THESIS FOR THE DEGREE OF DOCTOR OF TECHNOLOGY

Regularization of divergent integrals in complex geometry

Finite parts and residues of Archimedean zeta functions

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Abstract

A unifying theme of this thesis is the study of Archimedean zeta functions defined by complex-geometric data. Classically, Archimedean zeta functions over $\mathbb{C}$ are parameter-dependent integrals over a domain in $\mathbb{C}^n$, where the parameter is a complex power of the modulus of a holomorphic function. We consider a global complex-geometric generalization in which the integration is over a complex manifold or reduced analytic space, and where the modulus of a holomorphic function is replaced by the norm of a holomorphic section of a vector bundle.

The first two papers concern finite parts of divergent integrals on reduced complex analytic spaces. Given a singular differential form whose singularities are determined a holomorphic section of a vector bundle, a Hermitian metric induces a natural regularization of the divergent integral, giving rise to an Archimedean zeta function. A finite part of the divergent integral is defined as the constant term in the Laurent expansion of this zeta function about 0. Paper I establishes an explicit formula describing the dependence of the resulting finite part on the choice of Hermitian metric. Paper II develops a current calculus adapted to this setting and derives decomposition formulas that permit explicit computations of certain finite parts. We illustrate these formulas with a family of examples on projective space, where the resulting finite parts turn out to be multiple zeta values.

The second pair of papers studies Archimedean zeta functions arising as partition functions of Gibbs ensembles on compact Kähler manifolds. In Paper III, we consider systems of particles on the two-dimensional sphere, interacting through logarithmic pair potentials. Depending on the numerical values of the coupling constants, the resulting partition functions are either examples of Archimedean zeta functions or slight generalizations thereof. Using techniques from complex algebraic geometry, in particular the Fulton–MacPherson compactification of configuration space, we establish the meromorphic continuation of these partition functions and relate the location of their critical inverse temperatures to a discrete optimization problem governing both integrability and particle clustering.

Paper IV concerns Berman's probabilistic approach to Kähler–Einstein metrics on log Fano manifolds X. In this framework, the Kähler–Einstein geometry of X is encoded by a canonical random point process admitting a statistical-mechanical interpretation in terms of a family of Gibbs measures on the products X^N, whose associated partition functions define Archimedean zeta functions. The main contribution of the paper is an extension of this framework to log Fano manifolds with non-discrete automorphism groups. To this end, we propose a symmetry-breaking procedure based on a moment-map constraint for the Gibbs measures, and introduce an algebraic notion of Gibbs polystability, conjecturally equivalent to the existence of a Kähler–Einstein metric on X. Moreover, we conjecture that if X is Gibbs polystable, then the unique Kähler–Einstein metric with vanishing moment emerges when sampling N points on X subject to the moment-map constraint as $N \to \infty$. Inter alia, we verify several of our conjectures for log Fano curves.

Keywords: Archimedean zeta function, divergent integral, regularization, meromorphic continuation, finite part, current extension, partition function, Log gas

List of publications

The following papers are included in the thesis:

I. **Svensson, L.** On finite parts of divergent complex geometric integrals and their dependence on a choice of Hermitian metric. *J. Geom. Anal.* **34** (2024), Article 325. `doi: 10.1007/s12220-024-01773-9`.

II. **Svensson, L.** A calculus for finite parts and residues of some divergent complex geometric integrals. Preprint, `arXiv:2502.17991`.

III. Andreasson, R., **Svensson, L.** Critical temperatures and collapsing of two-dimensional Log gases. Submitted for publication. Preprint, `arXiv:2510.25312`.

IV. Andreasson, R., Berman, R. J., **Svensson, L.** Gibbs polystability of Fano manifolds, stability thresholds and symmetry breaking. Preprint, `arXiv:2511.16173`.

Author contributions

I. The problem was given by the supervisor. The main results are due to the author.

II. The central idea for the paper was developed in collaboration with the supervisor. The main results are due to the author.

III. The work was carried out in close collaboration with the coauthor, with roughly equal contributions.

IV. The original idea for the symmetry-breaking construction was due to the second author. The development of the main results was collaborative; the author's primary contributions were to Sections 2 and 8.

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Ludvig Svensson
Göteborg, June 2026

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1 Introduction

In this introductory chapter, we give a brief overview of the main themes of the thesis and a first indication of some of the central techniques and ideas. These will be developed in greater detail in later chapters. We also introduce the four appended papers, of which more detailed summaries are given in Chapter 4.

This thesis is broadly concerned with topics in complex analysis and complex geometry, with connections to mathematical physics. A common theme throughout the four appended papers is the central role played by so-called *Archimedean zeta functions* (defined over $\mathbb{C}$). Classically, these are meromorphic functions defined by an integral over a domain in $\mathbb{C}^n$ or $\mathbb{R}^n$ of the complex power of the absolute value of an analytic function.

A fundamental example of an Archimedean zeta function is the gamma function, defined by the integral

$$\Gamma(\lambda) = \int_0^\infty x^{\lambda-1} e^{-x} \mathrm{d}x. \tag{1.0.1}$$

The integral converges and is holomorphic as a function of λ, a priori only for $\mathfrak{Re}\,\lambda > 0$. However, it admits an analytic continuation to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, giving rise to a meromorphic function with poles at the non-positive integers. This perspective on the gamma function extends far beyond its original purpose as an interpolation of the factorial: Γ is a smooth function on $(0, \infty)$ and satisfies $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$. While such interpolations are certainly not unique, Γ is distinguished by its striking analytic and functional properties.

To see that Γ admits a *meromorphic continuation*, assume first that $\mathfrak{Re}\,\lambda$ is sufficiently large. Integration by parts then yields

$$\begin{aligned}
\Gamma(\lambda) &= \int_0^\infty \frac{1}{\lambda} \frac{\mathrm{d}}{\mathrm{d}x}\big(x^\lambda\big) e^{-x} \mathrm{d}x \\
&= \frac{1}{\lambda}\Big[x^\lambda e^{-x}\Big]_0^\infty + \frac{1}{\lambda}\int_0^\infty x^\lambda e^{-x} \mathrm{d}x \\
&= \frac{1}{\lambda}\int_0^\infty x^\lambda e^{-x} \mathrm{d}x
\end{aligned}$$

$$= \frac{1}{\lambda}\Gamma(\lambda+1),$$

whence we recover the functional equation $\lambda\Gamma(\lambda) = \Gamma(\lambda+1)$. We can repeat this relation iteratively: For any $m \in \mathbb{N}$, we obtain

$$\Gamma(\lambda) = \frac{1}{\lambda(\lambda+1)(\lambda+2)\dots(\lambda+m-1)}\Gamma(\lambda+m). \tag{1.0.2}$$

Now, for $\mathfrak{Re}\,\lambda > -m+1$, and provided $\lambda \notin \{0,-1,-2,\dots,m-1\}$, the right-hand side of (1.0.2) is defined and holomorphic as a function of λ, with simple poles at the points $\{0,-1,-2,\dots,m-1\}$. Since m can be chosen arbitrarily large, (1.0.2) yields a meromorphic continuation of Γ to all of $\mathbb{C}$, with poles precisely at the non-positive integers.

More generally, a typical Archimedean zeta function over $\mathbb{C}$ is of the form

$$\Gamma^{\xi}_{|f|}(\lambda) = i^{n^2}\int_{\mathbb{C}^n} |f|^{2\lambda}\xi\, \mathrm{d}z \wedge \mathrm{d}\bar{z}, \tag{1.0.3}$$

where λ is a complex parameter, a priori $\mathfrak{Re}\,\lambda \geq 0$, f is a holomorphic function and ξ is a *test function*, that is, a smooth function with compact support. The form $i^{n^2}\mathrm{d}z \wedge \mathrm{d}\bar{z} := i^{n^2}\mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_1 \wedge \dots \wedge \mathrm{d}\bar{z}_n$ is the standard Lebesgue measure on $\mathbb{C}^n$ (multiplied by 2^n). It turns out that (1.0.3) admits a meromorphic continuation to all of $\mathbb{C}$, with poles contained in a discrete subset of $\mathbb{Q}_-$.

The name Archimedean zeta functions reflects their place within the broader theory of local zeta functions. Namely they are defined over completions of number fields with respect to *Archimedean absolute values* arising from embeddings into $\mathbb{R}$ or $\mathbb{C}$, as opposed to non-Archimedean zeta functions (also known as *Igusa zeta functions*) which are defined over p-adic fields.

The terminology "zeta function" has to do with the role of such integrals as local factors in global *zeta integrals*, where, together with their non-Archimedean counterparts, they contribute to the analytic structure of *global arithmetic zeta functions* and L-functions. The fundamental example is provided by the Riemann zeta function, whose functional equation involves the gamma function, arising as the Archimedean local factor; see, e.g., [39].

The modern study of integrals of the form (1.0.3) originates in the work of Gel'fand–Shilov [31], and later Bernstein–Gel'fand [14] and independently Atiyah [2], who showed that such integrals admit meromorphic continuation in the parameter. *Bernstein–Sato theory* establishes the existence of a functional (differential) equation, analogous to (1.0.2), that provides a conceptual explanation for the meromorphic continuation, as well as information about the location of the poles, see, Section 3.1 below. At roughly the same time as

Bernstein initiated what is now known as Bernstein–Sato theory [13], Igusa [38] developed his theory of local zeta functions over non-Archimedean fields, commonly referred to as *Igusa zeta functions*, and given by similar integrals as (1.0.3), but instead defined over p-adic numbers.

Let us now give a brief overview of the settings and some main points of Papers I–IV. Terminology and concepts are explained in more detail in later chapters. The first two papers are devoted to the problem of extracting *finite parts* of divergent integrals in a global complex geometric setting. Roughly speaking, a finite part is a "reasonable" assignment of a finite value to an otherwise ill-defined divergent integral (or sum), typically obtained via a regularization procedure. This work is inspired by regularization and renormalization methods in quantum field theory and string theory, and in particular by earlier work of Felder–Kazhdan [26, 27] on finite parts of divergent integrals in an abstract global framework.

In the setting of Papers I and II, Archimedean zeta functions arise as regularizations of divergent integrals $\int_X \omega$ of a singular differential form ω over a compact complex manifold X (or, more generally, a reduced analytic space of pure dimension). We assume that ω is smooth outside the zero locus $\{s = 0\}$ of a holomorphic section $s\colon X \to E$ of a holomorphic vector bundle $E \to X$. Then, following classical ideas of Bernstein–Gel'fand and Atiyah, we consider regularized integrals of the form

$$\Gamma_{\|s\|}(\lambda, \omega) = \int_X \|s\|^{2\lambda}\omega, \tag{1.0.4}$$

where λ is a complex parameter, a priori with $\mathfrak{Re}\,\lambda \gg 0$, and $\|\cdot\|$ is a (smooth) Hermitian metric on E. Under the assumption that $\|s\|^{2M}\omega$ extends smoothly across $\{s = 0\}$ for some $M \in \mathbb{Q}_+$, $\Gamma_{\|s\|}(\lambda, \omega)$ is a holomorphic function of λ for $\mathfrak{Re}\,\lambda \gg 0$, and admits a meromorphic continuation to $\mathbb{C}$. In general, $\Gamma_{\|s\|}(\lambda, \omega)$ will have a pole at 0, reflecting the divergence of $\int_X \omega$. A natural finite part of $\int_X \omega$ is then defined as the constant term in the Laurent series expansion of $\Gamma_{\|s\|}(\lambda, \omega)$ about $\lambda = 0$.

Both Papers I and II rely heavily on the machinery of *currents*, with many of the ideas being closely related to the theory of *residue currents* in several complex variables, which also has roots in the work of Bernstein–Gel'fand and Atiyah. The singular differential form ω defines a distribution on $X \setminus \{s = 0\}$ by

$$\xi \mapsto \int_X \omega\,\xi,$$

for test functions ξ with compact support in $X \setminus \{s = 0\}$, thus "avoiding" the singularities of ω. From this perspective, the problem is to *extend* ω as a distribution to all of X. More generally, if ω is a singular differential form of arbitrary (bi)degree, it does not make sense to consider a (possibly divergent) integral of ω across X. However, we may still view ω as a current on $X \setminus \{s = 0\}$ via

$$\xi \mapsto \int_X \omega \wedge \xi,$$

for *test forms* (smooth differential forms with compact support) ξ on $X \setminus \{s = 0\}$ of complementary degree, and still seek a *current extension* of ω across X.

In Paper I, we study finite parts (and current extensions) in this setting. In particular, we consider the natural question of how the finite part depends on the choice of Hermitian metric used to define the regularization. Our main results include a general formula describing this dependence in terms of the associated current extensions of ω defined with respect to two different metrics.

In Paper II, we develop a current calculus adapted to a special class of singular forms, with the aim of systematically computing finite parts. Our main result is a decomposition formula for the finite part in terms of sums of products of explicit currents. We demonstrate the formula through examples on complex projective space. Interestingly, in these examples, the finite parts we compute all turn out to be integer linear combinations of *multiple zeta values*.

In Papers III and IV, Archimedean zeta functions appear in a different guise, as *partition functions* arising in a statistical mechanical framework associated with certain interacting particle systems on *Fano manifolds* (and more generally *log Fano pairs*). More precisely, we consider *canonical ensembles* of N interacting particles on a compact complex manifold X, where the interaction is governed by a potential energy function E of a particular form. Recall that the canonical ensemble describes an idealized system in thermal equilibrium at a fixed inverse temperature β. To such a system one associates a probability measure on the *configuration space* X^N, called the *Gibbs measure*:

$$\mu_N = \frac{1}{Z_N(\beta)} e^{-\beta E} \mathrm{d}V^{\otimes N},$$

where $\mathrm{d}V$ is a fixed volume form on X, and

$$Z_N(\beta) = \int_{X^N} e^{-\beta E} \mathrm{d}V^{\otimes N} \tag{1.0.5}$$

is the associated partition function. In the setting of Papers III and IV, the energy has the property that the resulting partition functions are examples of Archimedean zeta functions (or slight generalizations thereof), with the inverse temperature playing the role of the complex parameter λ, cf. (1.0.3).

In Paper III, we specialize to the case $X = \mathbb{P}^1 \simeq \mathbb{S}^2$, and consider canonical ensembles of N-particle systems defined by the *Coulomb potential*

$$E(p_1, \ldots, p_N) = - \sum_{1 \leq j < k \leq N} c_{jk} \log d(p_j, p_k)^2,$$

where $d(p_j, p_k)$ is the chordal distance between particles at positions p_j and p_k on $\mathbb{S}^2$, and where $(c_{jk}) \in \mathbb{R}^{N \times N}$ is a symmetric hollow[1] matrix of *coupling constants.* The resulting partition functions

$$Z_N(\beta) = \int_{(\mathbb{S}^2)^N} \prod_{1 \leq j < k \leq N} \|p_j - p_k\|^{2c_{jk}\beta} \mathrm{d}V^{\otimes N},$$

where $\mathrm{d}V$ is a smooth volume form on the sphere, closely resemble Archimedean zeta functions of the form (1.0.3), and exhibit many of the same analytic properties. Using tools from complex algebraic geometry, in particular the *Fulton–MacPherson compactification* of configuration space, we study the meromorphic continuation of $Z_N(\beta)$. In particular, we study clustering phenomena of the associated Gibbs ensembles near the *critical inverse temperatures*, corresponding to the integrability thresholds of the partition function.

Paper IV forms part of Berman's probabilistic program for *Kähler–Einstein* metrics. The guiding idea is the following: A Kähler–Einstein metric on a compact Kähler manifold X should emerge in the thermodynamical limit of a canonically defined statistical-mechanical system of identical interacting particles on X. The main purpose of the paper is to extend Berman's statistical-mechanical framework to the case of *Fano manifolds* (and more generally log *Fano pairs*) with non-discrete automorphism group. We do this by a certain "symmetry-breaking" procedure, realized via a moment-map constraint on the statistical-mechanical system. We develop this framework, and give tentative definitions of *(uniform) Gibbs polystability*, conjectured to be equivalent to the familiar notion of *(uniform) K-stability* in Kähler–Einstein geometry, which governs existence of Kähler–Einstein metrics on Fano varieties. We prove this conjecture in the case of log Fano curves and, in doing so, obtain a (sharp) refinement of the classical *logarithmic Hardy–Littlewood–Sobolev inequality* on $\mathbb{S}^2$, for probability measures with vanishing barycenter.

Before turning to the background material developed in the next two chapters, let us first pause to examine one of the basic ideas underlying Papers I and II, which may initially seem somewhat paradoxical: The notion of a finite part of a divergent integral.

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[1] A square matrix with zeros on the diagonal.

1.1 A first look at divergent integrals and distributions

Divergent integrals are, by definition, integrals that fail to converge to a finite value. Nevertheless, many such integrals admit meaningful regularizations with which one can define a distinguished *finite part* with practical use. To illustrate this, consider the simple initial value problem

$$u'(x) = f(x), \quad u(x_0) = 0, \tag{1.1.1}$$

for a given function f. Formally, the fundamental theorem of calculus provides a solution,

$$u(x) = \int_{x_0}^{x} f(t)\mathrm{d}t, \tag{1.1.2}$$

The solution is only formal since the integral need not converge for general f. If f does not behave well as $t \to x_0^-$, or if f is not integrable near some point $x_0 < t < x$, then (1.1.2) does not define a genuine solution. Nevertheless, it can still encode useful information.

As a concrete example, consider

$$xu'(x) = 1, \quad u(x_0) = 0, \tag{1.1.3}$$

where $x_0 < 0$. For $x \neq 0$, (1.1.3) can be rewritten as $u'(x) = 1/x$, suggesting the formal solution

$$u(x) = \int_{x_0}^{x} \frac{1}{t}\mathrm{d}t. \tag{1.1.4}$$

For $x < 0$, the right-hand side of (1.1.4) is defined and is equal to $\log|x| - \log|x_0|$. For $x > 0$, however, the integral diverges due to the singularity at $x = 0$.

A way to recover a meaningful solution for $x > 0$ is to interpret the integral in a regularized sense, namely as

$$u(x) := \lim_{\epsilon \to 0} \left(\int_{x_0}^{-\epsilon} \frac{1}{t}\mathrm{d}t + \int_{\epsilon}^{x} \frac{1}{t}\mathrm{d}t \right). \tag{1.1.5}$$

A straightforward computation shows that this limit exists and equals $\log x - \log|x_0|$. The limit is a natural finite part of (1.1.4); it is known as the *(Cauchy) principal value* of the integral in (1.1.4).

So far, the solution we obtained is only defined on $\mathbb{R} \setminus \{0\}$. To extend the solution to all of $\mathbb{R}$, one must relax the notion of solution. The obstacle is the singularity at $x = 0$, where $\log|x|$ fails to be differentiable. A natural framework for handling such singularities is provided by the theory of *distributions*. Instead of assigning values to points, a distribution u acts on *test functions* $\varphi \in \mathscr{C}_c^\infty(\mathbb{R})$, that is, smooth functions with compact support. We write $\langle u, \varphi \rangle$ for the action of u on φ. Every locally integrable function defines a distribution by integrating it against test functions, but the class of distributions is larger and allows for more singular objects.

A key feature of distributions is that they can be differentiated arbitrarily many times: The k^{th} distributional derivative $u^{(k)}$ of u is defined by

$$\langle u^{(k)}, \varphi \rangle = (-1)^k \langle u, \varphi^{(k)} \rangle, \quad \forall \varphi \in \mathscr{C}_c^\infty(\mathbb{R}). \tag{1.1.6}$$

Interpreting the derivative in (1.1.3) as a distributional derivative, we may seek a *weak* solution to the equation, that is, one that satisfies

$$\langle xu', \varphi \rangle = \langle 1, \varphi \rangle, \quad \forall \varphi \in \mathscr{C}_c^\infty(\mathbb{R}). \tag{1.1.7}$$

In this sense, $\log|x|$, which is locally integrable on all of $\mathbb{R}$, is a distributional solution of $xu' = 1$. Moreover, we have that

$$\frac{\mathrm{d}}{\mathrm{d}x} \log|x| = \operatorname{pv} \frac{1}{x}, \tag{1.1.8}$$

where $\operatorname{pv} 1/x$ denotes the *principal value distribution* associated to $1/x$, defined by

$$\left\langle \operatorname{pv} \frac{1}{x}, \varphi \right\rangle = \lim_{\epsilon \to 0^+} \int_{|x|>\epsilon} \frac{\varphi}{x} \mathrm{d}x, \tag{1.1.9}$$

cf. (1.1.5). We say that $\operatorname{pv} 1/x$ *extends* $1/x$ as a distribution across 0.

Under suitable regularity assumptions, weak solutions coincide with classical ones. In approaches to solving differential equations, one first constructs a weak solution, and then proves additional regularity to recover a classical solution. More generally, distributions provide a natural extension of functions that accommodates singularities; a fundamental example is the Dirac delta distribution, sometimes called the delta function, which, although not a function, plays a central role in analysis and physics.

Let us consider another similar example:

$$x^2 u'(x) = 1, \quad u(-\infty) = 0. \tag{1.1.10}$$

Proceeding as before, a formal solution for $x \neq 0$ is given by

$$u(x) = \int_{-\infty}^{x} \frac{1}{t^2} \mathrm{d}t. \tag{1.1.11}$$

As in the previous example, this integral diverges for $x > 0$. However, in contrast to that case, the principal value regularization fails: The truncated integration

$$\int_{-\infty}^{-\epsilon} \frac{1}{t^2} \mathrm{d}t + \int_{\epsilon}^{x} \frac{1}{t^2} \mathrm{d}t, \tag{1.1.12}$$

does not admit a finite limit as $\epsilon \to 0^+$.

To proceed in this case, we instead combine the identity

$$\frac{1}{x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\Big(-\frac{1}{x}\Big), \quad x \in \mathbb{R} \setminus \{0\},$$

with the fact that $\operatorname{pv} 1/x$ extends $1/x$ as a distribution to all of $\mathbb{R}$. Indeed, $u = -\operatorname{pv} 1/x$ is a weak solution to (1.1.10) on $\mathbb{R}$, that is,

$$\Big\langle x^2 \frac{\mathrm{d}}{\mathrm{d}x}\Big(-\operatorname{pv}\frac{1}{x}\Big), \varphi\Big\rangle = \langle 1, \varphi\rangle, \quad \forall \varphi \in \mathscr{C}_c^\infty(\mathbb{R}),$$

where the derivative is understood in the distributional sense. In particular, this construction provides a natural distributional extension of the non-locally integrable function $1/x^2$, given by $(\operatorname{pv} 1/x)'$. This distribution can in turn be used to define a natural finite part of the divergent integral in (1.1.11). Since $\operatorname{fp} 1/x^2$ agrees with $1/x^2$ away from the origin, one may define

$$\operatorname{fp}\int_{-\infty}^{x} \frac{\mathrm{d}t}{t^2} := \Big\langle \frac{\mathrm{d}}{\mathrm{d}x}\Big(-\operatorname{pv}\frac{1}{x}\Big), \chi\Big\rangle + \int_{-\infty}^{x} \frac{1-\chi}{t^2}\mathrm{d}t, \tag{1.1.13}$$

where $\chi \colon \mathbb{R} \to [0,1]$ is a smooth cutoff function equal to 1 near 0 with sufficiently small support. A straightforward computation shows that this quantity is independent of such a choice of χ and yields

$$\operatorname{fp}\int_{-\infty}^{x} \frac{1}{t^2}\mathrm{d}t = -\frac{1}{x}, \qquad x > 0. \tag{1.1.14}$$

The quantity in (1.1.14) is known as the *Hadamard finite part* of the divergent integral. It can equivalently be obtained via analytic continuation of an Archimedean zeta function or by subtracting the divergent terms in the asymptotic expansion of the truncated integral in (1.1.12) as $\epsilon \to 0^+$. These perspectives form a basis for the techniques in Papers I and II, where we study finite parts of divergent integrals in a global complex geometric setting.

1.1.1 Digression: Divergent integrals in quantum field theory

One of the main sources of inspiration for the study of finite parts of divergent integrals in Papers I and II comes from *quantum field theory (QFT)*, which provides a striking example of how divergent integrals can encode meaningful information. In broad terms, QFT synthesizes the principles of special relativity and quantum mechanics, and provides the theoretical framework underlying the *Standard model* of particle physics.

In the QFT framework, physical processes are described in terms of *transition amplitudes*: Complex-valued functions whose absolute value squared gives the probability of a transition between prescribed initial and final states. For interacting theories, these amplitudes cannot be computed exactly. Instead, one treats the interacting theory as a perturbation of a *free theory*[2], and expresses any given amplitude as formal power series in one or more coupling constants.

Each term in such an expansion then corresponds to a class of particle interactions and can be represented graphically by a set of *Feynman diagrams*. A Feynman diagram is a directed connected graph with half-edges, called *external legs*, representing incoming and outgoing particles with prescribed energy and momentum. These are connected by a number of internal edges and vertices: The internal edges represent the exchange of *virtual* particles, while the vertices represent interactions between the particles associated with the incident edges. Such a diagram encodes, via a set of *Feynman rules*, an associated integral expression. These rules assign factors to edges and vertices of the diagram and prescribe integration over internal momenta, subject to conservation laws.

The resulting expressions, known as *Feynman integrals*, can are organized according to the degree in the perturbative series, which coincides with the number of loops the corresponding Feynman diagrams contain. Diagrams without loops ("tree-level contributions") involve no integration over internal momenta. In contrast, higher-order contributions, known as *loop corrections*, typically involve integration over unbounded domains and frequently diverge due to the behavior of the integrand at large momenta (ultraviolet divergences) and/or at small momenta (infrared divergences). Such divergences must be addressed somehow in order to make meaningful predictions. This is achieved through a regularization procedure, not unlike ones considered above, followed by what is known as *renormalization*, which can, in a sense, be viewed as the extraction of a physically significant finite part.

1.2 Organization of the thesis

The next two chapters are devoted to background material. Chapter 2 introduces the necessary notions from complex analysis and geometry, while also fixing notation and conventions used throughout the text. In Chapter 3, we turn to more thesis-specific material, with particular emphasis on the Archimedean zeta functions that play a central role in the appended papers. Finally, Chapter 4 contains brief summaries of the constituent papers, highlighting some of their main ideas and results. This chapter assumes familiarity with much of the

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[2] A theory without interactions that, in general, can be solved explicitly.

material developed in the preceding chapters.

Reader's guide

Depending on the reader's background, different parts of Chapters Chapter 2 and Chapter 3 may be more or less relevant.

For Papers I and II, the most important sections are Sections 2.4 and 2.8, which introduce currents on complex manifolds and reduced analytic spaces, together with Sections 3.1 to 3.3 where the framework underlying the two papers is developed.

For Paper III, the key background is found in Section 2.9 and Section 3.4. The former introduces the principal geometric tool used in the proofs, while the latter describes the statistical-mechanical setting of the paper.

For Paper IV, in addition to the general background on complex algebraic geometry and Kähler manifolds, the essential section in Chapter 2 is Section 2.7, which introduces Kähler–Einstein metrics. The most relevant sections of Chapter 3 are Sections 3.4 and 3.5, where we outline the probabilistic approach to Kähler–Einstein geometry.

2 Complex geometry

In this section, we will fix notation and recall some of the fundamental concepts of complex analysis, as well as certain aspects of complex differential and algebraic geometry, important to the rest of the thesis. We assume familiarity with the basics of real analysis, commutative algebra, and point-set topology, as well as a working knowledge of smooth manifolds and standard concepts in differential geometry. This overview is far from a complete introduction and is uneven in its level of detail. Fortunately, there exist many excellent and comprehensive references for these topics, to which the interested reader may turn.

2.1 Elements of one-variable complex analysis

Let (x, y) denote coordinates on $\mathbb{R}^2$, and let $z = x + iy$ be the corresponding complex coordinate on $\mathbb{C}$. Any complex-valued function $f\colon \Omega \to \mathbb{C}$, defined on a domain[1] $\Omega \subset \mathbb{C}$, may be uniquely decomposed into its real and imaginary parts, $f = u + iv$, $u, v\colon \Omega \to \mathbb{R}$. Recall that if f is *holomorphic*, meaning that the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and is independent on the path $z \to z_0$ for each $z_0 \in \Omega$, then the partial derivatives of u and v exist in Ω and satisfy the *Cauchy–Riemann equations*

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}. \end{cases} \tag{2.1.1}$$

The converse is also true.

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[1] A non-empty, connected, open set in the standard topology.

Holomorphic functions are very rigid objects. In particular, complex differentiability implies smoothness. Moreover, holomorphic functions adhere to the *identity principle*: If $f, g\colon \Omega \to \mathbb{C}$ are holomorphic and if $f = g$ on some subset $S \subseteq \Omega$ which has an *accumulation point* in Ω, then $f = g$ on Ω. Recall that holomorphic functions are analytic, meaning that they are locally given by a convergent power series. A *meromorphic function* f on a domain $\Omega \subseteq \mathbb{C}$ is a function that is holomorphic on Ω except possibly at a discrete set of *poles.* A pole is a type of isolated singularity at which the function diverges, but in a controlled manner; a pole of f at $z_0 \in \Omega$ is equivalent to a zero at z_0 of $1/f$. The *order* of the pole at z_0 is defined to be the order of vanishing of $1/f$ at z_0. If $z_0 \in \Omega$ is a pole of f, then f has a *Laurent series* expansion about z_0: For $z \in \mathbb{D}^*(z_0, \epsilon)$, where $\mathbb{D}^{(*)}(z_0, \epsilon)$ denotes the (punctured) disc with radius ϵ centered at z_0, for some sufficiently small ϵ such that $\mathbb{D}(z_0, \epsilon) \subset \Omega$,

$$f(z) = \sum_{j=-\kappa_0}^{\infty} c_j (z - z_0)^j, \tag{2.1.2}$$

for some $\kappa_0 \in \mathbb{Z}_{\geq 0}$ which, if $c_{-\kappa_0} \neq 0$, is the order of the pole at z_0. The collection of negative degree terms in (2.1.2) is called the *principal part* of the Laurent series.

If f is holomorphic on Ω, then u and v are harmonic on Ω. Recall that a function $u\colon \Omega \to \mathbb{R}$ is *harmonic* if $u \in \mathscr{C}^2(\Omega, \mathbb{R})$ and $\Delta u = 0$, where $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$ denotes the Laplacian. Conversely, if u is a harmonic function on a simply connected domain Ω, then $u = \mathfrak{Re}\, f$ for some holomorphic function f on Ω, which is unique up to an additive constant. On a non-simply connected domain, however, a harmonic function need not be the real part of a globally defined holomorphic function.

It is often convenient to introduce the complex differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{2.1.3}$$

known as the *Wirtinger derivatives*, or the holomorphic and anti-holomorphic derivatives, respectively. If $\partial/\partial x$ and $\partial/\partial y$ are regarded as the natural basis of tangent vectors induced by the real coordinates (x, y), then $\partial/\partial z$ and $\partial/\partial \bar{z}$ form a natural complex basis for the tangent space after complexification. In this notation, the Cauchy–Riemann equations take the simple form

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Moreover, the Laplacian takes the form

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}. \tag{2.1.4}$$

The corresponding complex-valued differential 1-forms are defined by

$$\mathrm{d}z := \mathrm{d}x + i\mathrm{d}y \quad \text{and} \quad \mathrm{d}\bar{z} := \mathrm{d}x - i\mathrm{d}y, \tag{2.1.5}$$

These are dual to the complex tangent vectors $\partial/\partial z$ and $\partial/\partial \bar{z}$, in the sense that

$$\mathrm{d}z\left(\frac{\partial}{\partial z}\right) = 1 = \mathrm{d}\bar{z}\left(\frac{\partial}{\partial \bar{z}}\right), \quad \text{while} \quad \mathrm{d}z\left(\frac{\partial}{\partial \bar{z}}\right) = 0 = \mathrm{d}\bar{z}\left(\frac{\partial}{\partial z}\right).$$

It is then convenient, especially in higher dimensions and on complex manifolds, to introduce the differential operators ∂ and $\bar{\partial}$ by

$$\partial f := \frac{\partial f}{\partial z}\mathrm{d}z \quad \text{and} \quad \bar{\partial} f := \frac{\partial f}{\partial \bar{z}}\mathrm{d}\bar{z}, \tag{2.1.6}$$

respectively. These operators map functions to complex-valued 1-forms and satisfy

$$\partial + \bar{\partial} = \mathrm{d}, \tag{2.1.7}$$

where

$$\mathrm{d} = \frac{\partial}{\partial x}\mathrm{d}x + \frac{\partial}{\partial y}\mathrm{d}y,$$

is the usual exterior derivative on $\mathbb{R}^2$. Note that a function f is holomorphic if and only if $\bar{\partial} f = 0$.

2.1.1 Residues

Let $f\colon \Omega \to \mathbb{C}$ be a meromorphic function on a domain, and let $z_0 \in \Omega$. The *residue* $\mathrm{Res}_{z=z_0}\{f\}$ of f at z_0 is the coefficient c_{-1} of the Laurent series expansion of f about z_0. Clearly, with this definition, residues can be defined for a larger class of functions, namely functions that are holomorphic except at a discrete set of points, where it may have one or more *essential singularities*, characterized by the principal part being an infinite sum.

