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A Computer Formalisation of the Serre Finiteness Theorem

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Abstract

Few constructions in mathematics are as elusive as the homotopy groups of spheres. These groups, which intuitively measure n-dimensional loops on m-dimensional spheres, appear at first glance to be almost completely random – an unfortunate fact, seeing as they constitute one of the fundamental building blocks of algebraic topology and homotopy theory. However, the situation is not completely hopeless: in 1951, Serre proved his celebrated finiteness theorem, which says that these groups are almost always finite abelian groups, except in two classes of special cases when they also contain copies of the integers. In a recent paper, Barton and Campion proved a variation of this result in homotopy type theory (HoTT) – an extension of Martin-Löf type theory, particularly suitable for reasoning about and formalising algebraic topology and homotopy theory. Their result shows that the homotopy groups of spheres are all finitely presented – and constructively so. Prior to this proof, only low-dimensional homotopy groups of spheres had been computed in HoTT. This made it a major breakthrough for HoTT as a foundation and, as such, the immediate target of a full-scale formalisation project. In this paper, we present the outcome of this project: a complete formalisation of Barton and Campion's proof of the Serre finiteness theorem in Cubical Agda, a constructive proof assistant implementing a cubical flavour of HoTT. In the light of the constructivity of Cubical Agda, we discuss the prospect of running the algorithm provided by our formalisation in order to compute concrete homotopy groups of spheres.

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1 Introduction

Homotopy theory originated in algebraic topology, but is by now a fundamental tool in many branches of modern mathematics. A central notion are the *homotopy groups* of a space X, denoted $\pi_n(X)$. These groups constitute a homotopical invariant, making them a powerful tool for establishing whether two given spaces can, or cannot, be homotopy equivalent. The first two such groups of a space are easily understood: $\pi_0(X)$ characterises the connected components of X and $\pi_1(X)$ is the fundamental group, i.e. the group of equivalence classes of loops in X up to homotopy. This generalises to higher values of n, for which $\pi_n(X)$ consists of n-dimensional loops up to homotopy. For many spaces, these groups are increasingly esoteric and difficult to compute for large n. This is true also for seemingly tame spaces like spheres, for which $\pi_n(\mathbb{S}^m)$ in general is highly irregular when $n > m \geq 2$ (see [5, Figure 2.1] for a table of $\pi_n(\mathbb{S}^m)$ for small n and m).

In the early 1950s Jean-Pierre Serre proved a remarkable result: the homotopy groups of spheres, $\pi_n(S^m)$, are all finite, except for the main diagonal $n = m$ and when m is even, and $n = 2m - 1$, in which case they have a copy of $\mathbb{Z}$. This result was published in the Annals of Mathematics [31] and Serre was awarded the Fields medal a few years later. In 1957, Brown published a paper proving that these homotopy groups are in fact algorithmically computable [4], however his algorithm is so non-trivial that it cannot be used for practical computations. As these groups constitute one of the cornerstones of algebraic topology, with applications in many other areas of mathematics, computing them has been a major research programme in homotopy theory.

Homotopy type theory (HoTT) is a new foundation of mathematics based on type theory extended with Voevodsky's univalence axiom [37] and higher inductive types (HITs) [27]. This allows for a new *synthetic* approach to homotopy theory where spaces are represented using HITs while homotopical invariants, such as homotopy groups, are computed using univalence. By working synthetically, results about spaces can often be proved in a direct manner without ever having to talk about their topology or how they are encoded as subspaces of $\mathbb{R}^n$. This makes the approach especially amenable to computer formalisation using proof assistants with support for HoTT. Because of the central role that homotopy groups of spheres play in modern mathematics, formalising synthetic computations of these groups has been a major programme in HoTT, with landmark results including:

- $\pi_1(S^1) \cong \mathbb{Z}$ by Licata & Shulman [23],
- $\pi_2(S^2) \cong \pi_3(S^2) \cong \mathbb{Z}$ from the HoTT Book [35],
- $\pi_n(S^n) \cong \mathbb{Z}$ from the HoTT Book [35], alternatively by Licata & Brunerie [21],
- $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$, with corollary $\pi_4(S^2) \cong \pi_{4+n}(S^{3+n}) \cong \mathbb{Z}/2\mathbb{Z}$, by Brunerie [5], and
- $\pi_5(S^3) \cong \mathbb{Z}/2\mathbb{Z}$, with corollary $\pi_5(S^2) \cong \pi_{6+n}(S^{4+n}) \cong \mathbb{Z}/2\mathbb{Z}$, by Jack & Ljungström [18].

These computations capture many low-dimensional cases and the first few stable stems of homotopy groups of spheres. They have all been formally verified in proof assistants extended with support for HoTT, such as `Agda` [33], `Lean` 2 [11], and `Rocq` [34].

In 2022 Barton and Campion announced a proof of Serre's finiteness theorem in HoTT [3]. As opposed to the concrete computations discussed above, this provided the first synthetic result about *all* homotopy groups of *all* spheres. In particular, they showed that the homotopy groups of spheres are finitely presented, i.e., that they can (merely) be written on the form $\mathbb{Z}^k \oplus \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_l\mathbb{Z}$ for some numbers $k, p_1, \ldots, p_l$. In addition to being a landmark result in synthetic homotopy theory, their proof was remarkable for three major reasons:

1. it was fully constructive, hence subsuming Brown's 1957 result,
2. it used only homotopy invariant methods, making it formalisable in HoTT, and
3. it was quite elementary and did not rely on the advanced methods that Serre developed for his proof; in particular, no spectral sequences were involved in it.

Formalisation. This paper presents a formalisation of Barton and Campion's proof in `Cubical Agda` [36] – a HoTT proof assistant with computational support for univalence and HITs. A summary file linking the formalisation and paper can be found at `https://github.com/CMU-HoTT/serre-finiteness/blob/main/Summary.agda`. The development type checks with `Agda`'s `--safe` flag, which ensures that there are no admitted goals or postulates. In the process of the formalisation various simplifications to the proof were found. We discuss these in the paper, but the most surprising and significant such simplification was the realisation that homology could be completely avoided in the proof.

Outline. The paper is structured as follows: we begin, in Section 2, with the formal theorem statement in `Cubical Agda` and an overview of definitions and results from synthetic homotopy theory needed for the proof. We then, in Section 3, provide an outline of the proof, following Barton and Campion's proof [3], but focusing on being pedagogical rather than being complete. We then discuss the formalisation in Section 4 and the prospects of computing homotopy groups of spheres with it in Section 5.

2 The formal theorem statement and basics of HoTT

The goal of this section is to explain the statement of the main theorem and its formalisation in `Cubical Agda`. We also provide an overview of elementary HoTT definitions and results that are prerequisites for the proof of the Serre finiteness theorem. We will assume familiarity with dependent type theory, including standard type formers (Π-types, written `Agda` style $(x : A) \to B$, and Σ-types) and inductive types (empty type $\bot$, unit type $\mathbb{1}$, booleans Bool, natural number type $\mathbb{N}$).

As the paper concerns a formalisation in `Cubical Agda`, we will present all mathematics using `Agda` syntax. For the sake of readability, we do, however, take significant typesetting liberties, and write e.g. "$\pi_n\ X$" and "$\mathbb{S}^n$" rather than the actual `CubicalAgda` terms "$\pi\ n\ X$" and "$\mathbb{S}\ n$". For the same reason, our informal presentation in this paper does not always respect the indexing (regarding connectedness, h-levels, etc.) in our formalisation which usually start from 0. Instead, we follow the conventions of the HoTT book [35] where many things are indexed from -2.

2.1 The formal theorem statement

Theorems in `Agda` are introduced by first declaring their type and then its proof/construction. The Serre finiteness theorem (or SFT, as it is called here) and its proof looks as follows:

```
SFT : (n m : ℕ) → isFP (πₙ 𝕊ᵐ)
SFT n m = ThmA×ThmB→isFPπ𝕊 (ThmA n m) (ThmB n m)
```

The first line declares the type/statement which translates to:

> *For every n and m of type $\mathbb{N}$, the predicate* isFP *holds for π_n of $\mathbb{S}^m$*

or, in English:

> *For all natural numbers n and m, the nth homotopy group of the m-sphere is finitely presentable.*

Note that n is usually taken to be greater than 0, since 0th homotopy groups are not groups in general. However, the 0th homotopy group of spheres happen to be (abelian) groups, so the statement is technically correct. Furthermore, while π_1 are groups, they are not abelian in general, but for the spheres they happen to be. We hence state the theorem like this for the sake of uniformity.

