



## Wehrl Inequalities for Matrix Coefficients of Holomorphic Discrete Series

Downloaded from: <https://research.chalmers.se>, 2026-09-01 00:24 UTC

Citation for the original published paper (version of record):

van Haastrecht, R., Zhang, G. (2026). Wehrl Inequalities for Matrix Coefficients of Holomorphic Discrete Series. *Journal of Geometric Analysis*, 36(10).  
<http://dx.doi.org/10.1007/s12220-026-02573-z>

N.B. When citing this work, cite the original published paper.



# Wehrl Inequalities for Matrix Coefficients of Holomorphic Discrete Series

Robin van Haastrecht<sup>1</sup> · Genkai Zhang<sup>1</sup>

Received: 19 May 2025 / Accepted: 5 August 2026  
© The Author(s) 2026

## Abstract

Let  $G$  be a Hermitian Lie group. We prove Wehrl-type  $L^2(G) - L^p(G)$  inequalities for matrix coefficients of vector-valued holomorphic discrete series of  $G$ , for even integers  $p = 2n$ . The optimal constant is expressed in terms of Harish-Chandra formal degrees for the discrete series. We prove the maximizers are precisely the reproducing kernels.

**Keywords** Lie groups · Unitary representations · Hermitian symmetric spaces · Holomorphic discrete series · Harish-Chandra formal degree · Reproducing kernels · Matrix coefficients ·  $L^p$ -spaces · Wehrl inequality · Toeplitz operators

**Mathematics Subject Classification** 22E30 · 22E45 · 32A36 · 43A15

## 1 Introduction

In the present paper we shall study the  $L^2 - L^p$  optimal inequalities for matrix coefficients for holomorphic discrete series representations of Hermitian Lie groups. We start with a brief introduction on the main problem.

### 1.1 Background and Main Problem

Let  $(G, \pi, \mathcal{H})$  be a unitary irreducible representation of a Lie group  $G$  and assume that  $\pi$  is a discrete series relative to a homogeneous space  $G/H$  for a closed subgroup  $H \subset G$ , namely the square norms of the matrix coefficients  $\langle \pi(g)u, v \rangle$ ,  $g \in G$ ,  $u, v \in \mathcal{H}$  are well-defined as elements in  $L^2(G/H)$  for a certain  $G$ -invariant measure on  $G/H$ . The matrix coefficients are in  $L^\infty$  by the unitarity. It is a natural and important question to

---

Research by Genkai Zhang partially supported by the Swedish Research Council (Vetenskapsrådet).

✉ Genkai Zhang  
genkai@chalmers.se

<sup>1</sup> Mathematical Science, Chalmers University of Technology and University of Gothenburg, 412 96, Göteborg, Sweden

find the optimal estimates for the  $L^p$ -norm for  $p \geq 2$  as it is related to other questions and concepts.

The most studied case is when  $G$  is the Heisenberg group  $G = \mathbb{R} \ltimes \mathbb{C}^n$ , and the unitary representation  $(G, \pi, \mathcal{H})$  is on the Fock space  $\mathcal{H} = \mathcal{F}(\mathbb{C}^n)$ , or on  $\mathcal{H} = L^2(\mathbb{R}^n)$  in the Schrödinger model. The relevant optimal estimates are sometimes called Wehrl inequalities [29]. The matrix coefficients  $\langle \pi(g)f, f_0 \rangle$ , when restricted to  $\mathbb{C}^n = G/\mathbb{R}$ , are in the space  $L^2(\mathbb{C}^n)$ . The Fock space  $\mathcal{H} = \mathcal{F}(\mathbb{C}^n)$  has a reproducing kernel  $e^{\langle z, w \rangle}$ , which maximizes the evaluation at  $w$  among elements of fixed  $L^2$ -norm. Fix  $f_0 = 1$  as the reproducing kernel  $e^{\langle z, w \rangle}$  at  $w = 0$  (or the Gaussian function in the Schrödinger model). For each positive operator  $T \geq 0$  of unit trace,  $\text{Tr } T = 1$ , the matrix coefficients  $F(g) = \langle T\pi(g)f_0, \pi(g)f_0 \rangle = \text{Tr}(T\pi(g)f_0 \otimes (\pi(g)f_0)^*)$  define a probability measure on  $\mathbb{C}^n = G/\mathbb{R}$ ,  $\int_{\mathbb{C}^n} F(g)dg = 1$  by Weyl's Plancherel formula (up to a normalization). Wehrl [29] proposed the quantity  $-\int F(g) \ln F(g)dg$  as a classical entropy corresponding to the quantum entropy  $-\text{Tr}(T \ln T)$  of  $T$ . Wehrl investigated the question when the entropy is minimal. It is easy to see this must happen for some  $T = f \otimes f^*$  a pure tensor, by concavity of the function  $-x \ln x$ , so it is enough to consider these pure tensors. Wehrl conjectured the classical entropy is minimal for  $f = \pi(g)f_0$ , a translation of the function  $f_0 = 1$  (or the Gaussian function in the Schrödinger model) by an element  $g \in G$ . Lieb [15] studied a more general question on the optimal  $L^2(\mathbb{C}^n) - L^p(\mathbb{C}^n)$  boundedness,  $p \geq 2$  for the matrix coefficients  $\langle \pi(g)f, f_0 \rangle$ ,  $g \in \mathbb{C}^n$ , and proved that the maximizers are precisely achieved by  $f = \pi(g_0)f_0$  for some  $g_0 \in G$ ; the Wehrl conjecture becomes an immediate consequence by taking the derivative at  $p = 2$  of the inequality for  $p \geq 2$ .

When  $G$  is a compact semisimple Lie group any irreducible representation  $(\pi, \mathcal{H})$  is finite-dimensional, and there is also a preferred choice of the vector  $v_0$ , namely the highest weight vector or its translates under the action of  $G$ , similar to the function  $f_0 = 1$  above for the Heisenberg group. The Schur orthogonality computes  $L^2$ -norms of the matrix coefficients  $\langle \pi(g)f, f_0 \rangle$ ,  $g \in G$  using the dimension of  $\mathcal{H}$  and it is a natural problem to find  $L^2 - L^p$  optimal estimates. In [2] a statement on the  $L^2 - L^4$  optimal estimate was given with a sketch of the proof. For  $G = SU(2)$  the Wehrl  $L^2 - L^p$  inequality [29] was proved by Lieb and Solovej [16] more than 30 years later. They also proved the inequality [17] for  $G = SU(N)$  and for the symmetric tensor power  $S^m(\mathbb{C}^N)$  representations of  $G$ . They used methods quite different from the classical analytic method [15] by introducing quantum channel operators and proving more general results about the eigenvalue distribution of these operators.

The next interesting and challenging case is for real simple non-compact Lie groups  $G$  and their discrete series representations  $(\pi, \mathcal{H})$ . Harish-Chandra has generalized the Schur orthogonality relations for compact groups using the formal degree. It suggests that there should also be optimal  $L^2 - L^p$  estimates for the matrix coefficients,  $p \geq 2$ . When  $G = SU(1, 1)$  Lieb and Solovej [18] proved optimal  $L^2 - L^p$  estimates for the Bergman space as holomorphic discrete series representations of  $G$  for even integers  $p = 2n$  by using direct computations. This was generalized to all  $p \geq 2$  by Kulikov [14] using the isoperimetric inequality for the hyperbolic area of sublevel sets of the holomorphic functions (as sections of the cotangent bundle with the dual hyperbolic metric). In all these cases,  $G = \mathbb{R} \ltimes \mathbb{C}^n$ ,  $SU(2)$  and  $SU(1, 1)$ , the inequalities are

proved for any general positive convex function instead of the  $L^p$ -norm. A general systematic treatment is given by Frank [5].

## 1.2 Our Main Results and Methods

We consider now a Hermitian Lie group  $G$  and its holomorphic discrete series  $(\mathcal{H}_\Lambda, \pi_\Lambda)$  with highest weight  $\Lambda$ . The discrete series will be realized as the Bergman space of  $V_\Lambda$ -valued holomorphic functions on the bounded symmetric domain  $D = G/K$  of  $G$  with  $(V_\Lambda, \tau_\Lambda, K)$  the unitary representation of  $K$  with  $K$ -highest weight  $\Lambda$ . We will write  $\tau = \tau_\Lambda$  in the rest of the text if no confusion would arise. The holomorphic functions can be realized as sections of the holomorphic vector bundle over  $D$  with the Harish-Chandra realization of  $D$ , and the metric on the bundle can be expressed as  $\langle \tau_\Lambda(B(z, z)^{-1})v, v \rangle$  using the Bergman operator  $B(z, z)$ ; see Definition 2.1 below. The tensor product  $V_\Lambda^{\otimes n}$  has an irreducible component  $V_{n\Lambda}$  of multiplicity one, now let  $P = P_{n\Lambda} : V_\Lambda^{\otimes n} \rightarrow V_{n\Lambda}$  be the orthogonal projection. Write  $P(f^{\otimes n})(z) = P(f^{\otimes n}(z))$ , the point-wise orthogonal projection. Our main result is the following.

**Theorem 1.1** (Theorem 5.3 and Corollary 5.4) *Let  $n \geq 2$  be an integer,  $(V_\Lambda, \tau, K)$  be an irreducible representation of  $K$  with a unit highest weight vector  $v_\Lambda$  and  $\mathcal{H}_\Lambda$  the holomorphic discrete series realized as the Bergman space of  $V_\Lambda$ -valued holomorphic functions. Then*

$$\|P(f^{\otimes n})\|_{\mathcal{H}_\Lambda}^2 \leq c_G^{n-1} \frac{(d_\Lambda^H)^n}{d_{n\Lambda}^H} \|f\|_{\mathcal{H}_\Lambda}^{2n}, \quad (1)$$

and

$$\|F_f\|_{L^{2n}}^{2n} \leq c_G^{n-1} \frac{(d_\Lambda^H)^n}{d_{n\Lambda}^H} \|F_f\|_{L^2}^{2n} \quad (2)$$

for  $f \in \mathcal{H}_\Lambda$  and  $F_f(g) := \langle \pi(g)f, v_\Lambda \rangle$ ,  $g \in G$ . The equality holds if and only if  $f(z) = cK(z, w)\tau(k)v_\Lambda$  for some  $w \in D$ ,  $k \in K$ ,  $c \in \mathbb{C}$ .

The precise notation is found below. When  $\mathcal{H}_\Lambda$  is a scalar holomorphic discrete series this result is proved in [30].

We explain briefly our methods and some auxiliary results. First we consider the  $n$ -fold tensor power  $\mathcal{H}_\Lambda^{\otimes n}$  of the discrete series. The orthogonal projection  $Pf(z)^{\otimes n}$  of  $f(z)^{\otimes n} \in V_\Lambda^{\otimes n}$  onto the highest component (also called the Cartan component)  $V_{n\Lambda} \subset \otimes^n V_\Lambda$  defines a  $G$ -intertwining operator onto the discrete series  $H_{n\Lambda}$ . This follows from some general facts for holomorphic discrete series [23]. Thus there should be an inequality. The constant in the inequality is abstractly obtained by the Harish-Chandra formal degree. However the constant is only determined up to normalization, whereas our Bergman space is defined by the usual normalization. We then find the exact formula for the Harish-Chandra formal degree by using the evaluation of the Selberg Beta integral [1, 4]; see Proposition 4.3 below. As a consequence we find also in Theorem 4.4 the formula for the reproducing kernel under our normalization. To prove that the maximizers are achieved by the reproducing kernel we prove that they are eigenvectors of Toeplitz operators [30] and that they define the bounded

point evaluations. We finally use the earlier results in [2] about Wehrl inequalities for compact groups. However, we realized the proof in [2] is incomplete and we provide a full proof in Appendix A.

For the unit disk  $D = SU(1, 1)/U(1)$  we find in Theorem 6.2 an improved Wehrl inequality with a precise extra term added on the left hand side of the Wehrl inequality (1); the extra term involves first and second derivatives of  $f$ . Our result might lead to finding an improved Wehrl  $L^2 - L^p$ -inequality for the Bergman space on the unit disc [5, 14] and for the Fock space [6].

### 1.3 Further Questions

There are quite a few open questions related to the Wehrl inequality. The Wehrl  $L^2 - L^p$ -inequality for the Bergman space on the unit ball in  $\mathbb{C}^n$ ,  $n \geq 2$  is still open. In [17] the equality is proved for Bergman spaces of holomorphic sections of symmetric tangent bundles on the projective space  $\mathbb{P}^n = SU(n+1)/U(n)$  using quantum channels [16]. These channels can be defined [30] for the general holomorphic discrete series for  $SU(n, 1)$ . In [7, 8] the limit formulae for the functional calculus of the channels are found generalizing earlier results of [17]. It would be interesting to study the eigenvalue distributions of the channel operators for other representations of  $SU(n+1)$  and for the non-compact group  $SU(1, 1)$ . Kulikov [14] proved some subtle properties about the hyperbolic area of holomorphic functions in the Bergman space using isoperimetric inequalities. It might be important to study the volumes of sublevel sets for holomorphic functions in Bergman space in higher dimensions rather than isoperimetric problems for general sets. For a discrete series  $(\mathcal{H}, \pi, G)$  of a semisimple Lie group it seems a rather challenging problem to find the optimal  $L^2 - L^{2n}$  estimates.

### 1.4 Organization of the Paper

In Sect. 2 we recall some necessary known results on Hermitian symmetric spaces  $G/K$ , and in Sect. 3 we introduce holomorphic discrete series representations of  $G$  and their realizations as Bergman spaces of vector-valued holomorphic functions on  $D$ . We find in Sect. 4 the exact formula for the Harish-Chandra formal degrees under our (somewhat standard) normalization of the metric on  $G/K$ . The Wehrl inequalities are proved in Sect. 5. An improved Wehrl inequality for the unit disc is proved in Sect. 6. In Appendix A we give a complete proof for Wehrl inequalities for compact semisimple Lie groups and in Appendix B we prove that the bounded point evaluations for our Bergman space of vector-valued holomorphic functions are given by the points in  $D = G/K$ , and they are all needed to prove the Wehrl inequalities in Sect. 5.

### 1.5 Notation

For the convenience of the reader we add a list of the most common notation in the paper.

- (1)  $G$  is a simple Hermitian Lie group and  $G/K$  is a Hermitian symmetric space.