By Cauchy's residue theorem, if $\gamma \subset \Omega$ is a positively oriented simple closed curve enclosing z_0, and no other singularities of f lie on or inside γ, then

$$\frac{1}{2\pi i}\oint_\gamma f(z)\mathrm{d}z = \mathrm{Res}_{z=z_0}\{f(z)\}. \tag{2.1.8}$$

More generally, if $\gamma \subset \Omega$ is any positively oriented simple closed curve such that f is holomorphic along γ, then

$$\frac{1}{2\pi i}\oint_\gamma f(z)\mathrm{d}z = \sum_{z' \in S \cap \mathrm{int}(\gamma)} \mathrm{Res}_{z=z'}\{f(z)\},$$

where $S \subset \Omega$ is the singular locus of f and $\mathrm{int}(\gamma) \subset \Omega$ is the domain bounded by γ.

2.1.2 Analytic continuation

While we have already touched upon analytic (and meromorphic) continuation in the previous chapter, this is a convenient place to be slightly more precise. Analytic continuation refers to the extension of the domain of a holomorphic function in such a way that the extended function remains holomorphic. When such an extension exists, it is unique by the identity principle. In particular, a holomorphic function $f\colon \Omega \to \mathbb{C}$ is determined uniquely by its values on any non-empty open subset of its domain. Meromorphic continuation similarly refers to extending a holomorphic or meromorphic function to a larger domain where it is allowed to have poles, so that the resulting extension is meromorphic.

The uniqueness of analytic (and meromorphic) continuation provides a natural way to assign finite values to otherwise ill-defined or divergent sums and integrals, as illustrated by the following classical example.

Example 2.1.1. Consider the geometric series $\sum_{k=0}^{\infty} z^k$, which converges absolutely and defines a holomorphic function for z in the unit disk $\mathbb{D} \subset \mathbb{C}$, and diverges for $|z| \geq 1$. Within its domain of convergence, the identity

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \tag{2.1.9}$$

holds. The right-hand side of (2.1.9) defines a holomorphic function on $\mathbb{C} \setminus \{1\}$, and therefore provides the unique analytic continuation of the series beyond the unit disk. In this sense, the divergent series $\sum_{k=0}^{\infty} 2^k$, for example, may be assigned the regularized value $-1 = 1/(1-2)$.

2.2 Complex manifolds and analytic subvarieties

For brevity, in this section we assume some familiarity with the basics of smooth manifolds, and focus primarily on aspects related to the complex and holomorphic structures. For a more detailed introduction to complex manifolds and related aspects in complex algebraic geometry, see, for example, [32].

2.2.1 Holomorphic functions in several variables

Let $\mathbb{C}^n$, $n \geq 2$, denote the n-dimensional complex Euclidean space equipped with standard coordinates $z = (z_1, \dots, z_n)$, where $z_j = x_j + iy_j$ for each $1 \leq j \leq n$. Let $\Omega \subset \mathbb{C}^n$ be a domain and let $f\colon \Omega \to \mathbb{C}$ be a locally bounded function. As in the one-variable case, we define holomorphic and anti-holomorphic derivatives,

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \tag{2.2.1}$$

respectively, and 1-forms

$$\mathrm{d}z_j = \mathrm{d}x_j + i\mathrm{d}y_j \quad \text{and} \quad \mathrm{d}\bar{z}_j = \mathrm{d}x_j - i\mathrm{d}y_j, \tag{2.2.2}$$

for each $1 \le j \le n$. We then define the differential operators ∂ and $\bar{\partial}$ by

$$\partial f := \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\mathrm{d}z_j \quad \text{and} \quad \bar{\partial} f := \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}\mathrm{d}\bar{z}_j, \tag{2.2.3}$$

respectively; $\bar{\partial}$ is called the *Dolbeault operator* and ∂ the *conjugate Dolbeault operator*. Note that,

$$\mathrm{d}z_j\left(\frac{\partial}{\partial z_k}\right) = \delta_{jk} = \mathrm{d}\bar{z}_j\left(\frac{\partial}{\partial \bar{z}_k}\right) \quad \text{and} \quad \mathrm{d}z_j\left(\frac{\partial}{\partial \bar{z}_k}\right) = 0 = \mathrm{d}\bar{z}_j\left(\frac{\partial}{\partial z_k}\right),$$

where δ_{jk} denotes the Kronecker delta. We say that f is *holomorphic* if $\bar{\partial} f = 0$. This condition is equivalent to requiring that, for each $1 \le j \le n$, the function of one complex variable

$$z_j \mapsto f(z_1, \ldots, z_j, \ldots, z_n),$$

obtained by fixing all variables except z_j, is holomorphic in the usual one-variable sense.[2] A mapping $f = (f_1, \ldots, f_n)\colon \Omega \subset \mathbb{C}^m \to \mathbb{C}^n$ is holomorphic if each component $f_j\colon \Omega \to \mathbb{C}$ is holomorphic in the above sense. An invertible holomorphic mapping, whose inverse is also holomorphic, is called *biholomorphic*.

A function f in $\Omega \subset \mathbb{C}^n$ is called *meromorphic* if it is locally given as a quotient of two holomorphic functions. More precisely, f is meromorphic if, for each point $p \in \Omega$, there exists a neighborhood $U \ni p$ and holomorphic functions $g, h\colon U \to \mathbb{C}$, with $h \not\equiv 0$, such that

$$f(z) = \frac{g(z)}{h(z)} \quad \text{for } z \in U \setminus h^{-1}(0).$$

Since a meromorphic function f may fail to be defined on the zero set of the local holomorphic function h, it is not entirely accurate to regard f as being defined on *all* of Ω. In one complex variable, the possible singularities

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[2] In fact, the assumption that f is locally bounded may be omitted. A classical theorem of Hartogs asserts that a function that is holomorphic separately in each variable is automatically holomorphic; see, e.g, [37].

of a meromorphic function are poles at isolated points, and as such f may be viewed as an actual holomorphic map from Ω into the *Riemann sphere* $\mathbb{C} \cup \{\infty\} \cong \mathbb{P}^1$. In several complex variables, however, the situation is more subtle. The polar set of a meromorphic function is generally an *analytic subset* of complex *codimension* 1, rather than a discrete set of points, see Section 2.2.3 below. Moreover, there may be points where, locally, both the numerator and denominator vanish simultaneously, so that the quotient is not necessarily defined. These points form the so-called *indeterminacy set* of the meromorphic function. The indeterminacy set has complex codimension ≥ 2, explaining why it does not appear in the one-variable theory.

2.2.2 Complex manifolds

A *complex manifold* X of (complex) dimension n is a smooth manifold of (real) dimension $2n$ endowed with an equivalence class of *holomorphic atlases*, called a *complex structure*. A holomorphic atlas is a smooth atlas $\{(U_j, \varphi_j)\}$ where $\varphi_j(U_j) \subset \mathbb{C}^n$ and for any two charts $\varphi_j \colon U_j \to \mathbb{C}^n$ and $\varphi_k \colon U_k \to \mathbb{C}^n$, the transition maps

$$\varphi_j \circ \varphi_k^{-1} \colon \varphi_k(U_j \cap U_k) \to \varphi_j(U_j \cap U_k),$$

are biholomorphic. Such pairs (U_j, φ_j) are called holomorphic charts. Two holomorphic atlases $\{(U_j, \varphi_j)\}$ and $\{U'_j, \varphi'_j)\}$ are said to be equivalent if all maps $\varphi_j \circ {\varphi'}_k^{-1} \colon \varphi'_k(U_j \cap U'_k) \to \varphi_j(U_j \cap U'_k)$ are holomorphic, that is, if their union $\{(U_j, \varphi_j)\} \cup \{(U'_j, \varphi'_j)\}$ is again a holomorphic atlas. Each chart thus provides a local system of so-called holomorphic coordinates, and the biholomorphicity of the transition maps ensures that the notion of holomorphicity is independent of the choice of holomorphic chart. Accordingly, a function $f \colon X \to \mathbb{C}$ is said to be holomorphic if, for every chart $\varphi \colon U \subset X \to \mathbb{C}^n$, the local representative $f \circ \varphi^{-1}$ is holomorphic on $\varphi(U) \subset \mathbb{C}^n$.

Conversely, suppose we are given a collection of holomorphic functions $f_j \colon \Omega_j \to \mathbb{C}$, where $\Omega_j = \varphi_j(U_j) \subset \mathbb{C}^n$, satisfying the compatibility condition

$$f_k = f_j \circ \varphi_j \circ \varphi_k^{-1}$$

on $\varphi_k(U_j \cap U_k)$ for all j, k. Then the functions f_j glue to define a global holomorphic function on X. More generally, one may glue local holomorphic functions using non-trivial transition functions, that is, allowing the local functions to transform on overlaps by multiplication with nowhere-vanishing holomorphic functions, rather than agreeing exactly. This leads naturally to the notion of *holomorphic sections* of a *holomorphic line bundle*. Holomorphic functions correspond precisely to holomorphic sections of the trivial line bundle $\underline{\mathbb{C}} = X \times \mathbb{C}$, see Section 2.3 below.

While holomorphic functions in a given domain in $\mathbb{C}^n$ are abundant, the space of global holomorphic functions on a complex manifold is in general more restricted. For instance, on any compact complex manifold, there are no non-constant global holomorphic functions.

Example 2.2.1. A complex manifold of dimension 1 is called a *Riemann surface.* Classical examples include the complex plane $\mathbb{C}$, the Riemann sphere $\mathbb{P}^1$, and one-dimensional complex tori $\mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice.

Example 2.2.2. Complex projective space of dimension $n \geq 1$, denoted $\mathbb{P}^n$, is defined as the quotient $\mathbb{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/\sim$, where $z \sim w$ if $z = \lambda w$ for some $\lambda \in \mathbb{C}^*$. A point $p \in \mathbb{P}^n$ is represented by homogeneous coordinates $[Z_0 : \cdots : Z_n]$, where $Z_0, \ldots, Z_n \in \mathbb{C}$ are not all zero, and

$$[Z_0 : \ldots : Z_n] = [\lambda Z_0 : \ldots : \lambda Z_n], \quad \forall \lambda \in \mathbb{C}^*.$$

There is a natural holomorphic atlas $\{(U_j, \varphi_j)\}$, where $U_j = \{Z_j \neq 0\}$ and $\varphi_j \colon U_j \to \mathbb{C}^n$ is given by

$$\varphi_j \colon [Z_0 : \ldots : Z_n] \mapsto \left(\frac{Z_0}{Z_j}, \ldots, \frac{Z_{j-1}}{Z_j}, \frac{Z_{j+1}}{Z_j}, \ldots, \frac{Z_n}{Z_j}\right),$$

for $j = 0, \ldots, n$, that is, by normalizing the homogeneous coordinates so that the j^{th} coordinate $Z_j = 1$ and omitting it.

Example 2.2.3. A *complex Lie group* is a complex manifold G equipped with a group structure such that the multiplication map $G \times G \to G$ and inversion map $G \to G$ are holomorphic. Examples of abelian complex Lie groups include $\mathbb{C}^n$ and complex tori $\mathbb{C}^n/\Lambda$ with group operation induced by addition in $\mathbb{C}^n$. Non-abelian examples include matrix groups such as $\mathrm{GL}_n(\mathbb{C})$.

A map $f \colon X \to Y$ between complex manifolds X and Y is *holomorphic* if, for any holomorphic charts (U, φ) and (U', φ') of X and Y, respectively, the map

$$\varphi' \circ f \circ \varphi^{-1} \colon \varphi(f^{-1}(U') \cap U) \subset \mathbb{C}^{\dim_{\mathbb{C}} X} \to \varphi'(U') \subset \mathbb{C}^{\dim_{\mathbb{C}} Y}$$

is holomorphic. Two manifolds are said to be *biholomorphic* if there is an invertible map $f \colon X \to Y$ which is holomorphic and whose inverse is also holomorphic.

A complex *submanifold* Y of X is a subset $Y \subset X$ such that, for each point $p \in Y$, there exists a holomorphic chart (U, φ) of X with $U \ni p$ and $\varphi(p) = 0$ for which

$$\varphi(U \cap Y) = \{z = (z_1, \ldots, z_n) \in \varphi(U) \subseteq \mathbb{C}^n : z_{m+1} = \cdots = z_n = 0\},$$

for some $0 \leq m \leq n$. The integer m is the complex dimension of Y. In particular, every complex submanifold carries a natural structure of a complex manifold, whose holomorphic atlas is induced from charts of the above form.

2.2.3 Analytic subvarieties

In this section we will briefly introduce analytic subvarieties of complex manifolds, and more general analytic spaces. Analytic subvarieties may be regarded as complex submanifolds allowed to possess certain singularities. Let X be a complex manifold of dimension n. An *analytic subvariety* V of X is a closed subset such that, for each point $p \in V$, there exists a holomorphic chart (U, φ) with $p \in U$ for which $\varphi(U \cap V) \subset \mathbb{C}^n$ is the common vanishing locus of a finite set of holomorphic functions $f_1, \ldots, f_m \colon \varphi(U) \to \mathbb{C}$. More generally, if $V \subset X$ is an analytic subvariety, the analytic subvarieties of V are precisely the analytic subvarieties of X contained in V.

A point $p \in V$ is called *regular* if the functions $f_1, \ldots, f_m$ can be chosen such that the Jacobian matrix $(\partial f_j / \partial z_k)|_{\varphi(p)}$ has rank m. Equivalently, p is regular if there exists a neighborhood $U \ni p$ such that $U \cap V$ is a complex submanifold of X. We denote by V_{reg} the set of regular points of V. The complement $V_{\mathrm{sing}} := V \setminus V_{\mathrm{reg}}$ is the set of *singular points* of V.

An analytic subvariety V is called *(globally) irreducible* if it cannot be written as a union $V = V_1 \cup V_2$ of two proper analytic subvarieties $V_1, V_2 \subsetneq V$. An *irreducible component* of a subvariety V is a maximal (with respect to inclusion) irreducible analytic subvariety contained in V. The dimension of an irreducible variety is the complex dimension of its regular locus. More generally, an analytic subvariety V is said to be of *(pure) dimension* m if all of its irreducible components have dimension m. It is often convenient to refer instead to the *codimension* of V, defined as $\operatorname{codim} V := n - m$. Analytic subvarieties of codimension one are called *(analytic) hypersurfaces.*

The local defining equations of a subvariety need not be minimal. This leads to the following important notion. A subvariety V is called a *locally complete intersection* if, locally, it can be defined by exactly $m = \operatorname{codim} V$ holomorphic functions. For instance, hypersurfaces in a smooth ambient manifolds are always locally complete intersections. However, this may fail in singular ambient spaces. Locally complete intersections will play an important role later when discussing residue currents and current products, see Section 2.4.1.1 below.

2.3 Hermitian holomorphic vector bundles

Recall that a *fiber bundle* is the data of a tuple of topological spaces (E, B, F) together with a surjective map $\pi \colon E \to B$ called the *bundle projection* satisfying the following local triviality condition: Each point $b \in B$ has a neighborhood $U \ni b$ and a homeomorphism $\varphi \colon \pi^{-1}(U) \to U \times F$, such that $\pi = \mathrm{proj}_1 \circ \varphi$, where $\mathrm{proj}_1 \colon U \times F \to U$ is the natural projection onto the first factor. Such a pair (U, φ) is called a *trivializing neighborhood*. A collection of trivializing neighborhoods $\{(U_j, \varphi_j)\}$ such that $\{U_j\}$ is an open cover of B is called a *local trivialization* of the bundle. We will often denote the bundle simply by E or $E \to B$. We will work in the *smooth category*, meaning the spaces (E, B, F) will be assumed to be smooth manifolds, continuous maps will be smooth, homeomorphisms will be diffeomorphisms etc.

A homomorphism between fiber bundles $\pi\colon E \to B$ and $\pi'\colon E' \to B'$ is a pair of smooth maps $F\colon E \to E'$ and $f\colon B \to B'$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

commutes. If both F and f are diffeomorphisms, the bundles are said to be isomorphic. In the case of vector bundles, one additionally requires that the induced maps between fibers are linear. A fiber bundle $E \to B$ with fiber F is called *trivial* if it is isomorphic to the product bundle $\underline{F} = B \times F$.

A (real) vector bundle $E \to X$ over a manifold X is a fiber bundle whose fibers carry the additional structure of (finite-dimensional) real vector spaces and whose local trivializations are linear isomorphisms on each fiber. There is a number $m \in \mathbb{N}$, called the *rank* of the bundle, denoted $\operatorname{rank} E$, such that $F \simeq \mathbb{R}^m$. Complex vector bundles are simply vector bundles whose fibers are $\mathbb{C}$-vector spaces. The rank m of a complex vector bundle $E \to X$ is the $\mathbb{C}$-vector space dimension of the fiber, that is $F \simeq \mathbb{C}^m$. In particular, complex vector bundles of rank 1 are called complex *line bundles*.

Similar to how a manifold is determined by how local coordinate charts are glued together on overlaps, a vector bundle is determined by how local trivial bundles $U \times \mathbb{C}^m$ are glued together on overlaps. Let $E \to X$ be a complex vector bundle of rank m over a complex manifold X. Then, for any trivializing neighborhood (U, φ), the restriction of φ to each fiber defines a linear isomorphism

$$\pi^{-1}(\{p\}) \simeq \mathbb{C}^m,$$

Consequently, for any two trivializing neighborhoods (U_j, φ_j) and (U_k, φ_k), the mapping

$$\varphi_j \circ \varphi_k^{-1}\colon U_j \cap U_k \times \mathbb{C}^m \to U_j \cap U_k \times \mathbb{C}^m \tag{2.3.1}$$

is of the form

$$\varphi_j \circ \varphi_k^{-1}\colon (p, v) \mapsto (p, g_{jk}(p) \cdot v) \tag{2.3.2}$$

for some smooth function $g_{jk}\colon U_j \cap U_k \to \mathrm{GL}_m(\mathbb{C})$, where $g_{jk}(p) \cdot v$ denotes matrix multiplication. The function g_{jk} is called a transition function of the vector bundle. The transition functions satisfy $g_{jk} = g_{kj}^{-1}$ on $U_j \cap U_k$ and the *cocycle condition*:

$$g_{jk} \cdot g_{k\ell} \cdot g_{\ell j} = \mathrm{id}, \tag{2.3.3}$$

on $U_j \cap U_k \cap U_\ell$. Conversely, given an open cover $\{U_j\}$ of X and a collection $\{g_{jk} \colon U_j \cap U_k \to \mathrm{GL}_m(\mathbb{C})\}$ of smooth maps satisfying the above two conditions, there exists a complex vector bundle, unique up to isomorphism, with these as transition maps. Indeed, we obtain E as $\bigsqcup_j U_j \times \mathbb{C}^m / \sim$ where we identify $\{p\} \times \mathbb{C}^m$ in $U_j \times \mathbb{C}^m$ and $U_k \times \mathbb{C}^m$ via multiplication by $g_{jk}(p)$, that is, for $p \in U_j \cap U_k$, $(p, v) \sim (p, w)$ if $w = g_{jk}(p)v$.

A fundamental example of a vector bundle over a manifold is provided by the the tangent bundle, whose fiber over a point is given by the tangent space at that point. For a complex manifold X, there are several natural notions of a tangent space at a point $p \in X$. Let $z = (z_1, \ldots, z_n)$ be holomorphic coordinates defined in a neighborhood of p.

(1) The *real tangent space* $T_{\mathbb{R},p}X$ is the usual tangent space obtained by viewing X as a real $2n$-dimensional manifold. Writing $z_j = x_j + iy_j$, we denote the induced basis vectors by $\partial/\partial x_j$ and $\partial/\partial y_j$, $j = 1, \ldots, n$. Thus,

$$T_{\mathbb{R},p}X = \mathrm{Span}_{\mathbb{R}}\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right\}.$$

(2) The *complexified tangent space* is defined as the extension of scalars of $T_{\mathbb{R},p}X$:

$$T_{\mathbb{C},p}X = T_{\mathbb{R},p}X \otimes_{\mathbb{R}} \mathbb{C}.$$

This is a complex vector space of dimension $2n$ (real dimension $4n$), and can be written as

$$\begin{aligned} T_{\mathbb{C},p}X &= \mathrm{Span}_{\mathbb{C}}\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right\} \\ &= \mathrm{Span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_n}\right\}, \end{aligned}$$

where $\partial/\partial z_j$ and $\partial/\partial \bar{z}_j$ are related to $\partial/\partial x_j$ and $\partial/\partial y_j$ via (2.2.1).

(3) The *holomorphic* and *antiholomorphic tangent spaces*, denoted $T_p^{1,0}X$ and $T_p^{0,1}X$, respectively, are the n-dimensional $\mathbb{C}$-vector subspaces of $T_{\mathbb{C},p}X$ given by

$$T_p^{1,0}X = \mathrm{Span}_{\mathbb{C}}\left\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\right\} \quad \text{and} \quad T_p^{0,1}X = \mathrm{Span}_{\mathbb{C}}\left\{\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}\right\},$$

giving a direct sum decomposition $T_{\mathbb{C},p}X = T_p^{1,0}X \oplus T_p^{0,1}X$.

The real tangent spaces $\{T_{\mathbb{R},p}X\}_{p\in X}$ form the real tangent bundle $T_{\mathbb{R}}X$; likewise, the complexified tangent spaces $\{T_{\mathbb{C},p}X\}_{p\in X}$ form the complex vector bundle $T_{\mathbb{C}}X$. Note that the complexified tangent bundle $T_{\mathbb{C}}X = T_{\mathbb{R}}X \otimes \underline{\mathbb{C}}$, where the right hand-side is a tensor product of vector bundles.

The holomorphic and antiholomorphic tangent spaces form subbundles $T^{1,0}X$ and $T^{0,1}X$ of $T_{\mathbb{C}}X$, respectively. One thus obtains the direct sum decomposition

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X.$$

Moreover,

$$T^{0,1}X = \overline{T^{1,0}X},$$

where the right-hand side denotes the conjugate complex vector bundle, obtained by taking the complex conjugates of the fibers.

A *holomorphic vector bundle* over a complex manifold X is a complex vector bundle admitting local trivializations for which the transition functions

$$g_{jk} \colon U_j \cap U_k \to \mathrm{GL}_m(\mathbb{C})$$

are holomorphic. Equivalently, the total space E carries a natural structure of a complex manifold for which the projection $\pi \colon E \to X$ is holomorphic and the local trivializations are biholomorphisms.

Example 2.3.1. The holomorphic tangent bundle $T^{1,0}X$ carries a natural structure of a holomorphic vector bundle. Indeed, under a holomorphic change of coordinates

$$\varphi_j \circ \varphi_k^{-1} \colon (z_1, \dots, z_n) \mapsto (w_1, \dots, w_n),$$

the basis vectors $\partial/\partial z_j$ transform holomorphically according to the complex Jacobian matrix:

$$\frac{\partial}{\partial z_j} = \sum_{k=1}^{n} \frac{\partial w_k}{\partial z_j} \frac{\partial}{\partial w_k}.$$

In the sequel, all vector bundles will be assumed holomorphic unless explicitly stated otherwise. Nevertheless, some of the properties we present hold for general complex vector bundles.

Example 2.3.2. Let X be a complex manifold of dimension n and let $Y \hookrightarrow X$ be a complex submanifold of dimension m. The *(holomorphic) normal bundle* of Y in X, denoted $N_{Y/X}$ (or simply NY), is the holomorphic vector bundle on Y of rank $n - m$ defined as the quotient

$$N_{Y/X} := T^{1,0}X|_Y / T^{1,0}Y,$$

Intuitively, the normal bundle describes the directions in X that are transverse to Y.

2.3.1 Sections

For an open subset $U \subset X$, a smooth *local section* s of $\pi \colon E \to X$ over U is a smooth map $s \colon U \to E$ satisfying $\pi \circ s = \mathrm{id}_U$. Equivalently, for every trivializing neighborhood $(\widetilde{U}, \varphi)$ the map $\varphi \circ s|_{\widetilde{U} \cap U} \colon \widetilde{U} \cap U \to \widetilde{U} \cap U \times \mathbb{C}^m$ is of the form $p \mapsto (p, \sigma(p))$, where $\sigma \colon \widetilde{U} \to \mathbb{C}^m$ is a smooth map.

Unless otherwise specified, sections will be assumed to be smooth, and we will often omit this qualifier. The sections of a bundle form a *sheaf*, denoted $\mathscr{C}^\infty(E)$. We will not give the precise definition of a sheaf here; see, for example, [33] for details. Roughly speaking, a sheaf assigns data, such as sets, groups, rings, or modules, to open subsets of a topological space, subject to natural locality and gluing conditions.

In the present context, sheaves may be viewed as generalizations of vector bundles, in the sense that a vector bundle $E \to X$ is determined by its sheaf of smooth sections $\mathscr{C}^\infty(E)$, and vice versa. Let $\mathscr{C}^\infty_X = \mathscr{C}^\infty(\underline{\mathbb{C}})$ denote the sheaf of smooth complex-valued functions on X. Then $\mathscr{C}^\infty(E)$ is naturally a sheaf of $\mathscr{C}^\infty_X$-modules, and is *locally free of rank* $m = \operatorname{rank} E$. This means that for every point $p \in X$, there exists an open neighborhood $U \ni p$ such that $\mathscr{C}^\infty(E)|_U \simeq (\mathscr{C}^\infty_X|_U)^{\oplus m}$. A *global section* of E is a section $s \colon X \to E$.

A section $s \colon U \to E$ is *holomorphic* if, for every trivializing neighborhood $(\widetilde{U}, \varphi)$ the map $\varphi \circ s|_{\widetilde{U} \cap U} \colon \widetilde{U} \cap U \to \widetilde{U} \cap U \times \mathbb{C}^m$ is of the form $p \mapsto (p, f(p))$, where $f \colon \widetilde{U} \cap U \to \mathbb{C}^m$ is holomorphic. We denote the sheaf of holomorphic sections of a bundle E by $\mathcal{O}(E)$. The *structure sheaf* $\mathcal{O}_X$ of a complex manifold (or analytic space, see below) is defined as the sheaf of holomorphic functions on X, that is, $\mathcal{O}_X = \mathcal{O}(\underline{\mathbb{C}})$. We denote the subsheaf of nowhere-vanishing holomorphic functions by $\mathcal{O}^*_X$. As above, the sheaf $\mathcal{O}(E)$ is naturally a locally free sheaf of $\mathcal{O}_X$-modules of rank $m = \operatorname{rank} E$. We denote the $\mathbb{C}$-vector space of global holomorphic sections of E by $H^0(X, E)$.

More generally, one defines a *meromorphic section* s of $E \to X$ by allowing the local functions to be meromorphic rather than holomorphic. More precisely, a meromorphic section of E over an open subset $U \subset X$ is given, in each trivializing neighborhood $(\widetilde{U}, \varphi)$, by local expressions of the form $\varphi \circ s|_{\widetilde{U} \cap U}(p) = (p, f(p))$, where $f = (f_1, \dots, f_m)$ is a tuple of meromorphic functions on $\widetilde{U} \cap U$, compatible on overlaps. By abuse of notation, we write $s \colon U \to E$ also for meromorphic sections, even though such objects are not necessarily defined everywhere on U. Indeed, away from an analytic subset $V \subset U$ of codimension 1, a meromorphic section defines an honest holomorphic section.

Recall that if $\pi \colon E \to X$ is a vector bundle of rank m, a *local frame* for E over an open subset $U \subset X$ is a collection of sections $s_1, \dots, s_m \colon U \to E$, such that, for each $p \in U$, the vectors $s_1(p), \dots, s_m(p)$ form a basis for the fiber

$E_p = \pi^{-1}(p)$. A choice of local frames for E over an open cover of X determines a local trivialization of E, and conversely, every local trivialization determines a collection of local frames. Indeed, given a local frame $s_1, \dots, s_m$ over U, every vector $v \in E_p$, $p \in U$, can be written uniquely as

$$v = \sum_{j=1}^{m} a_j s_j(p),$$

with $a_j \in \mathbb{C}$, yielding a local trivialization $E|_U \simeq U \times \mathbb{C}^m$ given by

$$v \mapsto (p, (a_1, \dots, a_m)).$$

Such sections are often called *trivializing sections*, and we will use the terminology interchangeably. If a vector bundle $E \to X$ admits a global frame, equivalently a collection of global trivializing sections, then E is isomorphic to a trivial bundle: $E \simeq \underline{\mathbb{C}}^m$. In the special case of a line bundle, a single nowhere-vanishing section $s\colon U \to L$ already determines a local trivialization of L over U.

A *holomorphic local frame* is a local frame consisting of holomorphic sections. Holomorphic local frames are particularly useful since the corresponding transition functions of the bundle are holomorphic. For instance, relative to a holomorphic local frame, a holomorphic section is locally described by holomorphic functions. Moreover, in the case of line bundles, a meromorphic section can locally be written as $s = fs_0$, where s_0 is a nowhere-vanishing holomorphic section, that is, holomorphic trivializing section, and f is a meromorphic function.

Example 2.3.3. The coordinate vector fields $\{\partial/\partial z_1, \dots, \partial/\partial z_n\}$ defined by a system of holomorphic coordinates $z = (z_1, \dots, z_n)$ on an open subset $U \subset X$, determine a holomorphic local frame over U for $T^{1,0}X$. A holomorphic section s of $T^{1,0}X$ is called a *holomorphic vector field*. Relative to the above local frame, such a section can be written as

$$s = \sum_{j=1}^{n} f_j \frac{\partial}{\partial z_j},$$

where $f_j\colon U \to \mathbb{C}$ are holomorphic functions.

Before moving on, we briefly recall *pullbacks* of holomorphic vector bundles and their sections. This construction will be used repeatedly throughout the thesis, in particular when pulling back sections along resolutions of singularities, see Section 2.9 below.

Let $f\colon X \to Y$ be a holomorphic map, and let $\pi\colon E \to Y$ be a holomorphic vector bundle of rank m. The *pullback bundle* $f^*E \to X$ is the holomorphic vector bundle whose fiber over a point $x \in X$ is canonically identified with the $E_{f(x)}$. More precisely,

$$f^*E = \{(x, e) \in X \times E : \pi(e) = f(x)\}, \tag{2.3.4}$$

equipped with the natural projection $(x, e) \mapsto x$.

If $s\colon Y \to E$ is a section, then its *pullback* f^*s is a section of f^*E over X defined by $(f^*s)(x) = (x, s \circ f(x))$. In particular, if s is holomorphic, then f^*s is a holomorphic section of f^*E.

Example 2.3.4. Let $\pi_1\colon E_1 \to X_1$ and $\pi_2\colon E_2 \to X_2$ be complex vector bundles over complex manifolds X_1 and X_2, respectively, with $\operatorname{rank} E_1 = r_1$ and $\operatorname{rank} E_2 = r_2$. Then we can consider the *exterior tensor product bundle* $E_1 \boxtimes E_2 \to X_1 \times X_2$, which is a complex vector bundle of rank $r_1 r_2$, defined as the following product of pullback bundles

$$E_1 \boxtimes E_2 := \operatorname{proj}_1^* E_1 \otimes \operatorname{proj}_2^* E_2, \tag{2.3.5}$$

where $\operatorname{proj}_j\colon X_1 \times X_2 \to X_j$ is the natural projection onto the j^{th} factor, for $j = 1, 2$.

2.3.2 Hermitian vector bundles

A *Hermitian metric* h on a vector bundle $E \to X$ is an assignment of a Hermitian inner product h_p on each fiber E_p, depending smoothly on $p \in X$. Equivalently, h may be viewed as a smooth section of $\big(E \otimes \overline{E}\big)^* \to X$ satisfying $\overline{h_p(v, w)} = h_p(w, v)$ for all $v, w \in E_p$ and $h_p(v, v) > 0$ for all $v \in E_p \setminus \{0\}$, for each $p \in X$. The pair (E, h) is called a *Hermitian vector bundle*. We will often use the notation $\|\cdot\|$ for a Hermitian metric, emphasizing the associated norm: $\|s\|^2 := h(s, s)$ for sections $s \in \mathscr{C}^\infty(E)$.

An important special case is a Hermitian metric h on the holomorphic tangent bundle $T^{1,0}X$ of a complex manifold X. Such a pair (X, h) is called a *Hermitian manifold*. Note that such a metric enjoys a natural extension to the complexified tangent space $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$. Hermitian manifolds may be regarded as the complex-geometric analogue of Riemannian manifolds. Indeed, a Hermitian metric naturally induces a Riemannian metric for the underlying smooth manifold. In Section 2.6 below, we describe an equivalent point of view on Hermitian manifolds, in terms of a Riemannian metric compatible with the *(almost) complex structure* J.

