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Two Remarks about Game Semantics of Classical Logic

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We present and explain two unpublished remarks of Stefano Berardi connected to game semantics.

1 Introduction

Around 30 years ago, I had several discussions with Stefano Berardi on the topic of game semantics for classical logic, in particular connected to our work [2]. Stefano had several insightful remarks, unfortunately most of them unpublished, and the goal of this note is to report and comment on two of these remarks:

1. The first remark suggests a natural extension of the notion of views and debates [4, 5] to *transfinite* interaction sequences
2. The second remark shows that the game motivation of realization semantics [2] also validates *false* formulae, a remark which connects to recent work [10]

We first recall the main notions for the game semantics used in [2] and then present these remarks.

2 Game semantics of classical logic

The semantics is defined for an infinitary propositional calculus. The formulas of this calculus are defined inductively as: (i) 0 and 1 are (atomic) formulas, and (ii) if a_i $(i \in I)$ are formulas, where I is a countable set, then both $\wedge_i a_i$ and $\vee_i a_i$ are formulas. Note that each arithmetical formula can be represented as a formula of this infinitary propositional calculus in a natural way. Here, we regard atomic formulas as both universal and existential. We define $\neg a$ by induction on a, using de Morgan rules.

We can define intuitionistic validity, specifying the set $\mathscr{V}$ of intuitionistically valid formulas inductively: (i) $1 \in \mathscr{V}$, (ii) $\wedge_i a_i \in \mathscr{V}$ if $a_i \in \mathscr{V}$ for all i and (iii) $\vee_i a_i \in \mathscr{V}$ if $a_i \in \mathscr{V}$ for some i. We can consider the formula as specifying a perfect information game and then intuitionistic validity corresponds to a winning strategy for this game. Note that we have a winning strategy for $a \vee \neg a$ for any a, by induction on a, by a "copy-cat" strategy.

We then introduce the notion of classical validity by specifying the set $\mathscr{C}$ of classically valid formulas. $\mathscr{C}$ is defined inductively: (i) $1 \in \mathscr{C}$, (ii) $\wedge_i a_i \in \mathscr{C}$ if $a_i \in \mathscr{C}$ for all i, and (iii) $\vee_i a_i \in \mathscr{C}$ if there exists an i_0 such that either a_{i_0} is 1, or a_{i_0} is of the form $\wedge_j a_{i_0 j}$ with $a_{i_0 j} \vee \vee_i a_i \in \mathscr{C}$ for all $j \in J$.

Game theoretical semantics for this calculus is also given as a perfect information game over a formula between two players: ∃loise, who plays for existential formulas, and ∀belard, who plays for universal formulas. But the difference is now that ∃loise can backtrack. The game for a formula a is played as follows: If ∃loise (resp. ∀belard) has to play and a is atomic, then ∃loise (resp. ∀belard) wins if a is 1 (resp. 0). If a is universal of the form $\wedge_i a_i$, then ∀belard has to choose an $i \in I$ and ∃loise starts

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the game for a_i. If a is existential of the form $\vee_i a_i$, then ∃loise chooses an $i \in I$ and wins if a_i is 1, loses if a_i is 0. When a_i is universal of the form $\wedge_j a_{ij}$, ∃loise can start the game not for a_{ij} but for $a_{ij} \vee \vee_i a_i$ after ∀belard returns a $j \in J$. It is only ∃loise who is allowed to change her mind and backtracks in her choice. The intuition is that ∃loise learns from the environment ∀belard, by playing in this way.

For instance, if f is a function given as an oracle, the formula $\vee_x \wedge_y f(x) \leqslant f(y)$, stating that f takes a minimum value, is *not* intuitionistically valid. However ∃loise has a winning strategy for the classical game. She chooses first an arbitrary value $x = 0$. If ∀belard answers with a value $y = x_1$ such that $f(0) \leqslant f(x_1)$, then ∃loise wins. Otherwise $f(x_1) < f(0)$ and ∃loise backtracks by choosing $x = x_1$. If ∀belard answers with a value $y = x_2$ such that $f(x_1) \leqslant f(x_2)$, then ∃loise wins. Otherwise $f(x_2) < f(x_1)$ and ∃loise backtracks by choosing $x = x_2$, and so on. The game has to finish eventually since we have $f(x_n) < f(x_{n-1})$. Note that ∃loise may win without having found *the* actual minimum for f.

