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Scattering of electro-elastic waves by a piezoelectric circle

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ABSTRACT

The scattering by a piezoelectric circle of class orthorhombic 222 in an isotropic (nonpiezoelectric) surrounding is investigated for incident anti-plane (SH) elastic or transverse magnetic electromagnetic waves. Both the full piezoelectric equations and the quasi-static approximation are used. The approach employs a power series expansion in Cartesian coordinates inside the circle to derive recursion relations for the expansion coefficients. This expansion is then transformed to polar coordinates to satisfy the boundary conditions. For low frequencies it is possible to obtain explicit, simple expressions for the transition matrix elements. A numerical example is given for the far field scattering amplitude and it is seen that the effect of piezoelectricity on the scattering of an elastic wave is relatively small . Finally, it is concluded that the quasi-static approximation is valid for this scattering problem for low frequencies.

1. Introduction

Piezoelectric materials, which offer the possibility to transform mechanical energy into electrical energy or vice versa, have many important technical applications, particularly as sensors and actuators [1]. In this context piezoelectric composites have proven to be very useful [2] and for this reason a few investigations on the scattering by a piezoelectric obstacle in a purely elastic surrounding have been conducted. Thus, Moon [3] considers the scattering by a transversely isotropic piezoelectric cylinder in 2D and Qian et al. [4] solve the corresponding 3D problem. Due to the transverse isotropy the solution procedure can use separation of variables. This is also true for the scattering by a piezoelectric sphere in the special case when the anisotropy is spherically transversely isotropic [5] (with isotropy in the angular directions). The general situation with both the obstacle and the matrix piezoelectric is investigated by a polarization method (using a Green function calculated with a Radon transform) by Hu and Wang [6] in 2D and by Ma and Wang [7] in 3D.

The scattering by an anisotropic (nonpiezoelectric) circle in an isotropic surrounding is investigated in 2D by Boström [8,9] for both the anti-plane and in-plane elastodynamic cases. In this approach the interior field is expanded in trigonometric functions in the angular direction and powers in the radial direction. The equations of motion lead to recursion relations among the expansion coefficients and the boundary conditions then solve for the remaining unknowns. The approach is particularly well suited for low frequencies and explicit expressions for the transition (T-) matrix elements can be given. The method can be used also in 3D for the scattering by an anisotropic sphere [10]. The method can be extended to a piezoelectric circle, and the purpose here is to perform this generalization in the anti-plane (SH) elastic case. As there is coupling to electromagnetic waves the scattering by a transverse magnetic wave is considered at the same time. Furthermore, a modification is performed to the method so that the fields are first expanded in Cartesian coordinates which leads to a much simpler derivation of recursion relations. The expansion is then transformed to polar coordinates to apply the boundary conditions. Explicit expressions for the leading order T-matrix elements for low frequencies

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are given. Both the full piezoelectric constitutive equations and the quasi-static approximation are used and for low frequencies both methods give identical results. It is noted that some previous methods assume the quasi-static approximation without comment [4–7].

2. Scattering by a circle

Consider the 2D (in the xy plane) scattering of an anti-plane elastic or electromagnetic incident wave by a circular obstacle of radius a in an isotropic and homogeneous (nonpiezoelectric) elastic surrounding. Thus the main field variables are the z components of the displacement u_z and the magnetic field H_z. The material outside the circle has shear modulus μ and density ρ. Only time harmonic conditions are considered and the time factor $\exp(-\mathrm{i}\omega t)$, where ω is the frequency, is suppressed. The corresponding shear wavenumber is $k_s = \omega(\rho/\mu)^{1/2}$. The surrounding is assumed nonmagnetic so the only constitutive constants needed are the permittivity ϵ_0 and the permeability of vacuum μ_0. The corresponding electromagnetic wavenumber is $k_0 = \omega\sqrt{\epsilon_0\mu_0}$. The piezoelectric material inside the circle is described in the next section.

In this 2D case with the coupled scalar quantities u_z and H_z wavefunctions are defined according to the following [11]

$$\chi^0_{\sigma m}(k;\boldsymbol{r}) = \frac{\sqrt{\epsilon_m}}{2}\,\mathrm{J}_m(kr)\begin{Bmatrix}\cos m\varphi\\ \sin m\varphi\end{Bmatrix} \tag{1}$$

Here $m = 0,1,2,\dots$ and $\sigma = e$ ("e" for even) for the upper row (cosine function) and $\sigma = o$ ("o" for odd) for the lower row (sine function) with $m = 0$ excluded for $\sigma = o$. The Neumann factor $\epsilon_0 = 1$ and $\epsilon_m = 2$ for $m = 1,2,\dots$. The factor $(\epsilon_m)^{1/2}/2$ is included to get a symmetric T-matrix. The upper index 0 denotes the regular wavefunctions with a Bessel function; there is a corresponding out-going wavefunction, which is denoted $\chi^+_{\sigma m}(k;\boldsymbol{r})$, that contains a Hankel function instead of a Bessel function. The wavenumber in the wavefunctions can be either the elastic one k_s or the electromagnetic one k_0.

