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VSL Control with Monotone Traffic Model Bounds Based on T-norms

Mladen Čičić, Balázs Kulcsár

Abstract—In this paper, we propose a generic first-order traffic modeling framework based on cumulative vehicle numbers and t-norm operators, unifying several classical and kinetic traffic flow models within a common structure. Exploiting the monotonicity and cooperativeness properties of the resulting system, we construct simplified lower- and upper-bound dynamic models in order to enclose the trajectories of more complex and potentially unknown traffic dynamics. These bound systems enable designing Variable Speed Limits control laws with formal guarantees despite model uncertainty. We develop two control strategies: a simple proportional (myopic, approximate optimal) feedback controller and a minimum-time controller derived using the Pontryagin Maximum Principle. Both controllers are shown to preserve system monotonicity and to effectively reduce traffic density inhomogeneity. Numerical simulations demonstrate that the proposed approach accelerates congestion dissipation, while requiring very limited model information. The proposed framework introduces a systematic and analytically tractable approach to robust Variable Speed Limits control, using monotone system theory to define bounds to uncertain traffic flow models.

I. INTRODUCTION

Variable Speed Limit (VSL) control is one of the most studied forms of highway traffic control [1], often employed to harmonizing the traffic conditions, improving the safety and efficiency of the traffic flow [2]. As the traffic flow consists of unpredictable human-driven vehicles with different behaviors and reactions to VSL control signals, this model uncertainty needs to somehow be accounted for in all but the simplest control laws. Designing local traffic control laws robust to model uncertainty, relying directly on the information about the traffic state near the actuator position, is typically only done in case of bottleneck decongestion [3]. The need to consider the wider traffic situation often results in prediction-based VSL control, requiring good prediction models. One intriguing and underexplored option is to reformulate the model in terms of cumulative vehicle numbers, and design controllers local in the transformed state [4].

While there have been some efforts to formulate generic traffic models [5], there is a lack of simplified representations that can be studied to provide guarantees for complex and potentially uncertain traffic dynamics. Attempts to provide such guarantees have mostly relied on the theory of monotone systems [6], and mostly focused on routing in traffic networks [7]. In the recently proposed Traffic Reaction Model (TRM) [8], the fundamental properties of traffic model discretizations, such as their nonnegativity, are used

to establish links with positive kinetic systems, enabling the use of well-developed theory of chemical reaction networks and compartmental models.

Many traffic models, e.g., the Cell Transmission Model (CTM) [9], describe the traffic flow as the minimum of the *demand* and *supply* and downstream traffic conditions, respectively. TRM proposes a different way to combine demand and supply, as a product, consistent with mass action kinetics [10]. Notably, when rescaled appropriately, both of these operators are so-called *t-norms* [11], used in fuzzy logic to generalize intersection. This raises the question of whether other t-norms could be used in traffic models.

In this work we introduce a general first-order traffic modeling framework inspired by the TRM. The model is formulated using t-norms and nondecreasing functions, resulting in a structure with properties convenient for system analysis. By reformulating the model in terms of cumulative counts of vehicles, instead of traffic densities, we are able to use its monotone and cooperative nature to derive lower and upper bounds on trajectories of uncertain traffic dynamics, as well as give conditions for VSL controllers that preserve this property. Due to the simplicity of the bound systems, we are able to design control laws with guarantees for the original system. We design and compare a simple proportional controller and a minimum-time controller based on the Pontryagin Maximum Principle, and demonstrate their effectiveness in simulations.

We start the remainder of the paper by defining the traffic modeling framework in Section II. Then, in Section III, the cumulative vehicle number bound systems are defined, and used to design density homogenization VSL controllers. The modeling and control design results are demonstrated in simulations in Section IV, and we conclude in Section V.

II. MODEL

For simplicity, we assume a homogeneous road with no on- or off-ramps, split into I cells $i \in \mathcal{I}$, $\mathcal{I} = \{1, \dots, I\}$ of length Δx , stretching from $x = 0$ to $x = I\Delta x$. The traffic density in cell i at time t is denoted $\rho_i(t) \in [0, P]$, where P is the maximum (jam) density, and its dual, empty space on the road, is $\nu_i(t) = P - \rho_i(t) \in [0, P]$.

A. Base model

The first-order macroscopic traffic density conservation law (Lighthill-Whitham-Richards (LWR) model [12])

$$\partial_t \rho(x, t) + \partial_x Q(\rho(x, t)) = 0, \quad (1)$$

with concave flux function Q . This model may be reformulated into an inhomogeneous Hamilton-Jacobi PDE [13]

$$\partial_t N(x, t) - Q(-\partial_x N(x, t)) = Q(\rho_0(t)), \quad (2)$$

Mladen Čičić (mladen.cicic@centralesupelec.fr) is with Université Paris-Saclay, CentraleSupélec, CNRS, Laboratoire des Signaux et Systèmes, 91190, Gif-sur-Yvette, France. Balázs Kulcsár (kulcsar@chalmers.se) is with Department of Electrical Engineering, Chalmers University of Technology, 41296, Göteborg, Sweden.

with flux function Q as the Hamiltonian, and the source term $Q(\rho_0(t))$ effectively representing the traffic flow entering the stretch of the road. Function $N(x, t)$ represents the cumulated number of vehicles, i.e., the total number of vehicles in the position interval $[0, x]$, and we have $\rho(x, t) = \partial_x N(x, t)$. This representation can be seen as the modification of the Moskowitz function $M(x, t)$ [14], which represents the cumulative count of vehicles that passed by position x by time t , for which $\rho(x, t) = -\partial_x M(x, t)$.

