

On the Cauchy problem with large data for a space-dependent Boltzmann-Nordheim boson equation II.

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Abstract. This paper studies a space-inhomogeneous Boltzmann Nordheim equation with pseudo-Maxwellian forces. Strong solutions are obtained for the Cauchy problem with large initial data in an $L^1 \cap L^\infty$ setting. The main results are existence, uniqueness, stability and qualitative L^∞ features of solutions conserving mass, momentum and energy.

1 Introduction and main result.

In a previous paper [1], we have studied the Cauchy problem for the Boltzmann-Nordheim [11] boson equation in a slab with two-dimensional velocity space,

$$\partial_t f(t, x, v) + v_1 \partial_x f(t, x, v) = Q_0(f)(t, x, v), \quad f(0, x, v) = f_0(x, v), \quad (t, x) \in \mathbb{R}_+ \times [0, 1], \quad v = (v_1, v_2) \in \mathbb{R}^2.$$

Its solution was obtained as the limits when $\alpha \rightarrow 0$ of the kinetic equation for anyons,

$$\partial_t f(t, x, v) + v_1 \partial_x f(t, x, v) = Q_\alpha(f)(t, x, v), \quad f(0, x, v) = f_0(x, v), \quad (t, x) \in \mathbb{R}_+ \times [0, 1], \quad v = (v_1, v_2) \in \mathbb{R}^2.$$

The collision operator Q_α in [2] depends on a parameter $\alpha \in [0, 1]$ and is given by

$$Q_\alpha(f)(v) = \int_{\mathbb{R}^2 \times S^1} B(|v - v_*|, n) [f' f'_* F_\alpha(f) F_\alpha(f_*) - f f_* F_\alpha(f') F_\alpha(f'_*)] dv_* dn,$$

with the kernel B of Maxwellian type, f' , f'_* , f , f_* the values of f at v' , v'_* , v and v_* respectively, where

$$v' = v - (v - v_*, n)n, \quad v'_* = v_* + (v - v_*, n)n,$$

and the filling factor F_α

$$F_\alpha(f) = (1 - \alpha f)^\alpha (1 + (1 - \alpha)f)^{1-\alpha}.$$

In this paper, we solve in a direct way the Cauchy problem for the three-dimensional Boltzmann-Nordheim equation in a torus,

$$\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) = R_0(f)(t, x, v), \quad f(0, x, v) = f_0(x, v), \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3, \quad (1.1)$$

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where

$$R_0(f)(v) = \int_{\mathbb{R}^3 \times S^2} B(|v - v_*|, n) [f' f'_* (1 + f)(1 + f_*) - f f_* (1 + f')(1 + f'_*)] dv_* dn.$$

The Boltzmann-Nordheim equation (1.1) was initiated by Nordheim [11], Uehling and Uhlenbeck [13] using quantum statistical considerations. All quantum features appear at the level of the collision operator R_0 . For a gas of bosons, the quantum effects are taken into account by the probability of collision between two particles depending on the number of particles occupying the state after collision. Solutions to (1.1) satisfy an entropy principle and equilibrium states are the following entropy minimizers

$$\frac{1}{e^{\frac{|v-u|^2 - \mu}{2T}} - 1} + m_0 \delta_{v-u}, \quad (1.2)$$

where $u \in \mathbb{R}^3$, $\mu \leq 0$ is the chemical potential, $m_0 \geq 0$ and $\mu m_0 = 0$. It is expected that a global in time solution to (1.1) should converge to the equilibrium state (1.2) with the same mass, momentum and kinetic energy as its initial datum. This gives rise to a critical kinetic temperature T_c such that the initial distribution m_0 is different from zero if and only if $T < T_c$. It is a reason why one should only expect local in time existence results in L^∞ for (1.1), if no restriction on the temperature of the initial datum is made.

For the bosonic BN equation general existence results were first obtained by X. Lu in [7] in the space-homogeneous isotropic large data case. It was followed by a number of interesting studies in the same isotropic setting, by X. Lu [8, 9, 10], and by M. Escobedo and J.L. Velázquez [5, 6]. Results with the isotropy assumption removed, were recently obtained by M. Briant and A. Einav [3]. Finally a space-dependent case close to equilibrium has been studied by G. Royat in [12].

The papers [7, 8, 9, 10] by Lu, study the isotropic, space-homogeneous BN equation both for Cauchy data leading to mass and energy conservation, and for data leading to mass loss when time tends to infinity. Escobedo and Velázquez in [5, 6], again in the isotropic space-homogeneous case, study initial data leading to concentration phenomena and blow-up in finite time of the L^∞ -norm of the solutions. The paper [3] by Briant and Einav removes the isotropy restriction and obtain in polynomially weighted spaces of $L^1 \cap L^\infty$ type, existence and uniqueness on a time interval $[0, T_0)$. In [3] either $T_0 = \infty$, or for finite T_0 the L^∞ -norm of the solution tends to infinity, when time tends to T_0 . Finally the space-dependent problem is considered in [12] for a particular setting close to equilibrium, and well-posedness and convergence to equilibrium are proven.

The present paper studies a space-dependent, large data problem for the BN equation. The analysis is based on local in time estimates of the mass density.

