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Higher order Hölder approximation by solutions of second order elliptic equations

Grigori Rozenblum¹  · Nikolai A. Shirokov^{2,3}

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Abstract

For a given second order elliptic operation $\mathcal{L}$ in a domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, and a compact set $\mathbf{K} \subset \Omega$, order $N-2$ -Ahlfors-David regular, we define the space $\mathcal{H}_{\mathcal{L}}^{r+\omega}(\mathbf{K})$ of continuous functions $f(x)$, $x \in \mathbf{K}$, admitting, for any $\delta > 0$, a local approximation in the δ -neighborhood of any point $x \in \mathbf{K}$, with $\delta^r \omega(\delta)$ -error estimate, by solutions of the equation $\mathcal{L}u = 0$. For such functions, we prove the existence of a global approximation v_δ on $\mathbf{K}$ with the same order of error estimate, by a solution of the same equation in a δ -neighborhood of $\mathbf{K}$. A number of properties of these functions v_δ and their derivatives are established.

Keyword Approximation · Higher Hölder metric · Solutions of Elliptic Equations

Mathematics Subject Classification 41A30 · 35J15 · 41A63

1 Introduction

1.1 The approximation problem

Approximating 'bad' functions by 'good' ones is one of classical topics in Analysis. The qualitative direction has started with the Weierstrass Theorem on the possibility of polynomial approximation of continuous functions. An important further development here concerns approximating continuous functions by solutions of differential equations. A fundamental result for rather general differential equations (possessing a kind of unique continuation property) was obtained by Browder, [5, 6].

✉ Grigori Rozenblum
grigori@chalmers.se

Nikolai A. Shirokov
nikolai.shirokov@gmail.com

¹ Chalmers University of Technology, Gothenburg, Sweden

² St. Petersburg State University, Univ. Naberezhnaya, 7/9, 199034 St. Petersburg, Russia

³ National Research University Higher School of Economics, Kantemirovskaya Str. 3a, 194100 St. Petersburg, Russia

The studies in the quantitative direction began later. Generally speaking, quantitative approximation results can be expected to have the following common structure:

- (1) A class $\mathcal{F}$ of functions to be approximated is described;
- (2) A class $\mathcal{G}$ of functions used for approximation is proposed;
- (3) The result: a quantitative relation between the rate of approximation and the properties of the approximating function.

For example, the order of the error in the approximation of a continuous function by polynomials of a given degree is determined by the smoothness of this function, understood in a proper sense.

When considering approximation by solutions of elliptic equations, it is reasonable to consider as $\mathcal{F}$, a class of functions defined on a nowhere dense set $\mathbf{K}$. In fact, if, on the opposite, $\mathbf{K}$ possesses interior points, it is only solutions of the equation that can be approximated by solutions. So, we are interested in approximating a given continuous function f defined on a nowhere dense compact set $\mathbf{K} \subset \mathbb{R}^N$ by solutions of a second order elliptic equation. When the approximating functions are harmonic, and the set $\mathbf{K}$ is nice, say, a Lipschitz surface, there are many results in this direction, see, e.g. [1, 3, 4, 10–12, 15, 16] and many more.

When the conditions on $\mathbf{K}$ are less restrictive, one can cite [2, 24, 27]. Here, one needs to decide, which terms should be used to describe properties of the function f in order to determine the rate of approximation. If we only know that the given function f is continuous on a compact set $\mathbf{K}$, then the quality of this continuity, and consequently, the quality of approximation, can be described by the modulus of continuity of f . In this direction, in the paper [27], the authors considered the problem on approximating a continuous function f on $\mathbf{K} \subset \mathbb{R}^N$, possessing the continuity modulus $\omega(\delta)$, by solutions of a second order elliptic equation $\mathcal{L}u = 0$ ($\mathcal{L}$ -harmonic functions). It was established there that if the set $\mathbf{K}$ is Ahlfors-David $\mathbf{N}$ -2-regular (which means, almost exactly speaking, that it has one and the same Hausdorff dimension $\mathbf{N}$ -2 in any neighborhood of any of its points), then the function f can be, for any $\delta > 0$, approximated in $C(\mathbf{K})$ by a function v_δ , so that $|f(x) - v_\delta(x)| \leq c\omega(\delta)$ for all $x \in \mathbf{K}$, the function v_δ is $\mathcal{L}$ -harmonic in a δ -neighbourhood $\mathbf{K}_\delta$ of the set $\mathbf{K}$, moreover, the quality of this function v_δ is controlled by δ , namely, $|\nabla v_\delta(x)| \leq C \frac{\omega(\delta)}{\delta}$ in $\mathbf{K}_\delta$. This matches the general principle: the smaller δ , the better is the approximation, i.e., the smaller, is the approximation error, but the worse is the approximating function v_δ : it is $\mathcal{L}$ -harmonic on a smaller set, and its gradient may grow with δ decreasing. Moreover, a converse result was established: if a continuous function f on $\mathbf{K}$ can be approximated in the above sense, with some function $\omega(\delta)$, by solutions of a second order elliptic differential equation, then f possesses the continuity modulus majorated by $\omega(\delta)$.

It is natural to expect that if we wish to have a better approximation (the one better than with $O(\omega(\delta))$ error), with the same quality of the approximating functions, we should suppose some better properties of the given function f . If the set $\mathbf{K}$ were a smooth surface (of codimension 2), such 'better' properties would naturally involve a higher classical smoothness of f . However, if we only know that the set $\mathbf{K}$ is Ahlfors-David $\mathbf{N}$ -2-regular, some other terms should be used.

In the literature, there exist methods of defining spaces of 'nice' functions on arbitrary compacts. One of them is based upon describing classes of functions via their local approximations by polynomials or other sufficiently regular functions, see [7, 8, 28], and many sources afterwards.

So, the expected approximation results should sound like 'if a function admits local approximation of a certain kind, it admits the corresponding quality of global approximation' by $\mathcal{L}$ -harmonic functions.

This is, in fact, the contents of the present paper. Namely, in our main result, if f is a continuous function on $\mathbf{K}$, which can, for any $\delta > 0$, be locally, in a δ -neighborhood of any point $x \in \mathbf{K}$, approximated by a function $\Phi_{x,\delta}(y)$ which is a solution of the second-order elliptic equation $\mathcal{L}(y, \partial_y)u(y) = 0$ in a 2δ -neighbourhood of x , with error $O(\delta^r \omega(\delta))$, $r \geq 1$ (with some natural compatibility conditions concerning the functions $\Phi_{x,\delta}$ for different values of δ and different close-lying points x), then f can be approximated on the whole $\mathbf{K}$, with error of the same order, by a solution v_δ of the same equation in the δ -neighborhood of $\mathbf{K}$. Note that the above compatibility conditions, mentioned in parentheses, are unavoidable: they are proved to be necessary for the existence of the global approximation.

When comparing these results with our previous paper [27], where we established this kind of properties for $r = 0$, one can notice that an additional restricting condition appears: the locally approximating functions $\Phi_{x,\delta}(y)$ are required here to be solutions of the elliptic equation, while in [27] no such restriction has been imposed. This restriction is, unfortunately, unavoidable. An example we present in the paper demonstrates a function which admits a nice polynomial local approximation but does not admit a global approximation by harmonic functions. This effect is caused by a visible wildness of the set $\mathbf{K}$ in our example: it is easy to show that for a nicer $\mathbf{K}$, e.g., for a Lipschitz surface of codimension 2, such counter-examples are impossible and a local approximation by smooth functions is sufficient (and, of course, necessary) for existence of a global approximation by $\mathcal{L}$ -harmonic functions.

The elliptic differential operation $\mathcal{L}(x, \partial_x)$ is supposed to have coefficients of certain finite smoothness, $C^m(\Omega)$. The main approximation result, Theorem 1.2, is proved for $m = 3$. Under additional smoothness conditions, the main result can be somewhat strengthened: not only the approximating functions v_δ converge on $\mathbf{K}$ to the initial function f , but their derivatives $\partial^\alpha v_\delta$ (up to some order, depending on the smoothness of coefficients of $\mathcal{L}$) converge on $\mathbf{K}$ to some functions $f_{(\alpha)}$ which can be understood as generalized derivatives of the given function f . The greater m , to the higher order these surrogate derivatives of f can be defined, see Theorem 1.3.

1.2 The main results. 1.

We present here the exact formulation of our main approximation result. It is the following. Let $\omega(t)$, $t > 0$, be a continuity modulus, a growing concave function, $\omega(0) = 0$, satisfying the condition

$$\int_0^\tau \frac{\omega(t)}{t} dt + \tau \int_\tau^\infty \frac{\omega(t)}{t^2} dt \leq c\omega(\tau), \quad 0 < \tau < \infty,$$

(when $\omega(t) = t^\gamma$, $0 < \gamma < 1$, this is the standard Hölder continuity modulus.)

Let, further, $\mathbf{K}$ be a compact set in $\mathbb{R}^N$, N -2-Ahlfors-David regular (see, e.g., [9]). Let $\Omega \supset \mathbf{K}$ be a bounded open connected set, where a formally self-adjoint second-order elliptic operator

$$\mathcal{L}u(x) = - \sum_{j,j'} \partial_j(a_{jj'}(x)) \partial_{j'} u(x) \equiv -\nabla \cdot (\mathbf{a}(x) \nabla u(x)),$$

with C^m -coefficients $a_{jj'}$, $m \geq 3$, is defined. The operator is supposed to be uniformly elliptic, which, as usual, means that the matrix of the leading coefficients $a_{jj'}(x)$ is positive definite uniformly for $x \in \Omega$.

With the continuity modulus ω fixed, for an integer $\mathbf{r} \geq 0$, the local higher $\mathcal{L}$ -Hölder class $\mathcal{H}_{\mathcal{L}}^{\mathbf{r}+\omega}(\mathbf{K})$ is defined in the following way.

Definition 1.1 The continuous function $f(x)$, $x \in \mathbf{K}$, is said to belong to $\mathcal{H}_{\mathcal{L}}^{\mathbf{r}+\omega}(\mathbf{K})$, if there exist constants $\mathbf{c}_1 = \mathbf{c}_1(f)$, $\mathbf{c}_2 = \mathbf{c}_2(f)$, such that for any $x \in \mathbf{K}$ and any δ , $0 < \delta \leq 2 \operatorname{diam}(\mathbf{K})$, there exists a function $\Phi_{x,\delta}(y)$ defined in the ball $B_\delta(x)$ such that

$$\mathcal{L}_y \Phi_{x,\delta}(y) = 0, \quad y \in B_\delta(x),$$

$$|f(y) - \Phi_{x,\delta}(y)| \leq \mathbf{c}_1 \delta^{\mathbf{r}} \omega(\delta), \quad y \in B_\delta(x) \cap \mathbf{K}.$$

For close-lying points x_1, x_2 , the approximating functions should be consistent in the following sense: for some constants γ_1, γ_2 , $\frac{1}{8} \leq \gamma_1 \leq 1 \leq \gamma_2 \leq 8$, if $\gamma_1 \delta_1 \leq \delta_2 \leq \gamma_2 \delta_1$, given any points $x_1, x_2 \in \mathbf{K}$, such that the balls $B_{\delta_1}(x_1), B_{\delta_2}(x_2)$ are not disjoint, the inequality

$$|\Phi_{x_1,\delta_1}(y) - \Phi_{x_2,\delta_2}(y)| \leq \mathbf{c}_2 \delta_1^{\mathbf{r}} \omega(\delta_1). \quad (1.1)$$

must hold for all $y \in B_{\delta_1}(x_1) \cap B_{\delta_2}(x_2)$,

We recall the definition of Ahlfors-David regularity. The compact set $\mathbf{K}$ is called 'AD regular' of dimension $\varkappa$ if for some constants c', c'' , $0 < c' < c'' < \infty$, for any point $x \in \mathbf{K}$ and any $r \leq \operatorname{diam}(\mathbf{K})$, $c' r^\varkappa \leq \mu_\varkappa(B_r(x) \cap \mathbf{K}) \leq c'' r^\varkappa$, where $\mu_\varkappa$ is the Hausdorff measure of dimension $\varkappa$.

By $\mathbf{K}_\delta$ we denote the δ -neighborhood of $\mathbf{K}$. Our first main result is the following.

Theorem 1.2 *Let $\mathbf{K}$ be N -2-AD regular. Suppose that the coefficients of the operator $\mathcal{L}$ belong to C^3 . Then function f defined on $\mathbf{K}$ belongs to the class $\mathcal{H}_{\mathcal{L}}^{\mathbf{r}+\omega}(\mathbf{K})$ if and only if for any $\delta < \frac{1}{4} \operatorname{diam}(\mathbf{K})$, there exists an approximating function $v_\delta(x)$, $x \in \mathbf{K}_\delta$, such that, with some constant $c = c(f) > 0$,*

$$\mathcal{L}_y v_\delta(y) = 0, \quad y \in \mathbf{K}_\delta; \quad (1.2)$$

$$|v_\delta(x) - f(x)| \leq c(f) \delta^{\mathbf{r}} \omega(\delta), \quad x \in \mathbf{K};$$

$$|v_\delta(y) - v_{\delta/2}(y)| \leq c(f) \delta^{\mathbf{r}} \omega(\delta), \quad y \in \mathbf{K}_{\frac{\delta}{2}}.$$

1.3 The ideas of the proof

The proof of the main theorem is fairly technical, therefore we consider it reasonable to explain here its structure.