Following [8], the flux function is taken to be the form

$$Q(\rho) = \hat{q}q(\rho, \nu),$$

where q is a g-function, $g \in \mathcal{G}$, with $q(0, \nu) = q(\rho, 0) = 0$, and cell-based spatial discretization of (1) yields

$$\Delta x \dot{\rho}_i(t) = q(\rho_{i-1}(t), \rho_i(t)) - q(\rho_i(t), \rho_{i+1}(t)), \quad i \in \mathcal{I}, \quad (3)$$

where $q : [0, P]^2 \rightarrow \mathbb{R}_{\geq 0}$ describes the traffic flow over the boundary between two cells, and ρ_0 and ρ_{I+1} are boundary conditions. For notational simplicity, we omit writing the dependence on time in further text wherever unambiguous.

Definition 1. A **g-function** is a function $g : \mathbb{R}_{\geq 0}^n \rightarrow [0, 1]$ that is non-decreasing and locally Lipschitz-continuous. We denote the set of such functions $\mathcal{G}$.

In its most basic form, $q(\rho_-, \rho_+)$ can be written

$$q(\rho_-, \rho_+) = \hat{q}T(g_d(\rho_-), g_s(\nu_+)),$$

where $\hat{q}$ is the road capacity, $g_d, g_s \in \mathcal{G}$, $g_d(0) = g_s(0) = 0$, and T is some t-norm, as defined below.

Definition 2. Function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a **t-norm** if for $a, b, c, d \in [0, 1]$, we have $T(a, b) = T(b, a)$, $T(a, 1) = a$, $T(a, T(b, c)) = T(T(a, b), c)$, and if $a \leq c$ and $b \leq d$, $T(a, b) \leq T(c, d)$.

Lemma 1. If $g_1, g_2 \in \mathcal{G}$, and $g_3(x) = T(g_1(x), g_2(x))$, where T is a t-norm, then we have $g_3 \in \mathcal{G}$. Additionally, if for some x , $0 \leq x \leq 1$ componentwise, $g_1(x) = 0$ or $g_2(x) = 0$, then also $g_3(x) = 0$.

We may consider $g_d(\rho_-)$ to effectively be representing the *demand* of the upstream, and $g_s(\nu_+)$ the *supply* of the downstream cell. Let

$$g_d(\rho) = \min\left\{\frac{\rho}{\hat{\rho}}, 1\right\}, \quad g_s(\nu) = \min\left\{\frac{\nu}{\hat{\nu}}, 1\right\}, \quad (4)$$

with $\hat{\rho} > 0$ and $\hat{\nu} > 0$. Choosing T to be the minimum t-norm, $T(a, b) = T_{\min}(a, b) = \min\{a, b\}$, and setting $\hat{\rho} = \rho_{\text{cr}} < P$, $\hat{\nu} = P - \rho_{\text{cr}}$, and $\hat{q} = V\rho_{\text{cr}} = W(P - \rho_{\text{cr}})$ yields the Newell-Daganzo (triangular) flux function,

$$Q(\rho) = q(\rho, \rho) = \min\{V\rho, W(P - \rho)\},$$

with free flow speed V and congestion wave speed $W = V \frac{\rho_{\text{cr}}}{P - \rho_{\text{cr}}}$, replicating CTM dynamics [9]. Similarly, if we choose the product t-norm, $T_{\text{prod}}(a, b) = ab$, setting $\hat{\rho} = \hat{\nu} = \frac{P}{2}$, $\hat{q} = V \frac{P}{2}$, yields the Greenshields flux function $Q(\rho) = V\rho(1 - \frac{\rho}{P})$, and letting $\hat{\rho} = \hat{\nu} = P$ recovers the mass action kinetic flux decomposition from [8].

We adopt a state transformation to facilitate control design,

$$N_i = \Delta x \sum_{j=1}^i \rho_j, \quad \rho_i = \frac{N_i - N_{i-1}}{\Delta x},$$

which can be seen as the discretization of the Hamilton-Jacobi PDE (2). Writing $N = [N_1 \ \dots \ N_I]^\top$ jointly, the cumulated number of vehicles $N \in \mathcal{N}$, evolves according to

$$\dot{N} = r - q = F(N, u). \quad (5)$$

Here r_i is the cumulative difference between all inflows entering, and outflows leaving the road,

$$r_i = q_0 + \sum_{j=1}^i (r_j^{\text{on}} - r_j^{\text{off}}),$$

where q_0 is the traffic flow entering cell 1 at the upstream boundary of the road (equivalent to $Q(\rho_0(t))$ in (2)), and r_i^{on} and r_i^{off} are the on-ramp inflow to, and off-ramp outflow from cell i , respectively, both assumed to be zero in this work. The VSL-controlled traffic flows $q = [q_1 \ \dots \ q_I]^\top$ are specified in further text. The set of all possible N is defined by

$$\mathcal{N} = \{N \in \mathbb{R}_{\geq 0}^I \mid (\forall i \in \mathcal{I} \setminus \{0, I\}), \quad 0 \leq N_{i+1} - N_i \leq P\Delta x\}.$$

