



Quantum critical theories in a periodic potential: strange metallic thermoelectric and magnetotransport

Downloaded from: <https://research.chalmers.se>, 2026-08-31 06:51 UTC

Citation for the original published paper (version of record):

Nilsson, E., Schalm, K. (2026). Quantum critical theories in a periodic potential: strange metallic thermoelectric and magnetotransport. *Journal of High Energy Physics*, 2026(7).
[http://dx.doi.org/10.1007/JHEP07\(2026\)183](http://dx.doi.org/10.1007/JHEP07(2026)183)

N.B. When citing this work, cite the original published paper.

RECEIVED: January 13, 2026

REVISED: April 15, 2026

ACCEPTED: May 20, 2026

PUBLISHED: July 21, 2026

Quantum critical theories in a periodic potential: strange metallic thermoelectric and magnetotransport

E. Nilsson^a and K. Schalm^b

^a*Department of Physics and Astronomy, Chalmers University of Technology,
SE-412 96 Göteborg, Sweden*

^b*Instituut-Lorentz, ΔITP, Universiteit Leiden,
P.O. Box 9506, 2300 RA Leiden, The Netherlands*

E-mail: nieric@chalmers.se, kschalm@lorentz.leidenuniv.nl

ABSTRACT: We study DC and AC thermoelectric and magnetotransport in 2D quantum critical theories with strong translational symmetry breaking due to a varying chemical potential lattice with zero average $\bar{\mu} = 0$. The combination of quantum criticality and the absence of the average natural scale implies that such systems have idiosyncratic signatures that may apply more generally when the variance in the lattice potential far exceeds the average or for strong translational symmetry breaking in general. We model such theories holographically through near-extremal AdS black holes. We find that these systems (a) become *better* conductors. In a 2D lattice, this can be explained by currents flowing around obstacles; (b) exhibit bad-metal electrical transport with Drude-like thermal transport, though it is not Drude, and, notably, (c) display an approximately B -linear longitudinal magnetoresistance at large fields, similar to Effective Medium Theory. We comment on how these results may apply when $\bar{\mu} \neq 0$.

KEYWORDS: Holography and Condensed Matter Physics (AdS/CMT), Holography and Hydrodynamics

ARXIV EPRINT: [2512.19480](https://arxiv.org/abs/2512.19480)

Contents

1	Introduction	1
2	Review: holographic model of strongly correlated quantum criticality	4
2.1	Extracting quantum critical responses	5
2.2	Extracting AC transport	6
3	Transport expectations from hydrodynamics	7
4	Thermoelectric transport in 1D charge-neutral lattices	8
4.1	DC transport in 1D charge-neutral lattices	8
4.2	AC transport in 1D charge-neutral lattices	11
5	Thermoelectric transport in 2D charge-neutral lattices	15
5.1	DC transport in 2D charge-neutral lattices	15
5.2	AC transport in 2D charge-neutral lattices	19
5.3	DC magnetotransport in 2D charge-neutral lattices	21
6	Conclusion and outlook	23
A	Numerical method	25
A.1	Numerical convergence	25

1 Introduction

Essentially all critical points studied in real materials require fine tuning to a very special point among all its possible configurations. Nevertheless, this special point is of fundamental physical interest, due to universality: the physics at a critical point is independent of its microscopic origins. The particularities of the material in question no longer matter as long as one is able to tune to the special point. If the material is a (crystalline) solid, the crystal lattice is generically such a particularity. In the formal sense, the lattice is irrelevant under the renormalization group flow to the critical point. Precisely because of this fact, a lattice is often used in numerical analyses as a regulator.

Quantum critical theories, where the critical point exists at $T = 0$, have the added factor that the critical behavior extends fan-wise into the phase diagram at finite temperature T [1]. The temperature T acts as an IR regulator, and probes of the system at an energy scale less than T are controlled by collective hydrodynamic rather than pure scale-invariant and collisionless behavior. Importantly, the onset of the hydrodynamic regime halts the RG flow, and whatever remains of the lattice is imprinted in the response. A striking example is a weak chemical potential lattice, $\mu(x) = \bar{\mu} + \delta\mu \cos(Gx)$, with lattice momentum $G < \bar{\mu}$ [2]: in this case the IR is described by Umklapp hydrodynamics based on Bloch waves, rather than conventional hydrodynamics based on Fourier modes. When instead $G > \bar{\mu}$, the conventional notion that the lattice is irrelevant holds.

A chemical potential lattice also has a third scale $\delta\mu$, encoding the size of the lattice variations. Intuitively by real metals with a finite density of valence electrons, one normally studies the weak lattice regime, where the average chemical potential is parametrically larger than the size of these variations; $\bar{\mu} \gg \delta\mu$. For quantum critical theories placed in such a lattice, all changes w.r.t. the translational invariant theory are then encoded as functions of two dimensionless parameters $\bar{\mu}/G$ and $\delta\mu/G$.

Chesler, Lucas, and Sachdev realized that the study of the strong lattice regime $\delta\mu \gg \bar{\mu}$ in such critical theories becomes tractably simpler by taking the extremal value $\bar{\mu} = 0$ [3]. This eliminates a background scale, simplifying the structure of the solutions and opening the possibility of universal transport regimes exhibiting scaling collapse. This may therefore serve as a testing ground for strongly correlated physics near or at quantum criticality where translational symmetry breaking is strong. In this article we extend their findings of electronic transport in this special regime from 1D to 2D lattices, covering DC and AC thermal, thermoelectric, and magnetotransport. As Chesler, Lucas and Sachdev, we use holographic models of quantum criticality to compute these responses.

For an experimental realization of such physics, graphene at charge neutrality naturally comes to mind. Indeed, many studies have considered the scenario of a spatially periodic chemical potential imposed on $\bar{\mu} = 0$ charge neutral graphene either in terms of large scale ripples (a 1D periodic potential) or an egg-carton landscape (a 2D periodic potential), see e.g., [4–11]. Most of these adopt a non-interacting or weakly interacting point of view, even though charged excitations in graphene are arguably not weakly coupled. For 1D potentials, these weak coupling results agree qualitatively with the strong coupling approach in [3]. We shall show that this is not so for a 2D periodic potential, where the weak coupling approach surprisingly predicts no change in electrical transport [8]. Our approach at strong coupling does however show distinct physics that also differs from the 1D case; thus, both interactions and dimensionality matter.

Due to its quantum critical-like physics at charge neutrality, graphene is in fact an “almost strange metal” [12, 13]. Our findings are particularly relevant for the truly strange metal phase in high- T_c cuprate superconductors in light of the hypothesis that its phenomenology is described by such quantum critical physics in the presence of a 2D lattice; see e.g., [14, 15] for a review. This was recently exemplified by computations in a 2D Yukawa-SYK model, which has a holographic dual. This model reproduces the observed linear-in- T DC conductivity, an AC conductivity with apparent $\omega^{-2/3}$ mid-IR scaling, $T \ln T$ specific heat, Homes’ law relating the $T = 0$ superfluid density to $T_c \sigma_{\text{DC}}(T_c)$, and broad ARPES width from marginal Fermi liquid self-energies $\Sigma(\omega) \sim \omega \ln \omega$ [16]. The presence of translational symmetry breaking — in [16] mainly through disorder in addition to the lattice — is paramount in order to obtain finite and properly scaling transport properties. However, one important phenomenological observation is hitherto unexplained: the magnetotransport. No models as of yet are able to convincingly explain the observed Hall angle scaling $\frac{\sigma_{xx}}{\sigma_{xy}} \sim T^2$. In systems dominated by a single momentum relaxation rate, this rate determines both the Hall angle and DC resistivity and these must scale the same way with temperature. Anderson already pointed out that to explain this observed difference in the cuprate strange metal one needs a second relaxation rate, which is not trivial [17]. As one of us has recently shown [18], extending

Anderson’s earlier arguments to a full hydrodynamic analysis requires a system in which momentum relaxation through lattice Umklapp or disorder is *not* the only parametrically small relaxation rate, independently of whether the microscopics is strongly or weakly coupled. An important ingredient in reaching this conclusion is the AC response, in addition to the DC response. While analyses based solely on DC transport have been remarkably successful in describing strange metals [19, 20], such theories typically rely on a separation of mechanisms: the longitudinal DC resistivity is assumed to be of quantum critical origin, whereas the Hall conductivity is not. This effectively corresponds to scenarios in which the system behaves as if it were charge neutral, or exhibits strong translational symmetry breaking for longitudinal transport but weak symmetry breaking at finite charge density for Hall transport. This picture is *prima facie* in tension with measurements of the AC longitudinal optical conductivity, which display a clear Drude peak at low temperatures. The search for systems that exhibit both Drude-like AC optical conductivity and anomalous Hall transport therefore motivates our study of systems in the extreme limit of strong translational symmetry breaking.

Experiments on certain cuprates (and pnictides) also reveal a longitudinal magnetoresistance on the form [21–23]

$$\rho(B, T) - \rho(0, 0) \propto \sqrt{(\alpha k_B T)^2 + (\gamma \mu_B B)^2} \quad (1.1)$$

where α and γ are two dimensionless parameters. This leads to an unsaturating B -linear magnetoresistance at large magnetic fields, which stands similarly in stark contrast to the B^2 -scaling of conventional Fermi liquids that saturates to a constant at large fields [24]. B -linear magnetoresistance has been shown to be a general feature of systems with large spatial fluctuations in the conductivity, extending down to moderate magnetic fields in the case of strong inhomogeneities [25]. In that case, such systems can be modeled as random resistor networks, which has been shown to be equivalent to “Effective Medium Theory” (EMT) [26]. Within EMT, one considers the macroscopic response of a system constituting of multiple subdomains with varying conductive properties, and may be viewed as an extreme form of translational symmetry breaking. The case of two different types of subdomains at an equal area fraction yields an exactly B -linear magnetoresistance [27].¹ Whether all strange metals exhibit a magnetoresistance on quadrature form as in eq. (1.1) is questioned [29]; nevertheless, a roughly B -linear, unsaturating magnetoresistance appears to be a robust feature.

Both of these magnetotransport observations suggest that systems with strong translational symmetry breaking — whether due to an underlying lattice or disorder — should be considered to explain the phenomenology of cuprate strange metals. This provides an additional motivation to study quantum-critical systems in the charge-neutral regime, where the spatially averaged chemical potential vanishes ($\bar{\mu} = 0$) but fluctuates locally, $\delta\mu(\vec{x}) \neq 0$, beyond purely the fundamental theoretical interest in its universal response. Moreover, at charge neutrality, thermal and electrical transport naturally decouple. In such regimes there are therefore two different independent transport mechanisms that are important in the IR. The resulting universal features are most naturally captured within a holographic framework, which generalizes Landau-Ginzburg theory rather than relying on a specific

¹In holographic approaches there exist so-called DBI models which predict a resistivity of the form eq. (1.1) [28], but the physical origin is an underlying relativistic symmetry rather than an averaging.

SYK-like model [14]. This is the analysis we pursue in this paper. We provide a complete numerical analysis of the AC and DC thermoelectric response of quantum critical conformal field theories subject to a spatially modulating chemical potential with zero average. In particular, we consider both 1D (section 4) and 2D (section 5) “lattices” to find that the dimension of the lattice plays an important role, altering the IR fixed point of the theory, even though both are ultra-local theories evading any RG-like c -theorem as found by [3]. We find that electrical transport is completely incoherent, whereas the thermal transport is predominantly coherent and Drude-like. Turning on a magnetic field, we find preliminary evidence of B -linear magnetotransport at large fields. Our setup naturally suggests connections to Effective Medium Theory descriptions, and in section 6 we conclude with an outlook how a combination of this simplified regime with one dimensionless parameter voided provides an obvious direction for further study.

2 Review: holographic model of strongly correlated quantum criticality

To model a generic (large N) 2D strongly quantum critical system, we use its holographic dual description as dynamical gravity in 4D asymptotically anti-de-Sitter space.² Those familiar with this approach may skip directly to section 3. Including a Maxwell field dual to the global $U(1)$ of electromagnetism, we consider the AdS_4 action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \quad (2.1)$$

with $\Lambda = -3/L^2$ the negative cosmological constant; we set the AdS length scale $L = 1$ without loss of generality. The equations of motion in the bulk are the (trace-reversed) Einstein’s equations and Maxwell’s equations,

$$\begin{aligned} R_{\mu\nu}^H &= \kappa^2 \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right), & 2\kappa^2 T_{\mu\nu} &= F_{\mu}{}^{\rho} F_{\nu\rho} - g_{\mu\nu} \left(\frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} + 2\Lambda \right), \\ \nabla_{\mu} F^{\mu\nu} + \nabla^{\nu} \varphi &= 0, & \varphi &= \nabla_{\mu} (A^{\mu} - \bar{A}^{\mu}) + \xi^{\mu} (A_{\mu} - \bar{A}_{\mu}). \end{aligned} \quad (2.2)$$

We employ the de Turck trick to make the equations elliptic and the boundary value problem well-defined by substituting the Ricci tensor with the Harmonic Ricci tensor: $R_{\mu\nu}^H = R_{\mu\nu} - \nabla_{(\mu} \xi_{\nu)}$ [30]. This imposes the gauge $\xi^{\mu} = 0$, where $\xi^{\mu} = g^{\lambda\nu} (\Gamma_{\lambda\nu}^{\mu} - \tilde{\Gamma}_{\lambda\nu}^{\mu})$ is the de Turck vector and $\tilde{\Gamma}_{\nu\rho}^{\mu}$ is the Christoffel symbol for a suitably chosen background reference metric, which we take to be the analytical solution for an isotropic system ($\delta\mu = 0$). In the absence of a background magnetic field, this reference geometry is the AdS_4 -Reissner-Nordström solution, whereas for finite magnetic fields we consider the dyonic Reissner-Nordström black brane. We also add a similar gauge-fixing term to Maxwell’s equations in eq. (2.2), enforcing $\varphi = 0$, where $2\bar{A} = B(x dy - y dx)$ [31, 32]. This ensures that one can consistently periodically identify the spatial directions x and y as is necessary for the numerics, and imposes a modified Lorenz gauge for the periodic part $A_{\mu} - \bar{A}_{\mu}$ of the background gauge field.

²These quantum critical theories have a relativistic Lorentz invariance in the UV. This is the Lorentz invariance extended from quantum critical excitations with linear dispersion, where the emergent velocity ω/k plays the role of the speed of light. For most observables this Lorentz symmetry is broken in the IR.