- (2)  $\mathfrak{g}$  is the Lie algebra of  $G$ .
- (3)  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^-$  is the decomposition of the Lie algebra into eigenspaces of a central element of  $\mathfrak{k}^{\mathbb{C}}$ .
- (4)  $D = G/K$  is the bounded Hermitian symmetric domain of rank  $r$  realized in  $\mathfrak{p}^+ = \mathbb{C}^N$ .
- (5)  $\Delta$  are the roots of  $\mathfrak{g}^{\mathbb{C}}$  with respect to the Cartan subalgebra  $\mathfrak{h}^{\mathbb{C}}$  of  $\mathfrak{k}^{\mathbb{C}}$ , which is also a Cartan subalgebra of  $\mathfrak{g}^{\mathbb{C}}$ .
- (6)  $(V_{\Lambda}, \tau_{\Lambda}, K)$  is a representation of  $K$  of highest weight  $\Lambda$ .
- (7)  $(\mathcal{H}_{\Lambda}, \pi_{\Lambda}, G)$  is the holomorphic discrete series of  $G$  associated to the representation  $(V_{\Lambda}, \tau_{\Lambda}, K)$ .
- (8) The Haar measure of  $G$  is normalized by  $\int_G f(g)dg = \int_D \left( \int_K f(xk)dk \right) d\iota(x)$ , where  $\int_K dk = 1$  and  $d\iota$  is defined in (7).
- (9)  $d_{\Lambda}^H$  and  $d_{\Lambda}$  is the formal degree for a holomorphic discrete series  $(\mathcal{H}_{\Lambda}, \pi_{\Lambda}, G)$ , by different normalizations, see (11) and (13).
- (10)  $P_{\Lambda}$  is the projection onto an irreducible  $K$ -representation of highest weight  $\Lambda$ .
- (11)  $Q_0$  is the projection onto the Cartan component of highest weight  $n\Lambda$   $\mathcal{H}_{n\Lambda} \subseteq \mathcal{H}_{\Lambda}^{\otimes n}$ . For  $SU(1, 1)$ ,  $Q_k$  is the projection onto the irreducible component  $\mathcal{H}_{\mu+\nu+2k} \subseteq \mathcal{H}_{\mu} \otimes \mathcal{H}_{\nu}$ .

## 2 Hermitian Symmetric Spaces Realized as Bounded Domains

### $D = G/K$

We recall briefly some known facts on Hermitian symmetric spaces and related Lie algebras. We shall use the Jordan triple description; see [19] and [25, Chapter 2.5].

### 2.1 Hermitian Symmetric Spaces $G/K$ and the Lie Algebra $\mathfrak{g}$ of $G$

Let  $G$  be a connected simple Lie group of real rank  $r$ ,  $K$  its maximal compact subgroup, and  $G/K$  a Hermitian symmetric space of complex dimension  $N$ . Let  $\mathfrak{g}$  be the Lie algebra of  $G$  and  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  the Cartan decomposition with Cartan involution  $\theta$ . Then  $\mathfrak{k}$  has one-dimensional center, so  $\mathfrak{k} = [\mathfrak{k}, \mathfrak{k}] \oplus \mathbb{R}Z$ , where  $Z$  generates the center and is normalized so that  $J := \text{ad}(Z)$  defines a complex structure on  $\mathfrak{p}$ . This implies the existence of a Hermitian complex structure on the symmetric space  $D = G/K$ . Let  $\mathfrak{h} \subseteq \mathfrak{k}$  be a maximal Cartan subalgebra for  $\mathfrak{k}$ , then its complexification  $\mathfrak{h}^{\mathbb{C}} \subseteq \mathfrak{k}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$  is also a Cartan subalgebra for  $\mathfrak{g}^{\mathbb{C}}$  since  $\mathfrak{k}^{\mathbb{C}}$  and  $\mathfrak{g}^{\mathbb{C}}$  are of the same rank. The roots  $\Delta$  of  $\mathfrak{h}^{\mathbb{C}}$  in  $\mathfrak{g}^{\mathbb{C}}$  are  $\Delta = \Delta_c \cup \Delta_n$ , where  $\Delta_c$  are the compact roots  $\alpha$  with  $\mathfrak{g}_{\alpha} \subseteq \mathfrak{k}^{\mathbb{C}}$ , and  $\Delta_n$  the non-compact roots  $\alpha$  with  $\mathfrak{g}_{\alpha} \subseteq \mathfrak{p}^{\mathbb{C}}$ . We choose an ordering of roots so that  $J = \text{ad}(Z)$  acts on  $\Delta_n^{\pm} = \Delta^{\pm} \cap \Delta_n$  as  $\pm i$ . For every  $\alpha \in \Delta^+$  we fix an  $\mathfrak{sl}_2$ -triple such that  $h_{\alpha} \in i\mathfrak{h}$ ,  $e_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$  and

$$[h_{\alpha}, e_{\alpha}] = 2e_{\alpha}, \quad \theta(e_{\alpha}) = -e_{-\alpha}, \quad [e_{\alpha}, e_{-\alpha}] = h_{\alpha}.$$

We then have the decomposition  $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$  with  $\mathfrak{p}^+$  and  $\mathfrak{p}^-$  being the sum of the non-compact positive and negative roots, respectively and given by

$$\mathfrak{p}^\pm = \{v \mp iJv : v \in \mathfrak{p}\}.$$

Note that  $\overline{\mathfrak{p}^+} = \mathfrak{p}^-$ ,

$$[\mathfrak{p}^+, \mathfrak{p}^+] = [\mathfrak{p}^-, \mathfrak{p}^-] = 0,$$

and

$$[\mathfrak{p}^+, \mathfrak{p}^-] = \mathfrak{k}^\mathbb{C}.$$

Denote

$$D(u, \bar{v}) = [u, \bar{v}] \in \mathfrak{k}^\mathbb{C}$$

identified with its action on  $\mathfrak{p}^+ = \mathbb{C}^N$ ,

$$D(u, \bar{v})w := \text{ad}(D(u, \bar{v}))(w) = [D(u, \bar{v}), w], \quad u, v, w \in \mathfrak{p}^+ = \mathbb{C}^N.$$

Then the triple product  $D(u, \bar{v})w$  is symmetric in  $u$  and  $w$ . Let  $Q(u) : \mathfrak{p}^- = \overline{\mathbb{C}^N} \rightarrow \mathfrak{p}^+$  and  $Q(\bar{v}) : \mathfrak{p}^+ \rightarrow \mathfrak{p}^-$  be the quadratic maps

$$Q(u)\bar{v} = \frac{1}{2}D(u, \bar{v})u, \quad Q(\bar{v})u = \frac{1}{2}D(\bar{v}, u)\bar{v}, \quad u \in \mathfrak{p}^+, \bar{v} \in \mathfrak{p}^-,$$

See [19].

Let  $\{\gamma_i\}_{i=1}^r$  be the strongly orthogonal non-compact roots starting with the highest root  $\gamma_1$ , where  $r$  is the real rank of  $G$ . Denote the corresponding co-roots and root vectors of  $\gamma_j$  of by

$$h_j = h_{\gamma_j}, \quad e_{\pm j} = e_{\pm \gamma_j}$$

chosen as in [4] so that  $\overline{e_{\pm j}} = e_{\mp j}$ , and  $e_i$  is a tripotent [19],  $Q(e_i)\bar{e}_i = e_i$ . The root vectors  $\{e_i\}_{i=1}^r$  form a frame, i.e. a maximal orthogonal system of primitive tripotents of unit norm, in the sense of [19, Section 5.1]. Let  $\mathfrak{h}^- = \sum_{j=1}^r \mathbb{R}h_j$ , then  $\mathfrak{h} = \mathfrak{h}^- \oplus \mathfrak{h}^+$ , where  $\mathfrak{h}^+$  is the orthocomplement of  $\mathfrak{h}^-$ . Restricting the  $\mathfrak{h}^\mathbb{C}$  roots in  $\Delta_n^+$  to  $i\mathfrak{h}^-$  we obtain the  $\gamma_j$ ,  $1 \leq j \leq r$  with multiplicity 1,  $\frac{\gamma_i + \gamma_j}{2}$ ,  $1 \leq i < j \leq r$  all with the same multiplicity  $a$  and  $\frac{\gamma_j}{2}$  all with the same multiplicity  $b$  [13, 25].

Let

$$p := (r-1)a + b + 2, \quad n_1 = r + a \frac{r(r-1)}{2}.$$

The dimension  $N$  is then

$$N = n_1 + rb.$$

Note that the integer  $p$  can be computed as  $p = \text{Tr}(D(e_1^+, e_1^-)|_{\mathfrak{p}^+}) = \text{Tr}(D(e_j^+, e_j^-)|_{\mathfrak{p}^+})$  for any  $j$ . Now we normalize the  $K$ -invariant Euclidean inner product on  $\mathbb{C}^N$  by

$$\langle v, w \rangle = \langle v, w \rangle_{\mathfrak{p}^+} := \frac{1}{p} \text{Tr}(D(v, \bar{w})|_{\mathfrak{p}^+}), \quad (3)$$

so that  $\|e_j\| = 1$  for any  $j$  and the  $\{e_j\}_{j=1}^r$  are orthogonal.

## 2.2 The Harish-Chandra Realization of $G$ in $G^{\mathbb{C}} = P^+K^{\mathbb{C}}P^-$ and the Bergman Operator

The symmetric space  $D = G/K$  can be realized as a circular convex bounded domain in  $\mathbb{C}^N = \mathfrak{p}^+$  as follows, also called the Harish-Chandra realization. Consider the natural inclusion map followed by the quotient map

$$G \hookrightarrow G^{\mathbb{C}} = P^+K^{\mathbb{C}}P^- \rightarrow G^{\mathbb{C}}/K^{\mathbb{C}}P^- \cong P^+ \cong \mathfrak{p}^+.$$

Then  $K$  is mapped into the reference point  $0 \in \mathfrak{p}^+$  and it induces an injective holomorphic map and the Harish-Chandra realization of

$$D := G/K = G \cdot 0 \subseteq \mathfrak{p}^+.$$

To describe the action of  $G$  on  $D$  we need some quantities.

**Definition 2.1** The Bergman operator is defined as

$$B(x, \bar{y}) = I - D(x, \bar{y}) + Q(x)Q(\bar{y}) : \mathbb{C}^N \rightarrow \mathbb{C}^N.$$

It follows from [19, Theorem 8.11] that the element  $B(z, \bar{z})^{-1} \in K^{\mathbb{C}}$  for  $z \in D = G/K \subset \mathbb{C}^N$  and

$$B(z, \bar{z})^{-1} := K^{\mathbb{C}}\text{-part of } \exp(\bar{z}) \exp(z)$$

under the decomposition  $G^{\mathbb{C}} = P^+K^{\mathbb{C}}P^-$ .

We also have another norm on  $\mathbb{C}^N$ , the spectral norm  $|\cdot|$ , such that  $D$  is a unit ball with the norm,

$$D = \{z \in \mathbb{C}^N \mid |z| < 1\},$$

see [19, Theorem 4.1]. Furthermore, we have the following polar decomposition

$$D = \{\text{Ad}(k)(t_1 e_1 + \cdots + t_r e_r) \mid k \in K, t_i \in [0, 1)\};$$

see [19, Theorem 3.17].

We identify the holomorphic tangent space  $T_z^{(1,0)}(D)$  of  $D \subset \mathbb{C}^N$  at  $z \in D$  with  $\mathbb{C}^N$ ,  $T_z^{(1,0)}(D) = \mathfrak{p}^+$ . Denote  $J_g(z) = dg(z)$ , the Jacobian of the holomorphic map  $g : D \rightarrow D$  in local coordinates,

$$J_g(z) : \mathbb{C}^N = T_z^{(1,0)}(D) \rightarrow T_{gz}^{(1,0)}(D) = \mathbb{C}^N.$$

The identification of  $\mathbb{C}^N$  with  $T_z^{(1,0)}(D)$  is done by realizing  $D \subseteq \mathbb{C}^N$ . Now  $B(z, \bar{z})$  acts on  $\mathbb{C}^N$  by the adjoint action, and we have the following important transformation rule [19, Lemma 2.11]

$$J_g(z)^* B(g \cdot z, \overline{g \cdot z})^{-1} J_g(z) = B(z, \bar{z})^{-1}. \quad (4)$$

As  $B(0, 0) = I$  it then follows directly that

$$B(g \cdot 0, \overline{g \cdot 0}) = J_g(0)J_g(0)^*. \quad (5)$$

The Jacobian  $J_g(z)$  can be obtained from the more general canonical automorphy factor  $J(g, z)$  [25, Lemma 5.3] defined by

$$J(g, z) = K^{\mathbb{C}} - \text{part of } g \cdot \exp(z);$$

we have  $J_g(z) = \text{Ad}(J(g, z))$ . Since elements in  $K^{\mathbb{C}}$  are realized as linear maps on  $\mathfrak{p}^+$  via the adjoint action we can identify  $J_g(z)$  with  $J(g, z)$ , but it will be clear from context which one is meant. In particular we have  $J_k(z) = J(k, z) = k$ .

### 3 Holomorphic Discrete Series of $G$ Realized as Bergman Spaces of Vector-Valued Holomorphic Functions on $D$

#### 3.1 Bergman Space of Holomorphic Functions on $D$ : Invariant Measure

Let  $dm(z)$  be the Lebesgue measure defined by the inner product (3). The Bergman space of holomorphic functions  $f(z)$  on  $D$  such that

$$\int_D |f(z)|^2 dm(z) < \infty$$

has the reproducing kernel, up to a normalization constant (which will be determined below for general Bergman spaces),

$$\det B(z, w)^{-1} = h(z, w)^{-p} \quad (6)$$

where  $h(z, w)$  is an irreducible polynomial holomorphic in  $z$  and anti-holomorphic in  $w$  and of maximal bi-degree  $(r, r)$ ; see e.g. [4, 13]. Now by [19, Corollary 3.15] for  $z = \sum_{j=1}^r \lambda_j e_j$

$$h(z, z) = \prod_{j=1}^r (1 - |\lambda_j|^2).$$

Note that this actually describes  $h(z, z)$  for any  $z \in \mathbb{C}^N$  as

$$\mathbb{C}^N = \text{Ad}(K) \left( \sum_{i=1}^r \mathbb{R}_{\geq 0} e_i \right).$$

The Bergman metric on  $D$  at  $z \in D$  is given by

$$\langle v, w \rangle_z = \langle B(z, \bar{z})^{-1} v, w \rangle_{\mathbb{C}^N}$$

for  $v, w \in T_z D = \mathbb{C}^N$ . By the transformation property (4) the Bergman metric is invariant under  $G$ . We note that for  $k \in K$  we have  $B(k \cdot z, \overline{k \cdot z}) = k B(z, \bar{z}) k^{-1}$ , and thus for

$$z = \lambda_1 e_1 + \dots \lambda_r e_r$$

we get that

$$\det(B(k \cdot z, \overline{k \cdot z})^{-1}) = \det(B(z, \bar{z})^{-1}) = \prod_{j=1}^r (1 - |\lambda_j|^2)^{-p}.$$

The  $G$ -invariant Riemannian measure on  $D = G/K$  is obtained from the Bergman metric by

$$dt(z) = \det B(z, z)^{-1} dm(z) = h(z, z)^{-p} dm(z). \quad (7)$$

### 3.2 Bergman Space of Vector-Valued Holomorphic Functions

Let  $(V_\Lambda, \tau_\Lambda, K)$  be an irreducible unitary representation of  $K$  of highest weight  $\Lambda$ . It can be extended to a rational representation of  $K^\mathbb{C}$  on the space  $V_\Lambda$ . The  $K$ -unitary inner product on  $V_\Lambda$  will be denoted by  $\langle -, - \rangle_\tau$ .