B. Model modifications

More complex traffic dynamics may be represented by adopting a different form of $q(\rho_-, \rho_+)$. Allowing the parameters of q to vary across cells enables straightforward representation of stationary bottlenecks such as lane drops. The capacity drop phenomenon can be represented by

$$q(\rho_-, \rho_+) = \hat{q}T_q(g_D(\rho_-, \nu_-), g_s(\nu_+)),$$

$$g_D(\rho_-, \nu_-) = T_D(g_d(\rho_-), g_{\hat{q}}(\nu_-)),$$

where g_d and g_s are the same as before, $g_{\hat{q}} \in \mathcal{G}$, and T_q and T_D are potentially different t-norms. For example, adopting

$$g_{\hat{q}}(\nu) = \min\left\{1 - \alpha + \alpha \frac{\nu}{P - \rho_{\text{cr}}}, 1\right\},$$

where α is the maximum (at $\nu_- = 0$, $\rho_- = P$) capacity drop ratio, effectively introduces a linear reduction of capacity as ν decreases.

One particularly pertinent choice of exogenous control input is Variable Speed Limits (VSL) signals, whose influence on the traffic flow may be represented by adopting

$$g_{\hat{q}}(u) = \min\{u, 1\}.$$

Here the VSL control input u_i represents the relative speed limit in cell i expressed as a portion of the free flow speed V , and we have $u_i \in [\underline{u}, 1]$, $0 \leq \underline{u} \leq 1$, resulting in $g_{\hat{q}}(u_i) = u_i$. The flow from cell i to cell $i + 1$ can thus be written

$$q_i = \hat{q}T_q(T_D(g_d(\rho_i), u_i), g_s(P - \rho_{i+1})), \quad i \in \mathcal{I}, \quad (6)$$

for some t-norms T_q and T_D . Due to the monotonicity property of t-norms, we easily get upper-bounds for q_i ,

$$q_i \leq T_D(g_d(\rho_i), u_i) \leq g_d(\rho_i), \quad i \in \mathcal{I}.$$

C. Choice of t-norms

While minimum and product t-norms are by far the most common choice for T_q , other convenient options, such as the Łukasiewicz t-norm

$$T_L(a, b) = \max\{0, a + b - 1\},$$

may be chosen to simplify the model. Using this t-norm in particular, with the simple choice of g_d and g_s (4), yields

$$q_L(\rho_-, \rho_+) = \hat{q} \max\left\{0, \min\left\{\frac{\rho_-}{\hat{\rho}}, 1\right\} + \min\left\{\frac{\nu_+}{\hat{\nu}}, 1\right\} - 1\right\}.$$

Setting $\hat{v} = P$, and owing to the fact that we always have $\rho_+ \leq P$, this expression may be approximated by

$$\bar{q}(\rho_-, \rho_+) = \hat{q} \left(\frac{\rho_-}{\max\{\hat{\rho}, \rho_-\}} - \frac{\rho_+}{P} \right), \quad (7)$$

which is purely linear for $\rho_- \leq \hat{\rho}$. Additionally, while (7) may result in negative flows, it preserves $\rho_i(t) \in [0, P]$ for $t > 0$ if $\rho_i(0) \in [0, P]$, since using (7) in place of $q(\rho_-, \rho_+)$ in (3) with $\rho_{i\pm 1} \in [0, P]$ yields

$$\begin{aligned} \dot{\rho}_i|_{\rho_i=0} &= \frac{\hat{q}}{\Delta x} \left(\min \left\{ \frac{\rho_{i-1}}{\hat{\rho}}, 1 \right\} + \frac{\rho_{i+1}}{P} \right) \geq 0, \\ \dot{\rho}_i|_{\rho_i=P} &= \frac{\hat{q}}{\Delta x} \left(\min \left\{ \frac{\rho_{i-1}}{\hat{\rho}}, 1 \right\} - 2 + \frac{\rho_{i+1}}{P} \right) \leq 0. \end{aligned}$$

Furthermore, we assume that t-norm T_q that captures the actual traffic dynamics lies between the Łukasiewicz and the minimum t-norm,

$$a + b - 1 \leq T_L(a, b) \leq T_q(a, b) \leq T_{\min}(a, b), \quad (8)$$

allowing us to define bounds on traffic flow $q_i = q(\rho_i, \rho_{i+1})$,

$$\bar{q}(\rho_-, \rho_+) \leq q_L(\rho_-, \rho_+) \leq q(\rho_-, \rho_+) \leq q_{\min}(\rho_-, \rho_+).$$

While incomplete compliance with VSL controls is an important practical issue, in this work we restrict ourselves to the assumption that all drivers perfectly respect the speed limits. Setting cell i VSL control input u_i to some value in $[u, 1]$ effectively prohibits the speed of vehicles in that cell to exceed $u_i V$, upper-bounding the flow from this cell by $q_i \leq u_i V \rho_i$. This corresponds to using the product t-norm,

$$T_D(a, b) = ab. \quad (9)$$

Larger t-norms, $T_{\text{prod}}(a, b) \leq T_D(a, b)$, and similar functions, may be used to capture the effect of incomplete compliance. At the other extreme, if all drivers ignore VSL inputs u_i trivially have no influence on the traffic flow.

