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Bundles of skew-symmetric matrix pencils and their minimal degenerations ☆

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Small perturbations to the coefficients of a skew-symmetric matrix pencil can cause large changes in its complete eigenstructure, e.g., eigenvalues, their multiplicities, as well as minimal indices may change. In this manuscript, we state three results: (a) the characterization of inclusion between the closures of congruence bundles of skew-symmetric matrix pencils; (b) the necessary and sufficient condition for one congruence bundle of a skew-symmetric matrix pencil, P, to belong to the closure of the congruence bundle of another matrix pencil, Q, such that there is no matrix pencil, R, whose bundle contains the closure of the bundle of P and is contained in the closure of the bundle of Q; and (c) bundles are open in their closures.

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1. Introduction

In many applications, the eigenstructure of a matrix pencil plays a pivotal role. The eigenstructure includes eigenvalues, their multiplicities, as well as the left and right minimal indices. Yet these invariants are often sensitive to small perturbations: minor numerical changes in the entries of the matrix pencil can cause confluence or splitting of eigenvalues as well as other changes in canonical structure. These issues are studied with the help of miniversal deformations [1,14,18], and stratifications of orbits and bundles [20,21,25,33,34].

We recall that the set of matrix pencils with exactly the same complete eigenstructure forms an *orbit*. In turn, a bundle is a set of all matrix pencils with the same complete eigenstructure up to the values of distinct eigenvalues. To capture the confluence or splitting of eigenvalues under perturbations, we need to consider the stratifications [21, 22,25] of bundles (since in orbit stratification eigenvalues are fixed). Bundles of matrices under similarity were introduced in 1971 by Arnold [1] and generalized to, and studied for, matrices under (*)congruence [9,17,19,36], matrix pencils [5,10,22,24,25,33], and matrix polynomials [7,8,15,16,21,27]. Note that, a number of the references mentioned above deal with the matrix pencils and polynomials whose matrices have certain symmetries or structures.

One of the commonly studied symmetries is skew symmetry. Recall that a matrix pencil $A - \lambda B$ is skew symmetric if and only if $A = -A^T$ and $B = -B^T$. Skew-symmetric matrix pencils arise naturally in applications in mathematics, engineering, and physics. Notable examples include systems with bi-Hamiltonian structure [32], control theory [3,29,30], and multisymplectic partial differential equations [4]. Such matrix pencils also appear as structure-preserving linearizations of the corresponding matrix polynomials [8,13,16,31]. Skew-symmetric matrix pencils have attracted significant attention; in particular, people have studied their canonical forms [35,38], low-rank perturbations [2], miniversal deformations [12], computed the codimensions of their orbits [23], and developed their stratifications [5,13,22]. In this work, we develop the theory of bundles of skew-symmetric matrix pencils, which, in particular, allows us to complete the stratification of the bundles of skew-symmetric matrix pencils.

The main contributions of this work are complete answers to the following four fundamental questions concerning bundles of skew-symmetric matrix pencils (see Section 2.1 for definitions of closure and cover):

(1) If a skew-symmetric matrix pencil P_1 lies in the closure of the bundle associated with another matrix pencil P_2, does the entire bundle of P_1 also lie in the closure of the bundle of P_2?
(2) What are the necessary and sufficient conditions for the closure of the bundle of P_2 to contain the bundle of P_1?
(3) What are the necessary and sufficient conditions for the bundle of P_2 to *cover* the bundle of P_1?

(4) Is the bundle of a skew-symmetric matrix pencil open in its closure? (see the paragraph preceding Theorem 2.2 for the definition)

Question (2) is directly related to the perturbation theory of skew-symmetric matrix pencils. If P_1 lies in the closure of the bundle of P_2 ($P_1 \neq P_2$), then every neighbourhood of P_1 contains matrix pencils that share the same eigenstructure as P_2 (up to distinct eigenvalues), which differs from that of P_1. Consequently, an arbitrarily small perturbation of P_1 may result in a matrix pencil exhibiting the eigenstructure of P_2. To resolve question (2), one should first answer question (1). Question (3) forms the foundation for constructing bundle stratification graphs [13,20–22,25], in which each node represents a bundle and each edge corresponds to a cover relation between bundles. These graphs provide a precise description of how canonical structures change under perturbations. In [22], an algorithm for constructing such stratification graphs was proposed for skew-symmetric matrix pencils: it first builds the stratification graph of general matrix pencils under strict equivalence, then identifies the bundles corresponding to skew-symmetrizable matrix pencils, and finally examines the paths connecting them. Although effective, this approach is computationally demanding even for small matrix pencils. In this work, we provide a result that enables the stratification graph for skew-symmetric matrix pencils to be constructed directly, without relying on the stratification graph of general matrix pencils. Similar results have previously been obtained for orbits of skew-symmetric matrix pencils in [5]. Finally, a positive answer to question (4) tells that a bundle of a skew-symmetric matrix pencil is *generic* in its closure. Recall that a subset is generic in a set if it is both open and dense [15,16], and that a bundle is dense in its closure by construction.

The rest of the paper is organized as follows. In Section 3, we provide a characterization of inclusion between closures of skew-symmetric matrix pencils. More precisely, we obtain a necessary and sufficient condition for the closure of a bundle to be a subset of the closure of another skew-symmetric matrix pencil. Before doing so, we prove that if a skew-symmetric matrix pencil is in the bundle of another skew-symmetric matrix pencil of the same size then the bundle of the former is a subset of closure of the bundle of the latter. We will also see how closure of bundles of skew-symmetric matrix pencils can be written as a union of closure of orbits of skew-symmetric matrix pencils of the same size. In Section 4, we present a characterization for covering relation between closures of bundles of skew-symmetric matrix pencils. Finally, in Section 5 we prove that bundles of skew-symmetric matrix pencils under congruence are open in their closures.

2. Preliminary results

All matrices considered in this paper are over the field of complex numbers. The notations $\mathbb{Q}_+$, $\mathbb{C}$, $\mathbb{N}$ and $GL_n(\mathbb{C})$ represent set of positive rational numbers, set of complex numbers, set of positive integer numbers, and general linear group over $\mathbb{C}$, respectively. We introduce the $k \times (k+1)$ matrices

$$F_k := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}, \qquad G_k := \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \text{ for each } k = 0, 1, \dots \tag{1}$$

and the $k \times k$ matrices

$$I_k := \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \qquad J_k(\mu) := \begin{bmatrix} \mu & 1 & & \\ & \mu & \ddots & \\ & & \ddots & 1 \\ & & & \mu \end{bmatrix}, \text{ for each } k = 1, 2, \dots \tag{2}$$

All non-specified entries of $J_k(\mu), I_k, F_k$, and G_k are zeros.

2.1. Orbits and bundles for general matrix pencils

We start this subsection with defining strict equivalence for complex matrix pencils. An $m \times n$ matrix pencil $A - \lambda B$ is called strictly equivalent to the matrix pencil $C - \lambda D$ if and only if $Q(A - \lambda B)R = C - \lambda D$, for some non-singular matrices Q and R. The *orbit* of $A - \lambda B$ under the action of the group $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ on the space of all matrix pencils by strict equivalence is defined as follows:

$$\mathrm{O}^e(A - \lambda B) = \{Q(A - \lambda B)R \mid Q \in GL_m(\mathbb{C}), R \in GL_n(\mathbb{C})\}.$$

We recall the Kronecker Canonical Form (KCF) for matrix pencils under the strict equivalence transformation.

Theorem 2.1 *([26]). Each $m \times n$ matrix pencil $A - \lambda B$ is strictly equivalent to a direct sum, uniquely determined up to permutation of summands, of matrix pencils of the form*

$$E_k(\mu) := J_k(\mu) - \lambda I_k, \mu \in \mathbb{C}, \quad E_k(\infty) := I_k - \lambda J_k(0),$$
$$L_k := F_k - \lambda G_k, \quad \textit{and} \quad L_k^T := F_k^T - \lambda G_k^T.$$

The blocks $E_k(\mu)$, $E_k(\infty)$, L_k and L_k^T in the above theorem correspond to the finite eigenvalues, infinite eigenvalues, column minimal indices and row minimal indices, respectively, of $A - \lambda B$. The blocks L_k and L_k^T altogether form the singular part and the rest forms the regular part of the matrix pencil.

We define $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$. An integer partition of $n \in \mathbb{Z}_+$ is a non-increasing sequence of integers $\mathcal{P} = (p_1, p_2, p_3, \dots)$ such that each $p_i \geqslant 0$ and the sum of all p_i is n. The integer partitions called *Weyr characteristics* of a matrix pencil P (with eigenvalues $\mu_j \in \overline{\mathbb{C}}$) are:

- $\{\mathcal{J}_{\mu_j}(P) : j = 1, \dots, d\}$, where d is the number of distinct eigenvalues: for each distinct μ_j, $\mathcal{J}_{\mu_j}(P) = (j_1^{\mu_j}(P), j_2^{\mu_j}(P), \dots)$, the k^{th} position is the number of Jordan

blocks of P of the size greater than or equal to k (the position numeration starting from 1);
- $\mathcal{L}(P) = (l_0(P), l_1(P), \dots)$: the k^{th} position is the number of L^T-blocks with the indices greater than or equal to k (the position numeration starting from 0);
- $\mathcal{R}(P) = (r_0(P), r_1(P), \dots)$: the k^{th} position is the number of L-blocks with the indices greater than or equal to k (the position numeration starting from 0).

For integer partitions $\mathcal{P} = (p_1, p_2, \dots)$ and $\mathcal{Q} = (q_1, q_2, \dots)$, $\mathcal{P}+\mathcal{Q} = (p_1+q_1, p_2+q_2, \dots)$. The set of all integer partitions form a lattice with respect to the dominance order. By dominance order on integer partitions, we mean that for any integer partitions $\mathcal{P}$ and $\mathcal{Q}$, $\mathcal{P} \succcurlyeq \mathcal{Q}$ if and only if $p_1 + p_2 + \dots + p_i \geqslant q_1 + q_2 + \dots + q_i$, for $i = 1, 2, \dots$, and $\mathcal{P} \succ \mathcal{Q}$ if and only if $\mathcal{P} \succcurlyeq \mathcal{Q}$ and $\mathcal{P} \neq \mathcal{Q}$. In a lattice, we say $\mathcal{P}$ covers $\mathcal{Q}$ if and only if $\mathcal{P} \succ \mathcal{Q}$ and there exists no partition $\mathcal{Z}$ such that $\mathcal{P} \succ \mathcal{Z}$ and $\mathcal{Z} \succ \mathcal{Q}$.

By $\overline{\mathcal{X}}$ we denote the closure of a set $\mathcal{X}$ in the Euclidean topology. $\mathcal{X}$ is said to be open in its closure if $\mathcal{X}$ is open in $\overline{\mathcal{X}}$ with respect to the subspace topology. The normal rank of P, $nrk(P)$ is defined as the rank of the matrix pencil P over $\mathbb{C}(\lambda)$. It can also be formulated as $nrk(P) = n - r_0(P) = m - l_0(P)$, see [25]. The following theorem states the characterisation for a strict equivalence orbit of a matrix pencil to be in the closure of strict equivalence orbit of another matrix pencil.

Theorem 2.2 *([25,34]). Let P_1 and P_2 be two matrix pencils. $\overline{\mathrm{O}^e}(P_2) \subseteq \overline{\mathrm{O}^e}(P_1)$ if and only if the following relation holds for $h := nrk(P_1) - nrk(P_2)$:*

(i) $\mathcal{R}(P_2) \prec \mathcal{R}(P_1) + (h, h, \dots)$,
(ii) $\mathcal{L}(P_2) \prec \mathcal{L}(P_1) + (h, h, \dots)$,
(iii) $\mathcal{J}_{\mu_j}(P_1) \prec \mathcal{J}_{\mu_j}(P_2) + (h, h, \dots)$, *for all* $\mu \in \overline{\mathbb{C}}$.

The *bundle* $\mathrm{B}^e(P)$ under strict equivalence is a union of all orbits of matrix pencils strictly equivalent to P, up to distinct eigenvalues. We recall the notion of coalescing of eigenvalues as introduced in [10]. Let Ψ denotes the sets of all mappings from $\overline{\mathbb{C}}$ to itself.