Example 2.3.5. The holomorphic tangent bundle of complex projective space $\mathbb{P}^n$ carries a natural Hermitian metric, called the *Fubini–Study metric*, induced from the standard Hermitian inner product on $\mathbb{C}^{n+1}$. We will return to this metric in the discussion of Kähler manifolds below, see, Example 2.6.2 in particular.

On a line bundle $L \to X$, a Hermitian metric $\|\cdot\|$ on L can be described locally using a trivializing (nowhere-vanishing) holomorphic section s_0. If $s = fs_0$, where $f\colon U \to \mathbb{C}$ is a smooth function, then

$$\|s\|^2 = |f|^2\|s_0\|^2,$$

where $|\cdot|$ denotes the usual absolute value. Thus, relative to the trivializing section s_0, the metric is represented by the positive smooth function $\|s_0\|^2$, in contrast to higher-rank bundles, where metrics are represented locally by positive-definite Hermitian matrix-valued functions.

It is common in complex geometry and pluripotential theory to write

$$\|s_0\|^2 = e^{-\phi},$$

where ϕ is a smooth real-valued function, called a *local weight* of the metric. More generally, allowing $\phi \in L^1_{\mathrm{loc}}$ leads to the notion of a *singular Hermitian metric.* If s_0' is another holomorphic trivializing section on U, then $s_0' = gs_0$ for some nowhere vanishing $g \in \mathcal{O}_X(U)$. The corresponding local weight satisfies

$$\phi' = \phi - \log|g|^2.$$

In particular, if $\|\cdot\|_1$ and $\|\cdot\|_2$ are two Hermitian metrics on L with local weights ϕ_1 and ϕ_2, then the difference $\phi_1 - \phi_2$ is independent of the choice of local holomorphic trivialization. Thus, differences of local weights glue together to a globally defined smooth function on X.

2.4 Differential forms and currents

Recall that basis-independent linear algebra constructions, such as direct sums, tensor products, exterior products and dualization, can be performed fiberwise on vector bundles, thereby producing new vector bundles, such as cotangent bundles, and higher tensor bundles. Moreover, when applied to holomorphic vector bundles, these constructions again produce holomorphic vector bundles. The dual bundle $(T^{1,0}X)^*$ of $T^{1,0}X$ is called the *holomorphic cotangent bundle.* For any point $p \in X$, let $z = (z_1, \dots, z_n) = (x_1 + iy_1, \dots, x_n + iy_n)$ be holomorphic coordinates defined in a neighborhood $U \ni p$. Then, a holomorphic frame for $(T^{1,0}X)^*$ on U is given by the 1-forms induced by the holomorphic coordinates, $\mathrm{d}z_j = \mathrm{d}x_j + i\mathrm{d}y_j$, for $j = 1, \dots, n$. The *antiholomorphic cotangent bundle* $(T^{0,1}X)^*$ is defined analogously.

Recall that, for $m \geq 0$, *differential m-forms* are sections of the m^{th} exterior power of the real cotangent bundle $\bigwedge^m(T_{\mathbb{R}}X)^*$, with $\bigwedge^0(T_{\mathbb{R}}X)^* = \underline{\mathbb{R}}$. We denote by $\mathscr{E}^m_{X,\mathbb{R}}$ the sheaf of smooth sections of $\bigwedge^m(T_{\mathbb{R}}X)^*$.

On a complex manifold, the sheaf $\mathscr{E}_X^m = \mathscr{E}_{X,\mathbb{C}}^m$ of *(complex) differential m-forms*, that is, smooth sections of the complexified bundles $\bigwedge^m (T_{\mathbb{C}}X)^*$ enjoys a direct sum decomposition

$$\mathscr{E}_X^m = \bigoplus_{\substack{p,q\geq 0\\ p+q=m}} \mathscr{E}_X^{p,q}, \quad p,q \geq 0,$$

where

$$\mathscr{E}_X^{p,q} = \mathscr{C}^\infty\Big(\bigwedge^p (T^{1,0}X)^* \otimes \bigwedge^q (T^{0,1}X)^*\Big),$$

Complex conjugation interchanges the bidegrees: $\overline{\mathscr{E}_X^{p,q}} = \mathscr{E}_X^{q,p}$. In a coordinate neighborhood U, a *(p,q)-form* ω is a section of $\mathscr{E}_X^{p,q}$, and is given by

$$\omega = \sum_{J,K} f_{JK} \mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K, \tag{2.4.1}$$

where the sum ranges over multi-indices (J,K) of lengths p and q, respectively, the coefficients $f_{JK} \colon U \to \mathbb{C}$ are smooth functions, and

$$\mathrm{d}z_J = \mathrm{d}z_{J_1} \wedge \cdots \wedge \mathrm{d}z_{J_p}, \quad \mathrm{d}\bar{z}_K = \mathrm{d}\bar{z}_{K_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{K_q}.$$

Let $\mathrm{d} \colon \mathscr{E}_X^m \to \mathscr{E}_X^{m+1}$ denote the $\mathbb{C}$-linear extension of the exterior differential. It enjoys a natural decomposition as

$$\mathrm{d} = \partial + \bar{\partial},$$

where $\partial \colon \mathscr{E}_X^{p,q} \to \mathscr{E}_X^{p+1,q}$ and $\bar{\partial} \colon \mathscr{E}_X^{p,q} \to \mathscr{E}_X^{p,q+1}$. Similar to d, these operators satisfy the Leibniz rule: For $\alpha \in \mathscr{E}_X^{p,q}$ and $\beta \in \mathscr{E}_X^{p',q'}$,

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{p+q} \alpha \wedge \partial\beta,$$

and, similarly,

$$\bar{\partial}(\alpha \wedge \beta) = \bar{\partial}\alpha \wedge \beta + (-1)^{p+q} \alpha \wedge \bar{\partial}\beta.$$

On functions ($p = q = 0$), ∂ and $\bar{\partial}$ are locally defined by (2.2.3). On a general (p,q)-form ω, locally of the form (2.4.1), the operator ∂ acts only on the coefficient functions

$$\partial\omega = \sum_{\ell=1}^{n} \sum_{J,K} \frac{\partial f_{JK}}{\partial z_\ell} \mathrm{d}z_\ell \wedge \mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K, \tag{2.4.2}$$

and, similarly,

$$\bar{\partial}\omega = \sum_{\ell=1}^{n} \sum_{J,K} \frac{\partial f_{JK}}{\partial \bar{z}_\ell} \mathrm{d}\bar{z}_\ell \wedge \mathrm{d}z_J \wedge \mathrm{d}\bar{z}_K. \tag{2.4.3}$$

Similar to d, we have that $\partial^2 = 0 = \bar{\partial}^2$.

Recall that a differential form ω is called *closed* if $\mathrm{d}\omega = 0$, and *exact* if there exists a form η such that $\omega = \mathrm{d}\eta$. Since $\mathrm{d}^2 = 0$, every exact form is closed. Thus, the quotient of the space of closed m-forms by the subspace of exact m-forms is well-defined, giving rise to the familiar de Rham cohomology groups. Since $\bar{\partial}^2 = 0$, one may similarly define the *Dolbeault cohomology groups*[3]

$$H^{p,q}_{\bar{\partial}}(X) := \frac{\ker\big(\bar{\partial}\colon \mathscr{E}^{p,q}_X \to \mathscr{E}^{p,q+1}_X\big)}{\operatorname{im}\big(\bar{\partial}\colon \mathscr{E}^{p,q-1}_X \to \mathscr{E}^{p,q}_X\big)}.$$

A natural operation on differential forms is pullback under smooth maps. If $f\colon X \to Y$ is a holomorphic map between complex manifolds, then the usual induced pullback map $f^*\colon \mathscr{E}^m_Y \to \mathscr{E}^m_X$ preserves bidegree, that is, for each p, q,

$$f^*\colon \mathscr{E}^{p,q}_Y \to \mathscr{E}^{p,q}_X,$$

is well-defined. Moreover, pullback commutes with the exterior differential and the Dolbeault operators:

$$f^* \circ \mathrm{d} = \mathrm{d} \circ f^*, \qquad f^* \circ \partial = \partial \circ f^*, \qquad f^* \circ \bar{\partial} = \bar{\partial} \circ f^*.$$

In local holomorphic coordinates $w = (w_1, \dots, w_n)$ on an open subset $U \subset Y$, if

$$\omega = \sum_{J,K} g_{JK}\, \mathrm{d}w_J \wedge \mathrm{d}\bar{w}_K,$$

then $f^*\omega$ is locally given on $f^{-1}(U)$ by

$$f^*\omega = \sum_{J,K} (g_{JK} \circ f)\, \mathrm{d}(w_J \circ f) \wedge \mathrm{d}(\overline{w_K \circ f}).$$

Example 2.4.1. The *canonical line bundle* of an n-dimensional complex manifold X is the *determinant line bundle* of the holomorphic cotangent bundle, that is,

$$K_X := \bigwedge^n (T^{1,0}X)^*.$$

A holomorphic section $s\colon X \to K_X$ is locally, over a holomorphic chart U with coordinates $z = (z_1, \dots, z_n)$, of the form

$$s = f\, \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n, \tag{2.4.4}$$

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[3] The groups defined using ∂ are called the *conjugate Dolbeault cohomology groups.*

where $f\colon U \to \mathbb{C}$ is holomorphic, and $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$ is the local holomorphic frame for K_X over U induced by z. Similarly, a meromorphic section of K_X is locally of the form

$$s = f\,\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n, \tag{2.4.5}$$

where f is a meromorphic function on U. The dual line bundle $K_X^{-1} := K_X^*$ is called the *anticanonical bundle*. Equivalently, $K_X^{-1} \simeq \bigwedge^n T^{1,0}X$. Tensor powers $(K_X)^{\otimes k}$, for $k \in \mathbb{N}$, are called *pluricanonical bundles*. Similarly, $(K_X^{-1})^{\otimes k}$ are called *pluri-anticanonical bundles*. The (pluri)canonical and (pluri-)anticanonical bundles play important roles in complex and algebraic geometry, in particular in the context of *Kähler–Einstein metrics*, see Section 2.7 below.

More generally, for any holomorphic vector bundle $E \to X$, there is a distinguished $\mathbb{C}$-linear operator

$$\bar{\partial}_E\colon \mathscr{C}^\infty(E) \to \mathscr{E}_X^{0,1} \otimes \mathscr{C}^\infty(E), \tag{2.4.6}$$

called the *Dolbeault operator* on E, satisfying the Leibniz rule: For any $s \in \mathscr{C}^\infty(E)$ and $f \in \mathscr{C}^\infty(\underline{\mathbb{C}})$,

$$\bar{\partial}_E(fs) = \bar{\partial} f \otimes s + f\bar{\partial}_E s. \tag{2.4.7}$$

Moreover, $\bar{\partial}_E$ satisfies $\bar{\partial}_E^2 = 0$, where, by a slight abuse of notation,

$$\bar{\partial}_E^2\colon \mathscr{C}^\infty(E) \to \mathscr{E}_X^{0,2} \otimes \mathscr{C}^\infty(E).$$

Given a holomorphic local frame $s_1, \ldots, s_m$ for E over a trivializing neighborhood $U \subset X$, and a section s of E, locally written as

$$s = \sum_{j=1}^m f_j s_j,$$

then $\bar{\partial}_E$ is defined by

$$\bar{\partial}_E s := \sum_{j=1}^m \bar{\partial} f_j \otimes s_j,$$

where $\bar{\partial}$ denotes the Dolbeault operator on functions. One can check that this definition is independent of the choice of holomorphic frame, and thus gives rise to a global $\mathbb{C}$-linear operator on $\mathscr{C}^\infty(E)$ satisfying (2.4.7). For example, when $E = \mathscr{E}_X^{p,q}$, the operator $\bar{\partial}_E$ coincides with the usual $\bar{\partial}$, cf. (2.4.3). Moreover, $\bar{\partial}_E$ provides a global characterization of holomorphic sections: A section of E is holomorphic if and only if $\bar{\partial}_E s = 0$.

In contrast, there is in general no canonical operator

$$\partial_E\colon \mathscr{C}^\infty(E) \to \mathscr{E}_X^{1,0} \otimes \mathscr{C}^\infty(E)$$

satisfying the analogous properties. Such operators depend on additional geometric data, for instance a choice of Hermitian metric on E, see Section 2.4.2 below.

2.4.1 Currents

Currents play a similar role for differential forms as distributions do for functions. One way to think of a current is as a distribution-valued differential form, or equivalently, as a differential form whose coefficients are distributions, rather than smooth functions.

Let X be a smooth manifold of dimension n. For $0 \leq m \leq n$, let $\mathscr{D}^m(X)$ denote the space of smooth differential m-forms on X with compact support, called *test forms*. Moreover, we let $\mathscr{D}(X) = \bigoplus_{j=1}^n \mathscr{D}^j(X)$. The standard topology on $\mathscr{D}^m(X)$ is called the *canonical LF-topology*, where LF stands for inductive limit of *Fréchet spaces*. It is generated by a family of seminorms $\|\cdot\|_{K,k}$, indexed by compact subsets $K \subset X$ and integers $k \geq 0$.

Let $\xi \in \mathscr{D}^m(X)$. Using a partition of unity argument, we may reduce to the case where ξ is supported in a coordinate neighborhood U with coordinates $x = (x_1, \ldots, x_n)$, in which case

$$\xi = \sum_J \xi_J \mathrm{d}x_J,$$

where the sum runs over multi-indices $J \in \{1, \ldots, n\}^m$, $\mathrm{d}x_J = \mathrm{d}x_{J_1} \wedge \cdots \wedge \mathrm{d}x_{J_m}$ and where ξ_J are smooth compactly supported functions on U. We then define

$$\|\xi\|_{K,k} := \max_J \left\{ \max_{|\alpha| \leq k} \left\{ \sup_{x \in K} \left\{ |\partial^\alpha \xi_J(x)| \right\} \right\} \right\} \tag{2.4.8}$$

for compact subsets $K \subset U$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$\partial^\alpha \xi_J = \frac{\partial^{|\alpha|} \xi_J}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_n} x_n}.$$

Note that the definition of $\|\cdot\|_{K,k}$ a priori depends on the choices of local coordinates and of partition of unity. However, different choices yield equivalent seminorms. A sequence ξ_j converges to 0 in $\mathscr{D}^m(X)$ if all supports are contained in some fixed compact subset $K \subset X$, and $\|\xi_j\|_{K,k} \to 0$ for every $k \geq 0$.

An m-current is a continuous linear functional on $\mathscr{D}^{n-m}(X)$. The space of m-currents on X is denoted $\mathscr{D}'_m(X)$. Thus, a current $T \in \mathscr{D}'_m(X)$ assigns to each test form $\xi \in \mathscr{D}^{n-m}(X)$ a complex number, denoted $\langle T, \xi \rangle$. Continuity means that, for every compact subset $K \subset X$, there exist constants $C > 0$ and $k \in \mathbb{Z}_{\geq 0}$ such that

$$|\langle T, \xi \rangle| \leq C \|\xi\|_{K,k}, \tag{2.4.9}$$

for all test forms ξ supported in K. If the integer k can be chosen independently of K, then T is said to be of *finite order*. In that case, the minimal such k is called the *order* of T.

Remark 2.4.2. Equivalent to the above definition, T is continuous if, for any sequence $\xi_j \in \mathscr{D}^{n-m}(X)$ such that $\xi_j \to 0$, one has $\langle T, \xi_j \rangle \to 0$.

Remark 2.4.3. On smooth manifolds of (real) dimension n, currents of top degree n are in one-to-one correspondence with distributions. In particular, n-currents of order 0 correspond to (Radon) measures.

The exterior derivative extends to currents by duality: If T is an m-current, then

$$\langle \mathrm{d}T, \xi \rangle := (-1)^{m+1} \langle T, \mathrm{d}\xi \rangle,$$

for any $\xi \in \mathscr{D}^{n-m-1}(X)$. This gives rise to the notions of closed and exact currents.

If X is a complex manifold of dimension n, then the spaces of currents enjoy a decomposition by bidegree similar to that of differential forms. A (p, q)-*current* acts non-trivially on the space $\mathscr{D}^{n-p,n-q}(X)$ of test forms of bidegree $(n - p, n - q)$. The operator ∂ maps a (p, q)-current T to the $(p + 1, q)$-current ∂T, defined by

$$\langle \partial T, \xi \rangle := (-1)^{p+q+1} \langle T, \partial \xi \rangle,$$

for $\xi \in \mathscr{D}^{n-p-1,n-q}$ and, similarly for $\bar{\partial}$.

The *support* $\operatorname{supp} T$ of a current T is the smallest closed subset $F \subset X$ such that $\langle T, \xi \rangle = 0$ for every test form supported in $X \setminus F$.

Example 2.4.4. Let X be an n-dimensional complex manifold and let $V \subset X$ be an analytic subvariety of pure codimension m. The *current of integration* or *Lelong current* $[V]$ associated to V, introduced by Lelong in [45], is the (m, m)-current defined by

$$\langle [V], \xi \rangle := \int_V \xi, \quad \text{for } \xi \in \mathscr{D}^{n-m,n-m}(X). \tag{2.4.10}$$

For example, the Dirac distribution with mass at $p \in X$ is given by $\delta_p = [\{p\}]$. Lelong currents are closed, and, moreover, they are fundamental examples of *positive currents*. A real (m, m)-form α on X is said to be *positive* if for every collection of $(1, 0)$-forms $\eta_1, \dots, \eta_{n-m}$,

$$\alpha \wedge i\eta_1 \wedge \overline{\eta}_1 \wedge \cdots \wedge i\eta_{n-m} \wedge \overline{\eta}_{n-m}$$

is a non-negative (n,n)-form, that is, a non-negative multiple of the standard volume form in local coordinates. A real (m,m)-current T is said to be *positive* if $\langle T, \xi\rangle \geq 0$ for each real positive test form $\xi \in \mathscr{D}^{n-m,n-m}(X)$.

In Section 2.8 we define currents on reduced analytic spaces. In Section 3.2, we introduce the notion of *current-valued meromorphic functions* which will be useful in the formulation of the results in Papers I and II.

2.4.1.1 Residue and Principal value currents

Currents arise naturally in several complex variables in the form of *residue currents.* A naive theory of residues of meromorphic functions in several complex variables quickly encounters several difficulties. In one complex variable, Cauchy's residue theorem relies on the local nature of poles and on the fact that these can be enclosed by simple closed curves. In two or more dimensions, however, curves do not bound domains, and the polar set of a meromorphic function is often non-compact.

In [34], Herrera and Lieberman laid the groundwork for a residue theory in several variables by associating to a meromorphic function not a number, but a residue *current.* Let $f\colon \Omega \subseteq \mathbb{C}^n$ be a holomorphic function, not identically zero. Then, for any test form ξ of bidegree (n,n) in Ω, they define the *principal value current* U^f of $1/f$ by the limit

$$\langle U^f, \xi\rangle := \lim_{\epsilon\to 0}\int_{|f|>\epsilon}\frac{\xi}{f}, \tag{2.4.11}$$

which they showed indeed exists and defines a current. Proving the existence of such a limit in general can be done using Hironaka's theorem on the resolution of singularities, see Section 2.9 below. We notice that

$$fU^f = 1, \tag{2.4.12}$$

as currents.

The *residue current* R^f of $1/f$ is then defined by $R^f = \bar\partial U^f$. A straightforward computation, using the definition of $\bar\partial$ on currents and Stokes' theorem, shows that

$$\langle R^f, \xi\rangle = \lim_{\epsilon\to 0}\int_{|f|=\epsilon}\frac{\xi}{f}, \tag{2.4.13}$$

where ξ is a test of bidegree $(n,n-1)$. It follows that $\operatorname{supp} R^f \subseteq \{f=0\}$.

Example 2.4.5. Let $n=1$, and suppose a holomorphic function f has a zero at a point $z_0 \in \Omega$ and is non-zero otherwise. Letting $\chi\colon \Omega \to \mathbb{R}$ be a smooth bump function with $\chi \equiv 1$ in a neighborhood of z_0, then

$$\langle R^f, \chi \mathrm{d}z\rangle = 2\pi i \mathrm{Res}_{z=z_0}\{1/f\}.$$

One property of the residue current R^f is that its *annihilator ideal* $\operatorname{ann} R^f$, that is, the ideal of holomorphic functions h such that $hR^f = 0$, is equal to the principal ideal (f) generated by f. Indeed, if $h \in (f)$ it follows immediately from (2.4.13) that $hR^f = 0$. Conversely, suppose that $h \in \operatorname{ann} R^f$ and let $T = hU^f$. In view of (2.4.12), T is a $(0,0)$-current satisfying $fT = h$. Thus,

$$\bar\partial T = h\bar\partial U^f = hR^f = 0,$$

which, by the Weyl lemma for the $\bar\partial$-operator, implies that T is in fact a holomorphic function, and therefore $h = fT \in (f)$.

As a consequence, the residue currents yields a factorization of the Lelong current associated to the *divisor of* f:

$$2\pi i[\operatorname{div}(f)] = R^f \wedge \mathrm{d}f.$$

Here $\operatorname{div}(f)$ denotes the divisor of f, that is, the formal linear combination of irreducible components of $\{f = 0\}$ weighted by their multiplicities (see Section 2.5 below). The current $[\operatorname{div}(f)]$ is the associated Lelong current (see Example 2.5.6 below).

An alternative way to define the principal value current U^f, based on the work of Gel'fand–Shilov [GS64], Bernstein–Gelfand [BG69] and Atiyah [A70], is to consider the following parametric integral

$$\xi \mapsto \int_\Omega \frac{|f|^{2\lambda}}{f}\xi, \tag{2.4.14}$$

for $\xi \in \mathscr{D}^{n,n}(\Omega)$, where λ is a complex number. It turns out that, for any test form ξ, the right-hand side of (2.4.14) is holomorphic as a function of λ for $\mathfrak{Re}\,\lambda \gg 0$, and has an analytic continuation to a neighborhood of the half-space $\{\mathfrak{Re}\,\lambda \geq 0\}$. Furthermore, (2.4.14) defines a current for each λ in that neighborhood, where the value at $\lambda = 0$ is equal to $\langle U^f, \xi\rangle$. It follows that

$$\xi \mapsto \int_\Omega \frac{\bar\partial|f|^{2\lambda}}{f}\xi, \tag{2.4.15}$$

for $\xi \in \mathscr{D}^{n,n-1}(\Omega)$, is holomorphic in a neighborhood the half-space $\{\mathfrak{Re}\,\lambda \geq 0\}$, and its value at $\lambda = 0$ is equal to $\langle R^f, \xi\rangle$. Both (2.4.14) and (2.4.15) are examples of Archimedean zeta functions, cf. (3.0.1) in Chapter 3 below. In Papers I and II, we consider generalizations of (2.4.14) and (2.4.15), which are not necessarily holomorphic, but rather meromorphic, in a neighborhood of the origin.

One important generalization of the above is the association of a residue current to a collection of holomorphic functions. Let $f = (f_1, \ldots, f_m)\colon \Omega \subseteq$

$\mathbb{C}^n \to \mathbb{C}^m$ be a holomorphic mapping. If f defines a complete intersection, that is, if $f^{-1}(0)$ has codimension m, then Coleff and Herrera [20] introduced the current

$$R^f := R^{f_m} \wedge \cdots \wedge R^{f_1}, \tag{2.4.16}$$

called the *Coleff–Herrera product* associated to f.

Since there is no general notion of a product of currents, the right-hand side of (2.4.16) is not a priori well-defined. Nevertheless, under the complete intersection assumption, Coleff and Herrera constructed a product of residue currents that behaves, in many respects, like an honest wedge-product of the $(0,1)$-currents $R^{f_1}, \dots, R^{f_m}$. For instance, R^f is a $\bar{\partial}$-closed $(0,m)$-current and is antisymmetric with respect to the functions f_j. Moreover, its annihilator ideal $\operatorname{ann} R^f$ ideal coincides with the ideal $(f) = (f_1, \dots, f_m)$ generated by f. The inclusion $(f) \subseteq \operatorname{ann} R^f$ was proven by Coleff and Herrera in [20]. The reverse inclusion was established independently by Passare [52] and Dickenstein–Sessa [24].

Remark 2.4.6. The Coleff–Herrera product can also be defined without the complete intersection assumption on the tuple f. In this greater generality, however, the construction is no longer canonical, and it loses many of the properties that make it a suitable residue current associated with f. In [53], Passare–Tsich–Yger introduced Bochner–Martinelli type residue currents associated to arbitrary tuples of holomorphic functions. These currents remain explicit and enjoy better functorial properties than the non-complete-intersection Coleff–Herrera construction, although they do not retain all of the desirable features present in the complete intersection case. Subsequently, Andersson–Wulcan [1] constructed a residue current R^f canonically associated to the ideal (f). In particular, it satisfies

$$\operatorname{ann} R^f = (f).$$

The trade-off is that, unlike the Coleff–Herrera and Bochner–Martinelli constructions, the Andersson–Wulcan residue current is considerably less explicit.

In Paper II, we develop a current calculus with the goal of computing finite parts of certain divergent integrals of forms with singularities along a locally complete intersection, see the summary in Section 4.2. One of the main results in Paper II concerns products of certain currents and depends on the following result:

Theorem 2.4.7 (Samuelsson, Analytic continuation of residue currents)**.** *Let $\Omega \subseteq \mathbb{C}^n$ be a domain and let $f = (f_1, \dots, f_{p+q})\colon \Omega \to \mathbb{C}^{p+q}$ be a holomorphic mapping defining a complete intersection. For any positive integer N and test* *form* $\xi \in \mathscr{D}^{n,n-p}(\Omega)$*, the map*

$$(\lambda_1, \dots, \lambda_{p+q}) \mapsto \int_\Omega \frac{\bar{\partial}|f_1|^{2\lambda_1} \wedge \dots \wedge \bar{\partial}|f_p|^{2\lambda_p} |f_{p+1}|^{2\lambda_{p+1}} \dots |f_{p+q}|^{2\lambda_{p+q}}}{f_1^N \dots f_{p+q}^N} \wedge \xi, \tag{2.4.17}$$

apriori well-defined and holomorphic for $\mathfrak{Re}\,\lambda_1, \dots, \mathfrak{Re}\,\lambda_{p+q} \gg 0$*, is holomorphic in a neighborhood of the half-space* $\{(\lambda_1, \dots, \lambda_{p+q}) \in \mathbb{C}^{p+q} : \mathfrak{Re}\,\lambda_j \geq 0,\ 1 \leq j \leq p+q\}$*.*

In particular, evaluating the map (2.4.17) at $\lambda_1 = \dots = \lambda_{p+q} = 0$, with $q = 0$ and $N = 1$, one recovers the action of the Coleff–Hererra product (2.4.16) on the test form ξ.

2.4.2 Connections and curvature

Connections provide a way to differentiate sections of a vector bundle. They are fundamental tools in differential geometry and give rise to notions of curvature.

Let $E \to X$ be a complex vector bundle of rank m over a complex n-dimensional manifold X. A *connection* on E is a $\mathbb{C}$-linear map

$$\nabla \colon \mathscr{C}^\infty(E) \to \mathscr{E}^1_X \otimes \mathscr{C}^\infty(E),$$

mapping sections to 1-form-valued sections and satisfying the Leibniz rule

$$\nabla(fs) = \mathrm{d}f \otimes s + f \nabla s,$$

for all $f \in \mathscr{C}^\infty(X)$ and $s \in \mathscr{C}^\infty(E)$. In terms of a local frame $s_1, \dots, s_m$ for E over an open subset $U \subset X$, for any section s of E, locally written as $s = \sum_{k=1}^m f_k s_k$,

$$\nabla s = \sum_{j=1}^m \mathrm{d}f_j \otimes s_j + \sum_{j,k=1}^m f_j \theta_{kj} \otimes s_k,$$

where θ_{jk} is an $m \times m$ matrix of differential 1-forms, called the *connection matrix* of ∇ with respect to $s_1, \dots, s_m$, defined by

$$\nabla s_j = \sum_{k=1}^m \theta_{kj} \otimes s_k.$$

With respect to the decomposition $\mathscr{E}^1_X = \mathscr{E}^{1,0}_X \oplus \mathscr{E}^{0,1}_X$ we can write $\nabla = \nabla^{1,0} + \nabla^{0,1}$, with $\nabla^{1,0} \colon \mathscr{C}^\infty(E) \to \mathscr{E}^{1,0}_X \otimes \mathscr{C}^\infty(E)$, and similarly for $\nabla^{0,1}$.

If (E, h) is a Hermitian holomorphic vector bundle, then there exists a unique connection ∇_E compatible with both the holomorphic and Hermitian structures of E, called the *Chern connection.* The former means that $\nabla_E^{0,1} = \bar{\partial}_E$, where $\bar{\partial}_E$ is the Dolbeault operator on E, see Section 2.4 above. The latter means that, for any $s, t \in \mathscr{C}^\infty(E)$,

$$\mathrm{d}h(s,t) = h(\nabla_E s, t) + h(s, \nabla_E t), \tag{2.4.18}$$

where h on the right-hand side is extended *sesquilinerarly* ($\mathbb{C}$-linearly in the first argument and $\mathbb{C}$-antilinearly in the second argument) to E-valued forms.

Given a connection ∇, we define the *curvature operator*

$$\Theta \colon \mathscr{C}^\infty(E) \to \mathscr{E}_X^2 \otimes \mathscr{C}^\infty(E)$$

by $\Theta := \nabla^2$. For the *Chern* connection ∇_E, we denote the corresponding (Chern) curvature operator by $\Theta_E = \nabla_E^2$. A straightforward computation shows that Θ_E is $\mathscr{C}^\infty(X)$-linear, that is, $\Theta_E(fs) = f\Theta_E(s)$, for any $f \in \mathscr{C}^\infty(X)$ and $s \in \mathscr{C}^\infty(E)$. Hence, for each $p \in X$, the value $\Theta_E(s)(p)$ depends only on the value $s(p) \in E_p$. Thus Θ_E defines a smoothly varying fiberwise linear map $E_p \to \bigwedge^2(T_{\mathbb{C},p}X)^* \otimes E_p$, which is equivalent to it defining a smooth section of the bundle of *endomorphism-valued two-forms*

$$\mathscr{E}_X^2 \otimes \operatorname{Hom}(E,E) = \mathscr{E}_X^2 \otimes (E^* \otimes E).$$

In fact, Θ_E is of type $(1,1)$, that is, Θ_E takes values in $\mathscr{E}_X^{1,1} \otimes \operatorname{Hom}(E,E)$.

In general, the Chern curvature of a Hermitian holomorphic vector bundle is matrix-valued. However, in the special case of line bundles, the curvature reduces to a differential $(1,1)$-form. Let (L,h) be a Hermitian holomorphic line bundle and let s_0 be a local holomorphic frame with local weight ϕ, that is,

$$\|s_0\|^2 = e^{-\phi},$$

where $\|s_0\|^2 = h(s_0, s_0)$, cf. Section 2.3.2 above. Since the endomorphism bundle of a line bundle is canonically trivial, $L^* \otimes L \simeq \underline{\mathbb{C}}$, the curvature of the Chern connection may be viewed simply as a $(1,1)$-form. Using the defining properties of the Chern connection, namely $\nabla_L^{0,1} = \bar{\partial}_L$ and (2.4.18), one obtains that, with respect to the local holomorphic frame s_0, the Chern connection is locally given by

$$\nabla_L s_0 = -\partial\phi \otimes s_0.$$

Equivalently, for a local section $s = fs_0$,

$$\nabla_L s = (\partial f - f\partial\phi) \otimes s_0 + \bar{\partial} f \otimes s_0.$$

A direct computation then shows that the curvature form is given locally by

$$\Theta_L = \partial\bar{\partial}\phi.$$

We define the (normalized) *Chern form* by

$$c_1(L,h) = \frac{i}{2\pi}\Theta_L. \tag{2.4.19}$$

It is common to introduce $\mathrm{d}^c := i(\bar\partial - \partial)/4\pi$, in which case

$$c_1(L,h) = \mathrm{dd}^c\phi.$$

In view of the discussion in Section 2.3.2, on overlaps $U \cap U'$ of trivializing neighborhoods defined by holomorphic frames s_0 and s_0', respectively, the corresponding local weights satisfy

$$\phi' = \phi - \log|g|^2,$$

where g is a nowhere-vanishing holomorphic function on $U \cap U'$. Thus, since $\partial\bar\partial \log|g|^2 = 0$, the local forms $(i/2\pi)\partial\bar\partial\phi$ glue together to a globally defined $(1,1)$-form, namely the Chern form $c_1(L,h)$. Since $\mathrm{d}c_1(L,h) = 0$, $c_1(L,h)$ defines a de Rham cohomology class

$$[c_1(L,h)] \in H^2_{\mathrm{dR}}(X,\mathbb{R}),$$

called the *(first) Chern class* of L, and denoted $c_1(L)$. If h' is another Hermitian metric on L, then the corresponding Chern forms differ by an exact form. Hence the class $c_1(L)$ is independent of the choice of Hermitian metric and defines a topological invariant of the line bundle L. The subscript 1 reflects the fact that a higher-rank vector bundle E admits *higher Chern classes* $c_j(E)$ for $1 \le j \le \operatorname{rank} E$.