Finally, using the t-norm bounds (8) and (9), we formulate a lower and upper bound on cell i traffic flow q_i ,

$$q_i \geq \hat{q} \left(\min \left\{ \frac{\rho_i}{\hat{\rho}}, 1 \right\} u_i - \frac{\rho_{i+1}}{P} \right), \quad i \in \mathcal{I}, \quad (10)$$

$$q_i \leq \hat{q} \min \left\{ \frac{\rho_i}{\hat{\rho}}, 1 \right\} u_i, \quad i \in \mathcal{I}. \quad (11)$$

III. CONTROL

A cost function reflecting the total traffic density homogeneity can be written as

$$J = \sum_{i=1}^I J_i = \sum_{i=1}^I \left| N_i - \frac{i}{I} N_I \right|, \quad (12)$$

equivalent to 1-Wasserstein distance between the traffic density profile ρ_i , $i \in \mathcal{I}$ and a perfectly homogeneous density,

$$J = \Delta x \sum_{i=1}^I \left| \sum_{j=1}^i \rho_j - \frac{i}{I} \sum_{j=1}^I \rho_j \right|.$$

A. Bound systems

While the exact expressions for q_i may be highly nonlinear and potentially unknown, the expression of its lower and upper bounds (10)–(11) are significantly simpler. We define $\bar{N} \in \mathcal{N}$, evolving according to

$$\dot{\bar{N}} = \mathbf{1}_{q_0} - \bar{q} = \bar{F}(\bar{N}, \bar{u}), \quad (13)$$

with $\bar{q} = [\bar{q}_1 \ \dots \ \bar{q}_I]^\top$, $\bar{F}_i = q_0 - \bar{q}_i$, and

$$\bar{q}_i = \hat{q} \left(\min \left\{ \frac{\bar{N}_i - \bar{N}_{i-1}}{\Delta x \hat{\rho}}, 1 \right\} \bar{u}_i - \frac{\bar{N}_{i+1} - \bar{N}_i}{\Delta x P} \right), \quad i \in \mathcal{I}, \quad (14)$$

and $\underline{N} \in \mathcal{N}$, evolving according to

$$\dot{\underline{N}} = \mathbf{1}_{q_0} - \underline{q} = \underline{F}(\underline{N}, \underline{u}), \quad (15)$$

$$\underline{q}_i = \hat{q} \min \left\{ \frac{\underline{N}_i - \underline{N}_{i-1}}{\Delta x \hat{\rho}}, 1 \right\} \underline{u}_i, \quad i \in \mathcal{I}. \quad (16)$$

The dynamics of both bound systems may be written jointly,

$$\dot{N} = \mathbf{1}_{q_0} - \Gamma(N)u + s(N) = F(N, u), \quad i \in \mathcal{I}, \quad (17)$$

where $\Gamma(N)$ is a diagonal matrix of $\gamma_i(N) = \hat{q} \min \left\{ \frac{N_i - N_{i-1}}{\Delta x \hat{\rho}}, 1 \right\} \geq 0$, and $s(N)$ is a column vector of $s_i(N) = \hat{q} \frac{N_{i+1} - N_i}{\Delta x P}$ if we study the upper-bound system (13), (14), or $s(N) = 0$ for the lower-bound system (15), (16). This structural similarity allows us to jointly analyze the simplified dynamics (17) and use $\underline{N}$ and $\bar{N}$ as bounds on the cumulated number of vehicles N .

Theorem 1. Let the control law for system (5), (6) and both bound systems, (13), (14) and (15), (16), be given by $U : \mathbb{R}_{\geq 0}^I \rightarrow [u, 1]^I$, yielding $u_i = U_i(N)$, and $\bar{u}_i = U_i(\bar{N})$, $\underline{u}_i = U_i(\underline{N})$, respectively. Let $U(N)$ be off-diagonal nonincreasing, i.e., $\forall i \in \mathcal{I}$ control law $U_i(N)$ is monotone nonincreasing in all components of N except possibly N_i (for $N, \hat{N} \in \mathcal{N}, N \leq \hat{N}, N_i = \hat{N}_i$, we have $U_i(N) \geq U_i(\hat{N})$). Then systems (5), (6), (13), (14), and (15), (16), are all cooperative and monotone, and if at time t_0 we have $\underline{N}(t_0) = \bar{N}(t_0) = N(t_0)$, then

$$\underline{N}(t) \leq N(t) \leq \bar{N}(t), \quad \forall t \geq t_0.$$

Proof. Since U is off-diagonal nonincreasing, and the control is multiplied by a nonpositive $-\hat{q}g_d(\rho_i)$ in $\bar{F}$ and $\underline{F}$, it is straightforward to verify that $\bar{F}(\bar{N}, U(\bar{N}))$ and $\underline{F}(\underline{N}, U(\underline{N}))$ satisfy the Kamke-Müller conditions, i.e., $\forall i, \forall N \in \mathcal{N}$ and $\hat{N} \in \mathcal{N}$ such that $N \leq \hat{N}$, $N_i = \hat{N}_i$, we have $\bar{F}_i(N, U(N)) \leq \bar{F}_i(\hat{N}, U(\hat{N}))$ and $\underline{F}_i(N, U(N)) \leq \underline{F}_i(\hat{N}, U(\hat{N}))$. Similarly, due to the monotonicity property of t-norms, and the non-decreasing nature of g-functions, F also satisfies the Kamke-Müller conditions. Therefore, (13), (15), and (5) are all cooperative and monotone. Finally, due to (10) and (11), we have $F(N) \leq \bar{F}(N)$ and $\underline{F}(N) \leq F(N)$, respectively, and both (13) and (5) generate order-preserving flows [6]. $\square$

