set X and a partial action $\theta = ((D_g)_{g\in G}, (\theta_g)_{g\in G})$ on X, the transformation groupoid $S(G) \ltimes_{\widetilde{\theta}} X$ is ample. We have that the reduced partial crossed product of θ agrees with $C^*_\lambda(S(G) \ltimes_{\widetilde{\theta}} X)$ and is the completion of the convolution algebra $C_c(S(G) \ltimes_{\widetilde{\theta}} X)$ in the λ-norm; see Example 2.2.14, Definition 2.2.21, and [28]. Intuitively, one way to view the convolution algebra $C_c(S(G) \ltimes_{\widetilde{\theta}} X)$ is that it is generated by the unit space subalgebra $C(X)$ and a family $(u_g)_{g\in G}$ of normalizing partial isometries supported on the clopen bisections $S_g = \{[g]\} \times \mathrm{dom}(\theta_g) \subseteq S(G) \ltimes_{\widetilde{\theta}} X$ that implement the partial action.

We continue to introduce the analogous section algebra of a partial action. Since, for the purposes of Paper III, we are only dealing with finite groups, by [12, Theorem 20.7], both full and reduced completions of the section algebra coincide and can be described more easily in terms of the following algebra.

Definition 2.5.9. Let $\alpha = ((A_g)_{g\in G}, (\alpha_g)_{g\in G})$ be a partial action of a finite group G on a C*-algebra A. The associated *partial crossed product* is the set $A \rtimes_\alpha G$ of formal linear combinations of $a_g u_g$, for $g \in G$ and $a_g \in A_g$, with convolution and involution given by

$$(a_g u_g)(b_h u_h) := \alpha_g(\alpha_{g^{-1}}(a_g) b_h) u_{gh} \text{ and } (a_g u_g)^* := \alpha_{g^{-1}}(a_g^*) u_{g^{-1}}$$

for all $g, h \in G$, $a_g \in A_g$, $a_h \in A_h$. A *covariant representation* on a unital C*-algebra B is a pair (ρ, π) consisting of a $*$-homomorphism $\rho\colon A \to B$ and a partial representation $\pi\colon G \to B$ such that $\rho(\alpha_g(x)) = \pi_g \rho(x) \pi_{g^{-1}}$ for all $g \in G$ and $x \in A_{g^{-1}}$. We turn the partial crossed product into a C*-algebra by equipping it with the C*-norm

$$\Big\| \sum_{g\in G} a_g u_g \Big\| := \sup \Big\{ \Big\| \sum_{g\in G} \rho(a_g)\pi_g \Big\| \colon (\rho,\pi) \text{ is a covariant representation} \Big\}.$$

If the acting group G is abelian, the dual group $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$ is an interesting object, and we can define a dual action by the dual group on the partial crossed product.

Definition 2.5.10. Let $\alpha = ((A_g)_{g\in G}, (\alpha_g)_{g\in G})$ be a partial action by a finite abelian group G on a C*-algebra A. The *dual action* $\widehat{\alpha}\colon \widehat{G} \curvearrowright A \rtimes_\alpha G$ is given by

$$\widehat{\alpha}_\chi(a_g u_g) := \chi(g) a_g u_g$$

for all $\chi \in \widehat{G}$, $g \in G$ and $a_g \in A_g$.

The following are basic examples of dual actions in the global case.

Example 2.5.11. For a finite group G and the action $\mathrm{id}\colon G \curvearrowright \mathbb{C}$ that acts trivially by $\mathrm{id}_{\mathbb{C}}$ for all $g \in G$, the associated crossed product $G \ltimes_{\mathrm{id}} \mathbb{C}$ is canonically isomorphic to the reduced group C*-algebra $C^*_\lambda(G)$ in Definition 2.1.23 via $u_g \mapsto \lambda_g$ for all $g \in G$. If G is moreover abelian, then the Fourier transform in Example 2.1.25 implements a conjugacy between the dual action $\widehat{\mathrm{tr}}\colon \widehat{G} \curvearrowright G \ltimes_{\mathrm{id}} \mathbb{C}$ and the left translation $\mathtt{Lt}\colon \widehat{G} \curvearrowright C(\widehat{G})$ given by $\mathtt{Lt}_\chi(f)(\tau) = f(\overline{\chi}\tau)$ for all $\chi, \tau \in \widehat{G}$ and $f \in C(\widehat{G})$.