In more detail, the ingredients of the type/statement are formalised as follows:

- isFP : AbGrp → hProp is a predicate expressing that an abelian group is finitely presentable, i.e., for G : AbGrp, isFP G encodes that there *merely*[1] exist $r, g : \mathbb{N}$ and $\varphi : \mathbb{Z}^r \to \mathbb{Z}^g$ such that $G \simeq \mathbb{Z}^g/\mathsf{im}(\varphi)$. In particular, this means that G can be written using g generators and r relations. As we only have the mere existence of a presentation we say that G is "finitely presentable" instead of "finitely presented".
- π_n : Type$_\star$ → AbGrp is a family of abelian groups indexed over the universe of *pointed types* – the Σ-type consisting of types and designated basepoints. Note that in π_n, we fix $n : \mathbb{N}$, but technically π is also indexed over $\mathbb{N}$ and, in general, we need to require that $n \geq 2$. But as discussed above, for the spheres all π_n are abelian groups which justifies indexing by $\mathbb{N}$. We will return to the actual definition of π_n below.
- Finally, the type $\mathbb{S}^m$ denotes the m-sphere. This is defined as a *higher* inductive type (HIT) – an inductive type which allows for "higher" constructors. For now, let us give a simple special case of the spheres, namely the circle:

```
data 𝕊¹ : Type where
  base : 𝕊¹
  loop : base = base
```

Above, the type $=$ denotes the *identity type* of $\mathbb{S}^1$. The fundamental idea of HoTT is that identities can be interpreted as *paths*. Hence, by introducing base and loop : base = base above, we are saying that a circle is nothing but a basepoint together with a path from this basepoint to itself (which from a homotopical point of view captures a circle).
We should remark that identities/paths are implemented differently in `Cubical Agda` than in Book HoTT [35]. This is one of the key reasons `Cubical Agda` has native support for a general class of HITs [10, 8], but these intricacies are beyond the scope of this paper. All we need to know is that a HIT, just like an ordinary inductive type, comes with an elimination/induction principle. For instance, a map $f : \mathbb{S}^1 \to X$ is determined by a point $x : X$ to which it sends base and a loop $x = x$ to which it sends loop. Just as for regular inductive types in `Agda`, these can be written using pattern-matching in `Cubical Agda`.

The second line is the proof of the theorem. It consists of three things: two previously proved theorems ThmA and ThmB evidently saying something interesting about n and m, as well as a function taking the proofs of these theorems and converting them into a proof of the main theorem. In order to avoid too many technical details at this point, we defer the statement and proof of these key results to Section 3.

2.2 Elementary definitions and results from synthetic homotopy theory

We will now, very briefly, introduce some key concepts from synthetic homotopy theory that we will need in the coming sections. Experts on these topics can safely skip or skim this section. As remarked earlier, we will use a mix of `Cubical Agda` notation and prose. Most of the results here can be found in the HoTT book [35], but some notations may differ. We omit all proofs because of space constraints, but emphasise that all results have ben formalised.

[1] *Mere existence* captures that the existence of something is a property, not a structure. In the formalisation, this is represented using propositional truncation. However, since `Cubical Agda` enjoys canonicity, this does not prevent us from computing a concrete presentation (see Section 5 for details).

2.2.1 Basics

The following conventions and elementary definitions will be used in the paper.

- **Pointed types and functions:** We simply write $X : \mathsf{Type}_\star$ for pointed types and leave the basepoint $\star_X$ implicit. We write $X \to_\star Y$, i.e. the type of pairs of functions $f : X \to Y$ and proofs $\star_f : f(\star_X) = \star_Y$ which we always leave implicit.
- **Loop spaces:** $\Omega\, X$ is the loop space of a pointed type X, i.e., the type of paths $\star_X = \star_X$. It is pointed by the constant path (refl).
- **Fibres and equivalences:** Given a function $f : X \to Y$ and a point $y : Y$, we define the fibre of f over y by

$$\mathsf{fib}\ f\ y := \Sigma[\ x : X\ ]\ f\ x = y$$

  For the most part in this paper, Y will be pointed. In this case we simply write $\mathsf{fib}\ f$ rather than $\mathsf{fib}\ f\ \star_Y$. We say that a function is an equivalence if all its fibres are contractible, i.e, pointed by a unique point.
- ***n*-types:** For $n \geq -1$, we say that a type X is an n-type if for all $x : X$, the type $\Omega^{n+1}\, X$ is contractible (with x as basepoint of X). We give special names to (-1)-types and 0-types: these are called, respectively, *propositions* and *sets*.
- **Truncations:** We recall that any type X can be turned into an n-type via its n-truncation:

```
data ‖_‖ (X : Type) (n : ℕ) : Type where
  |_| : X → ‖ X ‖ₙ
  hub/spoke : ...
```

  The omitted hub and spoke constructors force $\| X \|_n$ to be an n-type, see [35, Chapter 7.3] for details. The caveat is that a map $X \to Y$ in general only lifts to a map $\| X \|_n \to Y$ if Y is an n-type. We take the opportunity to define $\exists[\ x : X\ ]\ P\, x := \| \Sigma[\ x : X\ ]\ P\, x \|_{-1}$. This type captures *mere* existence.
- **Connected types and functions:** A type X is n-connected if $\| X \|_n$ is contractible. Similarly, a function is n-connected if its fibres are n-connected.
- **Pushouts:** We define the pushout of a span $Y \xleftarrow{f} X \xrightarrow{g} Z$ by the following HIT.

```
data Pushout (f : X → Y) (g : X → Z) : Type where
  inl : Y → Pushout f g
  inr : Z → Pushout f g
  push : (x : X) → inl (f x) = inr (g x)
```

  We will need some special cases of pushouts.
  - **Cofibres:** In the special case when Y is the unit type, the pushout is the *cofibre* of g and is denoted cofib_f. The function $\mathsf{inr} : Z \to \mathsf{cofib}_g$, i.e. the universal map from the codomain of f into its cofibre, is often given a special name: cfcod. Cofibres are always taken to be pointed by the point from $\mathbb{1}$.
  - **Suspensions:** The suspension of a type X, denoted $\Sigma\, X$, is simply the cofibre of the unique map $X \to \mathbb{1}$. We remark that this construction is functorial – we overload notation and write $\Sigma\, f : \Sigma\, X \to \Sigma\, Y$ for the map induced by $f : X \to Y$.
  - **Spheres:** We define the n-sphere by iterated suspension of the empty type, $\bot$. That is: $\mathbb{S}^n = \Sigma^{n+1} \bot$. When $n = 1$, this is equivalent to the alternative definition of $\mathbb{S}^1$ given above.

- **Wedge sums:** Given a family of pointed types $B : X \to \mathsf{Type}_\star$, we define the wedge sum over B, denoted $\bigvee_{x:X}(B\,x)$ or simply $\bigvee_X B$ when B does not depend on X, to be the cofibre of the inclusion of the basepoint $\iota : X \to \Sigma[\,x : X\,](B\,x)$ defined by $\iota\ x = (x, \star_{B\,x})$. For us, X will always be a finite set. We write $B_0 \vee B_1$ for $\bigvee_X B$ when $X = \{0, 1\}$.
- **Smash products:** There is a canonical map $X \vee Y \to X \times Y$ for pointed types X and Y. The cofibre of this map is called the *smash product* of X and Y. It is denoted $X \wedge Y$.
- **Joins:** We define the join of two types X and Y, denoted $X * Y$, to be the pushout of the projection maps $X \xleftarrow{\mathsf{fst}} X \times Y \xrightarrow{\mathsf{snd}} Y$. This construction is also functorial. Here too, we overload the notation, writing $f * g : X_1 * X_2 \to Y_1 * Y_2$ for the map induced by $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$. This map is pointed if either f or g is.

- **Homotopy groups:** Given a pointed type X and $n \geq 1$, we define $\pi_n(X) = \| \mathbb{S}^n \to_\star X \|_0$. This is a group because it is equivalent to $\Omega^n \| X \|_n$ which has a group structure induced by composition of loops. These groups are abelian when $n \geq 2$ by the Eckmann-Hilton argument [35, Theorem 2.1.6], but also turn out to be abelian when $n < 2$ when X is a sphere. Any pointed function $f : X \to_\star Y$ induces a homomorphism on homotopy groups $f_* : \pi_n(X) \to \pi_n(Y)$ by post-composition.

2.2.2 Properties of connected maps and types

The notions of n-connected types and functions play a very important role in the characterisation of the homotopy groups in this paper. Intuitively, n-connectedness of a type tells us that is is homotopically uninteresting in dimension n and lower. For instance, $\pi_n(X)$ vanishes whenever X is n-connected (this is, in fact, often taken as a definition of connectedness in classical texts). On the other hand, n-connectedness of a (pointed) function $f : X \to_\star Y$ tells us that it induces an isomorphism on homotopy groups in dimension n and lower, i.e. $f_* : \pi_m(X) \cong \pi_m(Y)$ for $m \leq n$. In this case, f_* is also a surjection when $m = n + 1$. These facts become particularly important when combined with the following theorem:

▶ **Theorem 1** (The Freudenthal suspension theorem). *If X is an n-connected space, there is a $2n$-connected map $X \to \Omega\,(\Sigma\,X)$.*

This theorem implies in particular that $\pi_n(X) \cong \pi_{n+1}(\Sigma\,X)$ when X is sufficiently connected.

We will need to quickly recall some basic closure properties of n-connected functions. In what follows, let $f : X \to Y$ and $g : Y \to Z$.

