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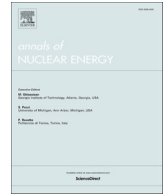
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The role of diffusion theory to describe multiplicity moments

M.M.R. Williams^a, Imre Pázsit^{b,*}

^a Mechanical Engineering Department, Nuclear Engineering Group, Imperial College of Science, Technology and Medicine, Exhibition Road, SW7 2AZ, UK

^b Chalmers University of Technology, Division of Subatomic, High Energy and Plasma Physics, SE-412 96 Göteborg, Sweden

ABSTRACT

The factorial moments of the neutrons emitted from a multiplying item due to one internal source event (also called multiplicity moments), are traditionally calculated in the so-called point model. Recently, interest arose in the calculation of these moments in a space-dependent model, using one-speed transport theory (Pázsit and Pál, 2021). In this work, a comparison of work in the space dependent model on multiplet evaluation (Pázsit et al., 2023, Williams, 2025) has been made using diffusion theory. The method of solution uses the scalar form of the transport equation reduced to diffusion theory form to calculate the singlet, doublet and triplet moments. It is well-known that diffusion theory is an approximation to transport theory but in general is much simpler to use. For the sake of completeness, we compare the results from transport theory with those of the corresponding diffusion theory solutions. As expected, the diffusion results differ from those of transport theory and we find that they are in significant error as the body radius increases; an explanation is advanced for this defect. At the same time, for large item sizes, the diffusion results still represent an improvement over the point model.

1. Introduction

The theory of multiplicity is used in nuclear safeguards to quantify the parameters of an item containing an unknown number of transuranic elements, such as Pu-239 and Pu-240 (Ensslin et al, 1998). The method is non-intrusive because it is based on the detection of neutrons emitted from the item in single, double and triple coincidences. These coincidences are induced due to an intrinsic source of spontaneous fissions from the even-even nuclei (e.g. Pu-240), multiplied in short induced fission chains before emission by the even-odd nuclei (such as Pu-239).

In order to unfold the source and internal multiplication properties of the item from the measured coincidence rates (called singles, doubles and triples), expressions and/or quantitative values of the factorial moments (the multiplicity moments) are needed.

Traditionally, these moments were calculated in the so-called point model, which ignores the detailed spatial variation of the multiplicities but uses a form of escape probability to account for neutron transfer out of the system (Böhnel, 1985; Pázsit and Pál, 2008). Recently, the theory was extended to account for the spatial transport of neutrons (Pázsit and Pál, 2021). These equations use the one speed form of the Pál-Bell equations and employ transport theory. Quantitative results were obtained for spheres (Pázsit and Pál, 2021) and finite cylinders (Pázsit and Dykin, 2022). In these first works, similar to the point model, only fission reactions were accounted for. Pázsit et al (2023) extended the theory by accounting for scattering and considering both solid spheres and spheres with a central cavity.

In all the above works, quantitative results were obtained by a collision number expansion (CN) method, i.e. a Neumann-series solution of backward type integral equations. Later Williams (2025) cast the equations into integro-differential form which enabled standard multi-group transport theory codes to be used for more general geometries and energy structure. Also, other quantitative methods of the one-speed forward transport theory of the integro-differential form were used to obtain accurate numerical solutions (Barichello and Pázsit, 2025). In all the works so far, energy dependence and the anisotropy of the scattering were neglected, which we will also follow in this paper. These approximations restrict the application of the results to pure metallic samples.

Diffusion theory is a simplification of transport theory, which is still capable of accounting for spatial neutron transport in a fissile and scattering medium. The objective of this paper is to derive the corresponding diffusion theory equations and compare the results with those obtained by the CN method and by Williams (2025).

2. General theory

To connect the present work with our previous publications, some clarifications of the terminology used are helpful. Because the master equations for the probability distributions, from which the moment equations are derived, are of the backward type, the derivation goes in two steps. In the first, the equations for the first three factorial moments of the number of neutrons emitted from the item due to a single source neutron with coordinates (r, μ) are derived. These factorial moments are

* Corresponding author.

E-mail addresses: mmrw@btinternet.com (M.M.R. Williams), imre@chalmers.se (I. Pázsit).

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denoted as $n(r,\mu)$, $m(r,\mu)$ and $w(r,\mu)$, respectively, and can be determined by solving the corresponding transport equation. To simplify the terminology in our previous paper (Williams 2025), these moments were re-named as singlets, doublets and triplets, or collectively, multiplet moments. In the second step, a formula is derived for the first three factorial moments of the total number of neutrons emitted from the sample due to one source event, usually a spontaneous fission, emitting a random number of neutrons with an isotropic angular distribution, the source being either uniformly distributed in the item, or concentrated in one point (usually the centre of the spherical item). These are denoted as N , M and W , and they can be calculated from the multiplet moments by simple quadrature. For simplicity, these were referred to as “total” singlets, doublets and triplets, and collectively as “total multiplet moments” by Williams (2025).¹

In this paper we will use the notation of Williams (2025). The derivation of the equations for the multiplets and the total multiplets in the form of backward type integral equations can be found in several of our previous publications hence we start directly with the multiplet equations. Following the procedure developed in Williams (2025) of converting them to an integro-differential form, we find the following transport equations for the singlet, doublet and triplet moments. It is convenient to consider the one-speed form of the transport equations for the multiplets, namely.