Definition 2.3 *([10]).* Let P be a matrix pencil, $\mu_1, .., \mu_s$ be distinct eigenvalues of P and $\psi \in \Psi$. Then, $\psi_c(P)$ is a matrix pencil of the same size as P and with the following properties:

- $\mathcal{R}(\psi_c(P)) = \mathcal{R}(P)$,
- $\mathcal{L}(\psi_c(P)) = \mathcal{L}(P)$,
- $\mathcal{J}_\mu(\psi_c(P)) = \bigcup_{\mu_i \in \psi^{-1}(\mu)} \mathcal{J}_{\mu_i}(P)$, for all $\mu_i \in \overline{\mathbb{C}}$.

The eigenvalues $\mu_{i_1}, \dots, \mu_{i_d}$ of P have coalesced to the eigenvalue μ in $\psi_c(P)$ if $\psi^{-1}(\mu) = \{\mu_{i_1}, \dots, \mu_{i_d}\} \cup S$, $S \cap \Lambda(P) = \phi$, where $\Lambda(P)$ is the set of all eigenvalues of P. The following theorem recalls a necessary and sufficient condition for a matrix pencil to be in the closure of the strict equivalence bundle of another matrix pencil.

Theorem 2.4 *([10]). Let P_1 and P_2 be two matrix pencils of the same size. Then $P_2 \in \overline{\mathrm{B}^e}(P_1)$ if and only if $P_2 \in \overline{\mathrm{O}^e}(\psi_c(P_1))$, for some map $\psi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$.*

We present the characterisation for inclusion of closure of strict equivalence bundles of two matrix pencils in the next theorem.

Theorem 2.5 *([10]). Let P_1 and P_2 be two matrix pencils of the same size. Then $\mathrm{B}^e(P_2) \subseteq \mathrm{B}^e(P_1)$ if and only if $P_2 \in \overline{\mathrm{O}^e}(\psi_c(P_1))$, for some map $\psi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$.*

2.2. Orbits and bundles of skew-symmetric matrix pencils

In this subsection, we recall results for skew-symmetric matrix pencils. An $n \times n$ skew-symmetric matrix pencil $A - \lambda B$ is said to be congruent to the pencil $C - \lambda D$ if and only if $M^T(A - \lambda B)M = C - \lambda D$, for some non-singular matrix M. The following theorem provides canonical form for skew-symmetric matrix pencils under congruence transformation.

Theorem 2.6 *([38]). Each skew-symmetric $n \times n$ matrix pencil $A - \lambda B$ is congruent to a direct sum of pencils of the following form, determined uniquely up to permutation of summands,*

$$H_h(\mu) = \begin{bmatrix} 0 & J_h(\mu) \\ -J_h(\mu)^T & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & I_h \\ -I_h & 0 \end{bmatrix}, \ \mu \in \mathbb{C},$$

$$K_k = \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & J_k(0) \\ -J_k(0)^T & 0 \end{bmatrix}, \ M_l = \begin{bmatrix} 0 & F_l \\ -F_l^T & 0 \end{bmatrix} - \lambda \begin{bmatrix} 0 & G_l \\ -G_l^T & 0 \end{bmatrix},$$

where F_k, G_k, I_k, and $J_k(\mu)$ are defined as in (1) and (2).

Remark 2.7. Note that $F_l - \lambda G_l$ and $F_l^T - \lambda G_l^T$ form the L-blocks and L^T-blocks, respectively, in Theorem 2.1.

The orbit of $A - \lambda B$ under the action of the group $GL_n(\mathbb{C})$ on the space of skew-symmetric matrix pencils by congruence is defined as follows:

$$\mathrm{O}^c(A - \lambda B) = \{M^T(A - \lambda B)M \mid M \in GL_n(\mathbb{C})\}.$$

We recall a lemma that connects the cover relation of strict equivalence orbits of general matrix pencils and congruence orbits of skew-symmetric matrix pencils.

Lemma 2.8 *([5]). Let P_1 and P_2 be two skew-symmetric matrix pencils such that $\mathrm{O}^c(P_1)$ covers $\mathrm{O}^c(P_2)$. Then there exist some $p \times q$ matrix pencils W_i and non-singular matrices Q_i such that $Q_i^T P_i Q_i = \begin{bmatrix} 0 & W_i \\ -W_i^T & 0 \end{bmatrix}$, for $i = 1, 2$ such that $\mathrm{O}^e_{W_1}$ covers $\mathrm{O}^e_{W_2}$.*

A necessary and sufficient condition for covering property between congruence orbits of skew-symmetric matrix pencils is presented in the following theorem.

Theorem 2.9 *([5]). Let P_1 and P_2 be two skew-symmetric matrix pencils. The congruence orbit $\mathrm{O}^c(P_1)$ covers the congruence orbit $\mathrm{O}^c(P_2)$ if and only if P_2 can be obtained from P_1 by applying one of the following four structure transitions to the canonical blocks of P_1:*

- *Rule 1. $M_j \oplus M_k \rightsquigarrow M_{j-1} \oplus M_{k+1}$, for $1 \leqslant j \leqslant k$ such that*

$$k - j = \min_{j_t \leqslant k, j \leqslant k_u} \big\{ \{k_1 - j, k_2 - j, \dots\} \cup \{k - j_1, k - j_2, \dots\} \big\};$$

- *Rule 2. $H_{j-1}(\mu) \oplus H_{k+1}(\mu) \rightsquigarrow H_j(\mu) \oplus H_k(\mu)$, $\mu \in \overline{\mathbb{C}}$, $1 \leqslant j \leqslant k$ such that*

$$k - j = \min_{j_t \leqslant k, j \leqslant k_u} \big\{ \{k_1 - j, k_2 - j, \dots\} \cup \{k - j_1, k - j_2, \dots\} \big\};$$

- *Rule 3. $M_{j+1} \oplus H_k(\mu) \rightsquigarrow M_j \oplus H_{k+1}(\mu)$, $j, k = 0, 1, 2, \dots$ and $\mu \in \overline{\mathbb{C}}$, such that k and $j+1$ are the sizes of the largest H-block and M-block in P_1, respectively, provided that $r_{j+1}(P_1) = l_{j+1}(P_1) = 1$;*
- *Rule 4. $\bigoplus_{i=1}^{t} H_{k_i}(\mu_i) \rightsquigarrow M_x \oplus M_y$, $x \geqslant z$, $y \geqslant z$, for all M_z in P_1; $|x - y| \leqslant 1$; $x + y + 1 = \sum_{i=1}^{d} k_i, k_i$ is the size of the largest Jordan block in P_1 corresponding to each distinct $\mu_i \in \overline{\mathbb{C}}$.*

The cover relation between two integer partitions has been defined before Theorem 2.2 and can be illustrated as follows: Place m coins in a table in a non-increasing sequence with n_i coins in column i, $\sum_i n_i = m$. Such a placement of coins is referred to as coin-diagram. An integer partition $\mathcal{Q}_1$ covers $\mathcal{Q}_2$ if $\mathcal{Q}_1$ may be obtained from $\mathcal{Q}_2$ by moving a coin upward one row or leftward one column such that the resulting integer partition remains monotonic. Such a coin move will be referred to as *minimum leftward coin move*. Similarly, *minimum rightward coin move* is a coin move downward one row or rightward one column, we obtain $\mathcal{Q}_2$ from $\mathcal{Q}_1$ (for more information, see e.g., [25]). The next theorem uses the above notion of coin move to provide a necessary and sufficient condition analogous to Theorem 2.9 for covering property between congruence orbits of skew-symmetric matrix pencils. Before that, we define *vertical pair of coins* to be a pair of coins positioned in the same column of the table and in the rows $1 + 2k$ and $2 + 2k$, for $k = 0, 1, \dots$ Fig. 1 illustrates the meaning.

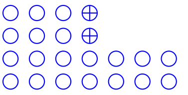

Fig. 1. Integer partition, $\mathcal{J} = \{4, 4, 4, 4, 2, 2, 2\}$, with vertical pair of coins marked as $\oplus$.

Theorem 2.10 *([5]). Let P_1 and P_2 be two skew-symmetric matrix pencils. Then $\mathrm{O}^c(P_1)$ covers $\mathrm{O}^c(P_2)$ if and only if the integer partitions, i.e., $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{J}_{\mu_i}$, corresponding to P_2 can be obtained by applying one of the four rules to the integer partitions of P_1:*

- *Rule I: Minimum rightward coin move from column j in $\mathcal{R}$ and the same coin move in $\mathcal{L}$, where $j \geqslant 1$;*
- *Rule II: If the rightmost column in each of $\mathcal{R}$ and $\mathcal{L}$ is a single coin, append these two coins (together) as a new rightmost column of some $\mathcal{J}_{\mu_i}$;*
- *Rule III: Minimum leftward coin move of a vertical pair of coins in any $\mathcal{J}_{\mu_i}$;*
- *Rule IV: Let $2k = (\sum_{i=1}^{t} 2k_i)$, where k_i denotes the total number of coins in each of the two longest (=lowest) row from each $\mathcal{J}_{\mu_i}$. Remove these $2k$ coins add two more coins to the set. The following two step distribution is done:*
  - *Distribute $k+1$ coins from the set of $2k+2$ coins to r_p, $p = 0, \ldots, t$ and l_q, $q = 0, \ldots, k-t-1$,*
  - *Distribute the remaining $k+1$ coins from the set to r_p, $p = 0, \ldots, k-t-1$ and l_q, $q = 0, \ldots, t$,*

  such that at least the existing columns of each of $\mathcal{R}(P_1)$ and $\mathcal{L}(P_1)$ receive 2 coins and $|p-q|$ is the minimum of all possible differences.

Now, we recall the notion of a bundle of skew-symmetric matrix pencils. Let Φ denote the set of one-to-one maps from $\overline{\mathbb{C}}$ to itself, and let $\varphi \in \Phi$. Let $\hat{P}$ be the congruence canonical form of the skew-symmetric matrix pencil P. Define $\varphi(P)$ as a skew-symmetric matrix pencil that is congruent to a matrix pencil whose canonical form is obtained from $\hat{P}$ by replacing each Jordan block corresponding to the eigenvalue $\mu \in \overline{\mathbb{C}}$ with a Jordan block of the same size corresponding to the eigenvalue $\varphi(\mu)$. A bundle of skew-symmetric matrix pencils under congruence transformation is defined analogously to that of general matrix pencils. Thus, for a skew-symmetric matrix pencil P,

$$\mathrm{B}^c(P) = \bigcup_{\varphi \in \Phi} \mathrm{O}^c(\varphi(P)). \tag{3}$$

3. Characterization of the inclusion between bundle closures of skew-symmetric matrix pencils

In this section, we answer questions (1) and (2) mentioned in the introduction. Thus, we aim to present a characterisation for inclusion of congruence bundles of skew-symmetric matrix pencils and their closures. We start with defining the notion of structure preserving coalescing of eigenvalues of a skew-symmetric matrix pencil.

Definition 3.1. Let P be an $n \times n$ skew-symmetric matrix pencil and $\psi \in \Psi$. Then, $\psi_c^{skew}(P)$ is a skew-symmetric matrix pencil of the same size as P such that it is congruent to $\begin{bmatrix} & \psi_c(P') \\ -\psi_c(P')^T & \end{bmatrix}$, for some $p \times (n-p)$ matrix pencil P'.

Note, the pencil in Definition 3.1 is not uniquely defined. The following lemma establishes a relationship between integer partitions of P and $\psi_c^{skew}(P)$.

Lemma 3.2. *Let P be a skew-symmetric matrix pencil with distinct eigenvalues μ_i, $i \in \{1, \ldots, s\}$ and $\psi \in \Psi$. Then,*

- $\mathcal{R}(\psi_c^{skew}(P)) = \mathcal{R}(P)$, *and* $\mathcal{L}(\psi_c^{skew}(P)) = \mathcal{L}(P)$,
- $\mathcal{J}_\mu(\psi_c^{skew}(P)) = \bigcup_{\mu_i \in \psi^{-1}(\mu)} \mathcal{J}_{\mu_i}(P)$, *for all* $\mu \in \overline{\mathbb{C}}$.

Proof. Without loss of generality, we assume that P is in the canonical form. By permutations of the rows and corresponding permutations of the columns, the matrix pencils P, can be written as

$$\widetilde{P} = Q^T P Q = \begin{bmatrix} 0 & W \\ -W^T & 0 \end{bmatrix}. \tag{4}$$

Since P is skew-symmetric, $\mathcal{J}_{\mu_j}(W) = \mathcal{J}_{\mu_j}(-W^T)$, for $\mu_j \in \overline{\mathbb{C}}$. Also, the right and left minimal indices of W are the left and right minimal indices of $-W^T$, respectively.