We now introduce the holomorphic discrete series.

**Definition 3.1** Let  $(V_\Lambda, \tau_\Lambda, K)$  be an irreducible representation of  $K$  with highest weight  $\Lambda$ . Let  $\mathcal{H}_\Lambda$  be the Hilbert space of holomorphic functions  $f : D \rightarrow V_\Lambda$  with the norm square

$$\|f\|_{\mathcal{H}_\Lambda}^2 := \int_D \langle \tau(B(z, \bar{z})^{-1}) f(z), f(z) \rangle_\tau dt(z) < \infty. \quad (8)$$

The holomorphic discrete series is  $(\mathcal{H}_\Lambda, \pi_\Lambda, G)$ , with the unitary representation

$$(\pi_\Lambda(g)f)(z) = \tau(J_{g^{-1}}(z)^{-1}) f(g^{-1} \cdot z), \quad (9)$$

provided  $\mathcal{H}_\Lambda$  is non-trivial.

Indeed, the space  $\mathcal{H}_\Lambda$  in Definition 3.1 could be trivial. The Harish-Chandra conditions give a characterization for  $\mathcal{H}_\Lambda$ ; see e.g. [9, Lemma 27, Paragraph 9], [13, equality (6)], [28, II, Theorem 6.5].

**Theorem 3.2** Let  $\Lambda$  be the highest weight of  $(V_\Lambda, \tau, K)$  and let  $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$  be the half sum of positive roots  $\Delta^+$ . If

$$(\Lambda + \rho)(h_1) < 0$$

then the Hilbert space  $\mathcal{H}_\Lambda \neq \{0\}$  and defines a discrete series of  $G$ .

The Hilbert space  $\mathcal{H}_\Lambda$  has reproducing kernel  $K_w(z) = K(z, w) = K_\Lambda(z, w)$  taking values in  $\text{End}(V_\Lambda)$ , holomorphic in  $z$  and anti-holomorphic in  $w$  such that for any  $v \in V_\Lambda$ ,  $f \in \mathcal{H}_\Lambda$ , we have  $K_w v \in \mathcal{H}_\Lambda$ , and

$$\langle f, K_w v \rangle_{\mathcal{H}_\Lambda} = \langle f(w), v \rangle_\tau.$$

The kernel  $K$  can be computed using the Bergman operator [13, Paragraph 4]: There is a constant  $C(\Lambda) > 0$ , to be evaluated in Theorem 4.4, such that

$$K(z, z) = C(\Lambda)\tau(B(z, \bar{z})) = C(\Lambda)\tau(J_g(0)J_g(0)^*) \quad (10)$$

where  $z = g \cdot 0$ . It follows by holomorphicity in  $z$  and anti-holomorphicity in  $w$  that

$$K(z, w) = C(\Lambda)\tau(B(z, \bar{w})).$$

Furthermore

$$K(g \cdot z, g \cdot w) = \tau(J_g(z))K(z, w)\tau(J_g(w))^*.$$

From (10) we see that for any  $z \in D$

$$K(z, 0) = C(\Lambda)I.$$

Thus for any  $v \in V_\Lambda$  the constant function  $v$  is in  $\mathcal{H}_\Lambda$ , as  $v = C(\Lambda)^{-1}K_0 v \in \mathcal{H}_\Lambda$ .

Furthermore, the space of  $V_\Lambda$ -valued polynomials is dense in  $\mathcal{H}_\Lambda$ , and as a representation of  $K$  it is  $\mathcal{P} \otimes V_\Lambda$  where  $\mathcal{P}$  is the space of scalar-valued polynomials; see e.g. [3].

## 4 The Formal Degree of the Holomorphic Discrete Series

The formal degree of the discrete series  $(\mathcal{H}, \pi, G)$  of a semisimple Lie group  $G$  is a proportionality constant between the  $|\langle u, v \rangle|^2$  for  $u, v \in \mathcal{H}$  and the  $L^2(G)$ -norm square of the matrix coefficient  $\langle \pi(g)u, v \rangle$ . Harish-Chandra [9] has computed the formal degree up to a normalization constant. We shall find the exact formula for the formal degree under our normalization (3) above. The formal degree will appear in the Wehrl inequality in the next section.

### 4.1 Definition of the Formal Degree

Harish-Chandra [9, Theorem 1] shows that for a holomorphic discrete series representation  $\mathcal{H}_\Lambda$  and  $f_1, f_2 \in \mathcal{H}_\Lambda$  there exists a positive number  $d_\Lambda$ , called the formal degree of  $\mathcal{H}_\Lambda$ , such that

$$\int_G |\langle \pi_\Lambda(g)f_1, f_2 \rangle_{\mathcal{H}_\Lambda}|^2 dg = d_\Lambda^{-1} \|f_1\|^2 \|f_2\|^2, \quad (11)$$

where all the inner products are in  $\mathcal{H}_\Lambda$ . We now normalize the Haar measure on  $G$  so that

$$\int_G f(g)dg = \int_D \left( \int_K f(xk)dk \right) d\iota(x),$$

where we realize  $G$  as the set  $D \times K$  with the invariant measure on  $D = G/K$  from (7) and the Haar measure on  $K$  is normalized so that  $\int_K dk = 1$ .

Harish-Chandra found a formula for the formal degree up to some normalization of the Haar measure on  $G$  [9, Theorem 4]. It is given by the following

$$d_\Lambda^H := (-1)^{\frac{\dim G - \text{rank } K}{2}} \prod_{\alpha \in \Delta^+} \frac{\Lambda(h_\alpha) + \rho(h_\alpha)}{\rho(h_\alpha)}, \quad (12)$$

where  $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ . (Harish-Chandra's formula was the absolute of the above formula without the sign  $(-1)^{\frac{\dim G - \text{rank } K}{2}}$ , and we take the sign with us to make it a polynomial in  $\Lambda$  and coincide with the absolute value for discrete series.)

It follows that there is a constant  $c_G$  such that

$$d_\Lambda = c_G \cdot d_\Lambda^H. \quad (13)$$

We shall find this constant by choosing scalar representations  $\tau$  of  $K$  and by evaluating both degrees.

## 4.2 Scalar Holomorphic Discrete Series

This series of representations is very well understood; see e.g. [3, 4, 28, 29]. Let  $\lambda \in \mathbb{Z}_+$  be an integer and let  $\tau(k) := \det(\text{Ad}(k)|_{\mathbb{C}^N})^{-\frac{\lambda}{p}}, k \in K$ . Then up to a covering of  $G$   $\tau$  defines a character of  $K$ , and the covering will have no effect on our results as we have fixed the integration of  $K$  so that  $\int_K dk = 1$ . We see that for  $H \in \mathfrak{h}$  the scalar highest weight  $\Lambda$  of  $\tau$  is given by

$$\Lambda(H) = \frac{d}{dt} \Big|_{t=0} \tau(e^{tH}) = \frac{d}{dt} \Big|_{t=0} e^{-t \frac{\lambda}{p} \text{Tr}(\text{ad}(H)|_{\mathbb{C}^N})} = -\frac{\lambda}{p} 2\rho_n(H),$$

where  $\rho_n = \frac{1}{2} \sum_{\alpha \in \Delta_n^+} \alpha$ . Thus we get for  $1 \leq j \leq r$  [13, (1.4)]

$$\Lambda(h_j) = -\lambda.$$

We shall identify the weight  $\Lambda$  with the scalar  $-\lambda$  and write the corresponding  $\tau_\Lambda$  as  $\tau_{-\lambda}$ . The condition in Theorem 3.2 becomes  $\lambda > p-1$ ; see [13, Section 4]. The Hilbert space  $\mathcal{H}_\Lambda$  is usually called the weighted Bergman space with weight  $\lambda - p > -1$ . The norm square (8) is now given by

$$\|f\|_{\mathcal{H}_\Lambda}^2 = \int_D |f(z)|^2 h(z, z)^\lambda d\iota(z) = \int_D |f(z)|^2 h(z, z)^{\lambda-p} dm(z),$$

with  $\tau_{-\lambda}(B(z, z))^{-1} = h(z, z)^\lambda$ , and the representation  $\pi_\Lambda$  becomes

$$\pi_\Lambda(g)f(z) = \det(J_{g^{-1}}(z))^{\frac{\lambda}{p}} f(g^{-1}z), \quad g \in G.$$

The following result follows easily from the definition and the mean value property of holomorphic functions.

**Lemma 4.1** *Let  $\lambda > p - 1$  and  $\Lambda$  be as above. For the representation  $\tau(k) = \tau_{-\lambda}(k)$  the formal dimension  $d_\Lambda$  of  $\mathcal{H}_\Lambda$  is given by*

$$d_\Lambda^{-1} = \int_G h(g \cdot 0, g \cdot 0)^\lambda d\iota(g) = \int_D h(z, z)^{\lambda-p} dm(z).$$

**Proof** We take  $f_1 = f_2 = 1$  in Eq. (11). The LHS becomes

$$\int_G |\langle \pi_\Lambda(g)1, 1 \rangle_{\mathcal{H}_\Lambda}|^2 dg,$$

and the integrand is by  $K$ -invariance

$$|\langle \pi_\Lambda(g)1, 1 \rangle_{\mathcal{H}_\Lambda}|^2 = |(\pi_\Lambda(g)1)(0)|^2 \iota_\lambda(D)^2,$$

where  $\iota_\lambda(D) = \int_D h(z, z)^\lambda d\iota(z)$ . Moreover by (5), and (6),

$$|(\pi_\Lambda(g)1)(0)|^2 = |\det(J_{g^{-1}}(0))|^{\frac{2\lambda}{p}} = |h(g^{-1} \cdot 0, g^{-1} \cdot 0)|^\lambda$$

so that the LHS is

$$\begin{aligned} \iota_\lambda(D)^2 \int_G h(g^{-1} \cdot 0, g^{-1} \cdot 0)^\lambda d\iota(g) &= \iota_\lambda(D)^2 \int_G h(g \cdot 0, g \cdot 0)^\lambda d\iota(g) \\ &= \iota_\lambda(D)^2 \int_D h(z, z)^\lambda d\iota(z), \end{aligned}$$

by our normalization of  $dg$ . The RHS of (11) is  $d_\Lambda^{-1} \iota_\lambda(D)^2$ , and our claims follows.  $\square$

### 4.3 Evaluation of the Constant $c_G$

We will use Lemma 4.1 to find the exact value of  $d_\Lambda$  for the scalar representation  $\tau_{-\lambda}$  and further for general discrete series. First we need some notation [4]. Let

$$\Gamma_a(\mathbf{s}) := \prod_{j=1}^r \Gamma(s_j - (j-1)\frac{a}{2})$$

be Gindikin's Gamma function associated with the root multiplicity  $a$  (without the factor  $(2\pi)^{\frac{n_1-r}{2}}$ ), for vectors  $\mathbf{s} = (s_1, \dots, s_r)$  and  $\Gamma_a(\lambda) := \Gamma_a((\lambda, \dots, \lambda))$ . Thus

$$\frac{\Gamma_a(\lambda - \frac{N}{r})}{\Gamma_a(\lambda)} = \prod_{j=1}^r \frac{\Gamma(\lambda - \frac{N}{r} - (j-1)\frac{a}{2})}{\Gamma(\lambda - (j-1)\frac{a}{2})}.$$

A sketch for the evaluation of the integral  $d_\Lambda^{-1}$  was given in [4, Theorem 3.6]; we give a detailed proof by using the known evaluation formula for the Selberg integral [1] as they are of importance for our main results.

**Proposition 4.2** *If  $\tau = \tau_{-\lambda}$  with  $\lambda > p - 1$  then we have*

$$d_\Lambda^{-1} = \int_D h(z, z)^{\lambda-p} dm(z) = \frac{\pi^N \Gamma_a(\lambda - \frac{N}{r})}{\Gamma_a(\lambda)}.$$

**Proof** The first equality is Lemma 4.1. We evaluate the integral by starting with the polar decomposition [10, Chapter I, Theorem 5.17] for  $\mathbb{C}^N$ ,

$$\begin{aligned} \int_{\mathbb{C}^N} f(z) dm(z) &= C \int_0^\infty \dots \int_0^\infty \int_K f(k \cdot (t_1 e_1 + \dots t_r e_r)) dk 2^r \\ &\quad \prod_j t_j^{2b+1} \prod_{j < k} |t_j^2 - t_k^2|^a dt_1 \dots dt_r, \end{aligned}$$

for some constant  $C$ . We calculate the exact value of this constant  $C$ . Let  $f$  be the  $K$ -invariant Gaussian function  $f(z) = e^{-\|z\|^2}$ , then

$$\begin{aligned} \pi^N &= \int_{\mathbb{C}^N} e^{-\|z\|^2} dm(z) \\ &= C \int_0^\infty \dots \int_0^\infty e^{-(t_1^2 + \dots + t_r^2)} 2^r \prod_j t_j^{2b+1} \prod_{j < k} |t_j^2 - t_k^2|^a dt_1 \dots dt_r \\ &= C \int_0^\infty \dots \int_0^\infty e^{-(s_1 + \dots + s_r)} \prod_j s_j^b \prod_{j < k} |s_j - s_k|^a ds_1 \dots ds_r. \end{aligned}$$

From [1, Corollary 8.2.2] we find

$$\begin{aligned} &\int_0^\infty \dots \int_0^\infty e^{-(s_1 + \dots + s_r)} \prod_j s_j^b \prod_{j < k} |s_j - s_k|^a ds_1 \dots ds_r \\ &= \prod_j \frac{\Gamma(b+1 + (j-1)\frac{a}{2}) \Gamma(1 + j\frac{a}{2})}{\Gamma(1 + \frac{a}{2})}. \end{aligned}$$

Hence we have

$$C = \pi^N \prod_j^r \frac{\Gamma(1 + \frac{a}{2})}{\Gamma(b+1 + (j-1)\frac{a}{2}) \Gamma(1 + j\frac{a}{2})}. \quad (14)$$

It follows that

$$\begin{aligned}
 & \int_D h(z, z)^{\lambda-p} dm(z) \\
 &= C \int_0^1 \dots \int_0^1 h(t_1 e_1 + \dots t_r e_r)^{\lambda-p} 2^r \prod_j t_j^{2b+1} \prod_{j < k} |t_j^2 - t_k^2|^a dt_1 \dots dt_r \\
 &= C \int_0^1 \dots \int_0^1 \prod_j (1-s_j)^{\lambda-p} \prod_j s_j^b \prod_{j < k} |s_j - s_k|^a ds_1 \dots ds_r.
 \end{aligned}$$

This is a Selberg integral and is evaluated by [1, Theorem 8.1.1]

$$\begin{aligned}
 & \int_0^1 \dots \int_0^1 \prod_j (1-s_j)^{\lambda-p} \prod_j s_j^b \prod_{j < k} |s_j - s_k|^a ds_1 \dots ds_r \\
 &= \prod_{j=1}^r \frac{\Gamma(b+1+(j-1)\frac{a}{2})\Gamma(\lambda-p+1+(j-1)\frac{a}{2})\Gamma(1+j\frac{a}{2})}{\Gamma(\lambda-p+b+2+(r+j-2)\frac{a}{2})\Gamma(1+\frac{a}{2})}. \quad (15)
 \end{aligned}$$

Hence

$$\begin{aligned}
 d_{\Lambda}^{-1} &= \int_D h(z, z)^{\lambda-p} dm(z) \\
 &= C \prod_{j=1}^r \frac{\Gamma(b+1+(j-1)\frac{a}{2})\Gamma(\lambda-p+1+(j-1)\frac{a}{2})\Gamma(1+j\frac{a}{2})}{\Gamma(\lambda-p+b+2+(r+j-2)\frac{a}{2})\Gamma(1+\frac{a}{2})} \\
 &= \pi^N \prod_{j=1}^r \frac{\Gamma(\lambda-p+1+(j-1)\frac{a}{2})}{\Gamma(\lambda-p+b+2+(r+j-2)\frac{a}{2})} \\
 &= \pi^N \frac{\Gamma_a(\lambda - \frac{N}{r})}{\Gamma_a(\lambda)}.
 \end{aligned}$$

□

Now we can finally find the exact value of the constant  $c_G$ . Recall the Pochhammer symbol  $(x)_k = x(x+1)\dots(x+k-1)$ .