$$-\Omega \cdot \nabla n(\mathbf{r}, \Omega) + (\Sigma_c(\mathbf{r}) + \Sigma_f(\mathbf{r}) + \Sigma_{str}(\mathbf{r}))n(\mathbf{r}, \Omega) = (\Sigma_{str}(\mathbf{r}) + \nu_{r,1}\Sigma_f(\mathbf{r}))\tilde{n}_0(\mathbf{r}) \quad (1)$$

$$-\Omega \cdot \nabla m(\mathbf{r}, \Omega) + (\Sigma_c(\mathbf{r}) + \Sigma_f(\mathbf{r}) + \Sigma_{str}(\mathbf{r}))m(\mathbf{r}, \Omega) = (\Sigma_{str}(\mathbf{r}) + \nu_{r,1}\Sigma_f(\mathbf{r}))\tilde{m}_0(\mathbf{r}) + \Sigma_f(\mathbf{r})\nu_{r,2}\tilde{n}_0^2(\mathbf{r}) \quad (2)$$

$$-\Omega \cdot \nabla w(\mathbf{r}, \Omega) + (\Sigma_c(\mathbf{r}) + \Sigma_f(\mathbf{r}) + \Sigma_{str}(\mathbf{r}))w(\mathbf{r}, \Omega) = (\Sigma_{str}(\mathbf{r}) + \nu_{r,1}\Sigma_f(\mathbf{r}))\tilde{w}_0(\mathbf{r}) + \Sigma_f(\mathbf{r})\left(\nu_{r,3}\tilde{n}_0^3(\mathbf{r}) + 3\nu_{r,2}\tilde{n}_0(\mathbf{r})\tilde{m}_0(\mathbf{r})\right) \quad (3)$$

with the boundary conditions

$$\begin{aligned} n(\mathbf{r}_B, \Omega) &= 1 \text{ for } \mathbf{n} \cdot \Omega > 0 \\ m(\mathbf{r}_B, \Omega) &= 0 \text{ for } \mathbf{n} \cdot \Omega > 0 \\ w(\mathbf{r}_B, \Omega) &= 0 \text{ for } \mathbf{n} \cdot \Omega > 0 \end{aligned} \quad (4a,b,c)$$

Here, the $\nu_{r,i}$ are the factorial moments of the number of neutrons generated in a reaction, and $\tilde{n}_0(\mathbf{r})$, etc. are the angularly averaged (“scalar”) multiplets. We have designated the transport cross section by Σ_{str} .

Converting the above to spherical symmetry we find

$$-\mu \frac{\partial}{\partial r} n(r, \mu) - \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu} n(r, \mu) + \Sigma(r)n(r, \mu) = (\Sigma_{str}(r) + \nu_{r,1}\Sigma_f(r))\tilde{n}_0(r) \quad (5)$$

with $\Sigma(r) = \Sigma_c(r) + \Sigma_f(r) + \Sigma_{str}(r)$

$$-\mu \frac{\partial}{\partial r} m(r, \mu) - \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu} m(r, \mu) + \Sigma(r)m(r, \mu) = (\Sigma_{str}(r) + \nu_{r,1}\Sigma_f(r))\tilde{m}_0(r) + \Sigma_f(r)\nu_{r,2}\tilde{n}_0^2(r) \quad (6)$$

$$-\mu \frac{\partial}{\partial r} w(r, \mu) - \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu} w(r, \mu) + \Sigma(r)w(r, \mu) = (\Sigma_{str}(r) + \nu_{r,1}\Sigma_f(r))\tilde{w}_0(r) + \Sigma_f(r)\left(\nu_{r,3}\tilde{n}_0^3(r) + 3\nu_{r,2}\tilde{n}_0(r)\tilde{m}_0(r)\right) \quad (7)$$

The boundary conditions are now

¹ These should not be confused with the standard safeguards terminology of “singles, doubles and triples”, which are the detection rates of one, two and three neutrons in coincidence.

$$\begin{aligned} n(a, \mu) &= 1 \text{ for } \mu > 0 \\ m(a, \mu) &= 0 \text{ for } \mu > 0 \\ w(a, \mu) &= 0 \text{ for } \mu > 0 \end{aligned} \quad (8a,b,c)$$

It should also be noted that

$$\begin{aligned} \tilde{n}_0(r) &= \frac{1}{2}n_0(r) = \frac{1}{2}\int_{-1}^1 n(r, \mu)d\mu, \quad \tilde{m}_0(r) = \frac{1}{2}m_0(r) = \frac{1}{2}\int_{-1}^1 m(r, \mu)d\mu \\ \tilde{w}_0(r) &= \frac{1}{2}w_0(r) = \frac{1}{2}\int_{-1}^1 w(r, \mu)d\mu \end{aligned}$$

which is the nomenclature used in the diffusion equations below.