Corollary 1.1. Consider two instances of system (5), (6), $\tilde{N}$ and $\hat{N}$, distinguished only by the choice of t-norm T_q , which is taken to be $\tilde{T}$ and $\hat{T}$, respectively. Let both systems be controlled by the same control law U , satisfying the conditions from Theorem 1, and let

$$\tilde{T}(a, b) \geq \hat{T}(a, b), \quad \forall a, b \in [0, 1].$$

Then if at time t_0 we have $\tilde{N}(t_0) = \hat{N}(t_0)$,

$$\tilde{N}(t) \leq \hat{N}(t), \quad \forall t \geq t_0.$$

B. Proportional (myopic) control

One simple feedback control law designed to greedily improve the traffic density homogeneity (12) is

$$U_i^p(N) = \max \left\{ u, \min \left\{ 1, 1 + k \left(N_i - \frac{i}{I} N_I \right) \right\} \right\}. \quad (18)$$

which if instead we consider $1 - U_i^p(N)$ to be the control,

$$1 - U_i^p(N) = \max \left\{ 0, \min \left\{ 1 - \underline{u}, k \left(\frac{i}{I} N_I - N_i \right) \right\} \right\}$$

is indeed simple saturated proportional feedback control, with gain k and reference $\frac{i}{I} N_I$. Since (18) is off-diagonal nonincreasing, it satisfies the conditions of Theorem 1, and we may calculate bounds on the evolution of N , $\underline{N}(t) \leq N(t) \leq \bar{N}(t)$, using two bound systems (17) which are easier to analyze and simulate.

C. Minimum-Time Optimal Control

We derive necessary optimality conditions for the minimum-time control of the two bound systems (17) using Pontryagin Maximum Principle (PMPs). Define the target set $\mathcal{N}^* \subset \mathcal{N}$ to be some neighborhood of $N_i = \frac{i}{I} N_I$, which would result in $J = 0$. The minimum-time problem is to determine $U(\cdot)$ such that

$$\begin{aligned} \min_{U(\cdot)} \quad & t_{\text{end}} \\ \text{s.t.} \quad & \dot{N}(t) = F(N(t), U(N(t))), \\ & N(0) = N^0, \quad N(t_{\text{end}}) \in \mathcal{N}^*, \\ & U(N) \in [\underline{u}, 1]^I. \end{aligned}$$

The Hamiltonian $\mathcal{H}$ can be written

$$\mathcal{H}(N, \lambda, U) = 1 + \lambda^\top F(N, U).$$

Theorem 2. Let $U^*(\cdot)$ be an optimal control and $N^*(\cdot)$ the corresponding trajectory. Then there exists an absolutely continuous $\lambda(\cdot) \neq 0$ such that, for almost all t ,

$$\dot{N}^* = F(N^*, U^*),$$

$$\dot{\lambda} = -\partial_N \mathcal{H}(N^*, \lambda, U^*),$$

$$U^*(t) \in \arg \min_{U \in [\underline{u}, 1]^I} \mathcal{H}(N^*, \lambda, U),$$

and $\mathcal{H}(\bar{N}^*(t), \lambda(t), U^*(t)) = 0$.

Moreover, U^* is bang-bang and satisfies, componentwise,

$$U_i^*(t) = \begin{cases} 1, & \lambda_i(t) > 0, \\ \underline{u}, & \lambda_i(t) < 0, \end{cases} \quad (19)$$

Proof. The result follows from the Pontryagin Maximum Principle for minimum-time problems. From (17), the Hamiltonian can be written as

$$\mathcal{H} = 1 + \sum_{i=1}^I \lambda_i q_0 - \hat{q} \sum_{i=1}^I \lambda_i (\gamma_i(N) U_i - s_i(N)),$$

affine in each U_i . Since $U_i \in [\underline{u}, 1]$, minimizers lie at the boundary. Moreover, we have $\partial_{U_i} \mathcal{H} = -\hat{q} \lambda_i \gamma_i(N)$, and since $\hat{q} \gamma_i(N) \geq 0$, the minimum time control is $U_i^* = 1$ if $\partial_{U_i} \mathcal{H} < 0$, and $U_i^* = \underline{u}$ if $\partial_{U_i} \mathcal{H} > 0$, yielding (19). The remaining conditions follow directly from PMP. $\square$

Remark 1. The switching function is $\sigma_i(N) = \lambda_i(N)$, thus along any optimal trajectory, there are no singular arcs, i.e., there does not exist a nontrivial interval on which $\sigma_i(t) \equiv 0$ for some $i \in I$ unless $\lambda_i(t) \equiv 0$. Finally, since $\mathcal{H}$ is affine in U , it is convex. Hence, any admissible pair (N^*, U^*) satisfying the PMP conditions and the pointwise minimization condition is optimal.