**Proposition 4.3** *With our normalization of the Haar measure the formal degree is given by*

$$d_{\Lambda} = c_G d_{\Lambda}^H,$$

where  $d_{\Lambda}^H$  is given by Eq. (12) and

$$c_G = \pi^{-N} r! \prod_{i=1}^{r-1} (2+i)_i.$$

for  $G = Sp(n, \mathbb{R})$ ,

$$c_G = \pi^{-N} \left(m - \frac{1}{2}\right) (2m - 2)!$$

for  $G = SO_0(2, 2m - 1)$ , and

$$c_G = \pi^{-N} \prod_{j=1}^r \frac{\Gamma\left(\frac{N}{r} + (j-1)\frac{a}{2} + 1\right)}{\Gamma\left(1 + (j-1)\frac{a}{2}\right)} = \pi^{-N} \prod_{j=1}^r \left(1 + (j-1)\frac{a}{2}\right)^{\frac{N}{r}}$$

for all other irreducible Hermitian symmetric spaces.

**Proof** The constant  $c_G$  is independent of  $\Lambda$  and can be found by choosing the representations  $\tau_{-\lambda}$  above. First we investigate  $d_\Lambda^H$  by evaluating  $\Lambda(h_\alpha)$  on the co-roots  $h_\alpha$ . From [13, (1.4)] we see that for any of the strongly orthogonal co-roots  $h_j$  we have

$$\Lambda(h_j) = -\frac{\lambda}{p} 2\rho_n(h_j) = -\lambda.$$

In fact, using that

$$\mathfrak{k} = \mathbb{R}Z \oplus [\mathfrak{k}, \mathfrak{k}]$$

is an orthogonal decomposition, and the fact that  $\Lambda|_{[\mathfrak{k}, \mathfrak{k}]} = 0$ , we get that

$$-\lambda = \Lambda(h_1) = \Lambda\left(\frac{\langle h_1, Z \rangle}{\langle Z, Z \rangle} Z\right) = \frac{2\gamma_1(Z)}{\langle \gamma_1, \gamma_1 \rangle \langle Z, Z \rangle} \Lambda(Z) = -\frac{2i\Lambda(Z)}{\langle \gamma_1, \gamma_1 \rangle \langle Z, Z \rangle}.$$

Thus for any root  $\alpha \in \Delta^+$  we get that

$$\Lambda(h_\alpha) = \Lambda\left(\frac{\langle h_\alpha, Z \rangle}{\langle Z, Z \rangle} Z\right) = \frac{2\alpha(Z)\Lambda(Z)}{\langle Z, Z \rangle \langle \alpha, \alpha \rangle} = -\frac{2i\Lambda(Z)}{\langle Z, Z \rangle \langle \alpha, \alpha \rangle} = -\lambda \frac{\langle \gamma_1, \gamma_1 \rangle}{\langle \alpha, \alpha \rangle}.$$

By [27] there are short and long roots and the strongly orthogonal roots are always long. Say they are of length  $l$ , then the short roots are of length  $\frac{l}{\sqrt{2}}$ . Then

$$\Lambda(h_\alpha) = -\lambda$$

if  $\alpha$  is long, and

$$\Lambda(h_\alpha) = -2\lambda$$

if  $\alpha$  is short.

We compare  $d_\Lambda^H$  with  $d_\Lambda$  as polynomials of  $\lambda$ . We divide  $\mathfrak{g}$  into three cases depending on the multiplicity being  $a = 1$ ,  $a > 1$  odd, and even. See e.g [3] for a list of the Lie groups associated with all irreducible Hermitian symmetric spaces.

Case 1:  $G = Sp(r, \mathbb{R})$ . Here  $a = 1$  and  $\mathfrak{g}^\mathbb{C} = \mathfrak{sp}(r, \mathbb{C})$  has  $\Delta^+ = \{2\epsilon_j\}_{j=1}^m \cup \{\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq r}$ , the Harish-Chandra roots are  $\{2\epsilon_j\}_{j=1}^m$ ,  $\rho = \sum_{i=1}^r (r+1-i)\epsilon_i$ , and

$\Lambda = -\frac{\lambda}{2}(\epsilon_1 + \cdots + \epsilon_1)$ . We have

$$d_\Lambda = \pi^{-N} \prod_{i=1}^r \frac{\Gamma(\lambda - \frac{1}{2}(i-1))}{\Gamma(\lambda - \frac{1}{2}(r+i))}$$

is a polynomial with leading coefficient  $\pi^{-N}$ . The Harish-Chandra formal degree is

$$d_\Lambda^H = \prod_{i=1}^r \frac{2\lambda - (r+1-i)}{r+1-i} \prod_{1 \leq i < j \leq r} \frac{\lambda - (r+1 - \frac{i+j}{2})}{r+1 - \frac{i+j}{2}}.$$

Comparing this with  $d_\Lambda$  we find

$$c_G = \pi^{-N} r! \prod_{1 \leq i < j \leq r} \left( r+1 - \frac{i+j}{2} \right) = \pi^{-N} 2^{-\frac{(r-1)r}{2}} r! \prod_{i=1}^{r-1} (2+i)_i.$$

Case 2:  $G = SO_0(2, 2m-1)$ . Here  $r = 2$ ,  $N = 2m-1$ ,  $a = 2m-3$  is odd, and  $\frac{N}{r}$  is not an integer. We get

$$d_\Lambda = \pi^{-N} \frac{\Gamma(\lambda) \Gamma(\lambda - \frac{2m-3}{2})}{\Gamma(\lambda - 1 - \frac{2m-3}{2}) \Gamma(\lambda - 2m+2)} = \pi^{-N} \left( \lambda - m + \frac{1}{2} \right) (\lambda - 2m+2) \cdots (\lambda-1),$$

The root system of  $\mathfrak{g}$  has  $\Delta^+ = \{\epsilon_j\}_{j=1}^m \cup \{\epsilon_i \pm \epsilon_j\}_{1 \leq i < j \leq m}$  with the Harish-Chandra strongly orthogonal roots being  $\epsilon_1 + \epsilon_2, \epsilon_1 - \epsilon_2$ . Now  $\rho = \frac{1}{2} \sum_{i=1}^m (2m+1-2i)\epsilon_i$  and comparing the two polynomials we find

$$c_G = \pi^{-N} \left( m - \frac{1}{2} \right) (2m-2)!.$$

Case 3: The remaining cases. All roots are of the same (long) length as the Harish-Chandra roots, with  $a$  being even and  $N/r$  an integer [3]. We have

$$\begin{aligned} d_\Lambda &= \pi^{-N} \frac{\Gamma_a(\lambda)}{\Gamma_a(\lambda - \frac{N}{r})} = \pi^{-N} \prod_{j=1}^r \frac{\Gamma(\lambda - (j-1)\frac{a}{2})}{\Gamma(\lambda - \frac{N}{r} - (j-1)\frac{a}{2})} \\ &= \pi^{-N} \frac{\Gamma_a(\lambda)}{\Gamma_a(\lambda - \frac{N}{r})} = \pi^{-N} \prod_{j=1}^r \frac{\Gamma(\lambda - (j-1)\frac{a}{2})}{\Gamma(\lambda - \frac{N}{r} - (j-1)\frac{a}{2})} \\ &= \pi^{-N} \prod_{j=1}^r \left( \lambda - (j-1)\frac{a}{2} - \frac{N}{r} \right)_{\frac{N}{r}} \end{aligned}$$

and as a polynomial of  $\lambda$  its zeros are all given and it has leading coefficient  $\pi^{-N}$ . Also,

$$d_{\Lambda}^H = \prod_{\alpha \in \Delta^+} \frac{\Lambda(h_{\alpha}) + \rho(h_{\alpha})}{\rho(h_{\alpha})} = \prod_{\alpha \in \Delta^+} \frac{-\lambda + \rho(h_{\alpha})}{\rho(h_{\alpha})}$$

is a polynomial with the coefficient of the leading term being  $\prod_{\alpha \in \Delta^+} \rho(h_{\alpha})^{-1}$  and zeros  $\rho(h_{\alpha})$ . It follows that the product of zeros is

$$\prod_{j=1}^r \frac{\Gamma\left(\frac{N}{r} + (j-1)\frac{a}{2} + 1\right)}{\Gamma\left(1 + (j-1)\frac{a}{2}\right)}.$$

Consequently

$$c_G = \pi^{-N} \prod_{j=1}^r \frac{\Gamma\left(\frac{N}{r} + (j-1)\frac{a}{2} + 1\right)}{\Gamma\left(1 + (j-1)\frac{a}{2}\right)} = \pi^{-N} \prod_{j=1}^r \left(1 + (j-1)\frac{a}{2}\right)^{\frac{N}{r}}.$$

This finishes the proof.  $\square$

As a corollary we can find  $\langle v, v \rangle_{\mathcal{H}_{\Lambda}}$  for  $v \in V_{\Lambda}$ , the constant  $C(\Lambda)$  and a precise formula for the reproducing kernel.

**Theorem 4.4** *Let  $\Lambda$  be as above. Then for any unit vector  $v \in V_{\Lambda}$*

$$\langle v, v \rangle_{\mathcal{H}_{\Lambda}} = d_{\Lambda}^{-1}.$$

*Furthermore, the reproducing kernel for the space  $\mathcal{H}_{\Lambda}$  is given by*

$$K(z, w) = d_{\Lambda} \tau(B(z, \bar{w})).$$

**Proof** From (10) we have  $K(z, w) = C(\Lambda) \tau(B(z, \bar{w}))$ , in particular  $K(z, 0) = C(\Lambda)I$  and

$$\langle f(0), v \rangle_{\tau} = C(\Lambda) \langle f, v \rangle_{\mathcal{H}_{\Lambda}}.$$

It follows by the reproducing kernel formula that for any  $v, w \in V_{\Lambda} \subset \mathcal{H}_{\Lambda}$ ,

$$\begin{aligned} d_{\Lambda}^{-1} |\langle v, w \rangle_{\tau}|^2 &= |C(\Lambda)|^2 d_{\Lambda}^{-1} |\langle v, w \rangle_{\mathcal{H}_{\Lambda}}|^2 = |C(\Lambda)|^2 \int_G |\langle \pi_{\Lambda}(g)v, w \rangle_{\mathcal{H}_{\Lambda}}|^2 dg \\ &= \int_G |\langle \pi_{\Lambda}(g)v(0), w \rangle_{\tau}|^2 dg = \int_G |\langle \tau(J_g^{-1}(0))^{-1}v, w \rangle_{\tau}|^2 dg \\ &= \int_G |\langle \tau(J_g(0))^{-1}v, w \rangle_{\tau}|^2 dg. \end{aligned} \tag{16}$$

Let  $v$  be a unit vector and  $\{v_i\}_{i=1}^d$  an orthonormal basis of  $V_\Lambda$ , where  $d = \dim(V_\Lambda)$ . We compare  $\langle v, v \rangle_{\mathcal{H}_\Lambda}$  with  $L^2$ -square norm of the corresponding matrix coefficients:

$$\begin{aligned} d\langle v, v \rangle_{\mathcal{H}_\Lambda} &= dC(\Lambda)^{-1} = C(\Lambda)^{-1} \sum_{i=1}^d \langle v_i, v_i \rangle_\tau = \sum_{i=1}^d \langle v_i, v_i \rangle_{\mathcal{H}_\Lambda} \\ &= \int_D \sum_{i=1}^d \langle \tau(B(z, \bar{z}))^{-1} v_i, v_i \rangle_\tau d\iota(z) = \int_D \text{Tr}(\tau(B(z, \bar{z}))^{-1}) d\iota(z) \\ &= \int_G \text{Tr}(\tau(J_g(0)J_g(0)^*)^{-1}) dg = \sum_{i=1}^d \int_G \langle \tau(J_g(0)J_g(0)^*)^{-1} v_i, v_i \rangle_\tau dg \\ &= \sum_{i,j=1}^d \int_G |\langle \tau(J_g(0))^{-1} v_i, v_j \rangle_\tau|^2 dg = d_\Lambda^{-1} \sum_{i,j=1}^d |\langle v_i, v_j \rangle_\tau|^2 = dd_\Lambda^{-1}, \end{aligned}$$

where the penultimate equality is by (16). Hence we get that for any unit vector  $v$

$$\langle v, v \rangle_{\mathcal{H}_\Lambda} = d_\Lambda^{-1}.$$

We also obtain that the constant  $C(\Lambda)$  is given by  $C(\Lambda) = d_\Lambda$ .  $\square$

**Remark 4.5** We note that as a consequence we obtain the following integral evaluation

$$\begin{aligned} &d_\Lambda \int_D \text{Tr}(\tau_\Lambda(B(z, z)^{-1})) d\iota(z) \\ &= Cd_\Lambda \int_{[0,1]^r} \text{Tr}(\tau_\Lambda \left( B \left( \sum_{j=1}^r t_j e_j, \sum_{j=1}^r t_j e_j, \right)^{-1} \right)) 2^r \prod_j t_j^{2b+1} \prod_{j < k} |t_j^2 - t_k^2|^a dt_1 \dots dt_r \\ &= \dim(V_\Lambda), \end{aligned}$$

for any unit vector  $v$ , where  $C$  is the constant (14). This might be viewed as a generalization of the Selberg integral (15); in other words, the result is a consequence of the Selberg integral evaluation and the Harish-Chandra formula for formal degree.