3. Diffusion theory

We may reduce the above transport equations to diffusion theory in the usual manner by expanding the angular dependence in spherical harmonics. This leads to the following equations for the multiplets, viz:

For the singlet

$$\nabla \cdot D(\mathbf{r})\nabla n_0(\mathbf{r}) + (\nu_{r,1}\Sigma_f(\mathbf{r}) - \Sigma_a(\mathbf{r}))n_0(\mathbf{r}) = 0 \quad (9)$$

subject to the boundary condition at \mathbf{r}_s

$$n_0(\mathbf{r}_s) + 2D(\mathbf{r}_s)\mathbf{n} \cdot \nabla n_0(\mathbf{r}_s) = 2 \quad (10)$$

For the doublet

$$\nabla \cdot D(\mathbf{r})\nabla m_0(\mathbf{r}) + (\nu_{r,1}\Sigma_f(\mathbf{r}) - \Sigma_a(\mathbf{r}))m_0(\mathbf{r}) + Q_2(\mathbf{r}) = 0 \quad (11)$$

subject to the boundary condition

$$m_0(\mathbf{r}_s) + 2D(\mathbf{r}_s)\mathbf{n} \cdot \nabla m_0(\mathbf{r}_s) = 0 \quad (12)$$

For the triplet

$$\nabla \cdot D(\mathbf{r})\nabla w_0(\mathbf{r}) + (\nu_{r,1}\Sigma_f(\mathbf{r}) - \Sigma_a(\mathbf{r}))w_0(\mathbf{r}) + Q_3(\mathbf{r}) = 0$$

subject to the boundary condition

$$w_0(\mathbf{r}_s) + 2D(\mathbf{r}_s)\mathbf{n} \cdot \nabla w_0(\mathbf{r}_s) = 0 \quad (13)$$

The source terms Q_2 and Q_3 are given by

$$Q_2(\mathbf{r}) = \frac{1}{2}\Sigma_f(\mathbf{r})\nu_{r,2}n_0^2(\mathbf{r}) \text{ and } Q_3(\mathbf{r}) = \frac{1}{4}\Sigma_f(\mathbf{r})(\nu_{r,3}n_0^3(\mathbf{r}) + 6\nu_{r,2}n_0(\mathbf{r})m_0(\mathbf{r})) \quad (14)$$

We recall that the diffusion coefficient is given by $D = 1/3\Sigma_{str}$.

It is our intention to use as an example the uniform solid sphere of radius a . Thus the above equations simplify to (with distance in units of total mean free path) for the singlet

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dn_0(r)}{dr} \right) + \alpha^2 n_0(r) = 0 \quad (15)$$

with boundary condition $n_0(a) + \lambda n_0'(a) = 2$ and $n_0'(0) = 0$

The symbol α is defined through²

$$c = (\nu_{r,1}\Sigma_f + \Sigma_s)/\Sigma, \quad (\nu_{r,1}\Sigma_f - \Sigma_a)/\Sigma = c - 1 \text{ and } \alpha^2 = 3(c - 1) \quad (16)$$

and the symbol λ is equal to $2/3$ or, to include a transport correction, equal to 0.7104 (but there is no significant change in using this).

For the doublet we have

² Not to be confused with the alpha factor of nuclear safeguards, which is zero in the cases considered in this paper and hence does not appear.

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dm_0(r)}{dr} \right) + \alpha^2 m_0(r) + \frac{3c_f}{2} \nu_{r2} n_0^2(r) = 0$$

with boundary conditions $m_0'(0) = 0$, $m_0(a) + \lambda m_0'(a) = 0$ and $c_f = \Sigma_f / \Sigma$

(17a)

and finally for the triplet equation

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dw_0(r)}{dr} \right) + \alpha^2 w_0(r) + \frac{3c_f}{4} (\nu_{r3} n_0^3(r) + 6\nu_{r2} n_0(r) m_0(r)) = 0$$

with boundary conditions $w_0'(0) = 0$ and $w_0(a) + \lambda w_0'(a) = 0$

(17b)

The above equations can be simplified if we consider the general equation

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi(r)}{dr} \right) + \alpha^2 \phi(r) + Q(r) = 0$$

(18a)

with boundary condition $\phi(a) + \lambda \phi'(a) = 0$ and $\phi'(0) = 0$

Then setting $\phi(r) = \psi(r)/r$ leads to

$$\frac{d^2\psi(r)}{dr^2} + \alpha^2 \psi(r) + rQ(r) = 0$$

with boundary conditions $(a - \lambda)\psi(a) + \lambda a\psi'(a) = 2pa^2$ and $\psi(0) = 0$

(18b)

For Eqns. (17) and (18) $p = 0$ and for Eqn. (15) $p = 1$.

The solution of the equations for $m_0(r)$ and $w_0(r)$ may be written as (Polyanin and Zaitsev, 1995)

$$\begin{aligned} \phi(r) = & \frac{\cos(\alpha r)}{\alpha r} \int_0^r dr' r' Q(r') \sin(\alpha r') + \frac{\sin(\alpha r)}{\alpha r} \int_r^a dr' r' Q(r') \cos(\alpha r') \\ & + \frac{a\lambda \sin(\alpha a) - (a - \lambda)\cos(\alpha a)}{a\lambda \cos(\alpha a) + (a - \lambda)\sin(\alpha a)} \frac{\sin(\alpha r)}{\alpha r} \int_0^a dr' r' Q(r') \sin(\alpha r') \end{aligned} \quad (19a)$$

When $\phi = m_0$ then $Q = Q_2$ and when $\phi = w_0$ then $Q = Q_3$. The terms Q_2 and Q_3 are defined in Eqn. (14).

