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Adiels, E., Malek, S., Williams, C. (2026). Michell structures made from two materials carrying their own weight. Proceedings of IASS-IWSS 2026 – September 14-18, 2026, Turin, Italy "R-Evolution of Shapes: Sustainability, Re-Use, and New Design Paradigms"

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Proceedings of IASS-IWSS 2026 – September 14-18, 2026, Turin, Italy
"R-Evolution of Shapes: Sustainability, Re-Use, and New Design Paradigms"
Annual Symposium of the International Association for Shell and Spatial Structures
3rd Italian Workshop on Shell and Spatial Structures
A. Manuello Bertetto, S. Gabriele, F. Marmo, A. Micheletti (eds.)

Michell structures made from two materials carrying their own weight

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Abstract

Let us imagine that we have a circular support of known radius in 2 dimensions and that we want to construct a minimum weight cantilever of a given span to support a mouse at the tip and the weight of the cantilever itself. The cantilever is made from 2 materials, one for tension and one for compression, with known strengths. We will make the same assumptions that Anthony Michell made in his seminal paper of 1904, 'The limits of economy of material in frame-structures', except that we will include the own weight of the structure, which Michell did not. However we follow Michell in ignoring buckling and thus the structure will consist of a fine network of pin ended struts and ties in which compression members can be very slender. This assumption means that if we replace our mouse by a rat that weights 10 times as much, then the whole structure will have to weight 10 times a much. We offer an alternative derivation of Michell's result, again using virtual power (also known as virtual work) as he did. Including the own weight of the structure in the formulation means that the equations almost certainly require a numerical solution, and we present numerical results for the cantilever, which we compare to Michell's for the special case of a weightless cantilever.

Keywords: Michell structures, calculus of variations, minimal structures, own weight, principal stress space

1. Introduction

Michell structures or Michell trusses were introduced in 1904 in a wonderfully concise, inventive and subtle paper by Anthony Michell [1] entitled *The limits of economy of material in frame-structures*. They consist of a fine network of pin ended struts and ties [2, 3, 4, 5, 6]. We will follow Michell in ignoring buckling, however we include the own weight of the structure, which Michell did not.

Michell starts by considering a discrete structure consisting of individual structural members. Quantities such as the total mass of the structure or the virtual work done during a virtual increment of deformation are found using summations. However Michell's final result is a continuum, in other words an infinitely fine array of members.

We shall assume a continuum from the start, and we also use virtual work, except that we use the phrase virtual power since we find it easier to imagine virtual velocities and rates of strain, rather than infinitesimal increments of displacement and strain.

In his paper Michell correctly shows that making a weightless structure from two materials, one for compression and one for tension, with different strengths does not affect the differential equation describing the geometry of the structure. However it does affect the boundary conditions at a support, at which the virtual velocity in the virtual power equation (or the virtual infinitesimal displacement in the virtual work equation) must be zero, even if the support for the real structure is flexible.

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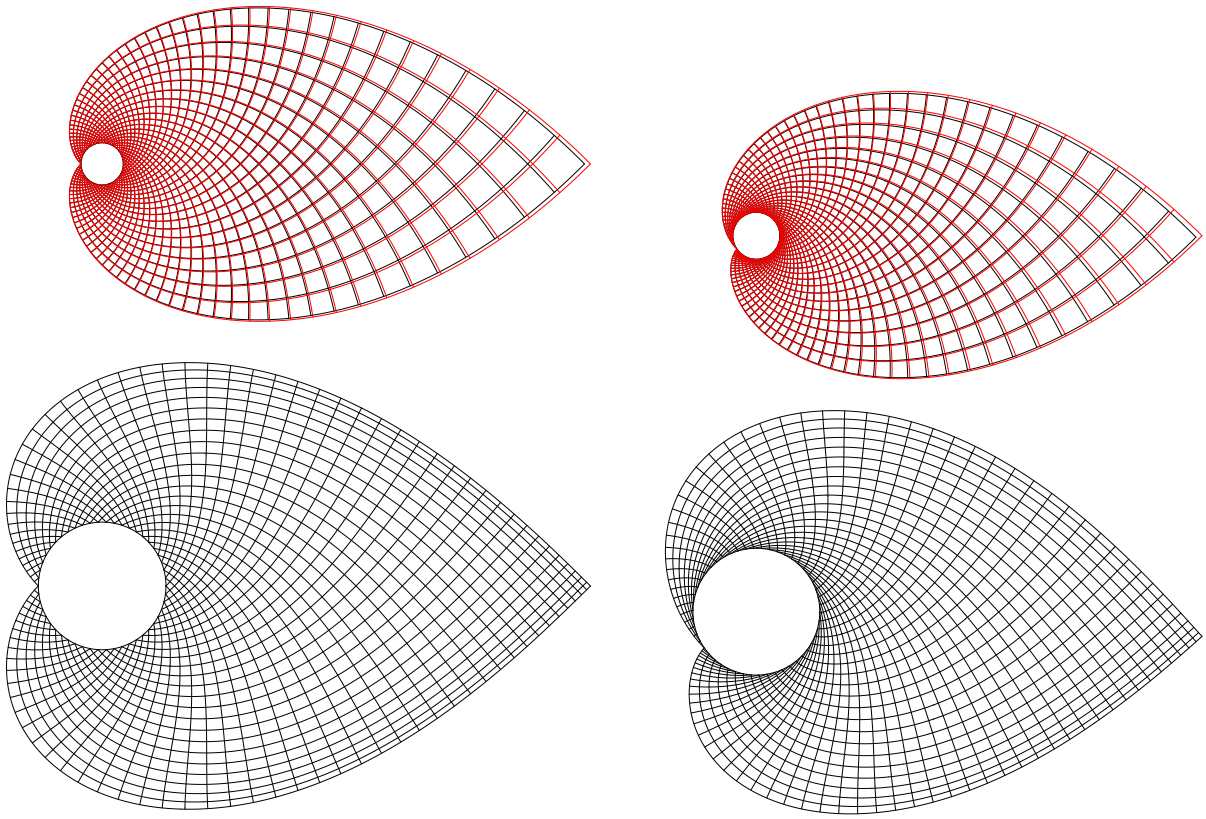

(a) Top: Cantilever as in Michell's paper with load at tip, made of one weightless material, analytic model in red, numerical in black.
Bottom: The same cantilever, with own weight, numerical model only.

(b) Top: Cantilever with load at tip, made of two weightless materials, analytic model in red, numerical in black.
Bottom: The same cantilever, with own weight, numerical model only.