Example 2.5.12. Let G be a finite abelian group and consider the left translation action $\mathtt{Lt}\colon \widehat{G} \curvearrowright C(\widehat{G})$ as in Example 2.5.11. To compute the dual action $\widehat{\mathtt{Lt}}\colon \widehat{\widehat{G}} \curvearrowright \widehat{G} \ltimes_{\mathtt{Lt}} C(\widehat{G})$, we can apply another Fourier-type C*-isomorphism $\widehat{G} \ltimes_{\mathtt{Lt}} C(\widehat{G}) \cong \mathcal{K}(\ell^2(G)) \cong M_{|G|}$ and find that $\widehat{\mathtt{Lt}}$, and thereby $\widehat{\widehat{\mathrm{id}}}$, is equivariantly isomorphic to the inner action $\mathrm{Ad}_\lambda\colon G \curvearrowright M_{|G|}$ given by conjugation with the left regular representation $\lambda\colon G \to \mathcal{U}(M_{|G|})$.

Intuitively, dualization takes free actions, such as $\mathtt{Lt}$, to inner actions and vice versa. Furthermore, the double-dual action is closely related to the original one via Takai duality. In essence, this intuition remains justified for partial actions; see also [1]. We end this section by defining the partial Rokhlin property that was first introduced and studied by Abadie, Gardella and Geffen in [2].

Definition 2.5.13. Let A be a C*-algebra, let $l^\infty(\mathbb{N}, A)$ be the algebra of norm-bounded A-valued sequences with entry-wise operations, and let $c_0(\mathbb{N}, A)$ be the ideal of sequences that converge to zero in norm. We define the *sequence* *algebra* as $A_\infty := l^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$. We identify A with the equivalence classes of constant sequences and define the *central sequence algebra* as its relative commutant $A_\infty \cap A'$. We further define the *multiplier algebra* of A as $M(A) := \{b \in A^{**} : bA \cup Ab \subseteq A\}$.

Let $\alpha = ((A_g)_{g\in G}, (\alpha_g)_{g\in G})$ be a partial action by a group G. Since A is an ideal in $M(A)$, we can regard the partial action α as a partial action on $M(A)$ as well. Applying α entry-wise, in turn, induces a partial action on A_∞ (or $M(A)_\infty$ respectively) that we denote by

$$\alpha^\infty := (((A_g)_\infty)_{g\in G}, (\alpha_g^\infty)_{g\in G}).$$

Definition 2.5.14. Let $\alpha = ((A_g)_{g\in G}, (\alpha_g)_{g\in G})$ be a partial action of a finite group G on a separable C*-algebra A. We say that the partial action α has the *Rokhlin property* if there is a unital $*$-homomorphism $\varphi\colon C(G) \to M(A)_\infty \cap A'$ such that

1. $\varphi(\delta_g)A_\infty \cup A_\infty\varphi(\delta_g) \subseteq (A_g)_\infty$ for all $g \in G$;
2. $\alpha_g^\infty(\varphi(f)x) = \varphi(\mathtt{Lt}_g(f))\alpha_g(x)$ for all $f \in C(G)$, $g \in G$, and $x \in A_{g^{-1}}$.

Actions with the Rokhlin property are relatively well-behaved, both because of their permanence properties and the dynamical advantages of having approximate G-partitions of A coming from sufficiently large entries of representatives of $((\varphi(\delta_g))_{g\in G}$. In this sense, the Rokhlin property facilitates the G-equivariant study of theories that rely on projections, such as the work by Connes and Jones on crossed product von Neumann algebras or the work by Kirchberg and Phillips on the K-theory of Kirchberg algebras.