The kernel $B(|v - v_*|, n)$ is assumed measurable with

$$0 \leq B \leq B_0, \quad (1.3)$$

for some $B_0 > 0$. It is also assumed to depend only on $|v - v_*|$ and $\frac{v - v_*}{|v - v_*|} \cdot n$ denoted by $\cos \theta$, and for some $\gamma > 0$, that

$$B(|v - v_*|, n) = 0 \quad \text{for} \quad |\cos \theta| < \gamma \quad \text{or} \quad |1 - \cos \theta| < \gamma. \quad (1.4)$$

These strong cut-off conditions on B are made for mathematical reasons and assumed throughout the paper. For a more general discussion of cut-offs in the collision kernel B , see [8]. Notice that

contrary to the classical Boltzmann operator where rigorous derivations of B from various potentials have been made, little is known about collision kernels in quantum kinetic theory (cf [14]).

Denote by

$$f^\sharp(t, x, v) = f(t, x + tv, v) \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3. \quad (1.5)$$

Strong solutions to the Cauchy problem with initial value f_0 associated to the Boltzmann Nordheim equation (1.1) are considered in the following sense.

Definition 1.1 *f is a strong solution to (1.1) on the time interval I if*

$$f \in \mathcal{C}^1(I; L^1(\mathbb{T}^3 \times \mathbb{R}^3)),$$

and

$$\frac{d}{dt} f^\sharp = (Q(f))^\sharp, \quad \text{on } I \times \mathbb{T}^3 \times \mathbb{R}^3. \quad (1.6)$$

The main result of this paper is the following.

Theorem 1.1 *Assume (1.3)-(1.4). Let $f_0 \in L_+^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ and satisfy*

$$(1 + |v|^2) f_0(x, v) \in L^1(\mathbb{T}^3 \times \mathbb{R}^3), \quad \int (1 + |v|^2) \sup_{x \in \mathbb{T}^3} f_0(x, v) dv = c_0 < \infty. \quad (1.7)$$

There exist a time $T_\infty > 0$ and a strong solution f to (1.1) on $[0, T_\infty)$ with initial value f_0 . For $0 < T < T_\infty$, it holds

$$f^\sharp \in \mathcal{C}^1([0, T_\infty); L^1(\mathbb{T}^3 \times \mathbb{R}^3)) \cap L^\infty([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3). \quad (1.8)$$

If $T_\infty < +\infty$ then

$$\limsup_{t \rightarrow T_\infty} \|f(t, \cdot, \cdot)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} = +\infty. \quad (1.9)$$

The solution is unique, depends continuously in L^1 on the initial value f_0 , and conserves mass, momentum, and energy.

Remarks.

A finite T_∞ may not correspond to a condensation. In the isotropic space-homogeneous case considered in [5, 6], additional assumptions on the concentration of the initial value are considered in order to obtain condensation.

Theorem 1.1 also holds for the classical Boltzmann equation with a similar proof.

To obtain Theorem 1.1 for the boson Boltzmann-Nordheim equation, we start from a fixed initial value f_0 bounded by 2^L with $L \in \mathbb{N}$. We shall prove that there are approximations $(f_\alpha)_{\alpha \in [0, 1]}$ to (1.1) and a time $T > 0$ independent of α , so that (f_α) is bounded by 2^{L+2} on $[0, T]$. We then prove that the limit f of (f_α) when $\alpha \rightarrow 0$ solves the bosonic Boltzmann-Nordheim Cauchy problem (1.1). Iterating the result from T on, it follows that f exists up to the first time T_∞ when (1.9) holds.

The paper is organized as follows. In the following section, approximations $(f_\alpha)_{\alpha \in [0, 1]}$ to the Cauchy problem (1.1) are constructed. In Section 3 the mass density of f_α is studied with respect to uniform control in α . Theorem 1.1 is proven in Section 4.

2 Approximations.

In this section and the following one, the initial datum f_0 is assumed to be continuous.

Approximations to the Cauchy problem (1.1) are built in the following way.

For $\alpha \in]0, 1]$, let χ_α be the characteristic function of $[0, \frac{1}{\alpha^2}]$ and

$$R_\alpha(f)(v) = \int_{\mathbb{R}^3 \times S^2} \chi_\alpha(|v|^2 + |v_*|^2) B(|v - v_*|, n) \left[\frac{f'}{1 + \alpha f'} \frac{f'_*}{1 + \alpha f'_*} \frac{1 + f}{1 + \alpha f} \frac{1 + f_*}{1 + \alpha f_*} - \frac{f}{1 + \alpha f} \frac{f_*}{1 + \alpha f_*} \frac{1 + f'}{1 + \alpha f'} \frac{1 + f'_*}{1 + \alpha f'_*} \right] dv_* dn.$$

Lemma 2.1

For every $\alpha \in]0, 1]$, there exists a strong nonnegative space periodic solution

$$f_\alpha \in C^1([0, \infty[; L^1(\mathbb{T}^3 \times \mathbb{R}^2))$$

to

$$\partial_t f_\alpha + v \cdot \nabla_x f_\alpha = R_\alpha, \quad f_\alpha(0, \cdot, \cdot) = f_0. \quad (2.1)$$

The solution is continuous and unique and conserves mass, momentum and energy.