One main difference with Lorenzen's approach is that we limit ourselves to $\wedge\vee$ formulae[1]. The strategies with backtracking correspond then to cut-free proofs, and the contribution w.r.t. Lorenzen's work is to analyse what corresponds to the process of *cut-elimination*.

In general, we represent a formula as a $\wedge\vee$ tree, possibly infinitely branching, with leaves being 0 or 1. For instance:

$$(\exists_n \forall_m f(n) \leqslant f(m)) \rightarrow \exists_u f(u) \leqslant f(u+1)$$

will be represented as a $\wedge\vee$ tree:

$$(\wedge_n \vee_m f(n) > f(m)) \vee \vee_u f(u) \leqslant f(u+1).$$

The strategy for ∃loise is the following, winning with at most two moves

- ∃loise asks for n.
- ∀belard answers $n = a$.
- If $f(a) \leqslant f(a+1)$, then ∃loise takes $u = a$.
- If $f(a) > f(a+1)$, then ∃loise takes $m = a+1$.

The strategies considered so far correspond to *cut-free proofs*, which describe how a proof behaves in an environment that does not change its mind. The cut-rule is interpreted as "cooperation" between proofs. For example: the strategy A for

$$\vee_n \wedge_m f(n) \leqslant f(m)$$

interacts with the strategy B for

$$(\wedge_n \vee_m f(n) > f(m)) \vee \vee_u f(u) \leqslant f(u+1)$$

to produce a proof of $\vee_u f(u) \leqslant f(u+1)$. The cut-formula is $\vee_n \wedge_m f(n) \leqslant f(m)$.

For a simple example, consider the function: $f(0) = 10,\ f(1) = 8,\ f(2) = 3,\ f(3) = 27, \ldots$

The interaction proceeds as follows[2]

1. B asks for n, with $\varphi(1) = 0$
2. A answers $n = 0$, with $\varphi(2) = 1$
3. B responds with $m = 1$, since $f(0) > f(0+1)$, with $\varphi(3) = 2$
4. A backtracks and answers by playing $n = 1$, with $\varphi(4) = 1$

[1] In this way, the intuitionistic strategies have a direct perfect information game interpretation.

[2] The map φ indicates to what moves we answer, with 0 as the start move.

5. B responds with $m = 2$, since $f(1) > f(1+1)$, with $\varphi(5) = 4$
6. A backtracks and answers by playing $n = 2$, with $\varphi(6) = 1$
7. B concludes by playing $u = 2$, since $f(2) \leqslant f(2+1)$

The cut-formula, viewed as a tree, serves as the "topic of the debate." The debate consists of:

- Arguments and counter-arguments.
- Two opponents who can *both* change their minds.
- At any point, they can *resume* the debate from a previous point.

In this example, A and B debate, learning both from this interaction, until B can produce a value for u. We can think of A as acting as the proof while B acts as a counter-proof. Note that, in this example, if e.g. $f(4) = 0$, it happens that the interaction stops before A finding the actual minimum of f.

Gentzen's cut-elimination corresponds to the fact that such a debate has to end eventually. In [4], reproduced in [6] and [3] (which also gives an alternative proof of termination), we gave an argument for termination, which we believe to be essentially different from the one of Gentzen[3]. This argument relies on a geometrical analysis of the interaction, in the form of an *interaction sequence*.

Definition 1. *An interaction sequence is given by a pair V,φ with $V(n) \subseteq [0,n[$ and $\varphi(n) \in V(n)$ for $n > 0$ with the conditions $V(1) = \{0\}$ and $V(n+1) = \{n\} \cup V(\varphi(n))$ for $n > 0$.*

Intuitively, $\varphi(n)$ records which earlier move the current move responds to.

For example, the interaction sequence (pointer structure) produced by this interaction between the strategy A and B above is

$$\varphi(1) = 0, \quad \varphi(2) = 1, \quad \varphi(3) = 2, \quad \varphi(4) = 1, \quad \varphi(5) = 4, \quad \varphi(6) = 1.$$

For a lively description of how such interaction sequence is obtained we refer to [6] and [1]. Let us define the segment $S(k)$ to be $[\varphi(k),k]$ if $k > 0$ and $S(0)$ to be $[0,0]$. We note that, for each $n > 0$, we have a partition of $[0,n[$ in segments $S(m_k)$ with $m_0 = n-1$ and $m_{k+1} = \varphi(m_k) - 1$. We have $V(n) = \{m_0, m_1, \dots\}$, and this is the "view" at stage n, notion introduced in [4] which has been later used in game semantics of programming languages[4].