Given a function $f \in \mathcal{H}_{\mathcal{L}}^{\mathbf{r}+\omega}(\mathbf{K})$ on the compact set $\mathbf{K}$, we construct its special extension f_0 to a fixed neighborhood Ω of $\mathbf{K}$ (the particular form of this neighbourhood is not essential, and we suppose further on that it is the unit ball containing the set $\mathbf{K}$ which is contained in the concentric ball with radius $\frac{1}{3}$). For this function f_0 , using the Green function $G^\circ(x, y)$ of the operator $\mathcal{L}$ in Ω , the integral representation is established:

$$f_0(x) = \int_{\Omega} \mathcal{L} f_0(y) G^\circ(x, y) dy, \quad x \in \Omega. \quad (1.3)$$

Although this representation looks quite usual if f_0 is sufficiently smooth, this is not the case for *our* function f_0 for which the derivatives may behave badly when approaching $\mathbf{K}$. Therefore, to justify (1.3), we need a detailed control of the behavior of $\mathcal{L} f_0(x)$ and of derivatives of $f_0(x)$ near $\mathbf{K}$. Obtaining this control requires complicated estimates of the Green function $G_{x,r}(x, y)$ for $\mathcal{L}$ in balls $B_r(x)$ centered at x , together with their derivatives, up to the third order, in the variables x, y , as well as in the additional variable ς on which the operator $\mathcal{L}$ depends as a parameter. Under the condition of a sufficient smoothness of coefficients of the operation $\mathcal{L}$, we derive some of these estimates directly, using Schauder-type approach, and borrow the other ones from the results by Ju. Krasovskii [18], and M. Grüter–K.-O. Widman [14]. Finally, having established the representation (1.3), we define the approximation function $v_\delta(x)$, looked for, by the integral

$$v_\delta(x) = \int_{\Omega \setminus \mathbf{K}_\delta} \mathcal{L} f_0(y) G^\circ(x, y) dy,$$

with addition of a collection of several compensatory $\mathcal{L}$ -harmonic terms of a simpler nature, see (4.17). The fact that v_δ is $\mathcal{L}$ -harmonic in $\mathbf{K}_\delta$ is obvious, it follows from the definition of the Green function $G^\circ(x, y)$, while the estimates producing the quality of the approximation follow from our estimates for the function $f_0(x)$ and its derivatives.

1.4 The main results. 2.

The second theorem describes the properties of the approximating functions v_δ : their derivatives, up to a prescribed order $k \leq \mathbf{r} + 1$ can be controlled. Moreover, we can define in a consistent way the generalized derivatives $f_{(\alpha)}$ of the initial function f on $\mathbf{K}$, so that the derivatives of v_δ approximate these derivatives of f . This property requires a certain additional smoothness of coefficients of the operator $\mathcal{L}$.

Theorem 1.3 *Suppose that $\mathbf{r} \geq 1$ and the coefficients $\mathbf{a}(x) = (a_{ij'}(x))_{j,j' \leq \mathbf{N}}$ belong to $C^{k_0+3}(\Omega)$ for a certain $k_0 \geq \mathbf{r}$. Let the function f , defined on the compact set $\mathbf{K}$, belong to the class $H_{\mathcal{L}}^{\mathbf{r}+\omega}(\mathbf{K})$ and v_δ be its approximation, as in (1.2). Then derivatives*

of v_δ satisfy

$$\|\nabla^{\mathbf{r}+1} v_\delta\|_{\mathbf{K}_{\delta/2}} \leq c \frac{\omega(\delta)}{\delta}. \quad (1.4)$$

moreover, surrogate derivatives $f_{(\alpha)}(x)$ can be defined, so that

$$|f_{(\alpha)}(x) - \partial^\alpha v_\delta(x)| \leq C \delta^{\mathbf{r}-|\alpha|} \omega(\delta), \quad x \in \mathbf{K}, \quad 1 \leq |\alpha| \leq k_0. \quad (1.5)$$

We consider it natural that the higher order derivatives are involved, the more smoothness is required for the coefficients: it stands to reason that solutions of an equation with coefficients of some finite smoothness are differentiable only a certain corresponding number of times. We are not completely sure that our smoothness conditions are non-improvable; checking their optimality or lack of it seems to be rather hard (and, honestly, not that exciting) task.

On the other hand, our results are, of course, valid for operators with constant coefficients. If we restrict ourselves to such operators, the results still seem to be new. The reasoning here can be somewhat simplified compared with the general case, especially, since the Green function can be effectively studied using the Fourier transform.

1.5 Structure of the paper

We start in Sect. 2 by presenting general material concerning certain geometry considerations, and formulate estimates of important integrals used in further analysis and of derivatives of the Green function, including the results of [14, 18]. In Sect. 3, we introduce the averaging kernel $K(x, y)$ and prove estimates of its derivatives. This is the most technical part of the paper. Next, in Sect. 4, we construct the extension function f_0 , derive its important properties and prove its integral representation, which results in presenting the required approximation of the given function $f(x) \in H_{\mathcal{L}}^{\mathbf{r}+\omega}(\mathbf{K})$, thus proving Theorem 1.2. In Sect. 5, we discuss generalized derivatives of the function f , and prove Theorem 1.3. Then, in Sect. 6, we present the example showing that for a wild set $\mathbf{K}$, the condition on local approximation cannot, generally, be relaxed.

Proofs of our estimates for derivatives of the Green function and of important integral inequalities are placed in the Appendix.

1.6 Conventions

In the course of the paper, we denote by the same symbol c or C various constants whose particular value is of no importance, as long as this does not cause confusion; sometimes, subscripts or superscripts are used in order to distinguish between such constants in the same formula. More important constants may be highlighted by a different font. By $f'_x = \partial_x f = \nabla_x f$ we denote the x -gradient of a function f ; for a vector function F , $\nabla_x F$ stands for the Jacobi matrix of F . The symbol $|\cdot|$ denotes the Euclidean norm of the vector involved, $\mathbf{E}$ denotes the unit matrix.

2 Some preparatory facts

2.1 Geometry considerations

Let b_N be the constant in the covering property of Ahlfors–David–regular sets of dimension $N-2$, see [20], Lemma 2.1, and [27], Corollary 2.2 there, namely,

Property 2.1 For any $\delta < \text{diam}(\mathbf{K})$, there exists a finite cover $\Upsilon(\delta)$ of $\mathbf{K}$ by open balls $B_\delta(x_\alpha)$ of radius δ ,

$$\mathbf{K} \subset \bigcup_{x_\alpha \in \mathbf{K}} B_\delta(x_\alpha) \equiv \mathbf{K}_{(\delta)},$$

such that for any $r \in [\delta, \text{diam}(\mathbf{K})]$ and any point $\xi_0 \in \mathbf{K}$, the quantity of points x_α in the ball $B_r(\xi_0)$ is not greater than $b_N \left(\frac{r}{\delta}\right)^{N-2}$.

Our aim at this moment is to associate, with each ball $B_\delta(x_\alpha)$ of the above cover, some new ball with radius 2δ , whose center is on the order δ distance from x_α and which is separated from $\mathbf{K}_{(\delta)}$, again, by an order δ distance.

We denote by σ_N the area of the unit sphere in $\mathbb{R}^N$. Next, we introduce some coefficient $A = A_N$, whose value will be determined later on in a special way. Thus, for any x , for the sphere $S_{\delta A}(x)$, its $N-1$ -dimensional surface measure equals $\sigma_N A^{N-1} \delta^{N-1}$.

We take some point x_{α_0} among centers of the balls in the cover $\Upsilon(\delta)$ in Property 2.1 and denote temporarily by $\xi_0 \equiv \xi_{\alpha_0}$ (one of) the point(s) in $\mathbf{K}$ closest to x_{α_0} (it may happen that these points coincide), so $|\xi_0 - x_{\alpha_0}| \leq \delta$. For the ball $\mathbf{B} := B_{\delta(A+4)}(\xi_0)$, there exist no more than $b_N(A+4)^{N-2}$ balls $B_\delta(x_\alpha)$ in the above cover $\Upsilon(\delta)$, whose centers x_α lie in $\mathbf{B}$. We place on the sphere $\mathbf{S} := S_{\delta A}(\xi_0)$, in an arbitrary way, a collection of $m \leq b_N(A+4)^{N-2}$ points ζ_k , and evaluate the area on the part of the sphere $\mathbf{S}$ covered by the union of balls with radii 6δ , centered at these points ζ_k ; we denote this area by $\sigma(\xi_0, \delta)$. This area is no greater than the sum of areas of spherical caps upon $\mathbf{S}$, covered by single balls, therefore,

$$\sigma(\xi_0, \delta) \leq 6^{N-1} \sigma_N b_N (A+4)^{N-2} \delta^{N-1}. \quad (2.1)$$

We denote by $\tilde{A}_N$ the largest positive root of the equation

$$6^{N-1} b_N (\tilde{A}_N + 4)^{N-2} = \frac{1}{2} \tilde{A}_N^{N-1}$$

and set

$$A = \max(\tilde{A}_N, 13). \quad (2.2)$$

It follows from (2.1), (2.2) that whatever points ζ_k , no more than $b_N(A+4)^{N-2}$ of them, we place on the sphere $\mathbf{S}$, at least a half of the area of this sphere is *not* covered by the concentric balls $B_{6\delta}(\zeta_k)$. We denote this, non-covered, part of the sphere by $\mathcal{V} \equiv \mathcal{V}_{x_0, \delta}(\{\zeta_k\})$, so,

$$\text{meas}_{N-1}(\mathcal{V}_{x_0, \delta}(\{\zeta_k\})) \geq \frac{1}{2} \text{meas}_{N-1} \mathbf{S}.$$

Next we consider the following geometrical construction. We choose the above points ζ_k in a special way. Denote by $\mathcal{P}$ the *closed* spherical annulus $\mathcal{P} = B_{A_N+4\delta}(\xi_0) \setminus B_{A_N-4\delta}(\xi_0)$ and consider *only* those points x_α which lie in $\mathcal{P}$. Suppose that a certain point x_α lies on the sphere $\mathbf{S}$. Then we set $\zeta_\alpha = x_\alpha$. If x_α *does not* lie on this sphere, we consider the straight ray, which we denote $[\xi_0, x_\alpha]$, starting at ξ_0 and passing through x_α , and accept as ζ_α the point where this ray hits the sphere $\mathbf{S}$. As explained above, the set $\mathcal{Y}$ is non-empty. We take an arbitrary point V in this set.

Our construction started with choosing a point $\xi_0 \equiv x_{\alpha_0}$ in Property 2.1. To reflect it, we mark the point V , just defined, as V_{α_0} , thus keeping α_0 fixed.

We are going to estimate from below the distance between x_α and V_{α_0} . For a point x_α in $\mathcal{P}$ and $\zeta_\alpha \in \mathbf{S}$, we have $|V_{\alpha_0} - \zeta_\alpha| \geq 6\delta$. The point x_α lies on the ray $[\xi_0, \zeta_\alpha]$, therefore, $|x_\alpha - V_{\alpha_0}|$ is not less than the length of the perpendicular dropped from V_{α_0} onto $[\xi_0, \zeta_\alpha]$. Since $|\xi_0 - \zeta_\alpha| = |\xi_0 - V_{\alpha_0}| \geq 13\delta$, the length of this perpendicular is not less than 4δ . Therefore, for x_α , we have

$$B_{2\delta}(x_\alpha) \cap B_{2\delta}(\xi_0) = \emptyset. \quad (2.3)$$

If, on the opposite, the point x_α does not lie in $\mathcal{P}$ then (2.3) obviously holds.

In this way, with each starting point x_{α_0} , we associate the point V_{α_0} such that $B_{2\delta}(x_\alpha) \cap B_{2\delta}(V_{\alpha_0}) = \emptyset$ for all $\alpha \neq \alpha_0$. Additionally,

$$|V_{\alpha_0} - x_{\alpha_0}| \leq |V_{\alpha_0} - \xi_0| + |\xi_0 - x_{\alpha_0}| \leq 14\delta,$$

and thus the point V_{α_0} is separated from $\mathbf{K}$

Finally, we introduce the notation

$$\mathbf{K}'_\delta = \bigcup_{x_\alpha} B_{2\delta}(x_\alpha);$$

it follows that $\mathbf{K}_\delta \subset \mathbf{K}'_\delta$

2.2 Estimates of some integrals

In the study of approximations, we will need estimates for some integrals involving the distance to the set $\mathbf{K}$. Here we give the formulations; proofs are placed in the Appendix.

Let $\mathbf{K} \subset \mathbb{R}^N$ be a compact set, N -2-regular, and let the point x lie outside $\mathbf{K}$, $d(x) \equiv \text{dist}(x, \mathbf{K}) \leq \delta$.

Lemma 2.2

$$\int_{B_{2\delta}(x)} \frac{d(y)^{r-2} \omega(d(y))}{|x - y|^{N-2}} dy \leq C \delta^r \omega(\delta). \quad (2.4)$$

Lemma 2.2, has a useful corollary.

Corollary 2.3 *Under the same conditions imposed on x ,*

$$\int_{B_{2\delta}(x)} d(y)^{\mathbf{r}-2} \omega(d(y)) dy \leq c \delta^{\mathbf{r}+\mathbf{N}-2} \omega(\delta). \quad (2.5)$$

Another important property concerns the integral

$$I_k(x) = \int_{B_{c_0\delta}(x)} \frac{d(y)^{k-2} \omega(d(y))}{|x-y|^{\mathbf{N}-2+k}} dy, \quad x \in \mathbf{K}, c_0 \geq 2. \quad (2.6)$$

Lemma 2.4 *For $k > 0$, the inequality holds*

$$I_k(x) \leq c \omega(\delta(x)). \quad (2.7)$$

There is a useful corollary:

Corollary 2.5 *For $1 \leq k < \mathbf{r}$,*

$$J_k = \int_{B_{c_0\delta}(x)} \frac{d(y)^{\mathbf{r}-2} \omega(d(y))}{|y-x|^{\mathbf{N}-2+k}} dy \leq c \delta^{\mathbf{r}-k} \omega(\delta). \quad (2.8)$$

In the next lemma we estimate the integral over the complement $CB_{c_0\delta}(x)$ of the ball.

Lemma 2.6 *For $k > 0$, we have*

$$\int_{CB_{c_0\delta}(x)} \frac{d(y)^{k-2} \omega(d(y))}{|y-x|^{\mathbf{N}-1+k}} dy \leq c \frac{\omega(\delta)}{\delta}. \quad (2.9)$$

Lemma 2.7 *Let $\mathbf{r} \geq 1$, $x_0 \in \mathbf{K}$, $x \in \mathbf{CK} \equiv \mathbb{R}^{\mathbf{N}} \setminus \mathbf{K}$ and $|x-x_0| \leq \delta/2$. Then*

$$\int_{B_\delta(x)} d(y)^{\mathbf{r}-2} \omega(d(y)) |x-y|^{2-\mathbf{N}} dy \leq C \delta^{\mathbf{r}} \omega(\delta)$$

$$\int_{B_\delta(x_0)} d(y)^{\mathbf{r}-2} \omega(d(y)) |x_0-y|^{2-\mathbf{N}} dy \leq C \delta^{\mathbf{r}} \omega(\delta).$$

Finally, we need the following estimate.