It is worth noting that if $c < 1$, then $\alpha = i\nu$ and (19a) becomes

$$\begin{aligned} \phi(r) = & \frac{\cosh(\nu r)}{\nu r} \int_0^r dr' r' Q(r') \sinh(\nu r') + \frac{\sinh(\nu r)}{\nu r} \int_r^a dr' r' Q(r') \cosh(\nu r') \\ & - \frac{a\lambda \sinh(\nu a) + (a - \lambda)\cosh(\nu a)}{a\lambda \cosh(\nu a) + (a - \lambda)\sinh(\nu a)} \frac{\sinh(\nu r)}{\nu r} \int_0^a dr' r' Q(r') \sinh(\nu r') \end{aligned} \quad (19b)$$

The solution for the singlet $n_0(r)$ subject to the further condition that $n_0'(0) = 0$ is

$$n_0(r) = A_0 \frac{\sin(\alpha r)}{r} \quad \text{with } A_0 = \frac{2a^2}{\lambda a \cos(\alpha a) + (a - \lambda)\sin(\alpha a)} \quad (20)$$

The total number of singlets, doublets and triplets for the uniformly distributed isotropic source are obtained through the equations (Williams, 2025)

$$N = \frac{3\nu_{s1}}{2a^3} \int_0^a dr r^2 n_0(r) = \frac{3\nu_{s1}}{a\alpha^2} \frac{\sin(\alpha a) - a\alpha \cos(\alpha a)}{a\lambda \cos(\alpha a) + (a - \lambda)\sin(\alpha a)} \quad (21)$$

$$M = \frac{3}{4a^3} \int_0^a dr r^2 (\nu_{s2} n_0^2(r) + 2\nu_{s1} m_0(r)) \quad (22)$$

$$W = \frac{3}{8a^3} \int_0^a dr r^2 (\nu_{s3} n_0^3(r) + 6\nu_{s2} n_0(r) m_0(r) + 4\nu_{s1} w_0(r)) \quad (23)$$

It is noted in each of the expressions for N , M and W that a term of the form $\int_0^a dr r^2 \phi(r)$ arises. This may be evaluated directly from Eqn. (19a) by reversing the order of integration over (r, r') and leads to

Table 1

First moments (total singlets N) for various values of the radius a of the sphere with both a point source and a uniformly distributed Source. CN = collision number expansion of transport theory.

a	Diffusion point source	Transport point source	N (CN) Uniform source	N(100) Transport uniform source	Diffusion uniform source
1.40	4.26	4.46	4.25125	4.25128	4.22
2.80	5.18	5.68	5.02995	5.03017	4.88
4.20	6.92	8.05	6.40794	6.40903	6.03
5.60	10.68	13.80	9.47754	9.48153	8.35
7.00	22.60	39.68	22.3943	22.4310	15.25

Table 2

Second moments (total doublets M) for various values of the radius a of the sphere with both a point source and a uniformly distributed source. CN = collision number expansion of transport theory.

a	Diffusion point source	Transport point source	M(CN) Uniform source	M(100) Transport uniform source	Diffusion uniform source
1.40	17.54	20.04	17.5213	17.5218	16.8
2.80	31.16	40.40	29.6169	29.6201	26.7
4.20	70.04	106.1	63.8354	63.8648	51.7
5.60	227.4	448.6	225.078	225.351	143
7.00	1707.2	7723.7	3531.89	3549.60	948

$$\int_0^a dr r^2 \phi(r) = \frac{1}{\alpha^2} \int_0^a dr r Q(r) \left[\frac{a^2 \sin(\alpha r)}{a\lambda \cos(\alpha a) + (a - \lambda)\sin(\alpha a)} - r \right] \quad (24)$$

When $\phi = m_0$ then $Q = Q_2$ and when $\phi = w_0$ then $Q = Q_3$.

In the case of the central point source the above equations become

$$N = \nu_{s1} n_0(0)/2 \quad (25)$$

$$M = \nu_{s1} m_0(0)/2 + \nu_{s2} n_0^2(0)/4 \quad (26)$$

$$W = \nu_{s1} w_0(0)/2 + \nu_{s3} n_0^3(0)/8 + 3\nu_{s2} n_0(0) m_0(0)/4 \quad (27)$$

These point source multiplets can be evaluated using Eqn. (19) for $r = 0$, viz:

$$\phi(0) = \int_0^a dr' r' Q(r') \left(\cos(\alpha r') + \frac{a\lambda \sin(\alpha a) - (a - \lambda)\cos(\alpha a)}{a\lambda \cos(\alpha a) + (a - \lambda)\sin(\alpha a)} \sin(\alpha r') \right) \quad (28)$$

For $c < 1$ and $\alpha = i\nu$ the appropriate transformations are used as in employed eq (19b).

Using Eqns. (24) and (28), some of the terms in (25)-(27) and (21)-(23) can be evaluated analytically.

4. Numerical study and discussion

The calculations we present are those which attempt to reproduce the results already given by Pázsit et al (2023) but now using diffusion theory.

Data used for the fissile components are

$$\begin{aligned} \nu_{r,1} &= 2.637, \quad \nu_{r,2} = 5.623, \quad \nu_{r,3} = 9.476 \\ \nu_{s,1} &= 3.757, \quad \nu_{s,2} = 11.962, \quad \nu_{s,3} = 31.812 \end{aligned}$$

$$\Sigma_f = 0.0621125 \text{ cm}^{-1}$$

Let β_s be the fractional contribution of scattering to the whole and thus we may write $\beta_s = \Sigma_s / (\Sigma_s + \Sigma_c + \Sigma_f)$ and also $\Sigma_s + \Sigma_c = \Sigma_f \beta_s / (1 - \beta_s)$.