Remark 2.4.8. The de Rham class $c_1(L)$ defined above in fact lies in the image of the natural map $H^2(X,\mathbb{Z}) \to H^2_{\mathrm{dR}}(X,\mathbb{R})$, where $H^2(X,\mathbb{Z})$ denotes the second integral cohomology group. Thus $c_1(L)$ may also be viewed as an integral cohomology class. From this perspective, the first Chern class defines a homomorphism of abelian groups

$$c_1 \colon \mathrm{Pic}(X) \to H^2(X,\mathbb{Z}),$$

where $\mathrm{Pic}(X)$ is the *Picard group* of (isomorphism classes of) holomorphic line bundles on X (see Section 2.5 below). Indeed, if $L \to X$ is a holomorphic line bundle and L^* is the dual line bundle to L, then

$$c_1(L^*) = -c_1(L).$$

Moreover, if $L' \to X$ is another holomorphic line bundle, then

$$c_1(L \otimes L') = c_1(L) + c_1(L').$$

A Hermitian holomorphic line bundle (L, h) is called *positive* if the associated Chern form $c_1(L, h)$ is positive definite, denoted $c_1(L, h) > 0$. Locally, in terms of a local weight ϕ and holomorphic coordinates $z = (z_1, \dots, z_n)$,

$$c_1(L, h) = \frac{i}{2\pi} \sum_{1 \le j,k \le n} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k. \tag{2.4.20}$$

Thus, $c_1(L, h)$ is positive if and only if the complex Hessian

$$(\phi_{j\bar{k}}) := \left(\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} \right)$$

of the local weight ϕ being positive definite at every point. A line bundle is said to be positive if it admits a Hermitian metric h such that (L, h) is positive; L is said to be *negative* if L^* is positive. Positive line bundles play a central role in complex and algebraic geometry, and will reappear throughout the rest of the thesis. Importantly, the Kodaira embedding theorem implies the existence of a positive line bundle L over a complex manifold X implies that X is projective, and hence *Kähler*, see Sections 2.5.2 and 2.6 below.

Remark 2.4.9. For higher-rank vector bundles, there are several inequivalent notions of positivity, such as *Griffiths positivity* and *Nakano positivity*, but we will not discuss these notions here. For the interested reader, see, e.g., [22, 44]. Nevertheless, an important special case arises from the tangent bundle of a Kähler manifold, where the curvature of the Chern connection gives rise to geometric quantities such as the *Ricci* and *scalar curvatures*. Recall that the Chern connection is the holomorphic analogue of the *Levi–Civita connection* on the real tangent space in Riemannian geometry. In general, the Ricci curvature is obtained by tracing the *Riemann curvature tensor* associated with the Levi–Civita connection with respect to the Riemannian metric. The scalar curvature is obtained by taking a further trace with respect to the metric. In *Kähler geometry*, the Chern and Levi–Civita connections coincide, and the Ricci curvature is naturally encoded by a closed real $(1, 1)$-form, called the *Ricci form*. In view of the above discussion, the Ricci form represents the first Chern class $c_1(K_X^*)$ of the anticanonical line bundle.

2.5 Divisors and holomorphic line bundles

Recall that compact complex manifolds X admit no non-constant global holomorphic functions. To capture global complex analytic information, one is therefore naturally led to consider holomorphic and meromorphic sections of line bundles over X. Holomorphic line bundles and their sections are intimately related to *divisors*, which are ubiquitous in complex algebraic geometry and provide a geometric way of recording the zero and pole structure of meromorphic objects.

We begin by recalling the basic definitions. A (Weil) divisor[4] on a complex manifold X is a locally finite linear combination of irreducible analytic hypersurfaces in X,

$$D = \sum_j a_j V_j.$$

Such hypersurfaces are called *prime divisors.* We say $\mathbb{Q}$-divisor, or $\mathbb{R}$-divisor, etc. to specify the possible values of the coefficients a_j. Generally, if unspecified, the coefficients a_j are assumed to be integers. If all of the coefficients are non-negative, the divisor is called *effective*, and we sometimes write $D \geq 0$. The *support* of D is the closed set $|D| := \bigcup_j V_j$, where it is assumed that V_j in $\sum_j a_j V_j$ are distinct.

Let $D = \sum_j a_j V_j$ be a divisor, where V_j are distinct, and let E be a prime divisor. The *order of vanishing* (or *coefficient*) of D along E is

$$\operatorname{ord}_E(D) := \begin{cases} a_j, & \text{if } E = V_j \text{ for some } j, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, $D = \sum_E \operatorname{ord}_E(D)E$, where the sum runs over all prime divisors on X.

Divisors arise naturally from meromorphic functions. Let V be an irreducible analytic hypersurface and h a holomorphic function defined in a neighborhood of a point $p \in V_{\mathrm{reg}}$. The *order of vanishing* $\operatorname{ord}_{V,p}(h)$ of h along V at p can be defined as the largest integer m such that locally $h = f^m g$, where f is a local defining function for V, and g is a holomorphic function not vanishing identically along V. It is a standard fact that $\operatorname{ord}_{V,p}(h)$ is independent of $p \in V_{\mathrm{reg}}$, and we therefore write $\operatorname{ord}_V(h)$.

For a meromorphic function $f = g/h$, with g and h holomorphic and relatively prime, we define the order of vanishing

$$\operatorname{ord}_V(f) := \operatorname{ord}_V(g) - \operatorname{ord}_V(h),$$

so that poles correspond to negative orders of vanishing. The associated divisor is

$$\operatorname{div}(f) := \sum_V \operatorname{ord}_V(f)V, \tag{2.5.1}$$

where the sum runs over irreducible components V of the zero and pole set of f. Thus $\operatorname{ord}_E(f) = \operatorname{ord}_E(\operatorname{div}(f))$. A divisor of the form (2.5.1) is called a *principal*

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[4] Here we restrict the discussion to divisors over complex manifolds, where the notions of *Weil* and *Cartier* divisors coincide.

divisor. Similarly, for a meromorphic section $s \colon X \to L$ of a holomorphic line bundle, one obtains a divisor $\operatorname{div}(s)$, locally defined by (2.5.1). If s is holomorphic, then $\operatorname{div}(s) \geq 0$.

The set of divisors on X forms an abelian group, and principal divisors form a subgroup. Two divisors D and D' are said to be *linearly equivalent*, denoted $D \sim D'$, if their difference $D - D' = \operatorname{div}(f)$ for some meromorphic function f. Linear equivalence defines an equivalence relation on the set of divisors, and the quotient group of linear equivalence classes of divisors is sometimes called the *divisor class group*.

Let us now describe the correspondence between divisors and holomorphic line bundles. A divisor D on a complex manifold X is locally defined by a single meromorphic function[5]. More precisely, there exists an open cover $\{U_j\}$ and meromorphic functions f_j on U_j such that $D|_{U_j} = \operatorname{div}(f_j)$. On overlaps $U_j \cap U_k$, the quotients $g_{jk} := f_j/f_k$ are necessarily holomorphic and nowhere vanishing, since f_j and f_k have the same order of vanishing along each hypersurface. Moreover, they satisfy $g_{jk} = g_{kj}^{-1}$, and, $g_{jk}g_{k\ell}g_{\ell j} = 1$ on $U_j \cap U_k \cap U_\ell$, for each j, k, ℓ. Thus, $\{g_{jk}\}$ defines a line bundle, in view of the discussion in the beginning of Section 2.3. By abuse of notation, we will denote this line bundle by $\mathcal{O}(D)$, which also denotes its sheaf of holomorphic sections. If $D = \operatorname{div}(f)$ is a principal divisor, then, tautologically, the restrictions $f_j = f|_{U_j}$ are local defining functions for D. It follows that $g_{jk} = 1$ for each j, k, and thus $\mathcal{O}(D) = \underline{\mathbb{C}}$.

The local defining functions f_j determine a distinguished meromorphic section s_D of $\mathcal{O}(D)$, given locally by f_j. By construction, $\operatorname{div}(s_D) = D$, and s_D is holomorphic if and only if D is effective.

Remark 2.5.1. The global holomorphic sections of $L = \mathcal{O}(D)$ can be identified with the set of meromorphic functions f on X satisfying

$$\operatorname{div}(f) + D \geq 0.$$

To see this, let s_D be the distinguished meromorphic section of L such that $\operatorname{div}(s_D) = D$. Any meromorphic section s of L can then be written uniquely as $s = f s_D$, for some meromorphic function f. Indeed, by choosing a trivializing cover $\{U_j\}$ of L, $s|_{U_j}$, $s_D|_{U_j}$ are meromorphic functions on U_j. On overlaps $U_j \cap U_k$, we have

$$s|_{U_j} = g_{jk} s|_{U_k}, \quad s_D|_{U_k} = g_{jk} s_D|_{U_k},$$

where g_{jk} are the transition maps of the trivialization. Thus, on overlaps $U_j \cap U_k$, the local quotients $f_j := s|_{U_j}/s_D|_{U_j}$, which are meromorphic functions, satisfy

$$f_j = \frac{g_{jk}}{g_{jk}} f_k = f_k.$$

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[5] On a reduced analytic space, we need to assume that D is a *Cartier* divisor.

It follows that the f_j glue together to define a global meromorphic function f on X, and $s = fs_D$. The section s is holomorphic if and only if it has no poles, which is equivalent to requiring that the poles of f are compensated by the zeros of s_D, and likewise, that the possible poles of s_D are compensated by zeros of f, that is, $\operatorname{div}(f) + D \geq 0$.

Holomorphic line bundles on X form an abelian group, known as the *Picard group* of X, denoted $\operatorname{Pic}(X)$, with identity element given by the trivial line bundle $\underline{\mathbb{C}}$, group multiplication given by tensor products, and inverses given by dual bundles. In the sequel, we will adopt the common additive notation for tensor products of line bundles:

$$\begin{aligned} L + L' &= L \otimes L', \\ -L &= L^*, \\ kL &= L^{\otimes k}. \end{aligned}$$

The mapping that sends a divisor D to the associated line bundle $\mathcal{O}(D)$ described above is actually a homomorphism with respect to this structure and the additive structure on divisors. Moreover, since principal divisors are mapped to the $0 = \underline{\mathbb{C}}$, we obtain a well defined homomorphism from the divisor class group to $\operatorname{Pic}(X)$. This map is, in fact, an isomorphism. Indeed, let $L \to X$ be a holomorphic line bundle and s a non-zero meromorphic section. Then one obtains a divisor $\operatorname{div}(s)$. If $s' \colon X \to L$ is any other meromorphic section, then

$$\operatorname{div}(s') = \operatorname{div}(s) + \operatorname{div}(s'/s),$$

where the quotient s'/s defines a global meromorphic function, cf. Remark 2.5.1. Thus, $\operatorname{div}(s'/s)$ is principal and, consequently, L determines a linear equivalence class of divisors.

Remark 2.5.2. When X is a compact Riemann surface, every divisor

$$D = \sum_j a_j p_j, \quad p_j \in X,$$

has a well-defined *degree* $\deg, D = \sum_j a_j$. Since principal divisors have degree zero, the degree depends only on the linear equivalence class of D, and thus only on the associated line bundle $L = \mathcal{O}(D)$. One therefore defines $\deg L = \deg D$. Equivalently, if s is a non-zero meromorphic section of a holomorphic line bundle $L \to X$, then $\deg L = \sum_p \operatorname{ord}_p(s)$, that is, the degree equals the number of zeros of s, counted with multiplicity, minus the number of poles (also counted with multiplicity). The degree of a line bundle controls the dimension of its space of holomorphic sections; the precise relation is given by the Riemann–Roch theorem. In higher dimension, analogous notions of degree are defined using intersection theory on linear equivalence classes of divisors.

A divisor $D = \sum_j a_j D_j$ on a complex manifold X is said to have *(simple) normal crossings*[6] if each D_j is smooth, and for every point $p \in |D|$, there exist local holomorphic coordinates $z = (z_1, \dots, z_n)$ centered at p such that

$$D = \sum_{j=1}^{r} a_j \{z_j = 0\},$$

in a neighborhood of p, for some $0 \leq r \leq n$. The support $|D| = \{z_1, \dots, z_r = 0\}$ is then a hypersurface with *normal crossings singularities.* Intuitively, $|D|$ locally looks like the intersection of some number of coordinate hyperplanes. Normal crossings divisors play a distinguished role in complex algebraic geometry. Indeed, by Hironaka's theorem on resolution of singularities, any divisor can, after a suitable sequence of *blowups*, be transformed into a divisor with normal crossings, see Section 2.9 below.

Example 2.5.3. The *tautological line bundle* $\mathcal{O}_{\mathbb{P}^n}(-1) \to \mathbb{P}^n$ is defined by declaring the fiber over a point $[Z] = [Z_0 : \dots : Z_n] \in \mathbb{P}^n$ to be the corresponding complex line $\mathbb{C}Z \subset \mathbb{C}^{n+1}$. Its dual bundle $\mathcal{O}_{\mathbb{P}^n}(1) := -\mathcal{O}_{\mathbb{P}^n}(-1)$ is called the *hyperplane bundle.* The global holomorphic sections of $\mathcal{O}_{\mathbb{P}^n}(1)$ are naturally identified with homogeneous polynomials of degree 1 in $n+1$ variables, and the divisor associated to any non-zero holomorphic section of $\mathcal{O}_{\mathbb{P}^n}(1)$ thus corresponds to a hyperplane in $\mathbb{P}^n$.

Example 2.5.4. For $k \in \mathbb{Z}$, one defines $\mathcal{O}_{\mathbb{P}^n}(k) := k\mathcal{O}_{\mathbb{P}^n}(1)$. For $k \geq 0$, the global holomorphic sections of $\mathcal{O}_{\mathbb{P}^n}(k)$ correspond to homogeneous polynomials of degree k in $n+1$ variables, while $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = 0$ for $k < 0$. Moreover, for each $n \geq 1$, $\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$ with generator $\mathcal{O}_{\mathbb{P}^n}(1)$. Thus every holomorphic line bundle on $\mathbb{P}^n$ is, in fact, isomorphic to $\mathcal{O}_{\mathbb{P}^n}(k)$ for some (unique) $k \in \mathbb{Z}$.

Example 2.5.5. The canonical bundle K_X determines a linear equivalence class of divisors called the *canonical class.* By abuse of notation, one often denotes both the bundle and the corresponding divisor class by K_X, and, frequently refers to a representative of this class as a *canonical divisor.* Similarly, the *anticanonical bundle* $-K_X$ determines the anticanonical class $-K_X$ of anticanonical divisors.

Example 2.5.6. Let $D = \sum_j a_j V_j$ be an $\mathbb{R}$-divisor on X. The *Lelong current* $[D]$ associated to D is defined by

$$[D] = \sum_j a_j [V_j],$$

where $[V_j]$ is the usual Lelong current of the analytic hypersurface V_j, see Example 2.4.4.

---

[6] Some authors distinguish between *normal crossings* and *simple normal crossings.* Throughout this thesis, *normal crossings* will always mean *simple normal crossings.*

Let $s\colon X \to L$ be a holomorphic section of a Hermitian holomorphic line bundle $(L, \|\cdot\|)$. The *Poincaré–Lelong* formula states that

$$\frac{i}{2\pi}\partial\bar{\partial}\log\|s\|^2 = [\operatorname{div}(s)] - c_1(L, \|\cdot\|), \tag{2.5.2}$$

where $c_1(L, \|\cdot\|)$ is the Chern form of $(L, \|\cdot\|)$, see Section 2.4.2. Note that the derivatives on the left-hand side should be interpreted in the weak sense, cf. Section 2.4.1.

2.5.1 $\mathbb{Q}$-line bundles

For a general $\mathbb{Q}$-divisor $D = \sum_j q_j D_j$, $q_j \in \mathbb{Q}$, on X, one may associate to D the corresponding $\mathbb{Q}$-*line bundle*:

$$\mathcal{O}(D) = \sum_j q_j \mathcal{O}(D_j) \in \operatorname{Pic}(X)_{\mathbb{Q}},$$

where $\operatorname{Pic}(X)_{\mathbb{Q}} := \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Equivalently, if $m \in \mathbb{N}$ is chosen such that mD is integral, then $\mathcal{O}(D) = m^{-1}\mathcal{O}(mD)$. Two $\mathbb{Q}$-line bundles D and D' are said to be $\mathbb{Q}$-linearly equivalent, denoted $D_1 \sim_{\mathbb{Q}} D_2$, if they determine the same $\mathbb{Q}$-line bundle. Completely analogously, one defines $\mathbb{R}$-line bundles and $\mathbb{R}$-linear equivalence by replacing $\mathbb{Q}$ with $\mathbb{R}$.

If L is a $\mathbb{Q}$-line bundle, a *holomorphic section* of L is understood as a formal m-root of a holomorphic section of mL, where m is such that mL is a genuine line bundle. In general such a section $s_L = (s_{mL})^{1/m}$ defines a multi-value, but its associated divisor $\operatorname{div}(s_L) := m^{-1}\operatorname{div}(s_{mL})$ is well defined.

Similarly, a smooth Hermitian metric on a $\mathbb{Q}$-line bundle L is understood via a smooth Hermitian metric on a sufficiently divisible multiple mL. If $\|\cdot\|_{mL}$ is a smooth Hermitian metric on mL, the corresponding metric on L is formally characterized by

$$\|(s_{mL})^{1/m}\|_L^2 = \|s_{mL}\|_{mL}^{2/m}.$$

2.5.2 Positivity and ampleness

An important principle in complex and algebraic geometry is that positivity of a line bundle, in the sense of Section 2.4.2, corresponds to an abundance of global holomorphic sections.

Recall that a holomorphic line bundle $L \to X$ is *ample* if, for some positive tensor power kL, is global holomorphic sections define a holomorphic embedding

$$\Phi_{kL}\colon X \hookrightarrow \mathbb{P}^N,$$

called a *Kodaira embedding.* More concretely, if $s_0, \ldots, s_N$ are global holomorphic sections of kL without common zeros, then they define a holomorphic map

$$\Phi_{kL}(p) := [s_0(p) : \cdots : s_N(p)],$$

and ampleness means that, for k sufficiently large, such a map is an embedding.

A fundamental result in complex geometry is the *Kodaira embedding theorem* which states that a holomorphic line bundle L defined over a compact complex manifold is ample if and only if L is positive, see, e.g., [32]. Thus, positivity provides a differential-geometric characterization of the algebro-geometric notion of projective embeddings. For a comprehensice treatment of positivity in algebraic geometry, see [44].

2.6 Kähler manifolds

Kähler manifolds form an important class of complex manifolds where the Hermitian structure, which consists of compatible complex and Riemannian structures, is compatible with an additional *symplectic* structure.

Let X be a complex manifold. The complex structure induces an endomorphism of the real tangent bundle

$$J\colon T_{\mathbb{R}}X \to T_{\mathbb{R}}X \tag{2.6.1}$$

satisfying

$$J^2 = -\mathrm{Id},$$

called the *almost complex structure* associated to the complex manifold structure on X. A smooth $2n$-dimensional manifold equipped with an almost complex structure is called an *almost complex manifold.* Every complex manifold naturally determines an almost complex manifold. The converse is false in general.

A Hermitian metric h on $T^{1,0}X$ may equivalently be viewed as a Riemannian metric g on $T_{\mathbb{R}}X$ that is compatible with J in the sense that $g(Ju, Jv) = g(u, v)$ for all tangent vectors u, v. Indeed, the almost complex structure J induces a canonical identification

$$\iota\colon T_{\mathbb{R}}X \to T^{1,0}X, \qquad u \mapsto \iota(u) = \frac{1}{2}(u - iJu). \tag{2.6.2}$$

In a coordinate neighborhood $z = (z_1, \ldots, z_n)$ with $z_j = x_j + iy_j$, the almost complex structure acts on the induced local frame for $T_{\mathbb{R}}X$ by

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \qquad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}.$$

Accordingly,

$$\iota\left(\frac{\partial}{\partial x_j}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial z_j},$$

and similarly

$$\iota\left(\frac{\partial}{\partial y_j}\right) = i\frac{\partial}{\partial z_j}.$$

The inverse of (2.6.2) is given by

$$T^{1,0}X \to T_{\mathbb{R}}X, \qquad u \mapsto u + \bar{u}.$$

Using this identification, the Riemannian metric g is recovered from h by

$$g(u,v) = 2\mathfrak{Re}\, h(\iota(u), \iota(v)).$$

Associated to such a compatible pair (g, J) is a real differential 2-form ω, defined by

$$\omega(u,v) := g(Ju, v), \tag{2.6.3}$$

called the *fundamental form* of the metric. Conversely, the Hermitian metric h is recovered from (g, J) by

$$h(\iota(u), \iota(v)) = \frac{1}{2}(g(u,v) - i\omega(u,v)).$$

From now on, we will generally consider the $\mathbb{C}$-bilinear extensions of g, ω, and J to the complexified tangent bundle $T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$. Under the identification of $T^{1,0}X$ with the $+i$-eigenspace of J, the relation

$$h = \frac{1}{2}(g - i\omega)$$

continues to hold after extending g, ω, and J $\mathbb{C}$-bilinearly to $T_{\mathbb{C}}X$. Moreover, it follows that $\omega(u,v) = 0$ whenever both $u, v \in T_{\mathbb{C}}X$ are of type $(1,0)$ or both of type $(0,1)$. Hence ω is a differential form of bidegree $(1,1)$.

A Hermitian manifold (X, h) is called *Kähler* if its fundamental form ω is closed,

$$\mathrm{d}\omega = 0. \tag{2.6.4}$$

In this case, ω is called a *Kähler form*, and h a *Kähler metric*. In later sections, rather than using the above definition of a Kähler manifold as a complex manifold equipped with a fixed Kähler structure, we will mostly refer to a Kähler manifold as a complex manifold that admits a Kähler structure.

Locally, in holomorphic coordinates $z = (z_1, \dots, z_n)$, by definition, h may be written as

$$h = \sum_{1 \le j,k \le n} h_{j\bar{k}}\, \mathrm{d}z_j \otimes \mathrm{d}\bar{z}_k$$

where $(h_{j\bar{k}})$ is a positive-definite Hermitian matrix-valued function. Consequently, ω may be written as

$$\omega = \frac{i}{2} \sum_{j,k=1}^{n} h_{j\bar{k}}\, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k. \tag{2.6.5}$$

Note that any real $(1,1)$-form on a complex manifold that is closed and positive, in the sense that everywhere locally ω is given by (2.6.5) with $h_{j\bar{k}}$ a positive definite Hermitian matrix, is a Kähler form, that is, furnishes a Kähler structure on X.

By a slight abuse of terminology, we will frequently identify a Kähler metric with its associated Kähler form, referring simply to "the Kähler metric ω". Since the metric and form determine one another uniquely, there is not ambiguity.

The condition (2.6.4) means precisely that ω defines a *symplectic form* on the underlying smooth manifold of X. As such, a Kähler manifold simultaneously carries compatible complex, Riemannian, and symplectic structures. Although the condition $\mathrm{d}\omega = 0$ might appear rather mild, it has profound geometric consequences. Since ω is closed, it defines a de Rham class $[\omega]_{\mathrm{dR}} \in H^2(X, \mathbb{R})$ called the *Kähler class* of ω (or of the associated metric).

Remark 2.6.1. While symplectic-geometric aspects are important in Kähler geometry in general, and Paper IV in particular, we will omit a general introduction; see, for example, [Lectures on Symplectic Geometry Ana Cannas da Silva] for a comprehensive reference.

A standard local $\partial\bar{\partial}$-lemma implies that every Kähler form ω can be locally written as

$$\omega = i\partial\bar{\partial}\phi.$$

where ϕ is a smooth real-valued function called a *Kähler potential.* The local function ϕ is determined only up to addition of the real part of a holomorphic function (which is annihilated by $\partial\bar{\partial}$).

An important source of Kähler forms is provided by positive holomorphic line bundles. Indeed, if $L \to X$ is a positive line bundle equipped with a Hermitian metric h, then its Chern form $c_1(L, h)$ is a positive $(1,1)$-form and hence a Kähler form, cf. Section 2.4.2. Consequently, every complex manifold admitting a positive line bundle is Kähler. Moreover, if L is positive, then Kähler metrics in the class $c_1(L)$ are in one-to-one correspondence with Hermitian metrics on L (up to multiplication by a positive constant).

Example 2.6.2. The fundamental example of a compact Kähler manifold is $\mathbb{P}^n$. The tautological bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ is negative, while its dual, the *hyperplane bundle* $\mathcal{O}_{\mathbb{P}^n}(1)$, is positive. The standard Hermitian metric on $\mathcal{O}_{\mathbb{P}^n}(1)$ is given in homogeneous coordinates $[Z] = [Z_0 : \cdots : Z_n]$ by the local weight

$$\phi = \log(|Z_0|^2 + \cdots + |Z_n|^2).$$

Its Chern form

$$\omega_{\mathrm{FS}} = \frac{i}{2}\partial\bar{\partial} \log\big(|Z_0|^2 + \cdots + |Z_n|^2\big), \tag{2.6.6}$$

is called the *Fubini–Study form.* The associated Kähler metric is the *Fubini–Study metric* on $\mathbb{P}^n$.

More generally, the weight

$$\phi_k = k \log(|Z_0|^2 + \cdots + |Z_n|^2)$$

defines a Hermitian metric on $\mathcal{O}_{\mathbb{P}^n}(k) \simeq k\mathcal{O}_{\mathbb{P}^n}(1)$, also referred to as the Fubini–Study metric on $\mathcal{O}_{\mathbb{P}^n}(k)$, whose Chern form is $k\,\omega_{\mathrm{FS}}$. Since

$$-K_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(n+1),$$

the induced Fubini–Study metric on the anticanonical bundle has curvature

$$(n+1)\omega_{\mathrm{FS}}.$$

Example 2.6.3. Any submanifold $\iota\colon X \hookrightarrow \mathbb{P}^N$ is Kähler: The pullback $\omega = \iota^*\omega_{\mathrm{FS}}$ of the Fubini–Study form on $\mathbb{P}^N$ via the inclusion furnishes a Kähler form on X.

2.7 Kähler–Einstein metrics

Kähler–Einstein metrics lie at the intersection of complex differential geometry, algebraic geometry, and, more recently, statistical mechanics and probability. The purpose of this section is to provide a brief introduction to the classical theory of Kähler–Einstein metrics, emphasizing aspects most relevant to Paper IV, such as the variational approach. Rather than aiming for a comprehensive treatment, we focus on a selection of ideas and results needed to place the probabilistic framework of Paper IV in a broader context. For a general introduction to Kähler–Einstein geometry, see, e.g., [56].

Kähler–Einstein metrics are special instances of *Einstein metrics*, a class of metrics that originated in Einstein's theory of General Relativity (GR). In GR, *spacetime* is modeled by a *pseudo-Riemannian manifold* (M, g) (of *Lorentzian signature*), and gravitation is interpreted as a manifestation of curvature. The geometry of spacetime is related to the distribution of matter and energy through *Einstein's field equations*:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}, \tag{2.7.1}$$

where $R_{\mu\nu}$ denotes the Ricci tensor, R the scalar curvature, Λ the *cosmological constant*, and $T_{\mu\nu}$ the stress energy tensor[7].

In the absence of matter and energy one obtains the *vacuum Einstein equations*, that is (2.7.1) with $T_{\mu\nu} = 0$. Taking the trace of the vacuum equations yields

$$R = \frac{2n}{n-2}\Lambda,$$

where n is the dimension of spacetime. Substituting this expression back into the vacuum equations shows that any *vacuum solution* satisfies

$$R_{\mu\nu} = \frac{2\Lambda}{n-2} g_{\mu\nu}.$$

In other words, the Ricci tensor is proportional to the metric.

This observation leads naturally to the differential-geometric notion of an *Einstein* metric: A (pseudo-)Riemannian metric g is called Einstein if

$$\mathrm{Ric} = \lambda g,$$

where Ric denotes the Ricci tensor of g, for some constant $\lambda \in \mathbb{R}$.

From a geometric perspective, Einstein metrics are natural candidates for canonical metrics on smooth manifolds. More generally, one often seeks distinguished metrics adapted to the underlying geometric structure of a manifold, with the hope that such metrics reflect important topological, analytic or algebraic properties of the space. This philosophy traces back to the uniformization theorem for Riemann surfaces, which asserts that every simply connected Riemann surface admits a canonical metric of constant curvature. *Kähler–Einstein metrics*, metrics that are simultaneously Kähler and Einstein, may be viewed as higher-dimensional analogues of such canonical metrics.

As we will see below, the existence of Kähler–Einstein metrics is intimately linked to the positivity properties of the canonical bundle K_X and its dual $-K_X$, and the study of such metrics therefore naturally lies at the intersection of complex differential geometry and algebraic geometry.

2.7.1 The Kähler–Einstein equation

Let X be a compact Kähler manifold of dimension n, that is, a compact complex manifold of dimension n admitting a Kähler structure. A Kähler form ω is called *Kähler–Einstein* if it solves the *Kähler–Einstein equation*:

$$\mathrm{Ric}\,\omega = -\beta\omega, \tag{2.7.2}$$

---

[7] The field equations (2.7.1) are written in component form, with respect to *Lorentz indices* μ, ν; equivalently, (2.7.1) may be viewed as a single equation for covariant 2-tensor fields.

for some constant $\beta \in \mathbb{R}$ (the minus sign is simply a notational convention), which corresponds to the Einstein condition in the previous section, specialized to the Kähler setting. The metric associated to such an ω is called a Kähler–Einstein metric. The *Ricci form* $\operatorname{Ric}\omega$ of ω is a real $(1,1)$-form, locally given by

$$\operatorname{Ric}\omega = -i\partial\bar{\partial}\log\omega^n. \tag{2.7.3}$$

Here $\log\omega^n$ denotes the logarithm of the local density of the volume form ω^n. More precisely, in local holomorphic coordinates $z = (z_1, \dots, z_n)$,

$$\omega^n = \frac{i^n}{2^n}\varrho\,\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n, \tag{2.7.4}$$

and $\log\omega^n$ is understood as $\log\varrho$.

Since $\operatorname{Ric}\omega$ is invariant if we rescale ω by a constant, we may normalize $\beta \in \{-1, 0, 1\}$.

On the level of cohomology, (2.7.3) implies that

$$[\operatorname{Ric}\omega] = 2\pi c_1(-K_X) = -2\pi c_1(K_X),$$

where $c_1(-K_X)$ is the first Chern class of the anti-canonical bundle, see Section 2.4.2. This can be understood from the fact that the volume form ω^n naturally defines a Hermitian metric on K_X. Indeed, in local holomorphic coordinates, the section $s = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$ trivializes K_X, and its squared norm with respect to the metric $\|\cdot\|_{\omega^n}$ induced by ω^n is precisely ϱ^{-1}. In view of (2.7.4),

$$\frac{i^{n^2}}{2^n} s \wedge \bar{s} = \|s\|^2_{\omega^n}\omega^n \implies \|s\|^2_{\omega^n} = \varrho^{-1}.$$

Thus, in the notation of Section 2.4.2, $\varrho^{-1} = e^{-\phi}$, so $\phi = \log\varrho$ is the local weight for the induced metric on K_X. Comparing (2.7.3) with (2.4.19), we see that $\operatorname{Ric}\omega$ is 2π times the Chern form of the dual metric on $-K_X$, whose local weight is $-\log\varrho$.

It is common in the Kähler–Einstein literature to absorb the factor 2π either into $\operatorname{Ric}\omega$ or the notation for the Chern classes, and write $[\operatorname{Ric}\omega] = c_1(-K_X) = -c_1(K_X)$. We will adopt this convention in the sequel.

Since any Kähler form $\omega > 0$, (2.7.2) thus imposes the cohomological condition on X that $c_1(K_X)$ either vanishes (whence $\beta = 0$) or has a definite sign (whence $\beta = \pm 1$). The existence problem for (2.7.2) is naturally divided into these three cases.

If $c_1(K_X) = 0$, X is called a Calabi–Yau manifold. In this case, (2.7.2) reduces to

$$\operatorname{Ric}\omega = 0 \tag{2.7.5}$$

and differs from the other two cases above in that the Kähler class $[\omega]$ can be chosen freely, rather than being fixed by $c_1(\pm K_X)$. Accordingly, Yau's celebrated solution [58] to the *Calabi conjecture* [17] states that, on a compact complex manifold X with $c_1(K_X) = 0$, for any fixed Kähler class $[\omega_0] \in H^2(X, \mathbb{R})$, there exists a unique $\omega \in [\omega_0]$ that is *Ricci flat*, that is, $\operatorname{Ric}\omega = 0$.

If $c_1(K_X) > 0$, then K_X is positive and X is said to be *canonically polarized.* Aubin [3] and Yau [58] proved independently that a unique Kähler–Einstein metric always exists in this case.