3 First remark

In [4], we proved, using classical logic, that if we have an infinite interaction sequence V,φ, then we can find an infinite sequence $n_k < n_{k+1}$ such that $\varphi(n_{k+1}) = n_k + 1$. The proof uses the following observation, combined with an induction of the depth of the formula. Define a segment $S(k)$ to be *definite* if k is not in the image of φ.

Lemma 1. *The definite segments form a nest structure: if we have to distinct definite segments then either they are disjoint or one is well inside the other.*

Stefano Berardi noticed that this result can be refined in the following way.

[3] For instance, F. Aschieri, a former student of Stefano Berardi, showed [1] that one could use this analysis to refine Gentzen's bounds on cut-elimination, by taking into account, not only the complexity of the cut-formula, but also the level of backtracking of the two strategies that are debating. In [5], we give a constructive version of termination.

[4] In [5], we show that we can define operations that have symmetry properties not simple to obtain with cut-eliminations; see [8] for a stochastic version of such a symmetric operation.

Theorem 1. *(classical) There is a* unique *sequence $n_k < n_{k+1}$ such that $n_k + 1 = \varphi(n_{k+1})$ and $S(n_k)$ is a* partition *of* $[0, \omega[$.

Uniqueness is essential since it indicates which of the two players can be considered as responsible for the infinite debate.

We don't give the proof, which is a simple variation of the argument in [4] relying on Lemma 1, but instead expand on the significance of this result. The infinite sequence n_k should be seen as the *view* at stage ω. In this sequence, all the n_k have the same parity. This means, intuitively, that if there is an infinite debate, then we can blame exactly one of the two players. This player has then a view $V(\omega)$ given by the set $\{n_k\}$, and it should then choose one $\varphi(\omega) = n_k$. We have then $V(\omega+1) = \{\omega\} \cup V(n_k)$ and we can then extend *transfinitely* the interaction sequence.

4 Second remark

In the work [2], we gave a modified realizability interpretation of classical HA^ω (with some simplified A-translation) extended with countable choice. This was motivated by an extension of the previous game interpretation where we allow to play *functions* and not only natural numbers. For instance, countable choice will be represented by a formula

$$\vee_f \wedge_x P(x, f(x)) \vee \vee_x \wedge_y \neg P(x, y)$$

The strategy for ∃loise for countable choice is then the following

- ∃loise plays an arbitrary function, for instance $f_0 = \lambda_n\ 0$
- ∀belard answers with a value $x = x_0$
- ∃loise backtracks and plays then $x = x_0$
- ∀belard answers with a value $y = y_0$
- ∃loise backtracks again and plays $f_1 = f_0, x_0 \mapsto y_0$ (that is f_0 updated with the value y_0 for x_0)
- ∀belard answers with a value $x = x_1$
- if $x_1 = x_0$, ∃loise wins by playing $y = y_0$ and then playing $\neg P(x_0, y_0)$ against $P(x_0, y_0)$ by "copy-cat" strategy; otherwise ∃loise backtracks and plays $x = x_1$, and so on

With this strategy, ∃loise updates successively the values of f

$$f_n = f_0,\ x_0 \mapsto y_0,\ x_1 \mapsto y_1, \ldots,\ x_{n-1} \mapsto y_{n-1}$$

by asking ∀belard what should be the value for y as an answer to $x = x_i$. If ever ∀belard answers to $f = f_n$ by playing a value $x_n = x_i$ which has already been answered, then ∃loise wins by playing $y_n = y_i$ and then playing $\neg P(x_i, y_i)$ against $P(x_i, y_i)$.

In [2], we remark that if ∀belard answers in a "continuous" way, i.e. proceeds using only a finite amount of information about the function f_n , it eventually has to answer to $f = f_n$ a value x which has already been answered for some $f_k,\ k < n$. By the discussion above, this means that ∃loise eventually wins in this case.

We used this strategy in [2] to provide a modified realizability interpretation of classical HA^ω extended with countable choice[5]. In [7], T. Hida used a similar justification (by a continuity argument) for

[5]Note that Spector [11] also provided a computational interpretation of this system, but it was using *Dialectica* interpretation and not modified realizability.

a strategy for the *axiom of determinacy* and could also produce a modified realizability interpretation of classical HA^ω extended with the axiom of determinacy[6].