To establish the accuracy of our method we consider the problem

Table 3

Third moments (total triplets W) for various values of the radius a of the sphere with both a point source and a uniformly distributed source. CN = collision number expansion of transport theory.

a	Diffusion point source	Transport point source	W(CN) Uniform source	W(100) Transport uniform source	Diffusion uniform source
1.40	76.8	101.5	77.7985	77.8022	56.5
2.80	247.9	404.5	242.619	242.666	186
4.20	1119.0	2329.8	1146.64	1147.60	920
5.60	8925	28,305	12384.8	12411.23	6596
7.00	284,880	3,578,575	1,574,870	1,588,295	130,797

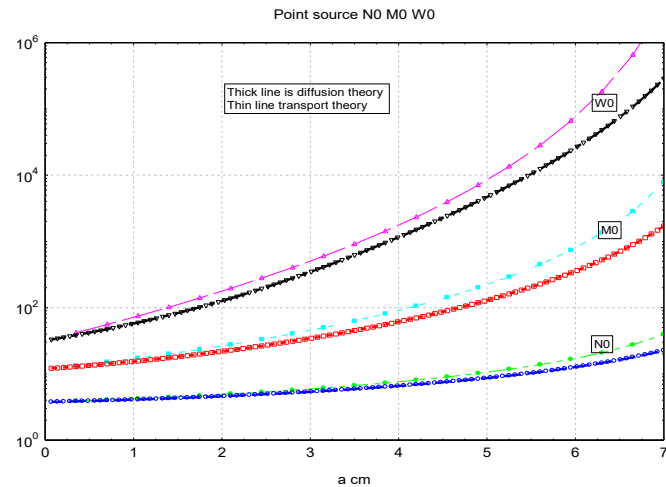


Fig. 1. Comparison of the total multiplets for the case of a point source in the centre of the sphere, calculated with diffusion theory (thick lines) and transport theory (thin lines).

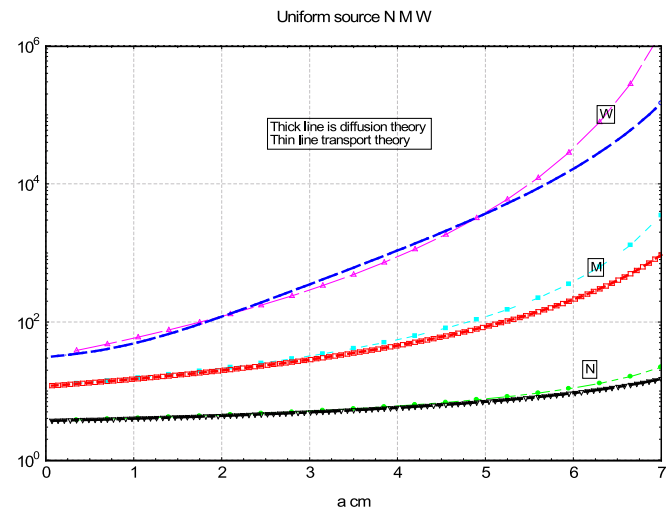


Fig. 2. Comparison of the total multiplets for the case of a uniformly distributed source, calculated with diffusion theory (thick lines) and transport theory (thin lines).

shown in Fig. 4 of (Pázsit et al., 2023). The radius is in the range up to 7 cm and uses the value of $\Sigma_f = 0.0621125 \text{ cm}^{-1}$ and $\Sigma_c = 0$ with scattering entering via $\Sigma_s = \Sigma_f \beta_s / (1 - \beta_s)$. The results are shown in Tables 1 and 3 for $\beta_s = 0, 0.25, 0.5, 0.75$ which compare our diffusion theory values with those of (Pázsit et al., 2023) and Williams (2025) for $\beta_s = 0.75$. We note that the diffusion theory results deteriorate rapidly as the

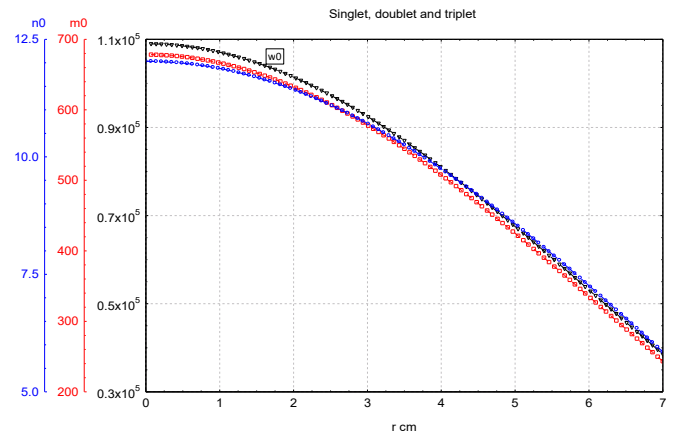


Fig. 3. Dependence of the multiplets in a sphere with given radius, as a function of the radial position of the single source neutron with isotropic angular distribution.