Figure 1: The scale of the drawings is only relevant for the bottom 2 drawings, when the scale depends upon the ratio of strength to density times gravity, which has the units of length.

2. Results

To explain the aim of our work, and to justify its mathematical complexity, we present the results at this stage, rather than wait until the end, as is the norm. The top left image in our figure 1 is a mirror image of figure 1 in Michell's paper and it shows the analytic result for an optimum weightless cantilever loaded at its tip, made of 1 material. The top right image is our analytic result for a weightless cantilever made of 2 materials. The bottom left image is our numerical result for a cantilever loaded at its tip made of 1 material including the own weight of the structure. Finally, the bottom right image shows our numerical result for a cantilever loaded at its tip made of 2 materials, including own weight.

In more detail, the top left image in figure 1 consists of two families of logarithmic spirals in red, and our numerical approximation in black. The reason for the numerical version is to examine its accuracy in comparison for the 'exact' analytic result. We can add other point loads to the body of the cantilever, away from the tip, without altering the form of the cantilever, *provided that the members in one direction remain in tension and the members in the other remain in compression*. The extra loads will modify the forces in the structural members, but not their layout.

In producing the lower left image in figure 1 we have assumed the structure is made of steel with density 7850 kg/m^3, strength 350 MPa and the acceleration due to gravity is 9.81 m/s^2. The structure is statically determinate and all the member forces can by found by resolving forces starting at the tip where a single point load is applied. Once we include the own weight of the structure its size becomes important and in this case the radius of the supporting circle is a fairly impressive 775 m and clearly this 2 dimensional structure will have to occupy a large volume of 3 dimensional space, with some very wide 'flanges'. Note that the own weight of the cantilever is unknown and therefore has to be included in the optimisation process.

We have only been able to model the case with own weight numerically because we have not succeeded in finding an analytic solution. The numerical procedure is described in §7. and the results are relatively independent of the fineness of the grid used, except the point at which the numerical results become unstable as the radius of the supporting circle is increased. The finer the grid, the smaller the radius of the supporting circle, *and therefore also the smaller the span of the cantilever itself*, and in figure 1 the numerical grid is 16 times finer than that shown, so that each point represent 256 points, and there are 320800 points in each analysis.

The top right image in 1 is again a weightless structure and again the analytic solution is in red and the numerical in black, but this time for 2 materials. We have assumed very high strength steel of density 7850 kg/m^3 and strength 2000 MPa in tension and low strength concrete of density 2500 kg/m^3 and strength 20 MPa in compression. Again, for the weightless structure the size of the structure is immaterial, however for the numerical solution including own weight shown in the bottom right image in figure 1, the radius of the supporting circle is a still reasonably impressive 340 m. Note that the form of the structure is only influenced by the ratio of the density to strength of each material.

3. Le principe des puissances virtuelles

The use of virtual power is central to Michell's paper. In France *Le principe des puissances virtuelles* or the principle of virtual power is used but in the English speaking counties the principle of virtual work is usually used, which is most unfortunate. This is because the principle only applies to an *increment of work done* during an *infinitesimal increment of displacement and strain*. In linear elasticity the words *increment* and *infinitesimal* are often left out, leading to confusion in non-linear applications. Use of power and rate of strain removes these problems.

The continuum form of the virtual power equation that we shall be using is

$$\int_V \boldsymbol{\sigma} : \boldsymbol{\epsilon} dV = \int_V \mathbf{p} \cdot \mathbf{v} dV + \int_{\partial V} d\mathbf{A} \cdot \boldsymbol{\sigma} \cdot \mathbf{v}, \tag{1}$$

in which the stress tensor

$$\boldsymbol{\sigma} = \sigma_x \mathbf{ii} + \tau_{xy} \mathbf{ij} + \tau_{xz} \mathbf{ik} + \tau_{yx} \mathbf{ji} + \sigma_y \mathbf{jj} + \tau_{yz} \mathbf{jk} + \tau_{zx} \mathbf{ki} + \tau_{zy} \mathbf{kj} + \sigma_z \mathbf{kk}, \tag{2}$$

the virtual rate of strain tensor

$$\boldsymbol{\epsilon} = \epsilon_x \mathbf{ii} + \frac{1}{2}\gamma_{xy} \mathbf{ij} + \frac{1}{2}\gamma_{xz} \mathbf{ik} + \frac{1}{2}\gamma_{yx} \mathbf{ji} + \epsilon_y \mathbf{jj} + \frac{1}{2}\gamma_{yz} \mathbf{jk} + \frac{1}{2}\gamma_{zx} \mathbf{ki} + \frac{1}{2}\gamma_{zy} \mathbf{kj} + \epsilon_z \mathbf{kk},$$

the applied body loads per unit volume and the virtual velocity are $\mathbf{p} = p_x \mathbf{i} + p_y \mathbf{j} + p_z \mathbf{k}$ and $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$ respectively. $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors in the x, y and z directions and $d\mathbf{A} = dA_x \mathbf{i} + dA_y \mathbf{j} + dA_z \mathbf{k}$ is the normal to an element of area of magnitude $|d\mathbf{A}|$ of the boundary ∂V surrounding the volume V.

The double scalar product (or double dot product or double inner product),

$$\begin{aligned} \boldsymbol{\sigma} : \boldsymbol{\epsilon} = \boldsymbol{\epsilon} : \boldsymbol{\sigma} &= \sigma_x \epsilon_x + \frac{1}{2}\tau_{xy}\gamma_{xy} + \frac{1}{2}\tau_{xz}\gamma_{xz} + \frac{1}{2}\tau_{yx}\gamma_{yx} + \sigma_y \epsilon_y + ... \\ &= \sigma_x \epsilon_x + \tau_{xy}\gamma_{xy} + \sigma_y \epsilon_y + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx} + \sigma_z \epsilon_z \end{aligned}$$

since $\tau_{xy} = \tau_{yx}$, $\gamma_{xy} = \gamma_{yx}$ and so on. The factors of $\frac{1}{2}$ in the definition of $\boldsymbol{\epsilon}$ are there because of the unfortunate error in defining rate of engineering shear strain (and shear strain itself), which results in, amongst other problems, having to use half the rate of engineering shear strain in the Mohr's circle for strain rate.