In Paper III, we characterize when a partial action by a finite abelian group has the Rokhlin property in terms of its dual action on the partial crossed product. Since partial actions with the Rokhlin property are equivariantly related to the action $\mathtt{Lt}$, it is reasonable to expect their dual actions to be close to inner. Indeed, for global actions with the Rokhlin property, Izumi showed in [21, Lemma 3.8] that the dual notion is approximate representability.

Definition 2.5.15. Let G be a finite abelian group, let B be a separable C*-algebra, and let $\beta\colon G \curvearrowright B$ be a global action. Let B^β denote the fixed point algebra. We say that β is *approximately representable* if there is a unitary representation $v\colon G \to \mathcal{U}(M(B^\beta)_\infty)$ such that $\beta_g(b) = v_g b v_{g^{-1}}$ for all $g \in G$ and $b \in B$.

Concretely, [21, Lemma 3.8] states that a global action has the Rokhlin property if and only if its dual is approximately representable, and, conversely, a global action is approximately representable if and only if its dual has the Rokhlin property. Generalizing Izumi's result to partial actions, however, requires us to dualize condition (1) in Definition 2.5.14 under the Fourier isomorphism as well. This leads to the notion of *dual approximate representability* and we obtain the following characterization as zero dimensional special cases of [20, Proposition A] and [20, Proposition B].

Proposition 2.5.16. Let $\alpha = ((A_g)_{g\in G}, (\alpha_g)_{g\in G})$ be a partial action of a finite group G on a separable C*-algebra A. Then α has the Rokhlin property if and only if $\widehat{\alpha}$ is dually approximately representable. Conversely, $\widehat{\alpha}$ has the Rokhlin property if and only if α is global and approximately representable in Izumi's sense.

3 Summary of results

This chapter presents and contextualizes the main results of the three papers included in the thesis. For a reader familiar with the research field, this summary facilitates the understanding of their respective contributions.

3.1 Paper I

Paper I, titled *Embeddings of L^p-operator algebras* [15], studies unital contractive homomorphisms between L^p-operator algebras arising from twisted étale groupoids with particular emphasis on rigidity phenomena for $p \neq 2$. The rigidity results rely on a detailed analysis of core normalizers and their automatic preservation. Concretely, using the notion of actors between groupoids, it is shown that under natural hypotheses, embeddings between reduced Weyl twist L^p-operator algebras can be described entirely in terms of morphisms of the underlying Weyl twists. In the paper, it is further shown that embeddings of reduced Weyl twist L^p-operator algebras induce embeddings of the associated topological full groups. The presented methods provide new tools for studying embeddability questions in the L^p-setting, and are particularly helpful when ruling out the existence of embeddings.

The study of Banach algebras arising from analytic, geometric, and dynamical constructions has a long history. Early on, questions concerning the structure and embeddability of subalgebras of operator algebras have played a central role. For example, Arveson showed in [3] that the study of (not necessarily self-adjoint) subalgebras of C*-algebras already leads to a rich and robust theory. Within this broad landscape, operator algebras on Hilbert spaces have played a central role, leading to deep classification results and a highly developed structural theory. At the same time, many naturally occurring Banach algebras are not C*-algebras and do not even admit contractive representations on Hilbert spaces, so that their analysis requires techniques that go beyond this setting.

In recent years, there has been growing interest in the systematic study of L^p*-operator algebras*, that is, Banach algebras admitting isometric representations on L^p-spaces for $1 \leq p < \infty$. These algebras are somewhere in between classical Banach algebras and C*-algebras. They retain enough analytic rigidity to support a meaningful structure theory, but they also exhibit genuinely new phenomena that cannot be expected in the case of operator algebras on Hilbert spaces. Early work in this direction includes L^p-analogues of group algebras, crossed products, as well as L^p-versions of some very well-studied C*-algebras such as the Cuntz algebras, AF algebras, and irrational rotation algebras. From a broader perspective, the study of L^p-operator algebras fits into a long-standing program concerned with embeddings and representations of nonselfadjoint operator algebras, going back at least to Arveson's work. In this sense, L^p-operator algebras provide a natural testing ground for understanding how far techniques from C*-algebra theory extend beyond Hilbert spaces.