Construct the matrix pencil $\psi_c^{skew}(P) = \begin{bmatrix} 0 & \psi_c(W) \\ -\psi_c(W)^T & 0 \end{bmatrix}$. By Definition 2.3, $\mathcal{R}(W) = \mathcal{R}(\psi_c(W))$, $\mathcal{L}(W) = \mathcal{L}(\psi_c(W))$, and $\mathcal{R}(-W^T) = \mathcal{R}(-\psi_c(W)^T)$, $\mathcal{L}(-W^T) = \mathcal{L}(-\psi_c(W)^T)$. Thus, the right and left minimal indices of $\psi_c(W)$ are the left and right minimal indices of $-\psi_c(W)^T$, respectively. Note, the right and left minimal indices of W and $-W^T$ together form the right and left minimal indices of P, respectively. Same argument holds for $\psi_c^{skew}(P)$. Thus, $\mathcal{R}(\psi_c^{skew}(P)) = \mathcal{R}(P)$ and $\mathcal{L}(\psi_c^{skew}(P)) = \mathcal{L}(P)$. Assume the eigenvalues $\mu_{i_1}, \ldots, \mu_{i_d}$ of W have coalesced to the eigenvalue μ in $\psi_c(W)$, i.e., $\psi^{-1}(\mu) = \{\mu_{i_1}, \ldots, \mu_{i_d}\} \cup S$, with $S \cap \Lambda(W) = \phi$. So, for $\mu \in \overline{\mathbb{C}}$,

$$\bigcup_{\mu_i \in \psi^{-1}(\mu)} \mathcal{J}_{\mu_i}(P) = \bigcup_{\mu_i \in \psi^{-1}(\mu)} 2\mathcal{J}_{\mu_i}(W) = 2\mathcal{J}_\mu(\psi_c(W)) = \mathcal{J}_\mu(\psi_c^{skew}(P)). \quad \square$$

Remark 3.3 *(Coalescing of eigenvalues).* From Definition 3.1 and Lemma 3.2, the distinct eigenvalues of $\mu_{i_1}, \ldots, \mu_{i_d}$ of P coalesc to the eigenvalue μ in $\psi_c^{skew}(P)$ if $\psi^{-1}(\mu) = \{\mu_{i_1}, \ldots, \mu_{i_d}\} \cup S$, where S has empty intersection with set of eigenvalues of P.

Before proceeding further, we state a few preliminary results.

Lemma 3.4 *([6]). Let $\{Z_k\}_{k\in\mathbb{N}}$ be a sequence of $m \times n$ complex matrix pencils converging to a matrix pencil D of the same size and $\sigma_\epsilon(D) = \bigcup_{\mu\in\Lambda(D)} B(\mu, \epsilon)$. Then for all suffi-*

ciently large k, there exists a nonnegative integer h_k such that the following conditions hold:

- $\mathcal{R}(D) \prec \mathcal{R}(Z_k) + (h_k, h_k, \dots)$;
- $\mathcal{L}(D) \prec \mathcal{L}(Z_k) + (h_k, h_k, \dots)$;
- $\mathcal{J}_\mu(Z_k) \prec \mathcal{J}_\mu(D) + (h_k, h_k, \dots)$, *for all* $\mu \in (\overline{\mathbb{C}} - \sigma_\epsilon(D)) \cup \Lambda(D)$.

Note that although Lemma 3.4 appears similar to Theorem 2.2, it does not imply any closure relationship between the orbits of D and Z_k. Lemma 3.4 along with the next theorem will allow us to develop a lemma that will play an important role in our characterization in this section.

Theorem 3.5 *([10]). Let $\{Z_k\}_{k\in\mathbb{N}}$ be a sequence of $m \times n$ complex matrix pencils such that:*

- *for some complex matrix pencil P, the sequence $\{Z_k\}_{k\in\mathbb{N}} \subset \mathrm{B}^e(P)$,*
- *$\lambda_{1,k},\dots,\lambda_{g,k} \in \overline{\mathbb{C}}$ are distinct eigenvalues of each Z_k and $\mathcal{J}_{\lambda_{m,k_1}}(Z_{k_1}) = \mathcal{J}_{\lambda_{m,k_2}}(Z_{k_2})$, for all $m = 1,\dots,g$ and all $k_1, k_2 \in \mathbb{N}$,*
- *$\{Z_k\}_{k\in\mathbb{N}}$ converges to a matrix pencil D, and*
- *the sequence $\{\lambda_{m,k}\}_{k\in\mathbb{N}}$ converges to $\mu \in \overline{\mathbb{C}}$, for all $m = 1,\dots,g$.*

Then, $\bigcup_{m=1}^{g} \mathcal{J}_{\lambda_{m,k}}(Z_k) \prec \mathcal{J}_\mu(D) + (h, h, \dots)$, where $h = nrk(P) - nrk(D)$.

The following lemma creates a bridge between closures of bundles of skew-symmetric matrix pencil under congruence and closures of bundles of general matrix pencil under strict equivalence.

Lemma 3.6. *Let P_1 and P_2 be two $n \times n$ skew-symmetric matrix pencils, such that $P_2 \subseteq \overline{\mathrm{B}^c}(P_1)$. Then there exist some $p\times(n-p)$ matrix pencils W_i, $i = 1, 2$ such that $\mathrm{B}^e(W_2) \subseteq \overline{\mathrm{B}^e}(W_1)$.*

Proof. Without loss of generality, assume P_i, $i = 1, 2$, are in the skew-symmetric canonical form in Theorem 2.6. The rows and columns of P_i can be permuted such that

$$\widetilde{P}_i = Q_i^T P_i Q_i = \begin{bmatrix} 0 & W_i \\ -W_i^T & 0 \end{bmatrix}, \quad i = 1, 2. \tag{5}$$

The block W_i in (5) is the direct sum of the top-right corner blocks of M-, H- and K-summands in the skew-symmetric canonical forms of P_1 and rest of the blocks form $-W_1^T$. The block W_2 in (5) is the direct sum of top-right corner blocks of H- and K-summands as well as the h largest L^T and $\left(r_0(P_2) - \dfrac{h}{2}\right)$ smallest L-blocks of P_2, where $h = nrk(P_2) - nrk(P_1)$.

By Theorem 2.5, to prove $\mathrm{B}^e(W_2) \subseteq \overline{\mathrm{B}^e}(W_1)$, it is sufficient to prove $W_2 \subseteq \overline{\mathrm{O}^e}(\psi_c(W_1))$, for some $\psi \in \Psi$, i.e., the majorizations in Theorem 2.2 are satisfied by W_2 and $\psi_c(W_1)$. To do so, first we find some sequence of matrix pencils in $\mathrm{B}^e(W_1)$ such that W_2 and this sequence or (if required) some of its subsequence satisfy the majorizations in Theorem 2.2 corresponding to integer partitions $\mathcal{R}$, $\mathcal{L}$ and the majorization in Theorem 3.5. Then, we construct $\psi_c(W_1)$. Using the former majorizations, we will prove the required majorizations.

Since $P_2 \in \overline{\mathrm{B}^c}(P_1)$, there exists a sequence of skew-symmetric matrix pencils $\{Z_k\}_{k\in\mathbb{N}} \subset \mathrm{B}^c(P_1)$ that converges to P_2. Define $\sigma_\epsilon(M) = \bigcup_{\mu\in\Lambda(M)} B(\mu,\epsilon)$ and $h = nrk(P_2) - nrk(P_1)$. From Lemma 3.4 and its proof, for all sufficiently large k, the matrix pencils $\{Z_k\}_{k\in\mathbb{N}}$ satisfy,

$$\mathcal{R}(P_2) \prec \mathcal{R}(Z_k) + (h,h,\dots) = \mathcal{R}(P_1) + (h,h,\dots) \tag{6}$$

$$\mathcal{J}_\mu(Z_k) \prec \mathcal{J}_\mu(P_2) + (h,h,\dots), \text{ for all } \mu \in (\overline{\mathbb{C}} - \sigma_\epsilon(P_2)) \cup \Lambda(P_2). \tag{7}$$

Since P_i, $i = 1,2$ and Z_k, $k \in \mathbb{N}$ are skew-symmetric matrix pencils, so (6) is also true for the integer partition $\mathcal{L}$. From the construction of W_i, $i = 1,2$, we get,

$$\mathcal{R}(W_1) = \mathcal{R}(P_1),\ \mathcal{L}(W_1) = 0,\ \mathcal{R}(W_2) + \mathcal{L}(W_2) = \mathcal{R}(P_2),$$
$$r_0(W_1) = r_0(P_1),\ l_0(W_1) = 0,\ r_0(W_2) = r_0(P_2) - \frac{h}{2},\ l_0(W_2) = \frac{h}{2},$$
$$nrk(W_1) = \frac{n - r_0(P_2) + h}{2} \text{ and } nrk(W_2) = \frac{n - r_0(P_2)}{2}.$$

Using the above equalities and following the computations related to integer partitions $\mathcal{R}$ and $\mathcal{L}$ in the proof method of Theorem 3.10 in [22], we get the following majorizations as a consequence of (6):

$$\mathcal{R}(W_2) \prec \mathcal{R}(W_1) + \frac{1}{2}(h,h,\dots) \text{ and } \mathcal{L}(W_2) \prec \mathcal{L}(W_1) + \frac{1}{2}(h,h,\dots) \tag{8}$$

For each k, Z_k has the same canonical congruence form as P_1, up to distinct eigenvalues. Thus, taking Z_k in canonical form represented as $\hat{Z}_k$, there exist some permutation matrix R_k, such that

$$\widetilde{Z}_k = R_k^T \hat{Z}_k R_k = \begin{bmatrix} 0 & \widetilde{W_k} \\ -\widetilde{W_k}^T & 0 \end{bmatrix}, k \in \mathbb{N}.$$

From construtcion of $\widetilde{W_k}$, $k \in \mathbb{N}$, the sequence $\{\widetilde{W_k}\}_{k\in\mathbb{N}} \subset \mathrm{B}^e(W_1)$. This leads to the following majorizations using (7) and (8), for sufficiently large k,

$$\mathcal{R}(W_2) \prec \mathcal{R}(\widetilde{W_k}) + \frac{1}{2}(h,h,\dots) \text{ and } \mathcal{L}(W_2) \prec \mathcal{L}(\widetilde{W_k}) + \frac{1}{2}(h,h,\dots). \tag{9}$$

$$\mathcal{J}_{\mu}(\widetilde{W_k}) \prec \mathcal{J}_{\mu}(W_2) + \frac{1}{2}(h, h, \dots), \text{ for all } \mu \in (\overline{\mathbb{C}} - \sigma_{\epsilon}(W_2)) \cup \Lambda(W_2). \tag{10}$$

Let $\{\lambda_{m,k}\}_{k\in\mathbb{N}}$ and λ_m denote sequences of eigenvalues of $\{\widetilde{W_k}\}_{k\in\mathbb{N}}$ and W_1, for $m = 1, \dots, g$, respectively. Thus, $\{\lambda_{m,k}\}_{k\in\mathbb{N}}$ and λ_m are also sequences of eigenvalues of $\{Z_k\}_{k\in\mathbb{N}}$ and P_1, for $m = 1, \dots, g$, respectively. Since $\{Z_k\}_{k\in\mathbb{N}} \subset \mathrm{B}^c(P_1)$, we get the following equality which is necessary for using Theorem 3.5:

$$\mathcal{J}_{\lambda_{m,k}}(Z_k) = \mathcal{J}_{\lambda_m}(P_1), \text{ for all } m = 1, \dots, g, \text{ and all } k \in \mathbb{C}. \tag{11}$$

Now, we see that, by taking a subsequence of $\{\widetilde{W_k}\}_{k\in\mathbb{N}}$ if necessary, we show that, for any $m = 1, \dots, g$, one of the following holds:

(I) $\{\lambda_{m,k}\}$ of $\widetilde{W_k}$ converges to an eigenvalue in W_2
(II) There is some $\epsilon > 0$ such that $\{\lambda_{m,k}\} \subset \overline{\mathbb{C}} - \sigma_{\epsilon}(W_2)$.