We solve these equations numerically. A well-converged numerical solution to the modified equations eq. (2.2) leads to a simultaneous solution of the Einstein-Maxwell equations and the gauge constraints. We verify the convergence of our numerical algorithm by an a posteriori check that φ and ξ^2 are numerically close to zero, as is detailed in appendix A. In the most general case of a 2D lattice and a finite magnetic field B , our ansatz for the background fields is

$$\begin{aligned} ds^2 &= \frac{L^2}{z^2} \left[-f(z) Q_{tt} \eta_t^2 + Q_{xx} \eta_x^2 + Q_{yy} \eta_y^2 + \frac{Q_{zz}}{f(z)} dz^2 \right], \\ \eta_t &= dt + Q_{tx} dx + Q_{ty} dy + Q_{tz} dz, \quad \eta_x = dx + Q_{xy} dy + Q_{xz} dz, \\ \eta_y &= dy + Q_{yz} dz, \quad f(z) = (1-z) \left(1 + z + z^2 - \frac{\bar{\mu}^2 + B^2}{4} z^3 \right), \\ A &= (1-z) a_t dt + \left(a_x - \frac{B}{2} \right) dx + \left(a_y + \frac{B}{2} \right) dy + a_z dz, \end{aligned} \quad (2.3)$$

where $Q_{\mu\nu} = Q_{\mu\nu}(x, y, z)$, $a_\mu = a_\mu(x, y, z)$ and z is an inverted radial coordinate, such that the AdS conformal boundary sits at $z = 0$. This ansatz describes a dyonic black hole of temperature

$$T = \frac{12 - \bar{\mu}^2 - B^2}{16\pi}, \quad (2.4)$$

where we have set the black hole horizon location $z_h = 1$. For nearly all the data presented, we limit ourselves to the universal regime where $\bar{\mu} = 0$. When $B = 0$, we reduce the complexity of the ansatz by directly setting $Q_{ti} = a_i = 0$, where $i = \{x, y, z\}$.³ In the UV ($z = 0$), we demand that the metric asymptotes to AdS_4 and that the time component of the gauge field equals the spatially modulating chemical potential, i.e.,

$$a_t = \mu(\vec{x}), \quad a_i = 0, \quad Q_{ti} = Q_{ij}|_{i \neq j} = 0, \quad Q_{\nu\nu} = 1, \quad \nu = \{t, x, y, z\}. \quad (2.5)$$

The IR boundary conditions ($z = z_h$) follow from algebraically solving the equations of motion on the horizon, which leads to the Dirichlet condition $(Q_{tt} - Q_{zz})|_{z=z_h} = 0$ and Robin-type boundary conditions for the other fields. The resulting non-linear equations of motion are solved numerically on the domain $x, y \in [0, 2\pi/G)$, $z \in [0, 1]$ with periodic boundary conditions in the transverse x and y directions. Further details on the numerical implementation can be found in appendix A.

2.1 Extracting quantum critical responses

Following refs. [33–36], we obtain the DC thermoelectric transport coefficients in the dual quantum critical theory by solving a Stokes flow problem on the horizon on the black hole for a given bulk geometry. This reduces the number of degrees of freedom needed to be solved for as compared to an AC calculation, drops the degree of the PDE by one, and avoids the need to take the numerical $\omega \rightarrow 0$ limit of AC data. The Stokes equations read [2, 35]

$$\begin{aligned} 2\sqrt{h} \nabla^j \nabla_{(i} v_{j)} + d\chi_{ij}^H Q^j + F_{ij}^H J^j + s^H T \zeta_i - \sqrt{h} \nabla_i p + n^H (E_i + \nabla_i w) &= 0, \\ \nabla_i Q^i &= 0, \quad \nabla_i J^i = 0, \end{aligned} \quad (2.6)$$

³For 1D lattices we also drop the y -dependence and set $Q_{xy} = Q_{yz} = 0$.

where $i = \{x, y\}$. These equations, sourced by an electric field E_i and a thermal gradient ζ_i , are to be solved for the four unknowns v_x, v_y, w and p , which are functions of x and y . The electric and heat currents J^i and Q^i and the transport coefficients depend on these functions as

$$\begin{aligned} J^i &= n^H v^i + \sqrt{h}(E^i + \nabla^i w + F_H^{ij} v_j), \quad Q^i = s^H T v^i, \\ s^H &= 4\pi\sqrt{h}, \quad n^H = \sqrt{h} \frac{a_t}{Q_{tt}}|_{z=z_h}, \quad \chi_i^H = Q_{ti}|_{z=z_h}, \quad F_{ij}^H = 2\partial_{(i} A_{j)}|_{z=z_h}, \end{aligned} \quad (2.7)$$

where indices are raised and lowered with respect to the horizon metric $h_{ij} = g_{ij}|_{z=z_h}$. The key idea of [35] is that the spatial averages $\bar{J}^i$ and $\bar{Q}^i$ (taken over the unit cell) do not renormalize as one moves from the horizon to the boundary. The DC thermoelectric response functions in the boundary theory may therefore be extracted from

$$\begin{pmatrix} \bar{J}_i \\ \bar{Q}_i \end{pmatrix} = \begin{pmatrix} \sigma_{ij} & T\alpha_{ij} \\ T\bar{\alpha}_{ij} & T\bar{\kappa}_{ij} \end{pmatrix} \begin{pmatrix} E^j \\ \zeta^j \end{pmatrix}. \quad (2.8)$$

In our numerics, we fix either the electric field E^i or the thermal gradient $\zeta^i = -\nabla^i T/T$ and study the extracted electric and heat currents J^i and Q^i . In the presence of a nonzero magnetic field B , the boundary electric and heat currents decompose into transport and magnetization pieces where the latter should be subtracted in order to obtain the physical transport currents that couple to the external electric field or thermal gradient [37, 38]. However, the magnetization currents vanish at the horizon [39], and as a result, the horizon currents computed via the Stokes flow method directly yield the physical DC conductivities.

2.2 Extracting AC transport

For transport at finite frequency, we consider linear perturbations of the background geometry. In the linearized versions of the bulk equations of motion eq. (2.2), the linearized de Turck term reads

$$\nabla_{(\mu} \tau_{\nu)}, \quad \text{where} \quad \tau_\mu = \nabla^\nu \delta(g_{\mu\nu}) - \frac{1}{2} \nabla_\mu \delta(g^\nu{}_\nu), \quad (2.9)$$

which imposes the de Donder gauge $\tau_\mu = 0$. Additionally, we add a gauge-fixing term to the left-hand side of the Maxwell equations as

$$\delta(\nabla_\nu \nabla^\nu A_\mu - \nabla_\mu \nabla_\nu A^\nu) + \nabla_\mu \nabla^\nu \delta(A_\nu) = 0, \quad (2.10)$$

imposing the Lorenz gauge $\nabla^\nu \delta(A_\nu) = 0$. This ensures that each of the fields in the ansatz below has a kinetic term (i.e., the equations of motion contain a term $\partial_z^2 \delta\Phi$ for all perturbations $\delta\Phi$). We make an ansatz for the perturbed metric and gauge field as

$$\delta(g_{\mu\nu}) = \frac{e^{-i\omega t}}{z^2} \delta g_{\mu\nu}(x, y, z), \quad \delta(A_\mu) = e^{-i\omega t} \delta A_\mu(x, y, z), \quad (2.11)$$

which in the most general case leads to 14 unknown functions $\{\delta g_{tt}, \dots, \delta g_{zz}, \delta A_t, \dots, \delta A_z\}$ to be solved for. The IR boundary conditions are obtained by factoring out an infalling factor and algebraically solving the equations of motion at the horizon. The UV boundary conditions impose either a source for the charge (fixing δA_x) or for the heat (fixing δg_{tx}),

setting the remaining fields to equal zero. Although the resulting system of equations is linear, it is in general much more numerically challenging to solve than the non-linear equations for the background.

We compute the frequency-dependent thermoelectric response from retarded two-point functions of the conserved $U(1)$ current J^μ and the energy-momentum tensor $T^{\mu\nu}$. The (boundary) heat current is defined by

$$Q^i = T^{ti} - \mu J^i. \quad (2.12)$$

With our conventions, we obtain

$$\begin{aligned} \sigma^{ij}(\omega) &= \frac{i}{\omega} \left[G_{J^i J^j}^R(\omega, k=0) - \text{Re } G_{J^i J^j}^R(\omega=0, k \rightarrow 0) \right], \\ T\alpha^{ij}(\omega) &= \frac{i}{\omega} \left[G_{J^i Q^j}^R(\omega, k=0) - \text{Re } G_{J^i Q^j}^R(\omega=0, k \rightarrow 0) \right], \\ T\bar{\kappa}^{ij}(\omega) &= \frac{i}{\omega} \left[G_{Q^i Q^j}^R(\omega, k=0) - \text{Re } G_{Q^i Q^j}^R(\omega=0, k \rightarrow 0) \right], \end{aligned} \quad (2.13)$$

and average the results over the unit cell. The subtraction of the zero-frequency real part removes potential contact terms [40, 41]. We restrict the AC analysis to zero background magnetic field, for which magnetization currents vanish and no subtraction is required.

3 Transport expectations from hydrodynamics

The periodic potential environment in which we wish to study the quantum critical response, is captured in the UV boundary conditions of our ansatz eq. (2.5). At finite density, i.e., finite average $\bar{\mu} \neq 0$, a simple (relativistic) hydrodynamic analysis extended with a momentum relaxation rate Γ parametrically smaller than any other rate ($\bar{\mu} \gg \delta\mu$ in the case of a chemical potential lattice) results in thermoelectric transport that is dominated by a Drude pole at $\omega = -i\Gamma \sim -i(\delta\mu)^2$, such that the AC transport coefficients take the general form [37, 41]

$$\begin{aligned} \sigma(\omega) &= \frac{n^2}{\varepsilon + p} \frac{1}{\Gamma - i\omega} + \sigma_Q, \\ \alpha(\omega) &= \frac{ns}{\varepsilon + p} \frac{1}{\Gamma - i\omega} - \frac{\mu}{T} \sigma_Q, \\ \bar{\kappa}(\omega) &= \frac{s^2 T}{\varepsilon + p} \frac{1}{\Gamma - i\omega} + \frac{\mu^2}{T} \sigma_Q, \end{aligned} \quad (3.1)$$

where n, s, ε and p denote the charge, entropy, energy and pressure densities, respectively. We will be studying the system in the opposite limit $\bar{\mu} \ll \delta\mu$, but the generic wisdom that translational symmetry breaking leads to finite DC conductivities should hold. In the extremal limit of interest to us, $\bar{\mu} = n = 0$, a simple exercise in hydrodynamics indicates that electric and thermal transport will decouple. We might therefore expect the electrical conductivity to be governed by the incoherent conductivity σ_Q . Similarly, the thermal conductivity should remain coherent (i.e., dominated by a single pole). In the absence of magnetic fields, we furthermore expect $\alpha = 0$, confirming that the mechanisms of electrical and thermal transport are essentially independent. We thus also expect no difference between the thermal conductivity at zero electric current $\kappa_{ij} = \bar{\kappa}_{ij} - T\bar{\alpha}_{ik}(\sigma^{-1})_{kl}\alpha_{lj}$ and $\bar{\kappa}_{ij}$.

At finite magnetic fields, the transport coefficients become tensorial in nature, and one expects $\sigma_{xy} \sim n/B$, as follows from a combination of parity and charge-conjugation symmetry. In a charge neutral system with $n = 0$, however, no Hall effect will occur, and the magnetic field will only affect longitudinal transport. Nevertheless, a finite magnetic field does allow for a Hall thermoelectric conductivity $\alpha_{xy} \sim s/B$ and we will be able to observe a non-zero contribution to the Nernst coefficient $e_N = -(\sigma^{-1}\alpha)_{xy}$, where from charged hydrodynamics one expects [37]

$$e_N = \frac{B}{T} \frac{1}{\Gamma \left(\frac{n^2}{\sigma_Q(\epsilon+p)\Gamma} + 1 \right)^2 + \omega_c^2}, \quad (3.2)$$

with the cyclotron frequency in natural units equal to $\omega_c = \frac{Bn}{\epsilon+p}$.⁴ For a charge neutral system this simplifies to a direct measurement of the momentum relaxation rate as $e_N = \frac{B}{T\Gamma}$.

There is one more expectation we can formulate. Our system is on average isotropic in the thermodynamic limit. In quantum critical theories (with a holographic dual), this implies that the DC electrical conductivity is bounded from below as $\sigma_{\text{DC}} \geq \sigma_\infty$ [44]. The electrical conductivity at charge neutrality in the absence of a lattice is given by that of the CFT dual to the Schwarzschild AdS_4 solution: $\sigma_{\text{DC}} = \sigma_\infty = 1$. This saturates the bound. Thus, the only thing the conductivity can do is to increase with increased disorder/increased lattice amplitude around $\bar{\mu} = 0$. We will indeed see this below.

4 Thermoelectric transport in 1D charge-neutral lattices

4.1 DC transport in 1D charge-neutral lattices

To build on the results of Chesler, Lucas and Sachdev [3], we first consider a 1D lattice at charge neutrality (i.e., that is charge-conjugation symmetric): we set the background chemical potential to be

$$\mu(\vec{x}) = \delta\mu \cos(Gx). \quad (4.1)$$

Recalling that the thermoelectric cross-conductivity α vanishes in a charge neutral setting, the remaining two DC transport coefficients, the electrical and thermal conductivity, σ_{DC} and $\bar{\kappa}_{\text{DC}}$, are presented in figure 1. A first conclusion is that the presence of a lattice *increases* the conductivity, as should be expected based on the bound $\sigma_{\text{DC}} \geq \sigma_\infty$ from [44]. This effect is strongest at low temperatures for 1D lattices, as is highlighted in figure 1(a). A similar temperature-behavior of the electrical conductivity σ_{DC} was recently observed in a holographic model with fully random, strong disorder at finite density [45]. However, we wish to stress that the underlying physics of our model is completely different: at finite $\bar{\mu} \geq \delta\mu$, the DC transport is governed by the Drude peak — leading to a greatly magnified overall scale — whereas our increase for $\bar{\mu} = 0$ stems from completely incoherent transport. Indeed, our observed scaling with respect to the disorder amplitude is inverted as compared with the results of ref. [45]. Holographically, this difference is visible in that a finite $\bar{\mu} \neq 0$ leads to an $\text{AdS}_2 \times \mathbb{R}^d$ near-horizon geometry, but that is not the case here.

⁴A review of charged hydrodynamics in the presence of weak translational symmetry breaking is given in [42].