Let $T > 0$ be given. We shall first prove by contraction that for $T_1 > 0$ and small enough, there is a unique solution f_α to (2.1) on $[0, T_1]$. Let

$$c_\alpha := \|f_0\|_\infty + \frac{16\pi^2 B_0 T}{3\alpha^7}.$$

Let the map $\mathcal{C}$ be defined on space periodic functions in

$$C\left([0, T] \times \mathbb{T}^3 \times \{v; |v| \leq \frac{1}{\alpha}\}\right) \cap \{f; f \in [0, c_\alpha]\}$$

by $\mathcal{C}(f) = g$, where g is the unique solution to

$$\partial_t g + v \cdot \nabla_x g = \int \chi_\alpha B \left[\frac{f'}{1 + \alpha f'} \frac{f'_*}{1 + \alpha f'_*} \frac{1 + f}{1 + \alpha f} \frac{1 + f_*}{1 + \alpha f_*} - \frac{g}{1 + \alpha f} \frac{f_*}{1 + \alpha f_*} \frac{1 + f'}{1 + \alpha f'} \frac{1 + f'_*}{1 + \alpha f'_*} \right] dv_* dn, \quad (2.2)$$

$$g(0, \cdot, \cdot) = f_0.$$

It follows from the linearity of the previous partial differential equation that it has a unique periodic solution g in $C([0, T] \times \mathbb{T}^3 \times \{v; |v| \leq \frac{1}{\alpha}\})$. Denote by

$$R_\alpha^+(f)(v) = \int \chi_\alpha B \frac{f'}{1 + \alpha f'} \frac{f'_*}{1 + \alpha f'_*} \frac{1 + f}{1 + \alpha f} \frac{1 + f_*}{1 + \alpha f_*} dv_* dn,$$

and

$$\nu_\alpha(f)(v) = \int \chi_\alpha B \frac{f_*}{1 + \alpha f_*} \frac{1 + f'}{1 + \alpha f'} \frac{1 + f'_*}{1 + \alpha f'_*} dv_* dn.$$

For f nonnegative, g takes its values in $[0, c_\alpha]$. Indeed,

$$g^\sharp(t, x, v) \geq f_0(x, v) e^{-\int_0^t (\frac{\nu_\alpha(f)}{1 + \alpha f})^\sharp(s, x, v) ds} \geq 0,$$

and

$$g^\sharp(t, x, v) \leq f_0(x, v) + \int_0^t (R_\alpha^+)^{\sharp}(s, x, v) ds \leq c_\alpha, \quad t \in [0, T].$$

$\mathcal{C}$ is a contraction in $C([0, T_1] \times \mathbb{T}^3 \times \{v; |v| \leq \frac{1}{\alpha}\}) \cap \{f; f \in [0, c_\alpha]\}$, for $T_1 > 0$ small enough only depending on α , since the partial derivatives of the maps

$$(r_i)_{1 \leq i \leq 4} \rightarrow \frac{r_3}{1 + \alpha r_3} \frac{r_4}{1 + \alpha r_4} \frac{1 + r_1}{1 + \alpha r_1} \frac{1 + r_2}{1 + \alpha r_2} \quad \text{and} \quad (r_i)_{1 \leq i \leq 4} \rightarrow \frac{1}{1 + \alpha r_1} \frac{r_2}{1 + \alpha r_2} \frac{1 + r_3}{1 + \alpha r_3} \frac{1 + r_4}{1 + \alpha r_4}$$

are bounded on $([0, +\infty])^4$ and the domains of integration in R_α^+ and ν_α are bounded. Let f_α be its fixed point, i.e. the solution of (2.1) on $[0, T_1]$.

The argument can be repeated and the solution can be continued up to $t = T$.

To obtain Theorem 1.1 for the boson Boltzmann-Nordheim equation, we start from a fixed initial value f_0 bounded by 2^L with $L \in \mathbb{N}$. We shall prove that there is a time $T > 0$ independent of $\alpha \in]0, 1]$, so that the solutions f_α to (2.1) are bounded by 2^{L+2} on $[0, T]$. We then prove that the limit f of the solutions f_α when $\alpha \rightarrow 0$ solves the corresponding bosonic Boltzmann-Nordheim problem. Iterating the result from T on, it follows that f exists up to the first time T_∞ when

$$\limsup_{t \rightarrow T_\infty} \|f_\alpha(t, \cdot, \cdot)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} = \infty.$$

We observe that

Lemma 2.2

Given $f_0 \leq 2^L$ and satisfying (1.7), there is for each $\alpha \in]0, 1]$ a time $T_\alpha > 0$ so that the solution f_α to (2.1) is bounded by 2^{L+2} on $[0, T_\alpha]$.

Proof of Lemma 2.2.