Assume that for some $m = 1, \dots, g$, (II) does not hold. Then, for any $\epsilon > 0$ there is a subsequence of $\{\lambda_{m,k}\}_{k\in\mathbb{N}}$ included in $\sigma_{\epsilon}(W_2)$. Since the numbers of eigenvalues of P_2 and W_2 are finite, a subsequence $\{\lambda_{m,k_j}\}$, or a finer subsequence if required, converges to an eigenvalue of W_2 for sufficiently small ϵ.

We choose all such m among $\{1, ..., g\}$ for which $\{\lambda_{m,k}\}$ or its subsequence satisfies (I). We get a subsequence of $\{\widetilde{W_k}\}_{k\in\mathbb{N}}$ whose eigenvalues converge to some eigenvalue of W_2. Assume eigenvalues λ_{m,k_j}, for $m = 1, \dots, t$, $t \leqslant g$, converge to distinct eigenvalues of W_2, say, $\mu_1, \dots, \mu_d$. We partition the set $T = \{1, .., t\}$ as a disjoint union $T = \bigsqcup_{y=1}^{d} I_y$, such that λ_{m,k_j} converge to μ_y whenever $m \in I_y$, for $y = 1, \dots, d$. Thus,

$$|\Lambda(W_2)| \geqslant d. \tag{12}$$

Since $\{Z_k\}_{k\in\mathbb{N}} \subset \mathrm{B}^c(P_1)$, then $\{Z_k\}_{k\in\mathbb{N}} \subset \mathrm{B}^e(P_1)$. Thus, (11) allows us to use Theorem 3.5 that leads to,

$$\bigcup_{m\in I_y} \mathcal{J}_{\lambda_{m,k_j}}(Z_{k_j}) \prec \mathcal{J}_{\mu_y}(P_2) + (h, h \dots).$$

From the construction of W_2 and $\{\widetilde{W_k}\}_{k\in\mathbb{N}}$, we get a similar majorization for W_2 and $\{\widetilde{W_k}\}_{k\in\mathbb{N}}$, i.e.,

$$\bigcup_{m\in I_y} \mathcal{J}_{\lambda_{m,k_j}}(\widetilde{W_{k_j}}) \prec \mathcal{J}_{\mu_y}(W_2) + \frac{1}{2}(h, h \dots). \tag{13}$$

Note, there might be eigenvalues of W_2 that do not arise from eigenvalues of $\widetilde{W_k}$. Since some eigenvalues of $\widetilde{W_k}$ may satisfy condition (II), we assume these eigenvalues to be $\lambda_{m,k}$, for $m = t + 1, \dots, g$.

We, now, construct $\psi_c(W_1)$ and establish the majorizations from Theorem 2.2 using the majorizations satisfied by a subsequence of $\widetilde{W}_k$ and W_2 in (9) and (13). Using (12), we define a function $\psi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that

$$\begin{cases} \psi(\lambda_k) = \mu_y, & \text{for } k \in I_y \text{ and } y = 1, \dots, d \\ \psi(\lambda_k) = \lambda_k, & \text{for } k = t+1, \dots, g \\ \psi(\mu) \notin \{\mu_1, \dots, \mu_d, \lambda_{t+1}, \dots, \lambda_g\}, & \text{for } \mu \notin \{\lambda_1, \dots, \lambda_g\} \end{cases}.$$

Then $\mu_1, \dots, \mu_d, \lambda_{t+1}, \dots, \lambda_g$ are distinct eigenvalues of $\psi_c(W_1)$. We first prove

$$\mathcal{J}_\mu(\psi_c(W_1)) \prec \mathcal{J}_\mu(W_2) + \frac{1}{2}(h, h, \dots), \text{ for all } \mu \in \overline{\mathbb{C}}. \tag{14}$$

Note, if μ is not an eigenvalue of $\psi_c(W_1)$, then clearly the identity (14) is satisfied. So we consider μ to be an eigenvalue of $\psi_c(W_1)$. From definition of $\psi_c(W_1)$, construction of $\widetilde{W}_k$ and (13),

$$\mathcal{J}_{\mu_y}(\psi_c(W_1)) = \bigcup_{k \in I_y} \mathcal{J}_{\lambda_k}(W_1) = \bigcup_{k \in I_y} \mathcal{J}_{\lambda_{m,k_j}}(\widetilde{W}_{k_j}) \prec \mathcal{J}_{\lambda_{m,k}}(W_2) + \frac{1}{2}(h, h, \dots).$$

For eigenvalues $\lambda_{t+1}, \dots, \lambda_g$, using (10) and (II),

$$\mathcal{J}_{\lambda_k}(\psi_c(W_1)) = \mathcal{J}_{\lambda_k}(W_1) = \mathcal{J}_{\lambda_{m,k}}(\widetilde{W}_k) \prec \mathcal{J}_{\lambda_{m,k}}(W_2) + \frac{1}{2}(h, h, \dots) = \frac{1}{2}(h, h, \dots),$$

$$\text{i.e., } \mathcal{J}_{\lambda_k}(\psi_c(W_1)) \prec \mathcal{J}_{\lambda_k}(W_2) + \frac{1}{2}(h, h, \dots).$$

It remains to note that (9) and the equalities $\mathcal{R}(\widetilde{W}_k) = \mathcal{R}(W_1) = \mathcal{R}(\psi_c(W_1))$ and $\mathcal{L}(\widetilde{W}_k) = \mathcal{L}(W_1) = \mathcal{L}(\psi_c(W_1))$ imply the majorizations:

$$\mathcal{R}(W_2) \prec \mathcal{R}(\psi_c(W_1)) + \frac{1}{2}(h, h, \dots) \text{ and } \mathcal{L}(W_2) \prec \mathcal{L}(\psi_c(W_1)) + \frac{1}{2}(h, h, \dots).$$

This completes the proof. □

The following theorem connects the closure of congruence bundle of a skew-symmetric matrix pencil P_1 with the closure of congruence orbit of a skew-symmetric matrix pencil obtained by coalescing eigenvalues of P_1.

Theorem 3.7. *Let P_1 and P_2 be two complex skew-symmetric matrix pencils. $P_2 \in \overline{\mathrm{B}^c}(P_1)$ if and only if $P_2 \in \overline{\mathrm{O}^c}(\psi_c^{skew}(P_1))$, for some map $\psi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$.*

Proof. If $P_2 \in \overline{\mathrm{B}^c}(P_1)$, then by Lemma 3.6, there exist some $p \times (n-p)$ matrix pencils W_i such that $\mathrm{B}^e(W_2) \subseteq \overline{\mathrm{B}^e}(W_1)$. Theorem 2.5 implies $W_2 \in \overline{\mathrm{O}^e}(\psi_c(W_1))$. Then, there exist non-singular matrices Q and R and arbitrary small (entrywise) matrix E such that $Q^{-1}(W_2 + E)R = \psi_c(W_1)$, and thus $R^T(-W_2 - E)^T Q^{-T} = -\psi_c(W_1)^T$. Combining the two equations above, we get,

$$\begin{bmatrix} 0 & \psi_c(W_1) \\ -\psi_c(W_1)^T & 0 \end{bmatrix} = \begin{bmatrix} Q^{-T} & 0 \\ 0 & R \end{bmatrix}^T \begin{bmatrix} 0 & W_2 + E \\ -W_2^T - E^T & 0 \end{bmatrix} \begin{bmatrix} Q^{-T} & 0 \\ 0 & R \end{bmatrix}. \tag{15}$$

Defining $\psi_c^{skew}(P_1) := \begin{bmatrix} 0 & \psi_c(W_1) \\ -\psi_c(W_1)^T & 0 \end{bmatrix}$, we conclude $P_2 \in \overline{\mathrm{O}^c}(\psi_c^{skew}(P_1))$.

For the converse, we prove that $\mathrm{O}^c(\psi_c^{skew}(P)) \subseteq \overline{\mathrm{B}^c}(P_1)$ by showing that each element of $\mathrm{O}^c(\psi_c^{skew}(P))$ can be approximated by a sequence in $\mathrm{B}^c(P_1)$. Without loss of generality, we assume P_1 is in canonical form given by $\tilde{P}_1 = \begin{bmatrix} 0 & W_1 \\ -W_1^T & 0 \end{bmatrix}$, where the block W_1 is chosen similar to that in (5). The proof of "only if" part of Theorem 9 in [10] shows that $\zeta_c(W_1) \in \overline{\mathrm{B}^e}(W_1)$, for any $\zeta \in \Psi$. Thus, we can find a sequence $\{\tilde{W}_k\}_{k\in\mathbb{N}}$, in $\mathrm{B}^e(W_1)$ that converges to $\psi_c(W_1)$. Then, the sequence $\{Q_k\}_{k\in\mathbb{N}}$, where $Q_k = \begin{bmatrix} 0 & \tilde{W}_k \\ -\tilde{W}_k{}^T & 0 \end{bmatrix}$, lies in $\mathrm{B}^c(P_1)$ and it converges to $\begin{bmatrix} 0 & \psi_c(W_1) \\ -\psi_c(W_1)^T & 0 \end{bmatrix}$. From the construction of W_1, definitions of skew-symmetric matrix pencil $\psi_c^{skew}(P_1)$ and matrix pencil $\psi_c(W_1)$,

$$\mathcal{R}(\psi_c^{skew}(P_1)) = \mathcal{R}(P_1) = \mathcal{R}(W_1) = \mathcal{R}(\psi_c(W_1));$$

$$\mathcal{L}(\psi_c^{skew}(P_1)) = \mathcal{L}(P_1) = \mathcal{R}(W_1) = \mathcal{R}(\psi_c(W_1)) \text{ and}$$

$$\mathcal{J}_\mu(\psi_c^{skew}(P)) = \bigcup_{\mu_i \in \psi^{-1}(\mu)} \mathcal{J}_{\mu_i}(P_1) = \bigcup_{\mu_i \in \psi^{-1}(\mu)} 2\mathcal{J}_{\mu_i}(W_1) = 2\mathcal{J}_\mu(\psi_c(W_1)),$$

for all $\mu \in \overline{\mathbb{C}}$. Thus, $\psi_c^{skew}(P)$ is congruent to $\begin{bmatrix} 0 & \psi_c(W_1) \\ -\psi_c(W_1)^T & 0 \end{bmatrix}$. From definition of $\psi_c^{skew}(P_1)$, $\mathrm{O}^c(\psi_c^{skew}(P)) \subseteq \overline{\mathrm{B}^c}(P_1)$, i.e., $\overline{\mathrm{O}^c}(\psi_c^{skew}(P)) \subseteq \overline{\mathrm{B}^c}(P_1)$. This completes our proof. □

From the proof of the converse of the above theorem, closure of the congruence bundle of a skew-symmetric matrix pencil can be represented as a union of closures of congruence orbits of other skew-symmetric matrix pencils obtained by coalescing eigenvalues of the former such that structure is preserved in the latter. This is illustrated in the following equality:

$$\overline{\mathrm{B}^c}(P) = \bigcup_{\psi \in \Psi} \overline{\mathrm{O}^c}(\psi_c^{skew}(P)), \text{ for a skew-symmetric complex matrix pencil } P.$$

Theorems 2.9 and 2.10 make use of the fact that if $P_2 \in \overline{\mathrm{O}^c}(P_1)$, then $\mathrm{O}^c(P_2) \subseteq \overline{\mathrm{O}^c}(P_1)$ which is straightforward. Similar statement for congruence bundles is true but not obvious. This is proven in the next theorem.

Theorem 3.8. *Let P_1 and P_2 be two skew-symmetric matrix pencils such that $P_2 \in \overline{\mathrm{B}^c}(P_1)$. Then, $\mathrm{B}^c(P_2) \subseteq \overline{\mathrm{B}^c}(P_1)$.*

Proof. Let P_2 be as in the statement. Then from Theorem 3.7, $P_2 \in \overline{\mathrm{O}^c}(\psi_c^{skew}(P_1))$, for some $\psi \in \Psi$. Let $Z \in \mathrm{B}^c(P_2)$. From definition of congruence bundle in (3), we have, $\mathrm{B}^c(P_2) = \bigcup_{\varphi\in\Phi} \mathrm{O}^c(\varphi(P_2))$. Then, Z is congruent to $\varphi(P_2)$, for some φ. But P_2 is obtained from $\psi_c^{skew}(P_1)$ using a sequence of the rules in Theorem 2.9. From definition of $\psi_c^{skew}(P)$ and $\varphi(P)$, for a matrix pencil P, it is clear that Z is obtained from $\varphi(\psi_c^{skew}(P_1))$.