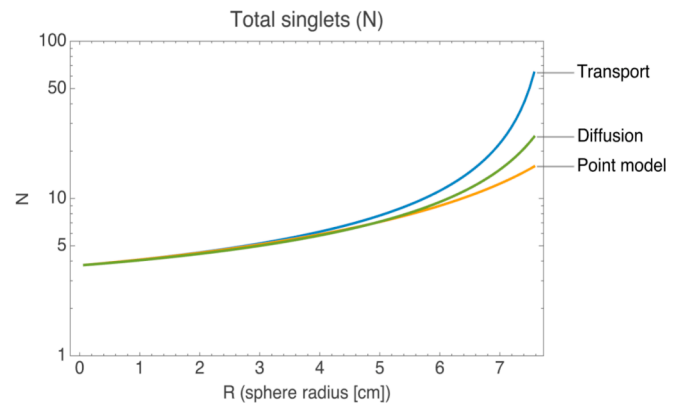


Fig. 4. Comparison of the total singlets as calculated by transport theory, diffusion theory and the traditional point model.

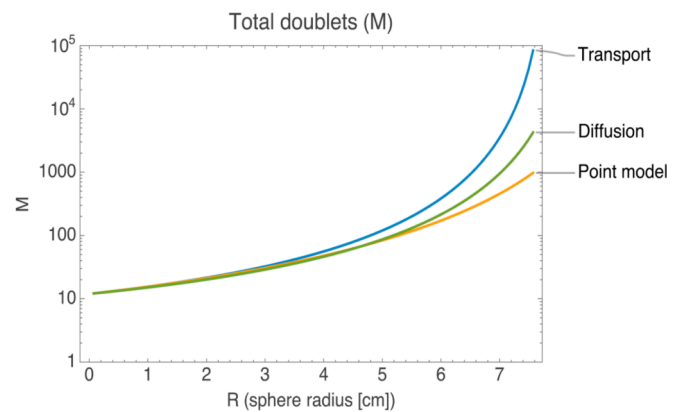


Fig. 5. Comparison of the total doublets as calculated by transport theory, diffusion theory and the traditional point model.

sphere radius increases, a behaviour that we discuss further below.

The Figs. 1 and 2 show uniform source and point source multiplets for $\beta_s = 0.75$. As expected the point source values are greater than those for a uniform source. Fig. 3 shows how the multiplets vary spatially across the system.

The results for a central point source are similarly inaccurate when compared with transport theory. On the other hand, for the uniform source, the diffusion theory results are better than those of the

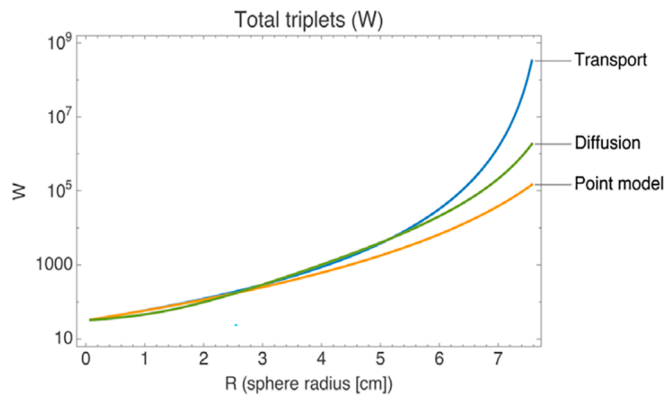


Fig. 6. Comparison of the total triplets as calculated by transport theory, diffusion theory and the traditional point model.

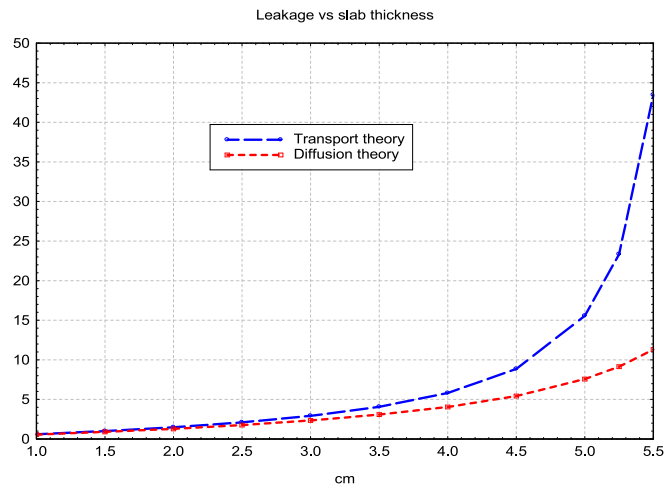


Fig. 7. Comparison of the leakage current in a slab as a function of the slab thickness, calculated by transport theory and diffusion theory.

traditional point model for large item sizes which are illustrated on Figs. 4–6.

We explain this behaviour by noting that the multiplets are a measure of the number of neutrons that leak out, i.e. a surface current and its fluctuations. The larger the size of the sphere the greater the distance the neutrons have to travel before leakage and the increase in, or accumulation of, the errors arising from diffusion theory. This is yet another example of the need to understand that the adjoint equation is more important in leakage than the normal equation.

As a check on the calculations we have also solved the set of differential equations (15), (17) and (18) using Chebyshev-Gauss-Lobatto collocation and by *Mathematica*. All results are identical to four significant figures.