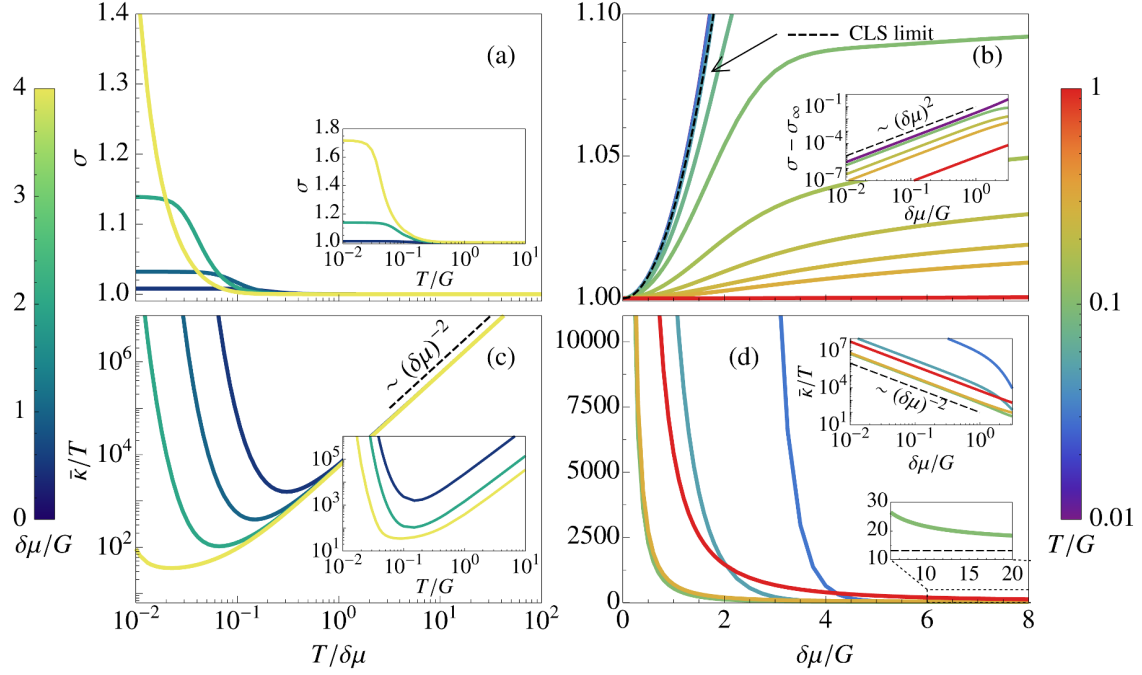


Figure 1. DC quantum critical transport in a 1D charge-neutral lattice, $\mu(x) = \delta\mu \cos(Gx)$. Note that the thermoelectric conductivity $\alpha = 0$ (numerically confirmed, not shown) and thus $\bar{\kappa} = \kappa$. **(a)** Electrical conductivity σ as a function of $T/\delta\mu$ for various $\delta\mu/G$ (left colorbar). At low temperatures, the lattice causes the conductivity to increase. Inset: electrical conductivity as a function of T/G . **(b)** Electrical conductivity σ as a function of lattice amplitude for various temperatures T/G (right colorbar). In the low-temperature limit for small $\delta\mu/G$, the scaling is that predicted by Chesler, Lucas and Sachdev [3], (eq. eq. (4.2) black dashed line). Inset: difference from the CFT value $\sigma_\infty = 1$ on a logarithmic scale, which grows as $(\delta\mu)^2$. **(c)** Thermal conductivity $\bar{\kappa}$ as a function of $T/\delta\mu$. There is a cross-over from a universal high- T regime to a strong thermal conductor-regime at low T . Inset: thermal conductivity $\bar{\kappa}$ as a function of T/G . **(d)** Thermal conductivity $\bar{\kappa}$ as a function of lattice amplitude. The thermal conductivity decreases most rapidly with increased lattice amplitude around $T/G \approx 0.1$, which corresponds with the minima seen in panel (c). Upper inset: for weak lattices, the thermal conductivity scales as $1/(\delta\mu)^2$. Lower inset: for strong lattices, the thermal conductivity approaches the bound $\kappa/T \geq 4\pi^2/3$ of [43].

This scaling of the electrical conductivity σ_{DC} as a function of lattice strength was elucidated by Chesler, Lucas and Sachdev [3]: deforming the UV with a 1D lattice with $\bar{\mu} = 0$ still results in an AdS_4 IR geometry at $T = 0$, except that in the natural UV coordinates the IR metric is anisotropic with regards to the x and y -directions. We highlight this effect in figure 2 (left column), that shows the RG flow (captured by the radial coordinate z), but halted due to the finite value of T , of the density vanishing in the IR and two metric components exhibiting the anisotropy. Absorbing the emergent anisotropy in the coordinates to bring the IR metric into standard AdS_4 form also rescales the length scales of the dual theory, resulting in

$$\sigma_{\text{DC}} = \sigma_\infty \left[1 + \frac{1}{2} \left(\frac{\delta\mu}{4G} \right)^2 \right]. \quad (4.2)$$

This perturbative result, valid for $T = 0$ and at $\delta\mu \ll G$, is shown as a black dashed line in figure 1(b) and which can be seen to asymptotically hold for $T \ll G$; in fact, the result extends to $\delta\mu/G = O(1)$ for low enough temperatures. At larger temperatures the deviation from σ_∞ is not as strong, but still scales as $(\delta\mu)^2$ (inset, panel (b)). Two comments about this strong coordinate anisotropy hiding translational symmetry mechanism are in order. Firstly, the explanation is particular to 1D, and will not apply to the 2D lattice as we shall see in section 5. Secondly, as explained by Chesler, Lucas and Sachdev, this emergent translational symmetry is an artifact of the holographic large N approximation [3]. There it was shown that by comparing to a system of N_f free Dirac fermions in 2+1D at charge neutrality, the true ground state is not translationally invariant, but is a lattice of electron and hole filled regions. Tunneling transitions between these regions change the spectrum such that new Dirac nodes emerge [6]. In the weak coupling description the enhancement of the DC electrical conductivity is due to the new low energy states in these new Dirac nodes.

In the opposite limit where $T \gg G$, the high temperature cuts off any RG flow so early that the lattice barely modulates the metric at all, and the near-horizon geometry is essentially that of the isotropic AdS_4 -Schwarzschild solution (figure 2, right column). The charge density, though strongly fluctuating on the horizon, is not able to influence the bulk (spatially averaged over a lattical cell) electrical conductivity σ_{DC} : Setting $ds^2 = ds_{\text{AdS}_4}^2$, the conservation of the heat current in the Stokes equations eq. (2.6) reduces to a conservation of the velocity field,

$$\nabla_i v^i = 0. \quad (4.3)$$

For 1D lattices where $i = x$, this enforces $v_x = \text{const.}$ and likewise, the continuity equation $\nabla_i J^i = 0$ together with the periodic boundary conditions constrain $\nabla^i w = -n^H v^i$. This amounts to a cancellation in the current eq. (2.7) such that $J^i = E^i$ and the conductivity is trivially $\sigma_{\text{DC}} = 1$. As should be clear from eq. (4.3), the fact that this equation has non-trivial solutions in 2D makes this story completely different for the 2D lattices discussed in section 5 below.

For the thermal conductivity $\bar{\kappa}_{\text{DC}}$, shown as a function of temperature in figure 1(c), we may conduct a similar analysis. At large temperatures — roughly $T \gtrsim 0.3G$ as can be deduced from the inset in figure 1(c) — a hydrodynamical analysis ought to be valid, such that the thermal transport is fully coherent with a Drude-like response. This is what the numerical data indeed appears to reflect, with a decrease of thermal conductivity with increasing T/G explainable by a thermal broadening of a Drude peak. However, a closer inspection of the AC thermal conductivity will reveal that this is not true. Before we explain there the underlying physics, we do observe that at large $T \gg G$ the result is perturbative in the lattice strength $\delta\mu$: The thermal conductivity scales as $\bar{\kappa}_{\text{DC}} \sim (\delta\mu)^{-2}$ suggesting again (incorrectly) a single decay rate $\Gamma \sim (\delta\mu)^2$ (cf. upper inset in figure 1(d)). Increasing $\delta\mu$ therefore means decreasing $\bar{\kappa}$. However, it can be shown that the thermal conductivity is also bounded from below as $\kappa/T \geq 4\pi^2/3$ [43], meaning that the thermal conductivity must eventually saturate. We indeed find that this is the case, as is shown in the lower inset in figure 1(d), where for the largest lattice strengths that we are able to probe ($\delta\mu = 20G$ at $T = 0.1G$) the thermal conductivity stays above the bound by about a factor of 1.4.

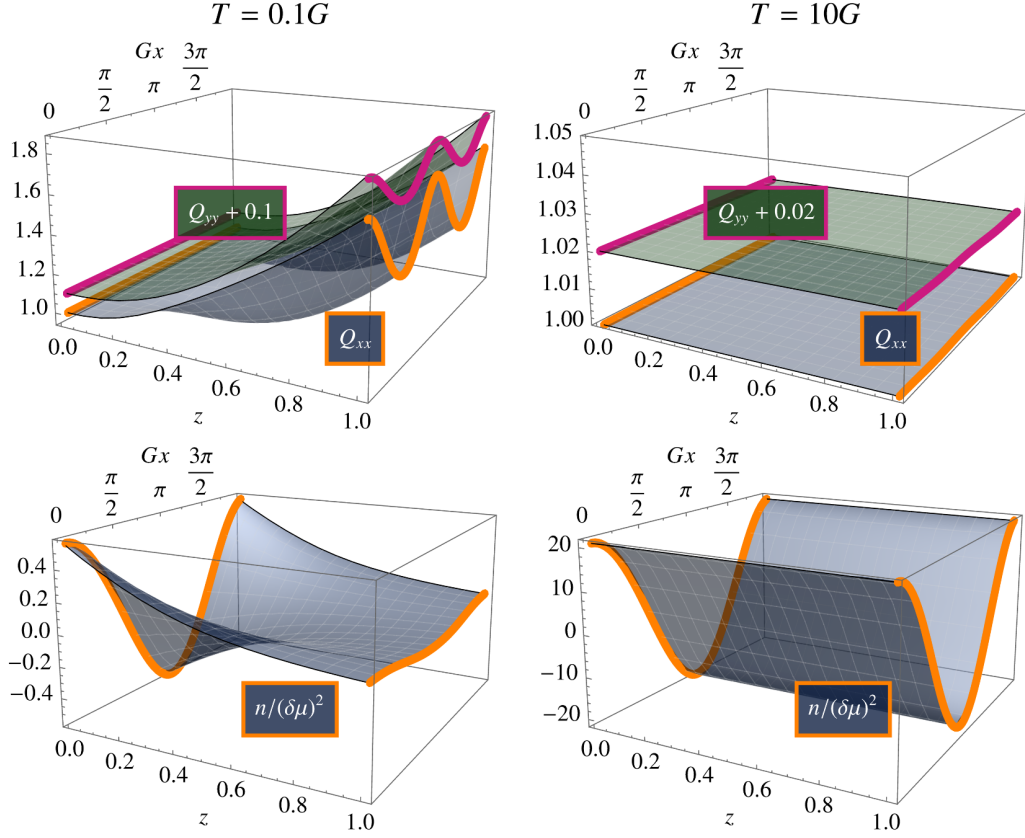


Figure 2. Metric components Q_{xx} and Q_{yy} (offset for clarity; top row) and local charge density $n(z) = F^{tz}$ (bottom row) for a 1D charge neutral lattice, $\mu(x) = \delta\mu \cos Gx$ with $\delta\mu/G = 2$. The evolution in the z coordinate captures the effect of RG flow. At low temperatures $T = 0.1G$ (left), the resulting x/y anisotropy at the horizon gives rise to the asymptotic scaling in eq. (4.2). At high temperatures $T = 10G$ (right), $ds^2 \approx ds_{\text{AdS}_4}^2$, and the renormalization of the charge density is cut off almost immediately. The horizon charge density entering the Stokes equations, eq. (2.6), thus becomes $n^H \approx \mu(x)$ for $T \gg G$.

The thermal conductivity $\bar{\kappa}_{\text{DC}}$ attains a minimum around $T/G \simeq 0.1$ (inset, figure 1(c)), and as one lowers the temperature further, it enters a highly thermally conducting regime where it increases exponentially and any semblance to (the $\omega \rightarrow 0$ limit of) the hydrodynamic Drude formula eq. (3.1) breaks down. This point coincides with the onset of the increased electrical conductivity driven by the emergent strong horizon anisotropy as identified by Chesler, Lucas and Sachdev. However, since this anisotropy is a coordinate artifact and the $T = 0$ state of the system is dual to pure translationally invariant AdS_4 , a perfect conductor is indeed expected to arise at the lowest temperatures.

4.2 AC transport in 1D charge-neutral lattices

Next, we study the finite frequency linear response. In the high temperature regime, $T \gg G$, $T \geq \delta\mu$ shown in figure 3(b) and (c), the AC electrical conductivity $\sigma(\omega)$ is completely dominated by an Umklapped sound mode, which sits at $\omega = v_{\text{sound}}G \approx G/\sqrt{2}$ and which becomes increasingly sharp at larger T/G . The identification is readily made by comparing

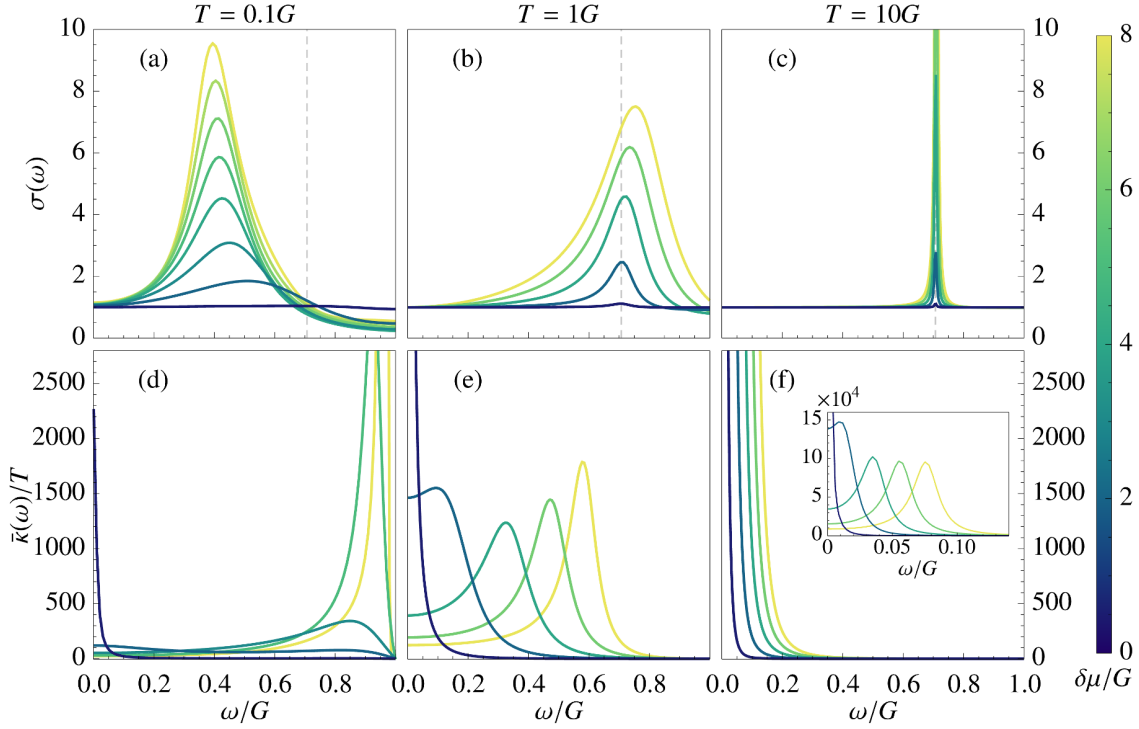


Figure 3. AC quantum critical transport in a 1D charge-neutral lattice, $\mu(x) = \delta\mu \cos(Gx)$ at three different temperatures, $T/G = 0.1, 1, 10$ (left to right). **(a-c)** The AC electrical conductivity $\sigma(\omega)$ exhibits an Umklapp sound peak at $\omega \approx G/\sqrt{2}$ (gray dashed line), which is dominant at high temperatures. At lower temperatures (panel (a)), there is a slight downward shift in the peak frequency. **(d-f)** The AC thermal conductivity $\bar{\kappa}(\omega)$ is dominated by a pair of poles that moves toward $\omega = G$, a process which happens more rapidly at low T/G . Inset (f): Same data on a larger vertical scale for small frequencies. The peak moves closer to the origin, but is not a single Drude peak centered at $\omega = 0$.

with systems at finite $\bar{\mu}$ [2, 46]. This may seem surprising, as in a charge-neutral system one expects electrical and thermal transport to decouple, with the sound mode residing entirely in the thermal sector. This decoupling between electrical and thermal transport is however only valid at leading order in the charge density. At subleading order (e.g., in terms of the density-density susceptibility χ_{nn}), the cross-coupling between sound and charge diffusion does survive [47].