The second important remark of Stefano Berardi is the following: there are examples of *false* formulae of HA^ω which have a strategy for ∃loise winning against any *continuous* opponent. One such example is the following

$$\vee_f \wedge_x \vee_y \; f(x) = 0 \wedge f(y) \neq 0 \qquad (*)$$

Surprisingly, this formula, though false, admits a strategy A_1 for ∃loise which is the following

- ∃loise plays the function $f_0 = \lambda_n 1$
- ∀belard answers with $x = x_0$
- ∃loise backtracks by playing $f_1 = f_0, 0 \mapsto 0$
- ∀belard answers with $x = x_1$
- if $x_1 < 1$ ∃loise wins by playing $y = 1$; otherwise ∃loise backtracks and plays $f_2 = f_1, 1 \mapsto 0$
- ∀belard answers with $x = x_2$
- if $x_2 < 2$ ∃loise wins by playing $y = 2$; otherwise ∃loise backtracks and plays $f_3 = f_2, 2 \mapsto 0$ and so on

If ∀belard plays in a continuous way, it has to play $x_n < n$ at some point, and then ∃loise wins by playing $y = n$.

But the formula $(*)$ is *false*, and we have a strategy B_1 for its negation

$$\wedge_f \vee_x \wedge_y \; f(x) \neq 0 \vee f(y) = 0$$

The strategy is as follows, after ∀belard has played $f = g$

- ∃loise plays an arbitrary value, e.g. $x = 0$
- ∀belard answers by playing $y = y_0$
- if $g(0) \neq 0$ or $g(y_0) = 0$ then ∃loise wins; otherwise $g(0) = 0$ and $g(y_0) \neq 0$ and ∃loise backtracks by playing $x = y_0$
- ∀belard answers by playing $y = y_1$, and ∃loise wins since $g(y_0) \neq 0$

If we let A_1 play against B_1, we get an infinite debate, which corresponds to the fact that we cannot expect to have cut-elimination.

1. A_1 plays $f_0 = \lambda_n 1$, with $\varphi(1) = 0$
2. B_1 answers $x = 0$, with $\varphi(2) = 1$
3. A_1 plays $f_1 = f_0, 0 \mapsto 0$, with $\varphi(3) = 0$
4. B_1 answers $x = 0$, with $\varphi(4) = 3$
5. A_1 answers $y = 1$, with $\varphi(5) = 4$
6. B_1 plays $x = 1$, with $\varphi(6) = 3$
7. A_1 plays $f_2 = f_1, 1 \mapsto 0$, with $\varphi(7) = 0$
8. B_1 plays $x = 0$, with $\varphi(8) = 7$

[6] For more connected recent work, see [9].

9. A_1 answers $y = 2$, with $\varphi(9) = 8$
10. B_1 answers $x = 2$, with $\varphi(10) = 7$
11. A_1 plays $f_3 = f_2, 2 \mapsto 0$, with $\varphi(11) = 0$, and so on.

This debate will create the infinite sequence of functions

$$f_0 = \lambda_n 1,\ f_1 = f_0,\ 0 \mapsto 0,\ f_2 = f_1, 1 \mapsto 0,\ \ldots.$$

It is thus remarkable that we could obtain a modified realizability interpretation for countable choice; we cannot do it for $(*)$, despite the fact that $(*)$ also has a winning strategy for ∃loise against any continuous opponent. This shows that continuity-based arguments are insufficient by themselves to characterise realizability semantics.

This example is to be compared with the work of S. Soloviev [10]. If f is the Ackermann function, and ∀belard is restricted to a primitive recursive strategy, then ∃loise has a winning strategy for the (false) formula

$$\vee_x \wedge_y y \leqslant f(x).$$

The strategy consists in playing successively the values $x = 0, 1, 2, \ldots$. Since ∀belard follows a primitive recursive strategy, it answers $y = y_0,\ y_1,\ y_2, \ldots$ in a primitive recursive way and we eventually should have $y_n \leqslant f(n)$. This example is philosophically similar to Stefano's second remark: if there is a asymmetry between the two players (e.g. in computing resources) the stronger player can lead the other player in believing a false statement.

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