## 5 Wehrl Inequality for Holomorphic Discrete Series

We prove our main results on Wehrl-type inequalities. We keep the previous notation. The tensor product below  $\mathcal{H}_1 \otimes \mathcal{H}_2$  of two Hilbert spaces of holomorphic functions on  $D$  will be realized as a space of holomorphic functions  $F(z, w)$  in two variables.

## 5.1 Tensor Products of Holomorphic Discrete Series and Intertwining Operators

We recall some known results on tensor product of holomorphic discrete series representations [23].

**Proposition 5.1** *Let  $(\mathcal{H}_\Lambda, \pi_\Lambda, G)$  and  $(\mathcal{H}_{\Lambda'}, \pi_{\Lambda'}, G)$  be two holomorphic discrete series representations of highest weights  $\Lambda$  and  $\Lambda'$ . Then  $\mathcal{H}_\Lambda \otimes \mathcal{H}_{\Lambda'}$  is a direct sum of representations of the form  $\pi_{\Lambda''}$  with finite multiplicities. The corresponding highest weights  $\Lambda''$  are of the form*

$$\Lambda'' = \Lambda_0 - (m_1\alpha_1 + \cdots + m_q\alpha_q),$$

where  $\Lambda_0$  is a weight of  $V_\Lambda \otimes V_{\Lambda'}$ ,  $m_i$  are nonnegative integers and the  $\alpha_i \in \Delta_n^+$ . In particular, there is an irreducible leading component

$$\mathcal{H}_{\Lambda+\Lambda'} \subseteq \mathcal{H}_\Lambda \otimes \mathcal{H}_{\Lambda'}$$

which is obtained by the intertwining map

$$J_0(F)(z) = P_{\Lambda+\Lambda'} F(z, z). \quad (17)$$

Here  $P_{\Lambda+\Lambda'} : V_\Lambda \otimes V_{\Lambda'} \rightarrow V_{\Lambda+\Lambda'} \subseteq V_\Lambda \otimes V_{\Lambda'}$  is the orthogonal projection. Moreover  $\mathcal{H}_{\Lambda+\Lambda'}$  appears in  $\mathcal{H}_\Lambda \otimes \mathcal{H}_{\Lambda'}$  with multiplicity one.

For any irreducible subrepresentation of  $V_\Lambda \otimes V_{\Lambda'}$  of  $K$  with highest weight  $\Lambda_0$  and the corresponding projection  $P_{\Lambda_0} : V_\Lambda \otimes V_{\Lambda'} \rightarrow V_{\Lambda_0}$  the map

$$F(z, w) \mapsto P_{\Lambda_0} F(z, z)$$

is an intertwining map onto an irreducible component of  $\mathcal{H}_\Lambda \otimes \mathcal{H}_{\Lambda'}$  of highest weight  $\Lambda_0$ .

We now find the exact constant  $C_{\Lambda, \Lambda'}$  such that  $C_{\Lambda, \Lambda'} J_0$  is a partial isometry.

**Proposition 5.2** *Let  $\mathcal{H}_\Lambda$ ,  $\mathcal{H}_{\Lambda'}$  and  $J_0$  be as in Proposition 5.1. Then  $C_{\Lambda, \Lambda'} J_0$  is a partial isometry, where*

$$C_{\Lambda, \Lambda'}^{-2} = c_G \left| \prod_{\alpha \in \Delta^+} \frac{(\Lambda(h_\alpha) + \rho(h_\alpha))(\Lambda'(h_\alpha) + \rho(h_\alpha))}{((\Lambda + \Lambda')(h_\alpha) + \rho(h_\alpha))\rho(h_\alpha)} \right|.$$

**Proof** The constant holomorphic function  $v$  is in  $\mathcal{H}_\Lambda$ , for any  $v \in V_\Lambda$  [13], and if  $v_\Lambda \in V_\Lambda$  and  $v_{\Lambda'} \in V_{\Lambda'}$  are highest weight vectors of unit length then so is  $P_{\Lambda+\Lambda'}(v_\Lambda \otimes v_{\Lambda'}) = v_\Lambda \otimes v_{\Lambda'}$ . Hence

$$\|J_0(v_\Lambda \otimes v_{\Lambda'})\|_{\mathcal{H}_{\Lambda+\Lambda'}} = \|P_{\Lambda+\Lambda'}(v_\Lambda \otimes v_{\Lambda'})\|_{\mathcal{H}_{\Lambda+\Lambda'}} = C_{\Lambda, \Lambda'}^{-1} \|v_\Lambda\|_{\mathcal{H}_\Lambda} \|v_{\Lambda'}\|_{\mathcal{H}_{\Lambda'}}.$$

Using Theorem 4.4 we see that

$$\begin{aligned} C_{\Lambda, \Lambda'}^{-2} &= \frac{\|P_{\Lambda+\Lambda'}(v_{\Lambda} \otimes v_{\Lambda'})\|^2}{\|v_{\Lambda}\|^2 \cdot \|v_{\Lambda'}\|^2} = \frac{d_{\Lambda} d_{\Lambda'}}{d_{\Lambda+\Lambda'}} = c_G \frac{d_{\Lambda}^H d_{\Lambda'}^H}{d_{\Lambda+\Lambda'}^H} \\ &= c_G \left| \prod_{\alpha \in \Delta^+} \frac{(\Lambda(h_{\alpha}) + \rho(h_{\alpha}))(\Lambda'(h_{\alpha}) + \rho(h_{\alpha}))}{((\Lambda + \Lambda')(h_{\alpha}) + \rho(h_{\alpha}))(\rho(h_{\alpha}))} \right|. \end{aligned}$$

□

## 5.2 Wehrl Inequality

We write  $Q_0 = C_{\Lambda, \Lambda'} J_0$ . We now prove our main result on the Wehrl-type inequality.

**Theorem 5.3** *The following Wehrl inequality holds for  $f \in \mathcal{H}_{\Lambda}$  and integers  $n \geq 2$ ,*

$$\begin{aligned} &\int_D \langle P_{n\Lambda}((\tau_{\Lambda}(B(z, \bar{z})^{-1})f(z))^{\otimes n}), P_{n\Lambda}(f(z)^{\otimes n}) \rangle_{\tau_{n\Lambda}} d\mu(z) \\ &\leq c_G^{n-1} \frac{(d_{\Lambda}^H)^n}{(d_{n\Lambda}^H)} \left( \int_D \langle \tau_{\Lambda}(B(z, \bar{z})^{-1})f(z), f(z) \rangle_{\tau_{\Lambda}} d\mu(z) \right)^n. \end{aligned}$$

*The equality holds if and only if  $f = K_u \tau(k)v_{\Lambda}$  for some  $u \in D$ ,  $k \in K$  and  $v_{\Lambda}$  a highest weight vector in  $V_{\Lambda}$ .*

**Proof** We have that

$$\mathcal{H}_{n\Lambda} \subseteq \mathcal{H}_{\Lambda}^{\otimes n},$$

and it appears with multiplicity one by Proposition 5.1. Now let

$$P_{n\Lambda} : V_{\Lambda}^{\otimes n} \rightarrow V_{n\Lambda}$$

be the projection. The operator

$$J_0 : \mathcal{H}_{\Lambda}^{\otimes n} \rightarrow \mathcal{H}_{n\Lambda}$$

defined by

$$J_0(f_1 \otimes \cdots \otimes f_n)(z) := P_{n\Lambda}(f_1(z) \otimes \cdots \otimes f_n(z)),$$

is then an intertwining map onto  $\mathcal{H}_{n\Lambda}$ ; this is Proposition 5.2 applied multiple times. Furthermore, by Proposition 5.2,

$$Q_0 = C_{\Lambda, n} J_0, \quad C_{\Lambda, n}^2 = c_G^{-n+1} \frac{d_{n\Lambda}^H}{(d_{\Lambda}^H)^n}$$

is a partial isometry. Applying this to the element  $f^{\otimes n}$  for  $f \in \mathcal{H}_\Lambda$  we get

$$\|P_{n\Lambda}(f(z)^{\otimes n})\|_{n\Lambda}^2 \leq c_G^{n-1} \frac{(d_\Lambda^H)^n}{d_{n\Lambda}^H} \|f^{\otimes n}\|^2 = c_G^{n-1} \frac{(d_\Lambda^H)^n}{d_{n\Lambda}^H} \|f\|^{2n},$$

or more explicitly

$$\begin{aligned} & \int_D \langle P_{n\Lambda}((\tau_\Lambda(B(z, \bar{z})^{-1} f(z))^{\otimes n}), P_{n\Lambda}(f(z)^{\otimes n}) \rangle_{\tau_{n\Lambda}} d\iota(z) \\ & \leq c_G^{n-1} \frac{(d_\Lambda^H)^n}{d_{n\Lambda}^H} \left( \int_D \langle \tau_\Lambda(B(z, \bar{z})^{-1} f(z), f(z) \rangle_{\tau_\Lambda} d\iota(z) \right)^n, \end{aligned}$$

proving the inequality.

We prove the rest of our Theorem for  $n = 2$ , and the same arguments are valid for general  $n$ . Note that by Proposition 5.1 we can write

$$f \otimes f = \bigoplus f_{\Lambda''},$$

where  $f_{\Lambda''} \in m_{\Lambda''} \mathcal{H}_{\Lambda''}$  and  $f_{2\Lambda} = Q_0(f \otimes f)$ . Now the inequality is an equality if and only if

$$f_{\Lambda''} \neq 0 \Leftrightarrow \Lambda'' = 2\Lambda.$$

This holds if and only if

$$f = Q_0^* Q_0(f). \quad (18)$$

This is clearly true if  $f = K_u \tau(k) v_\Lambda$ , because then if  $u = g \cdot 0$

$$\begin{aligned} f(z) &= K(z, u) \tau(k) v_\Lambda = \tau(J_g(g^{-1}z) K(g^{-1} \cdot z, 0) \tau(J_g(0))^* \tau(k) v_\Lambda \\ &= d_\Lambda \tau(J_{g^{-1}}(z))^{-1} \tau(J_g(0))^* \tau(k) v_\Lambda = d_\Lambda \pi_\Lambda(g) (\tau(J_g(0))^* \tau(k) v_\Lambda)(z). \end{aligned}$$

The identity (18) then follows by  $G$ -invariance of  $Q_0$  and the fact that the vector  $\tau(J_g(0))^* \tau(k) v_\Lambda$  is a translate of a highest weight vector.

Now suppose  $f \in \mathcal{H}_\Lambda$  is such that the equality (18) holds. By replacing  $f$  by  $\pi_\Lambda(g)f$  for some  $g \in G$  we may assume that  $f(0) \neq 0$  is a unit vector (note  $f$  is a reproducing kernel if and only if  $\pi_\Lambda(g)f$  is). We prove first that  $f(z) = K(z, u)v$  for some  $u \in D$  and  $v \in V_\Lambda$ , and then that the vector  $v$  has to be a translate  $\tau(k)v_\Lambda$  of the highest weight vector  $v_\Lambda \in V_\Lambda$ .

To prove that  $f(z) = K(z, u)v$  for some  $u$  and  $v$  we use the same idea as in [30]. Let  $z = (z_i)$  be the coordinates of  $z \in \mathbb{C}^N$  under some orthonormal basis. We consider the Toeplitz operator  $T_i$  by coordinate functions  $T_i f(z) = z_i f(z)$  on the space  $\mathcal{H}_\Lambda$ . First of all the operators  $T_i$  are bounded on  $\mathcal{H}_\Lambda$ ; indeed

$$\begin{aligned} \|T_i f\|^2 &= \int_D \langle \tau(B(z, \bar{z})^{-1} z_i f(z), z_i f(z) \rangle_{\tau} d\iota(z) = \int_D |z_i|^2 \langle \tau(B(z, \bar{z})^{-1} f(z), f(z) \rangle_{\tau} d\iota(z) \\ &\leq \int_D \|z\|^2 \langle \tau(B(z, \bar{z})^{-1} f(z), f(z) \rangle_{\tau} d\iota(z) \leq C_0 \|f\|^2 \end{aligned}$$

since  $D$  is bounded. Write  $T_{i,1}F(z, w) = z_i F(z, w)$  and  $T_{i,2}F(z, w) = w_i F(z, w)$  on the space  $\mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda$ . From the definition of  $Q_0$  we see that for any  $g \in \mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda$

$$Q_0((T_{i,1} - T_{i,2})g) = Q_0((z_i - w_i)g) = 0.$$

Thus

$$\begin{aligned} \langle f \otimes f, (T_{i,1} - T_{i,2})g \rangle_{\mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda} &= \langle Q_0^* Q_0(f \otimes f), (T_{i,1} - T_{i,2})g \rangle_{\mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda} \\ &= \langle Q_0(f \otimes f), Q_0((T_{i,1} - T_{i,2})g) \rangle_{\mathcal{H}_{2\Lambda}} = 0. \end{aligned}$$

Therefore  $(T_{i,1} - T_{i,2})^*(f \otimes f) = 0$ , which is the same as

$$(T_{i,1}^* f) \otimes f = f \otimes (T_{i,2}^* f).$$

This implies that there is a  $u_i \in \mathbb{C}$  such that

$$T_{z_i}^* f = u_i f.$$

We write  $u = (u_1, \dots, u_N)$ . This then implies that for any polynomial  $p$  (where  $\bar{p}$  is the polynomial where the coefficients are the complex conjugates of the original one) in  $D$  and  $v = f(0) \in V_\Lambda$

$$\begin{aligned} \langle pv, f \rangle_{\mathcal{H}_\Lambda} &= \langle p(T_1, \dots, T_n)v, f \rangle_{\mathcal{H}_\Lambda} = \langle v, \bar{p}(T_1^*, \dots, T_n^*)f \rangle_{\mathcal{H}_\Lambda} = \langle v, \bar{p}(u)f \rangle_{\mathcal{H}_\Lambda} \\ &= d_\Lambda \langle p(\bar{u})v, f(0) \rangle_{\tau_\Lambda}. \end{aligned}$$

Thus for any  $V_\Lambda$ -valued polynomial  $p$

$$\langle p, f \rangle_{\mathcal{H}_\Lambda} = d_\Lambda \langle p(\bar{u}), f(0) \rangle_{\tau_\Lambda}.$$

That is,  $p \rightarrow \langle p(\bar{u}), f(0) \rangle$  is a bounded evaluation and so by Lemma B.1  $\bar{u} \in D$ . Furthermore,  $f = K(z, \bar{u})v$  for some  $v \in V_\Lambda$ , where  $v = d_\Lambda^{-1} f(0)$ .