At these same temperatures $T \gg G$ the AC thermal conductivity $\bar{\kappa}(\omega)$ (panel (f)) seems at first to again be predominantly governed by a zero-frequency Drude peak, as expected from the hydrodynamic expression eq. (3.1). As forewarned, however, this is not the case upon closer inspection. The zoomed-in inset at low frequencies in figure 3(f) shows that there are in fact two symmetric peaks at infinitesimally small $\omega = \pm\omega_{\text{peak}}$. To elucidate this more clearly we computed the response for complex-valued frequencies to find the exact location of the poles in the complex frequency plane. Figure 4 shows indeed that there are two poles, each with a small but finite real part.

These two poles must have arisen from a collision of two diffusive poles constrained to the imaginary frequency axis. The realization that Umklapp sound appears in the electrical conductivity $\sigma(\omega)$ points to an answer for the origin of these two poles. In the presence of spatial periodic modulations, hydrodynamic excitations are not given by Fourier modes, but by Bloch waves. For weak interactions these can be interpreted as Umklapped versions of original hydrodynamic excitations. In the case of charged hydrodynamics, this Umklapp effect predicts that the single Drude mode gets accompanied by two Umklapp sound modes and one Umklapp charge diffusion mode [47]. Generically, both electrical and AC thermal transport will be dominated by these four modes at low frequencies. However, as first observed in [2], for certain parameters the (diffusive) Drude mode and the Umklapp charge diffusion mode can collide and move symmetrically off the imaginary axis, acquiring a propagating part. For finite $\bar{\mu}$ this happens as $T/\bar{\mu}$ is increased. By adding a small overall chemical potential such that we are back in the regime where both modes are diffusive, we see in figure 5 that the same physics happens as we tune $\bar{\mu}/T$ to zero instead. This explains these two poles. Umklapp hydrodynamics predicts four poles, but as $\bar{\mu} \rightarrow 0$ the residue of the two other poles — Umklapp sound — becomes negligible in the AC thermal conductivity $\bar{\kappa}(\omega)$ (cf. figure 4). This can be confirmed analytically [47]. Likewise, we see that at charge neutrality the residue of the Drude and charge diffusion pole in the AC electrical conductivity $\sigma(\omega)$ also vanishes. The set of four poles thus divide themselves between the electrical and thermal response at $\bar{\mu} = 0$; Umklapp sound is purely in the electrical sector, whereas the diffusive poles (after the collision) govern the thermal response.

These exotic pole collisions near $\bar{\mu}/T = 1$ are by now well known to occur in strongly coupled systems; for instance, it is also responsible for a vanishing plasma frequency for $\bar{\mu}/T \lesssim 1$ in models with dynamical electromagnetism [48]. For any finite $\delta\mu$, the thermal response is thus never truly Drude: unlike the hydrodynamic expressions eq. (3.1), the momentum relaxation rate does not enter through a single Drude pole, but always through this pair.

This two-pole anchoring of the AC thermal conductivity $\bar{\kappa}(\omega)$ can be seen more clearly at intermediate temperatures $T = G$ (figure 3(e)). The AC electrical response $\sigma(\omega)$ can still be seen to be dominated by Umklapped sound, but it has broadened significantly with respect to the high temperature regime. Qualitatively this broadening can be understood as an effect of stronger translational symmetry breaking as $\delta\mu/T$ has increased. The pinning of the Umklapp sound mode to the hydrodynamic prediction $\omega = v_{\text{sound}}G$ weakens as one approaches the viability-limit of the hydrodynamic approach $\omega \lesssim G$.

At even lower temperatures $T = 0.1G$ (figure 3, panels (a) and (d)), the resonance peak in the electrical conductivity $\sigma(\omega)$ starts to move towards lower frequencies, becoming slightly more sharply peaked at higher $\delta\mu/G$ as compared to moderate temperatures.⁵ As is evident from figure 4, this response is still dominated by the single pole that originates in Umklapp sound, though this regime is quite far from the Umklapp hydrodynamics regime $\delta\mu \ll T, \bar{\mu}$, and the imprint of the pole on the real axis is no longer close to $\omega = v_{\text{sound}}G$. This deformed Umklapp-sound peak remains absent in thermal transport: the physical mechanisms between

⁵Note that the increase in DC electrical conductivity with stronger lattices (figure 1) is barely visible in the $\omega \rightarrow 0$ limit in figure 3 due to the scale.

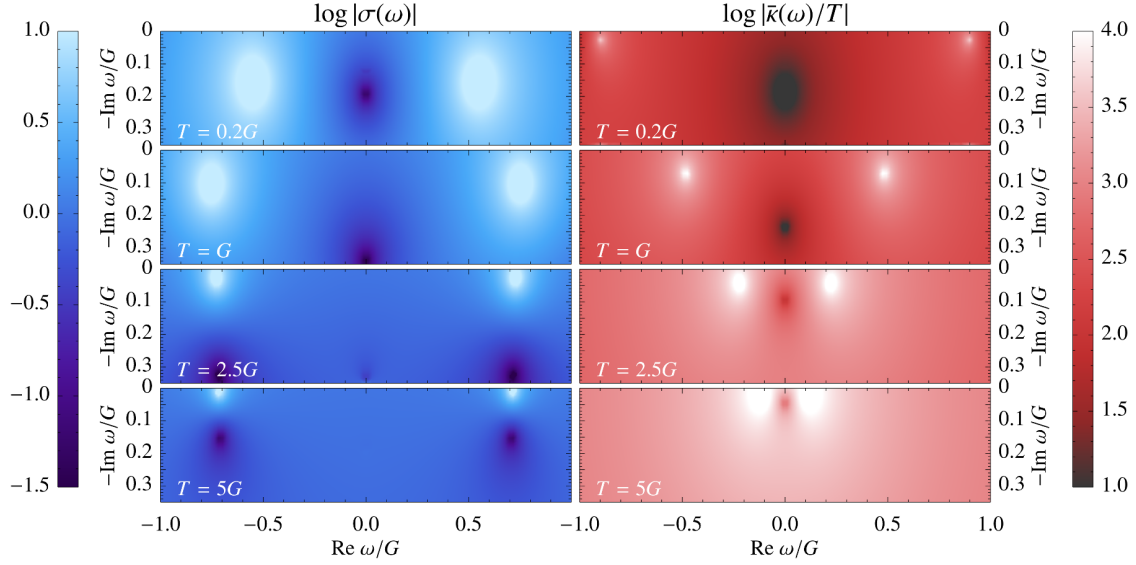


Figure 4. Pole structure of the AC response in a 1D charge-neutral lattice, $\mu(x) = \delta\mu \cos Gx$ for various T/G . In all panels $\delta\mu = 6G$. Left: logarithm of the complex electrical conductivity $\sigma(\omega)$. The Umklapp sound modes moves slightly toward the origin as T/G is decreased for fixed $\delta\mu/G$. There is no Drude pole or Umklapped charge diffusion pole present. Right: logarithm of the complex thermal conductivity $\bar{\kappa}(\omega)$. At all finite $\delta\mu$, transport is governed by a pair of modes that move in an arc toward $\omega = G$ as T/G is decreased. This pair of modes arises from a collision of the Drude pole with the Umklapp charge diffusion pole. At charge neutrality Umklapp sound has negligible weight in the thermal channel and is indeed not visible.

thermal and electrical transport stay distinct at low temperatures. In the AC thermal conductivity $\bar{\kappa}(\omega)$, on the other hand, the mid-IR peak originating in the collision of the Drude pole with the Umklapp charge diffusion pole sharpens and moves towards $\omega = G$ as T decreases. Though a peak at $\omega = G$ asks for an intuitive explanation in terms of the lattice and Umklapp, no such obvious explanation exists. Most such attempts will inadvertently rely on hydrodynamics, but for these lattice amplitudes one is outside its range of validity. A possible explanation is that this is a large N artifact denoting a “graviton-exchange” in the holographic gravitational description as postulated in [3]. Such a collective mode would travel at the speed of light, and can therefore give rise to an $\omega = G$ excitation after Umklapp. As this happens at $T < G$, an $\omega = G$ energy-mode is outside of the hydrodynamical regime, precluding an easy understanding from the quantum critical theory perspective.

In conclusion, the quantum critical model a charge neutrality in the presence of a 1D lattice exhibits two distinct regimes: at large T/G , we are in a regime with a fully incoherent DC electrical conductivity σ_Q , an AC electrical conductivity that is dominated by Umklapp scattering of the sound mode, and a thermal conductivity that at face value looks Drude-like, but is in fact dominated by a pair of modes slightly offset from zero frequency. Conversely, at small T/G , an anisotropic horizon geometry results in a novel strongly coupled response regime characterized by two distinct mid-IR peaks independently examinable in the AC electrical and thermal conductivity, whereas both DC conductivities are explainable in the restoration of translational invariance in the true $T = 0$ ground-state.

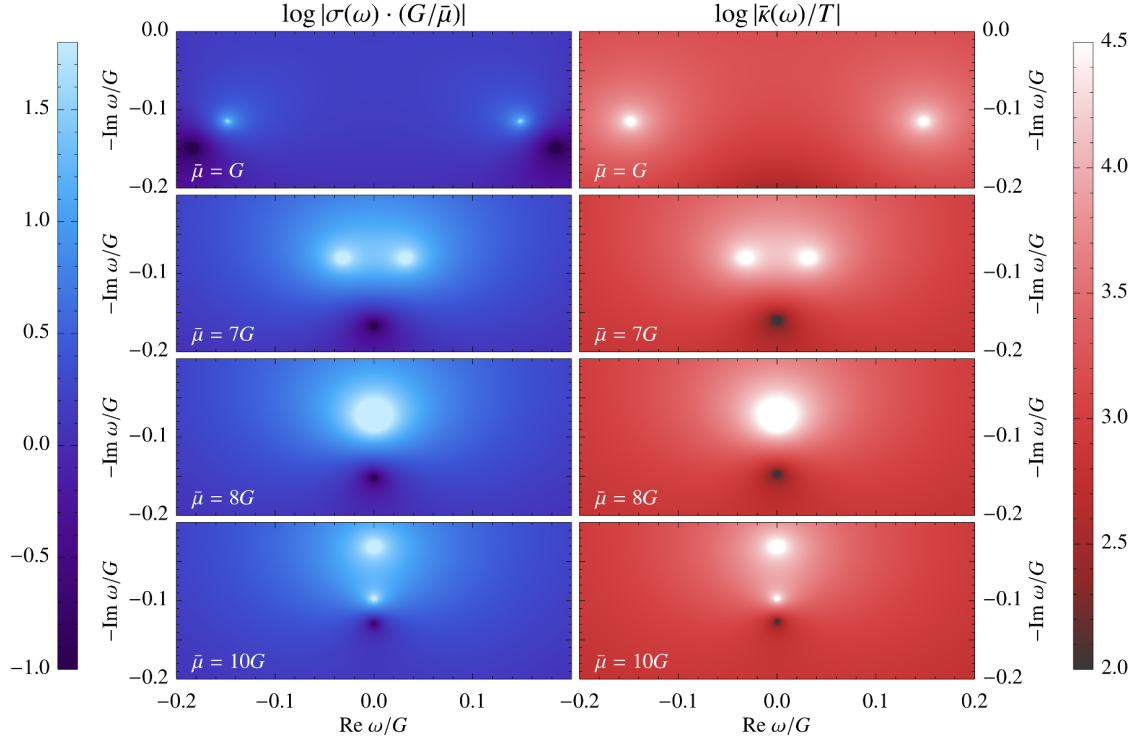


Figure 5. Pole structure of the AC response in a 1D lattice away from charge neutrality; $\mu(x) = \bar{\mu} + \delta\mu \cos Gx$. Left/right: logarithm of the complex electrical/thermal conductivity for various $\bar{\mu}/G$, at temperature $T = G$ and with $\delta\mu = 2G$. At large enough finite $\bar{\mu}$, there exists both a Drude pole and a charge diffusion pole along the imaginary axis in both the electrical and thermal sector. These eventually collide, turning into the pair of modes that govern the thermal transport at $\bar{\mu} = 0$. In the electrical sector, the residue of these poles vanishes as $\bar{\mu} \rightarrow 0$ due to a collision with a pair of zeros (top left panel).

5 Thermoelectric transport in 2D charge-neutral lattices

We now turn to quantum critical transport in 2D lattices, encoded in a chemical potential on the form

$$\mu(\vec{x}) = \frac{\delta\mu}{2} [\cos(Gx) + \cos(Gy)] . \quad (5.1)$$

We will show that there is a qualitatively novel contribution to the phenomenology of quantum critical transport in 2D charge-neutral lattices. In addition, the electrical conductivity has a modified RG-scaling from 1D. At the same time, the novel pole-collision physics behind 1D thermal transport does appear to continue to hold in 2D.