▶ **Proposition 2.** *If f and g are n-connected, then $g \circ f$ is n-connected.*

▶ **Proposition 3.** *If f and $g \circ f$ are n-connected, then g is n-connected.*

▶ **Proposition 4.** *If g is n-connected and $g \circ f$ is $(n-1)$-connected, then f is $(n-1)$-connected.*

2.2.3 (Co)fibre sequences

Some of the arguments in this paper use the machinery of fibre and cofibre sequences.

▶ **Definition 5** (Fibre sequences). *A* fibre sequence *is a sequence of pointed maps*

$$F \xrightarrow{p} E \xrightarrow{f} B$$

such that $(F, p) = (\mathsf{fib}\ f, \mathsf{fst})$ as elements of $\Sigma[\,X : \mathsf{Type}\,](X \to E)$, i.e. of the slice over E.

The main thing we need to know about fibre sequences is that they induce a long exact sequence of homotopy groups. The sequence from Definition 5 induces another sequence

$$\cdots \longrightarrow \pi_{n+1}(B) \longrightarrow \pi_n(F) \xrightarrow{p_*} \pi_n(E) \xrightarrow{f_*} \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow \cdots$$

usually referred to as the *long exact sequence of homotopy groups*.

Cofibre sequences have a significant presence in our formalisation.

▶ **Definition 6** (Cofibre sequences). *A* cofibre sequence *is a sequence of pointed maps*

$$X \xrightarrow{f} Y \xrightarrow{p} Z$$

such that $(Z, p) = (\mathsf{cofib}_f, \mathsf{cfcod})$ *as elements of* $\Sigma[\, A : \mathsf{Type} \,](Y \to A)$*, i.e. of the coslice under* Y*.*

There is a dual notion to the long exact sequence of homotopy groups associated with cofibre sequences, called the *Puppe sequence*:

$$\cdots \longrightarrow \Sigma^{n-1} Z \longrightarrow \Sigma^n X \xrightarrow{\Sigma^n f} \Sigma^n Y \xrightarrow{\Sigma^n p} \Sigma^n Z \longrightarrow \Sigma^{n+1} X \longrightarrow \cdots$$

Let us now mention some key results about cofibre sequences.

▶ **Proposition 7.** *For any pointed types* X*,* Y *and* Z*, if*

$$X \longrightarrow Y \longrightarrow Z$$

is a cofibre sequence then so is

$$Y \longrightarrow Z \longrightarrow \Sigma X$$

▶ **Corollary 8.** *For any pointed types* X*,* Y *and* Z*, if*

$$X \longrightarrow Y \longrightarrow Z$$

is a cofibre sequence, then so are the following subsequences of the Puppe sequence:

$$\Sigma^n X \longrightarrow \Sigma^n Y \longrightarrow \Sigma^n Z$$
$$\Sigma^n Y \longrightarrow \Sigma^n Z \longrightarrow \Sigma^{n+1} X$$
$$\Sigma^n Z \longrightarrow \Sigma^{n+1} X \longrightarrow \Sigma^{n+1} Y$$

▶ **Proposition 9.** *If there are two cofibre sequences* $X \to Y \to Z$ *and* $X' \to Y' \to Z'$ *and a commutative square:*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \longrightarrow & Y' \end{array}$$

then there is an induced map $Z \to Z'$*, and moreover if* f *is* n*-connected and* g *is* $(n+1)$*-connected, then this map* $Z \to Z'$ *is also* $(n+1)$*-connected.*

This result informs us, in particular, about the interaction of connected maps and suspensions.

▶ **Corollary 10.** *If $f : X \to Y$ is k-connected, then the induced map $\Sigma f : \Sigma X \to \Sigma Y$ is $(k+1)$-connected.*

A similar result which, although somewhat related, is more complicated to prove is the following:

▶ **Proposition 11.** *Suppose $f_1 : X_1 \to Y_1$ is n_1-connected, and $f_2 : X_2 \to Y_2$ is n_2-connected, and X_1 is m_1-connected and Y_2 is m_2-connected. Then we have a lower bound on the connectivity of $f_1 * f_2$, namely $\min(m_1 + n_2, m_2 + n_1) + 2$.*

3 Outline of the formalised proof

In this section, we go through Barton and Campion's proof of the Serre finiteness theorem [2]. Our presentation differs somewhat from that of Barton and Campion – it is informal at times and is meant to be pedagogical rather than complete. Barton and Campion's proof can be summarised by two key theorems which together imply the main result, and we will use these two theorems to guide our presentation of their proof and the formalisation. The first theorem concerns *connected covers*:

▶ **Definition 12** (Connected covers)**.** *Given a pointed type X, we define its* nth connected cover*, denoted $X\langle n\rangle$, to be the fibre of the map $X \to \| X \|_n$. This type is trivially n-connected.*

Barton and Campion's first theorem also uses the predicate *stably almost finite*. This predicate can be thought of as expressing that a given type has finitely presentable homology groups (without ever actually mentioning homology explicitly); let us black-box its definition for now.

▶ **Master Theorem A.** *If X is stably almost finite and 1-connected, then so is $X\langle n\rangle$ for $n \geq 1$.*

▶ **Master Theorem B.** *If X is stably almost finite and $(n-1)$-connected, then $\pi_n(X)$ is finitely presentable.*

Together, these theorems imply a general form of the Serre finiteness theorem:

▶ **Theorem 13.** *If X is stably almost finite and 1-connected, then $\pi_n(X)$ is finitely presentable for all $n > 1$.*

Proof. It is easy to see that $\pi_n(X) \cong \pi_n(X\langle n-1\rangle)$. Furthermore, $X\langle n-1\rangle$ is $(n-1)$-connected and hence 1-connected as $n > 1$. By Master Theorem A, it is also stably almost finite and thus, by Master Theorem B, has finitely presentable π_n. ◀

The Serre finiteness theorem follows immediately from Theorem 13 because spheres happen to (trivially) satisfy the mysterious "stably almost finite" predicate – indeed, so does any finite CW complex (we will define these soon). The plan now is to walk the reader through the proofs of the two master theorems, starting with Master Theorem A and thereafter moving on to Master Theorem B. In fact, the proof of Master Theorem A uses Master Theorem B, so in the following section we assume this theorem. This order of presentation might seem a bit strange, but it matches the formalisation better than the other way around.

3.1 A primer on homology

Before proving Master Theorem A and Master Theorem B, we make a brief digression on the topic of *homology*. Somewhat remarkably, Barton and Campion's proof the of Serre finiteness theorem does not actually rely on any explicit notion of homology; in fact, this idea crystallised during the production of the formalisation we present here, which originally was based on an earlier version of Barton and Campion's proof. Nevertheless, some definitions in the proof – in particular that of stably almost finite types – are still strongly inspired by the key ideas of homology and may come off as rather unmotivated to a reader who is not familiar with these constructions. For this reason, we will here give a very quick recap of the most crucial ideas. As this is not part of the actual formalisation, we will temporarily switch to a more classical (as opposed to type-theoretic) setting.

A homology theory is a (for our purposes) $\mathbb{N}$-indexed family of functors $H_n : \mathsf{Spaces} \to \mathsf{AbGroup}$ satisfying a list of axioms called the *Eilenberg–Steenrod axioms* [12]. Here, Spaces is meant to capture, informally, any reasonable notion of "space" one might be interested in. In this paper, we are primarily interested in the case when this refers to the category of finite CW complexes – an especially important family of inductively built spaces which includes all spheres. Unlike homotopy groups, these groups – called homology groups – are often easy to compute. For instance, the homology groups of the n-sphere vanish in all dimensions apart from n (and 0, depending on which notion of homology we are working with) in which case it is $\mathbb{Z}$. This is in stark contrast to their homotopy groups, which are unknown in general.

The first important property of homology groups is that they are finitely presentable for finite CW complexes. This, together with the following theorem (which we state here in a weak form that suffices for this presentation), should ignite some belief in the truth of the Serre finiteness theorem:

▶ **Theorem 14** (Hurewicz theorem). *Let X be an $(n-1)$-connected space. In this case $\pi_n^{\mathsf{ab}}(X) \cong H_n(X)$.*

Here, $\pi_n^{\mathsf{ab}}(X)$ denotes the abelianisation of $\pi_n(X)$, which agrees with $\pi_n(X)$ whenever $n > 1$. This theorem was, in fact, proved in HoTT by Christensen & Scoccola [9]. In light of it, the first nonzero homotopy group of any finite CW complex X coincides with the corresponding homology group, and therefore satisfies the conclusion of the Serre finiteness theorem. Unfortunately, the rest of the homotopy and homology groups of X generally do not coincide, so most of the proof of the Serre finiteness theorem boils down to constructing another space X' with $\pi_n(X) \cong \pi_n(X')$ such that X' *does* satisfy the connectedness assumption.

As stated before, the proof we have formalised does not use homology, but the above idea is still guiding the proof. Homology is avoided by introducing the "stably almost finite" predicate, which encodes the fact that the homology of a given space is finitely presentable without explicitly mentioning homology. To appreciate the definition of this predicate, we only need to know one more fact about homology groups, namely that they are *stable*. That is, they respect suspension: $H_n(X) \cong H_{n+1}(\Sigma X)$ for all n. We will now switch back to type theory but may refer back to the concepts mentioned here in order to build intuition.