A particularly important class of examples is provided by L^p-operator algebras associated with étale groupoids. Groupoids offer a flexible framework that simultaneously encodes topology, dynamics, and geometry, and they have proved to be a powerful tool in the theory of C*-algebras. In the L^p-context, groupoid algebras provide a large and tractable supply of examples that generalize both group L^p-operator algebras and algebras arising from dynamical systems. Moreover, many constructions that are familiar from the C*-theory admit meaningful analogues in this setting. In particular, the presence of a canonical diagonal subalgebra and its normalizers allows one to associate a Weyl groupoid to any L^p-operator algebra, extending Renault's reconstruction theory in the setting of Cartan subalgebras; see [8]. These extensions provide a bridge between the associated L^p-operator algebras and the underlying dynamics of the groupoid, which in the case $p \neq 2$ tend to be more rigid than what can be expected for C*-algebras.

In order to better understand a given class of Banach algebras, it is natural to study what the possible subalgebras of a given algebra are. This leads to the problem of *embeddability*. Given a Banach algebra A, one seeks to understand whether A embeds into a better understood or more rigid algebra, and what such an embedding reveals about the structure of A. In the setting of C*-algebras, embedding theorems have had a profound impact, most notably Kirchberg's $\mathcal{O}_2$-embedding theorem asserting that every separable, exact, unital C*-algebra embeds unitally into the Cuntz algebra $\mathcal{O}_2$, and the more recent result, due to Schafhauser [31], characterizing those UCT C*-algebras that embed into an AF algebra. These results, together with some predecessors, have influenced large parts of the modern classification program and have inspired analogous questions for other classes of algebras. For example, the failure of a counterpart to Kirchberg's $\mathcal{O}_2$-embedding theorem for $\mathbb{Z}$-algebras has been explored in [7], where it is shown that the Leavitt algebra $L_{2,\mathbb{Z}}$ does not contain its tensor square $L_{2,\mathbb{Z}} \otimes L_{2,\mathbb{Z}}$ unitally.

In contrast, embedding problems for L^p-operator algebras remain far less understood. The absence of adjoints and the absence of techniques available for Hilbert spaces produce significant technical obstacles, and many arguments from the C*-setting simply do not generalize. Nevertheless, recent progress has shown that groupoid methods can be adapted effectively to the L^p-framework, allowing one to prove striking rigidity and reconstruction results under suitable hypotheses. This suggests that embeddability questions for L^p-operator algebras should be approachable through a careful analysis of the canonical diagonal subalgebra, normalizers, or the topological full group, among others.

Paper I is devoted to the study of embeddings of L^p-operator algebras associated with twisted étale groupoids, with a particular emphasis on applications to embeddability questions for some well-studied algebras, such as irrational rotation L^p-operator algebras, spatial AF L^p-operator algebras, and L^p-Cuntz algebras. The article is based on a careful study of Weyl twists, core normalizers, and their behavior under algebra homomorphisms. By exploiting the notion of actors between groupoids developed by Meyer and Zhu [26], these methods allow one to analyze embeddings of algebras associated to groupoids directly at the level of the underlying groupoid data. This is surprising at first sight, since in the C*-algebraic setting, it is well-known that maps between groupoid C*-algebras do not have to be induced in any way at the groupoid level. The main theorem states that, under natural hypotheses, embeddings between certain L^p-operator algebras are, for $p \notin \{1, 2\}$, in one-to-one correspondence with free actor morphisms between the associated groupoids. To make this explicit, for an étale, Hausdorff groupoid $\mathcal{G}$, the canonical conditional expectation is denoted by $E_{\mathcal{G}} \colon F^p_\lambda(\mathcal{G}) \to C_0(\mathcal{G}^{(0)})$. The result reads as follows.