5.1 DC transport in 2D charge-neutral lattices

The difference in electrical transport is immediately evident in the DC conductivity σ_{DC} , as shown in figure 6 (panel (a)): the electrical conductivity now increases with temperature instead of decreasing monotonically back down to the translationally invariant quantum critical value σ_{∞} . This remarkable enhancement of the conductivity at higher temperatures

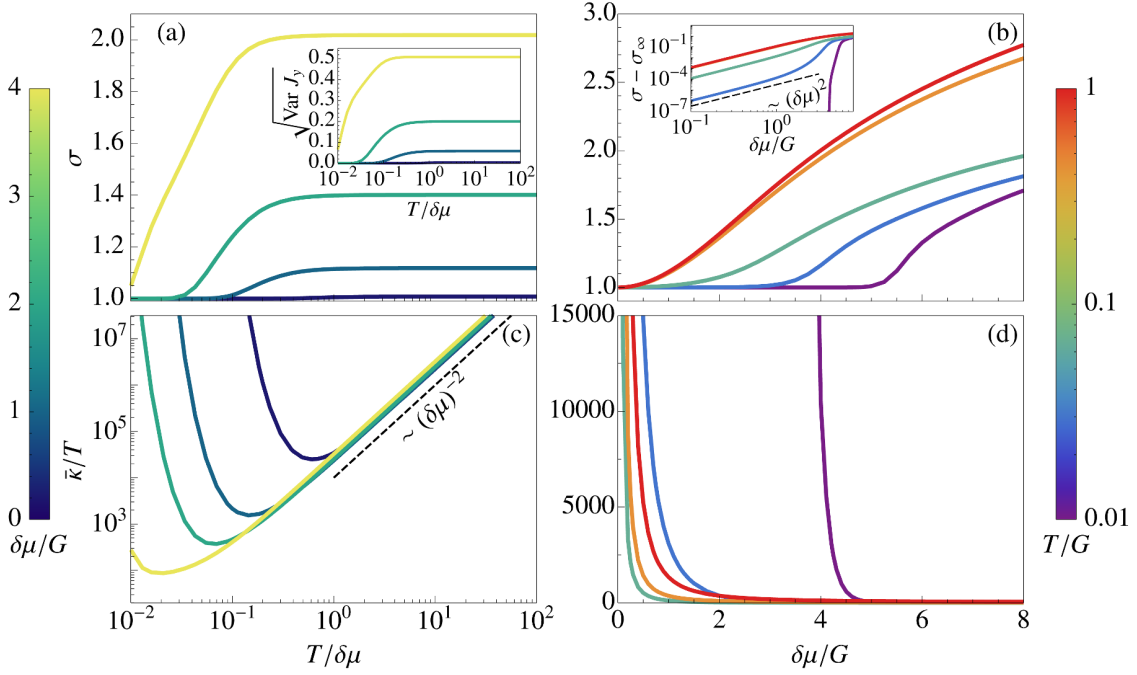


Figure 6. DC quantum critical transport in a 2D charge-neutral lattice, $\mu(\vec{x}) = \frac{1}{2}\delta\mu(\cos(Gx) + \cos(Gy))$. Note that the thermoelectric conductivity $\alpha = 0$ (numerically confirmed, not shown). **(a)** Electrical conductivity σ_{DC} as a function of $T/\delta\mu$ for various $\delta\mu/G$ (left colorbar). The lattice results in an increase in the conductivity at high temperatures. Inset: standard deviation of the fluctuations in $J_y(\vec{x})$ within a lattice cell, which correlates with the increased conductivity at high temperatures. **(b)** Electrical conductivity σ_{DC} as a function of lattice amplitude for various temperatures T/G (right colorbar). Inset: difference from the CFT value $\sigma_\infty = 1$ on a logarithmic scale, which grows as $(\delta\mu)^2$. The thermal conductivity $\bar{\kappa}_{DC}$ as a function of temperature **(c)** and lattice strength **(d)** remains similar as for 1D lattices.

can be understood if the lattice modulation acts as an irrelevant deformation. For a spatially dependent operator this is not straightforward to determine, but there is a heuristic argument that suggests that this is indeed so. One can make an analogy between a periodic source of translational symmetry breaking which vanishes averaged over a lattice cell $\int_{\vec{x}}^{\vec{x}+\vec{a}} \mu(\vec{x}) = 0$ but whose square does not $\int_{\vec{x}}^{\vec{x}+\vec{a}} \mu(\vec{x})\mu(\vec{x}) \neq 0$, and random disorder with $\langle g(\vec{x}) \rangle = 0$, $\langle g(\vec{x})g(\vec{x}') \rangle = g^2\delta(\vec{x} - \vec{x}')$. For such generic Gaussian random field disorder introduced through $\int d\vec{x}dt g(\vec{x})\mathcal{O}(\vec{x})$, the Harris criterion [49] reveals that it is relevant if the scaling dimension of $\mathcal{O}$ is $\Delta_{\mathcal{O}} < \frac{1}{2}(d+1)$. For a randomly disordered chemical potential modulation around a charge neutral lattice coupling to $\mathcal{O}(\vec{x}) = n(\vec{x})$ with dimension $[n] = d$, the modulation saturates this bound for $d = 1$, but violates it for $d \geq 2$. Such chemical potential disorder is thus marginal in $d = 1$ and irrelevant in higher dimensions. Although we here consider lattice modulations around zero average rather than random disorder, our numerics support this same division, suggesting that the vanishing of averages makes the lattice modulation act like disorder and that this is a good hint towards the underlying physics.⁶

⁶As can be seen in figure 2, the lattice modulation is marginally *irrelevant* for 1D lattices. The charge-density profile exhibits the same qualitative behavior for 2D lattices.

This is further supported by an analysis of the near-horizon geometry that determines the DC response. Similar to 1D lattices, for $T = 0$ the metric remains purely AdS_4 in the IR. However, for the square lattice we use, this is an isotropic AdS_4 and there is no enhancement expected of the quantum critical conductivity. Indeed our numerics show that the 2D DC electrical conductivity approaches σ_∞ as temperature decreases. This must be from a higher value $\sigma_T > \sigma_\infty$ by the bound [44], thus following an irrelevant RG flow. We do note that we are numerically limited to temperatures above $T/\delta\mu = 0.01$ while keeping $\delta\mu/G$ reasonably large.

At the same time, the arguments for a trivial conductivity $\sigma_{\text{DC}} = \sigma_\infty$ at *high temperatures* in the case of 1D must break down for 2D lattices, as is readily confirmed. In particular, from the holographic point of view, eq. (4.3) now allows for spatial variations in the horizon velocity field. The Stokes equations eq. (2.6) reduce in this limit to

$$\begin{aligned} v^i \partial_i \rho^H + \partial_i (E^i + \partial^i w) &= 0, \\ \partial^2 v_i + 4\pi T \zeta_i - \partial_i p + \rho^H (E^i + \partial^i w) &= 0, \end{aligned} \tag{5.2}$$

where we have used that all transport coefficients but ρ^H become trivial and that $\partial_i v^i = 0$. Thus, while the unit cell average of currents perpendicular to the electric field E_x must vanish ($\bar{J}_y = 0$), local variations of $J_y(\vec{x})$ need not be zero.⁷ In fact, at high temperatures it will in general fluctuate, whereas at low temperatures we find $J_y(\vec{x}) \rightarrow 0$. We show in the inset in figure 6(a) that the variance of $J_y(\vec{x})$ indeed follows the same general shape as the electrical conductivity. We may interpret this variance in $J_y(\vec{x})$ as the consequence of the current following a curved path through the potential landscape, in such a way that the conductivity is increased. Though a holographic computation in the strong coupling regime cannot reveal this, the natural mechanism prompting such perpendicular variations in a weakly coupled (Boltzmann transport) system that grow with temperature would be thermally assisted fluctuations or thermally assisted local scattering.

This picture is corroborated by a direct inspection of the current flows. Figure 7 shows both the unrenormalized lattice as well as the local electrical and thermal currents $J_i(\vec{x})$ and $Q_i(\vec{x})$ on the horizon, representing the quantities solved for in the Stokes equations eq. (2.6).⁸ From this, an interesting picture emerges, which on physical grounds must have its origins in that the collision-less to hydrodynamic crossover in quantum critical systems can happen at larger scales than one would expect: it seems that each patch in the transverse plane can be thought of as a local (electric) Reissner-Nordström geometry, i.e., locally it is a collective quantum critical system at finite density, even though the global charge density vanishes. If so, one can consider the hydrodynamic results for the conductivities eq. (3.1), assuming the expressions to be valid at some (suitably defined) local point $\vec{x}$, which should hold in an adiabatic approximation where the spatial variation is on a much larger scale

⁷Recall that $\bar{\mu} = 0$ causes the global Hall conductivity ($\sigma_{xy} = 0$) to vanish, but local variations are still allowed.

⁸Note that the extraction of the physical currents in the dual quantum critical theory from the horizon currents is based on the fact that the unit cell averages $\bar{J}_i, \bar{Q}_i$ do not renormalize. The horizon values of the local expressions $J_i(\vec{x})$ and $Q_i(\vec{x})$ do renormalize and are in general not identical to their boundary counterparts. Nevertheless, the Stokes equations describing the problem give insight into the underlying physics. We do in fact find a similar pattern in the boundary currents $J_i(\vec{x})$ and $Q_i(\vec{x})$ as computed by a (more numerically complicated) AC calculation for $\omega \rightarrow 0$.

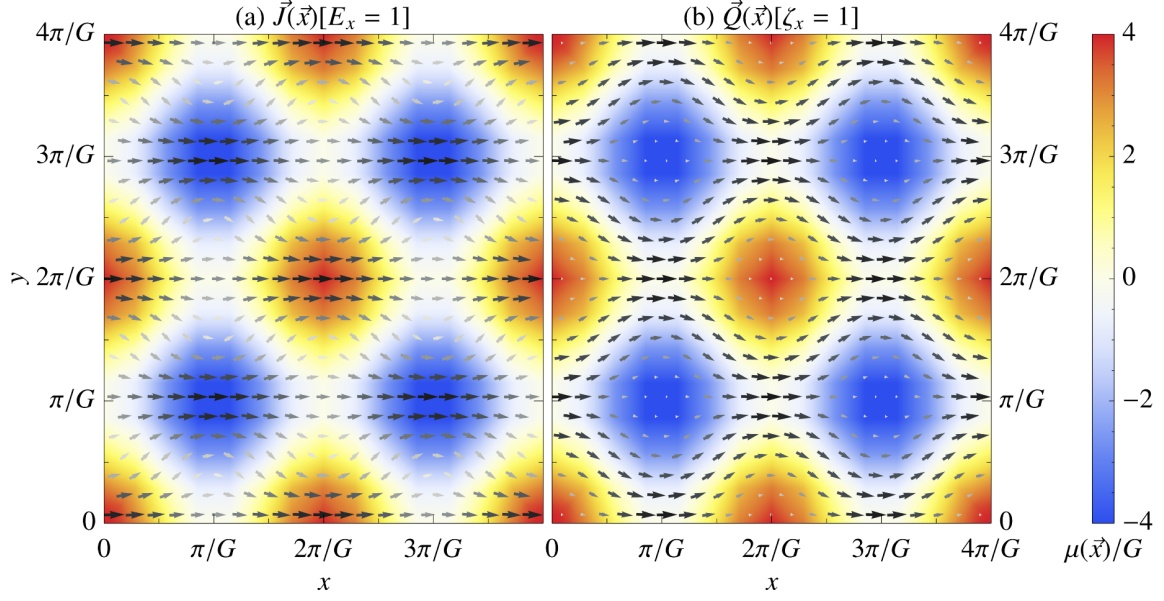


Figure 7. Vector plot of the electrical and thermal currents on the horizon for a 2D lattice with $\delta\mu/G = 4$ at a moderately high temperature $T = G$. Darker, larger vectors indicate greater magnitude. The underlying heat map shows the varying chemical potential, which is proportional to the horizon charge density. **(a)** Local electric current $\vec{J}(\vec{x})$ when applying a constant electric field in the x -direction, capturing the behavior of the electrical conductivity σ . The current flows from regions of $\mu(\vec{x}) \approx 0$ to where $|\mu(\vec{x})|$ is large. **(b)** Local thermal current $\vec{Q}(\vec{x})$ when applying a constant thermal gradient in the x -direction, capturing the behavior of the thermal conductivity $\bar{\kappa}$. The thermal conductivity is largest when $\mu(\vec{x}) \approx 0$.

than the onset of hydrodynamics:⁹

$$\begin{aligned}\sigma(\omega, \vec{x}) &= \frac{n^2(\vec{x})}{\varepsilon(\vec{x}) + p(\vec{x})} \frac{1}{\Gamma(\vec{x}) - i\omega} + \sigma_Q(\vec{x}), \\ \bar{\kappa}(\omega, \vec{x}) &= \frac{s^2(\vec{x})T}{\varepsilon(\vec{x}) + p(\vec{x})} \frac{1}{\Gamma(\vec{x}) - i\omega} + \frac{\mu^2(\vec{x})}{T} \sigma_Q(\vec{x}).\end{aligned}\tag{5.3}$$

Since for a finite density system, the electrical Drude weight is given by the (local) density squared, the local conductivity $\sigma_{\text{DC}}(\vec{x})$ is insensitive to the sign changes of the local chemical potential. This can be seen in figure 7(a): the electrical current density $J_i(\vec{x})$ concentrates in regions of large $|\mu(\vec{x})|$ independent of the sign, and evades regions where $\mu(\vec{x}) \approx 0$ (where the local conductivity is $\sigma = \sigma_\infty = 1$). This is reminiscent of Effective Medium Theory, where the current takes the path of least resistance through a material with regions of locally strongly varying conductivity [50].¹⁰

These same evading flow patterns are present in local probes of the thermal horizon current, albeit with an important twist. Figure 7(b) shows the thermal current $Q_i(\vec{x})$ when applying a constant thermal gradient in the x -direction. Keeping with our picture of local

⁹As the system is in thermal equilibrium, the temperature T will still be $\vec{x}$ -independent.

¹⁰Note that this means that locally, transport is governed by a Drude-like pole, but there is no global hydrodynamics where a Drude mechanism can explain transport.

Reissner-Nordström geometries and *local* hydrodynamic transport, the thermal conductivity $\bar{\kappa}(\vec{x})$ will be proportional to $\frac{s(\vec{x})^2 T}{s(\vec{x})T + \mu(\vec{x})n(\vec{x})}$ upon approximating $\epsilon + P = sT + \mu n$, cf. eq. (5.3). Similar to the electrical conductivity, the thermal conductivity is therefore insensitive to the local sign of the chemical potential: however, now regions of large $\mu(\vec{x})$ are regions of weak transport and vice versa. Hence one has a weaker thermal current at large local $|\mu(\vec{x})|$, and a stronger current in regions of small local $|\mu(\vec{x})|$, in contrast to the electrical conductivity.¹¹

Despite this far more intricate picture of the 2D flow patterns at sub-lattice length, the effective overall averaged DC thermal conductivity $\bar{\kappa}_{\text{DC}}$, shown in figure 6(c), exhibits the same phenomenology of the 1D periodic quantum critical DC thermal conductivity (figure 1(c)). At large T/G , the scaling is yet again perturbative in the lattice amplitude, suggestive of a momentum relaxation rate $\Gamma \sim \delta\mu^2$, transitioning to a strong thermal conductor at low T/G due to the irrelevancy of the lattice. As we shall see shortly, this suggestion is again deceptive, as the AC thermal conductivity $\bar{\kappa}(\omega)$ will reveal that also here multiple poles contribute.