The remaining case is when $c_1(-K_X) > 0$, in which case X is called a *Fano* manifold. Unlike the previous two cases, there are non-trivial obstructions to the existence of Kähler–Einstein metrics on Fano manifolds. A classical obstruction is provided by Matsushima's theorem [50], which asserts that the existence of a Kähler–Einstein metric implies that the complex Lie group $\mathcal{G} := \operatorname{Aut}_0(X)$ of (biholomorphic) automorphisms of X in the connected component of the identity is *reductive.* This is a strong algebraic condition generalizing semisimplicity, see, e.g., [56]; for instance, $\mathcal{G}$ is reductive if and only if it is the complexification of a maximal compact subgroup $\mathcal{K} \subset \mathcal{G}$. Another obstruction to the existence of a Kähler–Einstein metric is the *Futaki invariant*: A Fano manifold admitting a Kähler–Einstein metric must have vanishing Futaki invariant [30]. When discussing Fano manifolds X in the sequel we will always assume that the Futaki invariant of X vanishes, and, moreover, that $\mathcal{G} = \operatorname{Aut}_0(X)$ is reductive.

The problem of characterizing precisely when a Fano manifold admits a Kähler–Einstein metric remained open for several decades, and was ultimately solved through the resolution of the *Yau–Tian–Donaldson (YTD) conjecture* for Fano manifolds in [19]. More specifically, the YTD conjecture asserts that the existence of a Kähler–Einstein metric on a Fano manifold is equivalent to the algebro-geometric notion of *K-polystability.* This notion was first introduced by Tian [57] and later reformulated in a more algebro-geometric language by Donaldson [25]. Very roughly, K-stability is defined in terms of certain degenerations of X, called *test configurations.* These consist of (flat) families $\pi\colon \mathcal{X} \to \mathbb{C}$, equipped with a $\mathbb{C}^*$ action, such that the fibers $\pi^{-1}(t)$, for $t \neq 0$, are biholomorphic to X, together with a relatively ample line bundle restricting to the anticanonical bundle on the generic fibers. While we will omit the rigorous definition, see, e.g., [56] for an introduction, below we will discuss equivalent characterizations of K-(poly)stability in terms certain properties of functionals arising in a variational framework.

There is a hierarchy of stability notions:

$$\text{uniformly K-stable} \iff \text{K-stable} \implies \text{K-polystable} \implies \text{K-semistable}.$$

A manifold that is not K-semistable is called K-unstable. Roughly, K-stability corresponds to K-polystability together with discreteness of the automorphism group. Uniform K-stability, a priori a stronger notion than K-stability, is now known to be equivalent to K-stability for Fano manifolds [48]. In Paper IV, one of the motivating problems is to introduce, within Berman's probabilistic framework, a way to distinguish strictly K-polystable Fano manifolds, that is, K-polystable but not K-stable manifolds, from non-K-polystable ones, see Section 3.5.5 below.

Example 2.7.1. The *blowup* $\mathrm{Bl}_p\mathbb{P}^n$ $(n \geq 2)$ of complex projective space at a point p, see Section 2.9 below, is an example of a Fano manifold which has non-vanishing Futaki invariant, and therefore does not admit a Kähler–Einstein metric. Furthermore, $\mathrm{Aut}_0(\mathrm{Bl}_p\mathbb{P}^n)$ is the subgroup of $\mathrm{Aut}_0(\mathbb{P}^n) = \mathrm{PGL}_n(\mathbb{C})$ that fix the point p. This is not the complexification of a maximal compact subgroup and thus not reductive, see, e.g., [56].

2.7.2 Monge–Ampère equations and the variational approach

We will now turn briefly to some basic aspects of the variational approach to Kähler–Einstein metrics, which is an important part of the probabilistic framework underlying Paper IV. In this approach, Kähler–Einstein metrics arise as critical points, usually minimizers, of certain functionals defined on the space of Kähler metrics in a fixed Kähler class.

By the global $\partial\bar\partial$-lemma, every Kähler form in a fixed Kähler class $[\omega]$ may be obtained as $\omega_\varphi = \omega + i\partial\bar\partial\varphi$, for some $\varphi \in \mathscr{C}^\infty(X, \mathbb{R})$ such that $\omega_\varphi > 0$. The corresponding space

$$\mathcal{H}_\omega = \big\{\varphi \in \mathscr{C}^\infty(X, \mathbb{R}) \;\big|\; \omega_\varphi = \omega + i\partial\bar\partial\varphi > 0\big\}. \tag{2.7.6}$$

is called the space of Kähler potentials (for the class $[\omega]$).

Two particularly important functionals on $\mathcal{H}_\omega$ are the *Ding functional* $\mathcal{D}$ and the *Mabuchi K-energy* $\mathcal{M}$. The Ding functional is constructed so that its critical points are Kähler–Einstein metrics, while critical points of the Mabuchi K-energy are *constant scalar curvature Kähler (cscK)* metrics. However, cscK metrics in $c_1(\pm K_X)$ coincide with Kähler–Einstein metrics, making $\mathcal{M}$ relevant in the Kähler–Einstein setting. For precise definitions of $\mathcal{D}$ and $\mathcal{M}$, see, e.g., [56], or [8] a variational context. The Mabuchi K-energy is of particular importance to the probabilistic approach, due to its relationship with the *free energy functional* $\mathcal{F}$, see Section 2.7.4 below.

Using the parametrization by Kähler potentials, the Kähler–Einstein equation (2.7.2) may be reformulated as a scalar non-linear partial differential equation of *Monge–Ampère* type. Indeed, if $\psi \in \mathscr{C}^\infty(X, \mathbb{R})$ satisfies

$$\mathrm{Ric}\,\omega = -\beta\omega + i\partial\bar\partial\psi,$$

then, using the identity

$$\operatorname{Ric}\omega_\varphi = \operatorname{Ric}\omega - i\partial\bar\partial \log\left(\frac{\omega_\varphi^n}{\omega^n}\right),$$

the Kähler–Einstein equation for ω_φ can be rewritten as,

$$i\partial\bar\partial\Bigg(\beta\varphi + \psi - \log\left(\frac{\omega_\varphi^n}{\omega^n}\right)\Bigg) = 0.$$

The quotient $\omega_\varphi^n/\omega^n$ above should be interpreted as the unique (globally defined) function $\rho\colon X \to \mathbb{R}_+$ such that $\omega_\varphi^n = \rho\,\omega^n$. Since the only pluriharmonic functions on a compact manifold are constants, this implies (up to adding a constant to ψ) after exponentiation that

$$\omega_\varphi^n = e^{\beta\varphi+\psi}\omega^n. \tag{2.7.7}$$

This is a complex Monge–Ampère equation for the volume form ω_φ^n of the unknown Kähler–metric ω_φ.

Note that equations of the form

$$\omega_\varphi^n = \mathrm{d}V,$$

for some fixed volume form $\mathrm{d}V$ (corresponding to (2.7.7) in the case $\beta = 0$), are known as prescribed volume form equations. Yau's solution of the Calabi conjecture asserts that such equations admit unique solutions in a fixed Kähler class, up to normalization of the potential φ, provided that the total volumes agree:

$$\int_X \omega_\varphi^n = \int_X \mathrm{d}V.$$

In particular, when a Kähler–Einstein metric ω_{KE} exists, it is uniquely determined by its normalized volume form $\mathrm{d}V_{\mathrm{KE}} = \omega_{\mathrm{KE}}^n$. In the Fano case ($\beta = -1$), the metric can moreover be recovered directly from the volume form via

$$\omega_{\mathrm{KE}} = -i\partial\bar\partial \log \mathrm{d}V_{\mathrm{KE}},$$

cf. (2.7.3) and the surrounding discussion. Thus, once the Kähler–Einstein volume form is known, no further Monge–Ampère equation needs to be solved.

This observation is fundamental for the probabilistic approach to Kähler–Einstein metrics, in which the Kähler–Einstein volume form arises as a limit of empirical measures associated with a canonical random point process, see Section 3.5 below.

Remark 2.7.2. The variational approach was developed as an alternative to the so-called *continuity approach.* In the canonically polarized setting, the goal of the continuity method is to show that the set of $\beta \in [0,1]$ for which (2.7.7) has a solution (which is non-empty) is both open and closed, and thus must be all of $[0,1]$, where $\beta = 1$ corresponds to the Kähler–Einstein equation. In the Calabi–Yau and Fano cases, there exist similar continuity method approaches, see, e.g., [56]. However, in the Fano case, if a solution to (2.7.2) does not exist, then the solutions to (2.7.7) for $\beta < 0$ will blow up somewhere in the interval $(-1,0]$.

2.7.3 Coercivity and stability

Of central importance in any variational approach is whether or not the relevant functionals admit minimizers. In the finite-dimensional setting, a standard sufficient condition (assuming lower semi-continuity) is *coercivity*: A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called coercive if $|f| \geq \epsilon\|x\| - C$ for some constants $\epsilon > 0$ and $C \in \mathbb{R}$, where $\|\cdot\|$ denotes the euclidean norm on $\mathbb{R}^n$. Coercivity prevents minimizing sequences from escaping to infinity and, together with suitable compactness properties, implies the existence of minimizers.

In Kähler geometry, analogous notions of coercivity can be formulated for the Ding functional and the Mabuchi K-energy on the infinite dimensional space $\mathcal{H}_\omega$. The role of the distance from the origin is played by "energy-type" functionals on the space of Kähler potentials. One says that the Mabuchi K-energy $\mathcal{M}$ is coercive if there exist constants $\epsilon > 0$ and $C_\epsilon \in \mathbb{R}$ such that

$$\mathcal{M}(\varphi) \geq \epsilon J(\varphi) - C_\epsilon, \qquad \forall \varphi \in \mathcal{H}_\omega, \tag{2.7.8}$$

where J denotes Aubin's J-functional [4]. Equivalently, one may formula coercivity using other functionals comparable to J, such as Aubin's I-functional, the difference $I - J$, or the *Finsler distance* d_1 on $\mathcal{H}_\omega$, see, e.g., [21]. Doing this will change the constants ϵ and C_ϵ, but not when they do or do not exist. In the probabilistic setting, a natural choice turns out to be $I - J$.

Let X be a Fano manifold with reductive automorphism group and vanishing Futaki invariant. In [57], assuming $\mathrm{Aut}_0(X)$ is trivial, Tian showed that coercivity of $\mathcal{M}$ implies the existence of a unique Kähler–Einstein metric on X. Conversely, Darvas–Rubinstein showed that the existence of a unique Kähler–Einstein metric implies coercivity of $\mathcal{M}$, thereby establishing an equivalence between existence and coercivity of the Mabuchi K-energy [21]. This leads naturally to the notion of a stability threshold, measuring the optimal coercivity constant of the Mabuchi K-energy. The coercivity properties of $\mathcal{M}$ are closely related to algebro-geometric stability notions, notably K-stability. The optimal constant ϵ such that (2.7.8) holds is called the *coercivity threshold.* The *analytic* *δ-invariant* is defined by

$$\delta^{\mathrm{A}}(X) = 1 + \sup\bigl\{\epsilon \in \mathbb{R} \bigm| \exists C_\epsilon \in \mathbb{R} : \mathcal{M} \geq \epsilon(I - J) - C_\epsilon\bigr\},$$

normalized such that $\mathcal{M}$ is coercive if and only if $\delta^{\mathrm{A}}(X) > 1$. An algebraic counterpart was introduced by Fujita–Odaka [28]. Their *stability threshold* $\delta(X)$ is defined in terms of log canonical thresholds of certain anticanonical $\mathbb{Q}$-divisors on X. A theorem of Zhang [59] asserts that $\delta(X) = \delta^{\mathrm{A}}(X)$. Results of Fujita–Odaka, Blum–Jonsson and others show that the algebraic stability threshold $\delta(X)$ governs K-stability: $\delta(X) \geq 1$ is equivalent to K-semistability, and $\delta(X) > 1$ is equivalent uniform K-stability (which is equivalent to K-stability). Thus, the equality $\delta(X) = \delta^{\mathrm{A}}(X)$ provides a direct link between coercivity properties of the Mabuchi functional and algebro-geometric stability.

In the above discussion it is assumed that the automorphism group of X is discrete. Indeed, if the Futaki invariant of X vanishes, then the Mabuchi K-energy $\mathcal{M}$ is invariant under the action of $\mathcal{G} = \mathrm{Aut}_0(X)$. However, since $\mathcal{M}$ is $\mathcal{G}$-invariant, while $I - J$ is unbounded along non-compact $\mathcal{G}$-orbits, $\mathcal{M}$ cannot be coercive in the sense of (2.7.8) when $\mathcal{G}$ is non-trivial. This motivates a modified notion of coercivity taking automorphisms into account. One defines

$$(I - J)^{\mathcal{G}}(\varphi) = \inf_{g \in \mathcal{G}} (I - J)(g^*\varphi), \tag{2.7.9}$$

(and similarly for the other choices of functionals) and says that $\mathcal{M}$ is *$\mathcal{G}$-coercive* if there exist constants $\epsilon > 0$, $C_\epsilon \in \mathbb{R}$ such that

$$\mathcal{M}(\varphi) \geq \epsilon(I - J)^{\mathcal{G}}(\varphi) - C_\epsilon, \qquad \forall \varphi \in \mathcal{H}_\omega. \tag{2.7.10}$$

A theorem of Darvas–Rubinstein [21] asserts that $\mathcal{G}$-coercivity is equivalent to the existence of a Kähler–Einstein metric. This naturally leads to a *reduced analytic stability threshold* analogous to $\delta^{\mathrm{A}}(X)$:

$$\delta^{\mathrm{A}}(X)^{\mathcal{G}} := 1 + \sup\bigl\{\epsilon \in \mathbb{R} \bigm| \exists C_\epsilon \in \mathbb{R} : \mathcal{M} \geq \epsilon(I - J)^{\mathcal{G}} - C_\epsilon\bigr\}. \tag{2.7.11}$$

Equivalently, $\delta^{\mathrm{A}}(X)^{\mathcal{G}} > 1$ if and only if X admits a Kähler–Einstein metric.

Motivated by this picture, Paper IV introduces an algebraic invariant, the *reduced Gibbs stability threshold* $\gamma(X)^{\mathcal{G}}$ of X, and conjectures that $\gamma(X)^{\mathcal{G}} = \delta^{\mathrm{A}}(X)^{\mathcal{G}}$, see Section 3.5.5 below.

2.7.4 The free energy functional

The probabilistic counterpart to the Mabuchi K-energy is the *free energy* $\mathcal{F}$, defined on the space $\mathscr{P}(X)$ of probability measures on X. Under the correspondence furnished by the Monge–Ampère equation, the free energy is closely related to the Mabuchi K-energy $\mathcal{M}$. Indeed, for any normalized volume form $\mathrm{d}V$, let $\varphi \in \mathcal{H}_\omega$ be the normalized solution to the Monge–Ampère equation $\omega_\varphi^n = \mathrm{d}V$. Then, up to normalization,

$$\mathcal{F}(\mathrm{d}V) = \mathcal{M}(\varphi). \tag{2.7.12}$$

The free energy may thus be viewed as the Mabuchi K-energy under the *Monge–Ampère correspondence* $\varphi \leftrightarrow \omega_\varphi^n = \mathrm{d}V$. Moreover, the free energy admits a natural extension to arbitrary probability measures. This corresponds, under the extension of the Monge–Ampère correspondence, to extending the Mabuchi K-energy from smooth Kähler potentials to suitable classes of singular potentials; see, e.g., [12].

The free energy decomposes into an energy term and an entropy term:

$$\mathcal{F} = \beta E + \mathrm{Ent}, \tag{2.7.13}$$

where $E\colon \mathscr{P}(X) \to (-\infty, \infty]$ is an energy functional and $\mathrm{Ent}\colon \mathscr{P}(X) \to (-\infty, \infty]$ is a *relative entropy* (with respect to a fixed reference measure), see, e.g., [9] for details. This decomposition reflects the familiar energy–entropy balance of statistical mechanics: The energy term favors concentration near low-energy regions, while the entropy term favors disorder.

The sign of β plays a decisive role. Roughly speaking, both E and Ent admit favorable variational properties, such as lower semicontinuity and suitable lower bounds. Consequently, when $\beta = 1$, both terms contribute with the same sign, making the free energy amenable to minimization. In contrast, when $\beta = -1$, corresponding to the Fano setting, the energy and entropy compete. As a consequence, the existence of minimizers becomes more subtle. This mirrors the situation in Kähler geometry: Kähler–Einstein metrics always exist in the canonically polarized case, whereas in the Fano case their existence equivalent to K-polystability. As we shall see in Section 3.5, this distinction is reflected directly in the probabilistic approach through the sign of β.

In the probabilistic approach to Kähler–Einstein metrics, outlined in Section 3.5, the empirical measures associated to the canonical Gibbs ensembles on X are expected to satisfy a *large deviation principle* with *rate functional* given, up to an additive constant, by $\mathcal{F}$. Consequently, minimizers of the free energy describe the macroscopic equilibrium states of the particle system. Under the Monge–Ampère correspondence, these equilibrium measures correspond to Kähler–Einstein metrics.

2.7.5 Log Fano pairs

Paper IV concerns the probabilistic approach to Kähler–Einstein metrics on *log Fano pairs*. Although we shall mostly restrict the discussion in these introductory chapters to smooth Fano manifolds, it is useful to briefly introduce this more general setting.

For the purposes of this thesis, a *log pair* (X, Δ) consists of a compact Kähler manifold X together with an effective $\mathbb{Q}$-divisor Δ which plays the role of a boundary divisor on X, along which one allows prescribed singular behavior.[8]

The *log canonical line bundle* of X is the $\mathbb{Q}$-line bundle

$$K_{(X,\Delta)} \coloneqq K_X + \Delta,$$

where Δ is identified with the corresponding $\mathbb{Q}$-line bundle. A *log Fano pair* is a log pair (X, Δ) such that the *log anticanonical bundle* $-K_{(X,\Delta)}$ is ample.

Just as a Fano manifold carries a distinguished Kähler class $c_1(-K_X)$, a log Fano pair correspondingly determines the *log anticanonical class* $c_1(-K_{(X,\Delta)})$. Since the first Chern class extends linearly to $\mathbb{Q}$-line bundles, any $\mathbb{Q}$-divisor $\Delta = \sum_j a_j \Delta_j$ naturally determines a cohomology class

$$c_1(\Delta) \coloneqq c_1(\mathcal{O}(\Delta)) = \sum_j a_j c_1(\mathcal{O}(\Delta_j)) \in H^2(X, \mathbb{Q}),$$

and hence

$$c_1(-K_{(X,\Delta)}) = c_1(-K_X) - c_1(\mathcal{O}(\Delta)),$$

cf. Remark 2.4.8.

Roughly, a *log Kähler–Einstein metric* on a log Fano pair (X, Δ) is a representative of $c_1(-K_{(X,\Delta)})$ solving the equation

$$\operatorname{Ric} \omega = \omega + [\Delta],$$

in the weak sense of currents, where $[\Delta]$ denotes the Lelong current of Δ, see Example 2.5.6. The precise interpretation of this equation is subtle, requiring a suitable theory of singular Kähler metrics, Monge–Ampère operators, and Ricci curvature for currents. We shall not pursue these aspects further here; see, e.g., [11].

2.8 Currents on reduced analytic spaces

In this section we define smooth differential forms and currents on *reduced analytic spaces (of pure dimension).* These can be thought of as "complex manifolds with analytic singularities". The standard introduction uses the

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[8] In the literature, this is sometimes called *log smooth* log pairs (when Δ is assumed to have normal crossings); general log pairs allow X to have certain singularities.

algebro-geometric machinery of *locally ringed spaces*; for background, see, e.g., [22].

A locally ringed space is a pair $(X, \mathcal{O}_X)$, or simply X (with $\mathcal{O}_X$ understood) where X is a topological space and $\mathcal{O}_X$ is a sheaf of local rings on X, called the *structure sheaf.* An *analytic space* is a locally ringed space that is locally isomorphic to a *local model space.*

Let $\Omega \subseteq \mathbb{C}^N$ be a domain. A local model space is a ringed space $(Z, \mathcal{O}_Z)$, where $Z \subset \Omega$ is given by the common vanishing locus of a set of holomorphic functions $f_1, \ldots, f_m \colon \Omega \to \mathbb{C}$, and $\mathcal{O}_Z = \mathcal{O}_\Omega / \mathcal{J}_Z$. Here $\mathcal{O}_\Omega$ is the sheaf of rings of holomorphic functions on Ω induced by the standard structure sheaf $\mathcal{O}_{\mathbb{C}^N}$, and $\mathcal{J}_Z \subset \mathcal{O}_\Omega$ is the ideal sheaf generated by $f_1, \ldots, f_m$. This means that for any open $\widetilde{\Omega} \subseteq \Omega$, $\mathcal{J}_Z(\widetilde{\Omega})$ is the ideal in the ring $\mathcal{O}_\Omega(\widetilde{\Omega}) = \mathcal{O}_{\mathbb{C}^N}(\widetilde{\Omega})$ generated by $f_1|_{\widetilde{\Omega}}, \ldots, f_m|_{\widetilde{\Omega}}$. Locally isomorphic to a local model space means that each $x \in X$ admits a neighborhood U and an isomorphism of locally ringed spaces $\varphi \colon (U, \mathcal{O}_X|_U) \to (Z, \mathcal{O}_Z)$, where $(Z, \mathcal{O}_Z)$ is a local model space.

An analytic space X is called *reduced* if, for each local model Z, the corresponding ideal sheaf $\mathcal{J}_Z$ is radical (for each open $\widetilde{\Omega} \subset \Omega$ if $f^n \in \mathcal{J}_Z(\widetilde{\Omega})$ for some $n \in \mathbb{N}$ then $f \in \mathcal{J}_Z(\widetilde{\Omega})$) implying that $\mathcal{O}_Z$ is a *reduced* sheaf of rings.

We say that X has pure dimension n if every local model Z has pure dimension n in the sense of analytic subvarieties, see Section 2.2.3 above.

Example 2.8.1. Consider the analytic space

$$Z = \left\{ (z_1, z_2) \in \mathbb{C}^2 : z_1^2 = 0 \right\} \subset \mathbb{C}^2.$$

As a set, this is just the hyperplane $\{z_1 = 0\}$, but its structure sheaf is given by $\mathcal{O}_Z = \mathcal{O}_{\mathbb{C}^2}/(z_1^2)$. In this quotient, the function z_1 is not zero, but satisfies $z_1^2 = 0$ and is therefore nilpotent. Geometrically, Z can be thought of as a "thickened" version of $\{z_1 = 0\}$. Any holomorphic function on Z can be written locally as

$$f(z_1, z_2) = a(z_2) + b(z_2) z_1,$$

since higher powers of z_1 vanish in the quotient $\mathcal{O}_{\mathbb{C}^2}/(z_1^2)$. The term $a(z_2)$ is the restriction of f to the hyperplane $z_1 = 0$, while the coefficient $b(z_2)$ of z_1 can be thought of as describing some infinitesimal variation of f transverse to the hyperplane $\{z_1 = 0\}$.

Throughout this thesis, we restrict our attention to reduced analytic spaces of pure dimension. The reason is that the differential forms and currents considered here are geometric objects defined on the regular locus of the space. As such, they capture the complex-analytic geometry of the underlying reduced space, but do not detect nilpotent elements in the structure sheaf. While one can develop a theory of differential forms and currents on non-reduced spaces, this requires additional algebraic machinery and will not be needed here.

For a reduced analytic space X, the *regular locus*, denoted X_{reg}, is the set of points $p \in X$ such that X is a complex manifold in a neighborhood of p. The complement $X_{\mathrm{sing}} := X \setminus X_{\mathrm{reg}}$ is called the *singular locus* of X.

Since X_{reg} is a complex manifold, differential forms and currents on a reduced analytic space are defined by working on the regular locus and requiring compatibility with local embeddings into complex manifolds. Let $U \subset X$ be an open subset identified with a local model space $Z \subset \Omega \subset \mathbb{C}^N$. We define the sheaf $\mathscr{E}_X^m$ of smooth complex differential m-forms on X by

$$\mathscr{E}_Z^m := \mathscr{E}_\Omega^m / \mathscr{N}_{Z,\Omega}^m,$$

where $\mathscr{E}_\Omega^m$ denotes the sheaf of smooth complex differential m-forms on Ω, and $\mathscr{N}_{Z,\Omega}^m$ is the subsheaf consisting of forms whose pullback to Z_{reg} vanishes. In other words, two smooth forms on the ambient manifold Ω define the same differential form on Z if they agree on the regular locus Z_{reg}. Thus, a smooth form on Z may be viewed as a smooth form on Z_{reg}, modulo the ambiguity of how it is extended to the ambient manifold.

One checks that this definition is independent of the choice of local embedding and therefore defines a sheaf $\mathscr{E}_X^m$ on X. As in the manifold case, these sheaves admit a decomposition into forms of bidegree:

$$\mathscr{E}_X^m = \bigoplus_{p+q=m} \mathscr{E}_X^{p,q}.$$

More generally, the notions of complex and holomorphic vector bundles extend naturally to reduced analytic spaces by replacing local trivializations over manifolds with local trivializations over analytic spaces. In particular, holomorphic sections of holomorphic vector bundles over reduced analytic spaces are defined exactly as in the manifold case. In particular, $\mathscr{E}_X^m$ is a complex vector bundle over X for each m, on which the operators d, ∂ and $\bar\partial$ are naturally defined.

Similarly, the theory of currents on complex manifolds outlined in Section 2.4.1 extends naturally to reduced analytic spaces of pure dimension. We denote by $\mathscr{D}(X)$ the space of smooth differential forms with compact support on X, and write $\mathscr{D}^m(X)$ or $\mathscr{D}^{p,q}(X)$ when we want to emphasize the degree or bidegree. A *current* on X is a continuous (in the sense of Section 2.4.1 naturally extended to reduced analytic spaces) linear functional $T\colon \mathscr{D}(X) \to \mathbb{C}$. The space of currents is denoted by $\mathscr{D}'(X)$. A current of bidegree (p,q) on X is a continuous linear functional on $\mathscr{D}^{n-p,n-q}(X)$, where $n = \dim X$. The operators d, ∂ and $\bar\partial$ extend to currents by duality exactly as in the manifold case.

2.9 Blowups and resolution of singularities

Hironaka's celebrated theorem asserts that every reduced analytic space X (more generally, every variety over a field of characteristic zero) admits a *resolution of singularities* [35]; see also [43]. More precisely, a resolution of singularities of X means a non-singular analytic space $\widetilde{X}$ together with a *modification* $\pi\colon \widetilde{X} \to X$,. Moreover, Hironaka's construction proceeds by a finite sequence of *blowups* along smooth *centers*. The existence of such a resolution is fundamental for the results in Papers I and II. Moreover, in Section 2.9.1 below, we will introduce the *Fulton–MacPherson compactification* of configuration spaces, which is an explicit modification that can be constructed through an iterated sequence of blowups, and which plays an important role in Papers III and IV. We therefore begin by recalling the notions of modifications and blowups.

A holomorphic map $f\colon X \to Y$ between reduced analytic spaces is a modification if it is *proper*, that is, if $f^{-1}(K) \subset X$ is compact for every compact subset $K \subset Y$, and if there exists a subvariety $V \subset Y$ of codimension ≥ 1 such that $f^{-1}(V)$ is nowhere dense and the restriction

$$f\colon X \setminus f^{-1}(V) \to Y \setminus V$$

is a biholomorphism. We will refer to the set $f^{-1}(V)$ as the exceptional set of the modification.

An important feature of modifications is that they are biholomorphisms outside of a set of positive codimension. Since such subsets have Lebesgue measure zero, modifications preserve integrals of integrable forms of top degree. More precisely, if $f\colon X \to Y$ is a modification and ω is an integrable form of top degree on Y, then

$$\int_Y \omega = \int_X f^*\omega.$$

Thus, a modification may be viewed as a holomorphic change of variables formula. This property is what makes them so useful to consider in the study of singular integrals and Archimedean zeta functions.

Example 2.9.1. Let $f\colon X \to Y$ be a modification between complex manifolds. Then the *relative canonical bundle*

$$K_{X/Y} \coloneqq K_X - f^*K_Y$$

admits a distinguished divisor representative, called the *relative canonical divisor* and also denoted $K_{X/Y}$. Locally, it is defined by the vanishing of the Jacobian determinant of f, and supported on the exceptional set of the modification.

The model examples of modifications in complex algebraic geometry are *blowups* and compositions thereof. Let X be a reduced analytic space of pure dimension n, and let $V \subset X$ be a smooth analytic subvariety of codimension m. The *blowup* of X along V is a reduced analytic space $\mathrm{Bl}_V X$ together with a modification $\pi\colon \mathrm{Bl}_V X \to X$, which restricts to a biholomorphism $\mathrm{Bl}_V X \setminus \pi^{-1}(V) \to X \setminus V$. The subvariety V is called the center of the blowup. Geometrically, the blowup enriches each point of V with the data of the normal directions to V at p, see Remark 2.9.2, corresponding to the exceptional set $\mathrm{Exc}(\pi) := \pi^{-1}(V)$ which is a hypersurface, and called the *exceptional divisor* or the *exceptional hypersurface.* By abuse of notation, we will often identify the exceptional hypersurface with the corresponding effective divisor given by the formal sum of irreducible components of $\mathrm{Exc}(\pi)$.

For an analytic subvariety $W \subset X$, the *strict transform* $\mathrm{Strict}_\pi(W)$ of W (with respect to the blowup $\pi\colon \mathrm{Bl}_V X \to X$) is defined as the closure of $\pi^{-1}(W \setminus V)$ in $\mathrm{Bl}_V X$. We call the preimage $\pi^{-1}(W) \supset \mathrm{Strict}_\pi(W)$ the *total transform* of W. In Paper III, we also make use of the *dominant transform*, which interpolates between the strict and total transforms, see Paper III, Section 2.1.2.

The space $\mathrm{Bl}_V X \to X$ enjoys the following local description: For any $p \in V \cap X_{\mathrm{reg}}$, we can find a holomorphic chart (U, φ) with $z = (z_1, \dots, z_n)$ such that p corresponds to $z = 0$ and such that $\varphi(V \cap U) = \{z_1 = \cdots = z_m = 0\}$. Then, the blowup of X along V is locally given by

$$\mathrm{Bl}_V U = \big\{(z, [t]) \in U \times \mathbb{P}^{m-1} : z_j t_k - z_k t_j = 0 \text{ for each } 1 \leq j < k \leq m\big\}, \tag{2.9.1}$$

and the map π is locally given by the restriction to $\mathrm{Bl}_V U$ of the natural projection $\mathrm{proj}_1\colon U \times \mathbb{P}^{m-1} \to U$. In Paper III, Section 2.1.2 we give another local description of the blowup in terms of a covering by holomorphic charts.

In the case where V is the zero locus of a (non-identically vanishing) holomorphic section s of a holomorphic vector bundle $E \to X$, the blowup admits a global description in terms of the *projectivization* $\mathbb{P}(E)$ of E. Recall that the projectivization $\mathbb{P}(E) \to X$ of a rank-m vector bundle $E \to X$ is the fiber bundle whose fiber over $p \in X$ is the projective space $\mathbb{P}(E_p) \simeq \mathbb{P}^{m-1}$. Moreover, there is a tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathbb{P}(E)$ whose fiber over a point $[v] \in \mathbb{P}(E_p)$ is the complex line spanned by v inside E_p, cf. Example 2.5.3. The section s determines a holomorphic map $X \setminus V \to \mathbb{P}(E)$, and the blowup $\mathrm{Bl}_V X$ may be identified with the closure of its image in $\mathbb{P}(E)$.

Remark 2.9.2. If X is a manifold, then the exceptional divisor $\mathrm{Exc}(\pi)$ of a blowup $\pi\colon \mathrm{Bl}_V X \to X$ is naturally identified with the projectivized normal bundle $\mathbb{P}(N_{V/X}) \to V$. In particular, if V has codimension m, the fibers of $\mathrm{Exc}(\pi) \to V$ are projective spaces $\mathbb{P}^{m-1}$.

Remark 2.9.3. If X is a manifold and $V \subset X$ has codimension m, then

$$K_{\mathrm{Bl}_V X/X} = (m-1)\mathrm{Exc}(\pi). \tag{2.9.2}$$

A consequence of Hironaka's theorem is that, given a reduced analytic space X of pure dimension and a proper analytic subvariety $V \subset X$, there exists a modification $\pi\colon \widetilde{X} \to X$ such that

- $\widetilde{X}$ is non-singular, that is $\widetilde{X}$ is a complex manifold,
- π is a biholomorphism over $X \setminus (V \cup X_{\mathrm{sing}})$,
- the preimage $\pi^{-1}(V)$ of V is a normal crossings divisor on $\widetilde{X}$.

Moreover, π may be obtained as a finite composition of blowups along smooth centers.