Now, we prove that $\varphi(\psi_c^{skew}(P_1))$ is congruent to $(\varphi\circ\psi)_c^{skew}(P_1)$. From Definition 3.1 and Remark 3.3, $\psi_c^{skew}(P_1)$ is congruent to $\begin{bmatrix} & \psi_c(P') \\ -\psi_c(P')^T & \end{bmatrix}$, for some $p \times (n-p)$ matrix pencil P' and the distinct eigenvalues of $\mu_{i_1}, \dots, \mu_{i_d}$ of P_1 have coalesced to some eigenvalue μ in $\psi_c^{skew}(P_1)$ if $\psi^{-1}(\mu) = \{\mu_{i_1}, \dots, \mu_{i_d}\} \cup S$, where $S \cap \Lambda(P_1) = \phi$. Recalling from the paragraph above (3), the Jordan blocks of $\psi_c^{skew}(P_1)$ associated with eigenvalue $\hat{\mu} \in \overline{\mathbb{C}}$ are replaced with Jordan blocks of same size as that of the former corresponding to eigenvalue $\varphi(\hat{\mu})$ of $\varphi(\psi_c^{skew}(P_1))$. This implies that the Jordan blocks of $\psi_c(P')$ associated with eigenvalue $\hat{\mu} \in \overline{\mathbb{C}}$ are replaced with Jordan blocks of same size as that of the former corresponding to eigenvalue $\varphi(\hat{\mu})$ of $\varphi(\psi_c(P'))$. Let $\varphi(\mu) = \tilde{\mu}$. Then, $(\varphi\circ\psi)^{-1}(\tilde{\mu}) = \{\mu_{i_1}, \dots, \mu_{i_d}\} \cup S$, where $S \cap \Lambda(P') = \phi$. Thus, $\varphi(\psi_c(P'))$ is equivalent to $(\varphi\circ\psi)_c(P')$. This results in congruency between $\varphi(\psi_c^{skew}(P_1))$ is congruent to $(\varphi\circ\psi)_c^{skew}(P_1)$ which implies $Z \in \overline{\mathrm{O}^c}((\varphi\circ\psi)_c^{skew}(P_1))$. From definition of φ and ψ, $\varphi\circ\psi \in \Psi$. Thus, using Theorem 3.7, we get $Z \in \overline{\mathrm{B}^c}(P_1)$ which completes our proof. □

In Theorem 3.9, we state our main characterization. It is a direct consequence of Theorems 3.7 and 3.8.

Theorem 3.9. *Let P_1 and P_2 be two skew-symmetric matrix pencils. $P_2 \in \overline{\mathrm{O}^c}(\psi_c^{skew}(P_1))$, for some map $\psi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, if and only if $\mathrm{B}^c(P_2) \subseteq \overline{\mathrm{B}^c}(P_1)$.*

4. Characterization of the covering property between bundle closures of skew-symmetric matrix pencils

As a result of Theorem 3.9, we now find the rules, similar to Theorems 2.9 and 2.10, for congruence bundles. We provide a necessary and sufficient condition for a bundle of a skew-symmetric matrix pencil to cover another bundle. Recall that the bundle $\mathrm{B}^c(P_2)$ is said to cover $\mathrm{B}^c(P_1)$ if $\mathrm{B}^c(P_1) \subset \overline{\mathrm{B}^c}(P_2)$ and there exists no other skew-symmetric matrix pencil D such that $\mathrm{B}^c(P_1) \subset \overline{\mathrm{B}^c}(D) \subset \overline{\mathrm{B}^c}(P_2)$. For the sake of completeness, we state the necessary and sufficient condition for a strict equivalence bundle of a general matrix pencil to cover another bundle.

Theorem 4.1 *([25]). Let P_1 and P_2 be two matrix pencils such that $\mathrm{B}^e(P_1)$ covers $\mathrm{B}^e(P_2)$ if and only if P_2 can be obtained from P_1 by applying one of the five rules to the integer partition of P_1:*

- *Minimum rightward coin move in $\mathcal{R}$ (or $\mathcal{L}$) from column j, where $j \geqslant 1$;*

- *If the rightmost column in $\mathcal{R}$ (or $\mathcal{L}$) is single coin, use that coin to start a new set of coins for a new eigenvalue μ_i in $\mathcal{J}_{\mu_i}$;*
- *Minimum leftward coin move in any $\mathcal{J}_{\mu_i}$;*
- *Let k denote the total number of coins in all of the longest (=lowest) rows from all of the $\mathcal{J}_{\mu_i}$. If there is just one eigenvalue in the KCF or if all eigenvalues have at least 2 Jordan blocks and the total number of non-zero columns of $\mathcal{R}$ and $\mathcal{L}$ exceeds $k+1$, remove these k coins. Add one more coin to the set, and distribute $k+1$ coins to r_p, $p = 0, \dots, t$ and l_q, $q = 0, \dots, k-t-1$ such that at least all non-zero columns of $\mathcal{R}$ and $\mathcal{L}$ are given coins;*
- *Let any pair of eigenvalues coalesce, i.e., take the union of their sets of coins;*

The following theorem is an analogous result to Theorem 2.10 and is our main result in this section.

Theorem 4.2. *Let P_1 and P_2 be two skew-symmetric matrix pencils such that $\mathrm{B}^c_{P_1}$ covers $\mathrm{B}^c_{P_2}$ if and only if P_2 can be obtained from P_1 by applying one of five rules to the integer partition of P_1:*

- *Rule I´: Same as Rule I in Theorem 2.10.*
- *Rule II´: Same as Rule II in Theorem 2.10 but only append the two coins (together) for a new eigenvalue as a vertical pair of coins.*
- *Rule III´: Same as Rule III in Theorem 2.10.*
- *Rule IV´: Same as Rule IV in Theorem 2.10, but apply only if there is just one eigenvalue in the canonical structure or if all eigenvalues have at least 4 Jordan blocks of the same size.*
- *Rule V´: Any pair of eigenvalues coalesces.*

Proof. We assume P_1 and P_2 be two skew-symmetric matrix pencils such that $\mathrm{B}^c_{P_1}$ covers $\mathrm{B}^c_{P_2}$. Since bundles are unions of orbits, so the rules in Theorem 2.10 are all valid for bundles. All we need to prove is the existence of *Rule V´*, the extra restrictions on *Rules II´, IV´* and the non-existence of any sixth rule.

Theorem 3.9 explains coalescing of arbitrary number of eigenvalues. Nevertheless, such a coalescing can be done by coalescing a pair of eigenvalues repeatedly contradicting cover relation between $\mathrm{B}^c_{P_1}$ and $\mathrm{B}^c_{P_2}$. This implies *Rule V´*.

Now, we explain the extra restrictions on *Rules II´, IV´*.

Rule II´: If we do not append the two coins (together) for a new eigenvalue, the resultant partition after application of *Rule II´* followed by *Rule V´* is same as the partition we obtain by single application of *Rule II* from Theorem 2.10.

Rule IV´: If P_1 has at least 2 eigenvalues, use of *Rule V´* followed by *Rule IV* from Theorem 2.10 is equivalent to *Rule IV´*. Again, if all the eigenvalues have 4 Jordan blocks, a single application of *Rule IV´* can be replaced by *Rule V´* followed by *Rule IV* from Theorem 2.10.

Now, we prove that any sequence of *Rules I´-IV´* cannot replace *Rule V´*. Assume there exists some sequence of *Rules I´-IV´* that replaces *Rule V´*. Since *Rule IV´* changes normal rank and *Rule V´* does not change normal rank, the sequence cannot include *Rule IV´*. Note *Rule I´* changes the size of M-blocks, *Rule II´* creates a new H_1-block corresponding to some eigenvalue and the largest M-block disappears and *Rule III´* increases the size of H-blocks corresponding to the same eigenvalue. Thus, any sequence of *Rules I´-III´* can not replace *Rule V´*.

Assume a sixth rule exist using which we can obtain P_2 from P_1. By Lemma 3.6, taken in canonical form, the rows and columns of P_i can be permuted such that

$$\widetilde{P}_i = Q_i^T P_i Q_i = \begin{bmatrix} 0 & W_i \\ -W_i^T & 0 \end{bmatrix}, i = 1, 2, \tag{16}$$

and $\mathrm{B}^e(W_2) \subseteq \overline{\mathrm{B}^e}(W_1)$. Assume there exist W_k such that $\mathrm{B}^e(W_k)$ covers $\mathrm{B}^e(W_2)$ and $\mathrm{B}^e(W_1)$ covers $\mathrm{B}^e(W_k)$. By Theorem 4.1, W_2 can be obtained from W_k and W_k can be obtained from W_1 using one of the rules. Construct $P_k = \begin{bmatrix} 0 & W_k \\ -W_k^T & 0 \end{bmatrix}$. Then, P_2 can be obtained from P_k and P_k can be obtained from P_1 using one of the above rules contradicting the cover relation between $\mathrm{B}^e(P_2)$ and $\mathrm{B}^e(P_1)$ and proving that the sixth rule is equivalent to a sequence of two of the above rules and repetition of the same rule. The converse can be proven with contradiction. This completes our proof. □

The rules in the above theorem allow downward navigation in the bundle stratification graph. In the graph a downward path from a node representing P_1 to a node representing P_2 exists if and only if the canonical form P_2 can be obtained from the canonical form of P_1 by applying the rules stated in Theorems 2.10 and 4.2. *Codimension* of $\mathrm{B}^c(P)$ is *codimension* of $\mathrm{O}^c(P)$ minus the total number of distinct eigenvalues of P [22]. From now on, codimension will be denoted as *cod*. Below, we give two examples demonstrating the above result.

Example 4.3. In this example, all the bundles are considered under congruence transformation. In Fig. 2, we present the bundle stratification graph of 6×6 skew-symmetric matrix pencil with *Rules I´-V´* corresponding to each edge of the graph. *Rules I´* and *III´* in Theorem 4.2 are the same as *Rules I* and *III* in Theorem 2.10 which has been explicitly illustrated in Fig. 2 in [5]. We explain the rest of the rules from the figure, written in bold font, and explain the coin moves corresponding to each of them. The bullets (●) represent the coins that are involved in the coin moves and the empty circles (○) represent the coins that are not involved in the coin moves.

Rule II´: $M_0 \oplus M_1 \oplus H_1(\mu_1) \rightsquigarrow 2M_0 \oplus H_1(\mu_1) \oplus H_1(\mu_2)$, unlike orbit stratification graph of Figure 2 in [5], here, we can not move the coins from $\mathcal{R}$ and $\mathcal{L}$ to $\mathcal{J}_{\mu_1}$ (refer Rule III from Example 3.7 in [5]) because it is not empty.

Both $\mathcal{R}$ and $\mathcal{L}$ before *Rule II´*:	Both $\mathcal{R}$ and $\mathcal{L}$ after *Rule II´*:
○ ○ ●	○ ○
J^{μ_1} before *Rule II´*:	J^{μ_1} after *Rule II´*:
○ ○	○ ○
J^{μ_2} before *Rule II´*:	J^{μ_2} after *Rule II´*:
- -	● ●

Rule IV´: $H_1(\mu_1) \oplus H_2(\mu_1) \rightsquigarrow M_0 \oplus M_1 \oplus H_1(\mu_1)$, note, in orbit stratification graph of Figure 2 in [5], $M_0 \oplus M_1 \oplus H_1(\mu_1)$ is a closest neighbour to both $H_1(\mu_1) \oplus H_2(\mu_1)$ and $2H_1(\mu_1) \oplus H_1(\mu_2)$.

$\mathcal{R}$ and $\mathcal{L}$ before *Rule IV*	$\mathcal{R}$ and $\mathcal{L}$ after *Rule IV*
- -	● ● ●
J^{μ_1} before *Rule IV*	J^{μ_1} after *Rule IV*
○ ○ ● ● ● ●	○ ○

Rule V´: $2H_1(\mu_1) \oplus H_1(\mu_2) \rightsquigarrow H_1(\mu_1) \oplus H_2(\mu_1)$.