As already pointed out, the 2D electrical conductivity shows a very different behavior with a thermally activated quantum critical conductivity enhancement at larger T/G flowing to the purely homogeneous charge neutral quantum critical σ_∞ conductivity at low T/G (figure 6(a)).

5.2 AC transport in 2D charge-neutral lattices

For the AC transport in a 2D lattice, we are forced to restrict our focus to temperatures $T \gtrsim G$ due to numerical limitations. The AC response at $T = G$ (at which point the asymptotic high-temperature behavior in the DC response has set in, cf. figure 6) is shown in figure 8. By comparing with the 1D results (figure 3), the AC results show that this is nevertheless an intermediate temperature regime.

In the AC transport we encounter a qualitatively new contribution in 2D compared to 1D. Figure 8(a) shows that the AC electrical conductivity now displays classic bad metal behavior at low frequencies up to the Umklapp sound peak. It therefore appears that a new zero-frequency diffusive mode mixes in, which may be identified by studying the response at purely imaginary frequencies. In the inset in figure 8(a) we show the location of this pole along the imaginary frequency axis. For finite $\delta\mu$ it is well described by a quadratic function consistent with a perturbative origin in the lattice, aside from a finite offset at $\delta\mu = 0$. Such a discrete jump at $\delta\mu = 0$ is most likely attributable to a mode collision for tiny $\delta\mu \ll G, T$. Within our numerical resolution we could not resolve or identify another such mode within the electrical conductivity, but they are there as we shall see.¹² The bad metallic behavior persists at higher temperatures (figure 8(b)). As T increases, the peak height does not change, consistent with the results for DC transport in figure 6(a), where the DC value can be seen to attain a constant value for $T \gtrsim G$. The peak gets increasingly sharp, however, in contradistinction to thermal broadening, illustrating its unconventional origin.

¹¹The thermoelectric conductivity α — not shown here for brevity — is on the other hand sensitive to the sign of the local chemical potential and density (cf. eq. (3.1)) and indeed, we find that the local current $J_i(\vec{x})$ when applying a thermal gradient (or thermal current $Q_i(\vec{x})$ for an external electric field) flows in a direction set by $\text{sgn}(\mu(\vec{x}))$, leading to $\alpha = \bar{\alpha} = 0$ upon averaging (since $\bar{\mu} = 0$).

¹²An added subtlety is that the peak has a Fano lineshape. As can be seen in the 1D lattice pole landscape, figure 4, there also exists a zero of the electrical conductivity along the imaginary axis. The same is true in 2D, and at small enough $\delta\mu \ll G, T$ the pole and the zero appear to collide.

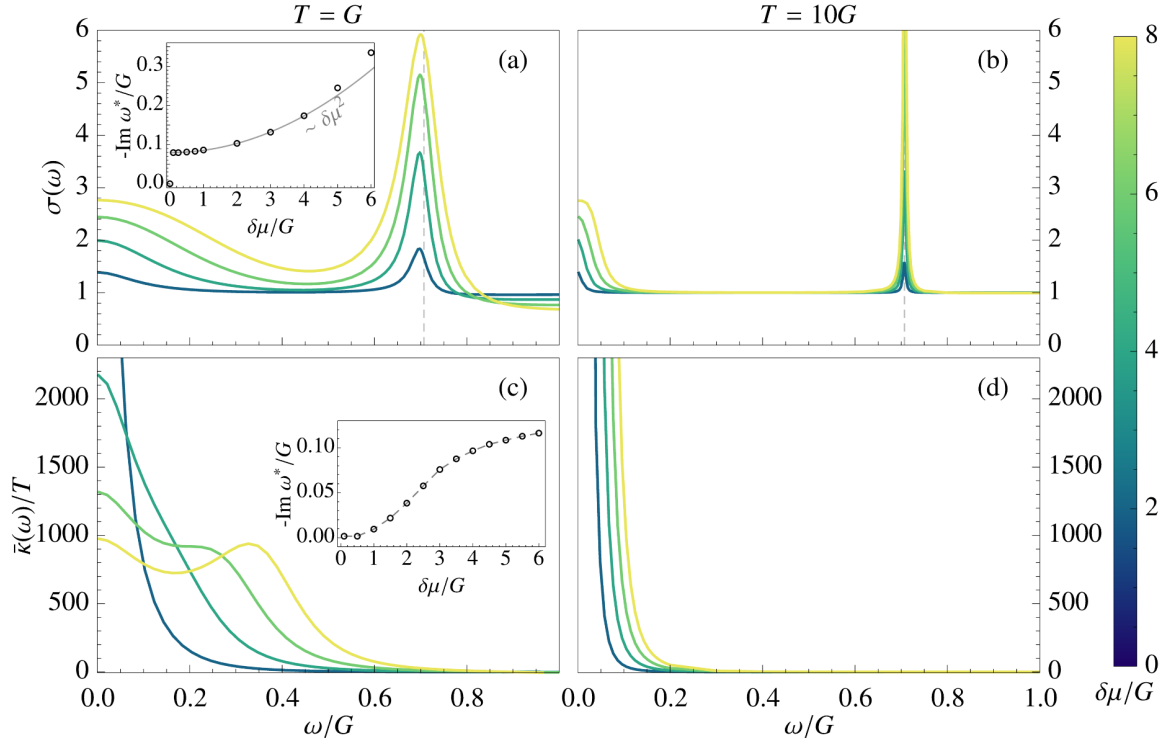


Figure 8. AC transport in a 2D charge neutral lattice, $\mu(\vec{x}) = \frac{1}{2}\delta\mu(\cos(Gx) + \cos(Gy))$. **(a-b)** In contrast to the 1D lattice, the 2D AC electrical conductivity $\sigma(\omega)$ exhibits a weak zero-frequency peak at large lattices, in addition to the Umklapp sound peak at $\omega \approx G/\sqrt{2}$ (gray dashed line). Inset: location of the new zero-frequency diffusive pole ω_{elec}^* along the imaginary axis. The solid gray line is a fit on the form $-\text{Im } \omega^* = a + b\delta\mu^2$ over the interval $\delta\mu/G \in (0, 4]$. **(c-d)** As for 1D lattices, the 2D AC thermal conductivity $\bar{\kappa}(\omega)$ shows a similar mid-IR peak at larger lattice strengths arising from the collision of the Drude pole with the Umklapp charge diffusion pole. There is also an additional novel zero-frequency peak as for the electrical conductivity. Inset: location of the new zero-frequency diffusive pole ω_{therm}^* along the imaginary axis. The gray dashed line is a guide to the eye. The scaling with $\delta\mu$ is different from the corresponding pole in the electrical conductivity, and is thus of a different origin.

A new weak zero-frequency peak is also visible in the 2D AC thermal transport (figure 8(c)) on top of the mid-IR mode arising from the collision of the Drude pole with the Umklapp charge diffusion pole. A first impulse might be to surmise that this novel mode is the same novel Drude-like mode as in the electrical conductivity, suggesting that some mixing between thermal and electrical conductivity is starting to occur. This is unlikely, however, as the AC thermoelectric conductivity $\alpha_{xx}(\omega)$ remains strictly zero within our numerical precision, indicating that the transport mechanisms of thermal and electrical transport remain distinct. The new pole in the AC thermal conductivity can therefore not be the same new pole as in AC electrical transport. It is indeed a different pole, as can be seen in the inset in figure 8(c): it does not exhibit the same scaling with $\delta\mu$ as the similar pole in the electrical sector. At larger temperatures, $T \gg G$ (figure 8 (panel (d))), its contribution to the DC thermal conductivity is subleading to that of the collided Drude and Umklapp charge diffusion poles, whose real part starts to get closer and closer to $\omega = 0$, showing a similar high- T scaling of the 1D and 2D DC thermal conductivities.

With our understanding of the DC conductivities through an Effective Medium flow through channels of least resistance, we can attempt a qualitative explanation of the underlying physics. The essence of this Effective Medium type point of view is that one essentially has a composite of independent materials — in this case particle-filled and hole-filled regions created by the periodic chemical potential. Each independent region has a single fluid hydrodynamic description, meaning that there are effectively as many Drude modes as there are local fluid regions that weakly interact with each other. This viewpoint is supported by similar findings in charge-neutral graphene, where in a periodic background one has new Dirac points that are remnants of the local Fermi surface in each pocket [3, 6]. In a collective interacting system, the same should happen for the emergent hydrodynamics provided the lattice period is large enough and the boundaries sharp enough. In a 1D charge neutral periodic lattice these modes act in unison, however; most of the local contributions either add coherently or cancel each other perfectly and a single global hydrodynamic description survives. In particular, there is no net charge flow leaving only a quantum critical electrical conductivity. In 2D, however, thanks to the variation of the flow allowed by the perpendicular direction, this cancellation of each of the local hydrodynamic charge flow is no longer exact and a remnant Drude-like mode ω_{elec}^* survives. This mode exists in addition to the global Drude mode, which happens to collide with the global Umklapp charge diffusion mode. This apparent excess of modes relative to the number of conserved quantities may be viewed as the hydrodynamic counterpart of the violation of the RG c -theorem in charge-neutral graphene in a periodic background. Naively, c counts the number of massless Dirac modes, which in charge-neutral graphene in a periodic potential increases due to the emergence of the new Dirac points. This violation is allowed because the c -theorem relies on translational invariance, which is no longer present.

Similarly, in the thermal sector a remnant Drude-like mode ω_{therm}^* survives from the mesoscopic physics. As we have seen this is a different mode than the remnant Drude-like mode in the electrical sector. Since all such novel Effective Medium type modes should interact, because they exist in region of non-zero local $\mu(x)$, simplicity argues that it is the level repulsion between this mode and the remnant Drude mode ω_{elec}^* that causes the offset of the latter at $\delta\mu = 0$. At the same time, it is the vanishing of the residue of each respectively that causes only one of them to arise in the respective transport currents.

On top of this novel emergent Drude-like mode of mesoscopic Effective Medium origin, we do recognize the same characteristics underlying 2D charge neutral quantum critical transport as in 1D. In particular, in the AC electrical conductivity $\sigma(\omega)$ at intermediate frequencies, we still see the Umklapped sound mode at $\omega = G/\sqrt{2}$.

5.3 DC magnetotransport in 2D charge-neutral lattices

A novel aspect of 2D transport is that we can now also consider transport in the presence of a background perpendicular magnetic field B_z . We limit our discussion to DC transport only, due to the complexity of the linearized equations of motion for the time-dependent perturbations in the case of $B \neq 0$.¹³

¹³A text file containing the simplified equations of motion and horizon boundary conditions for the perturbations totals about 40 MB.

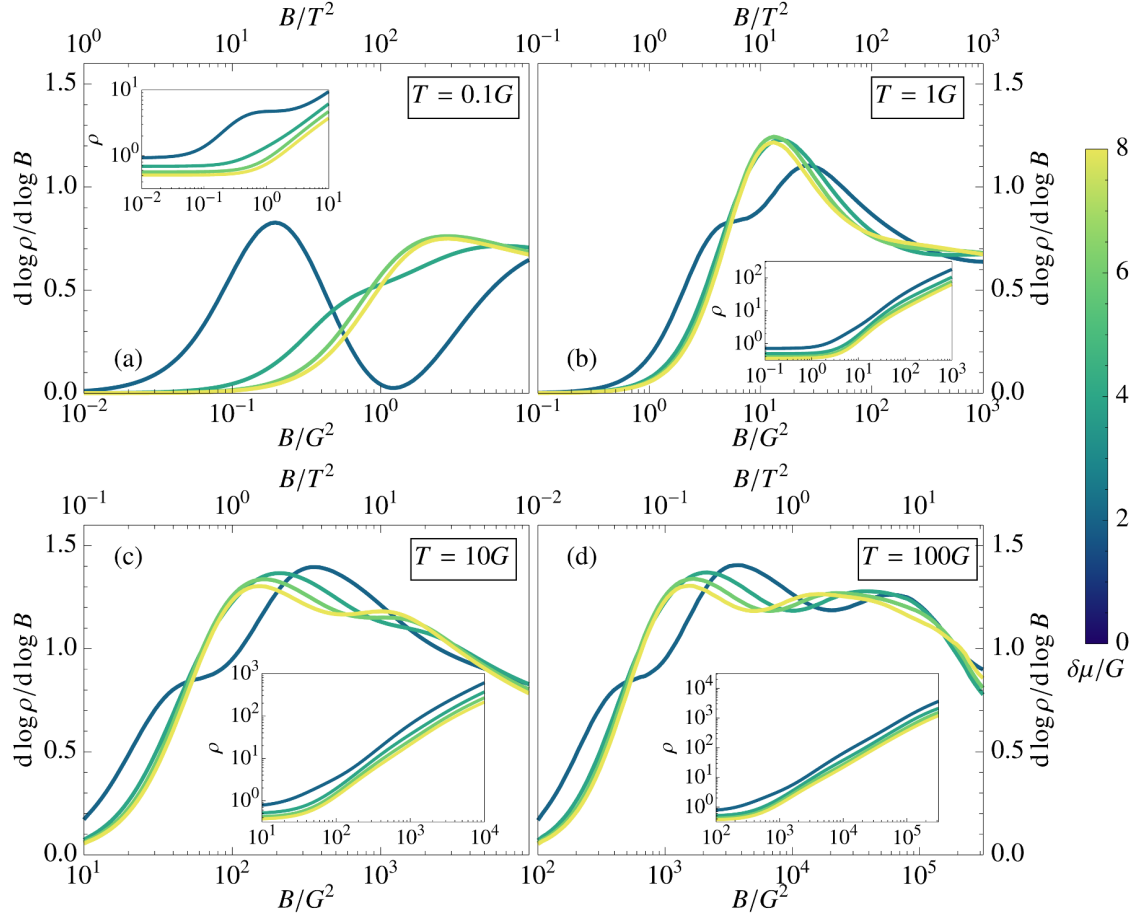


Figure 9. Logarithmic derivative of the longitudinal magnetoresistance ρ with respect to the magnetic field, capturing the local scaling exponent $\rho \sim B^\beta$. Insets: longitudinal magnetoresistance as a function of the magnetic field B .