3.2 Proving Master Theorem A

3.2.1 CW complexes and n-finite types

Before proving Master Theorem A, we better learn what "stably almost finite" means. To define this notion, the first thing we shall need is a definition of CW complexes. These were first studied in HoTT, in the context of cellular cohomology, by Buchholtz & Favonia [6]

and recently also by Ljungström & Pujet [25]. The proof of the Serre finiteness theorem relies on results from the latter paper, and our formalisation makes use of the formalisation accompanying it. For this reason, we adopt their definition of CW complexes. We only need the special case of finite complexes, so this is the one we introduce here. In what follows, $[n]$ denotes the finite set of size n.

▶ **Definition 15** (Finite CW structures). *A* finite CW structure *(of dimension n) is a sequence of types*

$$X_{-1} \xrightarrow{\iota^X_{-1}} X_0 \xrightarrow{\iota^X_0} X_1 \xrightarrow{\iota^X_1} \dots$$

equipped with the following data

- *(cell cardinalities) maps* $\mathsf{card}^X_{(-)} : \mathbb{N} \to \mathbb{N}$ *encoding the number of cells in each dimension,*
- *(attaching maps) maps* $\alpha^X_{i+1} : \mathbb{S}^i \times [\mathsf{card}^X_{i+1}] \to X_i$,
- *equivalences* $\mathsf{e}^X_i : X_{i+1} \simeq \mathsf{Pushout}\, \alpha^X_{i+1}\, \mathsf{snd}$, *i.e. equivalences witnessing that X_{i+1} is obtained as the pushout of α^X_i along the second projection $\mathbb{S}^i \times [\mathsf{card}^X_i] \to [\mathsf{card}^X_i]$,*
- *a proof that X_{-1} is empty, and*
- *(finiteness) a proof that $\iota^X_i : X_i \to X_{i+1}$ is an equivalence for each $i \geq n$.*

Given a CW structure $X_\bullet$, we can form its colimit (see [32] for details) which we will simply denote by X_∞.

▶ **Definition 16** (CW complexes). *A type X is a* finite CW complex of dimension n *if there merely exists a CW structure $X_\bullet$ of dimension n, s.t. $X_\infty \simeq X$. We say that X is a* finite CW complex *if there merely exists some n s.t. X is a finite CW complex of dimension n.*

We write FinCW_n for the universe of finite CW complexes of a fixed dimension n and, more generally, FinCW for the universe of all finite CW complexes. We are, in particular, interested in one example of CW complexes: suspensions.

▶ **Example 17.** Given a finite CW complex X with CW structure $X_\bullet$, we equip its suspension $\Sigma\, X$ with a (finite) CW structure $\Sigma\, X_\bullet$ defined as

$$\bot \longrightarrow \Sigma\, X_{-1} \xrightarrow{\Sigma\, \iota^X_{-1}} \Sigma\, X_0 \xrightarrow{\Sigma\, \iota^X_0} \Sigma\, X_1 \xrightarrow{\Sigma\, \iota^X_1} \dots$$

We set $\mathsf{card}^{\Sigma X}_0 = 2$ and $\mathsf{card}^{\Sigma X}_{n+1} = \mathsf{card}^X_n$. The attaching maps are obtained by taking suspensions of the attaching maps of X.

The fact that the above indeed defines a CW complex follows from the fact that FinCW is closed under pushouts [25]. Example 17 implies, in particular, that spheres are finite CW complexes. In addition to the above example, CW complexes are closed under many other important operations on types, e.g. binary products [2].

We can now use this definition to define what it means for a type to be *n-finite*. The idea is that an n-finite type X is one that is well-approximated up to dimension n by some finite CW complex C in the sense that there is an $(n-1)$-connected map from C to X. A for now provisional definition in `Cubical Agda` is the following:

```
is_-FiniteProv_ : ℕ → Type → Type₁
is n -FiniteProv X =
  ∃[ C ∶ FinCW ] Σ[ f ∈ ( C → X) ] (is (n - 1) -ConnectedFun f)
```

The reason that this is provisional is because we can make it even more informative. Indeed, because every finite CW complex has an n-dimensional skeleton with an $(n-1)$-connected map to the original complex, we immediately realise that the CW complex in question can equivalently be given the additional assumption of being n-dimensional. We thus go for the following definition:

▶ **Definition 18** (n-finite types). *A type X is* n-finite *if there merely exists some n-dimensional CW complex C equipped with an $(n-1)$-connected map $f : C \to X$. In* `Cubical Agda`*:*

```
is_-Finite_ : ℕ → Type → Type₁
is_-Finite_ n X =
  ∃[ C : FinCW n ] Σ[ f ∈ ( C → X) ] (is (n - 1) -ConnectedFun f)
```

It is immediate that any $(n+1)$-finite type is also n-finite. Let us now show some elementary closure properties of n-finite types.

▶ **Proposition 19.** *If Y is n-finite and $f : X \to Y$ is n-connected, then X is n-finite as well.*

Proof. By assumption, we (merely) have a diagram of the following form:

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ C & \longrightarrow & Y \end{array}$$

where the bottom arrow is $(n-1)$-connected, C is an n-dimensional CW complex and f is n-connected. It follows from [2, Lemma A.27], because f in particular is $(n-1)$-connected, that there (merely) exists a map $g : C \to X$ making the diagram commute. Then g must be $(n-1)$-connected as well, by Proposition 4. ◀

We can go a little further:

▶ **Proposition 20.** *If $X \to Y \to Z$ is a cofibre sequence, and X is $(n-1)$-finite and Y is n-finite, then Z is n-finite.*

Proof. By assumption, we obtain the following diagram:

$$\begin{array}{ccccc} C & & D & & \\ \downarrow & & \downarrow & & \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

A straightforward connectedness argument yields a map $C \to D$ forming a commutative square. Taking the cofibre of this map yields a CW complex C' with a map into Z that must be n-connected by Proposition 9. ◀

This implies, in particular, that if X is $(n-1)$-finite, then ΣX is n-finite. Let us finally consider the interaction between products and n-finiteness.

▶ **Proposition 21.** *If X and Y are n-finite, then so is $X \times Y$.*

Proof. Products of CW complexes are CW complexes, and products of k-connected maps are k-connected. The product complex will be $2n$-dimensional, but we can replace it by an n-dimensional complex using the observation before Definition 18. ◀

3.2.2 Stably n-finite types

So far, we have defined n-finiteness and proved that it satisfies some of the expected closure properties. However, n-finiteness is a rather strong notion. The idea of n-finiteness is, roughly, that types satisfying this property have finitely presentable homology groups through degree n. Since suspension induces an isomorphism $H_n(X) \cong H_{n+1}(\Sigma X)$ on homology (recall Section 3.1), we can still capture this idea by only requiring it to hold for *some iterated suspension* of the type in question. We take into account the shift in degree by requiring $(m+n)$-finiteness if we applied m suspensions to the type.

▶ **Definition 22** (Stably n-finite types). *A type is called* stably n-finite *if there merely exists an m such that $\Sigma^m X$, the m-fold suspension of X, is $(m+n)$-finite. In* `Cubical Agda`*:*

```
isStably_-Finite_ : ℕ → Type → Type₁
isStably_-Finite_ n X = ∃[ m : ℕ ] (is (m + n) -Finite (Σ^m X))
```

Every finite CW complex is stably n-finite for all n (take $m = 0$ above). It is immediate that if ΣX is stably n-finite, then X is stably $(n-1)$-finite, and vice versa. It follows from facts about n-finite types that if X is stably n-finite, then it is also stably $(n-1)$-finite.

By Corollary 8, we know that if $X \to Y \to Z$ is a cofibre sequence, then so are $(Y \to Z \to \Sigma X)$, $(Z \to \Sigma X \to \Sigma Y)$, $(\Sigma X \to \Sigma Y \to \Sigma Z)$, and so on. It then follows from facts about n-finite types that if X is stably $(n-1)$-finite and Y is stably n-finite, then Z is stably n-finite. It also follows that if X is stably $(n-1)$-finite and Z is stably $(n-1)$-finite, then Y is stably $(n-1)$-finite, and so on.

The notion of n-finite types also plays well with connectedness. Again, we can use facts about n-finite types to show that if $f : X \to Y$ is $(n-1)$-connected and X is stably n-finite, so is Y. Similarly, if f is n-connected, and Y is stably n-finite, so is X.

We finally mention the interaction between joins and stably n-finite types. This is a more advanced result.