Suppose $s\colon X \to E$ is a holomorphic section of a holomorphic vector bundle $E \to X$ such that $V = \{s = 0\}$. Then the modification $\pi\colon \widetilde{X} \to X$ above may be chosen so that the pullback ideal of the ideal generated by the components of s become *locally principal.* Equivalently, there exists a line subbundle $L \subset \pi^*E$ and a factorization

$$\pi^* s = s_0 \otimes s',$$

where s_0 is a holomorphic section of L such that $\mathrm{div}(s_0)$ has normal crossings, and s' is a nowhere-vanishing section of $L^{-1} \otimes \pi^*E$.

In local holomorphic coordinates $z = (z_1, \ldots, z_n)$ near any point $p \in \widetilde{X}$, one may write

$$s_0 = z_1^{m_1} \cdots z_r^{m_r} \sigma,$$

for $0 \leq r \leq n$ and positive integers $m_1, \ldots, m_r$, where σ is a local holomorphic frame for L. Thus, after passing to $\widetilde{X}$, the vanishing locus of s is locally given by a monomial equation.

In Paper III, in the special case where X is a manifold and $D = V$ is a divisor, we refer to a modification as above as an *embedded resolution* of the pair (X, D). In this setting, this is consistent with standard terminology. Likewise, it is consistent with the notion of a *log resolution*, which is used in Paper IV.

2.9.1 The Fulton–MacPherson compactification

In this section we introduce the *Fulton–MacPherson compactification* of the configuration space of a complex manifold (or reduced analytic space), which is essential for the results of Paper III, and which also sees use in Paper IV. We will omit many technical details and focus instead on the geometric intuition.

A more detailed treatment can be found in Paper III, Section 2.3, as well as in the original paper of Fulton and MacPherson [29]; see also, e.g., [49].

Let X be an n-dimensional complex manifold and let N be a positive integer. Recall that the N^{th} *configuration space* $F(X,N)$ is the open subset of X^N consisting of tuples of pairwise distinct points in X, that is,

$$F(X,N) := X^N \setminus \bigcup_{j<k} W_{\{j,k\}},$$

where

$$W_{\{j,k\}} = \big\{(x_1,\dots,x_N) \in X^N : x_j = x_k\big\},$$

denotes the *diagonal* corresponding to coincidence of the j^{th} and k^{th} points.

The Fulton–MacPherson compactification replaces the degenerate configurations represented by the union of diagonals $\bigcup_{j<k} W_{\{j,k\}}$ with additional geometric data recording the relative directions in which the points approach one another. More precisely, it consists of a complex manifold $X^{[N]}$ and a modification $\pi\colon X^{[N]} \to X^N$ such that

$$\pi^{-1}(F(X,N)) \simeq F(X,N),$$

while $\pi^{-1}(\bigcup_{j<k} W_{\{j,k\}})$ is a normal crossings divisor. It can be constructed as an iterated sequence of blowups along smooth centers determined by the collection of partial diagonals

$$W_J = \{(x_1,\dots,x_N) \in X^N : x_j = x_k \text{ for all } j,k \in J\},$$

where $J \subseteq \{1,\dots,N\}$.

The Fulton–MacPherson compactification thus furnishes an explicit modification of X^N, of the form discussed in the previous section. This makes it particularly useful in the study of the partition functions arising in Papers III and IV, see Section 3.4.1 below, allowing questions about their asymptotic and analytic behavior to be reduced to elementary model cases.

2.9.2 Log canonical thresholds

The log canonical threshold is a singularity invariant associated to an effective divisor. In the present context, it arises naturally as an integrability threshold for Archimedean zeta functions. For a more thorough introduction to log canonical threshold and related notions, see [44].

Let X be a compact complex manifold and let D be an effective $\mathbb{Q}$-divisor on X. Let $\pi\colon \widetilde{X} \to X$ be a log resolution of the pair (X,D), and let E be a prime divisor on $\widetilde{X}$. The *log discrepancy* of E with respect to (X, D) is

$$A_{(X,D)}(E) \coloneqq 1 + \operatorname{ord}_E(K_{\widetilde{X}/X} - \pi^* D), \tag{2.9.3}$$

where $K_{\widetilde{X}/X}$ denotes the relative canonical divisor, see Example 2.9.1 above. Roughly speaking, the log discrepancy measures the severity of the singularities of the pair (X, D) along the divisor E.

The pair (X, D) is called *log canonical* if $A_{(X,D)}(E) \geq 0$ for every prime divisor E on a log resolution of (X, D). It is called *Kawamata log terminal (klt)* if $A_{(X,D)}(E) > 0$ for every such divisor.

The *log canonical threshold* of D is

$$\operatorname{lct}_X(D) \coloneqq \inf \big\{ q \in \mathbb{Q} : (X, qD) \text{ is not log canonical} \big\}. \tag{2.9.4}$$

For the purposes of this thesis, the most important characterization of the log canonical threshold is as an integrability threshold: Let $m \in \mathbb{N}$ be such that mD is integral, and let $L_{mD} \to X$ be the associated line bundle, and $s_{mD} \colon X \to L_{mD}$ a holomorphic section of L_{mD} with $\operatorname{div}(s_{mD}) = mD$. Then the log canonical threshold of D can be defined as

$$\operatorname{lct}_X(D) = \sup \left\{ c > 0 : \int_X \|s_{mD}\|^{-2c/m} \mathrm{d}V < \infty \right\}, \tag{2.9.5}$$

for any choice of Hermitian metric $\|\cdot\|$ on L_{mD} and volume form $\mathrm{d}V$ on X. The value of $\operatorname{lct}_X(D)$ is independent of the choice of Hermitian metric on L_{mD}, volume form $\mathrm{d}V$ on X, and integer m for which mD is integral.

Formula (2.9.5) shows that the log canonical threshold is precisely the critical exponent governing the convergence of the corresponding Archimedean zeta function. Indeed,

$$\Gamma_{\|s_D\|}(\lambda) = \int_X \|s_D\|^{2\lambda} \mathrm{d}V,$$

converges for $\mathfrak{Re}\,\lambda > -\operatorname{lct}_X(D)$ and diverges for $\mathfrak{Re}\,\lambda < -\operatorname{lct}_X(D)$.

The theory extends naturally to pairs involving $\mathbb{R}$-divisors; see e.g., [42]. For instance, in the analytic formulation (2.9.5), one may replace the defining section by a meromorphic section, thereby allowing divisors with negative coefficients. More generally, analogous thresholds may be defined for $\mathbb{R}$-divisors. Such generalized thresholds arise in Paper III, where the interaction energy of a Coulomb gas is encoded by a coupling matrix that may produce $\mathbb{R}$-divisors with arbitrary real coefficients, see Section 3.4 below.

Log canonical thresholds also play a central role in Kähler geometry. For instance, the Fujita–Odaka stability threshold $\delta(X)$, discussed in Section 2.7.3 above, is defined in terms of log canonical thresholds of certain pluri-anticanonical divisors. There is also a valuative characterization of $\delta(X)$, due to Blum–Jonsson [16], which features in Paper IV but which we will not discuss here.

As aforementioned in Section 2.7.5, Paper IV is formulated in the more general setting of *log Fano pairs*, where the notions of log discrepancy and log canonical threshold admit natural extensions; see Paper IV, Section 2.

3 Archimedean zeta functions

The central objects of this thesis are certain meromorphic functions, obtained by regularizing (possibly) divergent integrals defined by global complex-geometric data. Borrowing terminology from number theory, such functions are often called *Archimedean zeta functions*, reflecting their role as the counterparts at the Archimedean places of the more classical *Igusa zeta functions* defined over p-adic places; see, e.g., [39].

The prototypical example is the Archimedean zeta function associated to a holomorphic function $f\colon \mathbb{C}^n \to \mathbb{C}$. For any test function $\xi \in \mathscr{D}(\mathbb{C}^n)$, let

$$\Gamma^{\xi}_{|f|}(\lambda) = i^{n^2} \int_{\mathbb{C}^n} |f(z)|^{2\lambda} \xi \, \mathrm{d}z \wedge \mathrm{d}\bar{z}, \tag{3.0.1}$$

A priori, $\Gamma^{\xi}_{|f|}(\lambda)$ is defined and holomorphic for $\mathfrak{Re}\, \lambda > 0$ (in fact, for $\mathfrak{Re}\, \lambda > -\epsilon$ for $\epsilon > 0$ sufficiently small). Indeed, $|f|^{2\lambda}$ is locally integrable for $\mathfrak{Re}\, \lambda > 0$, and holomorphicity follows from differentiation under the integral sign. As we have alluded to already in the introduction, and as we will soon see, (3.0.1) admits a meromorphic continuation to all of $\mathbb{C}$.

We will mostly encounter the following variant of (3.0.1):

$$\Gamma^{\xi}_{|f|e^{-\phi/2}}(\lambda; \omega) = \int_{\mathbb{C}^n} |f(z)|^{2\lambda} e^{-\lambda\phi} \omega \wedge \xi, \qquad \xi \in \mathscr{D}^{n-p,n-q}(\mathbb{C}^n), \tag{3.0.2}$$

where $\phi \in \mathscr{C}^\infty(\mathbb{C}^n, \mathbb{R})$, and

$$\omega = \frac{\widetilde{\omega}}{|f|^{2M}},$$

for some smooth (p,q)-form $\widetilde{\omega}$ and rational number $M \geq 0$. The additional factor $e^{-\lambda\phi}$ arises naturally in the global complex-geometric setting, but it does not affect the pole structure of the meromorphic continuation. In view of (3.0.2), the factor $|f|^{2\lambda}$ regularizes the singularity of the possibly non-integrable integrand $\omega \wedge \xi$. However, if we write $\widetilde{\omega} \wedge \xi = i^{n^2} \Psi \mathrm{d}z \wedge \mathrm{d}\bar{z}$, we see that, ignoring the factor $e^{-\lambda\phi}$, $\Gamma^{\xi}_{|f|}(\lambda; \omega) = \Gamma^{\Psi}_{|f|}(\lambda - M)$. Thus, we will restrict our focus to

$\Gamma^{\xi}_{|f|}(\lambda)$ in the following section where, using *Bernstein–Sato theory*, we will give a proof of the following classical result:

Theorem 3.0.1 (Atiyah [2], Bernšteĭn–Gel'fand [14]). *For every $\xi \in \mathscr{D}(\mathbb{C}^n)$, the function $\Gamma^{\xi}_{|f|}(\lambda)$ given by* (3.0.1)*, initially defined for $\mathfrak{Re}\,\lambda > 0$, admits a meromorphic continuation to $\mathbb{C}$, whose poles are contained in a discrete subset of the negative rational numbers.*

3.1 Bernstein–Sato polynomials

One of the fundamental properties of Archimedean zeta functions is that they admit meromorphic continuations. Although the results of this thesis rely primarily on resolution of singularities and local computations, it is instructive to note that meromorphic continuation may also be deduced from the *Bernstein–Sato polynomial.* We therefore briefly introduce this polynomial and explain how the associated functional equation yields the meromorphic continuation of Archimedean zeta functions of the form (3.0.1).

Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function. Then there exists a non-zero polynomial $b(\lambda) \in \mathbb{C}[\lambda]$ and a differential operator $P = \sum_j \lambda^j P_j$, where P_j are differential operators of the form

$$P_j = \sum_{J \in \mathbb{Z}^n_{\geq 0}} p_J(z) \frac{\partial^{|J|}}{\partial z^J}, \tag{3.1.1}$$

for $p_J(z) \in \mathbb{C}[z_1, \dots, z_n]$ and

$$\frac{\partial^{|J|}}{\partial z^J} = \frac{\partial^{|J|}}{\partial z_1^{J_1} \dots \partial z_n^{J_n}},$$

such that

$$P(\lambda) f^{\lambda+1} = b(\lambda) f^{\lambda}. \tag{3.1.2}$$

The set of all such polynomials contains a unique monic polynomial of minimal degree b_f, called the *Bernstein–Sato polynomial* associated to f. This result was originally proved by Bernstein for polynomial functions and later extended to germs of holomorphic functions on complex manifolds by Kashiwara using *D-module* techniques; see [15] for a modern treatment. Furthermore, it was shown by Kashiwara in [41] that the roots of the Bernstein–Sato polynomial b_f are all negative rational numbers.

The Bernstein–Sato functional equation (3.1.2) combined with Kashiwara's theorem on the roots of b_f gives rise to a meromorphic continuation of the

Archimedean zeta function (3.0.1). In view of (3.1.2), we have

$$\overline{P(\lambda)}\bar{f}^{\lambda+1} = b_f(\lambda)\bar{f}^{\lambda},$$

where we used that $\overline{b_f} = b_f$, since b_f has rational coefficients, and where $\overline{P(\lambda)}$ is obtained by replacing each P_j by $\overline{P_j}$; the parameter λ is regarded as an independent complex variable and is not conjugated. Multiplying the two formulas we thus obtain

$$P(\lambda)\overline{P(\lambda)}|f|^{2(\lambda+1)} = b_f(\lambda)^2|f|^{2\lambda}, \tag{3.1.3}$$

where we used that $f^{\lambda+1}\overline{P(\lambda)} = \overline{P(\lambda)}f^{\lambda+1}$, since $\overline{P(\lambda)}$ contains only antiholomorphic differential operators. Formally, we have the following equality

$$|f|^{2\lambda} = \frac{1}{b_f(\lambda)^2}P(\lambda)\overline{P(\lambda)}|f|^{2(\lambda+1)}, \tag{3.1.4}$$

which is completely rigorous for $\mathfrak{Re} > 0$. Thus, for $\xi \in \mathscr{D}(\mathbb{C}^n)$ and $\mathfrak{Re}\,\lambda > 0$, we can write

$$\begin{aligned}
\Gamma^{\xi}_{|f|}(\lambda) &= i^{n^2}\int_{\mathbb{C}^n}|f|^{2\lambda}\xi\,\mathrm{d}z\wedge\mathrm{d}\bar{z}\\
&= \frac{i^{n^2}}{b_f(\lambda)^2}\int_{\mathbb{C}^n}|P(\lambda)|^2|f|^{2(\lambda+1)}\xi\,\mathrm{d}z\wedge\mathrm{d}\bar{z}\\
&= \frac{i^{n^2}}{b_f(\lambda)^2}\int_{\mathbb{C}^n}|f|^{2(\lambda+1)}(|P(\lambda)|^2)^*\xi\,\mathrm{d}z\wedge\mathrm{d}\bar{z},
\end{aligned}$$

where $|P(\lambda)|^2 = P(\lambda)\overline{P(\lambda)}$, and where $(|P(\lambda)|^2)^*$ denotes the adjoint of $|P(\lambda)|^2$. Now, we see that the right-hand side is defined and holomorphic for $\mathfrak{Re}\,\lambda > -1$ except for at possible roots of $b_f(\lambda)$ in the interval $(-1,0)$. Furthermore, we can iterate (3.1.3) any number of times: For any $k \in \mathbb{N}$, we have

$$\begin{aligned}
|f|^{2\lambda} &= \frac{1}{b_f(\lambda)^2}|P(\lambda)|^2\frac{1}{b_f(\lambda+1)^2}|P(\lambda+1)|^2\cdots\frac{1}{b_f(\lambda+k)^2}|P(\lambda+k)|^2|f|^{2(\lambda+k+1)}\\
&= \frac{1}{b_f(\lambda)^2\cdots b_f(\lambda+k)^2}|P(\lambda)|^2\cdots|P(\lambda+k)|^2|f|^{2(\lambda+k+1)}.
\end{aligned}$$

Analogously, we can write

$$\Gamma^{\xi}_{|f|}(\lambda) = \frac{i^{n^2}}{h_f(\lambda)}\int_{\mathbb{C}^n}|f|^{2(\lambda+k+1)}(|P(\lambda+k)|^2)^*\cdots(|P(\lambda)|^2)^*\xi\,\mathrm{d}z\wedge\mathrm{d}\bar{z}, \tag{3.1.5}$$

where

$$h_f(\lambda) = b_f(\lambda)^2\cdots b_f(\lambda+k)^2.$$

The right-hand side of (3.1.5) is defined and holomorphic $\mathfrak{Re}\,\lambda > -k-1$, as long as one avoids any possible roots of $b_f(\lambda)\cdots b_f(\lambda+k)$ in the interval $(-k-1,0)$. Taking k arbitrarily large, this construction furnishes a meromorphic continuation of $\Gamma^{\xi}_{|f|}(\lambda)$ to all of $\mathbb{C}$, with poles of the form $r-k$ for $k\in\mathbb{Z}_{\geq 0}$, where $r\in\mathbb{Q}_-$ is a root of b_f.

In Paper I, we demonstrate an explicit construction of b_f and $P(\lambda)$ satisfying the relation (3.1.2) in the classical case where

$$f = z_1^{m_1}\cdots z_\kappa^{m_\kappa}, \tag{3.1.6}$$

for positive integers $m_1,\ldots,m_\kappa$. Furthermore, in Paper I we consider a global complex-geometric setting, where the holomorphic function is replaced by a holomorphic section of a holomorphic vector bundle, however, we can in principal always reduce to the case (3.0.2) locally. Historically, resolution of singularities of the locus $\{f=0\}$ was the key ingredient in the original proof of Theorem 3.0.1. After passing to local coordinates in which (the pullback of) f is of the form (3.1.6), the meromorphic continuation reduces to an elementary calculation, see the proof of Lemma 3.2 in Paper I.

3.2 Current-valued meromorphic functions

In this section we introduce the notion of *current-valued meromorphic functions*. Similar notions appear in the literature, see, e.g., [de Rham]. In the present setting, this notion is particularly convenient for the regularized divergent integrals, and their generalizations, appearing in Papers I and II, as well as for the Log gas partition functions featuring in Papers III and IV. Most of the following can be found in Paper II, Section 2.

Let $\Omega\subset\mathbb{C}$ be a domain, and let X be a reduced analytic space of pure dimension. *A current-valued holomorphic function* is a map $\mu\colon\Omega\to\mathscr{D}'(X)$ such that, for any test form $\xi\in\mathscr{D}(X)$, the pairing $\langle\mu(\lambda),\xi\rangle$ is a holomorphic function on Ω.

We define the *derivative* $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ of a current-valued holomorphic function μ by

$$\Big\langle\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(\lambda),\xi\Big\rangle = \frac{\mathrm{d}}{\mathrm{d}\lambda}\Big\langle\mu(\lambda),\xi\Big\rangle,\quad \lambda\in\Omega,\ \xi\in\mathscr{D}(X). \tag{3.2.1}$$

Lemma 3.2.1. *Let $\mu\colon\Omega\to\mathscr{D}'(X)$ be a current-valued holomorphic function. Then $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\colon\Omega\to\mathscr{D}'(X)$ is a current-valued holomorphic function.*

Proof. Fix $\lambda_0 \in \Omega$ and choose $R > 0$ such that $\overline{\mathbb{D}}(\lambda_0, R) \subset \Omega$. By Cauchy's integral formula,

$$\Big\langle \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(\lambda_0), \xi \Big\rangle = \frac{1}{2\pi i} \int_{|\lambda-\lambda_0|=R} \frac{\langle \mu(\lambda), \xi\rangle \, \mathrm{d}\lambda}{(\lambda-\lambda_0)^2}, \tag{3.2.2}$$

which is clearly linear in ξ.

To prove continuity, note that for each $\xi \in \mathscr{D}(X)$, the function $\lambda \mapsto \langle \mu(\lambda), \xi \rangle$ is continuous on the compact circle $|\lambda - \lambda_0| = R$. Hence the family of currents

$$\{\mu(\lambda) : |\lambda - \lambda_0| = R\}$$

is pointwise bounded on $\mathscr{D}(X)$. By the Banach–Steinhaus theorem, this pointwise bounded family of continuous linear functionals on $\mathscr{D}(X)$ is equicontinuous. In particular, for each compact subset $K \subset X$, there exist constants $k \in \mathbb{Z}_{\geq 0}$ and $C > 0$ such that, for each test form $\xi \in \mathscr{D}(X)$ supported in K,

$$|\langle \mu(\lambda), \xi \rangle| \leq C\|\xi\|_{K,k}$$

for every λ with $|\lambda - \lambda_0| = R$, see (2.4.8) and surrounding paragraphs. Thus,

$$\Big|\Big\langle \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(\lambda_0), \xi \Big\rangle\Big| \leq \frac{1}{2\pi} \int_{|\lambda-\lambda_0|=R} \frac{|\langle \mu(\lambda), \xi\rangle| |\mathrm{d}\lambda|}{R^2} \leq \frac{C}{R}\|\xi\|_{K,k},$$

whence $\xi \mapsto \langle \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(\lambda_0), \xi \rangle$ is continuous by definition, see Section 2.4.1 above.

Finally, for each $\xi \in \mathscr{D}(X)$, $\lambda \mapsto \langle \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}(\lambda), \xi \rangle = \frac{\mathrm{d}}{\mathrm{d}\lambda}\langle \mu(\lambda), \xi \rangle$ is holomorphic, since derivatives preserve holomorphicity. Thus, $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} : \Omega \to \mathscr{D}'(X)$ is a current-valued holomorphic function. □

If $\lambda \mapsto \mu(\lambda)$ is a current-valued holomorphic function on Ω, then, for each $\lambda_0 \in \Omega$, there exists $R > 0$ such that on $\mathbb{D}(\lambda_0, R) \subset \Omega$ one has the expansion

$$\mu(\lambda) = \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j \mu_j,$$

where

$$\mu_j = \frac{1}{j!} \frac{\mathrm{d}^j \mu}{\mathrm{d}\lambda^j}(\lambda_0).$$

This follows from the Taylor series expansion of $\langle \mu(\lambda), \xi \rangle$, together with Lemma 3.2.1.

In a similar vein, a *current-valued meromorphic function* on Ω as a map

$$\mu \colon \Omega \setminus P \to \mathscr{D}'(X),$$

where $P \subset \Omega$ is a discrete subset, such that μ is a current-valued holomorphic function on $\Omega \setminus P$, and such that for every $\xi \in \mathscr{D}(X)$, the pairing $\langle \mu(\lambda), \xi \rangle$ is a meromorphic function on Ω.

If $\lambda \mapsto \mu(\lambda)$ is a current-valued meromorphic function on Ω, then for each pole $\lambda_0 \in \Omega$, there exists $R > 0$ and $\kappa \in \mathbb{Z}_{\geq 0}$ such that on $\mathbb{D}^*(\lambda_0, R)$ one has the expansion

$$\mu(\lambda) = \sum_{j=-\kappa}^{\infty} (\lambda - \lambda_0)^j \mu_j, \tag{3.2.3}$$

where, for each $j \geq -\kappa$, $\mu_j \in \mathscr{D}'(X)$ such that for each $\xi \in \mathscr{D}(X)$, $\langle \mu_j, \xi \rangle$ is the j^{th} Laurent series coefficient of $\langle \mu(\lambda), \xi \rangle$ about λ_0. If $0 < r < R$, μ_j is defined by

$$\langle \mu_j, \xi \rangle = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = r} \frac{\langle \mu(\lambda), \xi \rangle \, \mathrm{d}\lambda}{(\lambda - \lambda_0)^{j+1}}, \tag{3.2.4}$$

and it follows from an argument analogous to that of Lemma 3.2.1 that μ_j indeed defines a current for each $j \geq -\kappa$. Equivalently,

$$\mu_j = \frac{1}{(j+\kappa)!} \frac{\mathrm{d}^{j+\kappa}}{\mathrm{d}\lambda^{j+\kappa}} \big(\lambda^{\kappa} \mu(\lambda)\big)\Big|_{\lambda = \lambda_0}, \qquad j \geq -\kappa. \tag{3.2.5}$$

The current coefficient μ_{-1} corresponds to the residue of the scalar pairings, that is,

$$\langle \mu_{-1}, \xi \rangle = \operatorname{Res}_{\lambda = \lambda_0} \langle \mu(\lambda), \xi \rangle.$$

The *order* of the pole of the current-valued meromorphic function $\mu(\lambda)$ at λ_0 is the minimal κ such that (3.2.3) holds, in the sense that $\mu_{-\kappa} \not\equiv 0$ and $\mu_j \equiv 0$ for each $j < -\kappa$. Then, clearly, for any fixed $\xi \in \mathscr{D}(X)$ the order of the pole of $\langle \mu(\lambda), \xi \rangle$ at λ_0 is at most κ, and there exists a test function $\xi' \in \mathscr{D}(X)$ such that $(\lambda - \lambda_0)^{\kappa} \langle \mu(\lambda_0), \xi \rangle \big|_{\lambda = \lambda_0} \neq 0$.

The upshot of all of this is that we can in many ways treat these families of currents as thought they were ordinary meromorphic functions.

Example 3.2.2. The main example of a current-valued meromorphic function to keep in mind is the one associated to an Archimedean zeta function (3.0.1). Indeed, let $f \colon \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function and consider the map $\lambda \mapsto \mu(\lambda)$ defined by

$$\langle \mu(\lambda), \xi \rangle = i^{n^2} \int_{\mathbb{C}^n} |f|^{2\lambda} \xi \, \mathrm{d}z \wedge \mathrm{d}\bar{z}, \quad \xi \in \mathscr{D}(\mathbb{C}^n). \tag{3.2.6}$$

Then $\mu(\lambda)$ defines a current-valued meromorphic function on $\mathbb{C}$, with poles in a discrete subset of $\mathbb{Q}_-$ determined by the roots of the Bernstein–Sato polynomial b_f.

3.3 Regularization of divergent complex-geometric integrals

In this section, we will introduce the class of global complex-geometric Archimedean zeta functions that are the main focus of the thesis, and, in particular, the focus of Papers I and II.

Let X be a reduced analytic space of pure dimension n. We are interested in possibly divergent integrals on X of the form

$$\int_X \omega,$$

where ω is a differential form of top degree on X that is smooth away from a proper analytic subset defined by a holomorphic section $s\colon X \to E$ of a holomorphic vector bundle $E \to X$. More precisely, we assume that ω defines a smooth form on $X \setminus \{s = 0\}$ and, for some smooth metric $\|\cdot\|$ on E, we assume that for each compact subset $K \subset X$, there exists a number $M \in \mathbb{Q}_{\geq 0}$ such that $\|s\|^{2M}\omega$ extends smoothly across $K \cap \{s = 0\}$. In other words, we assume that ω has at most algebraic singularities along the zero locus of s.

Let $\mathcal{A}^{n,n}_{s,\|\cdot\|}(X)$ be the space of all such forms, and let $\mathcal{A}^{n,n}_s(X)$ be the union of all such $\mathcal{A}^{n,n}_{s,\|\cdot\|}(X)$[1]. Following the terminology in [46, 47], we call $\mathcal{A}^{n,n}_s(X)$ the space of *quasi-meromorphic* forms (of top degree) on X with singularities defined by s (equivalently, by $\operatorname{div}(s)$). In the general setting of Papers I and II, and also in the sequel, we consider spaces $\mathcal{A}^{p,q}_{s,\|\cdot\|}(X)$ and $\mathcal{A}^{p,q}_s(X)$ of quasi-meromorphic forms of arbitrary bidegree, defined analogously as above. While the interpretation in terms of divergent integrals over X is specific to forms of top degree, the framework extends naturally to quasi-meromorphic (p,q)-forms with $p+q < 2n$. Lastly, we let $\mathcal{A}_{s,\|\cdot\|}(X) = \bigoplus_{0\leq p+q\leq 2n} \mathcal{A}^{p,q}_{s,\|\cdot\|}(X)$, and similarly for $\mathcal{A}_s(X)$.

To any $\omega \in \mathcal{A}^{p,q}_{s,\|\cdot\|}(X)$ we associate a current (also denoted by ω) on $X \setminus \{s = 0\}$, by

$$\langle \omega, \xi \rangle = \int_{X\setminus\{s=0\}} \omega \wedge \xi, \qquad \xi \in \mathscr{D}^{n-p,n-q}(X \setminus \{s = 0\}). \tag{3.3.1}$$

Now for $\mathfrak{Re}\,\lambda \gg 0$, the form $\|s\|^{2\lambda}\omega$ is locally integrable by assumption on ω, and hence it defines a current on X. We denote the action of this current on a

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[1] In the case where E is a line bundle, $\mathcal{A}^{n,n}_s(X) = \mathcal{A}^{n,n}_{s,\|\cdot\|}(X)$ for each smooth Hermitian metric $\|\cdot\|$.

test form $\xi \in \mathscr{D}^{n-p,n-q}(X)$ by $\Gamma^{\xi}_{\|s\|}(\lambda,\omega) := \langle \|s\|^{2\lambda}\omega, \xi\rangle$, that is,

$$\Gamma^{\xi}_{\|s\|}(\lambda,\omega) = \int_X \|s\|^{2\lambda}\omega \wedge \xi. \tag{3.3.2}$$

If ω is a top form and X is compact, we can let $\xi = 1$. Then $\Gamma^{1}_{\|s\|}(\lambda,\omega)$ can be regarded as a regularization of the divergent integral $\int_X \omega$. In Section 3.3.1, we will define a natural *finite part* of $\int_X \omega$ with respect to this regularization. More generally, if ω is not necessarily of top degree, $\|s\|^{2\lambda}\omega$ can be regarded as a regularization of the current defined by ω and a furnishes a natural *current extension.*

We refer to functions of the form (3.3.2) as Archimedean zeta functions, viewing them as global complex-geometric counterparts to the Archimedean zeta functions introduced in the beginning of the chapter. To illustrate this, assume first for simplicity that X is a complex manifold and that s is a section of a holomorphic line bundle. Then, choose a partition of unity ρ_j subordinate to a finite trivializing cover U_j of a neighborhood of $\operatorname{supp}\xi$, such that $\xi = \sum_j \xi_j$, where $\xi_j = \rho_j\xi$ has compact support in U_j, for each j. Thus, locally in U_j, we can write $\|s\|^{2\lambda} = |f_j|^{2\lambda}e^{-\lambda\phi_j}$, where $f_j\colon U_j \to \mathbb{C}$ is a holomorphic function, and $\phi_j \in \mathscr{C}^\infty(U_j,\mathbb{R})$ is a local weight for the Hermitian metric $\|\cdot\|$. Still assuming $\mathfrak{Re}\,\lambda \gg 0$, we have

$$\begin{aligned}\Gamma^{\xi}_{\|s\|}(\lambda,\omega) &= \int_X \|s\|^{2\lambda}\omega \wedge \sum_j \xi_j \\ &= \sum_j \int_{U_j} |f_j|^{2\lambda}e^{-\lambda\phi_j}\omega \wedge \xi_j.\end{aligned}$$

Moreover, locally in U_j we can find a number $M_j \geq 0$ such that $\omega = \widetilde{\omega}_j/|f_j|^{2M_j}$, where $\widetilde{\omega}_j$ is a smooth form. Each local contribution to $\Gamma^{\xi}_{\|s\|}(\lambda,\omega)$ is therefore precisely of the form (3.0.2).

More generally, when X is a reduced analytic space, and s is a section of a vector bundle, by Hironaka's theorem, we can always find a resolution of singularities $\pi\colon \widetilde{X} \to X$, where $\widetilde{X}$ is a complex manifold, and such that the pullback $\pi^*s = s_0 \otimes s'$, where s_0 is a holomorphic section of a holomorphic line bundle $L \to \widetilde{X}$ and s' is nowhere vanishing (and, moreover, such that $\operatorname{div}(s_0)$ is a normal crossings divisor), see Section 2.9 above. Since s' is nowhere vanishing, the singular behavior of $\|\pi^*s\|^{2\lambda}$ is entirely determined by s_0. Locally on $\widetilde{X}$, one may therefore write $\|\pi^*s\|^2 = |f|^2e^{-\phi}$, where f is holomorphic and $e^{-\phi}$ is smooth and positive. Since

$$\int_X \|s\|^{2\lambda}\omega \wedge \xi = \int_{\widetilde{X}} \|\pi^*s\|^{2\lambda}\pi^*\omega \wedge \pi^*\xi,$$

and $\pi^*\omega \in \mathcal{A}^{p,q}_{s_0}(\widetilde{X})$, after pulling back by π we thus reduce to the above setting. It follows from the local theory that $\Gamma^{\xi}_{\|s\|}(\lambda, \omega)$ is holomorphic for $\mathfrak{Re}\,\lambda$ sufficiently large, and that it admits a meromorphic continuation to all of $\mathbb{C}$ with poles in a discrete subset of $\mathbb{Q}$ (a proof can be found in Paper I, Section 3). Note that meromorphic continuation is unique, whence the Archimedean zeta function obtained is independent of the particular choice of resolution or modification.

3.3.1 Finite parts and current extensions

In view of Section 3.2, the mapping $\lambda \mapsto \|s\|^{2\lambda}\omega$ defines a current-valued meromorphic function on X. In particular, the Laurent series expansion of $\|s\|^{2\lambda}\omega$ about $\lambda = 0$ is given by

$$\|s\|^{2\lambda}\omega = \sum_{j=-\kappa}^{\infty} \lambda^j \mu_j^{\|s\|}(\omega), \tag{3.3.3}$$

where $\mu_j^{\|s\|}(\omega)$ are currents on X defined by (3.2.5) for some integer $\kappa \geq 0$. In fact, one has $\kappa \leq n$, and, in the line bundle case, κ is independent on the choice of metric $\|\cdot\|$. In Paper I, Theorem 4.1 (i), we give a description of the supports $\operatorname{supp}\mu_j^{\|s\|}(\omega)$ of the currents $\mu_j^{\|s\|}(\omega)$, for $-\kappa \leq j \leq 0$.