The sources of the waves are assumed to lie outside the obstacle. Often they are far away, and a very common assumption is that they are at infinity and the incident wave is a plane wave. As there are no sources in the vicinity of the obstacle the incident wave must be finite there and it must be possible to expand an incident SH wave in the regular wavefunctions

$$u^{\mathrm{in}}_z(\boldsymbol{r}) = \sum_{\sigma,m} b_{\sigma m}\chi^0_{\sigma m}(k_s;\boldsymbol{r}) \tag{2}$$

where the summation is over $\sigma = e, o$ and $m = 0,1,2,\dots$. This expansion is valid in a circle centred at the obstacle out to the sources of the incident wave. The coefficients $b_{\sigma m}$ are in principle known. For an incident electromagnetic wave the corresponding expansion is

$$H^{\mathrm{in}}_z(\boldsymbol{r}) = \sum_{\sigma,m} c_{\sigma m}\chi^0_{\sigma m}(k_0;\boldsymbol{r}) \tag{3}$$

with known coefficients $c_{\sigma m}$.

The SH wave scattered by the obstacle is carrying energy outwards from the obstacle (the radiation condition) and it is thus possible to expand it in the out-going wavefunctions

$$u^{\mathrm{sc}}_z(\boldsymbol{r}) = \sum_{\sigma,m} f_{\sigma m}\chi^+_{\sigma m}(k_s;\boldsymbol{r}) \tag{4}$$

Here the coefficients $f_{\sigma m}$ are to be determined. The corresponding expansion of the scattered magnetic field is

$$H^{\mathrm{sc}}_z(\boldsymbol{r}) = \sum_{\sigma,m} g_{\sigma m}\chi^+_{\sigma m}(k_0;\boldsymbol{r}) \tag{5}$$

with unknown coefficients $g_{\sigma m}$.

Another way to put this is to say that the T-matrix of the obstacle is to be determined, where this matrix is defined as the linear relation between the expansion coefficients of the incident and scattered waves

$$f_{\sigma m} = \sum_{\sigma',m'}\left(T^{mm}_{\sigma m,\sigma' m'} b_{\sigma' m'} + T^{me}_{\sigma m,\sigma' m'} c_{\sigma' m'}\right) \tag{6}$$

$$g_{\sigma m} = \sum_{\sigma',m'}\left(T^{em}_{\sigma m,\sigma' m'} b_{\sigma' m'} + T^{ee}_{\sigma m,\sigma' m'} c_{\sigma' m'}\right) \tag{7}$$

with m for "mechanical" and e for "electromagnetic". An advantage with calculating the T-matrix is that it fully characterizes the obstacle but is independent of the incident wave. If the material inside the circle is isotropic the T-matrix becomes diagonal in the m indices and independent of the σ indices. Often the wavefunctions are defined so that the T-matrix is fully symmetric, but in this case this is not so in that $T^{em}_{\sigma m,\sigma' m'} \neq T^{me}_{\sigma' m',\sigma m}$ due to the different dimensions of the elastic and magnetic fields.

As a comparison the scattering for low frequencies (radius small compared to wavelength) by an anisotropic elastic (but nonpiezoelectric) circle will be performed. To leading order for low frequencies only monopole ($m = 0$) and dipole ($m = 1$) terms contribute. For an elastic material with only the two shear stiffnesses C_{44} and C_{55} nonzero for SH wave propagation the relevant T-matrix elements are [11]

$$T^{ee}_{e0,e0} = (k_s a)^2\,\frac{\mathrm{i}\pi}{4}\left(\frac{\rho_1}{\rho} - 1\right) \tag{8}$$

$$T^{ee}_{e1,e1} = (k_s a)^2 \frac{\mathrm{i}\pi}{4} \frac{\mu - C_{55}}{\mu + C_{55}} \tag{9}$$

$$T^{ee}_{o1,o1} = (k_s a)^2 \frac{\mathrm{i}\pi}{4} \frac{\mu - C_{44}}{\mu + C_{44}} \tag{10}$$

Here ρ_1 is the density of the circle. The monopole scattering only depends on the density and this is due to the fact that $m = 0$ corresponds to a (anti-plane) rigid body motion to lowest order and thus involves no strain and therefore no stress either. This lowest order T-matrix element is expected to be the same also for a piezoelectric circle as the vanishing of the strain leads to no coupling to the electric field.