Denote f_α by f for simplicity. It holds that

$$\begin{aligned} \sup_{s \leq t} f^\sharp(s, x, v) &\leq f_0(x, v) + \int_0^t R_\alpha^+(f)(s, x + sv, v) ds \\ &= f_0(x, v) + \int_0^t \int \chi_\alpha B \frac{f^\sharp}{1 + \alpha f^\sharp}(s, x + s(v - v'), v') \\ &\quad \frac{f^\sharp}{1 + \alpha f^\sharp}(s, x + s(v - v'_*), v'_*) \frac{1 + f}{1 + \alpha f}(s, x + sv, v) \frac{1 + f}{1 + \alpha f}(s, x + sv, v_*) dv_* dnd s. \end{aligned} \quad (2.3)$$

Consequently,

$$\sup_{s \leq t} f^\sharp(s, x, v) \leq f_0(x, v) + \frac{t}{\alpha^2} \int B \sup_{(s, x) \in [0, t] \times \mathbb{T}^3} f^\sharp(s, x, v') \sup_{(s, x) \in [0, t] \times \mathbb{T}^3} f^\sharp(s, x, v'_*) dv_* dnd s. \quad (2.4)$$

With the change of variables $(v, v_*, n) \rightarrow (v', v'_*, -n)$,

$$\int \sup_{(s, x) \in [0, t] \times \mathbb{T}^3} f^\sharp(s, x, v) dv \leq c_0 + \frac{ct}{\alpha^2} \left(\int \sup_{(s, x) \in [0, t] \times \mathbb{T}^3} f^\sharp(s, x, v) dv \right)^2, \quad (2.5)$$

where

c_0 is defined in (1.7) and $c = 4\pi B_0$.

Denote by

$$M_1(t) = \int \sup_{(s,x) \in [0,t] \times \mathbb{T}^3} f^\#(s, x, v) dv.$$

It follows from (2.5) that

$$\frac{ct}{\alpha^2} M_1^2(t) - M_1(t) + c_0 \geq 0, \quad t \in [0, \frac{\alpha^2}{4c_0c}].$$

Hence

$$M_1(t) \leq \alpha \frac{\alpha - \sqrt{\alpha^2 - 4c_0ct}}{2ct} \quad \text{or} \quad M_1(t) \geq \alpha \frac{\alpha + \sqrt{\alpha^2 - 4c_0ct}}{2ct}, \quad t \in [0, \frac{\alpha^2}{4c_0c}]. \quad (2.6)$$

Moreover,

$$\alpha \frac{\alpha - \sqrt{\alpha^2 - 4c_0ct}}{2ct} \sim c_0 \quad \text{and} \quad \alpha \frac{\alpha + \sqrt{\alpha^2 - 4c_0ct}}{2ct} \sim \frac{\alpha^2}{ct}, \quad (2.7)$$

when t is a neighborhood of zero. By the continuity of M_1 and the behavior of the bounds (2.7), it follows from (2.6) that

$$M_1(t) \leq \alpha \frac{\alpha - \sqrt{\alpha^2 - 4c_0ct}}{2ct}, \quad t \in [0, \frac{\alpha^2}{4c_0c}].$$

And so,

$$M_1(t) \leq 2c_0, \quad t \in [0, \frac{\alpha^2}{4c_0c}]. \quad (2.8)$$

Coming back to (2.3), using the change of variables $v_* \rightarrow v'$ in the gain term of the right-hand side and denoting its Jacobian by β leads to

$$\begin{aligned} \|f_\alpha(t, \cdot, \cdot)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} &\leq 2^L + \tilde{c} \int_0^t M_1(s) \|f_\alpha(s, \cdot, \cdot)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} ds \\ &\leq 2^L + 2c_0\tilde{c} \int_0^t \|f_\alpha(s, \cdot, \cdot)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} ds, \quad t \in [0, \frac{\alpha^2}{4c_0c}], \end{aligned} \quad (2.9)$$

where $\tilde{c} = \frac{4\pi B_0 \max|\beta|}{\alpha^2}$. And so,

$$\begin{aligned} \|f_\alpha(t, \cdot, \cdot)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} &\leq 2^L \left(1 + e^{2c_0\tilde{c}t}\right) \\ &\leq 2^{L+2}, \quad t \in \left[0, \min\left\{\frac{\alpha^2}{4c_0c}, \frac{\ln 3}{2c_0\tilde{c}}\right\}\right]. \end{aligned} \quad (2.10)$$

The lemma follows. ■

3 Local control of the phase space density.

This section is devoted to obtaining a time $T > 0$, such that

$$\sup_{t \in [0, T], x \in \mathbb{T}^3} f_\alpha^\sharp(t, x, v) \leq 2^{L+2},$$

uniformly with respect to $\alpha \in]0, 1]$ when f_0 is continuous.

Lemma 3.1

For T_α such that $f_\alpha(t) \leq 2^{L+2}$, $t \in [0, T_\alpha]$ and c_0 defined in (1.7), let

$$\tilde{T}_\alpha = \min\{T_\alpha, \frac{1}{\pi c_0 2^{2L+6}}\}.$$

There is a constant c_1 independent of α , such that the solution f_α of (2.1) satisfies

$$\int (1 + |v|^2) \sup_{(t, x) \in [0, \tilde{T}_\alpha] \times \mathbb{T}^3} f_\alpha^\sharp(t, x, v) dv \leq c_1. \quad (3.1)$$

Proof of Lemma 3.1.