▶ **Proposition 23.** *Let X be an m-connected ($m \geq 0$) and stably 0-finite type, and Y be an $(m'-2)$-connected and stably n-finite type. Then $X * Y$ is stably $\min(m', n+m)$-finite.*

Proof. Since X is m-connected and $m \geq 0$, there merely exists a (-1)-connected map $\mathbb{1} \to X$. Since the proof we seek is of a mere proposition, we can apply elimination and it suffices to prove the claim about $X * Y$ from the following data:

$$\begin{array}{ccc} \mathbb{1} & & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & & \Sigma^k Y \end{array}$$

in which f is (-1)-connected, g is $(k+n-1)$-connected, and $\Sigma^k Y$ is $(k+m'-2)$-connected, and $\mathbb{1}$ is arbitrarily highly connected. It follows from Proposition 11 that $f * g$ is $(\min(k+m'-3, k+n+m-1)+2)$-connected, so, since $\mathbb{1} * C \simeq \mathbb{1}$, the join $X * \Sigma^k Y \simeq \Sigma^k (X * Y)$ is (at least, say) $\min(k+m', k+n+m)$-finite, which implies the result we were after. ◀

▶ **Proposition 24.** *Let X be $(m-2)$-connected for $m-1 \leq n$ and stably n-finite, and let Y be $(m'-2)$-connected and stably n'-finite. Then $X * Y$ is stably $\min(m'+n, m+n')$-finite.*

Proof. We can (merely) pick a k large enough and obtain the following two functions

$$\begin{array}{ccc} C & & D \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \Sigma^k X & & \Sigma^k Y \end{array}$$

where C and D are finite CW complexes, f is $(k+n-1)$-connected, g is $(k+n'-1)$-connected. Observe now that C must be at least $(k+m-2)$-connected, and thus, $f * g : C * D \to \Sigma^k X * \Sigma^k Y \simeq \Sigma^{2k}(X * Y)$ is at least $(\min(2k+m+n'-3, 2k+m'+n-3)+2)$-connected. Hence, $X * Y$ must be at least $\min(m+n', m'+n)$-finite as required. ◀

3.2.3 Stably almost finite types

Earlier, we remarked that stably n-finite types have finitely presentable nth homology groups and thus, if the Hurewicz theorem applies, also nth homotopy groups. To match the statement of the Serre finiteness theorem better, we would now like to express that a type satisfies this for *any* value of n. This motivates the definition of *stably almost finite types*:

▶ **Definition 25** (Stably almost finite types). *A type is called* stably almost finite *if it is stably n-finite for every n. In* `Cubical Agda`*:*

```
SAF : Type → Type₁
SAF X = (n : ℕ) → isStably n -Finite X
```

In particular, as remarked earlier, finite CW complexes are all stably almost finite.

Stably almost finite types immediately inherit the closure properties of stably n-finite types, e.g. closure under joins and cofibre sequences. Furthermore, we have the following additional closure principles:

▶ **Proposition 26.** *If X and Y are stably almost finite, then so are $X \times Y$, $X \vee Y$ and $X \wedge Y$.*

Proof. This follows immediately from the following cofibre sequences:

$$\begin{array}{ccccc} X & \longrightarrow & X \vee Y & \longrightarrow & Y \\ X \wedge Y & \longrightarrow & \mathbb{1} & \longrightarrow & X * Y \\ X \vee Y & \longrightarrow & X \times Y & \longrightarrow & X \wedge Y \end{array}$$

◀

3.2.4 Closure of stably almost finite types vis-à-vis fibre sequences

So far, we have shown that stably almost finite types enjoy several closure properties relating to colimit constructions – cofibre sequences, joins, smash products, and so on. However, since we are ultimately hoping that this notion will help us compute homotopy groups, what we really would like to know is whether similar closure properties also hold for *limit* constructions. In particular, as the main goal right now is to prove that connected covers are stably almost finite (i.e. Master Theorem A) and these are defined as *fibres* of truncation maps, we would like to know to what extent stably almost finite types are closed under *fibre* sequences. The key idea behind the translation from our previous results regarding cofibres to the corresponding results regarding fibres is the following theorem, called the *Ganea theorem* [13, Theorem 1.1]. We only state it and do not prove here (the proof is somewhat technical but straightforward, and we refer to the formalisation).

▶ **Theorem 27** (Ganea). *For any fibre sequence $F \xrightarrow{j} E \to B$, there is an associated fibre sequence $F * \Omega B \to C^1 \to B$, where C^1 is the cofibre of j.*

▶ **Corollary 28.** *Given a fibre sequence $F \to E \to B$, we can construct a diagram*

$$\begin{array}{ccccccc} F & & F*(\Omega B) & & F*(\Omega B)*(\Omega B) & & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ E & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ B & & B & & B & & \dots \end{array}$$

where each column is a fibre sequence and each "elbow"

$$\begin{array}{ccc} F*\cdots & & \\ \downarrow & & \\ C^n & \longrightarrow & C^{n+1} \end{array}$$

is a cofibre sequence.

The Ganea theorem will shed light on the relationship between stably almost finiteness of pointed types and their loop spaces. Concretely, we have the following two results:

▶ **Proposition 29.** *Let B be a connected pointed type. If ΩB is stably almost finite, then so is B.*

Proof. We instantiate Corollary 28 with the fibre sequence $\Omega B \to \mathbb{1} \to B$. This gives us the following sequence of fibre sequences:

$$\begin{array}{ccccccc} \Omega B & & \Omega B * \Omega B & & \Omega B * \Omega B * \Omega B & & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{1} & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ B & & B & & B & & \dots \end{array} \tag{1}$$

The exact construction of the increasingly involved types C^n is irrelevant to us. All we need to know is that each of them is stably almost finite – a fact that follows from Corollary 28 and cofibre sequence closure properties of stably almost finite types. As joins increase connectivity, the fibres $\Omega B * \cdots * \Omega B$ become highly connected and so also (by connectedness of B) does the map $C^n \to B$ for n large. Hence, for arbitrary k, this map is eventually $(k-1)$-connected which, because C^n is stably almost finite, implies that B is stably k-finite. Thus, B is stably almost finite. ◀

▶ **Proposition 30.** *Let B be 1-connected. If B is stably almost finite, then so is ΩB.*

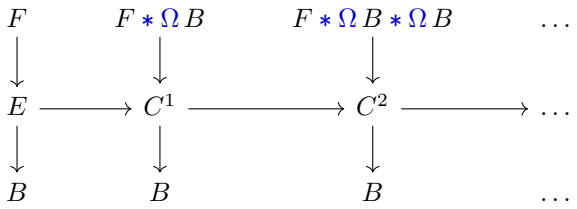
Proof. Let us again consider the sequence of fibre sequences in (1) from the proof of Proposition 29. We prove that ΩB is stably n-finite for all n by induction. When $n = 0$, we note that B is 1-connected, so ΩB is 0-connected. This means there is a map $\mathbb{1} \to \Omega B$ which is (-1)-connected, and so ΩB is (stably) 0-finite.

For the inductive step, we assume that ΩB is stably n-finite and seek to prove it then must be stably $(n+1)$-finite. Using Propositions 23 and 24, we see that each $\Omega B * \cdots * \Omega B$ must be stably $(n+2)$-finite, and so is C^k for k large: the maps from the C^k's to B (a stably almost finite type) get more and more connected. Now by cofibre sequence closure properties, we see that all preceding C^k's are stably $(n+2)$-finite, and finally, we observe that $C^1 = \Sigma \Omega B$ and it follows that ΩB must be stably $(n+1)$-finite. ◀

We continue to apply the Ganea theorem to get the desired closure properties under fibres. The proof of the following proposition is similar to the previous ones, so we only sketch it.

▶ **Proposition 31.** *Let $F \to E \to B$ be a fibre sequence with B 1-connected. If F and B are stably almost finite, then so is E.*

Proof sketch. Using the Ganea theorem, we construct a sequence of fibre sequences like before:



Each fibre in the sequence must be stably almost finite (by Proposition 30), and the maps get increasingly connected. Similar to the previous proof we just go along far enough until the lower-most map is connected enough that the corresponding total space is stably n-finite, then all the preceding total spaces are as well, by induction using the cofibre sequence closure properties. ◀

▶ **Proposition 32.** *Let $F \to E \to B$ be a fibre sequence with E and B 1-connected. If E and B are stably almost finite, then so is F.*

Proof. Apply Proposition 31 to the shifted fibre sequence $\Omega B \to F \to E$, in which ΩB is stably almost finite by Proposition 30. ◀

3.2.5 Stably almost finite types and finitely presentable groups

With Proposition 31, we are getting close to proving Master Theorem A. Indeed, the theorem concerns whether the type $X\langle n\rangle$ is stably almost finite, and as this type comes from the fibre sequence $X\langle n\rangle \to X \to \| X \|_n$, the theorem would follow if we knew that $\| X \|_n$ is stably almost finite whenever X is (for $n \geq 1$). This type is again best understood as sitting inside of a fibre sequence; this time, we consider the sequence $F \to \| X \|_n \to \| X \|_{n-1}$. In this sequence, the fibre F is an *Eilenberg–Mac Lane space*.

▶ **Proposition 33** (Eilenberg–Mac Lane spaces)**.** *Given an abelian group G and natural number n, there is a type $\mathsf{K}(G, n)$ called the nth Eilenberg–Mac Lane space of G with the following properties:*

- $\mathsf{K}(G, 0) = G$
- $\Omega\,\mathsf{K}(G, n) \simeq \mathsf{K}(G, n-1)$
- $\mathsf{K}(G, n)$ *is* $(n-1)$*-connected*

For the construction of these, we refer to Licata and Finster [22].