It seems that the corresponding T-matrix elements for the electromagnetic scattering by an anisotropic circle are not to be found in the literature.

3. 2D piezoelectricity

The general piezoelectric constitutive equations in tensor notation are

$$\sigma_{ij} = c_{ijkl} s_{kl} - e_{kij} E_k \tag{11}$$

$$D_k = \epsilon_{kl} E_l + e_{kij} s_{ij} \tag{12}$$

where σ_{ij} is the stress tensor, s_{ij} the strain tensor, D_k the electric displacement, and E_k the electric field. The constitutive parameters are the stiffness constants c_{ijkl}, the permittivity constants ϵ_{kl}, and the piezoelectric coupling constants e_{ijk}. The dynamic equations of motion are the elastodynamic one

$$\partial_j \sigma_{ij} = \rho_1 \partial_t^2 u_i \tag{13}$$

and two of Maxwell's equations

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B} \tag{14}$$

$$\nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D} \tag{15}$$

where the latter two are written in vector notation. Here $\boldsymbol{H}$ is the magnetic field and $\boldsymbol{B}$ is the magnetic induction. The material is assumed nonmagnetic so the magnetic induction is eliminated by the constitutive equation $\boldsymbol{B} = \mu_0 \boldsymbol{H}$.

To get a 2D situation all fields are assumed independent of the z coordinate and all z derivatives are then vanishing. In general the constitutive equations lead to a coupling between all the components of the fields, but for special choices of constitutive equations a decoupling into two subproblems is possible. One possibility is a piezoelectric material of class hexagonal 6 (with $6mm$ as a special case) with the symmetry axis along z, which leads to two isotropic subproblems in the xy plane, one for the anti-plane displacement component and one for the two in-plane components (Moon [3]). Another case of interest is a material of class orthorhombic 222, with a cubic material as a special case. Turning directly to the 2D case the constituent equations for this material become

$$\sigma_{xx} = C_{11} \partial_x u_x + C_{12} \partial_y u_y \tag{16}$$

$$\sigma_{yy} = C_{12} \partial_x u_x + C_{22} \partial_y u_y \tag{17}$$

$$\sigma_{zz} = C_{13} \partial_x u_x + C_{23} \partial_y u_y \tag{18}$$

$$\sigma_{yz} = C_{44} \partial_y u_z - e_{14} E_x \tag{19}$$

$$\sigma_{zx} = C_{55} \partial_x u_z - e_{25} E_y \tag{20}$$

$$\sigma_{xy} = C_{66} \left(\partial_y u_x + \partial_x u_y \right) - e_{36} E_z \tag{21}$$

$$D_x = \epsilon_1 E_x + e_{14} \partial_y u_z \tag{22}$$

$$D_y = \epsilon_2 E_y + e_{25} \partial_x u_z \tag{23}$$

$$D_z = \epsilon_3 E_z + e_{36} \left(\partial_y u_x + \partial_x u_y \right) \tag{24}$$

For the stiffness and piezoelectric coupling constants the reduced notation C_{IJ} and e_{iJ}, $I, J = 1,2,3,4,5,6$, $i = 1,2,3$, is used, see Auld [12]. The permittivity tensor is diagonal with entries $\epsilon_1 = \epsilon_{11}$, etc. There are thus eight independent stiffness constants, three permittivity constants, and three piezoelectric coupling constants.

With the strains eliminated the components of the elastodynamic equations of motion for the orthorhombic 222 material become

$$C_{11} \partial_x^2 u_x + C_{66} \partial_y^2 u_x + \left(C_{12} + C_{66} \right) \partial_x \partial_y u_y - e_{36} \partial_y E_z + \rho_1 \omega^2 u_x = 0 \tag{25}$$

$$C_{22}\partial_y^2 u_y + C_{66}\partial_x^2 u_y + \left(C_{12} + C_{66}\right)\partial_x\partial_y u_x - e_{36}\partial_x E_z + \rho_1\omega^2 u_y = 0 \tag{26}$$

$$C_{55}\partial_x^2 u_z + C_{44}\partial_y^2 u_z - e_{14}\partial_y E_x - e_{25}\partial_x E_y + \rho_1\omega^2 u_z = 0 \tag{27}$$