It holds that

$$\begin{aligned} \sup_{s \leq t} f^\sharp(s, x, v) &\leq f_0(x, v) + \int_0^t R_\alpha^+(f)(s, x + sv, v) ds = f_0(x, v) \\ &+ \int_0^t \int \chi_\alpha B \frac{f}{1 + \alpha f}(s, x + sv, v') \frac{f}{1 + \alpha f}(s, x + sv, v'_*) \frac{1 + f}{1 + \alpha f}(s, x + sv, v) \frac{1 + f}{1 + \alpha f}(s, x + sv, v_*) dv_* dn ds \\ &\leq f_0(x, v) + 2^{2L+6} t \int B \sup_{(s, x) \in [0, t] \times \mathbb{T}^3} f^\sharp(s, x, v') \sup_{(s, x) \in [0, t] \times \mathbb{T}^3} f^\sharp(s, x, v'_*) dv_* dn. \end{aligned}$$

With the change of variables $(v, v_*, n) \rightarrow (v', v'_*, -n)$ and (1.7),

$$\int (1 + |v|^2) \sup_{(s, x) \in [0, t] \times \mathbb{T}^3} f^\sharp(s, x, v) dv \leq c_0 + c 2^{2L} t \left(\int (1 + |v|^2) \sup_{(s, x) \in [0, t] \times \mathbb{T}^3} f^\sharp(s, x, v) dv \right)^2,$$

where $c = 2^8 \pi B_0$. Denote by

$$M_2(t) = \int (1 + |v|^2) \sup_{(s, x) \in [0, t] \times \mathbb{T}^3} f^\sharp(s, x, v) dv.$$

It follows from

$$c 2^{2L} t M_2^2(t) - M_2(t) + c_0 \geq 0, \quad t \in [0, \frac{1}{c_0 c 2^{2L+2}}],$$

that

$$M_2(t) \leq \frac{1 - \sqrt{1 - c_0 c 2^{2L+2} t}}{c 2^{2L+1} t} \quad \text{or} \quad M_2(t) \geq \frac{1 + \sqrt{1 - c_0 c 2^{2L+2} t}}{c 2^{2L+1} t}, \quad t \in [0, \frac{1}{c_0 c 2^{2L+2}}]. \quad (3.2)$$

By the continuity of M_2 and the behavior of the bounds

$$\frac{1 - \sqrt{1 - c_0 c 2^{2L+2} t}}{c 2^{2L+1} t} \sim c_0 \quad \text{and} \quad \frac{1 + \sqrt{1 - c_0 c 2^{2L+2} t}}{c 2^{2L+1} t} \sim \frac{1}{c 2^{2L} t}$$

for t in a neighborhood of zero, it follows from (3.2) that

$$M_2(t) \leq \frac{1 - \sqrt{1 - c_0 c 2^{2L+2} t}}{c 2^{2L+1} t}, \quad t \in \left[0, \frac{1}{c_0 c 2^{2L+2}}\right].$$

And so,

$$M_2(t) \leq 2c_0, \quad t \in \left[0, \frac{1}{c_0 c 2^{2L+2}}\right]. \quad (3.3)$$

Bounds on

$$\int (1 + |v|^2) \sup_{(s,x) \in \left[\frac{1}{c_0 c 2^{2L+2}} \left(1 + \frac{1}{2} + \dots + \frac{1}{2^n}\right), \frac{1}{c_0 c 2^{2L+2}} \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{n+1}}\right)\right] \times \mathbb{T}^3} f^\sharp(s, x, v) dv, \quad n \in \mathbb{N}^*,$$

are analogously obtained by induction.

And so, $M_2(t)$ is bounded up to the minimum of T_α and $\frac{1}{c_0 c 2^{2L+2}} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots\right) = \frac{1}{c_0 c 2^{2L+1}}$. ■

Lemma 3.2

Given $f_0 \leq 2^L$ and satisfying (1.7), there are c_2 independent on α and L , and $T > 0$ so that for all $\alpha \in]0, 1]$, the solution f_α to (2.1) is bounded by 2^{L+2} and

$$\int (1 + |v|^2) \sup_{(t,x) \in [0, T] \times \mathbb{T}^3} f_\alpha^\sharp(t, x, v) dv$$

is bounded by c_2 on $[0, T]$.

Proof of Lemma 3.2.

Given α , it follows from Lemma 2.2 that the maximum time T'_α for which $f_\alpha \leq 2^{L+2}$ on $[0, T'_\alpha]$ is positive. Moreover,

$$\begin{aligned} \sup_{s \leq t} f_\alpha^\sharp(s, x, v) &\leq f_0(x, v) + \int_0^t R_\alpha^+(f_\alpha)(s, x + sv, v) ds \\ &\leq f_0(x, v) + 2^{3L+8} \int_0^t \int B \frac{f_\alpha^\sharp}{1 + \alpha f_\alpha^\sharp}(s, x + s(v - v'), v') dv_* dnd s. \end{aligned}$$