The gradient of the virtual velocity, $\nabla\mathbf{v}$, is defined by $\delta\mathbf{v} = \delta\mathbf{r} \cdot \nabla\mathbf{v}$ where $\delta\mathbf{r} = \delta x\mathbf{i} + \delta y\mathbf{j} + \delta z\mathbf{k}$. $\delta\mathbf{v}$ is the change in $\mathbf{v}$ in moving from (x, y, z) to $(x + \delta x, y + \delta y, z + \delta z)$. Thus

$$\nabla\mathbf{v} = \frac{\partial v_x}{\partial x}\mathbf{ii} + \frac{\partial v_y}{\partial x}\mathbf{ij} + \frac{\partial v_z}{\partial x}\mathbf{ik} + \frac{\partial v_x}{\partial y}\mathbf{ji} + \frac{\partial v_y}{\partial y}\mathbf{jj} + \frac{\partial v_z}{\partial y}\mathbf{jk} + \frac{\partial v_x}{\partial z}\mathbf{ki} + \frac{\partial v_y}{\partial z}\mathbf{kj} + \frac{\partial v_z}{\partial z}\mathbf{kk}.$$

The virtual rate of strain tensor is the symmetric part of $\nabla\mathbf{v}$, that is $\boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right)$ in which $(\nabla\mathbf{v})^{\mathrm{T}}$ is the transpose of $\nabla\mathbf{v}$.

Finally we shall need the virtual angular velocity vector field, which is obtained from the antisymmetric part of $\nabla\mathbf{v}$,

$$\boldsymbol{\Omega} = \frac{1}{2}\boldsymbol{\Xi} : \nabla\mathbf{v} = \frac{1}{2}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{i} + \frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{j} + \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k}$$

in which the permutation pseudotensor, or the Levi-Civita pseudo tensor,

$$\boldsymbol{\Xi} = \mathbf{ijk} + \mathbf{jki} + \mathbf{kij} - \mathbf{kji} - \mathbf{ikj} - \mathbf{kji}. \tag{3}$$

The permutation pseudotensor is used in performing the vector product $\mathbf{a} \times \mathbf{b} = (\mathbf{ab}) : \boldsymbol{\Xi} = \mathbf{b} \cdot \boldsymbol{\Xi} \cdot \mathbf{a}$ between any two vectors $\mathbf{a}$ and $\mathbf{b}$. The permutation pseudotensor is often given the symbol $\boldsymbol{\epsilon}$, but we have used that for rate of strain and so we have used $\boldsymbol{\Xi}$, the upper case Greek letter, Xi.

$\boldsymbol{\Xi} \cdot \boldsymbol{\Omega} = \frac{1}{2}\boldsymbol{\Xi} \cdot \boldsymbol{\Xi} : \nabla\mathbf{v} = \frac{1}{2}\left(\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathrm{T}}\right)$ so that

$$\nabla\mathbf{v} = \boldsymbol{\epsilon} + \boldsymbol{\Xi} \cdot \boldsymbol{\Omega}. \tag{4}$$

The equilibrium equations, $\nabla \cdot \boldsymbol{\sigma} + \mathbf{p} = 0$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$ or $\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + p_x = 0$ etc. and $\tau_{xy} = \tau_{yx}$ etc. can be derived directly from the virtual power equation using the divergence theorem from pure mathematics. Alternatively we can derive the principal of virtual power from the equilibrium equations.

$\boldsymbol{\sigma}$ and $\mathbf{p}$ are in equilibrium and $\boldsymbol{\epsilon}$ and $\mathbf{v}$ are geometrically compatible. The equilibrium set and the compatibility set may occur simultaneously, but one or other of them may not actually occur, hence the word *virtual*.

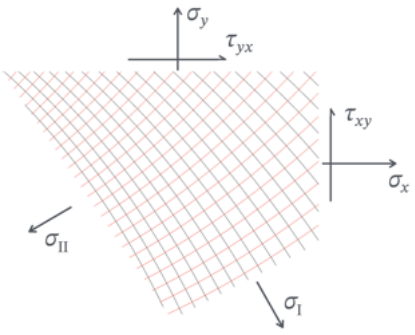

Figure 2: Structural members and Cartesian components of stress and principal stresses in 2 dimensions. One of the principal stresses is positive and the other is negative.

4. Structure mass per unit volume

As we have stated, we shall use a continuum approach from the outset. We assume a very fine network of members surrounded by voids or air, and that there is some known function

$$\mathscr{M} = \mathscr{M}(\boldsymbol{\sigma})$$

giving the total structure mass per unit total volume, including voids, in terms of the stress $\boldsymbol{\sigma}$ which is defined by its components, σ_x, τ_{xy}, τ_{xz}...τ_{zy} and σ_z.

$\boldsymbol{\sigma}$ is *the force per unit total cross-sectional area, including the area of the voids* as shown in figure 2. Thus because the structure may consist largely of voids, the stress in the individual structural members may be very much higher than $\boldsymbol{\sigma}$.

$\mathscr{M}$ is a scalar function of the stress $\boldsymbol{\sigma}$. Thus $\mathscr{M}$ is one quantity at each point in space derived from the 6 quantities representing $\boldsymbol{\sigma}$. Clearly if we double all the components of $\boldsymbol{\sigma}$, we expect $\mathscr{M}$ to double, since we are ignoring buckling. Also there must be some limit to $\boldsymbol{\sigma}$ since at some point the stress will be so high that there are no voids left and the stress cannot be increased. This is not so much a problem for 2 dimensional structures since we can introduce 'flanges' occupying the 3rd dimension.

If we know $\mathscr{M}(\boldsymbol{\sigma})$, all we have to do is to adjust $\boldsymbol{\sigma}$ in order to minimise the total mass given by the integral

$$M = \int_V \mathscr{M}(\boldsymbol{\sigma})\, dV \tag{5}$$

over the total volume V of structure, *subject to satisfying the equations of equilibrium as specified by the principle of virtual power*. The loads on the structure and the support locations are assumed to be known, except the own weight of the structure itself, which Michell did not include.

5. Minimising the structure mass using the calculus of variations

Let us conduct a thought experiment in which we continually vary $\boldsymbol{\sigma}$ so that the rate of change of the total mass is given by differentiating (5) with respect to time, $\dfrac{dM}{dt} = \displaystyle\int_V \frac{\partial \mathscr{M}}{\partial t} dV$. However $\mathscr{M}$ is a function of $\boldsymbol{\sigma}$ only so that

$$\frac{\partial \mathscr{M}}{\partial t} = \frac{d\mathscr{M}}{d\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} : \frac{d\mathscr{M}}{d\boldsymbol{\sigma}} = \frac{\partial \mathscr{M}}{\partial \sigma_x}\dot{\sigma}_x + \frac{\partial \mathscr{M}}{\partial \tau_{xy}}\dot{\tau}_{xy} + \frac{\partial \mathscr{M}}{\partial \tau_{xz}}\dot{\tau}_{xz} + \frac{\partial \mathscr{M}}{\partial \tau_{yx}}\dot{\tau}_{yx} + \frac{\partial \mathscr{M}}{\partial \sigma_y}\dot{\sigma}_y + ...$$

where $\dot{\boldsymbol{\sigma}} = \dfrac{\partial \boldsymbol{\sigma}}{\partial t}$ and $\dot{\sigma}_x = \dfrac{\partial \sigma_x}{\partial t}$, $\dot{\tau}_{xy} = \dfrac{\partial \tau_{xy}}{\partial t}$ etc.