Theorem 3.1.1. (See [15, Theorem A].) Let $\mathcal{G}$ and $\mathcal{H}$ be étale, effective, Hausdorff groupoids with compact unit spaces, let $p \in (1, \infty) \setminus \{2\}$ and let $\varphi \colon F^p_\lambda(\mathcal{G}) \to F^p_\lambda(\mathcal{H})$ be a unital contractive homomorphism. Then the following are equivalent:

1. $\varphi|_{C(\mathcal{G}^{(0)})}$ is injective and satisfies $\varphi \circ E_{\mathcal{G}} = E_{\mathcal{H}} \circ \varphi$.
2. φ is isometric and satisfies $\varphi \circ E_{\mathcal{G}} = E_{\mathcal{H}} \circ \varphi$.
3. φ is induced at the level of the groupoids, in the following sense: There are an étale, effective, Hausdorff groupoid $\mathcal{K}$ and groupoid homomorphisms

$$\mathcal{G} \xleftarrow{\pi} \mathcal{K} \xrightarrow{\iota} \mathcal{H}$$

such that π is surjective and fiberwise bijective and ι has open image, is injective and bijective on units, and $\varphi(f) = \mathbb{1}_{\iota(\mathcal{K})} \cdot (f \circ \pi \circ \iota^{-1})$ for $f \in C_c(\mathcal{G})$.

This result can be understood as a Cartan-type characterization of embeddings, providing new tools for ruling out the existence of maps between L^p-groupoid algebras.

In (1) and (2), the condition that φ intertwines the conditional expectations is often automatic, and this allows for various applications. One of the main applications is AF-embeddability. In the setting of C*-algebras, AF-embeddability has long been recognized as a strong regularity property, and its relationship with exactness, quasidiagonality, and the UCT has been extensively studied. Paper I investigates spatial AF-embeddability for reduced groupoid L^p-operator algebras, and establishes a strong rigidity result. Under mild assumptions on $\mathcal{G}$ and for $p \neq 2$, the algebra $F^p_\lambda(\mathcal{G})$ embeds into a spatial AF L^p-operator algebra if and only if $\mathcal{G}$ is itself an AF groupoid. The following consequence of this stands in contrast to the celebrated construction by Pismner-Voiculescu of an AF-embedding of the irrational rotation algebra.

Corollary 3.1.2. (See [15, Corollary B].) Let $p \in (1, \infty) \setminus \{2\}$, let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let $A^p_\theta := C(\mathbb{T}) \rtimes^p_{r_\theta} \mathbb{Z}$ be the associated L^p-irrational rotation algebra, where r_θ is the rotation homeomorphism by angle $2\pi\theta$. Then there is no unital contractive homomorphism from A^p_θ into a spatial AF L^p-operator algebra.

Another direction explored in Paper I is that of induced embeddings of topological full groups, at least in the setting of Weyl groupoids. Topological full groups play an important role in the study of dynamical systems and groupoids, and their interaction with operator algebras has been a source of fruitful connections, systematically studied ever since the work of Matui [25]. Since topological full groups capture subtle orbit and isotropy information of the groupoids in question, embeddings between them often reflect strong rigidity phenomena. Understanding how isometric embeddings reflect on the level of full groups thus provides a mechanism for transferring analytic information into a purely group-theoretic setting. The following theorem specifies this induced embedding of topological groups and provides the link between Paper I and the concept of Rubin embeddings in Paper II.

Theorem 3.1.3. (See [15, Theorem 6.6, Theorem 6.11].) Let $p, q \in [1, \infty) \setminus \{2\}$ and let $\mathcal{G}$ and $\mathcal{H}$ be Weyl groupoids with compact unit spaces. Then any unital contractive homomorphism $\varphi\colon F^p_\lambda(\mathcal{G}) \to F^q_\lambda(\mathcal{H})$ induces a group homomorphism $\psi\colon [\![\mathcal{G}]\!] \to [\![\mathcal{H}]\!]$. For minimal ample Weyl groupoids, this induced homomorphism ψ encodes a groupoid actor by itself, and the homomorphism $\psi\colon [\![\mathcal{G}]\!] \to [\![\mathcal{H}]\!]$ is injective as soon as $\varphi|_{C(\mathcal{G}^{(0)})}$ is injective.