Lemma 2.8 *Under conditions of Lemma 2.7,*

$$\mathcal{J}(x) = \int_{|x-y|>\delta} d(y)^{\mathbf{r}-2} \omega(d(y)) |x-y|^{1-\mathbf{N}} dy \leq C \delta^{-1} \omega(\delta). \quad (2.10)$$

2.3 Estimates of derivatives of the Green function

Here we present some results on the Green function and its derivatives.

Let $\mathcal{L} = \mathcal{L}(x, \partial_x)$ be a uniformly elliptic order 2 operator in a ball B_R of radius $R \leq 1$, with Dirichlet boundary conditions and $G(x, y) \equiv G^R(x, y)$ be the Green function for $\mathcal{L}$ in B_R .

Estimates of the first type concern the unit ball, $R = 1$, $B = B_1$, $G = G^\circ$, and we need them for all points $(x, y) \in (\bar{B} \times \bar{B})$, $x \neq y$. Our interest lies in estimating the singularity of the derivatives of $G(x, y)$ as x is close to y . Such estimates for $G(x, y)$ and derivatives $G_x(x, y)$, $G_y(x, y)$, $G_{xy}(x, y)$ were established in [14], under rather weak restrictions imposed on the coefficients, namely, if $\mathbf{a}(x) \in L_\infty(B)$ and is Dini continuous. Then

$$|\partial_x^\alpha \partial_y^\beta G(x, y)| \leq C|x - y|^{-N+2-|\alpha|-|\beta|}, \quad (2.11)$$

for $|\alpha|, |\beta| \leq 1$, with constant C depending on the norms of coefficients in the Dini class and on the ellipticity constant.

Estimates of derivatives of higher order require more regularity of coefficients. We need them only for derivatives $\partial_x^\alpha \partial_y^\beta G(x, y)$, $|\beta| \leq 1$. We cite here the result by Yu.Krasovskii, tailored for our particular case, [18], see Theorem 3.3. and its corollary.

Theorem 2.9 *Let the coefficients $\mathbf{a}(x)$ belong to $C^m(\bar{B})$, $m \geq 3$. Then*

$$|\partial_x^\alpha \partial_y^\beta G(x, y)| \leq C|x - y|^{-N+2-|\alpha|-|\beta|}, \quad |\alpha| < m - 1, \quad (2.12)$$

with constant C depending on C^m -norm of coefficients and the ellipticity constant.

We will also need such estimates of derivatives for the Green function $G^R(x, y)$ in a ball B_R with radius $R < 1$.

Corollary 2.10 *Under the conditions of Theorem 2.9, for a ball B_R , $R < 1$ estimate (2.12) holds for the Green function $G^R(x, y)$ in the ball B_R , with constant with constant C depending on C^m -norm of coefficients and the ellipticity constant, but not depending on R .*

Proof Having an operator $\mathcal{L}$ in the ball B_R , say, with center at the origin, we make a dilation $\mathbf{x} = R^{-1}x$ to the unit ball B_1 . Under this dilation, the C^0 -norm of a derivative of order α in x or y gains the factor $R^{|\alpha|} < 1$. Thus, the C^m norms of the coefficients do not grow. As follows from the chain rule, the Green function in x variables is R^{2-N} times the transformed Green function of the dilated operator in $\mathbf{x}$ variables,

$$G^R(x, y) = R^{2-N}G^\circ(\mathbf{x}, \mathbf{y}).$$

and the estimate for the Green function follows from the estimate in the unit ball. The same dilation takes care of derivatives of the Green function. $\square$

The second type of results concerns the Green function in a ball B_R with radius $R \leq 1$ for the points x, y well separated, namely, the point x lies in a small neighborhood

of the centerpoint $\mathbb{O}$ of the ball, while y lies in a neighborhood of some boundary point y° . The operator $\mathcal{L}$ depends on an additional parameter ς in a neighborhood of the point $\varsigma_0 = \mathbb{O} \in \mathbb{R}^N$, and we need estimates for the derivatives

$$|G_x|, |G_y|, |G_{xy}|, |G_{x\varsigma}|, |G_{xy\varsigma}|, |G_{xx\varsigma}|, |G_{yy\varsigma}|, |G_{x\varsigma\varsigma}|, |G_{xxy}| \leq C, \quad (2.13)$$

for such x, y , uniform in these variables and for $\varsigma = 0$.

Again, we start with the unit ball, $R = 1$, and the Green function, denoted $G^\circ(x, y; \varsigma)$.

Theorem 2.11 *Suppose that the operator $\mathcal{L}(\varsigma) = -\nabla(\mathbf{a}(x, \varsigma)\nabla)$ is uniformly elliptic and its coefficients satisfy $\partial_x^\alpha \partial_\varsigma^\gamma \mathbf{a}(x, \varsigma) \leq c_0$, for $|\alpha| + |\gamma| \leq 3$. Then for x near $\mathbb{O}$, y near $y^\circ \in \partial B_1$, $|x - y| > \frac{1}{2}$, the derivatives of the Green function $G^\circ(x, y, \varsigma)$, satisfy (2.13), with constant C depending on the norm of the coefficients in the above spaces and on the ellipticity constant.*

The proof is presented in Appendix B.

Again, a simple consequence of Theorem 2.11 is the following.

Corollary 2.12 *Under the conditions of Theorem 2.11, let $B_R \subset B_1$ be a ball with radius $R < 1$. Then for the Green function G^R for the operator $\mathcal{L}$ in B_R , the estimates hold.*

$$\begin{aligned} |G_\varsigma^R(x, y; \varsigma)|, |G_{\varsigma\varsigma}^R(x, y; \varsigma)| &\leq CR^{2-N}, \\ |G_x^R(x, y; \varsigma)|, |G_y^R(x, y; \varsigma)|, |G_{x\varsigma}^R(x, y; \varsigma)|, |G_{x\varsigma\varsigma}^R(x, y; \varsigma)| &\leq CR^{1-N}, \\ |G_{xy}^R(x, y; \varsigma)|, |G_{xy\varsigma}^R(x, y; \varsigma)|, |G_{xx\varsigma}^R(x, y; \varsigma)| &\leq CR^{-N}, \\ |G_{xyy}^R(x, y; \varsigma)| &\leq CR^{-1-N}, \end{aligned} \quad (2.14)$$

with constant C determined by $C^3(B_R \times B_\epsilon)$ -norms of coefficients of the operator $\mathcal{L}$ and its ellipticity constant.

Proof The proof repeats the above dilation reasoning. $\square$

3 The generating kernel and the extension

3.1 The extension operator

We are going to describe here the smooth extension to $\mathbb{R}^N$ of a given function $f \in \mathcal{H}_{\mathcal{L}}^{r+\omega}(\mathbf{K})$ with control of derivatives.

Let $\mathcal{Q} = \{Q\}$ be the Whitney decomposition into open cubes of the set $\mathbf{CK} = \mathbb{R}^N \setminus \mathbf{K}$. Recall that this means that $\bigcup_{Q \in \mathcal{Q}} \overline{Q} = \mathbf{CK}$ and different cubes in $\mathcal{Q}$ are disjoint. Moreover, if $\mathbf{a}_Q$ is the center of the cube $Q \in \mathcal{Q}$, $\delta(Q) = \text{dist}(\mathbf{a}_Q, \mathbf{K})$, then

$\frac{1}{8}\delta(Q) \leq \text{diam } Q \leq \frac{1}{4}\delta(Q)$. For $Q \in \mathcal{Q}$, we denote by $x_Q \in \mathbf{K}$ (one of) the closest to $\mathbf{a}_Q$ point(s) in $\mathbf{K}$, $d(\mathbf{a}_Q) = |x_Q - \mathbf{a}_Q|$, and for $y \in Q$, $\delta(Q) \leq 2 \text{diam } \mathbf{K}$, we define

$$\tilde{f}_0(y) = \Phi_{x_Q, 2\delta(Q)}(y) \quad (3.1)$$

where $\Phi_{x_Q, 2\delta(Q)}(y)$ are functions involved in the definition of the class $\mathcal{H}_{\mathcal{L}}^{r+\omega}(\mathbf{K})$, see Definition 1.1. If a point y does not belong to any of $Q \in \mathcal{Q}$ or it belongs to a far-lying cube, $\delta(Q) > 2 \text{diam } \mathbf{K}$, we set $\tilde{f}_0(y) = 0$. The function $\tilde{f}_0$, thus defined, is piecewise $\mathcal{L}$ -harmonic, however it may be discontinuous along the boundaries of cubes in $\mathcal{Q}$ in an uncontrollable manner. We use now the averaging kernel $K(x, y)$, to be constructed later in this section, and set

$$f_0(x) = \int_{\mathbb{R}^N} \tilde{f}_0(y) K(x, y) dy. \quad (3.2)$$

We will show that this function is a continuous extension of f to a neighborhood of $\mathbf{K}$, with controlled behavior of derivatives when approaching $\mathbf{K}$; the function $f_0(x)$ will later serve for constructing the required approximation.

3.2 Construction of the kernel K

Our reasoning will be constructive. The first step will be describing a proper averaging kernel. We denote by $d_0(x)$ the *regularized* distance from the point $x \in \mathbf{CK}$ to $\mathbf{K}$, namely $d_0(x) \in C^3(\mathbf{CK})$, $c \text{dist}(x, \mathbf{K}) \leq d_0(x) \leq c' \text{dist}(x, \mathbf{K})$, $c' < \frac{1}{4}$, $|\nabla^k d_0(x)| \leq c d_0(x)^{1-k}$, $k = 1, 2, 3$. Let $\mathbf{h}(t)$ be a function in $C^\infty(\mathbb{R}_+)$, $\mathbf{h}(t) \geq 0$, $\text{supp } \mathbf{h} \subset [\frac{1}{2}, 1]$, normalized by $\int_0^1 \mathbf{h}(t) dt = 1$. The scaled function $h_r(t) = r^{-1} \mathbf{h}(t/r)$ is normalized in such a way that $\int_0^\infty h_r(t) dt = 1$. Further on, for $x \in \mathbf{CK}$, $t < r = d_0(x)$, we denote by $S_t(x)$ the sphere $\{y : |y - x| = t\}$, and by $B_t(x)$ the corresponding open ball; they do not touch $\mathbf{K}$, moreover, they are on a controlled distance from $\mathbf{K}$.

Next, for our elliptic operator

$$\mathcal{L}(x, \partial_x) = - \sum_{j,j'} \partial_j a_{jj'}(x) \partial_{j'} \equiv -\nabla \cdot (\mathbf{a}(x) \nabla) \quad (3.3)$$

in the unit ball $\Omega \equiv B_1(0)$ containing $\mathbf{K}$, we construct the averaging kernel $K(x, y)$. It will act as a replacement, for the operator $\mathcal{L}$, of the mean value kernel, usual for the Laplacian: namely, for any point $x \in \mathbf{CK}$ and a function $\phi(y)$, $\mathcal{L}$ -harmonic in the ball $B_r(x)$, $\phi \in C(\overline{B_r(x)})$, $r = r(x) \leq d_0(x)$, the following representation holds:

$$\phi(x) = \int_{B_r(x)} K(x, y) \phi(y) dy.$$

The construction of $K(x, y)$ is performed in the following way. For a fixed point $z \in \mathbf{CK}$, we denote by $G_{z,t}(x, y)$ the Green function for the Dirichlet problem for $\mathcal{L}$ in the ball $B_t(z)$, so, in this notation, variables (z, t) in the subscript indicate the

domain, the ball, where the Green function is considered, while, further, (x, y) are the variables on which the value of the Green function depends. We are interested, especially, in $G_{x,t}(x, y)$, this means, the value of the Green function at the centerpoint of the sphere. In particular, we have $G_{x,t}(x, y) = 0$ on the sphere $|y - x| = t$. Since the operator $\mathcal{L}$ is Hermitian, we have $G_{x,t}(x, y) = G_{x,t}(y, x)$.

For $t \in [r/2, r]$, $y \in S_t(x)$, we denote by $\mathbf{v}_y(t)$ the external $\mathcal{L}$ -conormal to the sphere $S_t(x)$ at the point $y \in S_t(x)$; namely, for an operator $\mathcal{L}$ in (3.3), the components of the conormal are $(\mathbf{v}_y(t))_j = \sum_{j'} a_{j,j'}(y) \mathbf{n}_{j'}(y)$, where $\mathbf{n}_{j'}(y)$ are the components of the Euclidean normal to $S_t(x)$ at the point y (note that the conormal vector is not normalized).

By the classical representation formula for solutions of elliptic equations, the following equation is valid for a function ϕ , as long as $\phi \in C(\overline{B_t(x)})$, $\mathcal{L}\phi = 0$ in $B_t(x)$,

$$\phi(x) = \int_{S_t(x)} \partial_{\mathbf{v}_y} G_{x,t}(x, y) \phi(y) d\sigma_t(y), \quad (3.4)$$

where $d\sigma_t$ is the surface element on the sphere $S_t(x)$. We multiply (3.4) by $h_r(t)$, $r = d_0(x)$ and integrate in t . By our normalization, the left-hand side in (3.4) remains equal to $\phi(x)$ after integration, while the right-hand side takes the form of the integral over the spherical annulus $\mathcal{R}_r(x) = B_r(x) \setminus B_{\frac{r}{2}}(x)$:

$$\phi(x) = \int_{\mathcal{R}_r(x)} h_r(|x - y|) \partial_{\mathbf{v}_y} G_{x,|x-y|}(x, y) \phi(y) dy, \quad r = d_0(x). \quad (3.5)$$

It is the kernel in (3.5) that will be accepted as $K(x, y)$: we set

$$K(x, y) = h_{d_0(x)}(|x - y|) \partial_{\mathbf{v}_y} G_{x,|x-y|}(x, y),$$

for $y \in \mathcal{R}_r(x)$, and $K(x, y) = 0$ otherwise.

As follows from [14], the function $K(x, y)$ satisfies

$$|K(x, y)| \leq c d_0(x)^{-N}, \quad y \in \mathcal{R}_r(x).$$

In the following, we will also need estimates for the derivatives of the kernel $K(x, y)$ in the same domain $y \in \mathcal{R}_r(x)$, namely,

$$|\nabla_x K(x, y)| \leq c d_0(x)^{-N-1}, \quad (3.6)$$

and

$$|\nabla_x^2 K(x, y)| \leq c d_0(x)^{-N-2}. \quad (3.7)$$

These estimates are established further on.