Now, if we have managed to find a state in which the total mass M is minimum, then $\dfrac{dM}{dt} = 0$ so that

$$\int_V \dot{\boldsymbol{\sigma}} : \frac{d\mathscr{M}}{d\boldsymbol{\sigma}} dV = 0. \tag{6}$$

However $\boldsymbol{\sigma}$ must always be such that we continue to satisfy the equations of equilibrium as expressed by the virtual power equation (1). So now we need to differentiate the virtual power equation (1) with respect to time, assuming that only $\boldsymbol{\sigma}$ and $\mathbf{p}$ vary, and that $\mathbf{v}$ and $\boldsymbol{\epsilon}$ are independent of time. Thus

$$\int_V \dot{\boldsymbol{\sigma}} : \boldsymbol{\epsilon} dV = \int_V \dot{\mathbf{p}} \cdot \mathbf{v} dV + \int_{\partial V} d\mathbf{A} \cdot \dot{\boldsymbol{\sigma}} \cdot \mathbf{v}.$$

Now if any loads are kept constant, $\mathbf{p}$ only changes due to the change in weight of the structure itself, $\dot{\mathbf{p}} = -a_{\mathrm{g}} \dfrac{\partial \mathscr{M}}{\partial t}\mathbf{k} = -a_{\mathrm{g}} \dfrac{d\mathscr{M}}{d\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}}\mathbf{k}$ in which a_{g} is the acceleration due to gravity. Thus we have

$$\int_V \dot{\boldsymbol{\sigma}} : \left(\boldsymbol{\epsilon} + a_{\mathrm{g}} \frac{d\mathscr{M}}{d\boldsymbol{\sigma}} \mathbf{k} \cdot \mathbf{v} \right) dV = \int_{\partial V} d\mathbf{A} \cdot \dot{\boldsymbol{\sigma}} \cdot \mathbf{v}. \tag{7}$$

We now arrive at the conclusion that we have to satisfy both (6) and (7). Not only that but we have to satisfy (7) for any virtual velocity $\mathbf{v}$ that we care to choose, together with the corresponding virtual rate of strain $\boldsymbol{\epsilon}$.

At this stage we need to impose the condition $\int_{\partial V} d\mathbf{A}\cdot\dot{\boldsymbol{\sigma}}\cdot\mathbf{v}=0$ which can be satisfied by setting either $\dot{\boldsymbol{\sigma}}=0$ or $\mathbf{v}=0$ at all parts of the boundary ∂V. Setting $\dot{\boldsymbol{\sigma}}=0$ means setting applied loads equal to a constant on the boundary, however support reactions will vary with time, and so we must set the virtual $\mathbf{v}=0$ at all supports, at least in the direction of any support reaction. This applies even if the real structure is on flexible supports, which does not affect the stress since Michell structures are statically determinate.

Thus now we have to satisfy (6) for any stress such the 3 components of $\dot{\boldsymbol{\sigma}}$ satisfy the 2 equations of equilibrium as expressed by $\int_V \dot{\boldsymbol{\sigma}} : \left(\boldsymbol{\epsilon}+a_{\mathrm{g}}\dfrac{d\mathscr{M}}{d\boldsymbol{\sigma}}\mathbf{k}\cdot\mathbf{v}\right)dV=0$ for any distribution of the 2 components of $\mathbf{v}$ which satisfy the boundary conditions. This leads us to the conclusion that $c^2\dfrac{d\mathscr{M}}{d\boldsymbol{\sigma}}=\boldsymbol{\epsilon}+a_{\mathrm{g}}\dfrac{d\mathscr{M}}{d\boldsymbol{\sigma}}\mathbf{k}\cdot\mathbf{v}$ or

$$\boldsymbol{\epsilon}=\left(c^2-a_{\mathrm{g}}\mathbf{k}\cdot\mathbf{v}\right)\frac{d\mathscr{M}}{d\boldsymbol{\sigma}} \tag{8}$$

in which c is a constant. c can have any value we care to choose, and both $\mathbf{v}$ and $\boldsymbol{\epsilon}$ are proportional to c^2, which does not matter since both $\mathbf{v}$ and $\boldsymbol{\epsilon}$ are virtual.

We can write $\boldsymbol{\sigma}$ in terms of its Cartesian components as in (2) or we can write $\boldsymbol{\sigma}=\sigma_{\mathrm{I}}\mathbf{XX}+\sigma_{\mathrm{II}}\mathbf{YY}+\sigma_{\mathrm{III}}\mathbf{ZZ}$ where σ_{I}, σ_{II} and σ_{III} are the principal stresses in the directions of the orthogonal unit vectors $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ respectively. If we assume that $\mathscr{M}$ only depends upon the principal stresses, and not the principal stress directions, we can write $\dfrac{d\mathscr{M}}{d\boldsymbol{\sigma}}=\dfrac{\partial\mathscr{M}}{\partial\sigma_{\mathrm{I}}}\dfrac{d\sigma_{\mathrm{I}}}{d\boldsymbol{\sigma}}+\dfrac{\partial\mathscr{M}}{\partial\sigma_{\mathrm{II}}}\dfrac{d\sigma_{\mathrm{II}}}{d\boldsymbol{\sigma}}+\dfrac{\partial\mathscr{M}}{\partial\sigma_{\mathrm{III}}}\dfrac{d\sigma_{\mathrm{III}}}{d\boldsymbol{\sigma}}$ in which $\dfrac{d\sigma_{\mathrm{I}}}{d\boldsymbol{\sigma}}=\mathbf{XX}$, $\dfrac{d\sigma_{\mathrm{II}}}{d\boldsymbol{\sigma}}=\mathbf{YY}$ and $\dfrac{d\sigma_{\mathrm{III}}}{d\boldsymbol{\sigma}}=\mathbf{ZZ}$. Therefore

$$\epsilon_{\mathrm{I}}=\left(c^2-a_{\mathrm{g}}\mathbf{k}\cdot\mathbf{v}\right)\frac{\partial\mathscr{M}}{\partial\sigma_{\mathrm{I}}},\quad \epsilon_{\mathrm{II}}=\left(c^2-a_{\mathrm{g}}\mathbf{k}\cdot\mathbf{v}\right)\frac{\partial\mathscr{M}}{\partial\sigma_{\mathrm{II}}}\quad\text{and}\quad \epsilon_{\mathrm{III}}=\left(c^2-a_{\mathrm{g}}\mathbf{k}\cdot\mathbf{v}\right)\frac{\partial\mathscr{M}}{\partial\sigma_{\mathrm{III}}}, \tag{9}$$

and the principal directions associated with ϵ_{I}, ϵ_{II} and ϵ_{III} are the same as those associated with σ_{I}, σ_{II} and σ_{III} respectively. *Thus the principal strain rate directions are orthogonal to the surfaces* $\mathscr{M}=\text{constant}$ *in principal stress space, very much like the normality condition in plasticity* [7].