The above result allows one to deduce group-theoretical properties from Banach-algebraic data. One application is the study of tensor products of L^p-Cuntz algebras. In this direction, Paper I establishes further rigidity results which once again show a striking contrast with the setting of C*-algebras. Among other things, [15, Theorem 7.8] implies the following.

Corollary 3.1.4. (See [15, Corollary D].) Let $p \in [1, \infty) \setminus \{2\}$. Then there does not exist a unital contractive homomorphism $\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p \to \mathcal{O}_2^p$.

Besides being a significant strengthening of the result, obtained by Choi, Gardella and Thiel in [8], that there is no isometric isomorphism $\mathcal{O}_2^p \otimes_p \mathcal{O}_2^p \cong \mathcal{O}_2^p$ if $p \neq 2$, this result also shows that there is no reasonable analog of Kirchberg's $\mathcal{O}_2$-embedding theorem in the L^p-setting.

3.2 Paper II

Paper II, titled *An embedding version of Rubin's theorem* [19], studies group embeddings between groups that admit a Rubin action. Rubin's theorem asserts that if $\Gamma \curvearrowright X$ and $\Delta \curvearrowright Y$ are Rubin actions, then any group isomorphism $\Gamma \cong \Delta$ induces an equivariant homeomorphism $Y \cong X$. The paper provides an embedding version of Rubin's theorem in the sense that it highlights group embeddings that induce a spatial equivariant map of a certain form. It further showcases instances of such embeddings between generalized Brin-Thompson groups.

In the 1980s, Rubin studied rigidity phenomena of faithful group actions $\Gamma \curvearrowright X$ on various structured objects X. The theme is a reconstruction of these objects from their groups of transformations. Among other applications, his results also apply to topological spaces and highlight a class of faithful actions that allow for a reconstruction of the topological space and that are called *Rubin actions* in the following. Concretely, with the terminology of [6] and [25], Rubin's theorem [30, Corollary 3.5] states that if Γ is a group admitting Rubin actions $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$, then there is a unique Γ-equivariant homeomorphism $Y \cong X$.

Classes of groups that admit a canonical Rubin action include topological full groups of minimal effective étale groupoids with Cantor unit space, which received more and more attention recently, both in topological dynamics, operator algebras, and in the structure theory of ample groupoids in general.

Another way of interpreting Rubin's theorem is that, given two Rubin actions $\Gamma \curvearrowright X$ and $\Delta \curvearrowright Y$, a group isomorphism $\Phi \colon \Gamma \to \Delta$ induces a unique spatial homeomorphism $\rho \colon Y \to X$ such that the following diagram commutes for every $\gamma \in \Gamma$:

$$\begin{array}{ccc} X & \xleftarrow{\rho} & Y \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\Phi(\gamma)} \\ X & \xleftarrow{\rho} & Y. \end{array}$$

In this language, another question asked by Rubin on [30, p. 493] is whether there are any reasonable assumptions on a group *embedding* $\Phi\colon \Gamma \to \Delta$, instead of a group isomorphism, so that Φ still induces a Φ-equivariant spatial map in the sense of the commuting diagram above. Paper II addresses this question by using the framework and notation from [6] and [25]. Its main result reads as follows.