3.3 Estimates of derivatives of $K(x, y)$. First order derivatives

Since the dependence of K on x is rather implicit, the calculations of its derivatives are rather cumbersome and involve multiple applications of the chain rule. We would like to explain at this point, somewhat informally, how this calculation goes through and which derivatives of the Green function may appear; this facilitates better understanding of rigorous reasoning afterwards.

The easiest term to handle is the factor $h_{d_0(x)}(|x - y|)$; its derivatives are calculated directly. More trouble is caused by the factor $\partial_{\mathbf{v}_y} G_{x, |x-y|}(x, y)$. First, note that when the position of the point x changes infinitesimally, the sphere $\partial B_{|x-y|}(x)$ has its center moving, the sphere still passing through y , therefore the conormal vector $\mathbf{v}_y$ changes its length and direction, and in evaluating this change, the derivatives of G in x and y appear. Next, the size of the ball $B_{|x-y|}(x)$ changes, and this change should also be taken into account when differentiating in x . Finally, we must notice that when the point x moves, the operator $\mathcal{L}$ changes. To see this, we take an infinitesimally moved point $\xi = \xi(\varsigma)$, the new position of x , $\xi(0) = x$, and consider a linear conformal transformation $T(\varsigma) = \mathbf{E} + O(\varsigma)$ in $\mathbb{R}^N$ mapping the 'old' ball $B^{(0)} = B_{|x-y|}(x)$ onto the 'new' ball $B^{(\varsigma)} = B_{|\xi-y|}(\xi)$. Under this transformation, the operator $\mathcal{L}$ in the new ball $B^{(\varsigma)}$ transforms into a new operator $\mathcal{L}(\varsigma)$ in the initial ball $B^{(0)}$ with coefficients $\mathbf{a}(x, \varsigma) = \mathbf{a}(T(\varsigma)x)$, depending on the parameter ς ; in other words, instead of considering the 'old operator $\mathcal{L}$ ' in a 'new ball', we consider here the new operator $\mathcal{L}(\varsigma)$ in the old ball. Therefore, when we differentiate the Green function in x , we need to keep in mind this change of the operator and thus, for a fixed ball, the derivatives $\partial_\varsigma G(x, y; \varsigma)$, $\partial_\varsigma \partial_y G(x, y; \varsigma)$ and $\partial_\varsigma \partial_x G(x, y; \varsigma)$ arise, where $G(x, y; \varsigma)$ is the Green function of the operator $\mathcal{L}(\varsigma)$ in $B^{(0)}$.

Now we present rigorous calculations rendering concrete these hints.

We start with evaluating derivatives of an important auxiliary vector-function. $\mathbf{m}(x, z) = (x - z)d_0(x)^{-1}$, for $x, z \in \mathbf{CK}$, under the condition $|x - z| \leq d_0(x)$.

First, of course, $\nabla_z \mathbf{m}(x, z) = \mathbf{E}d_0(x)^{-1}$, therefore, $|\nabla_z \mathbf{m}(x, z)| = N^{\frac{1}{2}}d_0(x)^{-1}$. The calculation of the x -derivative is a little bit more laborious:

$$\begin{aligned} \nabla_x \mathbf{m}(x, z) &= (d_0(x)^{-1})'_x \otimes (x - z) + d_0(x)^{-1} \mathbf{E} = \\ &= d_0(x)^{-2} \nabla_x d_0(x) \otimes (z - x) + d_0(x)^{-1} \mathbf{E}, \end{aligned}$$

therefore,

$$|\nabla_x \mathbf{m}(x, z)| \leq cd_0(x)^{-1} \leq c'd(x)^{-1}, \quad (3.8)$$

and, similarly,

$$|\nabla_{xx}^2 \mathbf{m}(x, z)| \leq cd(x)^{-2}. \quad (3.9)$$

Now we estimate first order derivatives of the function $K(x, y)$ with respect to the x -variable. We fix a point $x \in \mathbf{CK}$ and consider ξ in the small ball $B_{d_0(x)/4}(x)$, $y \in S_t(x)$ for $t = |x - y|$. The function $K(\xi, y)$ depends on ξ in a complicated way; this dependence involves the change of the regularized distance $d_0(\xi)$, which leads to the change of the radius of the ball, and also change of the direction of the normal and

conormal vectors at y as ξ moves. Thus we encounter many terms requiring detailed analysis.

By the definition of $K(\xi, y)$, we have

$$K'_\xi(\xi, y) = (h_{d_0(\xi)}(|\xi - y|))'_\xi \partial_{\mathbf{v}_y(t)} G_{\xi, |\xi - y|}(\xi, y) + h_{d_0(\xi)}(|\xi - y|) (\partial_{\mathbf{v}_y(t)} G_{\xi, |\xi - y|}(\xi, y))'_\xi. \quad (3.10)$$

As follows from (3.8) and the obvious estimate

$$\left| (h_{d_0(\xi)}(|\xi - y|))'_\xi \right| \leq c d_0(\xi)^{-2}, \quad (3.11)$$

for the first term in (3.10), we have

$$\left| (h_{d_0(\xi)}(|\xi - y|))'_\xi \partial_{\mathbf{v}_y(t)} G_{\xi, |\xi - y|}(\xi, y) \right| \leq c d_0(\xi)^{-N-1}. \quad (3.12)$$

Next we consider the second term in (3.10). As explained earlier, while the point x , the center of the sphere, moves to the position ξ , under such movement, the point y remains at its place, but the sphere $S_{|x-y|}(x)$ moves to the new position $S_{|\xi-y|}(\xi)$, and this causes the change of the direction of the tangent plane at y and thus of the conormal vector $\mathbf{v}_y$.

To evaluate this change of direction, when calculating derivatives with respect to ξ variable, it is convenient to parametrize the point ξ in a special way. We need some elementary geometry here. Denote by $\mathcal{S}$ the unit sphere with center at the origin $\mathbb{O}$, consider the point $\lambda_0 = \frac{x-y}{|x-y|} \in \mathcal{S}$, and let ω be a neighborhood of λ_0 in $\mathcal{S}$: $\omega = \{\lambda = \frac{\xi-y}{|\xi-y|} : |\xi - x| \leq \frac{1}{4} d_0(x)\}$.

We set $\tau = |\xi - y|$ and consider the change of the operators of the conormal derivative, $\tilde{\partial} = \partial_{\mathbf{v}_y(\tau)}$, as ξ moves. The conormal derivatives can be understood as directional derivatives along the vectors $\mathbf{v}_y(t)$, not necessarily the unit ones. Since the position of the sphere $S_t(\xi)$ is determined by the parameter $\varsigma = \frac{\xi-y}{d_0(\xi)} \in \omega$, we include this parameter into the notation, suppressing temporarily t : $\mathbf{v}(y, \varsigma) := \mathbf{v}_y(t)$. In the lucky case, when $\mathbf{v}(y, \varsigma) \parallel \mathbf{v}(y, \lambda)$, the reasoning is quite simple since there is no change of the direction of the conormal vector. We consider the generic case $\mathbf{v}(y, \varsigma) \nparallel \mathbf{v}(y, \lambda)$. Let us draw the two-dimensional plane Π through the point y and the vectors $\mathbf{v}(y, \varsigma)$, $\mathbf{v}(y, \lambda)$. This plane cuts the sphere $S_\tau(x)$ along the circle that we denote by ϖ . We consider the intersection $\Pi \cap B_t(\xi)$. It is a two-dimensional disk.

Denote by $\nu_{\xi, y}$ the unit interior normal vector (the Euclidean one, note the font difference!) to ϖ . By definition, the Green function $G_{\xi, |\xi - y|}(\xi, y)$ vanishes for $y \in S_\tau(\xi)$, in particular, on ϖ . Therefore,

$$\partial_{\mathbf{v}(y, \lambda)} G_{\xi, |\xi - y|}(\xi, y) = |\mathbf{v}(y, \lambda)|^{-1} \partial_{\nu_{\xi, y}} \widehat{G_{\xi, |\xi - y|}(\xi, y)} \cos(\nu_{\xi, y}, \widehat{\mathbf{v}(y, \lambda)}), \quad (3.13)$$

and

$$\begin{aligned} \partial_{\mathbf{v}(y, \varsigma)} G_{\xi, |\xi-y|}(\xi, y) = \\ |\mathbf{v}(y, \varsigma)|^{-1} G_{\xi, |\xi-y|}(\xi, y) \cos(\nu_{\xi, y}, \widehat{\mathbf{v}(y, \lambda)}), \end{aligned} \quad (3.14)$$

where $\widehat{\nu, \nu'}$ denotes the angle between the vectors ν, ν' . Therefore, the ratio of the derivatives in (3.13) and (3.14), which we denote by $\mu(y, \lambda, \varsigma)$, satisfies

$$\mu(y, \lambda, \lambda) = 1, \quad |\nabla_{\varsigma} \mu(y, \lambda, \varsigma)| \leq C. \quad (3.15)$$

Now we are able to conclude the calculation of the change of the kernel K under the change of the point x , the center of the ball. Having ξ in a neighbourhood of x , due to the definition of the kernel K , we have

$$\begin{aligned} \partial_{\xi} K(\xi, y) &= \partial_{\xi} [h_{d_0(\xi)}(|\xi - y|) \partial_{\mathbf{v}(y, \varsigma)} G_{\xi, |\xi-y|}(\xi, y)] = \\ &\partial_{\xi} (h_{d_0(\xi)}(|\xi - y|)) \partial_{\mathbf{v}(y, \varsigma)} G_{\xi, |\xi-y|}(\xi, y) + \\ &h_{d_0(\xi)} \partial_{\xi} (\partial_{\mathbf{v}(y, \varsigma)} G_{\xi, |\xi-y|}(\xi, y)) \equiv I_1 + I_2. \end{aligned} \quad (3.16)$$

For the first term in (3.16), we use our estimate (3.11) for the derivative of h ,

$$|\partial_{\xi} (h_{d_0(\xi)}(|\xi - y|))| \leq c d_0(x)^{-2}.$$

For the derivatives of the Green function, in the first term on the right in (3.16), we use estimate (2.14), which gives $|\partial_{\mathbf{v}(y, \varsigma)} G(\xi, y)| \leq c d_0(x)^{1-N}$. Therefore, for the first term I_1 in (3.16), we have

$$|I_1| \leq C d_0(x)^{-1-N}. \quad (3.17)$$

Next, we estimate I_2 . For the last factor, i.e., for the second derivative of the Green function, we have

$$\begin{aligned} \partial_{\xi} (\partial_{\mathbf{v}(y, \varsigma)} G_{\xi, |\xi-y|}(\xi, y)) &= \\ \partial_{\xi} \left[\mu(y, \lambda, \varsigma) \Big|_{\varsigma = \frac{\xi-y}{d_0(x)}} \partial_{\mathbf{v}(y, \lambda)} G_{\xi, |\xi-y|}(\xi, y) \right] &= \\ = \partial_{\xi} \mu(y, \lambda, \varsigma) \Big|_{\varsigma = \frac{\xi-y}{d_0(x)}} \partial_{\mathbf{v}(y, \lambda)} G_{\xi, |\xi-y|}(\xi, y) + \mu(y, \lambda, \varsigma) \partial_{\xi} \partial_{\mathbf{v}(y, \lambda)} G_{\xi, |\xi-y|}(\xi, y) &= \\ I_3 + I_4. \end{aligned} \quad (3.18)$$

The term I_3 is again estimated using (3.10) and the bounds for the first and second order derivatives of the Green function in (2.11). This gives

$$\left| \partial_{\varsigma} \mu(y, \lambda, \varsigma) \partial_{\xi} \left[\frac{\xi - y}{d_0(\xi)} \right] \right| \leq c d_0(x)^{-1},$$

and, therefore,

$$\left| h_{d_0(\xi)}(|\xi - y|) \partial_{\varsigma} \mu(y, \lambda, \varsigma) \partial_{\xi} \left[\frac{\xi - y}{d_0(\xi)} \right] \partial_{\mathbf{v}(y, \lambda)} G_{\xi, |\xi - y|}(\xi, y) \right| \leq \quad (3.19)$$

$$cd_0(x)^{-1} \cdot d_0(x)^{-1} \cdot d_0(x)^{1-N} = cd_0(x)^{-1-N}.$$

The most troublesome is the evaluation of the term I_4 , since it requires tracing the behavior of derivatives of the Green function under a rotation of the co-ordinates system.

Consider the family of linear conformal mappings $\eta \mapsto T(\varsigma)\eta$ in $\mathbb{R}^N$, transforming $x - y$ to $\xi - y$, depending smoothly on ς (the parameter ς is concealed in ξ),

$$T(\varsigma) = \frac{|\varsigma|}{|\lambda|} B(\varsigma),$$

where $B(\varsigma)$ is an orthogonal transformation depending smoothly on ς , with $T(\lambda) = \mathbf{E}$. Next, we denote by $g(u, v, \varsigma)$ the rotated Green function, namely, its image under the transformation $T(\varsigma)$ of the ball $B(x, t)$ to the ball $B(\xi, \tau)$:

$$g(u, v, \varsigma) := \quad (3.20)$$

$$G_{T(\varsigma)(x-y)+y, |\varsigma|d_0(T(\varsigma)(x-y)+y)}(T(\varsigma)(v-y)+y, T(\varsigma)(u-y)+y).$$

It follows from (3.20) that the coefficients $a_{j,j'}(\cdot, \varsigma)$ of the operator $\mathcal{L}(\varsigma)$ obtained from $\mathcal{L}$ by the transformation $T(\varsigma)$ depend C^m -smoothly on the parameter ς .