5. Leakage current from surface of slab: transport theory vs diffusion theory

In Fig. 7 we have used an integral transport theory code (Williams, 2026) to calculate the leakage current from a slab of thickness a . This compares with the diffusion theory value of

$$L = \frac{6\alpha\lambda S_0 \sin(\alpha a/2)}{\alpha^2 (\cos(\alpha a/2) - \alpha\lambda \sin(\alpha a/2))} \quad (\lambda = 2/3, \alpha^2 = 3(c-1))$$

It is evident that the difference between the transport and diffusion values increases as the slab thickness increases. As the multiplets are related to the leakage current, we may infer that similar errors are

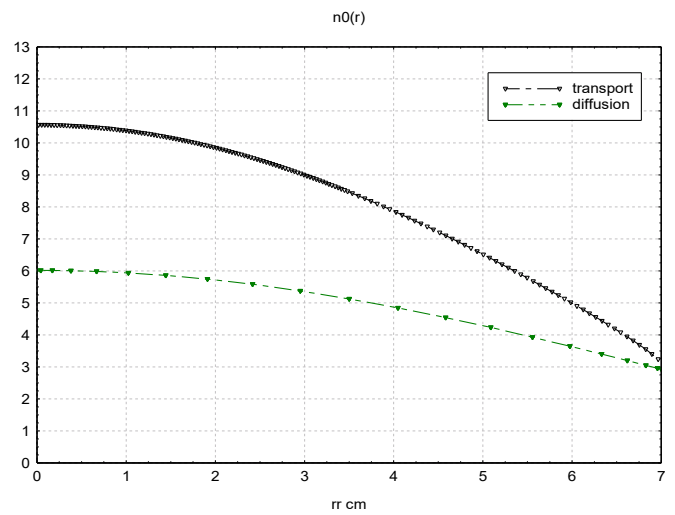


Fig. 8. Comparison between diffusion and transport theory of the dependence of the singlets on the starting position of the neutron for a sphere with outer radius $R = 7$ cm.

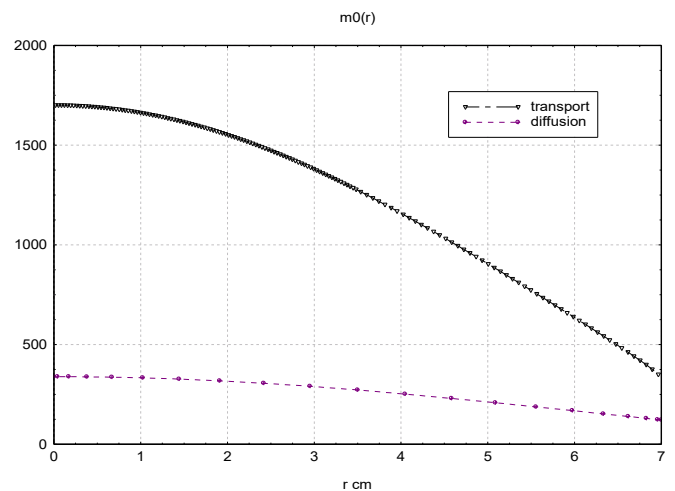


Fig. 9. Comparison between diffusion and transport theory of the dependence of the doublets on the starting position of the neutron for a sphere with outer radius $R = 7$ cm.

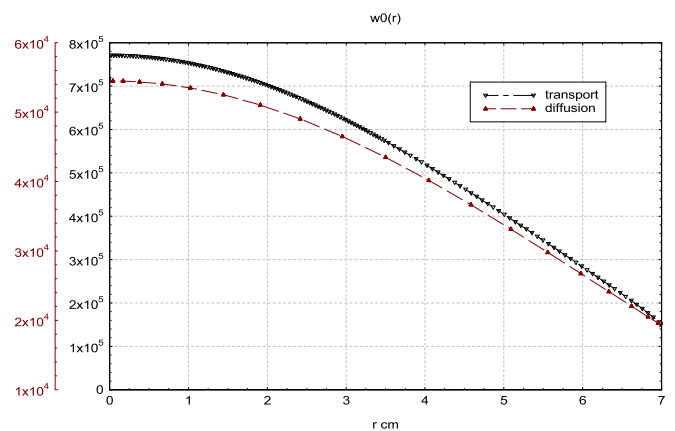


Fig. 10. Comparison between diffusion and transport theory of the dependence of the triplets on the starting position of the neutron for a sphere with outer radius $R = 7$ cm.

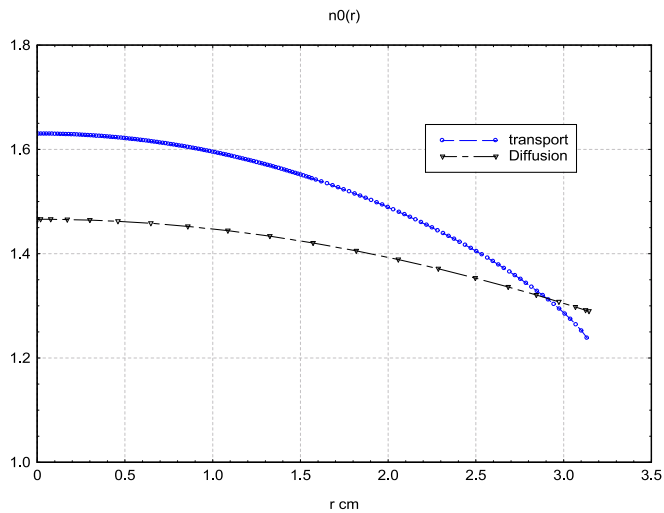


Fig. 11. Comparison between diffusion and transport theory of the dependence of the singlets on the starting position of the neutron for a sphere with outer radius $R = 3.134$ cm.