The logic in this section is perhaps a bit difficult to understand, but the same applies to Michell's paper, although he is more succinct. However, we shall see that we arrive at the same result.

For those unhappy with using virtual power, (8) can be derived using the calculus of variations in which we minimise M subject to satisfying the equilibrium equations. Then the components of $\mathbf{v}$ are simply the Lagrange multipliers.

Readers familiar with the concept of complementary strain energy [8] will immediately recognise the similarity of (8) with the equation $\boldsymbol{\epsilon}=\dfrac{dU^*}{d\boldsymbol{\sigma}}$ in which $\boldsymbol{\epsilon}$ is the strain tensor, not the rate of virtual strain tensor as used everywhere else in this paper, and U^* is the complementary strain energy.

6. Michell's function $\mathscr{M}(\boldsymbol{\sigma})$

Michell effectively assumed that $\mathscr{M}(\boldsymbol{\sigma})$ consists of 8 intersection planes in principal stress space:

$$\begin{aligned} \mathscr{M} &= \frac{\sigma_{\mathrm{I}}}{P} + \frac{\sigma_{\mathrm{II}}}{P} + \frac{\sigma_{\mathrm{III}}}{P}, \qquad & \mathscr{M} &= -\frac{\sigma_{\mathrm{I}}}{Q} - \frac{\sigma_{\mathrm{II}}}{Q} - \frac{\sigma_{\mathrm{III}}}{Q}, \\ \mathscr{M} &= \frac{\sigma_{\mathrm{I}}}{P} + \frac{\sigma_{\mathrm{II}}}{P} - \frac{\sigma_{\mathrm{III}}}{Q} \quad & &\text{plus 2 more planes permuting } \sigma_{\mathrm{I}}, \sigma_{\mathrm{II}} \text{ and } \sigma_{\mathrm{III}} \\ \text{and } \mathscr{M} &= -\frac{\sigma_{\mathrm{I}}}{Q} - \frac{\sigma_{\mathrm{II}}}{Q} + \frac{\sigma_{\mathrm{III}}}{P} \quad & &\text{plus 2 more planes permuting } \sigma_{\mathrm{I}}, \sigma_{\mathrm{II}} \text{ and } \sigma_{\mathrm{III}} \end{aligned} \tag{10}$$

in which P and Q are known positive constants, which for us represent the material strength divided by density in tension (P) and compression (Q), although Michell used P and Q simply for strength.

These 8 planes form an octahedron in principal stress space and we have an infinite number of nested octagons, each one representing a contour surface for $\mathscr{M}$ as shown in figure 3. However, for a given stress $\boldsymbol{\sigma}$, it is arbitrary which principal stress we call σ_{I}, which we call σ_{II} and which we call σ_{III}, which means that we have only 4 cases: all tensile, 2 tensile and 1 compressive, 1 tensile and 2 compressive and finally all compressive.

The principal strain rates can be found using using (9). Michell did not describe his assumption in this way, but we believe that his assumptions are equivalent to this.

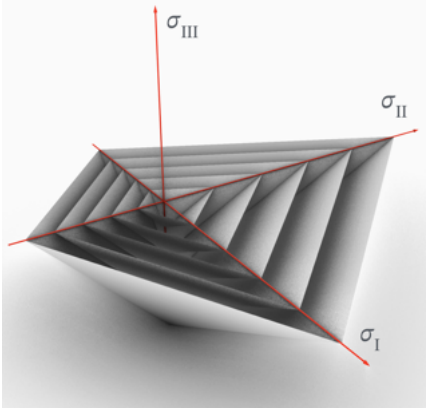

Figure 3: Michell's octahedral contour surfaces of constant $\mathscr{M}$ in principal stress space sliced by the plane $\sigma_{\mathrm{III}} = 0$

6.1. Michell structures in two dimensions

In 2 dimensions (10) becomes

$$\mathscr{M} = \frac{\sigma_{\mathrm{I}}}{P} + \frac{\sigma_{\mathrm{II}}}{P}, \quad \mathscr{M} = -\frac{\sigma_{\mathrm{I}}}{Q} - \frac{\sigma_{\mathrm{II}}}{Q}, \quad \mathscr{M} = \frac{\sigma_{\mathrm{I}}}{P} - \frac{\sigma_{\mathrm{II}}}{Q} \text{ and } \mathscr{M} = -\frac{\sigma_{\mathrm{I}}}{Q} + \frac{\sigma_{\mathrm{II}}}{P}.$$

Thus we end up with quadrilateral contour lines of $\mathscr{M}$ in $\sigma_{\mathrm{I}}, \sigma_{\mathrm{II}}$ space, which are obtained by intersecting the octahedral contour surfaces with the plane $\sigma_{\mathrm{III}} = 0$ as shown in figure 3. The cases when $\sigma_{\mathrm{I}} + \sigma_{\mathrm{II}}$ is equal to $\mathscr{M}P$ or $-\mathscr{M}Q$ are not particularly interesting because then (9) tell us that we have an isotropic strain rate, and therefore no information about the principal strain rate directions or the principal stress directions. The two cases with a mix of tension and compression are exactly equivalent, since we can interchange σ_{I} and σ_{II} and a point on one of the lines also appears on the other.