By the multiple usage of the chain rule, we express the derivatives of the function $g(y + T(\varsigma)(x - y), y, \varsigma)$ via the derivatives of $g(\xi, y, \varsigma)$; the latter is the Green function in the ball $B_{\tau}(\xi)$ for the transformed operator $\mathcal{L}(\varsigma)$. We obtain

$$\begin{aligned} & \partial_{\xi} \partial_{\mathbf{v}(y, \lambda)} g(y + T(\varsigma)(x - y), y, \varsigma) \\ &= \partial_{\mathbf{v}(y, \lambda)} \partial_{\xi} g(y + T(\varsigma)(x - y), y, \varsigma) = \partial_{\mathbf{v}(y, \lambda)} \partial_{\varsigma} g(y + T(\varsigma)(x - y)) \partial_{\xi} \varsigma \Big|_{\varsigma = \frac{\xi - y}{d_0(x)}} \\ &= \partial_{\mathbf{v}(y, \lambda)} \left(\partial_{\xi} g(y + T(\varsigma)(x - y), y, \varsigma) \partial_{\varsigma} T(\varsigma)(x - y) \right. \\ &\quad \left. + \partial_{\varsigma} g(y + T(\varsigma)(x - y), y, \varsigma) \partial_{\xi} \varsigma \right) \\ &= \partial_{\mathbf{v}(y, \lambda)} \partial_{\xi} g(y + T(\varsigma)(x - y), y, \varsigma) \partial_{\varsigma} T(\varsigma)(x - y) \\ &\quad + \partial_{\xi} g(y + T(\varsigma)(x - y), y, \varsigma) \partial_{\mathbf{v}(y, \lambda)} \partial_{\varsigma} T(\varsigma)(x - y) \partial_{\xi} \varsigma \\ &\quad + \partial_{\xi} g(y + T(\varsigma)(x - y), y, \varsigma) \partial_{\varsigma} T(\varsigma)(x - y) \partial_{\mathbf{v}} \partial_{\xi} \varsigma \\ &\quad + \partial_{\mathbf{v}(y, \lambda)} \partial_{\varsigma} g(\xi, y, \varsigma) \partial_{\xi} \varsigma + \partial_{\varsigma} g(\xi, u, \varsigma) \partial_{\mathbf{v}(y, \lambda)} \varsigma \Big|_{\varsigma = \frac{\xi - y}{d_0(x)}} \\ &= \partial_{\mathbf{v}(y, \lambda)} \partial_{\xi} g(y + T(\varsigma)(x - y), y, \varsigma) \partial_{\varsigma} T(\varsigma)(x - y) \partial_{\xi} \varsigma \\ &\quad + \partial_{\xi} g(y + T(\varsigma)(x - y), y, \varsigma) \partial_{\mathbf{v}(y, \lambda)} \partial_{\varsigma} T(\varsigma)(x - y) \partial_{\xi} \varsigma \\ &\quad + \partial_{\mathbf{v}(y, \lambda)} \partial_{\varsigma} g(\xi, y, \varsigma) \Big|_{\varsigma = \frac{\xi - y}{d_0(x)}}. \end{aligned} \quad (3.21)$$

When obtaining (3.21), we applied the fact that $\partial_{\mathbf{v}(y, \lambda)} \partial_{\xi} \frac{\xi - y}{d_0(x)} = 0$.

We collect now the estimates which we use to treat (3.21):

$$\begin{aligned} |\partial_{\mathbf{v}(y,\lambda)} \partial_{\xi} g(y + T(\lambda)(x - y), y, \varsigma)| &\leq C d_0(x)^{-N}; \\ |\partial_{\varsigma} T(\varsigma)(x - y)| &\leq c|x - y| \leq c d_0(x); \quad \left| \partial_{\xi} \frac{\xi}{d_0(x)} \right| \leq c d_0(x)^{-1}; \\ |\partial_{\mathbf{v}(y,\lambda)} \partial_{\varsigma} T(\varsigma)(x - y)| &\leq C; \quad |\partial_{\mathbf{v}(y,\lambda)} \partial_{\varsigma} g(\xi, y, \varsigma)| \leq C d_0(x)^{1-N}. \end{aligned} \quad (3.22)$$

After their substitution to (3.21), we obtain

$$|\partial_{\xi} \partial_{\mathbf{v}(y,\lambda)} g(y + T(\varsigma)(x - y), y, \varsigma)| \leq c d_0(x)^{-N}. \quad (3.23)$$

Taken together with (3.16), (3.17), (3.18), (3.19), this gives us the required estimate for the derivative of the kernel $K(\xi, y)$ for $\xi = x$:

$$|\partial_x K(x, y)| \leq c d_0(x)^{-1-N}. \quad (3.24)$$

Before we proceed, we review which estimates of derivatives of the Green function $G(x, y; \varsigma)$ we need. In estimating derivatives of the kernel $K(x, y)$, we applied the chain rule several times. In this calculations, some factors appeared related to variables changes and they did not involve the Green function, but only dealt with geometrical quantities. The only factors containing derivatives of the Green function were derivatives in x , accounting for the movement of the point x , see (3.19), and second derivatives in x , y and y, ς involved in the evaluation of the contribution of the change of direction of the conormal derivative, see (3.23), as well as the change of operator $\mathcal{L}$. Combined with other terms, they give estimate (3.24). We stress here that we use here derivatives of the Green function for a ball with radius $d_0(x)$ only at the points x, y , such that x lies at the center of the ball and y lies at its boundary, so, the distance between x, y is controlled from below by the radius of the ball.

3.4 Estimating the second-order derivatives, $K''_{xx}(x, y)$

We will also need estimates for the second-order derivatives of the kernel $K(x, y)$. The calculations go essentially in the same way as for the first derivative, however they are considerably more cumbersome, so we explain only their structure. Again, multiple application of the chain rule leads to a number of factors of geometrical nature, not depending on the Green function, as well as derivatives of the Green function. Of these derivatives, $\partial_{xx}^2 G(x, y; \varsigma)$ reflects the contribution of the movement of the point x , while $\partial_{xxy}^3 G(x, y; \varsigma)$, reflects the change of the direction of the conormal derivative. Additionally, as explained in the previous subsection, there are terms reflecting the change of the operator $\mathcal{L}$ when passing from the ball $B_{|x-y|}(x)$ to the ball $B_{|\xi-y|}(\xi)$; here, additionally the derivatives $\partial_{\varsigma\varsigma}^2 G(x, y; \varsigma)$, $\partial_{\varsigma\varsigma x}^3 G(x, y; \varsigma)$, $\partial_{\varsigma\varsigma y}^3 G(x, y; \varsigma)$, $\partial_{\varsigma xy}^3 G(x, y; \varsigma)$ appear.

In more detail, after differentiating (3.21), we obtain the sum of ten terms; we present them, recalling the notation (3.20) and setting symbolically for shorthand

$\Xi := (y + T(\varsigma)(\xi - y), y, \varsigma)$ and $\Psi = (\xi, y, \varsigma)$:

$$\begin{aligned} K''_{\xi, \xi}(\xi, y) = & \partial_{\mathbf{v}(y, \lambda)} g(\Xi) A_1(\Psi) + \partial_{\mathbf{v}(y, \lambda)} \partial_{\xi} g(\Xi) A_2(\Psi) + \\ & \partial_{\xi} g(\Xi) A_3(\Psi) + \partial_{\xi} \partial_{\xi} g(\Xi) A_4(\Psi) + \partial_{\mathbf{v}(y, \lambda)} \partial_{\varsigma} g(\Xi) A_5(\Psi) + \\ & \partial_{\xi} \partial_{\varsigma} g(\Xi) A_6(\Psi) + \partial_{\mathbf{v}(y, \lambda)} \partial_{\varsigma}^2 g(\Xi) A_7(\Psi) + \partial_{\xi}^2 \partial_{\mathbf{v}(y, \lambda)} g(\Xi) A_8(\Psi) + \\ & \partial_{\mathbf{v}(y, \lambda)} \partial_{\xi} \partial_{\varsigma} g(\Xi) A_9(\Psi) + \partial_{\mathbf{v}(y, \lambda)} \partial_{\varsigma}^2 A_{10}(\Psi). \end{aligned} \quad (3.25)$$

Here, in (3.25), as before, $\xi = y + T(\varsigma)(x - y)$, $\varsigma = \frac{\xi - y}{d_0(\xi)}$, and the expressions A_1 to A_{10} are combinations of the functions $h_{d_0(x)}(|x - y|)$, $\mu(\varsigma)$, $T(\varsigma)(x - y)$ and their derivatives, structurally similar to the expressions presented in (3.21), where the first order derivatives of $K(\xi, y)$ were being treated, as well as derivatives of G . These expressions can be estimated using relations (3.22). We keep in mind that, due to our choice of x, ξ, y , we have

$$|\xi - y| \geq \frac{1}{4}|x - y| = \frac{1}{4}d_0(x) \quad (3.26)$$

To make the explanation of these calculations more transparent, we introduce the following generic notations. By δg we denote any of first order derivatives of a function $g(\Xi)$ in the variables ξ, v ; by $\delta^2 g$ we denote any of second order derivatives in these variables, and, similarly, $\delta^3 g$ stands for third order derivatives. Relations (3.26) imply

$$|\delta g(\Xi)|, |\partial_{\varsigma} \delta g(\Xi)|, |\partial_{\varsigma}^2 \delta g(\Xi)| \leq C d_0(x)^{1-N};$$

$$|\delta^2 g(\Xi)|, |\partial_{\varsigma} g(\Xi)| \leq c d_0(x)^{-N};$$

$$|\delta^3 g(\Xi)| \leq c d_0(x)^{-1-N}.$$

On the other hand, we estimate the quantities A_1 – A_{10} using (3.22), (3.15), (3.10), (3.12), (3.11); their substitution into (3.25) gives

$$|K_{\xi, \xi}(\xi, y)| \leq c d_0(x)^{-2-N},$$

in particular, for $\xi = x$, we have

$$|K_{xx}(x, y)| \leq c d_0(x)^{-2-N}. \quad (3.27)$$

4 The approximating function; proof of Theorem 1.2

In this section, we construct the extension function $f_0(x)$ and establish its properties, with further construction of the approximating function v_{δ} .

4.1 An estimate for the local approximation

In order to study properties of the extension function f_0 , see (3.2), we need an estimate for derivatives of the local approximations $\Phi_{x,\delta}$.

Lemma 4.1 *Let $f(x)$, $x \in \mathbf{K}$ satisfy the conditions of Theorem 1.2, with $\mathbf{r} \geq 1$. Let $x \in \mathbf{K}$ and $\delta > 0$. Then for $y \in B_{\frac{\delta}{2}}$,*

$$|\nabla_y \Phi_{x,\delta}(y)| \leq C, \quad (4.1)$$

where the constant C may depend on $\mathbf{K}$, $\mathcal{L}$ and f , but for fixed $\mathbf{K}$, $\mathcal{L}$, f , does not depend on δ .

Proof Choose the integer N so that $2^{N-1}\delta < \text{diam}(\mathbf{K}) \leq 2^N\delta$. Then

$$\Phi_{x,\delta}(y) = \Phi_{x,2^N\delta}(y) - \sum_{k=1}^N (\Phi_{x,2^k\delta}(y) - \Phi_{x,2^{k-1}\delta}(y)).$$

We apply the classical property for the gradient of solutions of elliptic equations (see, e.g., [14]): if a function $\Psi(y)$ is a solution of the elliptic equation $\mathcal{L}\Psi(y) = 0$ in the ball $B_R(x)$, then in the smaller concentric ball $y \in B_{\frac{R}{2}}(x)$, the estimate holds

$$|\nabla \Psi(y)| \leq CR^{-1} \|\Psi\|_{C(B_R(x))},$$

with constant C not depending on R .

Due to our choice of N , for $y \in B_{\frac{\delta}{2}}(x)$,

$$|\nabla_y \Phi_{x,2^N\delta}(y)| \leq C,$$

since the function $\Phi_{x,2^N\delta}$, which serves as a 'local approximation' with error $2^N\delta$ on the whole of $\mathbf{K}$, is bounded in $B_{2^N\delta}(x)$ uniformly in $x \in \mathbf{K}$.

We may assume $\delta < 1$. Then Condition (1.1), for $k \geq 1$ implies

$$\|\Phi_{x,2^{k-1}\delta} - \Phi_{x,2^k\delta}\|_{C(B_{2^{k-1}\delta}(x))} \leq C \left(2^{k\mathbf{r}}\delta\right) \delta^{\mathbf{r}}\omega(2^k\delta).$$

Therefore,

$$\begin{aligned} & \|\nabla(\Phi_{x,2^{k-1}\delta} - \Phi_{x,2^k\delta})\|_{C(\overline{B_{\frac{\delta}{2}}(x)})} \leq \\ & C(2^k\delta)^{-1} \times 2^{k\mathbf{r}}\delta^{\mathbf{r}}\omega(2^k\delta) = c2^{k(\mathbf{r}-1)}\delta^{\mathbf{r}-1}\omega(2^k\delta), \quad \mathbf{r} \geq 1. \end{aligned}$$

It follows now that

$$\|\nabla \Phi_{x,\delta}\|_{C(\overline{B_{\frac{\delta}{2}}(x)})} \leq$$

$$\begin{aligned}
& \sum_{k=1}^N \|\nabla(\Phi_{x,2^{k-1}\delta} - \Phi_{x,2^k\delta})\|_{C(\overline{B_{\frac{\delta}{2}}(x)})} + \|\nabla\Phi_{x,2^N\delta}\|_{C(\overline{B_{\frac{\delta}{2}}(x)})} \leq \\
& c \sum_{k=1}^N 2^{k(r-1)} \delta^{r-1} \omega(2^k\delta) + c \leq \delta^{r-1} \int_0^N 2^{s(r-1)} \omega(2^s\delta) ds + c = \\
& c\delta^{r-1} \int_1^{2^N} t^{r-1} \omega(\delta t) \frac{dt}{t} + c = c\delta^{r-1} (\delta^{-1})^{r-1} \int_0^{2^N\delta} \tau^{r-2} \omega(\zeta) d\tau + c \\
& \leq C(\text{diam } \mathbf{K})^{r-1} \int_0^{2 \text{diam } \mathbf{K}} \omega(\tau) \tau^{-1} d\zeta + c \leq C.
\end{aligned}$$