The components of Maxwell's equations become

$$\partial_y E_z = \mathrm{i}\omega\mu_0 H_x \tag{28}$$

$$-\partial_x E_z = \mathrm{i}\omega\mu_0 H_y \tag{29}$$

$$\partial_x E_y - \partial_y E_x = \mathrm{i}\omega\mu_0 H_z \tag{30}$$

$$\partial_y H_z = -\mathrm{i}\omega\epsilon_1 E_x - \mathrm{i}\omega e_{14}\partial_y u_z \tag{31}$$

$$-\partial_x H_z = -\mathrm{i}\omega\epsilon_2 E_y - \mathrm{i}\omega e_{25}\partial_x u_z \tag{32}$$

$$\partial_x H_y - \partial_y H_x = -\mathrm{i}\omega\epsilon_3 E_z - \mathrm{i}\omega e_{36}\left(\partial_y u_x + \partial_x u_y\right) \tag{33}$$

From all these equations it is seen that there is a decoupling into two subproblems, the anti-plane problem for the components u_z, H_z, E_x, E_y and the in-plane problem for u_x, u_y, E_z, H_x, H_y. Here only the anti-plane problem is considered.

4. The piezoelectric circle

From Eqs. (31) and (32) the components E_x and E_y are eliminated and then there remain two equations of motion for u_z and H_z

$$C_5\partial_x^2 u_z + C_4\partial_y^2 u_z + \rho_1\omega^2 u_z + \frac{\mathrm{i}e_{25}}{\omega\epsilon_2}\partial_x^2 H_z - \frac{\mathrm{i}e_{14}}{\omega\epsilon_1}\partial_y^2 H_z = 0 \tag{34}$$

$$\frac{1}{\epsilon_2}\partial_x^2 H_z + \frac{1}{\epsilon_1}\partial_y^2 H_z + \mu_0\omega^2 H_z - \frac{\mathrm{i}\omega e_{25}}{\epsilon_2}\partial_x^2 u_z + \frac{\mathrm{i}\omega e_{14}}{\epsilon_1}\partial_y^2 u_z = 0 \tag{35}$$

Here the "stiffened" elastic constants are introduced

$$C_4 = C_{44}\left(1 + \alpha_4\right), \qquad \alpha_4 = \frac{e_{14}^2}{\epsilon_1 C_{44}} \tag{36}$$

$$C_5 = C_{55}\left(1 + \alpha_5\right), \qquad \alpha_5 = \frac{e_{25}^2}{\epsilon_2 C_{55}} \tag{37}$$

where α_4 and α_5 are two dimensionless constants measuring the degree of coupling between the mechanical and electromagnetic fields.

When the scattering by the circle is solved the boundary conditions on $r = a$ are needed. These are taken as the ordinary welded conditions which means that u_z, σ_{zr}, H_z, and E_φ are to be continuous. To this end σ_{zr} and E_φ are expressed in terms of u_z and H_z:

$$\sigma_{zr} = \cos\varphi\left(C_5\partial_x u_z + \frac{\mathrm{i}e_{25}}{\omega\epsilon_2}\partial_x H_z\right) + \sin\varphi\left(C_4\partial_y u_z - \frac{\mathrm{i}e_{14}}{\omega\epsilon_1}\partial_y H_z\right) \tag{38}$$

$$E_\varphi = \cos\varphi\left(-\frac{\mathrm{i}}{\omega\epsilon_2}\partial_x H_z - \frac{e_{25}}{\epsilon_2}\partial_x u_z\right) - \sin\varphi\left(-\frac{\mathrm{i}}{\omega\epsilon_1}\partial_y H_z - \frac{e_{14}}{\epsilon_1}\partial_y u_z\right) \tag{39}$$

Due to the symmetries of the anisotropic circle it is possible to divide the scattering problem into four subproblems which are even-even, even-odd, odd-even, and odd-odd with respect to x and y. In the following these are taken up in turn, starting with the odd-even problem, this being the simplest one (together with the almost identical even-odd one).

4.1. The odd-even problem

For this problem u_z and H_z should be odd with respect to x and even with respect to y, thus

$$u_z = u_{10}x + u_{30}x^3 + u_{12}xy^2 + \dots \tag{40}$$

$$H_z = H_{10}x + H_{30}x^3 + H_{12}xy^2 + \dots \tag{41}$$

where as many terms as needed are incorporated. This number depends on the frequency, but as mentioned in the introduction the focus is on low frequency scattering and then only the first term is actually needed. Inserting these expansions into the equations of motion (Eqs. (34) and (35)) two recursion relations result from the terms linear in x:

$$6C_5 u_{30} + 2C_4 u_{12} + \rho_1\omega^2 u_{10} + \frac{6\mathrm{i}e_{25}}{\omega\epsilon_2}H_{30} - \frac{2\mathrm{i}e_{14}}{\omega\epsilon_1}H_{12} = 0 \tag{42}$$

$$\frac{6}{\epsilon_2}H_{30}+\frac{2}{\epsilon_1}H_{12}+\mu_0\omega^2H_{10}-\frac{6\mathrm{i}\omega e_{25}}{\epsilon_2}u_{30}+\frac{2\mathrm{i}\omega e_{14}}{\epsilon_1}u_{12}=0 \tag{43}$$

These equations involve exactly those expansion coefficients given in Eqs. (40) and (41). More recursion relations are obtained if more terms are retained in the expansions. To lowest order only the terms in u_{10} and H_{10} are needed and then no recursion relation at all is obtained.