▶ **Proposition 34.** *The fibre of the map $X\langle n+1\rangle \to X\langle n\rangle$ is the Eilenberg–Mac Lane space $\mathsf{K}(\pi_{n+1}(X), n)$.*

We refer to [28] for the proof of this result. We would now like to know how the "stably almost finite" predicate interacts with Eilenberg–Mac Lane spaces.

▶ **Proposition 35.** *If G is a finitely presentable abelian group, then the Eilenberg–Mac Lane spaces $\mathsf{K}(G, n)$ are stably almost finite for all $n \geq 1$.*

Proof. Since $\mathsf{K}(G,n)$ is $(n-1)$-connected (and in particular $(n-2)$-connected), we know, by iteration of Proposition 29, that it is enough to check that $\Omega^{n-1}\,\mathsf{K}(G,n)$ is stably almost finite. This type is equivalent to $\mathsf{K}(G,1)$, so it suffices to prove the proposition in this case. By assumption, we have that $G \cong \mathbb{Z}^k \times \mathbb{Z}/p_1\mathbb{Z} \times \ldots \mathbb{Z}/p_l\mathbb{Z}$ for some natural numbers $k, p_1, \ldots, p_l$. Eilenberg–Mac Lane spaces are easily seen to commute with products and, since products of stably almost finite types are again stably almost finite, it suffices to check that $\mathsf{K}(\mathbb{Z},1)$ and $\mathsf{K}(\mathbb{Z}/n\mathbb{Z},1)$ are stably almost finite for any $n \geq 1$. We have that $\mathsf{K}(\mathbb{Z},1) \simeq \mathbb{S}^1$ which immediately is stably almost finite (as it is a CW complex). For $\mathsf{K}(\mathbb{Z}/n\mathbb{Z},1)$, we apply Proposition 29 again and check that its loop space is stably almost finite. Since $\Omega\,\mathsf{K}(\mathbb{Z}/n\mathbb{Z},1) \simeq [n]$ and finite types in particular are CW complexes, we are done. ◀

With this, we are finally ready to prove Master Theorem A. Recall that we are doing this under the assumption of Master Theorem B, which we will prove in the next section.

Proof of Master Theorem A. Let $n \geq 1$ and let X be a 1-connected and stably finite type. We use induction to prove the statement in question, namely that $X\langle n\rangle$ is stably almost finite *and*, in order to obtain a strong induction hypothesis, that $\| X \|_n$ is stably almost finite. When $n = 1$, we have $X\langle n\rangle \simeq X$ and $\| X \|_n \simeq \mathbb{1}$, which are both stably almost finite (by the hypothesis on X). For the inductive step, we assume that $X\langle n\rangle$ and $\| X \|_n$ are stably almost finite. We consider the following fibre sequence:

$$\mathsf{K}(\pi_{n+1}(X), n+1) \longrightarrow \| X \|_{n+1} \longrightarrow \| X \|_n$$

The first step is proving that the middle type is stably almost finite. By Proposition 31 and 1-connectedness of X (and hence also $\| X \|_n$), it suffices to show that the other two types are stably almost finite. For $\| X \|_n$, this holds by the induction hypothesis. For $\mathsf{K}(\pi_{n+1}(X), n+1)$, it suffices, by Proposition 35, to show that $\pi_{n+1}(X)$ is finitely presentable. We have $\pi_{n+1}(X) \cong \pi_{n+1}(X\langle n\rangle)$ and since $X\langle n\rangle$ is n-connected and stably almost finite (by the induction hypothesis), Master Theorem B tells us that $\pi_{n+1}(X\langle n\rangle)$ is finitely presentable.

To complete the inductive proof, it remains to prove that $X\langle n+1\rangle$ is stably almost finite. This now follows immediately from Proposition 32 because it sits in a fibre sequence $X\langle n+1\rangle \to X \to \| X \|_{n+1}$ where the other two types are 1-connected and stably almost finite. ◀

3.3 Proving Master Theorem B

Master Theorem B concerns the homotopy groups of stably almost finite types. As in the premise of the theorem, let X be an $(n-1)$-connected and stably almost finite type. Let us try to compute $\pi_n(X)$ and see what simplifications we can make before we get stuck. Because X is $(n-1)$-connected, it follows by the Freudenthal suspension theorem, Theorem 1, that $\pi_n(X) \cong \pi_{n+m}(\Sigma^m X)$ for any $m \geq 0$. Because X is stably almost finite, it is also stably $(n+1)$-finite and so we can find a number $m \geq 0$ and a finite CW complex C equipped with an $(n+m)$-connected map $C \to \Sigma^m X$. The existence of this map implies two things:

- $\pi_n(X) \cong \pi_{n+m}(\Sigma^m X) \cong \pi_{n+m}(C)$,
- Because $\Sigma^m X$ is $(n+m-1)$-connected, so is C.

Hence, we have proved the following lemma.

▶ **Lemma 36.** *Let $n \geq 1$. For any stably almost finite and $(n-1)$-connected type X, there merely exists a natural number $k \geq 1$ and a $(k-1)$-connected finite CW complex C s.t. $\pi_n(X) \cong \pi_k(C)$.*

In the light of Lemma 36, the following theorem is sufficient to prove Master Theorem B:

▶ **Theorem 37** (The weak Hurewicz theorem). *Let $n \geq 2$ and C be an $(n-1)$-connected finite CW complex. The homotopy group $\pi_n(C)$ is finitely presentable.*

We remark that this theorem also holds for $n = 1$ if we abelianise the homotopy group in question, but this case will not be needed here. This theorem follows from the full Hurewicz theorem [9] but we follow the direct approach taken in [25] as it allows us to avoid defining homology. The proof is rather involved but we will give the main steps. The key step in the proof is the following theorem (here stated for finite CW complexes).

▶ **Theorem 38** (The Hurewicz approximation theorem [25]). *Let $n \geq 0$ and let C be an $(n-1)$-connected CW complex. There is (merely) a CW structure $\widehat{C}_\bullet$ for C s.t.*

- $\widehat{C}_i \simeq \mathbb{1}$ *for* $0 \leq i < n$
- $\widehat{C}_n \simeq \bigvee_A \mathbb{S}^n$ *for A a finite set*
- $\widehat{C}_{n+1} \simeq \mathsf{cofib}_f$ *for some* $f : \bigvee_B \mathbb{S}^n \to \bigvee_A \mathbb{S}^n$ *where B is another finite set.*

This theorem provides us with a good first step in the proof of Theorem 37. Let C be as in the statement of Theorem 37, i.e. an $(n-1)$-connected finite CW complex with $n \geq 2$. Let us apply Theorem 38 to obtain a CW structure $\widehat{C}_\bullet$ for it with $\widehat{C}_{n+1} \simeq \mathsf{cofib}_f$ for some $f : \bigvee_B \mathbb{S}^n \to \bigvee_A \mathbb{S}^n$ between finite bouquets of n-spheres. Now, it is easy to see by the connectedness of the inclusion of skeleta $\widehat{C}_{n+1} \to C$ that $\pi_n(C) \cong \pi_n(\widehat{C}_{n+1})$. Hence $\pi_n(C) \cong \pi_n(\mathsf{cofib}_f)$. In other words, we have reduced the computation of the nth homotopy group of arbitrary $(n-1)$-connected finite CW complexes to those of the form cofib_f for a map f between finite bouquets of n-spheres. However, these can be computed.

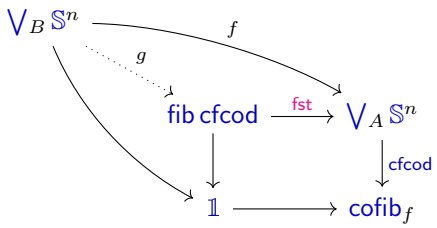
▶ **Proposition 39.** *Let $n \geq 2$ and $f : \bigvee_B \mathbb{S}^n \to \bigvee_A \mathbb{S}^n$ where A and B are finite sets. There is a map $\widehat{f} : \mathbb{Z}[B] \to \mathbb{Z}[A]$ such that:*

$$\pi_n(\mathsf{cofib}_f) \cong \mathbb{Z}[A] \,/\, \mathsf{Im}\, \widehat{f}$$

In particular, $\pi_n(\mathsf{cofib}_f)$ is finitely presentable.

The proof of this theorem requires machinery like the Blakers–Massey theorem [15, 1] which we have not quite introduced yet. We include a sketch of the proof for the expert reader but stress that it safely can be skipped.

Proof of Proposition 39. Since both the domain and codomain of f are $(n-1)$-connected, f is $(n-2)$-connected and the terminal map $\bigvee_B \mathbb{S}^n \to \mathbb{1}$ is $(n-1)$-connected. This implies, by the Blakers–Massey theorem, that the dashed map g below is $(2n-3)$-connected.