Now we prove  $f(0) = \tau(k)v_\Lambda$  for  $v_\Lambda$  a highest weight vector and for some  $k \in K$ . By Proposition 5.1 we see that for any irreducible representation  $V_{\Lambda_0} \subseteq V_\Lambda \otimes V_\Lambda$  the map

$$F(z, w) \mapsto P_{\Lambda_0}(F(z, z))$$

is an intertwining map  $\mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda \rightarrow \mathcal{H}_{\Lambda_0}$ . Thus if  $\Lambda_0 \neq 2\Lambda$  we have

$$P_{\Lambda_0}(f(z) \otimes f(z)) = 0, \quad z \in D.$$

In particular

$$P_{\Lambda_0}(f(0) \otimes f(0)) = 0,$$

for  $\Lambda_0 \neq 2\Lambda$ . This reduces to a condition for tensor product decomposition of finite-dimensional representations, and by Lemma A.1 we obtain that  $f(0) = \tau(k)v_\Lambda$  for  $v_\Lambda$  a highest weight vector.  $\square$

We reformulate the theorem as an  $L^2(G) - L^p(G)$ -estimate for matrix coefficients.

**Corollary 5.4** *Let  $\mathcal{H}_\Lambda$  be as above. Then we have the following  $L^2 - L^{2n}$ -estimates*

$$\int_G |\langle \pi(g)f, v_\Lambda \rangle_{\mathcal{H}_\Lambda}|^{2n} dg \leq c_G^{n-1} \frac{(d_\Lambda^H)^n}{(d_{n\Lambda}^H)^n} \left( \int_G |\langle \pi(g)f, v_\Lambda \rangle_{\mathcal{H}_\Lambda}|^2 dg \right)^n,$$

and equality holds if and only if  $f$  is as in Theorem 5.3 above.

**Proof** We realize  $V_{n\Lambda}$  as the leading component in the tensor product  $V_\Lambda^{\otimes n}$  as above, with  $v_\Lambda^{\otimes n} \in V_{n\Lambda} \subseteq V_\Lambda^{\otimes n}$ . By the proof of Theorem 5.3 the projection

$$Q_0 : \mathcal{H}_\Lambda^{\otimes n} \rightarrow \mathcal{H}_\Lambda^{\otimes n}$$

is given by

$$Q_0(f^{\otimes n})(z) = c_G^{-n+1} \frac{d_{n\Lambda}^H}{(d_\Lambda^H)^n} P_{n\Lambda} f(z)^{\otimes n}.$$

As  $v_\Lambda^{\otimes n} \in \mathcal{H}_{n\Lambda} \subseteq \mathcal{H}_\Lambda^{\otimes n}$  the  $L^{2n}$ -norm can be written, using (11), as

$$\begin{aligned} \int_G |\langle \pi_\Lambda(g)f, v_\Lambda \rangle_{\mathcal{H}_\Lambda}|^{2n} dg &= \int_G |\langle \pi_\Lambda^{\otimes n}(g)f^{\otimes n}, v_\Lambda^{\otimes n} \rangle_{\mathcal{H}_\Lambda}|^2 dg \\ &= \int_G |\langle \pi_{n\Lambda}(g)Q_0 f^{\otimes n}, Q_0(v_\Lambda^{\otimes n}) \rangle_{\mathcal{H}_{n\Lambda}}|^2 dg \\ &= d_{n\Lambda}^{-1} \|Q_0(f^{\otimes n})\|_{\mathcal{H}_{n\Lambda}}^2 \|Q_0(v_\Lambda^{\otimes n})\|_{\mathcal{H}_{n\Lambda}}^2, \end{aligned}$$

with

$$\int_G |\langle \pi_\Lambda(g)f, v_\Lambda \rangle_{\mathcal{H}_\Lambda}|^2 dg = d_\Lambda^{-1} \|f\|_{\mathcal{H}_\Lambda}^2 \|v_\Lambda\|_{\mathcal{H}_\Lambda}^2$$

for  $n = 1$ . Now by Theorem 5.3 we get

$$\begin{aligned} &d_{n\Lambda}^{-1} \|Q_0(f^{\otimes n})\|_{\mathcal{H}_{n\Lambda}}^2 \|Q_0(v_\Lambda^{\otimes n})\|_{\mathcal{H}_{n\Lambda}}^2 \\ &= \left( c_G^{-n+1} \frac{d_{n\Lambda}^H}{(d_\Lambda^H)^n} \right)^2 d_{n\Lambda}^{-1} \|P_{n\Lambda} f^{\otimes n}\|_{\mathcal{H}_{n\Lambda}}^2 \|P_{n\Lambda}(v_\Lambda^{\otimes n})\|_{\mathcal{H}_{n\Lambda}}^2 \\ &\leq c_G^{-n+1} \frac{d_{n\Lambda}^H}{(d_\Lambda^H)^n} d_{n\Lambda}^{-1} \|f\|_{\mathcal{H}_\Lambda}^{2n} \|P_{n\Lambda}(v_\Lambda^{\otimes n})\|_{\mathcal{H}_{n\Lambda}}^2 \\ &= \frac{1}{d_\Lambda^n} \|f\|_{\mathcal{H}_\Lambda}^{2n} \|P_{n\Lambda}(v_\Lambda^{\otimes n})\|_{\mathcal{H}_{n\Lambda}}^2 \\ &= \frac{\|P_{n\Lambda}(v_\Lambda^{\otimes n})\|_{\mathcal{H}_{n\Lambda}}^2}{\|v_\Lambda\|_{\mathcal{H}_\Lambda}^{2n}} \left( \int_G |\langle \pi_\Lambda(g)f, v_\Lambda \rangle_{\mathcal{H}_\Lambda}|^2 dg \right)^n, \end{aligned}$$

with equality if and only if  $f = K_u \tau(k) v_\Lambda$  for some  $u \in D$ ,  $k \in K$  and  $v_\Lambda$  a highest weight vector in  $V_\Lambda$ . We also know by Theorem 4.4

$$\|P_{n\Lambda}(v_\Lambda^{\otimes n})\|_{\mathcal{H}_{n\Lambda}}^2 = d_{n\Lambda}^{-1}, \quad \|v_\Lambda\|_{\mathcal{H}_\Lambda}^{2n} = d_{n\Lambda}^{-n}.$$

We conclude that

$$\int_G |\langle \pi(g)f, v_\Lambda \rangle_{\mathcal{H}_\Lambda}|^{2n} dg \leq \frac{d_\Lambda^n}{d_{n\Lambda}} \left( \int_G |\langle \pi(g)f, v_\Lambda \rangle_{\mathcal{H}_\Lambda}|^2 dg \right)^n,$$

with the constant  $\frac{d_\Lambda^n}{d_{n\Lambda}} = c_G^{n-1} \frac{(d_\Lambda^H)^n}{d_{n\Lambda}^H}$ . This completes the proof.  $\square$

**Remark 5.5** For the unit disc  $D = SU(1, 1)/U(1)$  a general inequality is proved in [5, 14] with the  $p$ -norm replaced by any positive convex function. A challenging problem would be to find optimal  $L^2 - L^p$  estimates for scalar holomorphic discrete series.

## 6 An Improved Wehrl Inequality for the Unit Disc

### 6.1 Irreducible Decomposition of Tensor Product Discrete Series of $SU(1, 1)$ and Differential Intertwining Operators

In this section we prove an improved  $L^2 - L^p$  Wehrl inequality for the holomorphic discrete series of  $SU(1, 1)$ , with  $p = 2n$  an even integer. For the Fock space  $\mathcal{F}(\mathbb{C})$  or equivalently the  $L^2(\mathbb{R})$ -space as representation space of the Heisenberg group  $\mathbb{R} \rtimes \mathbb{C}$  an improved Wehrl-type inequality (for any convex function instead of the  $p$ -norm) was recently obtained in [6]. Our result here might provide a method for obtaining a more precise remainder term for the improved  $L^2 - L^p$ -Wehrl inequalities for the Heisenberg group and  $SU(1, 1)$ .

Let  $\mathcal{H}_\nu$  be the weighted Bergman space of holomorphic functions  $f$  on the unit disk  $D \subset \mathbb{C}$  such that

$$\|f\|_{\nu,2}^2 := (\nu - 1) \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\nu \frac{dm(z)}{\pi(1 - |z|^2)^2} < \infty$$

where  $dm(z)$  as above is the Lebesgue measure. We also write

$$\|f\|_{\nu,p}^p := (\nu - 1) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^{\frac{p\nu}{2}} \frac{dm(z)}{\pi(1 - |z|^2)^2}.$$

Note that if  $\nu$  is an integer then  $\mathcal{H}_\nu$  is the holomorphic discrete series representation for the representation  $\tau_{-\nu}$  of  $U(1) \subseteq SU(1, 1)$

$$\tau_\nu \left( \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) = e^{-i\nu\theta}.$$

The tensor product of holomorphic discrete series of  $SU(1, 1)$  has a decomposition [22],

$$\mathcal{H}_\mu \otimes \mathcal{H}_\nu \cong \bigoplus_{k=0}^{\infty} \mathcal{H}_{\mu+\nu+2k},$$

where we normalize the inner product on all holomorphic discrete series so that  $\langle 1, 1 \rangle_{\mu+\nu+2k} = 1$ . Then we have partial isometries [20, 24]

$$Q_k^{\mu,\nu} : \mathcal{H}_\mu \otimes \mathcal{H}_\nu \rightarrow \mathcal{H}_{\mu+\nu+2k}$$

defined by

$$Q_k^{\mu,\nu} f(\xi) := C_{\mu,\nu,k} \sum_{j=0}^k (-1)^j \binom{k}{j} \frac{1}{(\mu)_j (\nu)_{k-j}} \partial_z^j \partial_w^{k-j} f|_{z=w=\xi}. \quad (19)$$

Here  $f \in \mathcal{H}_\mu \otimes \mathcal{H}_\nu$  is realized as a holomorphic function  $f(z, w)$  in  $(z, w) \in D^2$ , and the constant is determined by

$$C_{\mu,\nu,k}^{-2} = \frac{k!(\mu + \nu + k + 1)_k}{(\mu)_k (\nu)_k},$$

so that  $Q_k^{\mu,\nu}$  is a partial isometry [8]. We have also [24]

$$\mathcal{H}_\nu^{\otimes n} \cong \bigoplus_{k=0}^{\infty} \binom{n+k-2}{k} \mathcal{H}_{n\nu+2k},$$

where we call the projection onto the  $\binom{n+k-2}{k} \mathcal{H}_{n\nu+2k}$ -component  $Q_k$ .

**Lemma 6.1** *Let  $f \in \mathcal{H}_\nu$ . The projection  $Q_1(f^{\otimes n})$  of  $f^{\otimes n} \in \mathcal{H}_\nu^{\otimes n} \cong \bigotimes_{k=0}^{\infty} \binom{n+k-2}{n-2} \mathcal{H}_{n\nu+2k}$  onto the next leading component  $(n-1)\mathcal{H}_{n\nu+2}$  vanishes.*

**Proof** We do this by induction on  $n \geq 2$ . For  $n = 2$  we have

$$Q_1^{\nu,\nu}(f \otimes f) = C_{\nu,\nu,k} \left( \frac{1}{\nu} f'(\xi) f(\xi) - \frac{1}{\nu} f(\xi) f'(\xi) \right) = 0.$$

Now assume it is true for  $\mathcal{H}_\nu^{\otimes n}$ . We consider  $\mathcal{H}_\nu^{\otimes(n+1)} = \mathcal{H}_\nu^{\otimes n} \otimes \mathcal{H}_\nu$ , the first tensor factor is

$$\mathcal{H}_\nu^{\otimes n} = \bigoplus_{k=0}^{\infty} \binom{n+k-2}{n-2} \mathcal{H}_{n\nu+2k}$$

and  $Q_1 f^{\otimes n} = 0$ . We look for the projection  $F := Q_1 f^{\otimes(n+1)}$  of  $f^{\otimes(n+1)}$  onto the  $n\mathcal{H}_{(n+1)\nu+2}$ -isotypic component. It is obtained by

$$F = Q_1^{n\nu,\nu}(Q_0 f^{\otimes n} \otimes f) + Q_0^{n\nu+2,\nu}(Q_1 f^{\otimes n} \otimes f) = Q_1(Q_0 f^{\otimes n} \otimes f)$$

which furthermore is

$$F = C_{v,nv,1} \left( \frac{1}{v} f^n(\xi) f'(\xi) - \frac{1}{nv} n f'(\xi) f^n(\xi) \right) = 0,$$

completing the proof.  $\square$

## 6.2 An Improved Wehrl Inequality for the Unit Disc

Now we look at the projection onto the second factor  $\mathcal{H}_{nv+4}$  and obtain a stricter inequality.

**Theorem 6.2** *We have the following improved Wehrl  $L^2 - L^{2n}$  inequality*

$$\|f^n\|_{nv,2}^2 + \frac{2v^2(v+1)^2}{(2v+3)(2v+4)} \left\| \left( \frac{f''f}{(v)_2} - \frac{(f')^2}{v^2} \right) f^{n-2} \right\|_{nv+4,2}^2 \leq \|f\|_{v,2}^{2n}. \quad (20)$$

**Proof** We study the contribution to the component with highest weight  $\mathcal{H}_{nv+4}$  in the tensor product  $f^{\otimes n}$ . There is one contribution obtained from  $Q_{2v+4}^{v,v}(f \otimes f) \in \mathcal{H}_{2v+4} \subseteq \mathcal{H}_v \otimes \mathcal{H}_v$  and  $f^{\otimes(n-2)}$ ,

$$f^{\otimes n} \mapsto \left( z \mapsto f^{n-2}(z) Q_{2v+4}^{v,v}(f \otimes f)(z) \right).$$

Here  $Q_{2v+4}^{v,v}$  is the projection

$$\begin{aligned} Q_{2v+4}^{v,v}(f \otimes f) &= \frac{v(v+1)}{\sqrt{2(2v+3)(2v+4)}} 2 \left( \frac{1}{(v)_2} f''f - \frac{1}{v^2} (f')^2 \right) \\ &= \frac{\sqrt{2}v(v+1)}{\sqrt{(2v+3)(2v+4)}} \left( \frac{1}{(v)_2} f''f - \frac{1}{v^2} (f')^2 \right). \end{aligned}$$

Thus we obtain

$$\frac{\sqrt{2}v(v+1)}{\sqrt{(2v+3)(2v+4)}} \left( \frac{1}{(v)_2} f''f - \frac{1}{v^2} (f')^2 \right) f^{n-2} \in \mathcal{H}_{nv+4},$$

and

$$\|f\|_{v,2}^{2n} \geq \|f^n\|_{nv,2}^2 + \left\| \frac{\sqrt{2}v(v+1)}{\sqrt{(2v+3)(2v+4)}} \left( \frac{1}{(v)_2} f''f - \frac{1}{v^2} (f')^2 \right) f^{n-2} \right\|_{nv+4,2}^2,$$

completing the proof.  $\square$

We note that the second summand in (20) is vanishing exactly when  $\frac{1}{(v)_2} f''f - \frac{1}{v^2} (f')^2 = 0$ , i.e.

$$(f^{-\frac{1}{v}})'' = 0.$$

The solutions are exactly (multiples of) the reproducing kernels  $K_w$ , as  $f(z) = (c_1 + c_2 z)^{-\nu}$ . We have thus found a stronger inequality and identified the minimizer for the extra summand.