Holding all other parameters ($T/G, \delta\mu/G$) fixed, the most striking result is an approximately linear dependence of the longitudinal magnetoresistance $\rho_{xx} = \frac{1}{\sigma_{xx}}$ on the magnetic field B/T^2 beyond $B \sim T^2$ independent of the size of $\delta\mu/G$: in figure 9, we plot the logarithmic derivative of $\rho(B)$ to extract the local scaling exponent β for $\rho_{xx} \sim |B|^\beta$. In the “Effective Medium regime”, $T \gtrsim G$, we obtain roughly linear dependence on the magnetic field. As we increase the temperature further, we find that the scaling stays quasi-linear over a larger regime. Recalling that parity implies that the longitudinal resistivity can only be a function of B^2 , the explanation might therefore be a quadrature-type relation $\rho_{xx} \sim (\rho_{xx}(B=0))^{2/\beta} + \gamma\mu_B B^2)^{\beta/2}$ well-known to occur in Effective Medium Theory [21, 22, 25, 26, 29]. Notably, we obtain emergent B -linear scaling without an ad hoc addition of mesoscopic disorder on top of our microscopic model. As we argued in the previous section, the chemical potential modulation in a charge neutral background suffices to cause a flow to a ground state that consists of effectively independent particle-filled and hole-filled regions.

Although there is no Hall effect in a 2D charge-neutral lattice, the parity-breaking by the magnetic field can induce a Nernst effect. As recalled in eq. (3.2), for a charge neutral

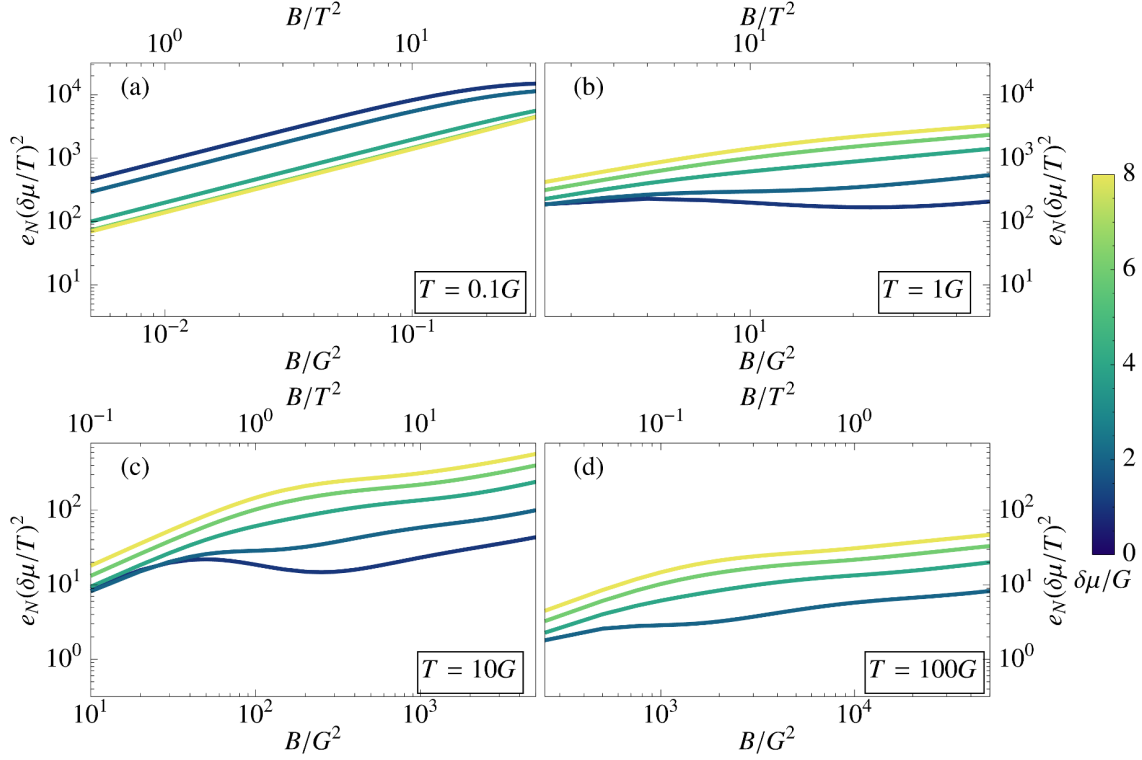


Figure 10. Nernst coefficient e_N at 4 different temperatures; $T = (0.1, 1, 10, 100)G$ (a-d).

system in singlefluid hydrodynamic regime at large $T/\delta\mu$ dominated by a single Drude mode, this measures the momentum relaxation rate directly as $e_N = B/TT$. Though our results have shown that this is not applicable here, we have shown that in this regime, transport is perturbative in $\delta\mu$. As to scale this out, figure 10 shows the rescaled Nernst coefficient $e_N \cdot \delta\mu^2/T^2$. This rescaled Nernst coefficient can be seen to exhibit two different scaling regimes (seen most clearly for high $T > G$ and $\delta\mu/G \geq 1$; panels (c) and (d) in figure 10). The transition between the two regimes coincides with the onset of the B -linear regime in the longitudinal resistivity. For lower $T < G$ (panel (a)) one is outside a perturbative regime: visually it appears one has reached a novel scaling regime with the added observation that an increase of lattice amplitude now decreases the Nernst effect. This again matches the behavior of the longitudinal magnetoresistance (figure 9(a)), which in this regime $T < G$ also becomes less sensitive to B than in the approximate B -linear regime at higher T/G .

6 Conclusion and outlook

Quantum critical systems can give rise to idiosyncratic transport signatures. This study of DC and AC thermoelectric and magnetotransport in 2D quantum critical theories with strong translational symmetry breaking due to a spatially varying chemical potential lattice with zero average exhibits this again. The most remarkable aspect is that the transport is never Drude-like in the sense that there is never a *single* longest-lived (momentum) relaxation mode controlling transport. In a 1D periodic potential electrical transport is purely quantum critical, whereas thermal transport is governed by two propagating modes originating in a

collision of the would-be Drude mode with the Umklapp charge-diffusion mode. In a 2D lattice, on the other hand, the low-energy physics resembles that of Effective Medium Theory: currents follow meandering spatial paths of least resistance in a heterogeneous system built out of local regions that resemble alternating particle-filled and hole-filled states. This results in new weak diffusive modes that are not the global momentum-relaxation mode, but probably better seen as remnants of the local Drude modes from the local charge-filled regions. This resemblance to results from Effective Medium Theory is also present in magnetotransport, where the longitudinal magnetoresistance shows a non-analytic unsaturating scaling behavior at large B that is approximately linear.

The context of this physics is familiar from studies of charge neutral graphene on various substrates with a periodicity larger than the intrinsic lattice. These studies have also shown the connection with Effective Medium Theory phenomenology [11, 51, 52]. Our findings that one can have to the eye Drude-like transport co-existing with unsaturating B -linear-like magnetoresistance may be also relevant to the strange metal phase of high- T_c cuprates. This provides a potential answer to the two-time scale puzzle needed to explain differing phenomenology of Hall and longitudinal transport, which has stubbornly eluded resolution since it was pointed out by Anderson. A purely charge-neutral system as studied here will however not apply; for one, the Hall effect vanishes. More fundamentally, theoretical studies in SYK models have shown that one must have an IR fixed point determined by $SL(2, R)$ time-reparametrization invariance to explain local quantum criticality. In terms of its holographic dual, the $T = 0$ near horizon geometry must be AdS_2 , whereas charge-neutral systems have an AdS_4 geometry. Once a finite density is added to the problem, the question to answer is whether a low-temperature regime with AdS_2 physics might still give Effective Medium Theory multi-time-scale signatures, as long as $T/G \gtrsim 1$.¹⁴ A very recent SYK-based computation [53] indeed finds different time-scales in ordinary and Hall transport, due to an effective increase of the Hall coefficient R_H with T — a feature that also follows from a phenomenological match of cuprate DC transport to hydrodynamics [19]. As $R_H = B/n$ perturbatively, this naively implies zero-density at $T = 0$, suggestively indicating that the fundamental physics is similar.

Acknowledgments

We thank Nicolas Chagnet for extensive discussions at the beginning of this project. E.N. is funded by the Nano Area of Advance at Chalmers University of Technology. K.S.'s research is supported in part by Dutch Research Council (NWO) NWA Consortium *Emergence at all scales* (NWA.1630.23.007) and NWO Summit Grant *Quantum Limits* (SUMMIT.1.016). The numerical computations were carried out in part on the ALICE-cluster of Leiden University. We are grateful for their help.

¹⁴One can already deduce that for the characteristic temperature of a cuprate strange metal $T = 100K$, the regime $T/G = 1$ corresponds roughly to spatial variations on the order of 500 nm assuming the relevant speed is the Fermi velocity $v_F \sim 5 \cdot 10^6$ m/s. The relevant modulation must therefore occur on scales far larger than the atomic lattice.

A Numerical method

The equations of motion were obtained in Mathematica with the help of the *xAct* package [54]. To solve the resulting linear/non-linear equations of motion, we have used the *PETSc* library in c [55, 56]. For a guide on implementation of the latter we refer the reader to ref. [57].

We have opted for a mix of pseudospectral (PS) and finite difference (FD) methods in the radial z -direction: we find an improved convergence at low temperatures/strong lattices with PS methods, at the cost of speed due to the global nature of the derivatives and use this to solve for the backgrounds. We do however find better numerical convergence with FD methods for the perturbations, along with a greatly appreciated speedup in the case of 2D lattices. In both cases we discretize the z -direction on a Chebyshev-Lobatto grid,

$$z_i = \frac{1}{2} \left[1 - \cos \left(\frac{\pi n}{N} \right) \right], \quad n = 0, 1, \dots, N, \quad (\text{A.1})$$

which clusters points near the boundary and the horizon. In the transverse xy -directions, we use 8th/4th order finite differences on uniform grids for 1D/2D lattices.¹⁵ This leads to additional sparsity in the matrix discretizing the problem, and most importantly, allows for efficient partitioning of the matrix on multiple processors. PETSc is built for finite differencing, so to implement the radial pseudospectral derivatives we force PETSc not to split the domain along the z -direction across the processors.¹⁶

To solve the non-linear equations for the backgrounds we used a GMRES non-linear solver, as the matrices are in general not symmetric due to the boundary conditions in the radial coordinate. To solve the linear equations (at each step of the non-linear algorithm, or for the linear equations of the perturbations), we use Schwarz domain decomposition implemented via an ASM preconditioner, distributing the calculation over 4–10 processors. As a sub-preconditioner on each block, we used ILU or LU factorization with varying levels of overlap, depending on the numerical difficulty. The linear equations for the perturbations often require full LU factorization with 2 points of overlap, whereas it generally suffices with ILU(0) with 1 point of overlap for the non-linear background equations, except for the very strongest lattices. The Stokes equations are easily solved with exact LU factorization on a single process. Note that since the Stokes equations eq. (2.6) only depend on the derivatives of p and w , one needs to fix a value for these functions at some point in the domain in order to remove a pair of zero modes of the matrix discretizing the equations.

A.1 Numerical convergence

The numerical difficulty increases as the temperature/lattice strength is reduced/increased. Throughout our calculations, we use units set by the horizon radius, which we fix to unity. In these units, the temperature is constant (for $B = 0$), so lowering T/G corresponds to increasing the lattice vector G . Maintaining a large ratio $\delta\mu/G$ therefore becomes increasingly

¹⁵In order to resolve the sharp Umklapp sound peak at large T/G in the 2D AC response, we must increase the finite difference order in the z -direction to 8.

¹⁶This is achieved by first lying to PETSc's DMDA routine that the stencil width is that of the finite difference stencil and then explicitly telling PETSc how many non-zero entries per row there are (otherwise a lot of time is lost on memory allocation).

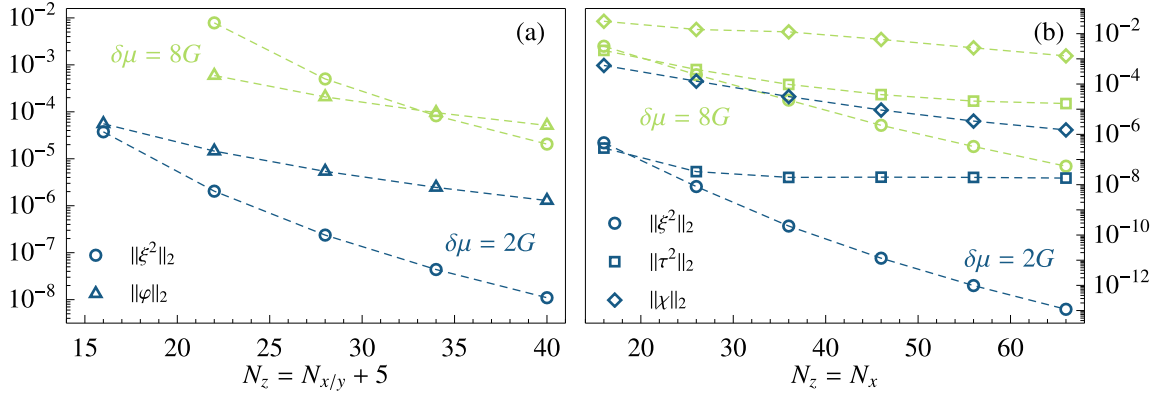


Figure 11. Convergence of the numerical algorithm. **(a)** Square of the de Turck vector ξ^2 and Maxwell gauge constraint scalar φ in a 2D lattice for increasing grid sizes, where $N_x = N_y = N_z - 5$, at a low temperature $T = 0.1G$ and magnetic field $B = 0.1G^2 = 10T^2$. We find an exponential convergence with increased grid sizes. **(b)** Square of the de Donder vector τ^2 and the Lorenz gauge scalar χ^2 , as well as the background de Turck vector, in a 1D lattice for increasing grid sizes. Here $N_x = N_z$, $T = 0.5G$ and $\omega = 0.2G$, and the boundary conditions source a charge.

challenging at low temperatures, since this requires $\delta\mu$ (in units of the horizon radius) to be correspondingly large.

To quantify the convergence of our numerical algorithm, we monitor the norm of the gauge constraints: the de Turck vector ξ^μ is spacelike, so it suffices to check that $\xi^2 \ll 1$. In figure 11(a) we show that the 2-norm $\|\xi^2\|_2$ converges exponentially with increased grid size, here using finite differences.¹⁷ The 2-norm of Maxwell gauge constraint scalar $\|\varphi\|_2$ (cf. eq. (2.2)) converges similarly. In figure 11(b), we illustrate the corresponding quantities for an AC calculation: the 2-norm of the square of the de Donder vector τ_μ and Lorenz scalar $\chi = \nabla^\mu \delta A_\mu$. Here we computed the backgrounds using pseudospectral methods in the radial direction (due to the low temperature/strong lattices), but used finite differences for the perturbations. In general, the perturbations exhibit weaker convergence with increased grid size. Both of these figures represent numerically challenging regions in the parameter space: at larger temperatures/weaker lattices, the convergence metrics look even better.