With the angular cut-off (2.2), $v_* \rightarrow v'$ is a change of variables. Using Lemma 3.1, the functions f_α satisfy for some constant $\bar{c}$,

$$\begin{aligned} \sup_{(s,x) \in [0, t] \times \mathbb{T}^3} f_\alpha^\sharp(s, x, v) &\leq f_0(x, v) + 2^{L-1} \bar{c} t \int \sup_{(s,x) \in [0, t] \times \mathbb{T}^3} f_\alpha(s, x, v') dv' \\ &\leq 2^L + 2^{L-1} \bar{c} c_1 t, \quad t \in \left[0, \min\left\{T'_\alpha, \frac{1}{\pi c_0 2^{2L+6}}\right\}\right]. \end{aligned}$$

And so,

$$\sup_{s \leq t} f_{\alpha}^{\#}(s, x, v) \leq 3(2^{L-1}), \quad t \in \left[0, \min\left\{T'_{\alpha}, \frac{1}{\pi c_0 2^{2L+6}}, \frac{1}{\bar{c} c_1}\right\}\right].$$

For all $\alpha \in]0, 1]$, it holds that

$$T'_{\alpha} \geq \min\left\{\frac{1}{\pi c_0 2^{2L+6}}, \frac{1}{\bar{c} c_1}\right\},$$

else T'_{α} would not be the maximum time such that $f_{\alpha}(t) \leq 2^{L+2}$ on $[0, T'_{\alpha}]$. Consequently,

$$\sup_{s \leq t} f_{\alpha}^{\#}(s, x, v) \leq 2^{L+2}, \quad t \in \left[0, \min\left\{\frac{1}{\pi c_0 2^{2L+6}}, \frac{1}{\bar{c} c_1}\right\}\right].$$

Let

$$T = \min\left\{\frac{1}{\pi c_0 2^{2L+6}}, \frac{1}{\bar{c} c_1}\right\}.$$

The proof of Lemma 3.1 is made again, with T_{α} replaced by T and leads to the bound c_2 of

$$\int (1 + |v|^2) \sup_{(t,x) \in [0,T] \times \mathbb{T}^3} f_{\alpha}^{\#}(t, x, v) dv.$$

■

4 Proof of Theorem 1.1.

We first prove the existence and uniqueness of a solution to (1.1) under the supplementary assumption that $f_0 \in C(\mathbb{T}^3 \times \mathbb{R}^3)$.

Let us first prove that the sequence (f_{α}) built in Section 2 is a Cauchy sequence in

$C([0, T]; L^1(\mathbb{T}^3 \times \mathbb{R}^3))$ with T of Lemma 3.2. Denote by F_{α} the function defined by $F_{\alpha}(x) = \frac{1+x}{1+\alpha x}$.

For any $(\alpha_1, \alpha_2) \in]0, 1]^2$, the function $g = f_{\alpha_1} - f_{\alpha_2}$ satisfies the equation

$$\begin{aligned} \partial_t g + v \cdot \nabla_x g &= \int \chi_{\alpha_1} B(f'_{\alpha_1} f'_{\alpha_1*} - f'_{\alpha_2} f'_{\alpha_2*}) F_{\alpha_1}(f_{\alpha_1}) F_{\alpha_1}(f_{\alpha_1*}) dv_* dn \\ &\quad - \int \chi_{\alpha_1} B(f_{\alpha_1} f_{\alpha_1*} - f_{\alpha_2} f_{\alpha_2*}) F_{\alpha_1}(f'_{\alpha_1}) F_{\alpha_1}(f'_{\alpha_1*}) dv_* dn \\ &\quad + \int \chi_{\alpha_1} B f'_{\alpha_2} f'_{\alpha_2*} \left(F_{\alpha_1}(f_{\alpha_1*}) (F_{\alpha_1}(f_{\alpha_1}) - F_{\alpha_1}(f_{\alpha_2})) + F_{\alpha_2}(f_{\alpha_2}) (F_{\alpha_1}(f_{\alpha_1*}) - F_{\alpha_1}(f_{\alpha_2*})) \right) dv_* dn \\ &\quad + \int \chi_{\alpha_1} B f'_{\alpha_2} f'_{\alpha_2*} \left(F_{\alpha_1}(f_{\alpha_1*}) (F_{\alpha_1}(f_{\alpha_2}) - F_{\alpha_2}(f_{\alpha_2})) + F_{\alpha_2}(f_{\alpha_2}) (F_{\alpha_1}(f_{\alpha_2*}) - F_{\alpha_2}(f_{\alpha_2*})) \right) dv_* dn \\ &\quad - \int \chi_{\alpha_1} B f_{\alpha_2} f_{\alpha_2*} \left(F_{\alpha_1}(f'_{\alpha_1*}) (F_{\alpha_1}(f'_{\alpha_1}) - F_{\alpha_1}(f'_{\alpha_2})) + F_{\alpha_2}(f'_{\alpha_2}) (F_{\alpha_1}(f'_{\alpha_1*}) - F_{\alpha_1}(f'_{\alpha_2*})) \right) dv_* dn \\ &\quad - \int \chi_{\alpha_1} B f_{\alpha_2} f_{\alpha_2*} \left(F_{\alpha_1}(f'_{\alpha_1*}) (F_{\alpha_1}(f'_{\alpha_2}) - F_{\alpha_2}(f'_{\alpha_2})) + F_{\alpha_2}(f'_{\alpha_2}) (F_{\alpha_1}(f'_{\alpha_2*}) - F_{\alpha_2}(f'_{\alpha_2*})) \right) dv_* dn \\ &\quad + \int (\chi_{\alpha_1} - \chi_{\alpha_2}) \left(f'_{\alpha_2} f'_{\alpha_2*} F_{\alpha_2}(f_{\alpha_2}) F_{\alpha_2}(f_{\alpha_2*}) - f_{\alpha_2} f_{\alpha_2*} F_{\alpha_2}(f'_{\alpha_2}) F_{\alpha_2}(f'_{\alpha_2*}) \right) dv_* dn. \end{aligned} \tag{4.8}$$