When applying the boundary conditions the fields are first transformed to polar coordinates and then one equation is obtained for each azimuthal order $m = 1, 3, \ldots$, but to lowest order only $m = 1$ is actually needed. Expanding the Bessel and Hankel functions to lowest order in the expansions of the incident and scattered fields in Eqs. (2)–(5), the four boundary conditions on u_z, H_z, σ_{zr}, and E_z give

$$b_{e1}\frac{k_s a}{2}-f_{e1}\frac{2\mathrm{i}}{\pi k_s a}=u_{10}a \tag{44}$$

$$\frac{1}{2}k_0 a-g_{e1}\frac{2\mathrm{i}}{\pi k_0 a}=H_{10}a \tag{45}$$

$$b_{e1}\frac{\mu k_s}{2}+f_{e1}\frac{2\mathrm{i}\mu}{\pi k_s a^2}=u_{10}C_5+H_{10}\frac{\mathrm{i}e_{25}}{\omega\epsilon_2} \tag{46}$$

$$-\frac{\mathrm{i}k_0}{2\omega\epsilon_0}+g_{e1}\frac{2}{\pi\omega\epsilon_0 k_0 a^2}=-H_{10}\frac{\mathrm{i}}{\omega\epsilon_2}-u_{10}\frac{e_{25}}{\epsilon_2} \tag{47}$$

The solution for the scattered wave coefficients becomes

$$\begin{aligned} f_{e1}&=\mathrm{i}(k_s a)^2\frac{\pi}{4}\frac{(\mu-C_{55})(\epsilon_2+\epsilon_0)-e_{25}^2}{(\mu+C_{55})(\epsilon_2+\epsilon_0)+e_{25}^2}b_{e1}\\ &\quad+k_s k_0 a^2\frac{\pi}{2}\frac{\epsilon_0 e_{25}}{\omega((\mu+C_{55})(\epsilon_2+\epsilon_0)+e_{25}^2)}c_{e1} \end{aligned} \tag{48}$$

$$\begin{aligned} g_{e1}&=k_0 k_s a^2\frac{\pi}{2}\frac{\mu\epsilon_0 e_{25}}{(\mu+C_{55})(\epsilon_2+\epsilon_0)+e_{25}^2}b_{e1}\\ &\quad+\mathrm{i}(k_0 a)^2\frac{\pi}{4}\frac{(\mu+C_{55})(\epsilon_2-\epsilon_0)+e_{25}^2}{(\mu+C_{55})(\epsilon_2+\epsilon_0)+e_{25}^2}c_{e1} \end{aligned} \tag{49}$$

To leading order at low frequencies there are thus four T-matrix elements

$$T^{mm}_{e1,e1}=\mathrm{i}(k_s a)^2\frac{\pi}{4}\frac{(\mu-C_{55})(\epsilon_0+\epsilon_2)-e_{25}^2}{(\mu+C_{55})(\epsilon_0+\epsilon_2)+e_{25}^2} \tag{50}$$

$$T^{me}_{e1,e1}=k_s k_0 a^2\frac{\pi}{2}\frac{\epsilon_0 e_{25}}{\omega((\mu+C_{55})(\epsilon_2+\epsilon_0)+e_{25}^2)} \tag{51}$$

$$T^{em}_{e1,e1}=k_0 k_s a^2\frac{\pi}{2}\frac{\mu\epsilon_0 e_{25}}{(\mu+C_{55})(\epsilon_2+\epsilon_0)+e_{25}^2} \tag{52}$$

$$T^{ee}_{e1,e1}=\mathrm{i}(k_0 a)^2\frac{\pi}{4}\frac{(\mu+C_{55})(\epsilon_2-\epsilon_0)+e_{25}^2}{(\mu+C_{55})(\epsilon_2+\epsilon_0)+e_{25}^2} \tag{53}$$

For a nonpiezoelectric material $e_{25}=0$ and the first of these reduces to the corresponding one in a purely elastic material given in Eq. (9). The difference between these two elements is of the order of α_5, which is a number that is usually around 0.1 or smaller.