In particular, g_* is a surjective on all homotopy groups in dimension lower than or equal to $2n-2$ and, since $n \geq 2$, on π_n. Now, the long exact sequence of homotopy groups associated with the cfcod gives us, in particular, a sequence

$$\pi_n(\mathsf{fib}\ \mathsf{cfcod}) \xrightarrow{\mathsf{fst}_*} \pi_n(\textstyle\bigvee_A \mathbb{S}^n) \longrightarrow \pi_n(\mathsf{cofib}_f) \longrightarrow \pi_{n-1}(\mathsf{fib}\ \mathsf{cfcod}).$$

The final group is easily seen to vanish by connectivity of cfcod (see e.g. [5, Proposition 2.3.10]). Hence $\pi_n(\mathsf{cofib}_f) \cong \pi_n(\bigvee_A \mathbb{S}^n) \,/\, \mathsf{Im}\, \mathsf{fst}_*$. However, since g_* is surjective, this is isomorphic to $\pi_n(\bigvee_A \mathbb{S}^n) \,/\, \mathsf{Im}\, (\mathsf{fst}_* \circ g_*)$. It is easy (via another application of the Blakers–Massey theorem) to see that $\pi_n(\bigvee_C \mathbb{S}^n) \cong \mathbb{Z}[C]$ for any finite set C – in particular A and B. This concludes the proof. ◀

As explained earlier, Proposition 39 establishes, via the Hurewicz approximation theorem, the Weak Hurewicz theorem (Theorem 37) and thus also Master Theorem B. Together with Master Theorem A, we may conclude the general version of the Serre finiteness theorem, i.e. Theorem 13. As a corollary, we get the theorem of interest.

▶ **Theorem 40** (The Serre finiteness theorem). *For $n, m \geq 0$, the abelian groups $\pi_n(\mathbb{S}^m)$ are finitely presentable.*

Proof. First, as S^m has two (when $m = 0$) or one (when $m > 0$) connected components, $\pi_0(\mathbb{S}^m)$ is obviously finitely presentable. Next, we consider the case when $n = 1$. We have

$$\pi_1(\mathbb{S}^m) \cong \begin{cases} \mathbb{1} & \text{if } m = 0 \\ \mathbb{Z} & \text{if } m = 1 \\ \mathbb{1} & \text{if } m > 1 \end{cases}$$

The first case follows from the fact that $\mathbb{S}^0$ is a set, the second case is well-known in HoTT [23], and the third case follows from the fact that $\mathbb{S}^m$ is 1-connected when $m > 1$. Hence $\pi_1(\mathbb{S}^m)$ is finitely presentable.

Let us now consider the case when $n > 1$. If $m \leq 1$, we have $\pi_n(\mathbb{S}^m) \cong \mathbb{1}$ since both $\mathbb{S}^0$ and $\mathbb{S}^1$ are (at least) 1-truncated, so let us assume that $m \geq 2$. In this case, we have that $\mathbb{S}^m$ is (at least) 1-connected. It is also stably almost finite (since it is a finite CW complex). Hence, by Theorem 13, it follows that $\pi_n(\mathbb{S}^m)$ is finitely presentable, and we are done. ◀

4 On the formalisation

In this section, we will discuss the formalisation, highlighting both difficulties and simplifications that arose during the process. The final step of the proof depends, as we've seen, on the following lemmas which we will (loosely) centre our discussion here around (the first two of which correspond roughly to Master Theorem A):

▶ **Proposition (31).** *If $F \to E \to B$ is a fibre sequence, and B is 1-connected, and F and B are stably almost finite, then so is E.*

▶ **Proposition (35).** *If A is a finitely presentable abelian group, then the Eilenberg–Mac Lane spaces $\mathsf{K}(A, n)$ are stably almost finite for $n \geq 1$.*

▶ **Theorem (B).** *If X is stably almost finite and $(n-1)$-connected, then $\pi_n(X)$ is finitely presentable.*

Here they are again, this time in `Cubical Agda`:

```
Proposition-31 : FibSeq F E B → is 1 -Connected B → SAF F → SAF B → SAF E

Proposition-35 : isFP A → n ≥ 1 → SAF K[ n , A ]

Theorem-B : SAF X → is (n - 1) -Connected X → isFP (πₙ X)
```

Propositions 31 and 35 were the first targets of the formalisation. Following a modular approach to formal proof development, we first introduced the definition of stably almost finite spaces as a "black box" and postulated a number of other crucial lemmas and definitions. Relying on these postulates, we formalised the final steps of the proof of Serre finiteness first.

In order to prove Propositions 31 and 35, we needed to know that loop spaces of 1-connected stably almost finite types are again stably almost finite, i.e. Proposition 30. We noticed that this proof is formally very similar to the proof of Proposition 31, and so, anticipating a similar level of difficulty, we chose to leave this proposition postulated too. With these postulates, however, we were confident that we could attack Proposition 35 in a straightforward and principled manner. This process of postulating results until we ended with something that felt feasible was an important part of "testing the waters" in the early stages of the formalisation project and helped us gauge, early on, how much work would be needed and which parts appeared to require special focus.

4.1 Getting started: finitely presentable abelian groups

In order to formalise Proposition 35, we needed to define what it means for an abelian group to be finitely presentable. However, also this felt premature, as we wished to coordinate our formalisation with Ljungström and Pujet's formalisation of CW complexes [25]. Their formalisation was to implement similar machinery but was not finished at the time. Yet, we still wished to work with finitely presentable abelian groups (and needed more information about these than a mere postulated definition). The solution was to postulate a definition of finitely presentable abelian groups *together* with an induction principle[2]:

▶ **Proposition 41** (Induction for finitely presentable abelian groups). *Let P be a proposition defined over all groups and suppose that P holds for all cyclic groups (i.e. for $\mathbb{Z}$ and the groups $\mathbb{Z}/n\mathbb{Z}$), and that if P holds for groups H and K, then it holds for $H \oplus K$. In this case, P holds for all finitely presentable abelian groups.*

This induction principle made the theory of finitely presentable abelian groups tractable for constructive, computer formalisation. In general it is clear that, where a classical mathematician working with pen-and-paper might find it most conceptually convenient to unfold a definition directly, for the constructive computer mathematician, an induction principle of this kind is far more convenient and allows for the principled and modular development of the proof to proceed. It effectively captures all the information relevant to us about the finitely presentable abelian groups without committing to a certain implementation.

Armed with this principle, Proposition 35 boiled down to showing that Eilenberg–Mac Lane spaces preserve products. However, this was easy to show since the theory of these spaces already had been extensively formalised in the `agda/cubical` library [26]. Naturally, we also needed to show that products of stably almost finite types are stably almost finite, i.e. Proposition 26. This became another item in our list of postulates.

4.2 Stably almost finite types and their closure properties

Once we had completed this part of the development, we started working our way backwards toward Proposition 31. We still did not have a concrete implementation of either finite CW complexes, or stably almost finite types. However, we noted that its proof relies heavily on

[2] It should be noted that the fact that the definition of finitely presentable abelian groups satisfies this induction principle is a consequence of the existence of the *Smith normal form*. Fortunately, this result was available to us in the `agda/cubical` library thanks to the efforts of Kang Rongji [19].

Ganea's theorem and iterated applications of it (i.e. results like Theorem 27 and Corollary 28). These results are, to a large extent, self-contained, so it made sense to make them our next target. They quickly turned out to be straightforward and the Ganea theorem could be formalised rather directly.

With this, it was time to consider applications – in particular Proposition 31. With the Ganea theorem formalised, the `Agda` proofs of Propositions 30 and 31 came to about 400 lines; this should be compared with the roughly 80 lines above spent on the proofs on paper – a ratio of 5-to-1. Obviously, the absence of diagrams makes the `Agda` proofs less legible, but it should be pointed out that many of the sub-lemmas that we used to break down the proofs each take up no more than about five lines (statement and proof). In particular, we never have to unfold any large terms. Overall, the proofs are structured in a rather principled way that is meant to impose a relatively light cognitive load on anyone hoping to read the formalised proof.

In this discussion, we are, however, brushing one important fact (which we more-or-less suppressed in the write-up above) under the rug, namely that the fibres in the sequence we get by applying Corollary 28 are stably k-finite; this is because they are joins of stably k'-finite spaces (for suitable k and k'). About 135 lines of code are spent on the proof that joins of stably almost finite spaces are stably almost finite (together with another 60 lines showing that joins of connected maps are connected). We can compare the number 135 with the 40 lines or so spent on the written proof above, the ratio is about 4-to-1, i.e. roughly the same as in the previous paragraph. However, we can report empirically that closure under joins (i.e. Propositions 23 and 24) generally took more time and effort to formalise than closure under fibre sequences (i.e. Propositions 31 and 32). That the proofs of closure under joins are more challenging is reflected in the denser and longer sub-lemmas that they are broken down into in the formalisation (in comparison with what is described above for closure under fibre sequences). It is also worth noting that some of these lemmas for closure under joins were incorrectly stated when they were originally postulated, and the final version of the closure property that appears in the Barton–Campion paper [2] was arrived at only in the later iterations of that paper.