For challenging perturbation calculations at low temperatures (e.g., panels (a) and (d) in figure 3) for 1D lattices, we discretize the equations on a 80×60 grid. For the backgrounds, this achieves a 2-norm of the de Turck vector $\|\xi^2\|_2 < 10^{-4}$ for the largest lattice amplitudes $\delta\mu/G = 6.0$, with an exponentially decreasing norm for weaker lattices. At high temperatures, for both 1D and 2D lattices, we are able to reach better convergence. Even on a moderately coarse $20 \times 20 \times 24$ grid, which we use for the AC response in figure 8, $\|\xi^2\|_2 < 10^{-6}$ for $\delta\mu/G = 8.0$. For the Stokes calculations of the DC response we can be more liberal with the grid sizes and employ up to $45 \times 45 \times 50$ grids where again $\|\xi^2\|_2 \lesssim 10^{-4}$ for the strongest lattices and lowest temperatures.

Data Availability Statement. This article has no associated data or the data will not be deposited.

¹⁷The 2-norm is taken over all grid points.

Code Availability Statement. This article has no associated code or the code will not be deposited.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] S. Sachdev, *Quantum Phases of Matter*, Cambridge University Press (2023) [[DOI:10.1017/9781009212717](https://doi.org/10.1017/9781009212717)].
- [2] F. Balm et al., *T-linear resistivity, optical conductivity, and Planckian transport for a holographic local quantum critical metal in a periodic potential*, *Phys. Rev. B* **108** (2023) 125145 [[arXiv:2211.05492](https://arxiv.org/abs/2211.05492)] [[INSPIRE](#)].
- [3] P. Chesler, A. Lucas and S. Sachdev, *Conformal field theories in a periodic potential: results from holography and field theory*, *Phys. Rev. D* **89** (2014) 026005 [[arXiv:1308.0329](https://arxiv.org/abs/1308.0329)] [[INSPIRE](#)].
- [4] C.-H. Park et al., *Anisotropic behaviours of massless Dirac fermions in graphene under periodic potentials*, *Nature Phys.* **4** (2008) 213 [[arXiv:0803.0306](https://arxiv.org/abs/0803.0306)].
- [5] J.H. Ho, Y.H. Chiu, S.J. Tsai and M.F. Lin, *Semimetallic graphene in a modulated electric potential*, *Phys. Rev. B* **79** (2009) 115427 [[arXiv:0809.1514](https://arxiv.org/abs/0809.1514)].
- [6] C.-H. Park et al., *New Generation of Massless Dirac Fermions in Graphene under External Periodic Potentials*, *Phys. Rev. Lett.* **101** (2008) 126804 [[arXiv:0809.3422](https://arxiv.org/abs/0809.3422)].
- [7] L.-G. Wang and S.-Y. Zhu, *Electronic band gaps and transport properties in graphene superlattices with one-dimensional periodic potentials of square barriers*, *Phys. Rev. B* **81** (2010) 205444 [[arXiv:1003.0732](https://arxiv.org/abs/1003.0732)].
- [8] P. Burset, A.L. Yeyati, L. Brey and H.A. Fertig, *Transport in superlattices on single-layer graphene*, *Phys. Rev. B* **83** (2011) 195434 [[arXiv:1101.5270](https://arxiv.org/abs/1101.5270)].
- [9] M. Yankowitz et al., *Emergence of superlattice Dirac points in graphene on hexagonal boron nitride*, *Nature Phys.* **8** (2012) 382 [[arXiv:1202.2870](https://arxiv.org/abs/1202.2870)].
- [10] L. Brey, T. Stauber, L. Martín-Moreno and G. Gómez-Santos, *Nonlocal Quantum Effects in Plasmons of Graphene Superlattices*, *Phys. Rev. Lett.* **124** (2020) 257401 [[arXiv:1912.05195](https://arxiv.org/abs/1912.05195)].
- [11] D.E. Fernandes, *Effective medium model for graphene superlattices with electrostatic and magnetic vector potentials*, *Phys. Rev. B* **107** (2023) 085119 [[arXiv:2301.02480](https://arxiv.org/abs/2301.02480)].
- [12] S. Sachdev and M. Mueller, *Quantum criticality and black holes*, *J. Phys. Condens. Matter* **21** (2009) 164216 [[arXiv:0810.3005](https://arxiv.org/abs/0810.3005)] [[INSPIRE](#)].
- [13] M. Mueller, J. Schmalian and L. Fritz, *Graphene: A Nearly Perfect Fluid*, *Phys. Rev. Lett.* **103** (2009) 025301 [[arXiv:0903.4178](https://arxiv.org/abs/0903.4178)] [[INSPIRE](#)].
- [14] J. Zaanen, Y.-W. Sun, Y. Liu and K. Schalm, *Holographic Duality in Condensed Matter Physics*, Cambridge Univ. Press (2015) [[DOI:10.1017/CB09781139942492](https://doi.org/10.1017/CB09781139942492)] [[INSPIRE](#)].
- [15] S.A. Hartnoll, A. Lucas and S. Sachdev, *Holographic quantum matter*, [arXiv:1612.07324](https://arxiv.org/abs/1612.07324) [[INSPIRE](#)].
- [16] C. Li et al., *Strange Metal and Superconductor in the Two-Dimensional Yukawa-Sachdev-Ye-Kitaev Model*, *Phys. Rev. Lett.* **133** (2024) 186502 [[arXiv:2406.07608](https://arxiv.org/abs/2406.07608)] [[INSPIRE](#)].

- [17] P.W. Anderson, *Hall effect in the two-dimensional Luttinger liquid*, *Phys. Rev. Lett.* **67** (1991) 2092 [[INSPIRE](#)].
- [18] N. Chagnet et al., *Natural anomalous cyclotron response in a hydrodynamic local quantum critical metal in a periodic potential*, [arXiv:2409.11095](#) [[INSPIRE](#)].
- [19] A. Amoretti et al., *Hydrodynamical description for magneto-transport in the strange metal phase of Bi-2201*, *Phys. Rev. Res.* **2** (2020) 023387 [[arXiv:1909.07991](#)] [[INSPIRE](#)].
- [20] M. Blake and A. Donos, *Quantum Critical Transport and the Hall Angle*, *Phys. Rev. Lett.* **114** (2015) 021601 [[arXiv:1406.1659](#)] [[INSPIRE](#)].
- [21] I.M. Hayes et al., *Scaling between Magnetic Field and Temperature in the High-Temperature Superconductor $\text{BaFe}_2(\text{As}_{1-x}\text{P}_x)_2$* , *Nature Phys.* **12** (2016) 916 [[arXiv:1412.6484](#)] [[INSPIRE](#)].
- [22] J. Ayres et al., *Incoherent transport across the strange-metal regime of overdoped cuprates*, *Nature* **595** (2021) 661 [[arXiv:2012.01208](#)].
- [23] P. Giraldo-Gallo et al., *Scale-invariant magnetoresistance in a cuprate superconductor*, *Science* **361** (2018) 479 [[arXiv:1705.05806](#)].
- [24] C. Kittel, *Introduction to solid state physics*, eighth ed., Wiley, Hoboken, NJ (2005) [ISBN: 9780471415268].
- [25] M.M. Parish and P.B. Littlewood, *Non-saturating magnetoresistance in heavily disordered semiconductors*, *Nature* **426** (2003) 162 [[cond-mat/0312020](#)].
- [26] N. Ramakrishnan et al., *Equivalence of effective medium and random resistor network models for disorder-induced unsaturating linear magnetoresistance*, *Phys. Rev. B* **96** (2017) 224203 [[arXiv:1703.05478](#)].
- [27] V. Guttal and D. Stroud, *Model for a macroscopically disordered conductor with an exactly linear high-field magnetoresistance*, *Phys. Rev. B* **71** (2005) 201304 [[cond-mat/0502162](#)].
- [28] E. Kiritsis and L. Li, *Quantum Criticality and DBI Magneto-resistance*, *J. Phys. A* **50** (2017) 115402 [[arXiv:1608.02598](#)] [[INSPIRE](#)].
- [29] C. Boyd and P.W. Phillips, *Single-Parameter Scaling in the Magnetoresistance of Optimally Doped $\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$* , *Phys. Rev. B* **100** (2019) 155139 [[arXiv:1905.02737](#)].
- [30] M. Headrick, S. Kitchen and T. Wiseman, *A new approach to static numerical relativity, and its application to Kaluza-Klein black holes*, *Class. Quant. Grav.* **27** (2010) 035002 [[arXiv:0905.1822](#)] [[INSPIRE](#)].
- [31] A. Donos and J.P. Gauntlett, *Minimally packed phases in holography*, *JHEP* **03** (2016) 148 [[arXiv:1512.06861](#)] [[INSPIRE](#)].
- [32] A. Donos, J.P. Gauntlett and C. Pantelidou, *Holographic Abrikosov Lattices*, *JHEP* **07** (2020) 095 [[arXiv:2001.11510](#)] [[INSPIRE](#)].
- [33] A. Donos and J.P. Gauntlett, *Thermoelectric DC conductivities from black hole horizons*, *JHEP* **11** (2014) 081 [[arXiv:1406.4742](#)] [[INSPIRE](#)].
- [34] A. Donos and J.P. Gauntlett, *The thermoelectric properties of inhomogeneous holographic lattices*, *JHEP* **01** (2015) 035 [[arXiv:1409.6875](#)] [[INSPIRE](#)].
- [35] E. Banks, A. Donos and J.P. Gauntlett, *Thermoelectric DC conductivities and Stokes flows on black hole horizons*, *JHEP* **10** (2015) 103 [[arXiv:1507.00234](#)] [[INSPIRE](#)].
- [36] A. Donos et al., *Holographic DC conductivity and Onsager relations*, *JHEP* **07** (2017) 006 [[arXiv:1704.05141](#)] [[INSPIRE](#)].

- [37] S.A. Hartnoll, P.K. Kovtun, M. Müller and S. Sachdev, *Theory of the Nernst effect near quantum phase transitions in condensed matter, and in dyonic black holes*, *Phys. Rev. B* **76** (2007) 144502 [[arXiv:0706.3215](#)] [[INSPIRE](#)].
- [38] M. Blake, A. Donos and N. Lohitsiri, *Magnetothermoelectric Response from Holography*, *JHEP* **08** (2015) 124 [[arXiv:1502.03789](#)] [[INSPIRE](#)].
- [39] A. Donos, J.P. Gauntlett, T. Griffin and L. Melgar, *DC Conductivity of Magnetised Holographic Matter*, *JHEP* **01** (2016) 113 [[arXiv:1511.00713](#)] [[INSPIRE](#)].
- [40] P. Kovtun, *Lectures on hydrodynamic fluctuations in relativistic theories*, *J. Phys. A* **45** (2012) 473001 [[arXiv:1205.5040](#)] [[INSPIRE](#)].
- [41] R.A. Davison and B. Goutéraux, *Dissecting holographic conductivities*, *JHEP* **09** (2015) 090 [[arXiv:1505.05092](#)] [[INSPIRE](#)].
- [42] A. Amoretti and D.K. Brattan, *On the hydrodynamics of $(2 + 1)$ -dimensional strongly coupled relativistic theories in an external magnetic field*, *Mod. Phys. Lett. A* **37** (2022) 2230010 [[arXiv:2209.11589](#)] [[INSPIRE](#)].
- [43] S. Grozdanov, A. Lucas and K. Schalm, *Incoherent thermal transport from dirty black holes*, *Phys. Rev. D* **93** (2016) 061901 [[arXiv:1511.05970](#)] [[INSPIRE](#)].
- [44] S. Grozdanov, A. Lucas, S. Sachdev and K. Schalm, *Absence of disorder-driven metal-insulator transitions in simple holographic models*, *Phys. Rev. Lett.* **115** (2015) 221601 [[arXiv:1507.00003](#)] [[INSPIRE](#)].
- [45] D. Arean, S. Grieninger and P.G. Romeu, *Disordered charged horizons*, *JHEP* **06** (2026) 013 [[arXiv:2509.08164](#)] [[INSPIRE](#)].
- [46] A. Donos and J.P. Gauntlett, *Holographic Q-lattices*, *JHEP* **04** (2014) 040 [[arXiv:1311.3292](#)] [[INSPIRE](#)].
- [47] N. Chagnet and K. Schalm, *Hydrodynamics of a relativistic charged fluid in the presence of a periodically modulated chemical potential*, *SciPost Phys.* **16** (2024) 028 [[arXiv:2303.17685](#)] [[INSPIRE](#)].
- [48] U. Gran, M. Tornsö and T. Zingg, *Exotic Holographic Dispersion*, *JHEP* **02** (2019) 032 [[arXiv:1808.05867](#)] [[INSPIRE](#)].
- [49] A.B. Harris, *Effect of random defects on the critical behaviour of Ising models*, *J. Phys. C* **7** (1974) 1671 [[INSPIRE](#)].
- [50] D. Stroud, *Generalized effective-medium approach to the conductivity of an inhomogeneous material*, *Phys. Rev. B* **12** (1975) 3368.
- [51] V.V. Cheianov, V.I. Fal'ko, B.L. Altshuler and I.L. Aleiner, *Random Resistor Network Model of Minimal Conductivity in Graphene*, *Phys. Rev. Lett.* **99** (2007) 176801 [[arXiv:0706.2968](#)].
- [52] E. Rossi, S. Adam and S. Das Sarma, *Effective medium theory for disordered two-dimensional graphene*, *Phys. Rev. B* **79** (2009) 245423 [[arXiv:0809.1425](#)].
- [53] D. Valentini, J. Schmalian, S. Sachdev and A.A. Patel, *Superlinear Hall angle and carrier mobility from non-Boltzmann magnetotransport in the spatially disordered Yukawa-Sachdev-Ye-Kitaev model on a square lattice*, [arXiv:2511.01030](#) [[DOI:10.1103/32ts-qh8d](#)].
- [54] D. Brizuela, J.M. Martín-García and G.A. Mena Marugán, *x Pert: Computer algebra for metric perturbation theory*, *Gen. Rel. Grav.* **41** (2009) 2415 [[arXiv:0807.0824](#)] [[INSPIRE](#)].

- [55] S. Balay, W.D. Gropp, L.C. McInnes and B.F. Smith, *Efficient management of parallelism in object-oriented numerical software libraries*, in *Modern software tools for scientific computing*, E. Arge, A.M. Bruaset and H.P. Langtangen eds., Birkhäuser, Boston (1997) p. 163–202 [[DOI:10.1007/978-1-4612-1986-6_8](#)].
- [56] S. Balay et al., *PETSc/TAO Users Manual (Rev. 3.20)*, ANL-21/39 Rev. 3.20, Argonne National Laboratory (2023) [[DOI:10.2172/2205494](#)].
- [57] E. Bueler, *PETSc for Partial Differential Equations: Numerical Solutions in C and Python*, Society for Industrial and Applied Mathematics (2020) [[DOI:10.1137/1.9781611976311](#)].