4.2. The even-odd problem

This problem is the same as the odd-even one, it is just to exchange some of the material constants. Thus replace C_{55} by C_{44}, ϵ_2 by ϵ_1, and e_{25} by e_{14}, and this gives the four T-matrix elements $T^{ij}_{o1,o1}$ with $i = m, e$ and $j = m, e$.

4.3. The even-even problem

This is the most complicated subproblem. The expansion of the interior fields are even in both x and y

$$u_z = u_{00} + u_{20}x^2 + u_{02}y^2 + \dots \tag{54}$$

$$H_z = H_{00} + H_{20}x^2 + H_{02}y^2 + \dots \tag{55}$$

All the terms included are necessary for a solution for low frequencies. The terms independent of both x and y in the equations of motion (Eqs. (34) and (35)) give the two recursion relations

$$2C_5u_{20} + 2C_4u_{02} + \rho\omega^2u_{00} + \frac{2\mathrm{i}e_{25}}{\omega\epsilon_2}H_{20} - \frac{2\mathrm{i}e_{14}}{\omega\epsilon_1}H_{02} = 0 \tag{56}$$

$$\frac{2}{\epsilon_2}H_{20} + \frac{2}{\epsilon_1}H_{02} + \mu_0\omega^2H_{00} - \frac{2\mathrm{i}\omega e_{25}}{\epsilon_2}u_{20} + \frac{2\mathrm{i}\omega e_{14}}{\epsilon_1}u_{02} = 0 \tag{57}$$

These two equations are the only ones needed for low frequencies, but if more terms are retained in the expansions in Eqs. (54) and (55) more recursion relations are obtained.

When applying the boundary conditions for low frequencies azimuthal order $m = 0$ and $m = 2$ are both needed, and this gives eight boundary conditions for u_z, σ_{zr}, H_z and E_φ. Together with the recursion relations this yields ten equations in the ten unknowns f_{e0}, f_{e2}, g_{e0}, g_{e2}, u_{00}, u_{20}, u_{02}, H_{00}, H_{20}, and H_{02}. These equations can be solved analytically and for the elastic part in fact gives the T-matrix element (which is now called $T^{mm}_{e0,e0}$) given in Eq. (8) as expected. To leading order the coupling elements $T^{me}_{e0,e0}$ and $T^{em}_{e0,e0}$ are vanishing, and this also applies to the scattering of electromagnetic to electromagnetic waves (the element $T^{ee}_{e0,e0}$).

4.4. The odd-odd problem

In this case the fields are odd in both x and y

$$u_z = u_{11}xy + \dots \tag{58}$$

$$H_z = H_{11}xy + \dots \tag{59}$$

In polar coordinates the lowest power of r is thus r^2 and this means that there is no monopole or dipole contribution, so the odd-odd problem does not give a leading order contribution for low frequencies.

5. The quasi-static approximation

As mentioned in the introduction the quasi-static approximation is often applied for wave propagation problems in piezoelectric media when the focus is on the mechanical part. Therefore, in this section the T-matrix elements involving the electromagnetic fields are not calculated. In the quasi-static approximation the electric field is assumed as the gradient of a scalar potential ϕ

$$\boldsymbol{E} = -\nabla\phi \tag{60}$$

It follows that the magnetic field vanishes. The differential equations determining u_z and ϕ are Eq. (27) and the vanishing (assuming no free charges) of the divergence of the electric displacement field

$$C_{55}\partial_x^2u_z + C_{44}\partial_y^2u_z + \rho_1\omega^2u_z + \left(e_{14} + e_{25}\right)\partial_x\partial_y\phi = 0 \tag{61}$$

$$-\epsilon_1\partial_x^2\phi - \epsilon_2\partial_y^2\phi + \left(e_{14} + e_{25}\right)\partial_x\partial_yu_z = 0 \tag{62}$$

The last equation is not hyperbolic, although it is still assumed that the potential has a harmonic time dependence.

There is still a division into four subproblems depending on the symmetries.

5.1. The odd-even problem

In this case the displacement and potential have the following expansions

$$u_z = u_{10}x + u_{30}x^3 + u_{12}xy^2 + \dots \tag{63}$$

$$\phi = \phi_{01}y + \phi_{03}y^3 + \phi_{12}x^2y + \dots \tag{64}$$

Due to the structure of the differential equations the potential has the opposite symmetries to the displacement. Inserting the expansions into the governing differential equations yields

$$6C_{55}u_{30} + 2C_{44}u_{12} + \rho_1\omega^2u_{10} + \left(e_{14} + e_{25}\right)\phi_{12} = 0 \tag{65}$$

$$-2\epsilon_1\phi_{30} - \frac{2}{\epsilon_1}H_{12} + \mu_0\omega^2 H_{12} - \frac{6\mathrm{i}\omega e_{25}}{\epsilon_2}u_{30} + \frac{2\mathrm{i}\omega e_{14}}{\epsilon_1}u_{12} = 0 \tag{66}$$

These recursion relations involve exactly those expansion coefficients given in Eqs. (63) and (64). To lowest order only the terms in u_{10} and ϕ_{01} are needed and then no recursion relations at all are obtained.