## Appendix A Wehrl Inequality for Matrix Coefficients of Representations of Compact Lie Groups

We have used the Wehrl-inequality for matrix coefficients of representations of compact Lie groups in our proof of the inequality for non-compact Hermitian symmetric spaces  $D = G/K$ . This inequality was stated in [26] for tensor powers  $V^{\otimes n}$  for general  $n \geq 2$ , referring further back to [2] for  $n = 2$ . However, the proof in [2] is incomplete. The precise gap is that the maximizer is proved to be an eigenvector of *one element* in the Cartan subalgebra but is claimed to be an eigenvector for *the whole Cartan subalgebra*. As we show below we do not actually need this fact, and the Cauchy-Schwarz inequality is needed as a critical step.

We recall the Casimir operator. Let  $\mathfrak{k}$  be the Lie algebra of a compact semisimple Lie group  $K$ ,  $\mathfrak{k}^{\mathbb{C}}$  the complexification and  $\kappa$  the Killing form. Let  $\{T_i\}$  be an orthonormal basis of  $i\mathfrak{k}$ . The Casimir element is

$$C = \sum_i T_i^2$$

and it acts on a representation  $(V_\tau, \tau, K)$  of  $K$  as

$$C = \sum_i \tau(T_i)^2.$$

If  $\tau$  is irreducible with highest weight  $\Lambda$  then  $C$  acts on  $V_\Lambda$  as the constant

$$\langle \Lambda + \rho, \Lambda + \rho \rangle - \langle \rho, \rho \rangle,$$

where  $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$  is the half-sum of the positive roots [11, Exercise 23.4]. (We have chosen  $T_i$  to be a basis of  $i\mathfrak{k}$  so that each  $\tau(T_i)$  is self-adjoint and  $C$  non-negative.)

Recall further that if  $V_{\Lambda_1}$  and  $V_{\Lambda_2}$  are two finite-dimensional irreducible representations of a semisimple Lie algebra with highest weights  $\Lambda_1$  and  $\Lambda_2$  then

$$V_{\Lambda_1} \otimes V_{\Lambda_2} \cong \bigoplus_{\Lambda} V_{\Lambda}, \quad (21)$$

where  $\Lambda \leq \Lambda_1 + \Lambda_2$  and  $V_{\Lambda_1 + \Lambda_2}$  appears exactly once in the decomposition. This is clear from the weight space decomposition [11, Chapter 21].

**Proposition A.1** (1) *Let  $K$  be a compact Lie group and  $(\tau_\Lambda, V_\Lambda)$  an irreducible representation of highest weight  $\Lambda$ . Consider the irreducible decomposition*

$$V_\Lambda^{\otimes n} \cong \bigoplus_{\Lambda'} m_{\Lambda'} V_{\Lambda'}$$

and

$$v^{\otimes n} = \sum_{\Lambda'} v_{\Lambda'}, \quad v_{\Lambda'} \in m_{\Lambda'} V_{\Lambda'}$$

for  $v \in V_{\Lambda}$  and  $n \geq 2$ . We have  $v^{\otimes n} \in V_{n\Lambda}$  if and only if  $v = \tau(k)v_{\Lambda}$  for some  $k \in K$  and a highest weight vector  $v_{\Lambda}$ .

(2) For any unit vector  $v \in V_{\Lambda}$  the following Wehrl inequality holds,

$$\int_K |\langle \tau(k)v, v_{\Lambda} \rangle|^{2n} dk \leq \frac{1}{\dim(V_{n\Lambda})}, \quad (22)$$

where  $v_{\Lambda}$  is a unit highest weight vector, and the equality holds if and only if  $v = \tau(k_0)v_{\Lambda}$  for some  $k_0 \in K$  and some highest weight vector  $v_{\Lambda}$ .

**Proof** We prove the proposition only for compact semisimple Lie groups and it implies the general result. We prove first that the second part is a consequence of the first one. Indeed, let

$$P_{n\Lambda} : V_{\Lambda}^{\otimes n} \rightarrow V_{n\Lambda}$$

be the projection. Then  $w_{n\Lambda} = P_{n\Lambda}(v_{\Lambda}^{\otimes n}) \in V_{n\Lambda}$  is a highest weight vector of unit length and we have that  $P_{n\Lambda}^* P_{n\Lambda}(v_{\Lambda}^{\otimes n}) = v_{\Lambda}^{\otimes n}$ , implying  $v_{\Lambda}^{\otimes n} \in V_{n\Lambda} \subseteq V_{\Lambda}^{\otimes n}$ , so

$$\begin{aligned} \int_K |\langle \tau_{\Lambda}(k)v, v_{\Lambda} \rangle_{V_{\Lambda}}|^{2n} dk &= \int_K |(\langle \tau_{\Lambda}(k)v \rangle^{\otimes n}, v_{\Lambda}^{\otimes n})_{V_{\Lambda}^{\otimes n}}|^2 dk \\ &= \int_K |\langle \tau_{n\Lambda}(k)P_{n\Lambda}(v^{\otimes n}), w_{\Lambda} \rangle_{V_{n\Lambda}}|^2 dk = \frac{1}{\dim(V_{n\Lambda})} \|P_{n\Lambda}(v^{\otimes n})\|_{V_{n\Lambda}}^2. \end{aligned}$$

This immediately implies the inequality in (22), and the equality holds if and only if  $v = \tau(k)v_{\Lambda}$  for a highest weight vector  $v_{\Lambda}$  as  $\|P_{n\Lambda}(v^{\otimes n})\|_{V_{n\Lambda}} = \|P_{n\Lambda}(v_{\Lambda}^{\otimes n})\|_{V_{n\Lambda}} = 1$  implies for any unit vector  $v$  that  $v \in V_{n\Lambda} \subseteq V_{\Lambda}^{\otimes n}$ . It follows also that

$$\int_K |\langle \tau(k)v, v_{\Lambda} \rangle|^{2n} dk \leq \int_K |\langle \tau(k)v_{\Lambda}, v_{\Lambda} \rangle|^{2n} dk, \quad (23)$$

with equality if and only if  $v = \tau(k)v_{\Lambda}$  for a highest weight vector  $v_{\Lambda}$ .

We now prove the first part. It follows from (21) that  $V_{n\Lambda} \subseteq V_{\Lambda}^{\otimes n}$  with multiplicity one, and that the other representations appearing in the decomposition of  $V_{\Lambda}^{\otimes n}$  have lower highest weights. Thus the sufficiency of the claim is clear, and we prove the necessity. We prove it first for  $n = 2$ , so we assume that  $v$  is a unit vector and  $v \otimes v \in V_{2\Lambda}$ .

Let  $C = \sum_i T_i^2$  be the Casimir element as above. Fix a choice of a Cartan subalgebra  $\mathfrak{h} \subseteq \mathfrak{k}$ . Consider the decomposition

$$V_{\Lambda} \otimes V_{\Lambda} \cong \bigoplus m_{\Lambda'} V_{\Lambda'},$$

the leading multiplicity being  $m_{2\Lambda} = 1$  and  $\Lambda' \leq 2\Lambda$ . Hence  $v \otimes v \in V_{2\Lambda} \subseteq V_{\Lambda}^{\otimes 2}$  if and only if

$$(\tau_{\Lambda} \otimes \tau_{\Lambda})(C)(v \otimes v) = (\langle 2\Lambda + \rho, 2\Lambda + \rho \rangle - \langle \rho, \rho \rangle)v \otimes v. \quad (24)$$

On the other hand

$$\begin{aligned} (\tau_{\Lambda} \otimes \tau_{\Lambda})(C) &= \sum_i (\tau \otimes \tau)(T_i)^2 = \sum_i (\tau(T_i) \otimes I + I \otimes \tau(T_i))^2 \\ &= \sum_i \tau(T_i)^2 \otimes I + I \otimes \tau(T_i)^2 + 2\tau(T_i) \otimes \tau(T_i) \\ &= \tau(C) \otimes I + I \otimes \tau(C) + 2 \sum_i \tau(T_i) \otimes \tau(T_i). \end{aligned}$$

Thus for any  $v$ ,

$$(\tau_{\Lambda} \otimes \tau_{\Lambda})(C)(v \otimes v) = 2(\langle \Lambda + \rho, \Lambda + \rho \rangle - \langle \rho, \rho \rangle)(v \otimes v) + 2 \sum_i \tau(T_i)v \otimes \tau(T_i)v. \quad (25)$$

Comparing (24) with (25) we find that  $v \otimes v \in V_{2\Lambda}$  if and only if

$$\sum_i \tau(T_i)v \otimes \tau(T_i)v = \langle \Lambda, \Lambda \rangle v \otimes v. \quad (26)$$

By taking the first tensor factor we see that (26) implies

$$\sum_i \langle \tau(T_i)v, v \rangle \tau(T_i)v = \langle \Lambda, \Lambda \rangle v. \quad (27)$$

Note that  $\tau(T_i)$  is self-adjoint and thus this gives an element

$$\sum_i \langle \tau(T_i)v, v \rangle T_i \in i\mathfrak{k}.$$

By [12, Theorem 4.34] there is a  $k \in K$  such that

$$\text{Ad}(k) \left( \sum_i \langle \tau(T_i)v, v \rangle T_i \right) \in i\mathfrak{h} \subseteq \mathfrak{h}^{\mathbb{C}},$$

and thus

$$\text{Ad}(k) \left( \sum_i \langle \tau(T_i)v, v \rangle T_i \right) = \sum_j a_j H_j \quad (28)$$

where  $H_j$  is an orthonormal basis of  $i\mathfrak{h}$  w.r.t. the Killing form  $\kappa$ . Note that

$$\begin{aligned} \sum_j a_j^2 &= \kappa \left( \sum_j a_j H_j, \sum_j a_j H_j \right) \\ &= \kappa \left( \text{Ad}(k) \left( \sum_i \langle \tau(T_i)v, v \rangle T_i \right), \text{Ad}(k) \left( \sum_i \langle \tau(T_i)v, v \rangle T_i \right) \right) \\ &= \kappa \left( \sum_i \langle \tau(T_i)v, v \rangle T_i, \sum_i \langle \tau(T_i)v, v \rangle T_i \right) = \sum_i \langle \tau(T_i)v, v \rangle^2 \\ &= \left\langle \sum_i (\tau(T_i) \otimes \tau(T_i))(v \otimes v), v \otimes v \right\rangle = \langle \Lambda, \Lambda \rangle. \end{aligned} \quad (29)$$

Applying  $\tau(k)$  to (27) and using (28) we find that  $w := \tau(k)v$  is an eigenvector of  $\tau(\sum_j a_j H_j)$ ,

$$\tau \left( \sum_j a_j H_j \right) w = \langle \Lambda, \Lambda \rangle w.$$

Decompose

$$w = \tau(k)v = \sum w_{\Lambda'}, \quad 0 \neq w_{\Lambda'} \in W_{\Lambda'}$$

as sum of weight vectors under the decomposition  $V_{\Lambda} = \bigoplus W_{\Lambda'}$  into weight subspaces of  $\mathfrak{h}^{\mathbb{C}}$ . Now we get

$$\langle \Lambda, \Lambda \rangle w = \tau \left( \sum_j a_j H_j \right) w = \sum_{\Lambda'} \left( \sum_j a_j \Lambda'(H_j) \right) w_{\Lambda'},$$

so that  $\sum_j a_j \Lambda'(H_j) = \langle \Lambda, \Lambda \rangle$ . The Cauchy–Schwarz inequality and (29) imply

$$\langle \Lambda, \Lambda \rangle = \sum_j a_j \Lambda'(H_j) \leq \left( \sum_j a_j^2 \right)^{\frac{1}{2}} \left( \sum_j \Lambda'(H_j)^2 \right)^{\frac{1}{2}} = \langle \Lambda, \Lambda \rangle^{\frac{1}{2}} \langle \Lambda', \Lambda' \rangle^{\frac{1}{2}}.$$

But  $\langle \Lambda', \Lambda' \rangle \leq \langle \Lambda, \Lambda \rangle$  with equality only for  $\Lambda' = \sigma \Lambda$  for some element  $\sigma$  in the Weyl group  $W$  by [12, Theorem 5.5]. So we find all  $w_{\Lambda'} = 0$  unless  $\Lambda' = \sigma \Lambda$  for some  $\sigma \in W$ . Hence (by taking into account the Weyl group element  $\sigma$ ) there is  $k \in K$  such that  $\tau(k)v = w = w_{\Lambda}$  is a highest weight vector.

Now we prove our claim for general  $n > 2$ , and we assume  $v$  is a unit vector,  $v^{\otimes n} \in V_{n\Lambda}$ . Observe that  $v^{\otimes n} = v^{\otimes 2} \otimes v^{\otimes (n-2)} \in V^{\otimes 2} \otimes V^{\otimes (n-2)}$  and the first factor has a decomposition

$$v^{\otimes 2} = \sum_{\Lambda' \leq 2\Lambda} v_{\Lambda'} \in \bigoplus_{\Lambda' \leq 2\Lambda} m_{\Lambda'} V_{\Lambda'}. \quad (30)$$

For any  $\Lambda' \leq 2\Lambda$ ,  $\Lambda' \neq 2\Lambda$  we have

$$V_{\Lambda'} \otimes V_{\Lambda}^{\otimes(n-2)} = \sum_{\Lambda''} m(\Lambda'') V_{\Lambda''}$$

with each  $\Lambda'' < n\Lambda$ . Thus  $V_{n\Lambda} \perp V_{\Lambda'} \otimes V_{\Lambda}^{\otimes(n-2)}$  in  $V^{\otimes n}$  and in particular  $P_{n\Lambda}(v_{\Lambda'} \otimes v^{\otimes(n-2)}) = 0$ . Therefore

$$P_{n\Lambda}(v^{\otimes n}) = P_{n\Lambda} \left( \left( \sum_{\Lambda'} v_{\Lambda'} \right) \otimes v^{\otimes(n-2)} \right) = P_{n\Lambda}(v_{2\Lambda} \otimes v^{\otimes(n-2)}).$$

This implies that

$$1 = \|v^{\otimes n}\| = \|P_{n\Lambda}(v^{\otimes n})\| = \|P_{n\Lambda}(v_{2\Lambda} \otimes v^{\otimes(n-2)})\| \leq \|v_{2\Lambda}\| \cdot \|v\|^{n-2} = \|v_{2\Lambda}\| \leq 1.$$

Thus all other components  $v_{\Lambda'}$  in (30) vanish and  $v^{\otimes 2} = v_{2\Lambda} \in V_{2\Lambda}$ . This reduces to the case  $n = 2$  and completes the proof.  $\square$

## Appendix B Bounded Point Evaluations for Bergman Spaces of Vector-Valued Holomorphic Functions

We prove that bounded point evaluations for our Bergman space of vector valued holomorphic functions on  $D$  are given by points in  $D$ . This might be a known fact for a larger class of Bergman spaces but we can not find some exact reference and we present here an elementary proof.