A natural choice of a current extension of ω across $\{s = 0\}$, given the regularization $\|s\|^{2\lambda}\omega$, is then the 0th order term $\mu_0^{\|s\|}(\omega)$ in (3.3.3). Recall that for a current $\mu \in \mathscr{D}'(X \setminus V)$, where V is an analytic subset, a current extension of μ across V is a current $\overline{\mu} \in \mathscr{D}'(X)$ such that $\overline{\mu}|_{X\setminus V} = \mu$.

Indeed, $\mu_0^{\|s\|}(\omega)$ is a current defined on X which clearly satisfies

$$\mu_0^{\|s\|}(\omega)|_{X\setminus\{s=0\}} = \omega.$$

Note that, similar to finite parts of divergent integrals, current extensions are generally not unique. The extension $\mu_0^{\|s\|}(\omega)$ is distinguished by the chosen regularization, but it still depends on the choice of Hermitian metric $\|\cdot\|$ on E. Moreover, when X is compact and $\omega \in \mathcal{A}^{n,n}_{s,\|\cdot\|}(X)$, a natural finite part $\operatorname{fp}\int_X \omega$ of $\int_X \omega$, distinguished by the regularization, is given by

$$\operatorname{fp}\int_X \omega := \langle \mu_0^{\|s\|}(\omega), 1\rangle. \tag{3.3.4}$$

Understanding the dependence of $\operatorname{fp}\int_X \omega$, and more generally $\mu_0^{\|s\|}(\omega)$, on the choice of metric is one of the main themes of Paper I, see the summary in Section 4.1 below.

Remark 3.3.1. One could also consider a change of holomorphic section defining the same analytic subvariety. Here we assume s is a part of the data of the problem, see Paper I, Example 5.1 for a discussion of a special case where a change of defining section can be realized as a change of Hermitian metric. A systematic study of the dependence on the choice of defining section could be an interesting direction for future work.

Another classical approach to regularization is to consider a *cut-off procedure*: For $\omega \in \mathcal{A}^{p,q}_{s,\|\cdot\|}(X)$, and $\epsilon > 0$ we define

$$\mathcal{I}^{\xi}_{\|s\|}(\epsilon, \omega) = \int_{\|s\|^2 \geq \epsilon} \omega \wedge \xi. \tag{3.3.5}$$

Since the integration domain avoids the singular locus of ω, $\mathcal{I}^{\xi}_{\|s\|}(\epsilon, \omega)$ is well-defined for every $\epsilon > 0$. In general, however, it diverges as $\epsilon \to 0^+$.

In Paper I, Section 6, we obtain an asymptotic expansion of $\mathcal{I}^{\xi}_{\|s\|}(\epsilon, \omega)$ as $\epsilon \to 0^+$. If X is compact and ω is of top degree, then a natural finite part of $\int_X \omega$, given the cut-off regularization procedure, is the limit of $\mathcal{I}^{\xi=1}_{\|s\|}(\epsilon, \omega)$ as $\epsilon \to 0^+$, after subtracting divergent terms from the asymptotic expansion. This finite part then also depends on the choice of metric $\|\cdot\|$ on E. A priori, the meromorphic regularization and the cut-off regularization yield two different notions of finite part. One of the main results of Paper I, shows that they coincide. The underlying reason is that $\mathcal{I}^{\xi}_{\|s\|}(\epsilon, \omega)$ and $\Gamma^{\xi}_{\|s\|}(\lambda, \omega)$ are related by the *Mellin transform*, allowing one to pass between asymptotic expansions in ϵ and Laurent expansions in λ. This is the subject of the following section.

3.3.2 The Mellin transform

The *Mellin transform* is an integral transform closely related to the Laplace transform, which features prominently in the study of asymptotic expansions.

Let $f\colon \mathbb{R}_+ \to \mathbb{C}$. The Mellin transform of f is the complex-valued function

$$\mathcal{M}\{f\}(\lambda) = \int_0^\infty \epsilon^{\lambda-1} f(\epsilon)\, \mathrm{d}\epsilon. \tag{3.3.6}$$

defined for those $\lambda \in \mathbb{C}$ for which the integral converges. It may be viewed as an integral over the multiplicative group $\mathbb{R}^\times_+$ with respect to the Haar measure $\mathrm{d}\mu = \mathrm{d}\epsilon/\epsilon$, with kernel ϵ^λ. Under the change of variables $\epsilon = e^{-t}$, the Mellin transform of f becomes

$$\mathcal{M}\{f\}(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(e^{-t})\, \mathrm{d}t,$$

and may therefore be regarded as a (two-sided) Laplace transform (of $f \circ e^{-t}$).

For the Mellin transform to converge, f needs to be locally integrable on $\mathbb{R}_+$, and satisfy suitable growth conditions near 0 and ∞. For example, if

$$f(\epsilon) = \mathcal{O}(\epsilon^{-a}) \quad \text{as } \epsilon \to 0, \quad f(\epsilon) = \mathcal{O}(\epsilon^{-b}) \quad \text{as } \epsilon \to \infty,$$

for some $a < b \in \mathbb{R}$, then $\mathcal{M}\{f\}$ converges absolutely in the strip $a < \mathfrak{Re}\,\lambda < b\} \subset \mathbb{C}$. When defined, the Mellin transform is holomorphic in its domain of convergence.

Conversely, under suitable decay and regularity assumptions, a holomorphic function φ on the strip $\{a < \mathfrak{Re}\,\lambda < b\}$ admits an *inverse Mellin transform*, given by

$$\mathcal{M}^{-1}\{\varphi\}(\epsilon) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \epsilon^{-\lambda} \varphi(\lambda) \mathrm{d}\lambda. \tag{3.3.7}$$

As already mentioned in the previous section, and as is demonstrated in Paper I, Section 6, we have that

$$\mathcal{M}\{\mathcal{I}^{\xi}_{\|s\|}(\epsilon, \omega)\}(\lambda) = \frac{1}{\lambda} \Gamma^{\xi}_{\|s\|}(\lambda, \omega), \tag{3.3.8}$$

where $\Gamma^{\xi}_{\|s\|}(\lambda, \omega)$ and $\mathcal{I}^{\xi}_{\|s\|}(\epsilon, \omega)$ are given in (3.3.2) and (3.3.5), respectively, and where $\omega \in \mathcal{A}_{\|\cdot\|,s}(X)$. This fact is used to derive a formula for the asymptotic expansion $\mathcal{I}^{\xi}_{\|s\|}(\epsilon, \omega)$ as $\epsilon \to 0^+$, and to show that the natural finite part defined with respect to the cut-off regularization agrees with (3.3.4).

3.4 Statistical mechanics and partition functions

In this section we introduce the statistical mechanical systems of interacting particles that feature in Papers III and IV. We begin with a general framework adapted to the complex-geometric setting and then discuss Coulomb gases on the sphere, which are the main objects of study in Paper III and also feature in Paper IV.

Let X be a compact complex manifold and consider a system of N particles with positions $p = (p_1, \dots, p_N)$ interacting through an energy function

$$E_N \colon X^N \to \mathbb{R} \cup \{+\infty\}.$$

When N is large, tracking the behavior of individual particles becomes intractable. One therefore seeks a statistical description of the system.

We then consider the *canonical ensemble* or *Gibbs ensemble*, a standard statistical mechanical model for a system of N interacting particles in thermal equilibrium at inverse temperature β. In this picture, microscopic states of the system are distributed according to the *Gibbs measure*

$$\mu_\beta^{(N)} := \frac{1}{Z_N(\beta)} e^{-\beta E_N} \mathrm{d}V^{\otimes N}, \tag{3.4.1}$$

where $\mathrm{d}V$ is a fixed volume form on X, and

$$Z_N(\beta) = \int_{X^N} e^{-\beta E_N} \mathrm{d}V^{\otimes N}, \tag{3.4.2}$$

is the *partition function.*

The partition function plays a central role, as it contains global information about the statistical properties of a given system. In particular, many thermodynamical quantities, such as the average energy, entropy and free energy, can expressed in terms of $Z_N(\beta)$. One is often primarily interested in the *thermodynamic limit* $N \to \infty$, where collective phenomena emerge. A fundamental question is then whether the particles admit a limiting distribution on X as N becomes large. Since the Gibbs measures $\mu_\beta^{(N)}$ are defined on different spaces X^N, this question must be formulated in terms of suitable objects derived from $\mu_\beta^{(N)}$. We will discuss the probabilistic framework needed to make this precise in the next section.

We mainly consider systems whose interaction energies are encoded by complex-geometric data. More precisely, we consider logarithmic interaction energies of the form

$$E_N = -\log \|s\|^2,$$

where s is a holomorphic section of a holomorphic line bundle $L \to X^N$, and $\|\cdot\|$ is a Hermitian metric on L (and slight generalizations thereof, see Section 3.4.1). In this case,

$$Z_N(\beta) = \int_{X^N} \|s\|^{2\beta} \mathrm{d}V^{\otimes N}, \tag{3.4.3}$$

which precisely corresponds to an Archimedean zeta function of the type introduced in Section 3.3, cf. (3.3.2).

3.4.1 The Coulomb gas on the sphere

In Paper III, we fix $X = \mathbb{P}^1 = \mathbb{S}^2$, and, for any $N \geq 2$, consider the interaction energy defined by the *Coulomb potential*

$$E_N = -\sum_{1\leq j<k\leq N} c_{jk} \log d(p_j, p_k)^2, \tag{3.4.4}$$

where $(c_{jk}) \in \mathbb{R}^{N\times N}$ is a hollow symmetric matrix of *coupling constants*, and where $d(p_j, p_k)$ denotes the chordal distance on the sphere. Following standard terminology, we will often refer to such a system as a *Coulomb gas* or *Log gas*. The special case $(c_{jk}) = (q_j q_k)$ corresponds to a system of charged particles with charges $q_1, \dots, q_N$ interacting through the Coulomb potential. Such systems arise naturally in statistical mechanics and mathematical physics, including plasma models and Onsager's theory of point vortices; see, e.g., [55, 51].

The interaction energy (3.4.4) admits an algebro-geometric interpretation. In particular, if $c_{jk} = 1$ for each $j \neq k$, then $E_N = -\log \|s\|^2$ where s is a holomorphic section of the holomorphic line bundle $\mathcal{O}_{(\mathbb{P}^1)^N}(N-1)$ (cf. Section 3.5.2 below) vanishing on the union of diagonals

$$\bigcup_{1\le j<k\le N} \{p_k = p_j\} \subset (\mathbb{P}^1)^N,$$

and $\|\cdot\|$ is the associated Fubini–Study metric, cf. Example 2.6.2 above.

Remark 3.4.1. More generally, for arbitrary $(c_{jk}) \in \mathbb{R}^{N\times N}$, s can be viewed as a multi-valued section corresponding to the $\mathbb{R}$-divisor on $(\mathbb{P}^1)^N$ given by

$$\sum_{1\le j<k\le N} c_{jk} W_{jk},$$

where W_{jk} is the prime divisor corresponding to the diagonal hypersurface $\{p_j = p_k\} \subset (\mathbb{P}^1)^N$. Nevertheless, the norm $\|s\|^2$ of s will still be single-valued.

In a dense affine chart $\mathbb{C}^N \subset (\mathbb{P}^1)^N$ with holomorphic coordinates $z = (z_1, \dots, z_N)$, we can write

$$E_N = -\sum_{1\le j<k\le N} c_{jk} \log \frac{|z_j - z_k|^2}{(1+|z_j|^2)(1+|z_k|^2)}.$$

Moreover, fixing $\mathrm{d}V$ to be the standard normalized volume form on the sphere, corresponding to the Fubini–Study form ω_{FS} on $\mathbb{P}^1$, cf. (2.6.6), then the partition function becomes

$$Z_N(\beta) = \int_{\mathbb{C}^N} \prod_{1\le j<k\le N} |z_j - z_k|^{2c_{jk}\beta} e^{-\beta U} \omega_{\mathrm{FS}}(z_1) \wedge \cdots \wedge \omega_{\mathrm{FS}}(z_N), \qquad (3.4.5)$$

where

$$U = \sum_{1\le j<k\le N} c_{jk} \log{(1+|z_j|^2)(1+|z_k|^2)},$$

and

$$\omega_{\mathrm{FS}}(z_j) = \frac{i}{2} \frac{\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j}{(1+|z_j|^2)^2}.$$

For the special choice

$$c_{jk} = 1/(N-1), \qquad j \neq k,$$

the partition function (3.4.5) coincides with that of the Gibbs ensembles on $\mathbb{P}^1$ appearing in the probabilistic approach to Kähler–Einstein metrics, which will be discussed in Section 3.5 below.

More generally, if $(c_{jk}) \in \mathbb{Q}^{N\times N}$, after a suitable rescaling of the parameter β, the partition function may be interpreted as an Archimedean zeta function associated to a meromorphic section rather than a holomorphic one. Consequently, $Z_N(\beta)$ is initially defined only in a strip $\{\beta \in \mathbb{C} : \beta^- < \mathfrak{Re}\,\beta < \beta^+\}$, but nevertheless admits a meromorphic continuation to all of $\mathbb{C}$; its poles form a discrete subset of $\mathbb{Q}$, see Paper III, Section 3. The same phenomenon persists for arbitrary real coupling matrices $(c_{jk}) \in \mathbb{R}^{N\times N}$, although, the poles are then contained in a discrete subset of $\mathbb{R}$ rather than $\mathbb{Q}$.

In contrast to Paper IV, where the main interest lies in the large-N limit, Paper III is primarily concerned with understanding the system in the finite-N regime. One of our primary goals is to understand how the coupling matrix (c_{jk}) governs the integrability thresholds and meromorphic continuation of the partition function (3.4.5). Viewing β as a complex parameter, there are in general at most two real values of β at which the partition function ceases to be defined by an absolutely convergent integral; we refer to these as the *critical inverse temperatures* of the N-particle system. Using the Fulton–MacPherson compactification of the configuration space of $\mathbb{P}^1$, we relate these critical temperatures to a discrete optimization problem. We also describe the behavior of the Gibbs measure as one approaches a critical inverse temperature from the integrable regime.

3.5 A probabilistic approach to Kähler–Einstein metrics

In this final section of the chapter, we briefly introduce the probabilistic approach to Kähler–Einstein metrics, introduced by Berman [6, 7]. The central idea is that a Kähler–Einstein metric on a Kähler manifold X should emerge in the thermodynamical limit of certain Gibbs ensembles constructed from algebro-geometric data of X.

More precisely, for each sufficiently large integer k, one constructs a canonical symmetric probability measure $\mu^{(N_k)}$ on the configuration space X^{N_k}, where N_k grows with k. These measures may be interpreted as Gibbs ensembles of N_k identical interacting particles on X. The central prediction is that, as $k \to \infty$, the corresponding particle systems exhibit a deterministic thermodynamic limit described by the Kähler–Einstein geometry of X. In the case of canonically polarized manifolds, this picture has been established by Berman [7]; see Section 3.5.3 below. To formulate this result, and the corresponding conjectural framework in the Fano setting, we first recall some basic notions from probability theory and the theory of *Large Deviations*.

3.5.1 Empirical measures and large deviations

For this section, we assume some familiarity with measure theory and probability theory. Background for the following can be found in any standard textbook on probability theory; see, e.g., [40]. For background on the theory of large deviations, see, e.g., [23]. For simplicity, we will specialize to the setting of point processes on compact Kähler manifolds, even though much of the following material can be stated much more generally.

Let X be a compact Kähler manifold and let $N \in \mathbb{N}$. A symmetric[2] probability measure $\mu^{(N)}$ on the N-fold product X^N defines an *N-point process* on X. We refer to a sequence of N-point processes on X, for increasing N, simply as a point process.

A *random measure* on a space Ω is a random variable taking values in the space $\mathscr{P}(\Omega)$ of probability measures on Ω. The *empirical measure* of an N-point process $(X^N, \mu^{(N)})$ is defined as the following random measure:

$$\delta_N \colon X^N \to \mathscr{P}(X), \qquad \delta_N(x_1, \ldots, x_N) := \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j}. \tag{3.5.1}$$

Thus, δ_N encodes the distribution of the N particles as a single probability measure on X.

The *law* of δ_N is the probability measure $(\delta_N)_* \mu^{(N)}$ on $\mathscr{P}(X)$ defined by the pushforward of the measure $\mu^{(N)}$ with respect to δ_N. We note that $(\delta_N)_* \mu^{(N)}$ is an element of $\mathscr{P}(\mathscr{P}(X))$. The sequence δ_N of random measures on X is said to *converge in law* to a deterministic element $\mu^\star \in \mathscr{P}(X)$ if the corresponding laws $(\delta_N)_* \mu^{(N)}$ converge weakly to the Dirac measure $\delta_{\mu^\star} \in \mathscr{P}(\mathscr{P}(X))$.

Since X is compact, the space $\mathscr{P}(X)$, equipped with the weak topology, is a compact *Polish space*[3]. In particular, for any metric d inducing the weak topology (for example, a Wasserstein metric), convergence in law to a deterministic measure $\mu^\star$ is equivalent to *convergence in probability*: The

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[2] Invariant under the symmetric group of permutations of the N copies of X.

[3] A complete separable metric space.

sequence δ_N *converges in probability* to $\mu^\star$ if, for every $\epsilon > 0$,

$$\lim_{N\to\infty} \mu^{(N)}\big(\{x \in X^N : d(\delta_N(x), \mu^\star) > \epsilon\}\big) = 0.$$

A *Large Deviation Principle (LDP)* formalizes the notion of exponentially fast convergence in probability. Sticking to our specialized setting, the sequence of probability measures $(\delta_N)_*\mu^{(N)}$ on $\mathscr{P}(X)$ is said to satisfy a LDP with *speed* $r_N \to \infty$ and *rate functional* $I\colon \mathscr{P}(X) \to [0,\infty]$ if

$$\limsup_{N\to\infty} \frac{1}{r_N} \log{(\delta_N)_*\mu^{(N)}(\mathscr{A})} \le -\inf_{\mu\in\mathscr{A}} I(\mu)$$

for every closed subset $\mathscr{A} \subset \mathscr{P}(X)$ and

$$\liminf_{N\to\infty} \frac{1}{r_N} \log{(\delta_N)_*\mu^{(N)}(\mathscr{B})} \ge -\inf_{\mu\in\mathscr{B}} I(\mu)$$

for every open subset $\mathscr{B} \subset \mathscr{P}(X)$.

The following is a standard consequence of the LDP upper and lower bounds (specialized to laws of empirical measures of point processes); see, e.g., [23].

Lemma 3.5.1. *If the laws $(\delta_N)_*\mu^{(N)} \in \mathscr{P}(\mathscr{P}(X))$ of δ_N satisfy a LDP with speed $r_N = N$ and a rate functional I which admits a unique minimizer $\mu^\star$, then the empirical measures δ_N converge in law to the deterministic element $\mu^\star$. Moreover, for each $\epsilon > 0$ there exist constants $C_\epsilon, c_\epsilon > 0$ such that*

$$\mu^{(N)}\big(\{x \in X^N : d(\delta_N(x), \mu^\star) \ge \epsilon\}\big) \le C_\epsilon e^{-c_\epsilon N}.$$

In particular, if the rate functional admits a unique minimizer $\mu^\star$, then empirical measures concentrate exponentially fast around $\mu^\star$. In the probabilistic approach to Kähler–Einstein metrics, this minimizer is the volume form associated to a Kähler–Einstein metric and the rate functional is the Free energy $\mathcal{F}$ on $\mathscr{P}(X)$.

3.5.2 Canonical point processes from pluri-(anti)canonical sections

Let X be a compact complex manifold of dimension n such that βK_X is ample, where $\beta = \pm 1$. The cases $\beta = 1$ and $\beta = -1$ corresponds to canonically polarized and Fano manifolds, respectively[4]. A key observation due to Berman is that the spaces of pluricanonical or pluri-anticanonical sections (global holomorphic

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[4] A discussion on the Calabi–Yau case can be found in [9].

sections of $\pm kK_X$, respectively) naturally determine a sequence of canonical symmetric probability measures on configuration spaces X^{N_k}, where $N_k \to \infty$.

For $k \in \mathbb{N}$, let $N_k := \dim H^0(X, k\beta K_X)$. Since βK_X is ample, $N_k \to \infty$ as $k \to \infty$.

Choose a basis $s_1, \dots, s_{N_k}$ of $\dim H^0(X, k\beta K_X)$. The associated *Slater determinant* is the section

$$\det S^{(k)} \in H^0(X^{N_k}, (k\beta K_X)^{\boxtimes N_k}),$$

defined by

$$\det S^{(k)}(x_1, \dots, x_{N_k}) := \det s_j(x_k).$$

Using the natural identification

$$K_{X^{N_k}} \simeq (K_X)^{\boxtimes N_k},$$

the section $\det S^{(k)}$ may be viewed as a holomorphic section of the line bundle $k\beta K_{X^{N_k}}$[5]. Consequently, if $z = (z_1, \dots, z_{nN_k})$ are holomorphic coordinates in a neighborhood $U \subset X^{N_k}$, we may write

$$\det S^{(k)} = F_U(z)\big(\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_{nN_k}\big)^{\otimes \beta k},$$

locally, where $F_U \colon U \to \mathbb{C}$ is a holomorphic function. It follows from the transformation law for (pluri)canonical forms that $\det S^{(k)}$ induces a measure on X^{N_k}, denoted

$$\left|\det S^{(k)}\right|^{2\beta/k} = \left(\det S^{(k)} \wedge \overline{\det S^{(k)}}\right)^{\beta/k}.$$

More concretely, in local holomorphic coordinates $z = (z_1, \dots, z_{nN_k})$,

$$\left|\det S^{(k)}\right|^{2\beta/k} = \left|F_U(z)\right|^{2\beta/k} \frac{i^{(nN_k)^2}}{2^{nN_k}} \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_{nN_k} \wedge \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_{nN_k}, \quad (3.5.2)$$

and these local expressions glue together to define a measure on X^{N_k}.

A canonical symmetric probability measure $\mu^{(N_k)}$ on X^{N_k} is then defined as

$$\mu^{(N_k)} = \frac{1}{Z_{N_k}} \left|\det S^{(k)}\right|^{2\beta/k}, \qquad Z_{N_k} = \int_{X^{N_k}} \left|\det S^{(k)}\right|^{2\beta/k}. \quad (3.5.3)$$

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[5] Notice that we use additive notation for tensor powers of line bundles and $\boxtimes$ for external tensor products. Indeed,

$$(k\beta K_X)^{\boxtimes N_k} = \big((\beta K_X)^{\otimes k}\big)^{\boxtimes N_k} \simeq \big((\beta K_X)^{\boxtimes N_k}\big)^{\otimes k} \simeq (\beta K_{X^{N_k}})^{\otimes k} = k\beta K_{X^{N_k}}.$$

Note that under a change of basis, $\det S^{(k)}$ only changes by a multiplicative constant, namely the determinant of the base-change matrix, and this merely rescales both the numerator and the denominator in (3.5.3) by the same positive constant. Hence $\mu^{(N_k)}$ is independent of the choice of basis, and therefore canonically associated to X.

When $\beta = 1$, the partition function is always finite, and the measures $\mu^{(N_k)}$ are therefore well-defined. In contrast, in the Fano case when $\beta = -1$, the factor $\left|\det S^{(k)}\right|^{-2/k}$ may fail to be integrable along the zero locus of $\det S^{(k)}$. The resulting integrability problem is closely related to algebro-geometric stability and is of central importance in the Fano case discussed below.

The measures $\mu^{(N_k)}$ also admit a natural interpretation from the viewpoint of statistical mechanics. Indeed, comparing (3.5.3) with the Gibbs measures introduced in Section 3.4, one is led to regard the parameter $\beta = \pm 1$ as an inverse temperature.

Remark 3.5.2. Alternatively, to avoid negative inverse temperatures, one can interpret the sign of K_X as the sign of the coupling constants. With this perspective, $K_X > 0$ and $-K_X > 0$ generalize to repulsive and attractive *one-component plasmas*, respectively.

This suggests introducing Gibbs ensembles for arbitrary values of β, however, one has to be a bit careful. Let $\alpha \in \{\pm 1\}$ be chosen such that αK_X is ample. If $\det S^{(k)}$ is a section of $\alpha k K_{X^{N_k}}$, then for $\beta \neq \pm 1$, the local expressions defined by $\left|\det S^{(k)}\right|^{\beta/k}$, cf. (3.5.2), no longer glue to a globally well-defined measure on X^{N_k}. Instead, we proceed as follows: Fix a choice of Hermitian metric $\|\cdot\|$ on αK_X; this metric then induces a volume form $\mathrm{d}V$ on X, as well as Hermitian metrics on $\alpha k K_X$ which we shall denote also by $\|\cdot\|$. Then we define the *temperature-deformed Gibbs measure*

$$\mu^{(N_k)}_{\|\cdot\|}(\beta) = \frac{1}{Z_{N_k,\|\cdot\|}(\beta)} \left\| \det S^{(k)} \right\|^{2\beta/k} \mathrm{d}V^{\otimes N_k}, \tag{3.5.4}$$

where

$$Z_{N_k,\|\cdot\|}(\beta) = \int_{X^{N_k}} \left\| \det S^{(k)} \right\|^{2\beta/k} \mathrm{d}V^{\otimes N_k}, \tag{3.5.5}$$

is called the *temperature-deformed partition function.* Clearly, $\mu^{(N_k)}_{\|\cdot\|}(\beta)$ is not canonical, as is depends on a choice of metric $\|\cdot\|$ on αK_X. However, we have that

$$\mu^{(N_k)}_{\|\cdot\|}(\alpha) = \mu^{(N_k)}.$$

Indeed, when $\beta = \alpha$ the metric factors cancel[6] and one recovers the canonical

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[6] It is important here that we used the volume form $\mathrm{d}V$ induced by $\|\cdot\|$ in the definition of $\mu^{(N_k)}_{\|\cdot\|}(\beta)$.

measure defined in (3.5.3). Thus, the canonical point process may be viewed as a distinguished Gibbs ensemble at inverse temperature $\beta = \alpha$. This point of view plays an important role in Berman's proof establishing a LDP in the case $K_X > 0$. It is also the point of view adopted in Paper IV.

3.5.3 Convergence for canonically polarized manifolds

As aforementioned, when X is canonically polarized, the Gibbs measures $\mu^{(N_k)}$ are well-defined, since the corresponding partition functions Z_{N_k} are finite. Moreover, Berman proved the following LDP for the associated empirical measures:

Theorem 3.5.3 (Berman [7]). *Let X be a compact Kähler manifold with $K_X > 0$. Then the laws of the empirical measures δ_{N_k} of the N_k-point process defined by $\mu^{(N_k)}$ satisfy a LDP with speed N_k and rate functional*

$$\mu \mapsto \mathcal{F}(\mu) - \inf_{\nu \in \mathscr{P}(X)} \mathcal{F}(\nu),$$

where $\mathcal{F}$ is the free energy functional on $\mathscr{P}(X)$.

The so-called *one-point correlation measure* of the process is obtained by pushing forward $\mu^{(N_k)}$ under the natural projection onto any factor. Thus, if $\mathrm{proj}_1 \colon X^{(N_k)} \to X$ denotes the projection onto the first coordinate, the measure

$$\nu_k := (\mathrm{proj}_1)_* \mu^{(N_k)} = \frac{1}{Z_{N_k}} \int_{X^{N_k - 1}} \left| \det S^{(k)}(-, x_2, \ldots, x_{N_k}) \right|^{2/k} \mathrm{d}V^{\otimes(N_k - 1)} \tag{3.5.6}$$

defines a probability measure on X. Since the free energy admits a unique minimizer, namely the normalized volume form $\mathrm{d}V_{\mathrm{KE}}$ of the Kähler–Einstein metric, the LDP implies that the empirical measures concentrate around $\mathrm{d}V_{\mathrm{KE}}$. In particular, the one-point correlation measures ν_k converge weakly to $\mathrm{d}V_{\mathrm{KE}}$ as $k \to \infty$.

Moreover, one may associate to ν_k the current

$$\omega_k := i\partial\bar{\partial} \log \nu_k,$$

where ν_k is identified with its local density in local holomorphic coordinates, cf. (2.7.3) and the surrounding discussion above. As a further consequence of the LDP, ω_k converges weakly to the Kähler–Einstein metric ω_{KE} as $k \to \infty$.

Remark 3.5.4. At first sight, the above weak convergence of the measures ν_k appears to be substantially weaker than establishing a large deviation principle. However, no direct proof of this convergence is currently known. Instead, the convergence is obtained as a consequence of the stronger large deviation result.

3.5.4 Fano manifolds and Gibbs stability

Let X be a Fano manifold. As previously mentioned in Section 3.5.2, the canonical measure,

$$\mu^{(N_k)} = \frac{1}{Z_{N_k}} \Big| \det S^{(k)} \Big|^{-2/k},$$

is not automatically well-defined since $\big|\det S^{(k)}\big|^{-2/k}$ may fail to be integrable along the hypersurface $\{\det S^{(k)} = 0\}$. The integrability problem is a feature of the Fano case and has no analogue in the canonically polarized setting. Recall from Section 2.7 that X admits a unique Kähler–Einstein metric on X if and only if it is K-stable, equivalently if $\delta(X) = \delta^{\mathrm{A}}(X) > 1$. As a probabilistic analogue of K-stability, Berman introduced the notion of *Gibbs stability*, defined by the condition

$$Z_{N_k} < \infty,$$

for all k sufficiently large. He conjectured that Gibbs stability is equivalent to K-stability and that Gibbs stability should imply a large deviation principle for the associated empirical measures, analogous to that in the canonically polarized case. In particular, the unique minimizer of the corresponding rate functional should be the normalized volume form $\mathrm{d}V_{\mathrm{KE}}$ of the Kähler–Einstein metric ω_{KE}.

The above definition of Gibbs stability admits a manifestly algebro-geometric reformulation. Define the *microscopic stability threshold at level* k by

$$\gamma^{(N_k)}(X) := \mathrm{lct}\Big(X^{N_k}, k^{-1}\mathrm{div}\big(\det S^{(k)}\big)\Big). \tag{3.5.7}$$

The invariants $\gamma^{(N_k)}(X)$ furnish definitions of Gibbs stability, *Gibbs semistability*, *uniform Gibbs stability* and *Gibbs instability*, mirroring the corresponding notions in K-stability theory. More precisely, X is Gibbs semistable if

$$\gamma(X) := \liminf_{k\to\infty} \gamma^{(N_k)}(X) \geq 1,$$

and X is Gibbs unstable otherwise; X is Gibbs stable if $\gamma^{(N_k)}(X) > 1$ for N_k sufficiently large. Indeed, by the analytic characterization of log canonical thresholds discussed in Section 2.9.2, $\gamma^{(N_k)}(X)$ is precisely the integrability threshold of the temperature-deformed partition function $Z_{N_k,\|\cdot\|}(\beta)$; the condition $\gamma^{(N_k)}(X) > 1$ is thus equivalent to the finiteness of Z_{N_k}. Lastly, we say that X is uniformly Gibbs stable if $\gamma(X) > 1$. Analogously to the fact that K-stability and uniform K-stability are equivalent for Fano manifolds, it is conjectured that Gibbs stability and uniform Gibbs stability are equivalent.

The relationship between Gibbs stability and K-stability is not yet fully understood. Fujita and Odaka showed in [28] that $\delta_k(X) \geq \gamma^{(N_k)}(X)$ for each k, where $\lim_{k\to\infty} \delta_k(X) = \delta(X)$, from which it immediately follows that uniform Gibbs stability implies uniform K-stability and thus K-stability. Subsequently, Berman has given a direct proof that uniform Gibbs stability implies the existence of a Kähler–Einstein metric, without involving uniform K-stability, see [10]. The converse statement remains open in general.

Likewise, the conjectural large deviation principle has not been established in the Fano setting, however, the problem has been reduced to showing suitable bounds on the partition functions Z_{N_k}, together with a certain *zero-free* property of the temperature-deformed partition functions $Z_{N_k,\|\cdot\|}(\beta)$, viewed as meromorphic functions of the complex parameter β; see [9].

Finally, we note that if the automorphism group of X is non-discrete, then X is automatically Gibbs unstable. Since such Fano manifolds may nevertheless admit Kähler–Einstein metrics, it is natural to seek a refined notion of Gibbs stability, analogous to K-polystability, in which they appear as polystable objects.