Remark 2. The feedback laws satisfying the conditions in Theorem 1 may be interpreted as bang-bang *approximations* (state feedback) of the minimum-time optimal control

characterized in Theorem 2. Consider the proportional state feedback controller $U^p(N)$ defined in (18). For sufficiently large gain k , the control law $U_i^p(N)$ approaches a saturated bang-bang structure, i.e., $U_i^p(N) \approx [\underline{u}, 1]$ for most $N \in \mathcal{N}$. Letting $q_I = q_0$, for both the bound system, we have

$$\dot{J}_i = \begin{cases} q_0 + s_i(N) - \gamma_i(N), & N_i > \frac{i}{I} N_I, \\ -q_0 - s_i(N) + \gamma_i(N) \underline{u}, & N_i < \frac{i}{I} N_I, \end{cases}$$

with $u_i = U_i^p(N)$ effectively myopically maximizing the decrease in J_i for all system dynamics in the specified range.

Moreover, the switching condition in $U_i^p(N)$ is determined by the sign of the deviation $N_i - \frac{i}{I} N_I$, while the optimal controller $U_i^*(t)$ in Theorem 2 switches according to the sign of the costate $\lambda_i(t)$. However, due to the monotonicity and cooperative structure of the system (cf. Theorem 1), $\text{sign}(\lambda_i(t)) = \text{sign}(\frac{i}{I} N_I(t) - N_i(t))$, $\forall i \in I$, along the optimal trajectories. Consequently, the switching structure of the proportional controller $U^p(N)$ is qualitatively aligned with that of the optimal control $U^*(t)$, in the sense that both apply the same control action (up to saturation) whenever the above signs coincide.

IV. SIMULATION RESULTS

We simulated a $I = 100$ cell stretch of road. For simplicity, we normalize the space x and time t so that we have the normalized cell length $\Delta x = 1$, free-flow speed $V = 2$, congestion wave speed $W = 1$, and discretization time step of $\Delta t = \frac{\Delta x}{V+W+0.1}$. We normalize the traffic density ρ that the jam density is $P = 1$, and set $\hat{\rho} = \frac{1}{3}$ and $\hat{v} = 1$ in (4). Finally, we assume the minimum VSL control input is $\underline{u} = 0.5$, and that $u_1 = u_I = 1$. We run the simulation for 150 time steps, setting the inflow at the upstream road boundary to $q_0 = 0.3$. The initial density was set to

$$\rho_i(0) = \rho_j^0, \quad \underline{i}_j < i \leq \bar{i}_j, \quad j = 1, \dots, 8,$$

with parameters ρ_j^0 and $\underline{i}_j$ given in Table I, $\bar{i}_j = \underline{i}_{j+1}$, $j = 1, \dots, 7$, and $\bar{i}_8 = 100$, yielding the initial total (normalized) number of vehicles $\Delta x \sum_i \rho_i(0) = N_I(0) = 30$.

We run two sets of simulations, first comparing the dynamics and illustrating the trajectory ordering of two selected traffic model instances, the lower-bound, and the upper-bound system, in Fig. 1–3. We then demonstrate the effectiveness of the two designed controller, in Fig. 4–6.

A. Model comparison

We first compare four traffic model instances:

- lower-bound model (11), assuming free flow conditions,
- CTM-like model using the minimum t-norm,
$$q_i^{\min} = \hat{q} \min \{ g_d(\rho_i^{\min}) u_i^{\min}, 1 - \rho_{i+1}^{\min} \}, \quad (20)$$
- TRM-inspired model based on the product t-norm, similar to using the Greenshields flux function,
$$q_i^{\text{prod}} = \hat{q} g_d(\rho_i^{\text{prod}}) u_i^{\text{prod}} (1 - \rho_{i+1}^{\text{prod}}),$$
- upper-bound model (10), based on approximating the Łukasiewicz t-norm.

j	1	2	3	4	5	6	7	8
ρ_j^0	0.15	0	0.15	1	0	0.3	0.8	0.15
$\underline{i}_j$	0	10	20	30	40	50	70	80

TABLE I: Initial density parameters.

All four models use the same g-functions (4), and since $\bar{\nu} = P$, $g_s(P - \rho) = \frac{P - \rho}{P} = 1 - \rho$. The first two models align with the general form of q_i (6). We first perform the comparison in the uncontrolled case, setting $u_i = 1, \forall i \in \mathcal{I}$, and then demonstrate that the trajectory ordering is preserved under control (18).

The evolution of traffic density ρ_i is shown color-coded (with warmer colors indicating higher density) in Fig. 1. As expected, the density profiles of all but the lower-bound system (in Fig. 1a, exhibiting no rarefaction) are qualitatively similar, with the least diffusion present in $\rho^{\min}$, the most in $\bar{\rho}$, and ρ^{prod} between them.