However we shall now replace P by $1/S_{\mathrm{I}}$ and Q by $-1/S_{\mathrm{II}}$ so that S_{II} is negative and $\mathscr{M} = \sigma_{\mathrm{I}}/P - \sigma_{\mathrm{II}}/Q$ becomes $\mathscr{M} = S_{\mathrm{I}}\sigma_{\mathrm{I}} + S_{\mathrm{II}}\sigma_{\mathrm{II}}$, and $\dfrac{\partial \mathscr{M}}{\partial \sigma_{\mathrm{I}}} = S_{\mathrm{I}}$ and $\dfrac{\partial \mathscr{M}}{\partial \sigma_{\mathrm{II}}} = S_{\mathrm{II}}$. The reason for doing this is to make σ_{II} and S_{II} have the same sign and so we have

$$\epsilon_{\mathrm{I}} = S_{\mathrm{I}}\left(c^2 - a_{\mathrm{g}}\mathbf{j}\cdot\mathbf{v}\right) \quad \text{and} \quad \epsilon_{\mathrm{II}} = S_{\mathrm{II}}\left(c^2 - a_{\mathrm{g}}\mathbf{j}\cdot\mathbf{v}\right) \tag{11}$$

in which we now take the unit vector $\mathbf{j}$ to be in the vertical direction. This is exactly the same conclusion that Michell comes to, except he did not include the own weight of the structure, so he effectively took $a_{\mathrm{g}} = 0$. We have two equations from which we can find the 2 components of $\mathbf{v}$ and hence the principal strain rates and their corresponding principal directions. Hence we can find the principal stress directions, and then resolve forces to find the principal stresses.

Let us again introduce the orthogonal unit vectors, now in 2 dimensions,

$$\mathbf{X} = \cos\lambda\mathbf{i} + \sin\lambda\mathbf{j} \quad \text{and} \quad \mathbf{Y} = -\sin\lambda\mathbf{i} + \cos\lambda\mathbf{j} \tag{12}$$

in the principal stress and strain rate directions so that

$$\boldsymbol{\epsilon} = \epsilon_{\mathrm{I}}\mathbf{XX} + \epsilon_{\mathrm{II}}\mathbf{YY} = \left(S_{\mathrm{I}}\mathbf{XX} + S_{\mathrm{II}}\mathbf{YY}\right)\left(c^2 - a_{\mathrm{g}}\mathbf{j}\cdot\mathbf{v}\right). \tag{13}$$

In two dimensions the permutation pseudotensor, $\boldsymbol{\Xi}$, in (3) becomes simply $\boldsymbol{\Xi} = \mathbf{ij} - \mathbf{ji} = \mathbf{XY} - \mathbf{YX}$ and the angular velocity vector has only the one component $\boldsymbol{\Omega} = \Omega\mathbf{k}$ about an axis perpendicular to the 2 dimensional plane. Thus (4) becomes

$$\nabla\mathbf{v} = \boldsymbol{\epsilon} + \Omega\boldsymbol{\Xi} = \boldsymbol{\epsilon} + \Omega\left(\mathbf{XY} - \mathbf{YX}\right). \tag{14}$$

We can form the gradient of the gradient of $\mathbf{v}$,

$$\nabla\nabla\mathbf{v} = \left(S_{\mathrm{I}} - S_{\mathrm{II}}\right)\nabla\lambda\left(\mathbf{XY} + \mathbf{YX}\right)\left(c^2 - a_{\mathrm{g}}\mathbf{j}\cdot\mathbf{v}\right) - a_{\mathrm{g}}\nabla\mathbf{v}\cdot\mathbf{j}\left(S_{\mathrm{I}}\mathbf{XX} + S_{\mathrm{II}}\mathbf{YY}\right) + \nabla\Omega\boldsymbol{\Xi}$$

since S_{I} and S_{II} are constants and $\nabla\mathbf{X} = \nabla\lambda\mathbf{Y}$ and $\nabla\mathbf{Y} = -\nabla\lambda\mathbf{X}$. The gradient of the gradient of any vector is a 3rd order tensor that is symmetric in its first two parts, and so $\boldsymbol{\Xi} : \nabla\nabla\mathbf{v} = 0$ and

$$\left(S_{\mathrm{I}} - S_{\mathrm{II}}\right)\nabla\lambda\cdot\left(\mathbf{XX} - \mathbf{YY}\right)\left(c^2 - a_{\mathrm{g}}\mathbf{j}\cdot\mathbf{v}\right) + \left(a_{\mathrm{g}}\nabla\mathbf{v}\cdot\mathbf{j}\right)\cdot\left(S_{\mathrm{I}}\mathbf{YX} - S_{\mathrm{II}}\mathbf{XY}\right) - \nabla\Omega = 0$$

since $\boldsymbol{\Xi}\cdot\boldsymbol{\Xi} = -\mathbf{ii} - \mathbf{jj} = -\mathbf{I}$ where $\mathbf{I}$ is the unit tensor in 2 dimensions.

We therefore end up with 2 equations, one in the $\mathbf{X}$ direction and one in the $\mathbf{Y}$ direction:

$$\begin{aligned}\left(S_{\mathrm{I}} - S_{\mathrm{II}}\right)\nabla\lambda\cdot\mathbf{X}\left(c^2 - a_{\mathrm{g}}\mathbf{j}\cdot\mathbf{v}\right) + a_{\mathrm{g}}S_{\mathrm{I}}\mathbf{Y}\cdot\nabla\mathbf{v}\cdot\mathbf{j} - \nabla\Omega\cdot\mathbf{X} = 0\\ \left(S_{\mathrm{I}} - S_{\mathrm{II}}\right)\nabla\lambda\cdot\mathbf{Y}\left(c^2 - a_{\mathrm{g}}\mathbf{j}\cdot\mathbf{v}\right) + a_{\mathrm{g}}S_{\mathrm{II}}\mathbf{X}\cdot\nabla\mathbf{v}\cdot\mathbf{j} + \nabla\Omega\cdot\mathbf{Y} = 0\end{aligned} \tag{15}$$

It is not obvious how to proceed from this point, without using numerical integration which we shall consider in §7.

6.2. Weightless Michell structures in two dimensions

However, if the structure is weightless, as Michell assumed, then $a_{\mathrm{g}} = 0$ in (15), so that $c^2\left(S_{\mathrm{I}} - S_{\mathrm{II}}\right)\nabla\lambda\cdot\mathbf{X} - \nabla\Omega\cdot\mathbf{X} = 0$ and $c^2\left(S_{\mathrm{I}} - S_{\mathrm{II}}\right)\nabla\lambda\cdot\mathbf{Y} + \nabla\Omega\cdot\mathbf{X} = 0$. Now if we introduce curvilinear coordinates [9], θ^1 and θ^2, such that $\mathbf{X}$ is in the direction of increasing θ^1, and $\mathbf{Y}$ is in the direction of increasing θ^2, then $c^2\left(S_{\mathrm{I}} - S_{\mathrm{II}}\right)\dfrac{\partial\lambda}{\partial\theta^1} - \dfrac{\partial\Omega}{\partial\theta^1} = 0$ and $c^2\left(S_{\mathrm{I}} - S_{\mathrm{II}}\right)\dfrac{\partial\lambda}{\partial\theta^2} + \dfrac{\partial\Omega}{\partial\theta^2} = 0$. Hence $\dfrac{\partial^2\Omega}{\partial\theta^1\partial\theta^2} = 0$ and $\dfrac{\partial^2\lambda}{\partial\theta^1\partial\theta^2} = 0$. Michell obtains this last result using a diagram and some very brief and intense discussion.