3.5.5 Non-trivial automorphisms and Gibbs polystability

In this final section we arrive at the setting of Paper IV. One of the main goals of the paper is to extend the conjectural framework surrounding the probabilistic approach to Kähler–Einstein metrics on Fano manifolds to the case where $\mathcal{G} := \mathrm{Aut}_0(X)$ is nontrivial. As discussed in Section 2.7, the presence of automorphisms fundamentally changes the Kähler–Einstein problem. Indeed, if ω_{KE} is a Kähler–Einstein metric on X, then so is every pullback $g^*\omega_{\mathrm{KE}}$ for $g \in \mathcal{G}$. Consequently, Kähler–Einstein metrics are no longer isolated objects, but rather occur in $\mathcal{G}$-orbits. The isometry group of any Riemannian metric, in particular any Kähler–Einstein metric, on a compact manifold is compact. However, when $\mathcal{G} = \mathrm{Aut}_0(X)$ is nontrivial and reductive, $\mathcal{G}$ is the complexification of a maximal compact subgroup $\mathcal{K} \subset \mathcal{G}$, and is therefore non-compact. Thus a Kähler–Einstein metric cannot be invariant under the full group $\mathcal{G}$, but only under a maximal compact subgroup.

The same phenomenon appears on the probabilistic side and causes it to break down. The canonical probability measures $\mu^{(N_k)}$ are $\mathcal{G}$-invariant by construction. Since a Kähler–Einstein metric is generally not invariant under the full group $\mathcal{G}$, there is no distinguished Kähler–Einstein volume form towards which the associated empirical measures could converge. In this sense, the canonical ensembles possess too much symmetry. A manifestation of this obstruction is that the partition functions Z_{N_k} are divergent for every k. Thus, unlike the case of discrete automorphism group discussed above, the canonical Gibbs ensembles are not even defined and X is Gibbs unstable.

This situation is closely analogous to the variational approach. There, the presence of automorphisms forces one to work with functionals modulo the action of $\mathcal{G}$. In particular, the ordinary stability threshold $\delta^{\mathrm{A}}(X)$ is replaced by a $\mathcal{G}$-reduced stability threshold $\delta^{\mathrm{A}}(X)^{\mathcal{G}}$, see Section 2.7.3 above.

Paper IV introduces the *reduced microscopic stability thresholds* $\gamma^{(N_k)}(X)^{\mathcal{G}}$ defined analogously to the microscopic stability thresholds $\gamma^{(N_k)}(X)$, but on the *semistable locus* of the diagonal $\mathcal{G}$-action on X^{N_k} in the sense of *Geometric Invariant Theory (GIT)*, see, e.g., [56]. Accordingly, we define X to be *Gibbs polystable* if X is Gibbs semistable and $\gamma^{(N_k)}(X)^{\mathcal{G}} > 1$ for each k sufficiently large. Moreover, we define the *reduced Gibbs stability threshold*

$$\gamma(X)^{\mathcal{G}} := \liminf_{k\to\infty} \gamma^{(N_k)}(X)^{\mathcal{G}},$$

and say that X is *uniformly Gibbs polystable* if X is Gibbs semistable and $\gamma(X)^{\mathcal{G}} > 1$. These invariants play, in the presence of automorphisms, the same role as the microscopic stability thresholds $\gamma^{(N_k)}(X)$ do when $\mathrm{Aut}_0(X)$ is trivial. In particular, we conjecture that

$$\gamma(X)^{\mathcal{G}} = \delta^{\mathrm{A}}(X)^{\mathcal{G}}, \tag{3.5.8}$$

which would imply that uniform Gibbs polystability is equivalent to uniform K-polystability, and thus to the existence of a Kähler–Einstein metric, in analogy with the conjectural relationship between uniform Gibbs stability and uniform K-stability. Moreover, (3.5.8) would provide an algebro-geometric characterization of the reduced stability threshold $\delta^{\mathrm{A}}(X)^{\mathcal{G}}$.

In Paper IV we also propose a refinement of the probabilistic framework. The basic idea is to explicitly break part of the symmetry arising from automorphisms. Since this necessarily requires making a choice, we fix a maximal compact subgroup $\mathcal{K} \subset \mathcal{G}$ and a $\mathcal{K}$-invariant Kähler form $\omega_0 \in c_1(-K_X)$. Associated to these choices is a *moment map* $\boldsymbol{m}\colon X \to \mathfrak{k}^*$ for the $\mathcal{K}$-action on X, and an induced moment map

$$\boldsymbol{m}_{N_k} = \frac{1}{N_k} \sum_{j=1}^{N_k} \mathrm{proj}_j^* \boldsymbol{m}\colon X^{N_k} \to \mathfrak{k}^*, \tag{3.5.9}$$

for the diagonal $\mathcal{K}$-action on X^{N_k}, where $\mathfrak{k}^*$ denotes the dual of the Lie algebra of $\mathcal{K}$. Moment maps are fundamental objects in symplectic geometry. In our setting, their role is to single out a subset $\{\boldsymbol{m}_{N_k} = 0\} \subset X^{N_k}$, on which some of the redundant symmetry has been removed (indeed, $\{\boldsymbol{m}_{N_k} = 0\}$ is $\mathcal{K}$-invariant but not $\mathcal{G}$-invariant). More specifically, the zero locus $\{\boldsymbol{m}_{N_k} = 0\}$ serves as a representative of the *semistable quotient*: The closure of every *semistable* $\mathcal{G}$-orbit meets $\{\boldsymbol{m}_{N_k} = 0\}$, and this intersection is unique up to the action of $\mathcal{K}$; see, e.g., [56] for background.

In Paper IV, we explore different approaches to "restricting" or "reducing" the infinite Gibbs measures $\left|\det S^{(k)}\right|^{-2/k}$ to the locus $\{\boldsymbol{m}_{N_k} = 0\} \subset X^{N_k}$, and study the resulting partition functions. By introducing temperature-deformed versions of these reduced Gibbs measures and partition functions, cf. Section 3.5.2, these systems may be interpreted as canonical ensembles of N_k interacting particles subject to the moment-map constraint $\boldsymbol{m}_{N_k} = 0$.

A main motivation for this approach is the following fact: After fixing $\mathcal{K} \subset \mathcal{G}$ and ω_0 as above, if X admits a Kähler–Einstein metric then there exists a unique Kähler–Einstein metric ω such that

$$\int_X \boldsymbol{m}\,\omega^n = 0;$$

see, [5]. It is this distinguished Kähler–Einstein metric that the reduced Gibbs measures are conjectured to recover in the large N_k-limit. Analogously to the cases of canonically polarized manifolds and Fano manifolds with discrete automorphism group, we conjecture that Gibbs polystability implies that the empirical measures associated to the reduced Gibbs measures satisfy a LDP, whose rate functional is given, up to normalization, by the free energy.

4 Summary of appended papers

In this final chapter we give brief summaries of the appended papers, highlighting some of the main results and ideas.

4.1 Summary of Paper I

In Paper I, we assume the general setting of Section 3.3, namely, that of a reduced analytic space X of pure dimension n and a holomorphic section $s\colon X \to E$ of a holomorphic vector bundle $E \to X$. Let $\omega \in \mathcal{A}^{p,q}_{s,\|\cdot\|}(X)$ be a quasi-meromorphic form, where $\|\cdot\|$ is a smooth Hermitian metric on E.

We study the Archimedean zeta function $\Gamma^{\xi}_{\|s\|}(\lambda, \omega)$ from (3.3.2), for $\xi \in \mathscr{D}^{n-p,n-q}(X)$, and equivalently the current-valued meromorphic function $\|s\|^{2\lambda}\omega$, see Section 3.2. Following classical ideas, using a principalization of the ideal (sheaf) generated by s, we show that $\Gamma^{\xi}_{\|s\|}(\lambda, \omega)$, and equivalently $\|s\|^{2\lambda}\omega$, admit meromorphic continuation to all of $\mathbb{C}$, with poles contained in a discrete subset of $\mathbb{Q}$. Moreover, we show that, in a neighborhood of $\lambda = 0$,

$$\|s\|^{2\lambda}\omega = \sum_{j=-\kappa}^{\infty} \lambda^j \mu_j^{\|s\|}(\omega),$$

where $\kappa \leq n$, and the currents $\mu_j^{\|s\|}(\omega)$ are characterized by the property that, for every test form ξ, the pairing $\langle \mu_j^{\|s\|}(\omega), \xi \rangle$ is the j^{th} Laurent series coefficient of $\Gamma^{\xi}_{\|s\|}(\lambda, \omega)$ at $\lambda = 0$, for each $j \geq -\kappa$[1]. Moreover, we show that

$$\operatorname{supp} \mu_{-n}^{\|s\|}(\omega) \subseteq \operatorname{supp} \mu_{-n+1}^{\|s\|}(\omega) \subseteq \cdots \subseteq \operatorname{supp} \mu_{-1}^{\|s\|}(\omega) \subseteq \operatorname{supp} \mu_0^{\|s\|}(\omega) = \overline{\operatorname{supp} \omega}.$$

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[1] Notice here that we here use the notational convention found in Paper II, where we label the coefficients by the order in the Laurent series, which differs by a sign to the convention used in Paper I.

In particular, $\operatorname{supp}\mu_{-1}^{\|s\|}(\omega) \subseteq \{s=0\}$. Moreover, if the ideal sheaf generated by s is radical, $\operatorname{supp}\mu_j^{\|s\|}(\omega) \subseteq \{s=0\}_{\mathrm{sing}} \cup (X_{\mathrm{sing}} \cap \{s=0\})$ for $j \leq -2$, see Paper I, Theorem 4.1. In particular, $\mu_0^{\|s\|}(\omega)$ is a current extension of ω across $\{s=0\}$. In the case where X is compact and ω is of top degree, we define a finite part of the divergent integral $\int_X \omega$ by

$$\mathrm{fp}\int_X \omega := \langle \mu_0^{\|s\|}(\omega), 1\rangle. \tag{4.1.1}$$

In general, the currents $\mu_j^{\|s\|}(\omega)$ depend on the choice of Hermitian metric $\|\cdot\|$ on E. Even when the underlying zero locus $V=\{s=0\}$ is fixed, the currents generally depend on the choice of defining section s. In Paper I, however, the section s is regarded as part of the input data, and our focus is on the dependence on the Hermitian metric. The following theorem, which is the main result of Paper I, describes how the currents $\mu_j^{\|s\|}(\omega)$ transform under a change of metric.

Theorem 4.1.1 (Paper I, Theorem 1.1). *Let $\omega \in \mathcal{A}_s^{p,q}(X)$. For any two Hermitian metrics $\|\cdot\|$ and $|\cdot|$ on E, we have that*

$$\mu_j^{|s|}(\omega) = \sum_{\ell=0}^{n+j} \frac{1}{\ell!}\left(\log\frac{|s|^2}{\|s\|^2}\right)^{\ell} \mu_{j-\ell}^{\|s\|}(\omega), \tag{4.1.2}$$

for $j \geq -\kappa$.

A subtlety in (4.1.2) is that the factors $\big(\log\frac{|s|^2}{\|s\|^2}\big)^{\ell}$ are locally bounded but need not be smooth, so the products on the right-hand side are not a priori well-defined. The key observation is that, after pulling back to a suitable modification, these products are canonically defined. Indeed, if s defines a (Cartier) divisor, then $|s|^2/\|s\|^2$ is smooth and positive.

In the same setting as above, we also consider the cut-off regularization $\mathcal{I}^{\xi}_{\|s\|}(\lambda,\omega)$, see (3.3.5). Using the relation (3.3.8) between $\mathcal{I}^{\xi}_{\|s\|}(\lambda,\omega)$ and $\Gamma^{\xi}_{\|s\|}(\lambda,\omega)$, we obtain an asymptotic expansion of $\mathcal{I}^{\xi}_{\|s\|}(\lambda,\omega)$ as $\epsilon \to 0^+$ of the form

$$\begin{aligned}\mathcal{I}^{\xi}_{\|s\|}(\lambda,\omega) = \langle \mu_0^{\|s\|}(\omega),\xi\rangle &+ \sum_{j=1}^{\kappa}\frac{1}{j!}\big(\log\epsilon^{-1}\big)^j \langle \mu_{-j}^{\|s\|}(\omega),\xi\rangle \\ &+ \sum_{p\in P_+}\operatorname*{Res}_{\lambda=p}\Big\{\epsilon^{-\lambda}\lambda^{-1}\Gamma^{\xi}_{\|s\|}(\lambda,\omega)\Big\} + \mathcal{O}(\epsilon^{\delta}),\end{aligned} \tag{4.1.3}$$

for some $\delta > 0$, where P_+ denotes the set of poles of $\Gamma^{\xi}_{\|s\|}(\lambda,\omega)$ in the half-plane $\{\mathfrak{Re}\,\lambda > 0\}$, see Paper I, Theorem 6.1. In particular, when X is compact and ω

has top degree, we find that the distinguished finite part extracted from the cut-off regularization coincides with the finite part defined by (4.1.1).

4.2 Summary of Paper II

In Paper II, we remain in the general setup of Paper I. The main objective of Paper II is to develop a systematic approach to computing finite parts of divergent complex-geometric integrals. The starting point is the identification of a subclass of quasi-meromorphic forms for which it is possible to compute finite parts explicitly.

More precisely, let $s\colon X \to L$ be a holomorphic section of a holomorphic line bundle equipped with a Hermitian metric $\|\cdot\|$. We define the *elementary quasi-meromorphic form* $\omega \in \mathcal{A}^{1,1}_{s,\|\cdot\|}(X)$ associated to $(s, \|\cdot\|)$ by

$$\omega = \bar\partial \log \|s\|^2 \wedge \partial \log \|s\|^2. \tag{4.2.1}$$

We show that the Laurent series expansion of the current-valued meromorphic function $\|s\|^{2\lambda}\omega$ about $\lambda = 0$ always has a simple pole at $\lambda = 0$, that is

$$\|s\|^{2\lambda}\omega = \sum_{j=-1}^{\infty} \lambda^j \mu_j^{\|s\|}(\omega).$$

Furthermore, the expansion can be computed explicitly. More precisely, we have the following formula for the current coefficients:

$$\mu_j^{\|s\|}(\omega) = \frac{1}{(j+2)!}\bar\partial\partial\big(\log\|s\|^2\big)^{j+2} + \frac{2\pi i}{(j+1)!}\big(\log\|s\|^2\big)^{j+1}c_1(L, \|\cdot\|), \tag{4.2.2}$$

for each $j \geq -1$, where $c_1(L, \|\cdot\|)$ is the Chern form of the Hermitian line bundle $(L, \|\cdot\|)$. Note that $\partial\bar\partial$ are distributional derivatives of the locally integrable functions $\big(\log\|s\|^2\big)^{\ell}$, for $\ell \geq 0$.

In particular, cf. (2.5.2),

$$\mu_{-1}^{\|s\|}(\omega) = 2\pi i[\mathrm{div}(s)], \tag{4.2.3}$$

where $[\mathrm{div}(s)]$ denotes the Lelong current of the divisor $\mathrm{div}(s)$, .

The main result of the paper is a decomposition formula expressing the Laurent coefficients of certain products of elementary quasi-meromorphic forms in terms of the Laurent coefficients of the individual factors.

Theorem 4.2.1 (Paper II, Theorem 1.1)**.** *For each $1 \leq j \leq \kappa$, where $1 \leq \kappa \leq n$, let $s_j\colon X \to L_j$ be a holomorphic section of a holomorphic line bundle L_j,* *such that* $s_{j_1}, \dots, s_{j_\ell}$ *is a locally complete intersection for each* $1 \le j_1 < \cdots < j_\ell \le \kappa$. *Furthermore, let* $\| \cdot \|_j$ *be a Hermitian metric on* L_j *and let* $\omega_j = \bar\partial \log \|s_j\|_j^2 \wedge \partial \log \|s_j\|_j^2$. *Let* $s = s_1 \otimes \cdots \otimes s_\kappa$ *and let* $\| \cdot \|$ *be the Hermitian metric on* $L = L_1 + \cdots + L_\kappa$ *defined by* $\|s\|^2 = \|s_1\|_1^2 \cdots \|s_\kappa\|_\kappa^2$. *Then we have that*

$$\sum_{j=-\kappa}^{\infty} \lambda^j \mu_j^{\|s\|}(\omega) = \left(\sum_{j_1=-1}^{\infty} \lambda^{j_1} \mu_{j_1}^{\|s_1\|_1}(\omega_1) \right) \wedge \cdots \wedge \left(\sum_{j_\kappa=-1}^{\infty} \lambda^{j_\kappa} \mu_{j_\kappa}^{\|s_\kappa\|_\kappa}(\omega_\kappa) \right). \tag{4.2.4}$$

The main technical difficulty is to show that the products of currents appearing on the right-hand side of (4.2.4) admit a natural interpretation and are well-defined.

As a corollary of the above, we find that

$$\mu_{-\kappa}^{\|s\|}(\omega) = (2\pi i)^\kappa [\mathrm{div}(s_1)] \wedge \cdots \wedge [\mathrm{div}(s_\kappa)],$$

where the product of Lelong currents on the right-hand side is well-defined under the locally complete intersection assumption and coincides with the Lelong current associated to the *proper intersection* $\mathrm{div}(s_1) \cdot \mathrm{div}(s_2) \cdots \mathrm{div}(s_\kappa)$.

Although the assumptions of Theorem 4.2.1 are restrictive, the decomposition formula can nevertheless be applied in a much broader context. Indeed, after pulling back to a suitable modification, one may reduce to the case where X is a complex manifold and ω has singularities along a normal crossings divisor. Combining this reduction with the cohomological nature of the finite part (4.1.1) and a generalization of Paper I, Theorem 1.1, we show that any finite part $\mathrm{fp} \int_X \omega$, where $\omega \in \mathcal{A}_s^{n,n}(X)$, can, in principle, be expressed in terms of Laurent coefficients associated to finite sums of products of elementary quasi-meromorphic forms multiplied by smooth forms; see Paper II, Section 5 (see formula (5.8) in particular).

While the resulting expression is generally too complicated for direct computation and involves non-constructive elements, it simplifies considerably in certain geometric situations. In particular, we apply the formula to compute finite parts for a family of examples on $\mathbb{P}^n$ see Paper II, Section 6.

4.3 Summary of Paper III

In Paper III, we assume the setting of Section 3.4.1, that is, we consider Gibbs ensembles of N-particle systems on $\mathbb{P}^1 \simeq \mathbb{S}^2$, governed by the Coulomb energy

$$E_N = - \sum_{1 \le j < k \le N} c_{jk} \log d(p_j, p_k)^2,$$

where $d(p_j, p_k)$ is the chordal distance between particles at positions p_j and p_k on the sphere, and $c_{jk} \in \mathbb{R}$ determines the strength and sign of their interaction. The statistical-mechanical canonical ensemble of the system is governed by a Gibbs measure, see (3.4.1), defined with respect to a volume form on $\mathbb{S}^2$. For simplicity, we fix the standard normalized volume form on $\mathbb{S}^2$, which corresponds to the Fubini–Study form on $\mathbb{P}^1$.

In this general setup, we show that the associated partition function $Z_N(\beta)$, see (3.4.2), admits a meromorphic continuation in the inverse temperature parameter β with poles lying in a discrete subset of $\mathbb{R}$ (or $\mathbb{Q}$ if $(c_{jk}) \in \mathbb{Q}^{N\times N}$).

The integral expression for the partition function is defined in some strip $\beta^- < \mathfrak{Re}\,\beta < \beta^+$ (where, possibly, $\beta^\pm = \pm\infty$), and we call the integrability thresholds $\beta^\pm$ the *critical inverse temperatures.* The first result of Paper III is a characterization of $\beta^\pm$ in terms of a discrete optimization problem.

Theorem 4.3.1 (Paper III, Theorem 1.3)**.** *The partition function $Z_N(\beta)$ is finite for $\beta^- < \mathfrak{Re}\,\beta < \beta^+$, where*

$$\beta^- = \begin{cases} 1/T^- & \textit{if } T^- < 0, \\ -\infty & \textit{otherwise}, \end{cases} \tag{4.3.1}$$

where

$$T^- = -\max\left\{ \frac{1}{|S|-1} \sum_{j,k\in S:\, j<k} c_{jk} \;\middle|\; S \subset \{1,\dots,N\},\ |S| \geq 2 \right\}. \tag{4.3.2}$$

Similarly,

$$\beta^+ = \begin{cases} 1/T^+ & \textit{if } T^+ > 0, \\ -\infty & \textit{otherwise}, \end{cases} \tag{4.3.3}$$

where

$$T^+ = -\min\left\{ \frac{1}{|S|-1} \sum_{j,k\in S:\, j<k} c_{jk} \;\middle|\; S \subset \{1,\dots,N\},\ |S| \geq 2 \right\}. \tag{4.3.4}$$

In particular, when $c_{jk} \geq 0$ for all $j < k$, then β^- computes the log canonical threshold associated to the $\mathbb{R}$-divisor

$$\sum_{1\leq j<k\leq N} c_{jk}\{p_j = p_k\}.$$

on $(\mathbb{S}^2)^N \simeq (\mathbb{P}^1)^N$.

The theorem reduces the determination of the critical inverse temperatures to a finite combinatorial optimization problem. Although solving this problem explicitly for arbitrary coupling matrices (c_{jk}) and large values of N is generally difficult, the formulas readily yield effective lower bounds for β^- and upper bounds for β^+ by simply evaluating the expressions in (4.3.2) and (4.3.4), respectively, on suitably chosen $S \subset \{1, \dots, N\}$.

The second main result concerns the behavior of the Gibbs measure $\mu(\beta)$, see (3.4.1), as β approaches one of the critical inverse temperatures. The limiting measure is supported on a union of partial diagonals corresponding to the subsets that optimize the expressions appearing in (4.3.2) and (4.3.4). More precisely, one associates to the collection of optimizing subsets certain maximal nested families ("nests"), and the support of the limiting measure is determined by the corresponding collision loci. To state the result, let $\mathcal{N}^\pm$ denote the collections of maximal nests associated to the optimizers of (4.3.2) and (4.3.4), see Paper III, Section 1 for details.

Theorem 4.3.2 (Paper III, Theorem 1.5)**.** *As $\beta \to \beta^\pm$, provided the endpoint is finite, $\mu(\beta) \to \mu^\pm$ weakly, where $\mu^\pm$ is a probability measure on $(\mathbb{S}^2)^N$ supported on the following locus*

$$\bigcup_{K \in \mathcal{N}^\pm} \bigcap_{S \in K} \{(p_1, \dots, p_N) \in (\mathbb{S}^2)^N : p_{j_1} = \cdots = p_{j_k},\ S = \{j_1, \dots, j_k\}\}. \tag{4.3.5}$$

Thus, the same combinatorial optimization problem that determines the critical inverse temperatures also determines the support of the limiting measures.

We illustrate the above two theorems by looking at some classical examples. One such example which we consider is the *two-component plasma* on $\mathbb{S}^2$: Fix positive integers N_1, N_2 such that $N_1 + N_2 = N$. For $Z_1, Z_2 \in \mathbb{R}_+$, let $c_{jk} = q_j q_k$, where

$$q_j = \begin{cases} Z_1 & \text{if } 1 \leq j \leq N_1, \\ -Z_2 & \text{if } N_1 + 1 \leq j \leq N. \end{cases}$$

The number q_j should be thought of as the charge of the j^{th} particle; the first N_1 particles have charge Z_1 and the remaining N_2 particles have charge $-Z_2$. Thus, particles with the same charge repel and particles with different charge attract.

In this example, we find that any subset $S = \{j, k\}$ where $1 \leq j \leq N_1$ and $N_1 + 1 \leq k \leq N$ attain the minimum value in (4.3.4) and that the corresponding critical inverse temperature $\beta^+ = 1/(Z_1 Z_2)$. Moreover, these are the only subsets attaining the minimum. Thus, the limiting measure μ^+ is supported on the subvariety

$$\bigcup_{\substack{I \subset \{N_1+1, \dots, N\} \\ |I| = N_1}} \bigcap_{j=1}^{N_1} \{p_j = p_{I_j}\}, \tag{4.3.6}$$

where the union runs over all ordered tuples $I \subset \{N_1+1, \dots, N\}$ of size N_1, where we assume, without loss of generality, that $N_2 \geq N_1$. The support can be understood as the union over all possible configurations of points where the maximal number of positively charged and negatively charged particles are paired up.

The proofs of the above results rely on viewing the un-normalized Gibbs measure as a distribution on $(\mathbb{P}^1)^N$, that is, we consider the mapping

$$\xi \mapsto Z_N^\xi(\beta) = \int_{(\mathbb{P}^1)^N} \prod d(p_j, p_k)^{2c_{jk}\beta} \xi \, \mathrm{d}V^{\otimes N}, \qquad \xi \in \mathscr{D}(X).$$

This defines a generalized Archimedean zeta function. Relative to the zeta functions considered in Papers I and II, the novelty is that the exponents c_{jk} are allowed to be arbitrary real numbers of either sign.

We study the pole structure of this integral by resolving the singularites of the divisor

$$\sum_{1 \leq j < k \leq N} c_{jk}\{p_j = p_k\}$$

using the explicit compactification of configuration spaces due to Fulton–MacPherson, see Section 2.9.1 above.

4.4 Summary of Paper IV

In Paper IV, we assume the setting of Section 3.5.5 generalized to log Fano pairs. For most of this summary, however, we restrict attention to the case of a Fano manifold X. Moreover, we assume that the Futaki invariant of X vanishes, and that the identity component of the automorphism group $\mathcal{G} = \mathrm{Aut}_0(X)$ of X is reductive and non-trivial.

As we briefly explained in Section 3.5.5, by fixing a maximal compact subgroup $\mathcal{K} \subset \mathcal{G}$, and a $\mathcal{K}$-invariant Kähler form $\omega_0 \in c_1(-K_X)$, one obtains a moment map $\boldsymbol{m}\colon X \to \mathfrak{k}^*$ for the $\mathcal{K}$-action, where $\mathfrak{k}^*$ denotes the dual of the Lie algebra of $\mathcal{K}$. Letting $N_k = \dim H^0(X, -kK_X)$, for any $k \in \mathbb{N}$, $\boldsymbol{m}$ induces a moment map $\boldsymbol{m}_{N_k} = N_k^{-1} \sum_{j=1}^{N_k} \mathrm{proj}_j^* \boldsymbol{m}$ on the N_k-fold product X^{N_k}.

Assuming that 0 is a regular value of $\boldsymbol{m}_{N_k}$, a Gelfand–Leray type construction produces from the infinite measure $\left|\det S^{(k)}\right|^{-2/k}$ a measure $\nu_0^{(N_k)}$ on X^{N_k} supported on the locus $\{\boldsymbol{m}_{N_k} = 0\}$, see Paper IV, Section 6. This leads to the *reduced Gibbs measure*

$$\mu_0^{(N_k)} = \frac{1}{Z_{N_k,0}} \nu_0^{(N_k)}, \qquad Z_{N_k,0} = \int_{X^{N_k}} \nu_0^{(N_k)}, \tag{4.4.1}$$

which, whenever $Z_{N_k,0} < \infty$, is a well-defined probability measure on X^{N_k}, supported on $\{\boldsymbol{m}_{N_k} = 0\}$, and depending only on the maximal compact subgroup $\mathcal{K} \subset \mathcal{G}$ and the Kähler form ω_0. We refer to $Z_{N_k,0}$ as the *reduced partition function.*

As in the $\mathcal{G}$-trivial setting, finiteness of $Z_{N_k,0}$ is equivalent to an algebraic condition. We introduce the *reduced microscopic stability threshold at level k*

$$\gamma^{(N_k)}(X)^{\mathcal{G}} := \operatorname{lct}\Big((X^{N_k})_{\mathrm{ss}}, k^{-1}\operatorname{div}\big(\det S^{(k)}\big)\Big), \tag{4.4.2}$$

where $(X^{N_k})_{\mathrm{ss}}$ denotes the *semistable locus* (in the sense of GIT) of X^{N_k} with respect to the diagonal $\mathcal{G}$-action. Then, $Z_{N_k,0} < \infty$ if and only if $\gamma^{(N_k)}(X)^{\mathcal{G}} > 1$, cf. (3.5.7) and the surrounding discussion.

The invariants $\gamma^{(N_k)}(X)^{\mathcal{G}}$ and their limit $\gamma(X)^{\mathcal{G}} := \liminf_{k\to\infty} \gamma^{(N_k)}(X)^{\mathcal{G}}$, called the reduced Gibbs stability threshold, furnish a new algebro-geometric stability condition for X, which we call Gibbs polystability, see Section 3.5.5.

We make the following two conjectures:

Conjecture 4.4.1 (Paper IV, Conjecture 1.1)**.** *Let (X, Δ) be a log Fano manifold. Then the following are equivalent:*

- *(X, Δ) is Gibbs polystable,*
- *(X, Δ) is uniformly Gibbs polystable,*
- *(X, Δ) is K-polystable.*

Conjecture 4.4.2 (Paper IV, Conjecture 1.2)**.** *On any log Fano manifold (X, Δ) with vanishing Futaki invariant, the invariants $\gamma(X, \Delta)^{\mathcal{G}}$ and $\delta^{\mathrm{A}}(X, \Delta)^{\mathcal{G}}$ coincide.*

As evidence for these conjectures, we introduce a stronger notion, called *strong uniform Gibbs stability.* This notion admits both an algebro-geometric formulation in terms of log canonical thresholds and an analytic formulation in terms of the finiteness of a "thickened" reduced partition function

$$Z_{N_k,\epsilon_{N_k}} = \frac{1}{\epsilon_{N_k}^{\dim \mathfrak{k}}} \int_{X^{N_k} \cap \{|\boldsymbol{m}_{N_k}| < \epsilon_{N_k}\}} \left|\det S^{(k)}\right|^{-2/k}, \tag{4.4.3}$$

which depends on a suitable sequence ϵ_{N_k}. We prove that strong uniform Gibbs polystability implies K-polystability (Paper IV, Theorem 1.4).

The thickened construction also makes sense when 0 is not a regular value of the moment map. In that case we define

$$\mu_{\epsilon_{N_k}}^{(N_k)} = \frac{1}{\epsilon_{N_k}^{\dim \mathfrak{k}} Z_{N_k,\epsilon_{N_k}}} \mathbf{1}_{\{|\boldsymbol{m}_{N_k}| < \epsilon_{N_k}\}} \left|\det S^{(k)}\right|^{-2/k}, \tag{4.4.4}$$

where $\mathbf{1}_{\{|\boldsymbol{m}_{N_k}|<\epsilon_{N_k}\}}$ is the characteristic function, and $Z_{N_k,\epsilon_{N_k}}$ is given by (4.4.3). We show that, when 0 is a regular value of $\boldsymbol{m}_{N_k}$, then

$$\lim_{\epsilon_{N_k}\to 0} \mu^{(N_k)}_{\epsilon_{N_k}} = \mu^{(N_k)}_0,$$

for fixed N_k.

In dimension one, the only Fano manifold is $\mathbb{P}^1$. To obtain a richer class of examples, we therefore consider log Fano curves. These are all of the form $(\mathbb{P}^1, \Delta)$, where $\Delta = \sum_{j=1}^r w_j\{p_j\}$, for $p_j \in \mathbb{P}^1$, $w_j \in [0,1)$ such that $2 - \sum_{j=1}^r w_j > 0$. When $\mathcal{G} = \mathrm{Aut}_0(X,\Delta)$ (the subgroup of $\mathrm{Aut}_0(X)$ that preserves Δ) is non-trivial, then it is well-known that (X,Δ) is K-polystable if and only if $\Delta = w(\{p_1\}+\{p_2\})$, for $p_1 \neq p_2$, $0 \leq w < 1$.

In the one-dimensional case, $\det S^{(k)}$ vanishes precisely along the union of the diagonals $\{p_j = p_k\} \subset (\mathbb{P}^1)^{N_k}$. Consequently, the analysis becomes closely related to that of Paper III. Furthermore, the semi-stable locus $((\mathbb{P}^1)^{N_k})_{\mathrm{ss}}$ with respect to the diagonal $\mathcal{G}$-action is explicit in this case. It consists of the configurations $p = (p_1, \ldots, p_{N_k}) \in (\mathbb{P}^1)^{N_k}$ for which at most half of the p_j coincide. Using this explicit description of the semi-stable locus, together with the Fulton–MacPherson compactification, we are able to compute the invariants $\gamma^{(N_k)}(\mathbb{P}^1,\Delta)^{\mathcal{G}}$, see Paper IV, Theorem 1.3. In particular, we compute $\gamma(\mathbb{P}^1)^{\mathcal{G}} = 2$ and $\delta^{\mathrm{A}}(\mathbb{P}^1)^{\mathcal{G}} = 2$, thereby verifying Conjecture 4.4.2 in this case.

In the process, we make use of a thermodynamic characterization of $\delta^{\mathrm{A}}(\mathbb{P}^1)^{\mathcal{G}}$ as the optimal lower bound for the free energy on the subspace $\mathscr{P}(X)_0 \subset \mathscr{P}(X)$ of probability measures with vanishing moment, that is,

$$\mathscr{P}(X)_0 = \left\{\mu \in \mathscr{P}(X) : \int_X \boldsymbol{m}\mu = 0\right\}.$$

This yields a refinement of the classical logarithmic Hardy–Littlewood–Sobolev inequality on $\mathbb{S}^2$ (Paper IV, Theorem 1.6).

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