Next, we demonstrate that the ordering of cumulative vehicle numbers of different models is preserved, $\underline{N} \leq N^{\min} \leq N^{\text{prod}} \leq \bar{N}$, as provided by Theorem 1, showing the differences between different pairs in Fig. 2. It can be seen that for $t = 0$, the differences for all i start at zero, and remain nonnegative throughout the simulation run. Notably, while the maximum difference between $\bar{N}$ and $\underline{N}$ grows over time, reaching $\|\bar{N} - \underline{N}\|_{\infty} = 10.7962$ at the end of the simulation, the difference between $\bar{N}$ and N^{prod} remains rather small ($\|\bar{N} - N^{\text{prod}}\|_{\infty} = 1.8948$, or around 6.3% of the initial total number of vehicles), in spite of the lower-bound system being significantly simpler to analyze and simulate. As shown in Fig. 3, the ordering is preserved even when a control law satisfying the conditions of Theorem 1, (18), is used, with an even smaller maximum gap between the two bound systems $\bar{N}$ and $\underline{N}$, and a notably small difference between $N^{\min}$ and $\underline{N}$ (less than 10% of $N_I(0)$).

B. Density homogenization VSL control

Finally, we demonstrate the effectiveness of the designed controllers in improving the traffic density homogeneity (reducing (12)). We compare three scenarios:

- no control, $u = 1$,
- proportional control (18), $u = U^P(N)$,
- minimum-time control (19), $u = U^*(N)$.

For this comparison, we assume the traffic dynamics are given by the CTM-like model (20), but that the exact model is unknown. While the proportional controller assumes no model knowledge, for the minimum-time control we use the upper-bound system as the best guess of the actual model.

The minimum-time controller was implemented employing receding horizon forward-backward sweeping, with horizon length of $t_{\text{hor}} = 15$ time steps and a maximum of $l_{\text{max}} = 20$ iterations. The control signal was initialized at $u^{(0)} = 1$, and the terminal value for λ was set to

$$\lambda_i^{(l)}(t + t_{\text{hor}}) = N_i^{(l)}(t + t_{\text{hor}}) - \frac{i}{I} N_I^{(l)}(t + t_{\text{hor}}).$$

The control signal was iterated with relaxation according to

$$u_i^{(l+1)}(t) = \begin{cases} (1 - \alpha)u_i^{(l)}(t) + \alpha, & \lambda_i^{(l)}(t) > 0, \\ (1 - \alpha)u_i^{(l)}(t) + \alpha\underline{u}, & \lambda_i^{(l)}(t) < 0, \end{cases}$$

with $\alpha = 0.2$, and the first time step of the final iteration is applied in a receding horizon manner. Note that due to the numerical errors, approximate model, and imperfect initialization of λ , the actual applied control law $U^*(N)$ is neither optimal, nor bang-bang, but achieves a good reduction of cost function (12) nonetheless.

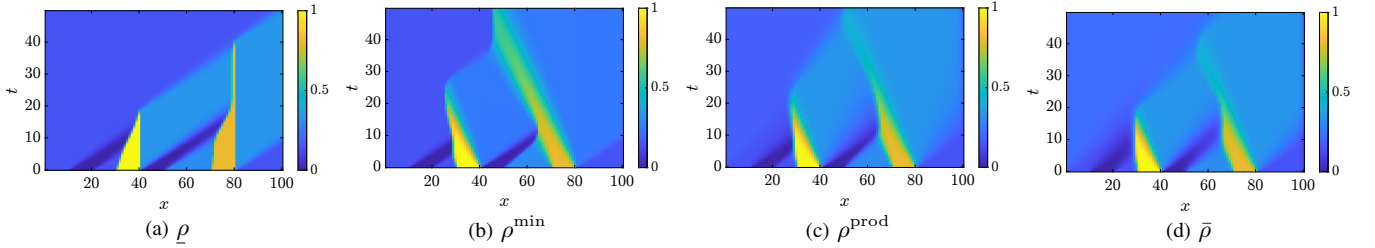


Fig. 1: Traffic density ρ evolution under the four compared models without control ($u = 1$).

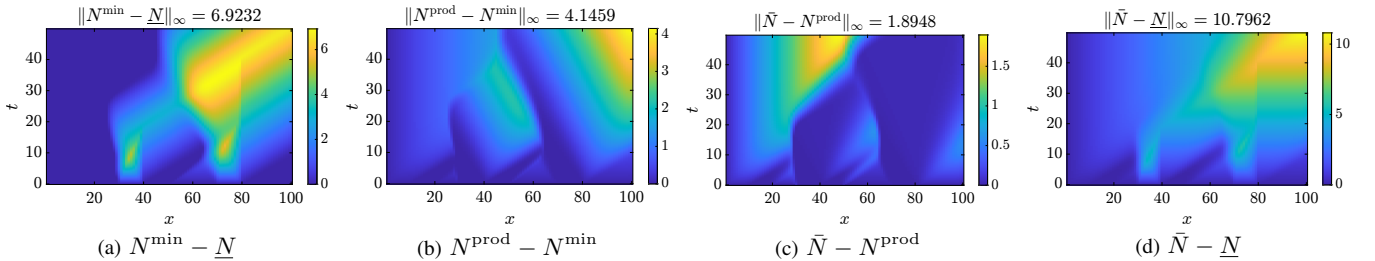


Fig. 2: Differences in cumulated vehicle numbers between the four compared models without control ($u = 1$).