Thus we come to the conclusion that

$$\lambda = f\left(\theta^1\right) + F\left(\theta^2\right) \quad \text{and} \quad \Omega = c^2\left(S_{\mathrm{I}} - S_{\mathrm{II}}\right)\left(f\left(\theta^1\right) - F\left(\theta^2\right)\right) + \omega, \tag{16}$$

where ω is a constant. Remembering that S_{I} is positive and S_{II} is negative. (14) and (13) now give us

$$\nabla\mathbf{v} = c^2S_{\mathrm{I}}\mathbf{XX} + c^2S_{\mathrm{II}}\mathbf{YY} + \Omega\left(\mathbf{XY} - \mathbf{YX}\right). \tag{17}$$

6.3. Analytical solution for a two material weightless cantilever with circular support

Consider the coordinate system $\mathbf{r} = \dfrac{R}{\sqrt{\cosh 2\phi}}\left(\sinh\mu\mathbf{X} + \cosh\mu\mathbf{Y}\right)$ where $\lambda = -\theta^1 + \theta^2 + \phi$ and $\mu = \theta^1 + \theta^2 + \psi$ and in which R, ϕ and ψ are constants. $\mathbf{X}$ and $\mathbf{Y}$ are the unit vectors (12) and the circular support is at $\theta^1 + \theta^2 = 0$, when $|\mathbf{r}| = R$.

The covariant base vectors [9] are $\mathbf{a}_1 = 2R\cosh\mu\mathbf{X}/\sqrt{\cosh 2\phi}$ and $\mathbf{a}_2 = 2R\sinh\mu\mathbf{Y}/\sqrt{\cosh 2\phi}$, and the covariant components of the metric tensor, $a_{11} = 4R^2\cosh^2\mu/\cosh 2\phi$, $a_{12} = 0$ and $a_{22} =$

$4R^2 \sinh^2 \mu / \cosh 2\phi$. We also have $\lambda_{,12} = 0$ and so the coordinate system is a suitable candidate for the principal virtual strain and stress directions of a Michell structure.

From (16) we have $\Omega = -c^2 \left(S_\mathrm{I} - S_\mathrm{II}\right) \left(\theta^1 + \theta^2\right) + \omega$ where ω is a constant, and so (17) becomes

$$\nabla \mathbf{v} = c^2 S_\mathrm{I} \mathbf{X}\mathbf{X} + c^2 S_\mathrm{II} \mathbf{Y}\mathbf{Y} + \left(-c^2 \left(S_\mathrm{I} - S_\mathrm{II}\right) \left(\theta^1 + \theta^2\right) + \omega\right) \left(\mathbf{X}\mathbf{Y} - \mathbf{Y}\mathbf{X}\right) = \mathbf{a}^1 \mathbf{v}_{,1} + \mathbf{a}^2 \mathbf{v}_{,2}$$

in which the contravariant base vectors, $\mathbf{a}^1 = \sqrt{\cosh 2\phi}\mathbf{X} / \left(2R \cosh \mu\right)$ and $\mathbf{a}^2 = \sqrt{\cosh 2\phi}\mathbf{Y} / \left(2R \sinh \mu\right)$. Thus

$$\mathbf{v}_{,1} = \left(c^2 S_\mathrm{I} \mathbf{X} + \left(-c^2 \left(S_\mathrm{I} - S_\mathrm{II}\right) \left(\theta^1 + \theta^2\right) + \omega\right) \mathbf{Y}\right) \frac{2R \cosh \mu}{\sqrt{\cosh 2\phi}}$$

$$\mathbf{v}_{,2} = \left(c^2 S_\mathrm{II} \mathbf{Y} - \left(-c^2 \left(S_\mathrm{I} - S_\mathrm{II}\right) \left(\theta^1 + \theta^2\right) + \omega\right) \mathbf{X}\right) \frac{2R \sinh \mu}{\sqrt{\cosh 2\phi}}$$

which are satisfied by

$$\mathbf{v} = \frac{c^2 R}{\sqrt{\cosh 2\phi}} \begin{pmatrix} \left(-\left(S_\mathrm{I} - S_\mathrm{II}\right) \left(\theta^1 + \theta^2\right) + \dfrac{\omega}{c^2}\right) \left(-\cosh \mu \mathbf{X} + \sinh \mu \mathbf{Y}\right) \\ + S_\mathrm{II} \sinh \mu \mathbf{X} + S_\mathrm{I} \cosh \mu \mathbf{Y} \end{pmatrix}.$$

When $\theta^1 + \theta^2 = 0$ so that $\mu = \phi$ and for $\mathbf{v}$ to be zero at the support, $\omega \cosh \phi + c^2 S_\mathrm{II} \sinh \phi = 0$ and $\omega \sinh \phi + c^2 S_\mathrm{I} \cosh \phi = 0$ so that $\tanh \phi = \sqrt{-S_\mathrm{I}/S_\mathrm{II}}$. ψ is chosen so that the tip of the cantilever is at the same level as the centre of the circular support.

7. Numerical integration

The analytic solution in §6.3. only applies to a weightless cantilever, $a_\mathrm{g} = 0$. When this is not the case it appears that we have to find a numerical solution. The equations which we have to satisfy are $a_{12} = 0$, the equations (11) in which S_I, S_II, a_g and c^2 are known constants, and the equations (14). The unknowns are the positions $\mathbf{r}\left(\theta^1, \theta^2\right)$ and the virtual velocities $\mathbf{v}\left(\theta^1, \theta^2\right)$.