Theorem 3.2.1. (See [19, Theorem A].) Let X and Y be locally compact Hausdorff spaces with no isolated points, let $\Gamma \curvearrowright X$ be a Rubin action, let $\Delta \curvearrowright Y$ be a faithful action, and let $\Phi\colon \Gamma \to \Delta$ be an injective group homomorphism. The following are equivalent:

1. $\Phi(\Gamma) \curvearrowright Y$ has no global fixed points, is locally dense on saturated subsets, and satisfies $\Phi(\Gamma_{\mathrm{rsupp}_X(\gamma)}) = \Phi(\Gamma)_{\mathrm{rsupp}_Y(\Phi(\gamma))}$ for all $\gamma \in \Gamma$.

2. There is a unique continuous, surjective and Φ-equivariant spatial map $\rho\colon Y \to X$ such that $\rho^{-1}(\mathrm{rsupp}_X(\gamma)) = \mathrm{rsupp}_Y(\Phi(\gamma))$ for all $\gamma \in \Gamma$.

An embedding Φ as in (1) is then called a *Rubin embedding*, and the map ρ as in (2) its *anchor map*.

3.3 Paper III

Paper III, titled *Duality of partial Rokhlin dimension* [20], generalizes two duality results for actions with finite Rokhlin dimension from global to partial actions. For this, the notion of representability dimension is extended to partial actions and the notion of dual representability dimension is introduced for global actions by finite abelian groups. The first result of the paper is that the Rokhlin dimension of a partial action by a finite abelian group agrees with the dual representability dimension of the dual action on the partial crossed product. The second result is that the representability dimension of a partial action agrees with the Rokhlin dimension of its dual.

Finite group actions on C*-algebras with the Rokhlin property have a long history that dates back to the late 1970s. Even before this notion had been established in its modern form, Kishimoto studied instances of Rokhlin actions in [23]. The dynamical system of an action with the Rokhlin property is particularly well-behaved. For example, the Rokhlin property allows one to pass on many properties from the C*-algebra both to the resulting fixed point algebra and to the crossed product. Among others, this includes being simple, a Kirchberg algebra, having real rank zero, or having finite stable rank or nuclear dimension. Furthermore, the Rokhlin property passes to tensor product actions and quotients, and facilitates K-theoretical and Cuntz semigroup computations. This is why the Rokhlin property is used early on for generalization attempts of the classification oriented work of Kirchberg and Phillips to an equivariant framework.

Among many other results, Izumi characterizes in [21, Lemma 3.8] when a global action by a finite abelian group has the Rokhlin property in terms of the dual action. The dual notion is approximate representability allowing to check certain examples of action with the Rokhlin property more easily. The theory of Rokhlin actions ramified in many directions since then. Among others, one branch of generalizations is from finite to compact groups, from separable to not necessarily separable C*-algebras, from the Rokhlin property to Rokhlin dimensions, and from global to partial actions. Following this branch, Izumi's result generalizes accordingly to compact groups in [5, Theorem 4.27], and to Rokhlin and representability dimensions in [16, Theorem 1.14].

Paper III generalizes this theme to partial actions in the sense that the Rokhlin dimension of a partial action agrees with the dual representability dimension of its dual. Using terminology from Landstad duality, the dual representability dimension is the representability dimension in [16, Definition 1.10] with an additional compatibility relation for duals of partial actions that is automatic for duals of global actions.

Proposition 3.3.1. (See [20, Proposition A].) Let G be a finite abelian group, and let $\alpha = ((A_g)_{g\in G}, (\alpha_g)_{g\in G})$ be a partial action on a separable C*-algebra A. Then $\dim_{\mathrm{Rok}}(\alpha) = \dim_{\widehat{\mathrm{rep}}}(\widehat{\alpha})$.

Similarly, there is a generalization of representability dimension to partial actions leading to a complementary characterization: The representability dimension of a partial action by a finite abelian group agrees with the global Rokhlin dimension of the dual action.

Proposition 3.3.2. (See [20, Proposition B].) Let G be a finite abelian group, and let $\alpha = ((A_g)_{g\in G}, (\alpha_g)_{g\in G})$ be a partial action on a separable C*-algebra A. Then $\dim_{\mathrm{Rok}}(\widehat{\alpha}) = \dim_{\mathrm{rep}}(\alpha)$.

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