□

4.2 Estimating $\mathcal{L}f_0(x)$

Now, using our estimates for derivatives of the kernel $K(x, y)$, obtained in the previous section, we establish estimates for the function $f_0(x)$ constructed in Section 3.1, see (3.2), and for the result of the action of the operator $\mathcal{L}$ on this function. For a fixed point $x_0 \in \mathbf{CK}$, we consider the open cube Q in the Whitney cover $\mathcal{Q}$, whose closure contains x_0 . Let x_Q be the point in $\mathbf{K}$, closest to the center of this cube (or one of such points). Recall that the function $\tilde{f}_0(x)$ (defined in (3.1)) equals 0 on the boundary of Q . By construction, $\mathcal{L}_x \Phi_{x_Q, 2\delta(Q)}(x) = 0$, for x in the ball $B_{2\delta(Q)}(x_Q)$, in particular, this holds in a small neighborhood of the ball $\overline{B_{d_0(x_0)}(x_0)}$. Therefore, for x in a small neighborhood of the point x_0 , we obtain, recalling the definition of the kernel $K(x, y)$ (which acts as an $\mathcal{L}$ -replacement for the mean value kernel):

$$\begin{aligned}
|\mathcal{L}_x f_0(x)| &= |\mathcal{L}_x(f_0(x) - \Phi_{x_Q, 2\delta(Q)}(x))| = \tag{4.2} \\
& \left| \mathcal{L}_x \left(\int_{\mathbb{R}^N} \tilde{f}_0(y) K(x, y) dy - \int_{\mathbb{R}^N} \Phi_{x_Q, 2\delta(Q)}(y) K(x, y) dy \right) \right| = \\
& \left| \mathcal{L}_x \left(\int_{\mathbb{R}^N} (\tilde{f}_0(y) - \Phi_{x_Q, 2\delta(Q)}(y)) K(x, y) dy \right) \right| \leq \\
& C \int_{B_{d_0(x_0)+\varepsilon}(x_0)} |\tilde{f}_0(y) - \Phi_{x_Q, 2\delta(Q)}(y)| |\nabla_{xx}^2 K(x, y)| dy \leq \\
& C d_0(x)^{-N-2} \int_{B_{d_0(x_0)+\varepsilon}(x_0)} |\tilde{f}_0(y) - \Phi_{x_Q, 2\delta(Q)}(y)| dy,
\end{aligned}$$

using, on the last step, our estimates for derivatives of the kernel $K(x, y)$. We apply estimate (3.27) now. For $x \in \overline{B_{d_0(x_0)+\varepsilon}(x_0)}$, the difference $\tilde{f}_0(y) - \Phi_{x_Q, 2\delta(Q)}(y)$ has the form $\Phi_{x_{Q_1}, 2\delta(Q_1)}(y) - \Phi_{x_Q, \delta(Q)}(y)$, for a certain cube Q_1 , and satisfies the

conditions of the main theorem with parameters $\delta(Q)$, $\delta(Q_1)$. Therefore,

$$|\tilde{f}_0(y) - \Phi_{x_Q, 2\delta(Q)}(y)| = |\Phi_{x_{Q_1}, 2\delta(Q_1)}(y) - \Phi_{x_Q, \delta(Q)}(y)| \leq cd_0(x_0)^{\mathbf{r}} \omega(d_0(x_0)). \quad (4.3)$$

We set $x = x_0$ in (4.2), (4.3) and obtain the required estimate for $\mathcal{L}f_0$ outside $\mathbf{K}$:

$$|(\mathcal{L}f_0)(x_0)| \leq cd_0^{\mathbf{r}-2}(x_0) \omega(d_0(x_0)). \quad (4.4)$$

We stress here that by (4.4), the larger $\mathbf{r}$ in the conditions of the Theorem, the faster the function $\mathcal{L}f_0(x)$ decays as the point x approaches $\mathbf{K}$. This fact will be essentially used further on.

4.3 The integral representation of the function $f_0(x)$.

The construction of an approximating functions goes similarly to the one in [27], with natural modifications. Let, again, the point $x \in \mathbf{CK}$ be fixed, $\delta_0 := \text{dist}(x, \mathbf{K}) > 0$. We fix a number $\delta \in (0, \frac{\delta_0}{2})$ and construct a finite cover of $\mathbf{K}$ by balls with radius δ , possessing Property 2.1, as this was done in [27], see Corollary 2.2 there. This implies that for $R_0 = \text{diam}(\mathbf{K})$ and $r < R_0$, there exists a collection Υ_δ of disjoint balls $B_r(x_\alpha)$ centered in $\mathbf{K}$, such that the concentric balls $B_{5r}(x_\alpha)$ form a cover of $\mathbf{K}$. Moreover, for any $R \in (\delta, R_0)$ and any $\mathbf{x} \in \mathbf{K}$, the quantity of points x_α in the ball $B_R(\mathbf{x})$ is no greater than $\mathbf{b}_\mathbf{N}(R/r)^{\mathbf{N}-2}$, with constant $\mathbf{b}_\mathbf{N}$ not depending on the radii r, R ; we apply this result for $r = \frac{\delta}{5}$.

We denote by $\mathbf{K}_{(\delta)}$ the union of balls in Υ_δ . The boundary of $\mathbf{K}_{(\delta)}$ is piecewise smooth: it consists of a finite union of parts of spheres with radius r . Let $G^\circ(x, y)$ be the Green function for $\mathcal{L}$ in the domain Ω° containing $\mathbf{K}$ (recall that Ω° is taken to be the unit ball), where the operator $\mathcal{L}$ is defined. Therefore, in the integral representation of f_0 in the domain $\Omega_\delta = \Omega^\circ \setminus \overline{\mathbf{K}_{(\delta)}}$, the integrals over $\partial\Omega$ vanish, and this integral representation takes the form (with some coefficient $c_\mathbf{N}$)

$$\begin{aligned} f_0(x) = c_\mathbf{N} \int_{\partial(\mathbf{K}_\delta)} f_0(y) \partial_{\mathbf{v}_y} G^\circ(x, y) dS(y) \\ - c_\mathbf{N} \int_{\partial(\mathbf{K}_\delta)} \partial_{\mathbf{v}_y} f_0(y) G^\circ(x, y) dS(y) + c_\mathbf{N} \int_{\Omega_\delta} \mathcal{L}f_0(y) G^\circ(x, y) dy, \quad x \in \Omega_\delta; \end{aligned} \quad (4.5)$$

here dS denotes the $\mathbf{N}$ -1-dimensional surface measure on the piecewise smooth surface $\partial(\mathbf{K}_\delta)$ and $\mathbf{v}_y$ is the conormal vector associated with operator $\mathcal{L}$.

We consider the behavior of each of the terms in (4.5) as $\delta \rightarrow 0$, having the point x fixed. In the first term, since the distance between x and $\partial(\mathbf{K}_\delta)$ is separated from zero, the Green function $G^\circ(x, y)$ is bounded uniformly in y . At the same time, the $\mathbf{N}$ -1-dimensional measure of $\partial\mathbf{K}_\delta$ tends to zero as $\delta \rightarrow 0$. To see this, recall that the number of balls in Υ_δ is no greater than $C\delta^{2-\mathbf{N}}$, and the area of the boundary of each ball is $c\delta^{\mathbf{N}-1}$. Therefore, the first term in (4.5) tends to zero.

To estimate the second term, we let y belong to the closure of some cube Q in the Whitney cover, this means, $y \in \partial(\mathbf{K}_\delta) \cap \overline{Q}$. Then, according to the definition of f_0 ,

$$f_0(y) = \Phi_{x_Q, 2\delta(Q)}(y) + \int_{\mathbb{R}^N} K(y, w)(\tilde{f}_0(w) - \Phi_{x_Q, 2\delta(Q)}(w))dw;$$

further on,

$$|\partial_y f_0(y)| \leq |\partial_y(f_0(y) - \Phi_{x_Q, 2\delta(Q)}(y))| + |\partial_y \Phi_{x_Q, 2\delta(Q)}(y)| \leq \quad (4.6)$$

$$\left| \int_{\mathbb{R}^N} \partial_y K(y, w)(\tilde{f}_0(w) - \Phi_{x_Q, 2\delta(Q)}(w))dw \right| + C,$$

with some absolute constant C , since, by (4.1), see Lemma 4.1, the functions $\Phi_{x_Q, 2\delta(Q)}$ are uniformly bounded.

According to the definition of the kernel K , the integral in (4.6) is spread only over the ball $B_{d_0(y)}(y)$; for w in this ball, we have, by the conditions of the main theorem,

$$|\tilde{f}_0(w) - \Phi_{x_Q, 2\delta(Q)}(w)| = |\Phi_{x_{Q_1}, 2\delta(Q_1)}(w) - \Phi_{x_Q, 2\delta(Q)}(w)| \leq \quad (4.7)$$

$$c\delta(Q)^{\mathbf{r}}\omega(\delta(Q)) \leq C\delta^{\mathbf{r}}\omega(\delta),$$

where Q_1 is the cube in the Whitney cover, containing w , and x_{Q_1} is the point in $\mathbf{K}$ closest to its center. Now we recall the estimates we obtained for derivatives of the kernel K : $|\partial_y K(y, w)| \leq cd_0(y)^{-1-N} \leq c\delta^{-1-N}$; therefore, (4.6) and (4.7) imply

$$\left| \int_{\mathbb{R}^N} \partial_y K(y, w)(\tilde{f}_0(w) - \Phi_{x_Q, 2\delta(Q)}(w))dw \right| \leq c\delta^{-1-N}\delta^{\mathbf{r}}\omega(\delta)\delta^{\mathbf{N}} \leq c\frac{\omega(\delta)}{\delta}.$$

As a result, we obtain

$$|\partial_y f_0(y)| \leq c\frac{\omega(\delta)}{\delta},$$

(the same estimate that we had in [27]). Thus, the second term in (4.5) tends to zero as $\delta \rightarrow 0$ and we arrive at the integral representation for the function $f_0(x)$:

$$f_0(x) = c_{\mathbf{N}} \int_{\Omega \setminus \mathbf{K}} \mathcal{L}f_0(y)G^\circ(x, y)dy. \quad (4.8)$$

Since $\mathbf{K}$ has zero Lebesgue measure, we can treat the integral in (4.8), as spread over the whole Ω ,

$$f_0(x) = c_{\mathbf{N}} \int_{\Omega} \mathcal{L}f_0(y)G^\circ(x, y)dy, \quad x \notin \mathbf{K}. \quad (4.9)$$

Finally, we establish that the integral in (4.9) is continuous on $\mathbf{K}$ as well. To show this, for a point $x_0 \in \mathbf{K}$, we consider the integral

$$F(x_0) = c_{\mathbf{N}} \int_{\Omega} \mathcal{L}f_0(y)G(x_0, y)dy$$

and estimate $f_0(x) - F(x_0)$, $x \notin \mathbf{K}$, in order to show that $F(x) - F(x_0) \rightarrow 0$ as $x \rightarrow x_0$. So, for a given δ , we suppose that $|x - x_0| \leq \frac{\delta}{2}$. We can represent the difference $f_0(x) - F(x_0)$ as

$$\begin{aligned} f_0(x) - F(x_0) &= c_N \int_{B_\delta(x)} \mathcal{L} f_0(y) G(x, y) dy - c_N \int_{B_\delta(x_0)} \mathcal{L} f_0(y) G(x_0, y) dy \\ &\quad + \sum_{j=1}^{\infty} \int_{\mathcal{A}_j} \mathcal{L} f_0(y) (G(x, y) - G(x_0, y)) dy \\ &\equiv I_0(x) - I_0(x_0) + \sum_{j=1}^{\infty} I_j(x, x_0), \end{aligned} \quad (4.10)$$

where $\mathcal{A}_j = B_{2^j\delta}(x) \setminus B_{2^{j-1}\delta}(x)$. Using our estimate for $\mathcal{L} f_0(y)$, we have for the terms $I_0(x)$, $\tilde{I}_0(x_0)$ on the first line in (4.10),

$$\begin{aligned} |\tilde{I}_0(x_0)| &\leq C \int_{B_\delta(x_0)} \frac{d(y)^{\mathbf{r}-2} \omega(d(y))}{|x_0 - y|^{\mathbf{N}-2}} dy \\ &\leq C \delta^{\mathbf{r}} \int_{B_{2\delta}(x_0)} \frac{d(y)^{-2} \omega(d(y))}{|x_0 - y|^{\mathbf{N}-2}} dy \leq c \omega(\delta), \end{aligned} \quad (4.11)$$

according to Lemma 2.4, for $k = 0$. Similarly,

$$|I_0(x)| \leq C \int_{B_\delta(x)} \frac{d(y)^{\mathbf{r}-2} \omega(d(y))}{|x - y|^{\mathbf{N}-2}} dy \leq C \delta^{\mathbf{r}} \omega(\delta), \quad (4.12)$$

by Lemma 2.2. The term $I_j(x, x_0)$ on the second line in (4.10) is estimated in the following way. For $j \geq 1$, $y \in \mathcal{A}_j(x)$, we have $|G(x, y) - G(x_0, y)| \leq C \delta (2^j \delta)^{1-\mathbf{N}}$, therefore,

$$|I_j(x, x_0)| \leq c \delta (2^j \delta)^{1-\mathbf{N}} \int_{\mathcal{A}_j(x)} d(y)^{\mathbf{r}-2} dy = c \delta (2^j)^{\mathbf{r}-1} \omega(2^j \delta).$$

Using now (2.6), we obtain

$$|I_j(x, x_0)| \leq c \delta (2^j \delta)^{1-\mathbf{N}} (2^j \delta)^{\mathbf{r}+\mathbf{N}-2} \omega(2^j \delta).$$

In our case, since $\mathbf{r} \geq 1$ and the domain Ω° is bounded (it is the unit disk), the sum in (4.10) is finite, it contains only terms with $2^{j-1} \delta < 1$, $j \leq N_0$, therefore, $(2^j \delta)^{\mathbf{r}-1} \leq C (2^j \delta)^{-1}$. This gives for $I_j(x, x_0)$ the estimate

$$|I_j(x, x_0)| \leq C \delta (2^j \delta)^{-1} \omega(2^j \delta) = c 2^{-j} \omega(2^j \delta).$$

It follows that

$$\begin{aligned} \sum_j |I_j(x, x_0)| &\leq c \sum_j 2^{-j} \omega(2^j \delta) \\ &\leq c \int_0^\infty \frac{\omega(2^t \delta)}{2^t} dt \\ &= c' \int_1^\infty \frac{\omega(\tau \delta)}{\tau^2} d\tau \leq c'' \omega(\delta). \end{aligned} \quad (4.13)$$

Taken together, estimates (4.10), (4.11), (4.12), (4.13) give $|f_0(x) - F(x_0)| < C\omega(\delta)$. This means that $F(x)$ converges to $f_0(x_0)$ as $x \rightarrow x_0 \in \mathbf{K}$.

Since both parts in (4.9) are continuous on $\mathbf{K}$, we see that the representation (4.9) is valid for all $x \in \mathbf{K}$, and therefore on the whole Ω° .

4.4 Construction of the approximating function v_δ

Before giving a detailed description of the formula (4.7) below, we would like to compare this construction with the one used in [27] for the case $\mathbf{r} = 0$. In that paper, only the integral term in [27] was present, and it provided approximation with error $\omega(\delta)$. The extra terms which appear in (4.7) are $\mathcal{L}$ -harmonic and they improve the quality of approximation when $\mathbf{r} > 0$.