Both $\mathcal{R}$ and $\mathcal{L}$ before *Rule V´*:	Both $\mathcal{R}$ and $\mathcal{L}$ after *Rule V´*:
-	-
J^{μ_1} before *Rule V´*:	J^{μ_1} after *Rule V´*:
● ● ● ●	● ● ● ● ● ●
J^{μ_2} before *Rule V´*:	J^{μ_2} after *Rule V´*:
● ●	- -

Example 4.4. In Fig. 3 the bundle stratification graph of 7×7 skew-symmetric matrix pencil with *Rules I´-V´* corresponding to each edge of the graph along with the codimensions of each bundle is mentioned.

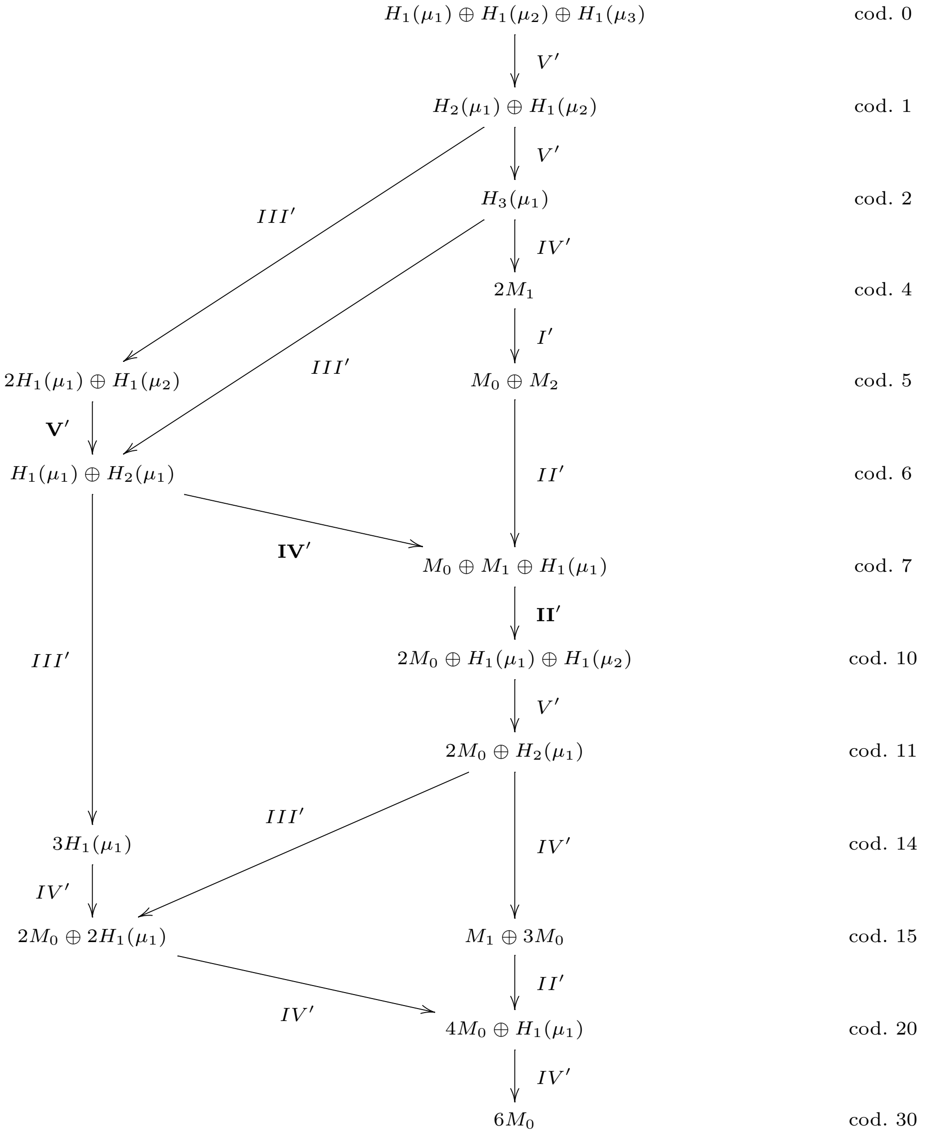

Fig. 2. Bundle stratification for 6×6 skew-symmetric matrix pencils along with the codimension. Rules from Theorem 4.2 used for each structure transition are indicated near the corresponding edge.

5. Bundles are open in their closure

In this section, our main aim is to prove that congruence bundles of skew-symmetric matrix pencils are open in their closures. We start with recalling a formula that computes the codimension of congruence orbits of skew-symmetric matrix pencils. Note that

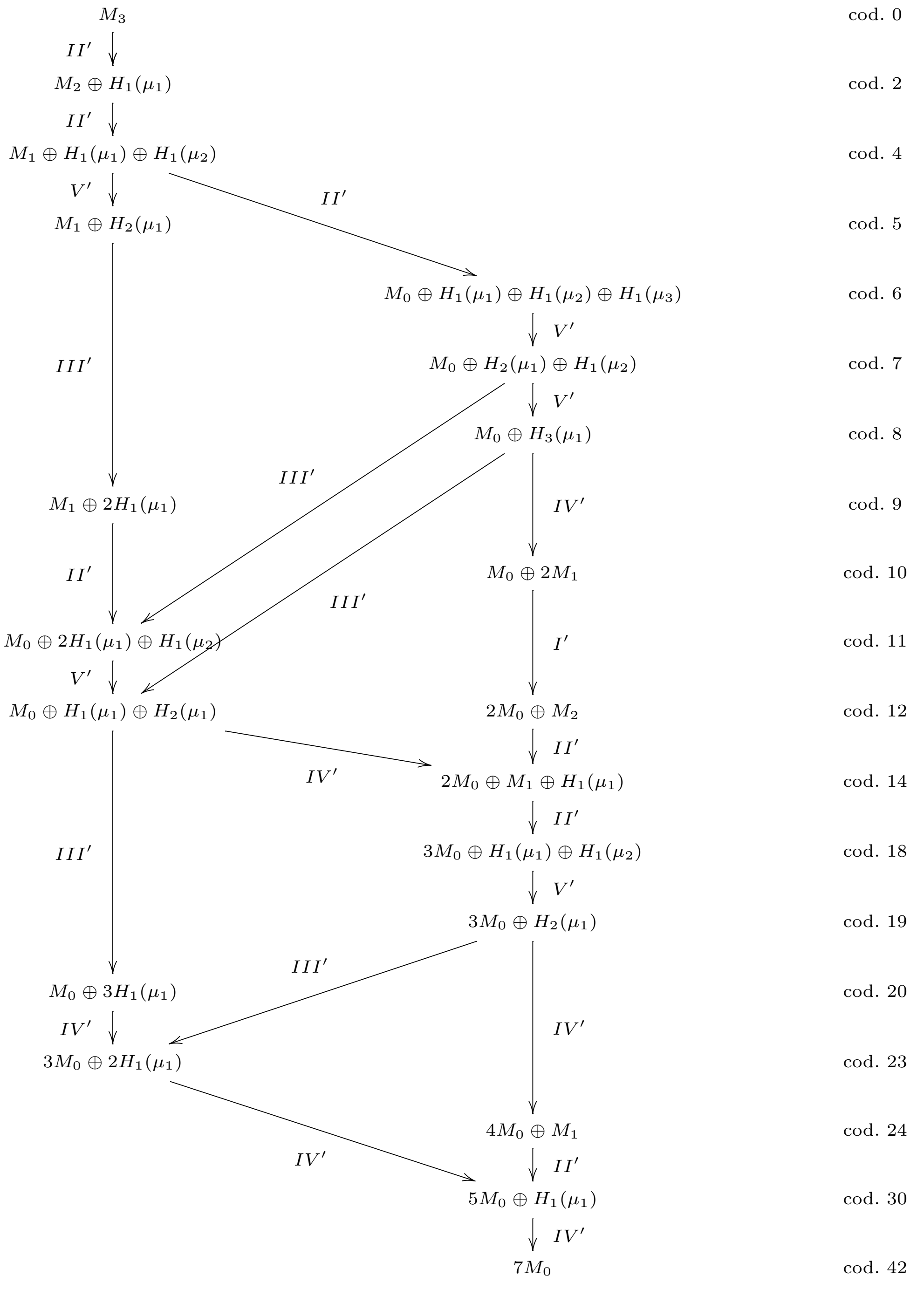

Fig. 3. Bundle stratification for 7×7 skew-symmetric matrix pencils along with the codimension. Rules from Theorem 4.2 used for each structure transition are indicated near the corresponding edge.

codimensions were computed for various cases by different authors, e.g., for matrices and matrix pencils [11], structured matrix pencils [9,18,19], and generalised matrix products [28].

Theorem 5.1 *([23]). The codimension of the congruence orbit of an $n \times n$ complex skew-symmetric matrix pencil P, whose congruence canonical form is the direct sum of blocks from Theorem 2.6, is equal to,*

$$cod\, \mathrm{O}^c(P) = d_H + d_K + d_L + d_{HH} + d_{KK} + d_{LL} + d_{HL} + d_{HK} + d_{KL} - n.$$

The summands in the above identity correspond to

- *the direct summands in Theorem 2.6:*

$$d_H := 3\sum_{i=1}^{a} h_i, \quad d_K := 3\sum_{j=1}^{b} k_j, \quad d_L := c + 2\sum_{r=1}^{c} l_r;$$

- *the pairs of direct summands in Theorem 2.6 of the same type:*

$$d_{HH} := 4\sum_{i<i', \lambda_i=\lambda_{i'}} min(h_i, h_{i'}), \quad d_{KK} := 4\sum_{j<j'} min(k_j, k_{j'}),$$

$$d_{LL} := \sum_{r<r'} (2max(l_r, l_{r'}) + \varepsilon_{rr'}) \quad where\ \varepsilon_{rr'} := \begin{cases} 2 & if\ l_r = l_{r'} \\ 1 & if\ l_r \neq l_{r'} \end{cases};$$

- *the pairs of direct summands in Theorem 2.6 of different types:*

$$d_{HK} := 0, \quad d_{HL} := 2c\sum_{i} h_i, \quad d_{KL} := 2c\sum_{j} k_j,$$

where c is the total number of M-blocks in the canonical form.

Now we develop a few lemmas needed to prove our result. In the next lemma, we present a formula for codimension of the orbit of a skew-symmetric matrix pencil P_2 obtained from a skew-symmetric matrix pencil P_1 using *Rule 3* in Theorem 2.9 such that $|\Lambda(P_2)| = |\Lambda(P_1)| + 1$.

Lemma 5.2. *Let P_1 and P_2 be two $n \times n$ complex skew-symmetric matrix pencils. If P_2 is obtained from P_1 using the structure transition $M_j \rightsquigarrow M_{j-1} \oplus H_1(\mu)$, $j = 1, 2, \dots$ and $\mu \in \overline{\mathbb{C}}$, such that j is the largest index of M-block in P_1, provided $r_j(P_1) = l_j(P_1) = 1$, then $cod\, \mathrm{O}^c(P_2) = cod\, \mathrm{O}^c(P_1) + 2 + r_{j-1}(P_1)$.*

Proof. Let P_1 and P_2 be as in the statement of the lemma. Let

$$cod\, \mathrm{O}^c(P_1) = d_H + d_K + d_L + d_{HH} + d_{KK} + d_{LL} + d_{HL} + d_{HK} + d_{KL} - n, \text{ and}$$

$$cod\, \mathrm{O}^c(P_2) = \tilde{d}_H + \tilde{d}_K + \tilde{d}_L + \tilde{d}_{HH} + \tilde{d}_{KK} + \tilde{d}_{LL} + \tilde{d}_{HL} + \tilde{d}_{HK} + \tilde{d}_{KL} - n.$$

We find the relation between each summand for P_1 and P_2. Without loss of generality, we assume the newly formed block, corresponding to some eigenvalue, be an H-block. Note, after the application of the rule the total number of M-blocks remains the same but the total number of H-blocks increases by 1. The index of the largest M-block reduces by 1 and the new H-block has index 1. Also, recall that the total number of M-blocks is r_0. Therefore, using Theorem 5.1 we get the following equalities

$$\tilde{d}_H = d_H + 3, \quad \tilde{d}_K = d_K, \quad \tilde{d}_L = d_L - 2; \tag{17}$$

$$\tilde{d}_{HK} = d_{HK}, \quad \tilde{d}_{KL} = d_{KL}, \quad \tilde{d}_{HL} = d_{HL} + 2r_0; \tag{18}$$

$$\tilde{d}_{KK} = d_{KK}, \quad \tilde{d}_{HH} = d_{HH}. \tag{19}$$

We consider two cases. The case of $r_{j-1}(P_1) > r_j(P_1)$, i.e., there exist at least one M_{j-1}-blocks in the canonical form of P_1. The other case when $r_{j-1}(P_1) = r_j(P_1) = 1$, i.e., there exist no M_{j-1}-blocks in the canonical form of P_1.