**Lemma B.1** *Let  $u \in \mathfrak{p}^+$ ,  $0 \neq v_0 \in V_{\Lambda}$  be a fixed vector, and consider the evaluation of polynomials  $p \in \text{Pol}(\mathbb{C}^N) \otimes V_{\Lambda} \subset \mathcal{H}_{\Lambda}$ ,*

$$p \mapsto \langle p(u), v_0 \rangle_{\tau}.$$

*It is bounded on the Hilbert space  $\mathcal{H}_{\Lambda}$  if and only if  $u \in D$ .*

**Proof** Obviously the evaluation map is bounded if  $u \in D$  by the reproducing kernel property, as

$$|\langle p(u), v_0 \rangle_{\tau}| = |\langle p, K_u v_0 \rangle_{\mathcal{H}_{\Lambda}}| \leq \|K_u v_0\|_{\mathcal{H}_{\Lambda}} \|p\|_{\mathcal{H}_{\Lambda}}.$$

Now we prove the converse. Recall [19] that  $\sum_{j=1}^r \mathbb{R}(e_j + e_{-j}) \subseteq \mathfrak{p}$  is a maximal Abelian subalgebra of  $\mathfrak{p}$  and if  $a(t) := \exp(\sum_{j=1}^r t_j(e_j + e_{-j}))$  then

$$t \mapsto a(t) \cdot 0 = \sum_{j=1}^r \tanh t_j e_j$$

in  $D = G/K$ . The space  $\mathbb{C}^N$  is a disjoint union of  $D = \{u \mid |u| < 1\}$ , the boundary  $\partial D = \{u \mid |u| = 1\}$  and the complement  $\overline{D}^c = \{u \mid |u| > 1\}$ . We assume first that  $u \in \partial D$  and prove that the evaluation

$$p \mapsto \langle p(u), v_0 \rangle_\tau$$

is unbounded. By the  $K$ -equivariance we can assume that  $u = u_1 e_1 + \cdots + u_r e_r$ , with  $u_1 = 1$  and  $0 \leq u_j \leq 1$ . Write  $x = x(t) = a(t) \cdot 0 = \sum_{j=1}^r x_j e_j$ ,  $x_j = x_j(t) = \tanh t_j$ ,  $t_j \geq 0$ , as above. Now if  $\{v^s\}_s$  is an orthonormal basis of weight vectors for  $V_\Lambda$  with weights  $\Lambda^s$  then [13]

$$\tau(J_{a(t)}(0))v^s = \prod_j (1 - x_j^2)^{\frac{1}{2}\Lambda^s(h_j)} v^s.$$

As

$$B(g \cdot 0, \overline{g \cdot 0}) = J_g(0)J_g(0)^*,$$

we see that the reproducing kernel acts on  $v^s$  as

$$K(x, x)v^s = d_\Lambda \prod_j (1 - x_j^2)^{\Lambda^s(h_j)} v^s.$$

By using  $K$ -equivariance we have the same is true for  $z_1 e_1 + \cdots + z_r e_r \in D$ ,  $z_1, \dots, z_r \in \mathbb{C}$ , namely

$$K(z, z)v^s = d_\Lambda \prod_j (1 - |z_j|^2)^{\Lambda^s(h_j)} v^s.$$

As  $K(z, w)$  is holomorphic in the first coordinate and antiholomorphic in the second, we see that if  $z = \sum_{j=1}^r z_j e_j$ ,  $w = \sum_{j=1}^r w_j e_j \in D$  then

$$K(z, w)v^s = d_\Lambda \prod_j (1 - z_j \overline{w_j})^{\Lambda^s(h_j)} v^s.$$

We have also for  $0 < \delta < 1$ ,

$$K_{\delta w}(z) = K(z, \delta w) = K(\delta z, w) = K_w(\delta z)$$

for any  $z, w \in D$  and that the function  $z \mapsto K_w(z)v$  has norm

$$\|K_w v\|_{\mathcal{H}_\Lambda}^2 = \langle K(w, w)v, v \rangle_\tau.$$

Thus the function

$$f_{\delta u} : z \mapsto \langle K(\delta u, \delta u)v, v \rangle_\tau^{-\frac{1}{2}} K_{\delta u}(z)v$$

is of unit norm and can be analytically extended to a bigger set

$$D_\epsilon = \{z \in \mathbb{C}^N \mid d(z, D) < \epsilon\},$$

containing  $\overline{D}$  where the distance  $d(z, D)$  is defined using spectral norm. Now we choose a unit weight vector  $v = v^s$  such that  $|\langle v^s, v_0 \rangle| > 0$  and we get

$$\begin{aligned} |\langle f_{\delta u}(u), v_0 \rangle_\tau| &= \left| \left\langle K(\delta u, \delta u) v^s, v^s \right\rangle_\tau^{-\frac{1}{2}} K_{\delta u}(u) v^s, v_0 \right\rangle_\tau \Big| \\ &= d_\Lambda^{\frac{1}{2}} \left( \prod_j (1 - \delta^2 |u_j|^2)^{\Lambda^s(h_j)} \right)^{-\frac{1}{2}} \prod_j (1 - \delta |u_j|^2)^{\Lambda^s(h_j)} |\langle v^s, v_0 \rangle_\tau| \\ &= d_\Lambda^{\frac{1}{2}} \prod_j \left( \frac{1 - \delta |u_j|^2}{(1 - \delta^2 |u_j|^2)^{\frac{1}{2}}} \right)^{\Lambda^s(h_j)} |\langle v^s, v_0 \rangle_\tau|. \end{aligned}$$

Now by the Harish-Chandra condition in Theorem 3.2 we have that

$$\Lambda^s(h_1) < 1 - p \leq -1.$$

Also,  $u_1 = 1$  so

$$\lim_{\delta \rightarrow 1} \frac{1 - \delta |u_1|^2}{(1 - \delta^2 |u_1|^2)^{\frac{1}{2}}} = \lim_{\delta \rightarrow 1} \frac{1 - \delta}{(1 - \delta^2)^{\frac{1}{2}}} = 0.$$

It follows that

$$\lim_{\delta \rightarrow 1} \langle f_{\delta u}(u), v_0 \rangle_\tau = \lim_{\delta \rightarrow 1} d_\Lambda^{\frac{1}{2}} \prod_j \left( \frac{1 - \delta |u_j|^2}{(1 - \delta^2 |u_j|^2)^{\frac{1}{2}}} \right)^{\Lambda^s(h_j)} = \infty. \quad (31)$$

Now the domain  $D$  is convex, so  $D$  and its closure  $\overline{D}$  are polynomially convex. It follows by the Oka–Weil theorem [21, Theorem VI.1.5] that for every  $\epsilon > 0$  there are  $V_\Lambda$ -valued polynomials  $p_k$  on  $\mathbb{C}^N$  such that

$$\sup_{z \in \overline{D}} \|f_{\delta u}(z) - p_k(z)\|_\tau \rightarrow 0, k \rightarrow \infty.$$

By the dominated convergence theorem we then also have

$$\|p_k\|_{\mathcal{H}_\Lambda}^2 \rightarrow \|f_{\delta u}\|_{\mathcal{H}_\Lambda}^2 = 1, k \rightarrow \infty.$$

We can then use (31) to prove that  $\langle p(u), v_0 \rangle_\tau$  is unbounded for polynomials  $p$ .

Finally we prove if  $u \notin \overline{D}$  then evaluation is unbounded. Clearly  $\frac{u}{|u|} \in \partial D$  where  $|u|$  is the spectral norm. Let  $M > 0$  be arbitrarily large. By the previous result for  $\frac{u}{|u|}$

there is a polynomial  $p = p_M$  such that  $\|p\| < 2$  and  $\langle p(\frac{u}{|u|}), v_0 \rangle_\tau > M$ . Denote  $q(z) = p(\frac{z}{|z|})$ . Then for  $z \in D$

$$\|q(z)\|_\tau = \left\| p\left(\frac{z}{|z|}\right) \right\|_\tau = \left\langle p, K_{\frac{z}{|z|}} \frac{q(z)}{\|q(z)\|_\tau} \right\rangle_{\mathcal{H}_\Lambda} \leq \|p\|_{\mathcal{H}_\Lambda} \left\| K_{\frac{z}{|z|}} \right\|_{\mathcal{H}_\Lambda}.$$

This must be bounded as  $\|p\| \leq 2$  and  $z \mapsto \|K_{\frac{z}{|z|}}\|$  is bounded on  $D$  as  $|u| > 1$ . Thus  $\|q\|$  is also bounded irrespective of  $M$ . However,

$$\langle q(u), v_0 \rangle > M,$$

so evaluation in  $u$  is unbounded. This completes the proof.  $\square$

**Acknowledgements** We would like to thank Jan Frahm, Rupert Frank, Paul Nelson and Bent Ørsted for some stimulating discussions. We would also like to thank the anonymous referee for carefully reading the manuscript and for several suggestions.

**Funding** Open access funding provided by Chalmers University of Technology.

**Data availability** Not applicable. No datasets were generated or analyzed during the study.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

## References

1. Andrews, G.E., Askey, R., Roy, R.: Special Functions, Encyclopedia of Mathematics and Its Applications, vol. 71. Cambridge University Press (1999)
2. Delbourgo, R., Fox, J.: Maximum weight vectors possess minimal uncertainty. *J. Phys. A* **10**, L233–L235 (1977)
3. Enright, T., Howe, R., Wallach, N.: A classification of unitary highest weight modules. In: Representation Theory of Reductive Groups, Progress in Mathematics, vol. 40, pp. 97–144. Birkhäuser, Boston (1983)
4. Faraut, J., Korányi, A.: Function spaces and reproducing kernels on bounded symmetric domains. *J. Funct. Anal.* **88**(1), 64–89 (1990)
5. Frank, R.L.: Sharp inequalities for coherent states and their optimizers. *Adv. Nonlinear Stud.* **23**(1), 20220050 (2023)
6. Frank, R.L., Nicola, F., Tilli, P.: The generalized Wehrl entropy bound in quantitative form. *J. Eur. Math. Soc.* (2025). <https://doi.org/10.4171/jems/1674>
7. van Haastrecht, R.: Limit formulas for the trace of the functional calculus of quantum channels for  $SU(2)$ . *J. Lie Theory* **34**(3), 653–676 (2024)
8. van Haastrecht, R.: Functional calculus of quantum channels for the holomorphic discrete series of  $SU(1, 1)$ . *J. Funct. Anal.* **289**(6), 111036 (2025)
9. Harish-Chandra: Representations of semisimple Lie Groups VI: integrable and square-integrable representations. *Am. J. Math.* **78**(3), 564–628 (1956)

10. Helgason, S.: Groups and Geometric Analysis. Mathematical Surveys and Monographs, vol. 83. American Mathematical Society (1984)
11. Humphreys, J.E.: Introduction to Lie Algebras and Representation Theory. Graduate Texts in Mathematics, vol. 9. Springer (1972)
12. Knapp, A.W.: Lie Groups Beyond an Introduction. Progress in Mathematics, vol. 140. Birkhäuser (2002)
13. Korányi, A.: A simplified approach to the holomorphic discrete series. *Indag. Math. (N.S.)* **36**(1), 29–41 (2025)
14. Kulikov, A.: Functionals with extrema at reproducing kernels. *Geom. Funct. Anal.* **32**(4), 938–949 (2022)
15. Lieb, E.: Proof of an entropy conjecture of Wehrl. *Commun. Math. Phys.* **62**(1), 35–41 (1978)
16. Lieb, E., Solovej, J.P.: Proof of an entropy conjecture for Bloch coherent spin states and its generalizations. *Acta Math.* **212**(2), 379–398 (2014)
17. Lieb, E., Solovej, J.P.: Proof of the Wehrl-type entropy conjecture for symmetric  $SU(N)$  coherent states. *Commun. Math. Phys.* **348**(2), 567–578 (2014)
18. Lieb, E., Solovej, J.P.: Proof of a Wehrl-Type Entropy Inequality for the Affine  $AX + B$  Group. EMS Series of Congress Reports, pp. 301–314. EMS Press, Berlin (2021)
19. Loos, O.: Bounded symmetric domains and Jordan pairs. University of California, Irvine (1977)
20. Peng, L., Zhang, G.: Tensor products of holomorphic representations and bilinear differential operators. *J. Funct. Anal.* **210**(1), 171–192 (2004)
21. Range, R.M.: Holomorphic Functions and Integral Representations in Several Complex Variables. Graduate Texts in Mathematics, vol. 108. Springer (1986)
22. Repka, J.: Tensor products of unitary representations of  $SL_2(\mathbb{R})$ . *Am. J. Math.* **100**(4), 747–774 (1978)
23. Repka, J.: Tensor products of holomorphic discrete series representations. *Can. J. Math.* **31**(4), 836–844 (1979)
24. Rosengren, H.: Multivariate orthogonal polynomials and coupling coefficients for discrete series representations. *SIAM J. Math. Anal.* **30**(2), 232–272 (1999)
25. Satake, I.: Algebraic Structures of Symmetric Domains. Kanô Memorial Lectures, vol. 4. Iwanami Shoten/Princeton University Press, Tokyo/Princeton (1980)
26. Sugita, A.: Proof of the generalized Lieb-Wehrl conjecture for integer indices larger than one. *J. Phys. A* **35**(42), L621–626 (2002)
27. Schlichtkrull, H.: One-dimensional  $K$ -types in finite-dimensional representations of semisimple Lie groups: a generalization of Helgason's theorem. *Math. Scand.* **54**(2), 279–294 (1984)
28. Wallach, N.R.: The analytic continuation of the discrete series. I, II. *Trans. Am. Math. Soc.* **251**, 1–17, 19–37 (1979)
29. Wehrl, A.: On the relation between classical and quantum-mechanical entropy. *Rep. Math. Phys.* **16**(3), 353–358 (1979)
30. Zhang, G.: Wehrl-type inequalities for Bergman spaces on domains in  $\mathbb{C}^d$  and completely positive maps. In: The Bergman Kernel and Related Topics. Springer Proceedings in Mathematics & Statistics, vol. 447, pp. 343–355. Springer, Singapore (2024)