We start our integration with a series of points on the support, whose position we know and whose virtual velocity is zero. We then progress row by row away from the support along the principal strain rate directions, and each row has one fewer point than the previous row until we arrive at the tip. Now let us suppose that we know $\mathbf{r}_{i-1,j}$ and $\mathbf{r}_{i,j-1}$ in figure 4 as well as $\mathbf{v}_{i-1,j}$ and $\mathbf{v}_{i,j-1}$ and want to calculate $\mathbf{r}_{i,j}$ and $\mathbf{v}_{i,j}$. We have

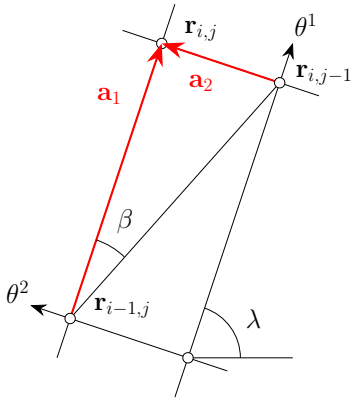

Figure 4: Points in the numerical integration

$$\mathbf{a}_1 + \mathbf{a}_2 = \cos 2\beta \left(\mathbf{a}_1 - \mathbf{a}_2\right) + \sin 2\beta \left(\mathbf{k} \times \left(\mathbf{a}_1 - \mathbf{a}_2\right)\right) \tag{18}$$

and $a_{11} + a_{22} = \left(\mathbf{a}_1 - \mathbf{a}_2\right) \cdot \left(\mathbf{a}_1 - \mathbf{a}_2\right)$ together with $\sqrt{a_{11}} = \sqrt{\left(a_{11} + a_{22}\right)} \cos \beta$ and $\sqrt{a_{22}} = \sqrt{\left(a_{11} + a_{22}\right)} \sin \beta$, plus $\cos 2\beta = \left(a_{11} - a_{22}\right) / \left(a_{11} + a_{22}\right)$ and $\sin 2\beta = 2\sqrt{a_{11} a_{22}} / \left(a_{11} + a_{22}\right)$. We also have

$$\mathbf{v}_{,1} = \sqrt{a_{11}} \left(\epsilon_\mathrm{I} \frac{\mathbf{a}_1}{\sqrt{a_{11}}} + \Omega \frac{\mathbf{a}_2}{\sqrt{a_{22}}}\right) \quad \text{and} \quad \mathbf{v}_{,2} = \sqrt{a_{22}} \left(-\Omega \frac{\mathbf{a}_1}{\sqrt{a_{11}}} + \epsilon_\mathrm{II} \frac{\mathbf{a}_2}{\sqrt{a_{22}}}\right). \tag{19}$$

Let us introduce $D_\parallel = \left(a_{11} + a_{22}\right) \left(\frac{1}{2}\left(\epsilon_\mathrm{I} + \epsilon_\mathrm{II}\right) + \frac{1}{2}\left(\epsilon_\mathrm{I} - \epsilon_\mathrm{II}\right) \cos 2\beta\right)$ and from $D_\parallel$ we can find the strain rate in the direction of $\left(\mathbf{a}_1 - \mathbf{a}_2\right)$ which is equal to

$$\epsilon_\parallel = \frac{D_\parallel}{\left(\mathbf{a}_1 - \mathbf{a}_2\right) \cdot \left(\mathbf{a}_1 - \mathbf{a}_2\right)} = \frac{1}{2}\left(\epsilon_\mathrm{I} + \epsilon_\mathrm{II}\right) + \frac{1}{2}\left(\epsilon_\mathrm{I} - \epsilon_\mathrm{II}\right) \cos 2\beta \tag{20}$$

as we would expect from the Mohr's circle for strain rate. If we assume that we know ϵ_{I} and ϵ_{II} we can find $\cos 2\beta$ and then $\sin 2\beta = \sqrt{1-\cos^2 2\beta}$, which we know must be positive.

Let us also introduce $D_\perp = (\mathbf{k} \times (\mathbf{a}_1 - \mathbf{a}_2)) \cdot (\mathbf{v}_{,1} - \mathbf{v}_{,2}) = (a_{11} + a_{22}) \left(\frac{1}{2} (\epsilon_{\mathrm{I}} - \epsilon_{\mathrm{II}}) \sin 2\beta + \Omega \right)$ from which we can find Ω. The strain in the direction perpendicular to $(\mathbf{a}_1 - \mathbf{a}_2)$ is

$$\epsilon_\perp = \frac{D_\perp}{(\mathbf{a}_1 - \mathbf{a}_2) \cdot (\mathbf{a}_1 - \mathbf{a}_2)} - \Omega = \frac{1}{2} (\epsilon_{\mathrm{I}} - \epsilon_{\mathrm{II}}) \sin 2\beta \tag{21}$$

again as we as we would expect from the Mohr's circle for strain rate.

All these equations are differential equations, which we now have to put in the finite difference form by writing $\mathbf{a}_1 = (\mathbf{r}_{i,j} - \mathbf{r}_{i-1,j}) / \delta\theta$, $\mathbf{a}_2 = (\mathbf{r}_{i,j} - \mathbf{r}_{i,j-1}) / \delta\theta$, $\mathbf{v}_{,1} = (\mathbf{v}_{i,j} - \mathbf{v}_{i,j}) / \delta\theta$ and $\mathbf{v}_{,2} = (\mathbf{v}_{i,j} - \mathbf{v}_{i,j-1}) / \delta\theta$ where the increments $\delta\theta^1 = \delta\theta^2 = \delta\theta$. Finally we have $\mathbf{r}_{i.j} = (\mathbf{r}_{i-1,j} + \mathbf{r}_{i,j-1}) / 2 + (\mathbf{a}_1 + \mathbf{a}_2)\, \delta\theta/2$ and $\mathbf{v}_{i.j} = (\mathbf{v}_{i-1,j} + \mathbf{v}_{i,j-1}) / 2 + (\mathbf{v}_{,1} + \mathbf{v}_{,2})\, \delta\theta/2$. However we know $\mathbf{a}_1 - \mathbf{a}_2$ and $\mathbf{v}_{,1} - \mathbf{v}_{,2}$ but not $\mathbf{a}_1 + \mathbf{a}_2$ and $\mathbf{v}_{,1} + \mathbf{v}_{,2}$. But we can find $\mathbf{a}_1 + \mathbf{a}_2$ from (18) using the value of $\cos 2\beta$ from (20) and hence $\mathbf{r}_{i,j}$. Thus, if we know $\mathbf{a}_1 - \mathbf{a}_2$ and $\mathbf{a}_1 + \mathbf{a}_2$ then we know $\mathbf{a}_1$ and $\mathbf{a}_2$. We can find Ω from (21) and hence we can find $\mathbf{v}_{,1} + \mathbf{v}_{,2}$ from (19).

Acknowledgements

Emil Adiels gratefully acknowledges the funding received from Lars Erik Lundberg Scholarship Foundation (grant ID no. 1001904) and Åke och Greta Lissheds Stiftelse (grant ID no. 2025-029), both of which have provided essential support for his contributions to this research.

The views expressed in this paper are those of the author(s) and do not reflect the official policy or position of the U.S. Naval Academy, Department of the Navy, the Department of Defense, or the U.S. Government.

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