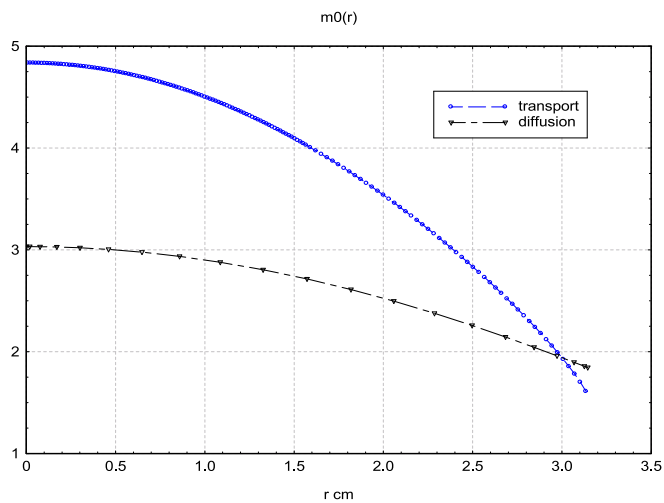


Fig. 12. Comparison between diffusion and transport theory of the dependence of the doublets on the starting position of the neutron for a sphere with outer radius $R = 3.134$ cm.

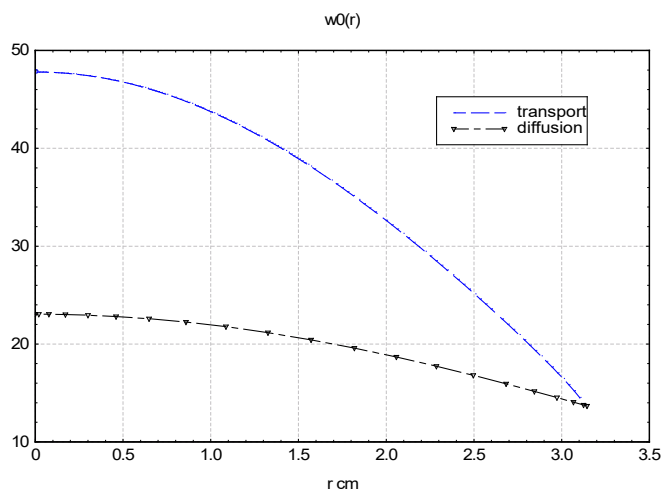


Fig. 13. Comparison between diffusion and transport theory of the dependence of the triplets on the starting position of the neutron for a sphere with outer radius $R = 3.134$ cm.

present in them.

The critical slab thickness in diffusion theory is given by $a_c = \frac{2}{\alpha} \tan^{-1} \left(\frac{1}{\alpha l} \right)$ which for the data above with $\beta_s = 0.75$ leads to $a_c = 6.790$ cm. The critical thickness from transport theory is 5.814 cm. Similar differences arise for spheres.

We have Figs. 8–10 for the multiplets with outer radius 7 cm, Figs. 11–13 for outer radius 3.134 cm. It is clear that the error for the large radius is much greater than that for the smaller one. This is consistent with Fig. 7. We also note that the values in Fig. 3 for the multiplicities are twice those in Figs. 8–10 because of the notation $\tilde{n}_0(r) = n_0(r)/2$. In our transport theory case we have used $\tilde{n}_0(r)$ whereas in the diffusion case it is $n_0(r)$.

6. Summary and conclusions

Diffusion theory was used to calculate the multiplicity moments of nuclear safeguards, i.e. the factorial moments of the number of particles emitted from a multiplying system, due to a single source event (also called “total multiplets” in this paper). The original master equations for the full probability distribution are of backward type integral equations, but the equations for the multiplets can be converted into an integro-differential form (Williams, 2025). This integro-differential form is suitable for the derivation of a diffusion theory approximation. The diffusion theory results were evaluated quantitatively for spheres of various sizes and were compared with the exact transport theory solutions, obtained in previous work. It was found that the diffusion theory solution was in substantial error as compared to the exact transport theory results, although they were better than the results from the point model.

The error of the diffusion theory results increases with increasing size of the spherical item, which is contrary to expectations based on traditional diffusion theory of reactor physics, which breaks down for small systems, but works reasonably well for large systems. An explanation was offered by noting that the statistics of the neutrons emitted from the item is related to the surface current, and by demonstrating (for slabs) that the error of diffusion theory in calculating the surface current increases with increasing system size.

One can also put it in another way. The common wisdom of diffusion theory being applicable in large systems arises from the traditional forward formalism of neutron transport. The problem of neutron emission statistics considered in this paper is based on the backward (adjoint) formalism, to which the intuitive concepts developed by forward thinking are not transferable.

The quantitative work shows that the diffusion theory results, although showing an improvement as compared to those of the point model, are inferior to those of exact transport theory, especially for large items, which are the concern of nuclear safeguards. Given that transport solvers are available for fast and accurate calculation of the multiplicity moments, the use of these is recommended.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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