Case 1: Let $d_{LL} = m + p + h + s + q$, where m, p, h, s and q correspond to pairs of blocks $(M_j,\, M_{j-1})$, $(M_j,\, M_z)$, $(M_{j-1},\, M_{j-1})$, $(M_{j-1},\, M_z)$, and the remaining pairs of M-blocks of P_1, respectively, where $z < j-1$. Note, total number of M_{j-1}-blocks and M_z-blocks of P_1 are $r_{j-1}(P_1) - 1$ and $r_0(P_1) - r_{j-1}(P_1)$, respectively. Also, total number of distinct pairs of blocks $(M_{j-1},\, M_{j-1})$ are $\dfrac{(r_{j-1}(P_1) - 1)(r_{j-1}(P_1) - 2)}{2}$. Then, from Theorem 5.1,

$$m = (r_{j-1}(P_1) - 1)(2j + 1), \quad p = (r_0(P_1) - r_{j-1}(P_1))(2j + 1), \tag{20}$$

$$h = (r_{j-1}(P_1) - 1)(r_{j-1}(P_1) - 2)j, \;\; s = (r_0(P_1) - r_{j-1}(P_1))(r_{j-1}(P_1) - 1)(2j - 1). \tag{21}$$

Assume, $\tilde{d}_{LL} = \tilde{h} + \tilde{s} + \tilde{q}$, where $\tilde{h}$, $\tilde{s}$ and $\tilde{q}$ correspond to all pairs of M_{j-1}-blocks, the new (M_{j-1}, M_z), $z < j-1$, and the remaining pairs of M-blocks of P_2, respectively. Note that there exists no M_j-block in P_2. The total number of M_{j-1}-blocks is $r_{j-1}(P_2)$. Thus, total number of distinct pairs $(M_{j-1},\, M_{j-1})$ are $\dfrac{r_{j-1}(P_2)(r_{j-1}(P_2) - 1)}{2}$. Then

$$\widetilde{h} = r_{j-1}(P_2)(r_{j-1}(P_2) - 1)j \text{ and } \widetilde{s} = r_{j-1}(P_2)(r_0(P_2) - r_{j-1}(P_2))(2(j-1) + 1).$$

After application of the rule in the statement of the lemma, the total number of M-blocks and total number of M-blocks with index greater than or equal to $j-1$ remain same, i.e., $r_{j-1}(P_2) = r_{j-1}(P_1)$ and $r_0(P_2) = r_0(P_1)$. Thus, the above identity can be re-written as

$$\widetilde{h} = r_{j-1}(P_1)(r_{j-1}(P_1) - 1)j \text{ and } \widetilde{s} = r_{j-1}(P_1)(r_0(P_1) - r_{j-1}(P_1))(2j - 1). \tag{22}$$

Using expressions from (20)-(22) and definitions of q and $\tilde{q}$, we get,

$$m + h = \tilde{h} + (r_{j-1}(P_1) - 1), \text{ and } p + s = \tilde{s} + 2(r_0(P_1) - r_{j-1}(P_1)) \text{ and } \tilde{q} = q.$$

Therefore,

$$\tilde{d}_{LL} = d_{LL} - (r_{j-1}(P_1) - 1) - 2(r_0(P_1) - r_{j-1}(P_1)). \tag{23}$$

Using the formula for codimension computation, (17)-(19) and (23), we have, $cod\,\mathrm{O}^c(P_2)$ $= d_H + 3 + d_K + d_L - 2 + d_{HH} + d_{KK} + d_{LL} - (r_{j-1}(P_1) - 1) - 2(r_0(P_1) - r_{j-1}(P_1)) +$ $d_{HL} + 2r_0(P_1) + d_{KL} - n$. Expanding the equation, we conclude that

$$cod\,\mathrm{O}^c(P_2) = cod\,\mathrm{O}^c(P_1) + 2 + r_{j-1}(P_1).$$

Case 2: Note, when $r_{j-1}(P_1) = r_j(P_1) = 1$, m h, $\tilde{h}$ and s are absent in the formula for computation of codimension. So, $d_{LL} = m + p + q$, and $\tilde{d}_{LL} = \tilde{h} + \tilde{s} + \tilde{q}$, where the definition of each of the summands remain the same as that of the previous case. Expressions in (20) and (22) remain same. Thus,

$$p = \widetilde{s} + 2(r_0(P_1) - 1), \quad q = \tilde{q}.$$

The rest of the calculations are the same as that of the previous case. □

In the following result, we witness how the codimensions of congruence bundles of two skew-symmetric matrix pencils change, given that one of them is in the closure of the congruence orbit of the other.

Lemma 5.3. *If P_1 and P_2 are two skew-symmetric matrix pencils of the same size such that $P_2 \in \overline{\mathrm{O}^c}(P_1)$, then cod $\mathrm{B}^c(P_1) \leqslant cod\,\mathrm{B}^c(P_2)$. Moreover, cod $\mathrm{B}^c(P_1) <$ cod $\mathrm{B}^c(P_2)$ if and only if $P_2 \notin \mathrm{O}^c(P_1)$.*

Proof. Assume that $P_2 \in \overline{\mathrm{O}^c}(P_1)$. We know, if $X \in \mathrm{O}^c(P_1)$, $\mathrm{B}^c(X) = \mathrm{B}^c(P_1)$ and thus they have the same codimension. So, we prove that for $P_2 \in \overline{\mathrm{O}^c}(P_1)$ the smallest *cod* $\mathrm{B}^c(P_2)$ is attained only if $P_2 \in \mathrm{O}^c(P_1)$. We assume P_2 is such a skew-symmetric matrix pencil in $\overline{\mathrm{O}^c}(P_1)$ that its bundle has the smallest codimension among the codimensions of bundles of all other skew-symmetric matrix pencils in $\overline{\mathrm{O}^c}(P_1)$. Since $P_2 \in \overline{\mathrm{O}^c}(P_1)$, $P_2 \in \mathrm{O}^c(M) \subseteq \overline{\mathrm{O}^c}(P_1)$. We, first, prove our result when $\mathrm{O}^c(M)$ is covered by $\mathrm{O}^c(P_1)$.

Under the assumption of covering relation, from Theorem 2.9, M can be obtained from P_1 by using one of *Rules 1-4*. When $nrank(M) \neq nrank(P_1)$, M is obtained from P_1 using *Rule 4* which reduces the number of H-blocks corresponding to all distinct eigenvalues of M leading to $\Lambda(M) \subseteq \Lambda(P_1)$, i.e., $|\Lambda(M)| \leqslant |\Lambda(P_1)|$. Now, we deal with the situation when $nrank(M) = nrank(P_1)$. In this situation, M can be obtained from P_1 using one of *Rules 1-3*. We try to study separately the difference in the cardinality of spectrum of P_1 and M after application of the rules:

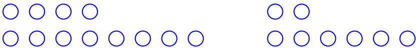

Fig. 4. Integer partitions $\mathcal{Q}^{\mu_1}(P)$ and $\mathcal{Q}^{\mu_2}(P)$ presented in terms of coin-diagram.

Rule 1: It changes the sizes of M-blocks of P_1 and thus, the eigenvalues remain unchanged, i.e., $\Lambda(M) = \Lambda(P_1)$, implying $|\Lambda(M)| = |\Lambda(P_1)|$.
Rule 2: It changes the sizes of H-blocks of P_1 corresponding to the same eigenvalue and thus, the eigenvalues remain unchanged, i.e., $|\Lambda(M)| = |\Lambda(P_1)|$.
Rule 3: This rule reduces the size of an M-block of P_1 and either increases the size of an H-block of P_1 or creates a new H-block of index 1 corresponding to new eigenvalue of P_1. This leads to $|\Lambda(M)| = |\Lambda(P_1)|$ in the former case and $|\Lambda(M)| = |\Lambda(P_1)| + 1$ in the latter case. By our assumption of covering of orbits of P_1 and M, *cod* $\mathrm{O}^c(P_1) <$ *cod* $\mathrm{O}^c(M)$.

From definition of codimension of congruence bundle, if $|\Lambda(M)| \leqslant |\Lambda(P_1)|$,

$$\mathit{cod}\ \mathrm{B}^c(M) = \mathit{cod}\ \mathrm{O}^c(M) - |\Lambda(M)| > \mathit{cod}\ \mathrm{O}^c(P_1) - |\Lambda(P_1)| = \mathit{cod}\ \mathrm{B}^c(P_1)$$

Similarly, if $|\Lambda(M)| = |\Lambda(P_1)| + 1$, using Lemma 5.2 and the definition of codimension of congruence bundles, we get,

$$\mathit{cod}\ \mathrm{B}^c(M) = \mathit{cod}\ \mathrm{O}^c(P_1) + 2 + r_{j-1}(P_1) - |\Lambda(P_1)| - 1 = \mathit{cod}\ \mathrm{B}^c(P_1) + 1 + r_{j-1}(P_1)$$

$$\text{i.e., } \mathit{cod}\ \mathrm{B}^c(M) > \mathit{cod}\ \mathrm{B}^c(P_1).$$

Since $P_2 \in \mathrm{O}^c(M)$, $\mathrm{B}^c(M) = \mathrm{B}^c(P_2)$. This leads to *cod* $\mathrm{B}^c(P_2) >$ *cod* $\mathrm{B}^c(P_1)$ in both the above cases contradicting the minimality of codimension of congruence bundle of P_2. Thus, for $\mathrm{O}^c(P_1)$ that covers $\mathrm{O}^c(P_2)$, we have *cod* $\mathrm{B}^c(P_1) <$ *cod* $\mathrm{B}^c(P_2)$.

Now, if $\mathrm{O}^c(M)$ is not covered by $\mathrm{O}^c(P_1)$, then there exist a finite number of skew-symmetric matrix pencils $Q_1, \ldots, Q_f$ such that $\mathrm{O}^c(Q_1)$ covers $\mathrm{O}^c(M)$, $\mathrm{O}^c(Q_{k+1})$ covers $\mathrm{O}^c(Q_k)$, $k = 1, \ldots, f-1$ and $\mathrm{O}^c(P_1)$ covers $\mathrm{O}^c(Q_f)$. Therefore, for $P_2 \in \overline{\mathrm{O}^c}(P_1)$, we have *cod* $\mathrm{B}^c(P_1) \leqslant$ *cod* $\mathrm{B}^c(P_2)$. □

We recall partial multiplicities of an eigenvalue, the Segre characteristics and how Segre characteristics changes when two eigenvalues coalesc. This will help us to prove our next lemma. The partial multiplicities of a matrix pencil P at eigenvalue λ are the sizes of the Jordan blocks associated with λ in the KCF of P. For each eigenvalue of a matrix pencil P, its partial multiplicities form a non-increasing sequence of non-negative integers. These sequences are known as Segre characteristics. For example, consider a skew-symmetric matrix pencil $P = 2H_2(\mu_1) \oplus 2H_1(\mu_1) \oplus H_2(\mu_2) \oplus 2H_1(\mu_2)$. The Segre characteristics of P corresponding to eigenvalues μ_1 and μ_2 are $\mathcal{Q}^{\mu_1}(P) = (2, 2, 2, 2, 1, 1, 1, 1)$ and $\mathcal{Q}^{\mu_2}(P) = (2, 2, 1, 1, 1, 1)$ respectively. As explained in the paragraph before Theorem 2.10, we obtain coin-diagram corresponding to Segre characteristics by placing m coins in a table with q_i coins in column i. Fig. 4 illustrates $\mathcal{Q}^{\mu_1}(P)$ and $\mathcal{Q}^{\mu_2}(P)$.

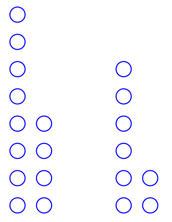

Fig. 5. Integer partitions $\mathcal{J}_{\mu_1}(P)$ and $\mathcal{J}_{\mu_2}(P)$ presented in terms of coin-diagram.