Using Lemma 3.2 and taking $\alpha_1 < \alpha_2$,

$$\begin{aligned} & \int \chi_{\alpha_1} B \left(|f_{\alpha_1} f_{\alpha_1*} - f_{\alpha_2} f_{\alpha_2*}| F_{\alpha_1}(f'_{\alpha_1}) F_{\alpha_1}(f'_{\alpha_1*}) \right)^{\sharp} dx dv dv_* dn \\ & \leq c 2^{2L} \left(\int \sup_{x \in \mathbb{T}^3} f_{\alpha_1}^{\sharp}(t, x, v) dv + \int \sup_{x \in \mathbb{T}^3} f_{\alpha_2}^{\sharp}(t, x, v) dv \right) \int |(f_{\alpha_1} - f_{\alpha_2})^{\sharp}(t, x, v)| dx dv \\ & \leq c c_2 2^{2L} \int |g^{\sharp}(t, x, v)| dx dv. \end{aligned}$$

We similarly obtain

$$\int \chi_{\alpha_1} B \left(f'_{\alpha_2} f'_{\alpha_2*} F_{\alpha_1}(f_{\alpha_1*}) |(F_{\alpha_1}(f_{\alpha_2}) - F_{\alpha_2}(f_{\alpha_2*}))| \right)^{\sharp} dx dv dv_* dn \leq c c_2 2^{2L} |\alpha_1 - \alpha_2|,$$

and

$$\int \chi_{\alpha_1} B \left(f_{\alpha_2} f_{\alpha_2*} F_{\alpha_1}(f'_{\alpha_1*}) |F_{\alpha_1}(f'_{\alpha_1}) - F_{\alpha_1}(f'_{\alpha_2})| \right)^{\sharp} dx dv dv_* dn \leq c c_2 2^L \int |g^{\sharp}(t, x, v)| dx dv.$$

Moreover,

$$\begin{aligned} & \left| \int (\chi_{\alpha_1} - \chi_{\alpha_2}) \left(f'_{\alpha_2} f'_{\alpha_2*} F_{\alpha_2}(f_{\alpha_2}) F_{\alpha_2}(f_{\alpha_2*}) - f_{\alpha_2} f_{\alpha_2*} F_{\alpha_2}(f'_{\alpha_2}) F_{\alpha_2}(f'_{\alpha_2*}) \right) dx dv dv_* dn \right| \\ & \leq c 2^{2L} \int_{|v| > \frac{1}{\sqrt{2\alpha_1}} \text{ or } |v_*| > \frac{1}{\sqrt{2\alpha_1}}} f_{\alpha_2}(t, x, v) f_{\alpha_2}(t, x, v_*) dx dv dv_* \\ & \leq c c_2 2^{2L} \int_{|v| > \frac{1}{\sqrt{2\alpha_1}}} f_{\alpha_2}(t, x, v) dx dv \\ & \leq c c_2 2^{2L} \alpha_1^2 \int |v|^2 f_{\alpha_2}(t, x, v) dx dv. \end{aligned}$$

The remaining terms are estimated in the same way. It follows

$$\frac{d}{dt} \int |g^{\sharp}(t, x, v)| dx dv \leq c c_2 2^{2L} \left(\int |g^{\sharp}(t, x, v)| dx dv + |\alpha_1 - \alpha_2| + \alpha_1^2 \right).$$

Hence

$$\lim_{(\alpha_1, \alpha_2) \rightarrow (0,0)} \sup_{t \in [0, T]} \int |g^{\sharp}(t, x, v)| dx dv = 0.$$

And so (f_{α}) is a Cauchy sequence in $C([0, T]; L^1(\mathbb{T}^3 \times \mathbb{R}^3))$. Denote by f its limit. Analogously,

$$\lim_{\alpha \rightarrow 0} \int |Q(f) - Q(f_{\alpha})|(t, x, v) dt dx dv = 0.$$

Hence f is a strong solution to (1.1) on $[0, T]$ with initial value f_0 .

If there were two solutions, their difference denoted by G would with similar arguments satisfy

$$\frac{d}{dt} \int |G^{\sharp}(t, x, v)| dx dv \leq c c_2 2^{2L} \int |G^{\sharp}(t, x, v)| dx dv,$$

hence be identically equal to its initial value zero. And so there exists a unique solution to (1.1).

Let us prove the existence and uniqueness of a solution to (1.1) for any $f_0 \in L_+^{\infty}(\mathbb{T}^3 \times \mathbb{R}^3)$ satisfying (1.7).