When applying the boundary conditions an expansion of the potential of the scattered electric field is needed This is taken as the solution of the electromagnetic Helmholtz equation

$$\phi^{\mathrm{sc}}(\boldsymbol{r}) = g_1 H_1(k_0 r)\sin\varphi \tag{67}$$

The boundary conditions on $r = a$ are the continuity of u_z, σ_{zr}, ϕ, and the normal component of the electric displacement field

$$D_r = \cos\varphi\left(-\epsilon_1\partial_x\phi + e_{14}\partial_y u_z\right) + \sin\varphi\left(-\epsilon_2\partial_y\phi + e_{25}\partial_x u_z\right) \tag{68}$$

At low frequencies only the azimuthal order $m = 1$ is needed and expanding the Bessel and Hankel functions for small arguments the boundary conditions give

$$b_{e1}\frac{k_s a}{2} - f_{e1}\frac{2\mathrm{i}}{\pi k_s a} = u_{10}a \tag{69}$$

$$-g_{e1}\frac{2\mathrm{i}}{\pi k_0 a} = \phi_{01}a \tag{70}$$

$$b_{e1}\frac{\mu k_s}{2} + f_{e1}\frac{2\mathrm{i}\mu}{\pi k_s a^2} = u_{10}C_{55} + \phi_{01}e_{25} \tag{71}$$

$$g_{e1}\frac{2}{\pi\omega\epsilon_0 k_0 a^2} = -H_{10}\frac{\mathrm{i}}{\omega\epsilon_2} - u_{10}\frac{e_{25}}{\epsilon_2} \tag{72}$$

Solving these equations gives exactly the same T-matrix element as given in Eq. (50) using the full piezoelectric equations.

5.2. The even-odd problem

The even-odd part is essentially the same as the odd-even one, it is just to exchange some material parameters.

5.3. The even-even problem

This is the most complicated subproblem. The expansion of u_z is even in both x and y

$$u_z = u_{00} + u_{20}x^2 + u_{02}y^2 + \ldots \tag{73}$$

$$\phi = \phi_{11}xy + \ldots \tag{74}$$

All the terms included are here necessary for a solution for low frequencies. Only one recursion relations is obtained from the equations of motion

$$2C_{55}u_{20} + 2C_{44}u_{02} + \rho\omega^2 u_{00} + \left(e_{25} + e_{14}\right)\phi_{11} = 0 \tag{75}$$

This is the only equation needed for low frequencies.

Expanding the scattered potential and applying the boundary conditions again yields the T-matrix element given in Eq. (8).

5.4. The odd-odd problem

In this case u_z is odd in both x and y

$$u_z = u_{11}xy + \ldots \tag{76}$$

$$\phi = \phi_{20}x^2 + \phi_{02}y^2 + \ldots \tag{77}$$

As in the case when the quasi-static approximation is not applied, the odd-odd problem does not give a leading order contribution at low frequencies.

In summary this section shows that the quasi-static approximation gives identical T-matrix elements for low frequencies as the full piezoelectric equations. It is expected that this is also true for those T-matrix elements involving electromagnetic fields.

Table 1
Material parameters of Rochelle salt (Auld [12]).

$\rho_1 = 1767\,\mathrm{kg/m^3}$	$c_{44} = 0.666 \cdot 10^{10}\,\mathrm{N/m^2}$	$c_{55} = 0.285 \cdot 10^{10}\,\mathrm{N/m^2}$
$e_{14} = 2.23\,\mathrm{C/m^2}$	$e_{25} = 0.154\,\mathrm{C/m^2}$	
$d_{14} = 3.45 \cdot 10^{-10}\,\mathrm{C/N}$	$d_{25} = 0.54 \cdot 10^{-10}\,\mathrm{C/N}$	
$\epsilon_1^T = 1.815 \cdot 10^{-9}$ F/m	$\epsilon_2^T = 8.85 \cdot 10^{-11}$ F/m	
$\epsilon_1^S = 1.022 \cdot 10^{-9}$ F/m	$\epsilon_2^S = 8.02 \cdot 10^{-11}$ F/m	
$\alpha_4 = 0.0731$	$\alpha_5 = 0.1038$	