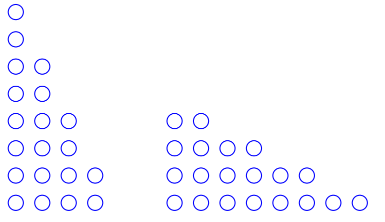

Fig. 6. Integer partition $\mathcal{J}_{\mu_3}(\psi_c^{skew}(P))$ and $\mathcal{Q}^{\mu_3}(\psi_c^{skew}(P))$ presented in terms of coin-diagram, respectively.

We know, rotating the above coin-diagram, corresponding to $\mathcal{Q}^{\mu}(T)$, for an eigenvalue μ of a matrix pencil T, anti-clockwise and flipping it vertically, we obtain Weyr characteristic corresponding to the eigenvalue. From Lemma 3.2, coalescing of eigenvalues is a union of Weyr characteristics corresponding to eigenvalues. Such a union results to a new coin-diagram corresponding to Weyr characteristics such that the heights of the columns of the new diagram are in a non-increasing order. This results in a new coin-diagram corresponding to Segre characteristics. Weyr characteristics corresponding to the eigenvalues μ_1 and μ_2 are $\mathcal{J}_{\mu_1}(P) = (8, 4)$ and $\mathcal{J}_{\mu_2}(P) = (6, 2)$, respectively, shown in Fig. 5.

Assume $\mathcal{J}_{\mu_3}(P)$ is empty, μ_1 and μ_2 of P are coalesced to μ_3 in $\psi_c^{skew}(P)$. Weyr characteristic corresponding to eigenvalue μ_3 is $\mathcal{J}_{\mu_3}(\psi_c^{skew}(P)) = (8, 6, 4, 2)$ is shown in the first diagram in Fig. 6.

Rotating the first coin-diagram in Fig. 6 clockwise and flipping it horizontally, we get the new coin-diagram, shown in the second diagram in Fig. 6 corresponding to $\mathcal{Q}^{\mu_3}(\psi_c^{skew}(P)) = (4, 4, 3, 3, 2, 2, 1, 1)$.

We now develop another lemma which states that congruence orbits of two skew-symmetric matrix pencils, such that one is obtained by coalescing eigenvalues of the other, have the same codimension. This along with the other lemmas will help us in proving the main result of this section.

Lemma 5.4. *For a complex skew-symmetric matrix pencil P, $cod\,\mathrm{O}^c(P) = cod\,\mathrm{O}^c(\psi_c^{skew}(P))$, for $\psi \in \Psi$.*

Proof. Let the distinct finite eigenvalues $\mu_{i_1}, \ldots, \mu_{i_d}$ of P be coalesced to the eigenvalue μ in $\psi_c^{skew}(P)$, i.e., $\psi^{-1}(\mu) = \{\mu_{i_1}, \ldots, \mu_{i_d}\} \cup S$, where $S \cap \Lambda(P) = \phi$. From the formula of codimension computation in Theorem 5.1, the only summands that might get affected by coalescing is d_{HH}. We check the relation of d_{HH} with the heights of the columns of coin-diagram and then analyze the changes in the new coin-diagram after coalescing of eigenvalues of P. Since P is a skew-symmetric matrix pencil and each element of Segre characteristic is a partial multiplicity of P taken in a non-increasing sequence, so $q_j = q_{j+1}$, for odd number j. Thus, from definition of Segre characteristics, $d_{HH} := 4\sum_{i<i',\lambda_i=\lambda_{i'}} min(h_i, h_{i'}) = 4\sum_{1\leqslant m<i}\sum_{\lambda_{2m}=\lambda_{2i}} min(q_{2m}, q_{2i})$, for each eigenvalue. Since Segre characteristic is non-increasing and $1 \leqslant m < i$, $min(q_{2m}, q_{2i})$ is q_{2i}.

Without loss of generality, for an eigenvalue ρ of a skew-symmetric matrix pencil T, assume $\mathcal{Q}^\rho(T) = (q_1^\rho(T), \dots, q_l^\rho(T)) = (q_1^\rho(T), \dots, q_l^\rho(T), 0, 0, \dots)$. We now calculate the summand d_{HH} of the new matrix pencil $\psi_c^{skew}(P)$, for a skew-symmetric matrix pencil P, after the coalescing of a pair of eigenvalues μ_1 and μ_2 to an eigenvalue μ_3, using the expression for heights of columns of Segre characteristics. Note, for each column i in coin-diagram corresponding to Segre characteristic for eigenvalue μ_3, $q_i^{\mu_1}(P) + q_i^{\mu_2}(P) = q_i^{\mu_3}(\psi_c^{skew}(P))$. Thus, $\sum_{1\leqslant m<i} min(q_{2m}^{\mu_3}(\psi_c^{skew}(P)), q_{2i}^{\mu_3}(\psi_c^{skew}(P))) = \sum_{1\leqslant m<i} min(q_{2m}^{\mu_1}(P) + q_{2m}^{\mu_2}(P), q_{2i}^{\mu_1}(P) + q_{2i}^{\mu_2}(P))$. But, for four positive integers u, v, x, y such that $x \geqslant u$ and $y \geqslant v$, $min(x+y, u+v) = u+v = min(x,u) + min(y,v)$. This leads to $\sum_{1\leqslant m<i} min(q_{2m}^{\mu_3}(\psi_c^{skew}(P)), q_{2i}^{\mu_3}(\psi_c^{skew}(P))) = \sum_{1\leqslant m<i} min(q_{2m}^{\mu_2}(P), q_{2i}^{\mu_2}(P)) + \sum_{1\leqslant m<i} min(q_{2m}^{\mu_1}(P), q_{2i}^{\mu_1}(P))$. Since Segre characteristics corresponding to other eigenvalues remain unaffected, $d_{HH}(P) = d_{HH}(\psi_c^{skew}(P))$. If the total number of coalesced eigenvalues is more than 2, then the coalescing can be done using 2 eigenvalues at a time. This completes the proof. □

In the following theorem, we prove that the closure of congruence bundle of a skew-symmetric matrix pencil $\mathrm{B}^c(P)$ is a union of the bundle itself along with the other bundles whose closures are strictly included in $\overline{\mathrm{B}^c}(P)$.

Theorem 5.5. *Let P be a skew-symmetric matrix pencil. Then, there is a finite number of different skew-symmetric matrix pencils, P_i, $i \in \{1, \dots, d\}$ with $P_1 = P$, satisfying $\mathrm{B}(P_i) \neq \mathrm{B}(P_j)$, for $i \neq j$, such that*

$$\overline{\mathrm{B}^c}(P) = \bigcup_{i=1}^{d} \mathrm{B}^c(P_i).$$

Moreover, $\overline{\mathrm{B}^c}(P_i) \subset \overline{\mathrm{B}^c}(P)$ and cod $\mathrm{B}^c(P) <$ cod $\mathrm{B}^c(P_i)$, for $i \neq 1$.

Proof. Since, total number of different bundles of complex skew-symmetric matrix pencils with a fixed size is finite, let these be $\mathrm{B}^c(P_i)$, for skew-symmetric matrix pencils P_i, $i \in \{1, \dots, l\}$. Reordering the pencils, if needed, we may assume, $P_i \in \overline{\mathrm{B}^c}(P)$, $i \in \{1, \dots, d\}$ and $P_{d+1}, \dots, P_l \notin \overline{\mathrm{B}^c}(P)$. By Lemma 3.8, $\bigcup_{i=1}^{d} \mathrm{B}^c(P_i) \subseteq \overline{\mathrm{B}^c}(P)$. Conversely, assume $\tilde{P} \in \overline{\mathrm{B}^c}(P)$. Again by Lemma 3.8, $\mathrm{B}^c(\tilde{P}) \subseteq \overline{\mathrm{B}^c}(P)$, then $\mathrm{B}^c(\tilde{P}) = \mathrm{B}^c(P_i)$, for some $i \in \{1,\dots,d\}$, else $\mathrm{B}^c(\tilde{P}) = \mathrm{B}^c(P_i)$, for some $i \in \{d+1,\dots,l\}$, contradicting its existence in $\overline{\mathrm{B}^c}(P)$. This establishes the reverse inclusion which leads to the equality of $\bigcup_{i=1}^{d} \mathrm{B}^c(P_i)$ and $\overline{\mathrm{B}^c}(P)$.

For $\overline{\mathrm{B}^c}(P_i) \subset \overline{\mathrm{B}^c}(P)$, assume a skew-symmetric matrix pencil $M \in \mathrm{B}^c(P_i)$, for some $i \neq 1$. Then $M \notin \mathrm{B}^c(P_1)$ but $M \in \overline{\mathrm{B}^c}(P_1)$. This justifies the strict inclusion. From characterization of inclusion of bundles in Theorem 3.9, there exists a map $\psi \in \Psi$ such that $P_i \in \overline{\mathrm{O}^c}(\psi_c^{skew}(P))$, $i \neq 1$. Now, we consider two possible cases, namely, ψ being injective and not injective over the set of eigenvalues of P.

Injective: If $P_i \in \mathrm{O}^c(\psi_c^{skew}(P))$, $\mathrm{B}^c(P_i) = \mathrm{B}^c(\psi_c^{skew}(P))$, for $i \neq 1$. But, ψ being injective over the set of eigenvalues of P, $\mathrm{B}^c(\psi_c^{skew}(P)) = \mathrm{B}^c(P)$. Combining both the equalities we get, $\mathrm{B}^c(P_i) = \mathrm{B}^c(P)$, for $i \neq 1$, a contradiction to the above result. Thus, $P_i \notin \mathrm{O}^c(\psi_c^{skew}(P))$ and our required result follows by Lemma 5.3.

Not injective: $cod\, \mathrm{O}^c(P) = cod\, \mathrm{O}^c(\psi_c^{skew}(P))$ from Lemma 5.4. Since ψ is not injective, $|\Lambda(P)| > |\Lambda(\psi_c^{skew}(P))|$ resulting to $cod\ \mathrm{B}^c(P) < cod\ \mathrm{B}^c(\psi_c^{skew}(P))$. From Lemma 5.3, $cod\ \mathrm{B}^c(\psi_c^{skew}(P)) \leqslant cod\ \mathrm{B}^c(P_i)$ and hence the result follows. □

Now, we state the main result of the section.

Theorem 5.6. *For a skew-symmetric complex matrix pencil P, $\mathrm{B}^c(P)$ is open in its closure.*

Proof. Assume $Z \in \mathrm{B}^c(P)$ and $\mathrm{B}^c(P)$ is not an open set in its closure. Then, there exists an open neighbourhood of Z in the closure that is not entirely contained in $\mathrm{B}^c(P)$. So, we assume there exists a sequence of skew-symmetric matrix pencils $\{Z_i\}_{i\in\mathbb{N}}$ converging to Z such that $\{Z_i\}_{i\in\mathbb{N}} \subset \overline{\mathrm{B}^c}(P)$ but $\{Z_i\}_{i\in\mathbb{N}} \not\subset \mathrm{B}^c(P)$. Since $\overline{\mathrm{B}^c}(P) = \bigcup_{i=1}^{d} \mathrm{B}^c(P_i)$, $P_1 = P$, we consider a subsequence (if needed) of $\{Z_i\}_{i\in\mathbb{N}}$, $\{Z_{i_k}\} \subset \mathrm{B}^c(P_k)$, for some $k \in \{2,\dots,d\}$, say $k = 2$. But by convergence of $\{Z_{i_k}\}$, $Z \in \overline{\mathrm{B}^c}(P_2)$. Thus, $\mathrm{B}^c(P) \subseteq \overline{\mathrm{B}^c}(P_2)$. By definition of closures, $\overline{\mathrm{B}^c}(P) \subseteq \overline{\mathrm{B}^c}(P_2)$. This contradicts the strict inclusion of $\overline{\mathrm{B}^c}(P_2)$ in $\overline{\mathrm{B}^c}(P)$ in Lemma 5.5. Thus, $\mathrm{B}^c(P)$ is open in its closure. □

6. Conclusion

In this paper, we have presented a characterization for closures of bundles of skew-symmetric pencils. We have also presented a necessary and sufficient condition for one bundle of a skew-symmetric matrix pencil to cover the bundle of another skew-symmetric matrix pencil. We conclude our paper by proving that bundles are open in their closures. Extending these results to skew-symmetric matrix polynomials and implementing all the results in the Stratigraph software [37] remain open problems.

Declaration of generative AI and AI-assisted technologies in the writing process

Generative AI was used only in the writing process for editing to improve the readability of the paper.

Declaration of competing interest

The authors declare there are no competing interests.

Data availability

No data was used for the research described in the article.

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