If f_1 (resp. f_2) is a solution to (1.1) with the continuous initial value f_{10} (resp. f_{20}), then similar arguments lead to

$$\frac{d}{dt} \int |(f_1 - f_2)^\sharp(t, x, v)| dx dv \leq c c_2 2^{2L} \int |(f_1 - f_2)^\sharp(t, x, v)| dx dv,$$

so that

$$\| (f_1 - f_2)(t, \cdot, \cdot) \|_{L^1(\mathbb{T}^3 \times \mathbb{R}^3)} \leq e^{c c_2 2^{2L} T} \| f_{10} - f_{20} \|_{L^1(\mathbb{T}^3 \times \mathbb{R}^3)}, \quad t \in [0, T].$$

Consider f_0 as the limit in $L^1(\mathbb{T}^3 \times \mathbb{R}^3)$ of a sequence $(f_{0,n})_{n \in \mathbb{N}}$ of continuous functions satisfying (1.7). Let $(f_n)_{n \in \mathbb{N}}$ be the solutions to (1.1) associated to the initial data $(f_{0,n})_{n \in \mathbb{N}}$. It can similarly be proven that $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $C([0, T]; L^1(\mathbb{T}^3 \times \mathbb{R}^3))$ and that its limit f is the unique solution in $C([0, T]; L^1(\mathbb{T}^3 \times \mathbb{R}^3))$ to (1.1).

Finally, if f_1 (resp. f_2) is the solution to (1.1) with initial value f_{10} (resp. f_{20}), then similar arguments lead to

$$\frac{d}{dt} \int |(f_1 - f_2)^\sharp(t, x, v)| dx dv \leq c c_2 2^{2L} \int |(f_1 - f_2)^\sharp(t, x, v)| dx dv,$$

so that

$$\| (f_1 - f_2)(t, \cdot, \cdot) \|_{L^1(\mathbb{T}^3 \times \mathbb{R}^3)} \leq e^{c c_2 2^{2L} T} \| f_{10} - f_{20} \|_{L^1(\mathbb{T}^3 \times \mathbb{R}^3)}, \quad t \in [0, T],$$

i.e. stability holds.

If

$$\sup_{(x,v) \in \mathbb{T}^3 \times \mathbb{R}^3} f(T, x, v) < 2^{L+2},$$

then the procedure can be repeated, i.e. the same proof can be carried out from the initial value $f(T)$. It leads to a maximal interval denoted by $[0, \tilde{T}_1]$ on which $f(t, \cdot, \cdot) \leq 2^{L+2}$. By induction there exists an increasing sequence of times $(\tilde{T}_n)$ such that $f(t, \cdot, \cdot) \leq 2^{L+2n}$ on $[0, \tilde{T}_n]$. Let

$$\tilde{T}_\infty = \lim_{n \rightarrow +\infty} \tilde{T}_n.$$

Either $\tilde{T}_\infty = +\infty$ and the solution f is global in time, or $\tilde{T}_\infty$ is finite and then the limes superior of the solution is infinity in the L^∞ -norm at $\tilde{T}_\infty$. ■

Lemma 4.1 *The solution f to (1.1) with initial value f_0 conserves mass, momentum and energy.*

Proof of Lemma 4.1.

The conservation of mass and first momentum of f will follow from the boundedness of the total energy. The energy is non-increasing since the approximations f_α conserve energy and

$$\lim_{\alpha \rightarrow 0} \int_{\mathbb{T}^3} \int_{|v| < V} |(f - f_\alpha)(t, x, v)| |v|^2 dx dv = 0, \quad \text{for all } t \in [0, T_\infty[\text{ and positive } V.$$

Energy conservation will be satisfied if the energy is non-decreasing. Taking $\psi_\epsilon = \frac{|v|^2}{1 + \epsilon |v|^2}$ as approximation for $|v|^2$, it is enough to bound

$$\int R_0(f)(t, x, v) \psi_\epsilon(v) dx dv = \int B \psi_\epsilon \left(f' f'_* (1 + f)(1 + f_*) - f f_* (1 + f')(1 + f'_*) \right) dx dv dv_* dn$$

from below by zero in the limit $\epsilon \rightarrow 0$. Similarly to [8],

$$\begin{aligned}
\int R_0(f) \psi_\epsilon dx dv &= \frac{1}{2} \int B f f_*(1 + f')(1 + f'_*) \left(\psi_\epsilon(v') + \psi_\epsilon(v'_*) - \psi_\epsilon(v) - \psi_\epsilon(v_*) \right) dx dv dv_* dn \\
&\geq - \int B f f_*(1 + f')(1 + f'_*) \frac{\epsilon |v|^2 |v_*|^2}{(1 + \epsilon |v|^2)(1 + \epsilon |v_*|^2)} dx dv dv_* dn \\
&\geq -c \epsilon 2^{2L} \int |v|^2 \sup_{(t,x) \in [0,T] \times \mathbb{T}^3} f^\sharp(t, x, v) dv \int |v_*|^2 f(t, x, v_*) dx dv_* \\
&\geq -c c_2 \epsilon 2^{2L}.
\end{aligned}$$

This implies that the energy is non-decreasing, and bounded from below by its initial value. That completes the proof of the lemma. $\blacksquare$

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