Now we pass to the description of our approximation. We fix a point $\mathbf{O} \in \mathbf{K}$, which will serve as the starting point of our construction, for all values of the parameter δ , $0 < \delta \leq \text{diam}(\mathbf{K})$. For a given δ , we consider the cover Υ_δ , as in Property 2.1, by balls $B_\delta(x_\alpha)$. We enumerate the points x_α in the following way: the starting numbers go to the points $x_\alpha \in \overline{B_{2\delta}(\mathbf{O})}$, the following ones go to the points $x_\alpha \in \overline{B_{4\delta}(\mathbf{O})} \setminus \overline{B_{2\delta}(\mathbf{O})}$, and further on, along the expanding spherical annuli. The points, with new numbering, will be denoted x_ν , $\nu = 1, \dots, N$. We introduce disjoint sets, $W_1 = \overline{B_{2\delta}(x_1)}$, $W_2 = \overline{B_{2\delta}(x_2)} \setminus \overline{B_{2\delta}(x_1)}$, $W_3 = \overline{B_{2\delta}(x_3)} \setminus (\overline{B_{2\delta}(x_2)} \cup \overline{B_{2\delta}(x_1)})$, and so on. If it turns out that for some ν , the set W_ν is void, $W_\nu = \emptyset$, we just delete it and compress the numeration, so that, as a result, we have the sequence of nonempty sets W_ν . We define now the sequence of weights ρ_ν :

$$\rho_\nu = (\text{meas}_{\mathbf{N}} B_{2\delta}(\mathbf{O}))^{-1} \int_{W_\nu} \mathcal{L} f_0(x) dx. \quad (4.14)$$

Definition (4.14) and estimates (4.4), (2.5) imply

$$|\rho_\nu| \leq c\delta^{-\mathbf{N}} \delta^{\mathbf{r}-2+\mathbf{N}} \omega(\delta) = c\delta^{\mathbf{r}-2} \omega(\delta). \quad (4.15)$$

Next, we define the function

$$F_\nu(x) = c_{\mathbf{N}} \rho_\nu \int_{B_{2\delta}(V(x_\nu, 2\delta))} G^\circ(x, y) dy, \quad x \in \Omega; \quad (4.16)$$

here $V(x_\nu, 2\delta)$ is the point constructed, with $\mathbf{O} = x_0$ in place of x_ν , in the end of Sect. 2.1 and $c_{\mathbf{N}}$ is defined in (4.9). This function is $\mathcal{L}$ -harmonic outside the ball

$B_{2\delta}(V(x_\nu, 2\delta))$, and, therefore, inside $\mathbf{K}_{(\delta)}$. Finally, for $x \in \mathbf{K}_{2\delta}$, we define

$$v_\delta(x) = c_N \int_{\Omega \setminus \mathbf{K}'_\delta} G(x, y) \mathcal{L} f_0(y) dy + \sum_{\nu=1}^N F_\nu(x). \quad (4.17)$$

This function is $\mathcal{L}$ -harmonic in $\mathbf{K}_{(\delta)}$; it will serve as the required approximation.

4.5 Estimates for $f_0 - v_\delta$

We recall that f_0 is a smooth extension of the given function f from the set $\mathbf{K}$ to the enveloping domain Ω° with controlled behavior of derivatives and of $\mathcal{L} f_0(x)$ as x approaches $\mathbf{K}$. Thus, on $\mathbf{K}$, in fact, estimates for $f_0 - v_\delta$ coincide with estimates for $f(x) - v_\delta(x)$, this means, they give the quantity we are interested in.

Using (4.9) and (4.17), we can represent the difference $f_0(x) - v_\delta(x)$, $x \in \mathbf{K}$, as

$$\begin{aligned} f_0(x) - v_\delta(x) &= c_N \int_{\Omega \setminus \mathbf{K}'_\delta} G^\circ(x, y) \mathcal{L} f_0(y) dy + c_N \int_{\mathbf{K}'_\delta} G^\circ(x, y) \mathcal{L} f_0(y) dy \\ &\quad - c_N \int_{\Omega \setminus \mathbf{K}'_\delta} G^\circ(x, y) \mathcal{L} f_0(y) dy - \sum_{\nu=1}^N F_\nu(x) \\ &= c_N \int_{\mathbf{K}'_\delta} G^\circ(x, y) \mathcal{L} f_0(y) dy - \sum_{\nu=1}^N F_\nu(x) \\ &= c_N \sum_{\nu=1}^N \left(\int_{W_\nu} G^\circ(x, y) \mathcal{L} f_0(y) dy - \rho_\nu \int_{B_{2\delta}(V(x_\nu, 2\delta))} G^\circ(x, y) dy \right). \end{aligned} \quad (4.18)$$

In the transformation in (4.18), we used the fact that the sets W_ν are disjoint, $W_\nu \subset \overline{B_{2\delta}(x_\nu)}$, and their union is $\mathbf{K}'_\delta$. We choose the number M so that

$$\text{diam}(\mathbf{K}) < 2^M \delta \leq 2 \text{diam}(\mathbf{K}).$$

With this notation, the last expression in (4.18) can be transformed in the following way:

$$\begin{aligned} &\sum_{\nu=1}^N \left(\int_{W_\nu} G^\circ(x, y) \mathcal{L} f_0(y) dy - \rho_\nu \int_{B_{2\delta}(V(x_\nu, 2\delta))} G^\circ(x, y) dy \right) \\ &= \sum_{\nu: x_\nu \in B_{4\delta}(x)} \left(\int_{W_\nu} G^\circ(x, y) \mathcal{L} f_0(y) dy - \rho_\nu \int_{B_{2\delta}(V(x_\nu, 2\delta))} G^\circ(x, y) dy \right) \\ &= \sum_{l=3}^M \sum_{\nu}^{(l)} \left(\int_{W_\nu} G^\circ(x, y) \mathcal{L} f_0(y) dy - \rho_\nu \int_{B_{2\delta}(V(x_\nu, 2\delta))} G^\circ(x, y) dy \right) \\ &\stackrel{\text{def}}{=} \mathcal{I}_0 + \sum_{l=3}^M \mathcal{I}_l, \end{aligned} \quad (4.19)$$

where the superscript (l) in $\sum^{(l)}$ indicates the summation over those ν for which $x_\nu \in B_{2^l\delta}(x) \setminus B_{2^{l-1}\delta}(x)$. The term $\mathcal{I}_0$ contains no more than $4^{N-2}b_N$ summands. For each of them, this means, for $x_\nu \in B_{4\delta}(x)$, we use estimate (4.4), the general estimate $|G(x, y)| \leq c|x - y|^{2-N}$, and the estimate of the integral (2.4). This gives us

$$\left| c_N \int_{W_\nu} G^\circ(x, y) \mathcal{L}f_0(y) dy \right| \leq c \int_{B_{2\delta}(x_\nu)} d_0(y)^{r-2} \omega(d_0(y)) |x - y|^{2-N} dy \leq C\delta^r \omega(\delta). \quad (4.20)$$

Next, for $y \in B_{2\delta}(V(x_\nu, 2\delta))$, we have $|G^\circ(x, y)| \leq c|x - y|^{2-N} \leq c\delta^{2-N}$, therefore, (4.15) implies

$$\left| c_N \rho_\nu \int_{B_{2\delta}(x_\nu)} G^\circ(x, y) dy \right| \leq c\delta^{r-2} \omega(\delta) \times \delta^{2-N} \delta^N = c\delta^r \omega(\delta). \quad (4.21)$$

Estimates (4.20), (4.21) produce the bound for $\mathcal{I}_0$:

$$|\mathcal{I}_0| \leq c\delta^r \omega(\delta).$$

Next, we consider the term $\mathcal{I}_l$, $l \geq 3$ in (4.19), this means, the sum over such ν for which $2^{l-1}\delta \leq |x_\nu - x| \leq 2^l\delta$. There are no more than $2^{l(N-2)}b_N$ points x_ν in this spherical annulus. Considering one of these points, we choose arbitrarily two additional points $y_{\nu 1} \in W_\nu$ and $y_{\nu 2} \in B_{2\delta}(V(x_\nu, 2\delta))$. Then we have

$$\begin{aligned} & \int_{W_\nu} G^\circ(x, y) \mathcal{L}f_0(y) dy - \rho_\nu \int_{B_{2\delta}(V(x_\nu, 2\delta))} G^\circ(x, y) dy \\ &= \int_{W_\nu} \mathcal{L}f_0(y) (G^\circ(x, y) - G^\circ(x, y_{\nu 1})) dy + G^\circ(x, y_{\nu 1}) \int_{W_\nu} \mathcal{L}f_0(y) dy \\ & \quad - \rho_\nu \int_{B_{2\delta}(V(x_\nu, 2\delta))} (G^\circ(x, y) - G^\circ(x, y_{\nu 2})) dy - \rho_\nu G^\circ(x, y_{\nu 2}) \int_{B_{2\delta}(V(x_\nu, 2\delta))} dy \\ &= \int_{W_\nu} \mathcal{L}f_0(y) (G^\circ(x, y) - G^\circ(x, y_{\nu 1})) dy \\ & \quad - \rho_\nu \int_{B_{2\delta}(V(x_\nu, 2\delta))} (G^\circ(x, y) - G^\circ(x, y_{\nu 2})) dy \\ & \quad + (G^\circ(x, y_{\nu 1}) - G^\circ(x, y_{\nu 2})) \int_{W_\nu} \mathcal{L}f_0(y) dy. \end{aligned} \quad (4.22)$$

When performing transformations in (4.22), we used the definition of the coefficient ρ_ν in (4.14).

We pass to estimating separate terms in (4.22). For $y \in \overline{B_{2^l\delta}(x)} \setminus B_{2^{l-1}\delta}(x)$, we have $|\nabla_y G^\circ(x, y)| \leq |x - y|^{1-N} \leq C(2^l\delta)^{1-N}$. Therefore,

$$|G^\circ(x, y) - G^\circ(x, y_{v1})| = \left| \int_0^1 D_t(G^\circ(x, y + t(y_{v1} - y))) dt \right| \leq c\delta|x - y|^{1-N} \leq c\delta(2^l\delta)^{1-N}. \quad (4.23)$$

In a similar way, using the estimate for the derivatives of the Green function and taking into account the position of the points y, y_{v1}, y_{v2} , we obtain

$$|G^\circ(x, y) - G^\circ(x, y_{v2})| \leq \delta(2^l\delta)^{1-N}, \quad (4.24)$$

$$|G^\circ(x, y_{v1}) - G^\circ(x, y_{v2})| \leq \delta(2^l\delta)^{1-N}. \quad (4.25)$$

Adding up inequalities (4.22)–(4.25), we arrive at the estimate for a single term in $\mathcal{I}_l$:

$$\begin{aligned} c_N \left| \int_{W_v} G^\circ(x, y) \mathcal{L}f_0(y) dy - \rho_v \int_{B_{2\delta}(V(x_v, 2\delta))} G^\circ(x, y) dy \right| \\ \leq c\delta(2^l\delta)^{1-N} \left(\int_{W_v} |\mathcal{L}f_0(y)| dy + \rho_v \operatorname{meas}_N B_{2\delta}(\mathbf{O}) \right) \\ \leq c\delta(2^l\delta)^{1-N} \left(\int_{W_v} d_0(y)^{\mathbf{r}-2} \omega(d_0(y)) dy + \delta^{\mathbf{r}-2+N} \omega(\delta) \right) \\ \leq c\delta^{\mathbf{r}-2+N} \omega(\delta) \delta(2^l\delta)^{1-N}; \end{aligned} \quad (4.26)$$

in transformations in (4.26), we used (4.15) and (2.5). Since in the spherical annulus $B_{2^l\delta}(x) \setminus B_{2^{l-1}\delta}(x)$, there are no more than $c2^{l(N-2)}$ points x_v , we obtain the required estimate for $\mathcal{I}_l$:

$$|\mathcal{I}_l| \leq c2^{l(N-2)} \delta^{\mathbf{r}} \omega(\delta) \cdot \frac{1}{2^{l(N-1)}} = C2^{-l} \delta^{\mathbf{r}} \omega(\delta).$$

Now we sum over l and arrive at

$$|f_0(x) - v_\delta(x)| \leq c\delta^{\mathbf{r}} \omega(\delta) (1 + \sum_{l \geq 3} 2^{-l}) = C\delta^{\mathbf{r}} \omega(\delta).$$

This inequality proves the first statement of the main theorem. The second part follows easily from the first one since:

$$|v_\delta(x) - v_{\frac{\delta}{2}}(x)| \leq |v_\delta(x) - f_0(x)| + |v_{\frac{\delta}{2}}(x) - f_0(x)| \leq c\delta^{\mathbf{r}} \omega(\delta). \quad (4.27)$$

Thus, the 'only if' part of the Theorem is proved for $x \in \mathbf{K}'_\delta \supset \mathbf{K}_\delta$.

The 'if' statement follows by setting $\Phi_{x,\delta}(y) = v_\delta(y)$, $y \in B_\delta(x)$, for all $x \in \mathbf{K}$, this means, we take the single function $v_\delta(y)$ as the local approximates for f at all points $x \in \mathbf{K}$. The required property of the function $\Phi_{x,\delta}$ follows from (4.27) and the equality

$$v_\delta - v_{2^{-3}\delta} = \sum_{s=0}^2 (v_{2^{-s}\delta} - v_{2^{-s-1}\delta}).$$

5 Generalized derivatives of $f(x)$, $x \in \mathbf{K}$

In this section, we *define* generalized derivatives of the function $f_0(x)$ at points $x \in \mathbf{K}$, namely, points, where the usual derivatives, generally, do not exist. These derivatives are used to define surrogates of derivatives of the initial function f . We show here that this definition is self-consistent, and then we prove that the derivatives of the approximating functions v_δ converge to these generalized derivatives of f on $\mathbf{K}$ as $\delta \rightarrow 0$. Naturally, the higher derivatives we consider, the more smoothness we require from the coefficients of the operator $\mathcal{L}$.

5.1 Definitions

Let α be a multi-index, $1 \leq |\alpha| = k \leq \mathbf{r}$, c_N is the constant in (4.9), where the representation for f_0 is found.