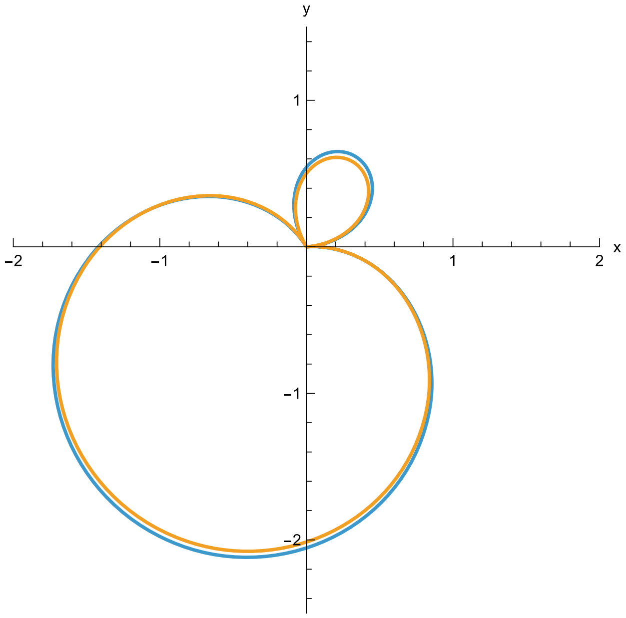

Fig. 1. The absolute value of the far field scattering amplitude for direction of incidence $\varphi_0 = \pi/3$; blue curve is with piezoelectricity and yellow curve is without piezoelectricity. The amplitude is normalized with the factor $(k_s a)^2$.

6. Numerical example

To investigate the impact of the piezoelectric effect on the scattering of elastic waves a numerical example is now given. Only the low frequency limit is considered and then the far field scattering amplitude is given by

$$F(\varphi) = T_{e0,e0} + 2T_{e1,e1} \cos\varphi \cos\varphi_0 + 2T_{o1,o1} \sin\varphi \sin\varphi_0 \tag{78}$$

Here φ_0 specifies the direction of incidence.

There do not seem to be many piezoelectric materials of class 222 that are of technical importance. Here Rochelle salt is chosen, because it is quite strongly anisotropic and has a relatively strong piezoelectric effect. The material parameters are given in Table 1 (Auld [12]). It is noted that the permittivity is given for both constant strain (ϵ^S) and constant stress (ϵ^T). It is the one for constant strain that is used in Eq. (12) and all the following, although this is not written out explicitly in the notation. Auld [12] only gives the permittivities for constant stress, but the ones for constant strain given in Table 1 are calculated from

$$\epsilon_{ij}^S = \epsilon_{ij}^T - d_{iI} c_{IJ} d_{Jj} \tag{79}$$

where d_{iI} are the piezoelectric strain constants (as opposed to the piezoelectric stress constants e_{iI} [12]). The two dimensionless piezoelectric coupling constants α_4 and α_5 are also given and indicate a fairly strong piezoelectric effect. The isotropic and nonpiezoelectric material surrounding the circle is assumed to be polyethylene with density $\rho = 990\,\mathrm{kg/m^3}$, shear modulus $\mu = 2.6 \cdot 10^8\,\mathrm{N/m^2}$, and permittivity $\epsilon = 1.992 \cdot 10^{-11}$ F/m.

Fig. 1 shows the absolute value of the far field scattering amplitude when the circle is piezoelectric (blue curve) and when piezoelectricity is ignored (yellow curve; the coupling constants $e_{14} = e_{25} = 0$). The direction of incidence is $\varphi_0 = \pi/3$, but very similar results are obtained for other directions. The curves are normalized with the factor $(k_s a)^2$, which gives the frequency and radius dependence for low frequencies. The two curves are quite close together, the difference is about 3%, so the influence of piezoelectricity is small, at least for low frequencies.

7. Concluding remarks

The scattering of anti-plane elastic and electromagnetic waves in 2D by a piezoelectric circle of symmetry class 222 in an isotropic surrounding is investigated. The method leads to a simple derivation of the leading order T-matrix elements for low frequencies. The quasi-static approximation is also investigated and it is concluded that this approximation gives identical result to the full piezoelectric equations for low frequencies. A numerical example indicates that the effects due to piezoelectricity are small for the scattering of elastic waves (with an incident elastic wave). However, most interest of piezoelectricity stems from the ability of transformation of electrical energy into mechanical energy or vice versa and then the parts of the T-matrix coupling elastodynamic and electromagnetic waves come into play, for example when considering a piezoelectric composite [2].

The present method has previously been used for the scattering by an anisotropic sphere [10]. It should thus be possible to extend the method to the scattering by a piezoelectric sphere.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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