One of the closure properties we mentioned before is that stably almost finite types are closed under products – a result that follows from the corresponding theorem for finite CW complexes. At this point, we had instantiated the definition of stably almost finite types, but the definition of CW complexes was still only postulated. We also knew that the formalisation of CW complexes by Ljungström and Pujet would not contain this result, so we would have to formalise it ourselves. There is an "obvious" way of constructing the product of two CW complexes (see e.g. [14, Theorem A.6]), but this would be rather cumbersome to formalise (and especially difficult without a concrete definition of CW complexes in mind). Faced with this problem, we were reminded of our previous successes with the induction principle for finitely presentable groups; the idea now was to phrase a similar principle for *finite* CW complexes. This gave rise to the following result (which also ended up in the most recent version of Barton and Campion's paper [2] – this is a situation where the formalisation inspired the pen-and-paper proof):

▶ **Proposition 42** (Induction for finite CW complexes). *Let* $P : \mathsf{FinCW} \to \mathsf{Type}$ *be a family of propositions such that*

- $P\,\bot$ *and* $P\,\mathbb{1}$ *hold, and*
- *for every span of finite CW complexes* $B \xleftarrow{f} A \xrightarrow{g} C$ *s.t.* $P\,A$, $P\,B$ *and* $P\,C$ *hold, we also have that* $P\,(\mathsf{Pushout}\,f\,g)$ *holds.*

In this case, P *holds for all finite CW complexes.*

This proposition falls out immediately from the definition of finite CW complexes but still makes many proofs incredibly straightforward. In particular, using that pushouts commute with products, it can be used to give a short proof of the fact the class of finite CW complexes is closed under products. Naturally, this requires that this class also is closed under pushouts – a particularly involved result. Fortunately for us, the formalisation of this result would be made available to us via the work of Ljungström and Pujet [25].

4.3 Finishing up: CW complexes and the weak Hurewicz theorem

The final stretch of the formalisation concerned merging the then-recently finished formalisation of CW complexes and the Hurewicz theorem by Ljungström and Pujet. At this point, the two remaining postulates were the definitions of CW complexes and of finitely presentable abelian groups. We chose to implement these definitions following Ljungström and Pujet, and proving the previously mentioned induction principles for CW complexes and finitely presentable abelian groups was straightforward. Hence, all that remained now was to formalise Master Theorem B.

Because homology groups of finite CW complexes can be computed relatively directly (see [25]), the original plan for the formalisation was to derive the necessary facts about the homotopy groups of connected finite CW complexes from the Hurewicz theorem (Theorem 14).

▶ Remark 43. As stated earlier, the Hurewicz theorem (in full generality) had already been proved in HoTT by Christensen and Scoccola [9] who also formalised their proof. However, their formalisation took place in the `Rocq` proof assistant [34], and we expected that importing their formalisation directly into `Cubical Agda` would take significantly more time than using the special case formalised by Ljungström and Pujet. In addition, Christensen and Scoccola's definition of homology has not been proved to satisfy the Eilenberg–Steenrod axioms, without which it is unclear how we could show that the homology groups of finite CW complexes are finitely presented. Another reason for choosing to work with the formalisation of [25] is that it, to a certain extent, is designed with computation in mind, which is important if one hopes to use the Serre finiteness theorem to run homotopy group computations.

Work began to complete the proof of the Serre finiteness theorem by appealing to the anticipated Hurewicz theorem formalisation. However, when doing this, we quickly realised that the "weak" Hurewicz theorem is all we need and, upon inspecting Ljungström and Pujet's initial formalisation closer, this theorem was hidden away in a where-clause and was only implemented as a lemma for proving the full Hurewicz theorem. This is another case where our formalisation had an impact on another paper/formalisation, as it made Ljungström and Pujet restructure their proof, putting greater emphasis on this result (that they originally had not fully realised was useful in its own right). To clarify: in general, for an arbitrary type, the weak Hurewicz theorem is a corollary to the Hurewicz theorem above. However, in the special case of (finite) CW complexes, the weak Hurewicz theorem can be proved directly, and the full Hurewicz theorem follows as a corollary. This is the approach taken, for example, by Allen Hatcher in his algebraic topology textbook [14, Example 4.29, Theorem 4.32]. The more general proof of the Hurewicz theorem given by [9] used much more high-tech, heavy-duty machinery (a careful analysis of the smash product of maps, Ω-prespectra, ...). Instead, our approach led to the total avoidance of homology, which is in sharp contrast to the original proof of Serre, and other proofs that have appeared subsequently (see e.g. [29]). As a matter of fact, homology for CW complexes was formalised as our work was in progress, but it need not have been. In this sense, the finiteness of the homotopy groups is a "purely homotopical" phenomenon, and does not depend upon the

auxiliary concept of homology for its proof. This shows how our modular – to be frank, somewhat lazy – approach to formalisation (i.e. working backwards whenever a theorem, such as the Hurewicz theorem, appears difficult to formalise, and working with postulates for as long as possible) actually led to an unlooked-for discovery about the mathematics we were formalising.

5 Using our formalisation to compute homotopy groups of spheres

One of the most exciting prospects of using `Cubical Agda` to formalise the Serre finiteness theorem is its computational support. Indeed, the Serre finiteness theorem promises, for each $\pi_n(\mathbb{S}^m)$, a list of integers $k, p_1, \ldots, p_l$ s.t. $\pi_n(\mathbb{S}^m) \cong \mathbb{Z}^k \oplus \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_l\mathbb{Z}$ (this version of finite presentability follows immediately from Proposition 41) and, if we are interested in finding out exactly what this list looks like, we should simply be able to *normalise* it in `Cubical Agda`. The idea of being able to compute (in the computer scientist's sense of the word) the homotopy groups of spheres is of course not new: Brown showed already in 1957 that the homotopy groups of spheres are computable [4]. However, he also writes:

> "*It must be emphasised that although the procedures developed for solving these problems are finite, they are much too complicated to be considered practical.*"

Other more modern approaches exist. Indeed, the computations of *stable* homotopy groups (using the machinery of spectral sequences) have been studied in e.g. [20] and [17]. One must also mention the `Kenzo` software [30] which is capable of computing a large class of homotopy groups in a surprisingly effective way.

What sets a possible `Cubical Agda` computation apart from these traditional approaches is, first and foremost, that the `Cubical Agda` computation is a *certified* computation – if we believe that our proof assistant is correctly implemented, then we better also believe that the output of such a computation is correct. In addition, results proved in HoTT constitute vast generalisations of their classical counterparts, since they hold in arbitrary ∞-toposes.[3] A successful computation in a HoTT-based language only makes use of constructive and entirely synthetic constructions, and it is remarkable that these are enough to prove a theorem as strong as the Serre finiteness theorem.

Unfortunately, despite these prospects, our current attempts at using the Serre finiteness theorem for homotopy group computations have been unsuccessful. We tried the simplest non-trivial example possible: computing the number of generators in the finite presentation of $\pi_2(\mathbb{S}^2)$. In `Cubical Agda`, we can write the definition of the number of generators computed using the SFT as in the following snippet, which we, to avoid technicalities, have typeset in a rather liberal manner omitting a propositional truncation. As cubical type theory satisfies canonicity, the propositional truncation of a natural number should compute to the application of the propositional truncation constructor to a numeral [16]. The propositional truncation does hence not make much of a difference for closed terms like this and we can safely omit it from this informal discussion.

```
nGens-π₂S² : ℕ
nGens-π₂S² = isFP→nGens (SFT 2 2)
```

[3] It should be mentioned here that the status of the interpretability of the cubical type theory implemented by `Cubical Agda` in ∞-toposes is not settled; this is an active area of research.

Despite our best efforts, even this "simple" example does not compute. This is perhaps not surprising to anyone who has followed previous attempts at similar computations in `Cubical Agda`, e.g. the (in)famous Brunerie number [5] (i.e. the number n s.t. $\pi_4(\mathbb{S}^3) \cong \mathbb{Z}/n\mathbb{Z}$) which, just like the number above, was given a constructive implementation in `Cubical Agda` but has still not been successfully normalised using Brunerie's original definition, even 10 years after its publication. However, as shown by Ljungström and Mörtberg [24], there is still hope: despite the Brunerie number not computing in its original form, it is still possible to simplify its definition until it does compute. We hope that similar simplifications can be made to the Serre finiteness theorem – at the very least to low-dimensional special cases of it – in order to improve the situation.

6 Conclusion and future work

In this paper, we have presented a full formalisation of the Serre finiteness theorem in `Cubical Agda`. We have also seen how the formalisation, in addition to constituting a significant contribution to the field of formalised homotopy theory in HoTT-based proof assistants, also provided insights on how the pen-and-paper proof could be substantially simplified.

Future work

An obvious direction for future work is to try harder to use the formalisation to compute presentations. As briefly discussed above, the propositional truncation does not make much a difference when attempting to compute closed terms, but it would still be theoretically interesting to eliminate it and prove the stronger result that all homotopy groups of spheres are finitely presented (as a structure). This should be possible as the Smith normal form is unique (up to multiplication by units). This result has previously been formalised in `Rocq` [7] and it should be possible to port it to `Cubical Agda`.

Another direction of future work is to prove Serre's original formulation of the theorem which would provide even more information about homotopy groups of spheres in HoTT. For instance, it tells us when these groups are actually finite. This would most likely involve developing some rational homotopy theory in HoTT, which would be interesting in its own right.

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