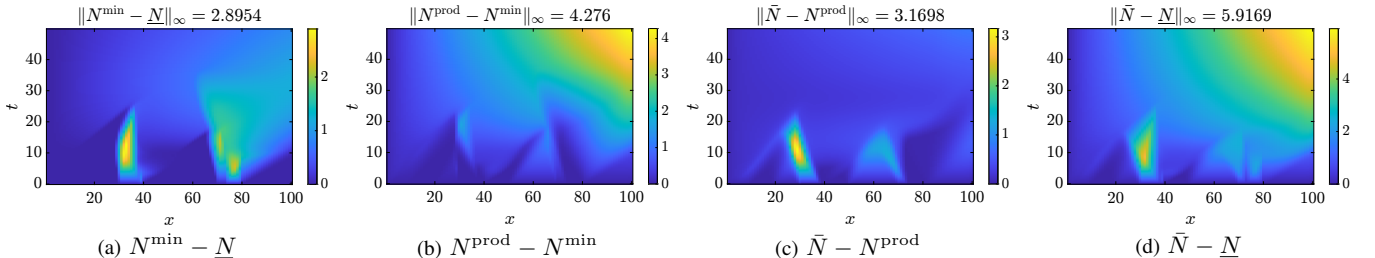


Fig. 3: Differences in cumulated vehicle numbers between the four compared models with control (18).

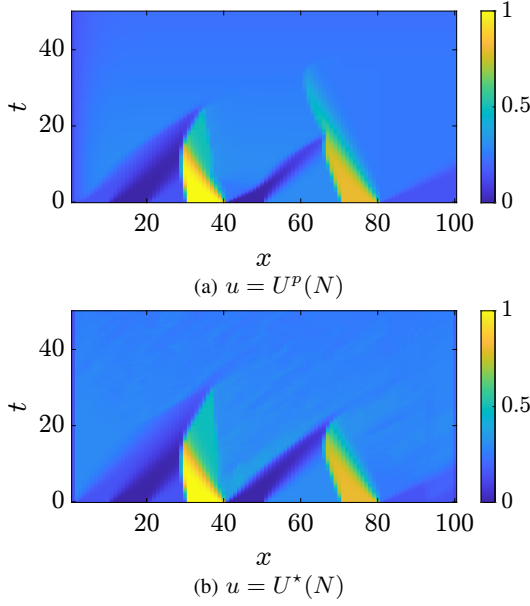


Fig. 4: Traffic density ρ under the two controlled scenarios.

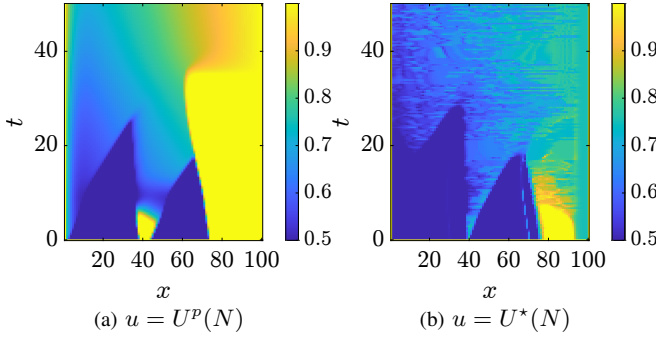


Fig. 5: VSL control signal for the two controlled scenarios.

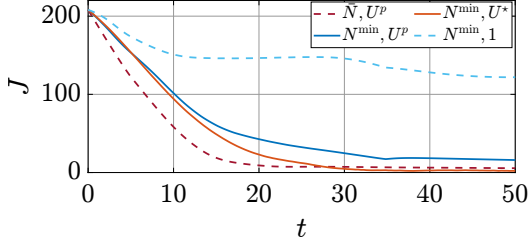


Fig. 6: Traffic density homogeneity cost function J over time.

Traffic density profiles of the controlled scenarios are shown in Fig. 4 (the uncontrolled one is in Fig. 1b), and the VSL control signals are in Fig. 5. Both controllers accelerate the dissipation of congestion and density homogenization, at the cost of slowing down the overall traffic. This is confirmed in Fig. 6, presenting the value of J over time, where U^* achieves the faster decrease in spite of its noisy control signal. Also shown in the figure are the evolution of the cost function in the no control scenario (as an indicative upper bound), and for an upper-bound system controlled by U^P (as an indicative lower bound for the initial time period).

V. CONCLUSION

We propose a general traffic modeling approach based on t-norms, capable of capturing the most commonly used space-discretized kinetic traffic flow models. Considering cumulative vehicle numbers instead of densities, we are able

to leverage monotone systems theory to use simplified lower- and upper-bound systems to derive bounds for the trajectories of the original system with unknown dynamics. These bound systems are used to design an optimal and an approximate optimal VSL controller that balances traffic density inhomogeneity and help dissipate congestion efficiently.

In the future, we plan to refine the designed VSL controller to capture spatio-temporal constraints and other performance aspects. The real-world issue of limited driver compliance with VSL controls is to be addressed through extending the modeling framework to multi-class (compliant and non-compliant) traffic.

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