Definition 5.1 For $x \in \mathbf{K}$, we define the generalized derivative $f_{(\alpha)}(x)$ by

$$f_{(\alpha)}(x) := c_N \int_{\Omega^\circ} \mathcal{L} f_0(y) \partial_x^\alpha G^\circ(x, y) dy + \sum_v \partial^\alpha F_v(x), \quad (5.1)$$

where $G^\circ(x, y)$ is the Green function for $\mathcal{L}$ in Ω° .

This means that we define derivatives of f_0 by, still formal, differentiation of the representation (4.9).

To justify the definition, we need first to prove that the integral in (5.1) converges. In fact, for the function $f_0(x)$, as defined earlier, we have the estimate

$$|\mathcal{L} f_0(y)| \leq c d(y)^{\mathbf{r}-2} \omega(d(y)).$$

Suppose that the coefficients of the operator $\mathcal{L}$ belong to $C^{3+|\alpha|}$. Since, by (2.12),

$$|\partial_x^\alpha G^\circ(x, y)| \leq c |x - y|^{-(\mathbf{N}-2+k)}, \quad k = |\alpha|,$$

we have

$$|\mathcal{L} f_0(y) \partial_x^\alpha G^\circ(x, y)| \leq C \frac{d(y)^{\mathbf{r}-2} \omega(d(y))}{|x - y|^{\mathbf{N}-2+k}},$$

and, since $k = |\alpha| \leq \mathbf{r}$, we can use the results of Lemma 2.4, 2.7, therefore, the integral in (5.1) converges.

5.2 Proof of Theorem 1.3

In this section, $\|\cdot\|_{\mathbf{K}_{\frac{\delta}{2}}}$ denotes the norm in $C(\mathbf{K}_{\frac{\delta}{2}})$ etc. We prove the estimate (1.4) first. For some $\Delta > 0$, fix a point $x^0 \in \mathbf{K}_{\Delta/2}$. Then, for $x \in \mathbf{K}_{\Delta}$, we have $\mathcal{L}v_{\Delta}(x) = \mathcal{L}v_{2\Delta}(x) = 0$. Consider the Green function $G_{\Delta}(x, y)$ for $\mathcal{L}$ in the ball $B_{\Delta} \equiv B_{\Delta}(x^0)$. Then

$$v_{\Delta}(x) - v_{2\Delta}(x) = c_{\mathbf{N}} \int_{\partial B_{\Delta}} (v_{\Delta}(y) - v_{2\Delta}(y)) \partial_{\mathbf{v}_y} G_{\Delta}(x, y) dS(y).$$

We differentiate this equality $\mathbf{r} + 1$ times in x :

$$\nabla^{\mathbf{r}+1}(v_{\Delta}(x) - v_{2\Delta}(x)) = c_{\mathbf{N}} \int_{\partial B_{\Delta}} (v_{\Delta}(y) - v_{2\Delta}(y)) \partial_{\mathbf{v}_y} \nabla_x^{\mathbf{r}+1} G_{\Delta}(x, y) dS(y).$$

Now we use the estimate for the derivative of the Green function and obtain, for $x \in B_{\Delta/2}$, see Corollary 2.10:

$$\begin{aligned} |\nabla^{\mathbf{r}+1}(v_{\Delta}(x) - v_{2\Delta}(x))| &\leq C \Delta^{\mathbf{r}} \omega(\Delta) \int_{\partial B_{\Delta}} |x - y|^{-\mathbf{N}+1-\mathbf{r}} dS(y) \\ &\leq C \Delta^{-1} \omega(\Delta). \end{aligned} \quad (5.2)$$

With δ fixed, we take an integer M so that $\text{diam}(\mathbf{K}) < 2^M \delta \leq 2 \text{diam}(\mathbf{K})$, and we write (5.2) for $\Delta = 2^{j-1} \delta$, $1 \leq j \leq M$. Adding the corresponding estimates, we obtain and

$$\begin{aligned} \|\nabla^{\mathbf{r}+1} v_{\delta}\|_{\mathbf{K}_{\frac{\delta}{2}}} &\leq \sum_{j=1}^M \|\nabla^{\mathbf{r}+1}(v_{2^{j-1}\delta} - v_{2^j\delta})\|_{\mathbf{K}_{\frac{\delta}{2}}} + \\ \|\nabla^{\mathbf{r}+1}(v_{2^M\delta})\|_{\mathbf{K}_{\frac{\delta}{2}}} &\leq \|\nabla^{\mathbf{r}+1}(v_{2^M\delta})\|_{\mathbf{K}_{\frac{\delta}{2}}} + O(1). \end{aligned} \quad (5.3)$$

due to the third property of the approximating function in (1.2) and Lemma 4.1.

After this, the sum $\sum_{j=1}^M \delta^{-1} 2^{-j} \omega(2^j \delta)$ is estimated similarly to the sums in Sect. 2.2, via the integral

$$\delta^{-1} \int_1^M 2^{-\tau} \omega(2^{\tau} \delta) d\tau \leq c \int_{2\delta}^{\infty} \frac{\omega(s)}{s^2} ds \leq c \frac{\omega(\delta)}{\delta},$$

and this proves the inequality.

Now we prove the approximation property. It follows from (4.17) that

$$\partial^{\alpha} v_{\delta}(x) = c_{\mathbf{N}} \int_{\Omega \setminus \mathbf{K}_{2,\delta}} \mathcal{L} f_0(y) \partial_x^{\alpha} G^{\circ}(x, y) dy + \sum_{v=1}^N \partial^{\alpha} F_v(x), \quad x \in \mathbf{K}, \quad (5.4)$$

where $F_v(x)$ are functions constructed for the given δ as in Sect. 4.4.

Next, from the definition (5.1), and (5.4), an estimate for the approximation of derivatives follows,

$$\begin{aligned} f_{(\alpha)}(x) - \partial^\alpha v_\delta(x) &= c_N \int_{\mathbf{K}_{2,\delta}} \mathcal{L} f_0(y) \partial_x^\alpha G^\circ(x, y) dy - \sum_{v=1}^N \partial^\alpha F_v(x) \\ &= \sum_{v=1}^N c_N \int_{W_v} \mathcal{L} f_0(y) \partial_x^\alpha G^\circ(x, y) dy - \sum_{v=1}^N \partial^\alpha F_v(x). \end{aligned} \quad (5.5)$$

Now we can use again the estimates of $\mathcal{L}f$ obtained in Sect. 4.2 and of derivatives of the Green function, which gives

$$\left| \int_{W_v} \mathcal{L} f_0(y) \partial_x^\alpha G^\circ(x, y) dy \right| \leq c \int_{B_{2\delta}(x_v)} \frac{d(y)^{\mathbf{r}-2} \omega(d(y))}{|x - y|^{\mathbf{N}-2+|\alpha|}} dy.$$

If the point x_v is close to x , namely, $x_v \in B_{4\delta}(x)$, we can use estimates (A.1) and (4.20), which give

$$\int_{B_{2\delta}(x_v)} \frac{d(y)^{\mathbf{r}-2} \omega(d(y))}{|x - y|^{\mathbf{N}-2+k}} dy \leq c \delta^{\mathbf{r}-k} \omega(\delta),$$

since here $B_{2\delta}(x_v) \subset B_{6\delta}(x)$.

For the derivative $\partial^\alpha F_v$ the estimate in this domain is easier. It follows from (4.15) that

$$\partial^\alpha F_v(x) = c_N \rho_v \int_{B_{2\delta}(V_{x_v, \delta})} \partial_x^\alpha G(x, y) dy. \quad (5.6)$$

By our estimate (4.16),

$$|\rho_v| \leq c \delta^{\mathbf{r}-2} \omega(\delta). \quad (5.7)$$

For $x_v \in \overline{B_{4\delta}(x)}$, relations (5.6), (5.7) imply

$$\begin{aligned} |\partial^\alpha F_v(x)| &\leq c \delta^{\mathbf{r}-2} \omega(\delta) \int_{B_{2\delta}(V_{x_v, 2\delta})} |\partial_x^\alpha G^\circ(x, y)| dy \leq \\ &c \delta^{\mathbf{r}-2} \omega(\delta) \int_{B_{2\delta}(V_{x_v, 2\delta})} |x - y|^{-\mathbf{N}+2-|\alpha|} dy. \end{aligned} \quad (5.8)$$

Since $|V(x_{v, 2\delta}) - x_v| \geq 12\delta$, we have $|y - x| \geq 12\delta - 2\delta - 4\delta = 6\delta$ for $y \in B_{2\delta}(V_{x_v, 2\delta})$ and $x_v \in B_{4\delta}(x)$, therefore, we obtain from (5.8):

$$|\partial^\alpha F_v(x)| \leq c \delta^{\mathbf{r}-2} \omega(\delta) \cdot \delta^{\mathbf{N}} \delta^{-\mathbf{N}+2-|\alpha|} = c \delta^{\mathbf{r}-|\alpha|} \omega(\delta).$$

We consider now those x_v which lie outside the ball $B_{4\delta}(x)$, i.e., $x_v \in B_{2^{l+1}\delta}(x) \setminus B_{2^l\delta}(x)$ for some $l \geq 2$. In this case, for $y \in B_{2\delta}(x_v)$, we have $|x - y| \asymp 2^l \delta$, and,

again, using Green functions estimates (2.12) we have

$$\begin{aligned} |S_v| &:= \left| \int_{W_v} \mathcal{L} f_0(y) \partial_x^\alpha G(x, y) dy \right| \leq C \int_{B_{2\delta}(x_v)} \frac{d(y)^{\mathbf{r}-2} \omega(d(y))}{|x - y|^{\mathbf{r}-2+k}} dy \\ &\leq c \frac{\delta^{\mathbf{N}-2+\mathbf{r}} \omega(\delta)}{(2^l \delta)^{\mathbf{N}-2+k}} = c \delta^{\mathbf{r}-k} \omega(\delta) 2^{-(\mathbf{N}-2+k)j}, \quad k = |\alpha|. \end{aligned} \quad (5.9)$$

We recall that in the proof of the estimate for $f(x) - v_\delta(x)$ we used the fact that the spherical annulus $B_{2^{l+1}\delta}(x) \setminus B_{2^l\delta}(x)$ contains no more than $c 2^{(\mathbf{N}-2)l}$ points x_v , therefore, inequality (5.9) implies the estimate

$$\begin{aligned} \sum_{v: x_v \in B_{2^{l+1}\delta}(x) \setminus B_{2^l\delta}(x)} |S_v| &\leq \\ c 2^{(\mathbf{N}-2)l} \delta^{\mathbf{r}-k} \omega(\delta) \frac{1}{2^{(\mathbf{N}-2+k)l}} &= c 2^{-lk} \delta^{\mathbf{r}-k} \omega(\delta). \end{aligned} \quad (5.10)$$

To obtain the estimate for $\sum_{v=1}^\infty \partial^\alpha F_v$ a similar, but much simpler calculation works. We collect estimates (5.5)–(5.10) to obtain the required inequality

$$|f_{(\alpha)}(x) - \partial^\alpha v_\delta(x)| \leq c \delta^{\mathbf{r}-k} \omega(\delta) \sum_{l=1}^\infty 2^{-l} + c \delta^{\mathbf{r}-k} \omega(\delta) = c \delta^{\mathbf{r}-k} \omega(\delta).$$

5.3 Taylor remainder estimates

We have defined, for a function $f \in \mathcal{H}_{\mathcal{L}}^{\mathbf{r}+\omega}(\mathbf{K})$ admitting local approximation by $\mathcal{L}$ –harmonic functions, certain surrogates of derivatives. The same kind of calculations as in the previous subsection, which established the convergence of derivatives of the approximating functions v_δ to the generalized derivatives of f , enables one to prove that in a certain sense, these generalized derivatives may be used to construct a Taylor type formula for f and its derivatives. We give here only the formulation.

Theorem 5.2 *For the function f and the compact set $\mathbf{K}$ satisfying the conditions of this paper, provided the coefficients of $\mathcal{L}$ belong to $\mathbf{C}^{\mathbf{r}+4}$, the following inequality holds, with some constant c not depending on $x_1, x_2 \in \mathbf{K}$:*

$$\begin{aligned} &\left| f_{(\alpha)}(x_2) - f_{(\alpha)}(x_1) - \sum_{|\beta| \geq 1, |\alpha|+|\beta| \leq \mathbf{r}} (\beta!)^{-1} f_{(\alpha+\beta)}(x_1) (x_2 - x_1)^\beta \right| \\ &\leq c |x_2 - x_1|^{\mathbf{r}-k} \omega(|x_2 - x_1|), \end{aligned}$$

for $1 \leq |\alpha| = k < \mathbf{r}$. As a limit case,

$$|f_{(\alpha)}(x_1) - f_{(\alpha)}(x_2)| \leq c \omega(|x_1 - x_2|), \quad |\alpha| = \mathbf{r}.$$

The, rather technical, proof, is based upon the Taylor expansion of the Green function substituted into the expression for f_0 .

6 The counter-example

In this counter-example, we show that if we relax the local approximation condition imposed on the function f defined on the compact set $\mathbf{K}$ in the main theorem, namely, if we only suppose that the approximating functions $\Phi_{x,\delta}$ are just smooth, without requiring that they are $\mathcal{L}$ -harmonic, then the global approximation by solutions of this equation may fail. This effect is surely caused by a pathological structure of the set $\mathbf{K}$. We note that such counter-examples are possible only in the case $\mathbf{r} \geq 1$: in our paper [27], we have shown that for a minor smoothness, i.e., $\mathbf{r} = 0$, this means, for the approximation with quality $\omega(\delta)$, the requirement that the locally approximating functions are $\mathcal{L}$ -harmonic does not arise.

The compact set $\mathbf{K}$ in our example looks as a dish-brush with dense $\mathbf{N}$ -2-dimensional bristles looking in different directions. So, for a harmonic function v_δ approximating on $\mathbf{K}$ the given function f , all second derivatives of v_δ should approximate all second derivatives of f , therefore, if f is a trace on $\mathbf{K}$ of a *non-harmonic* function, such approximation is impossible. On the opposite, if $\mathbf{K}$ were more regular, say a C^2 -surface of codimension 2, the global approximation by harmonic functions v_δ would impose restriction only upon *some* partial derivatives of f , not causing a contradiction.

Now we render concrete the above description.

We introduce here a special notation for $\mathbf{N}$ -2-dimensional balls in $\mathbb{R}^{\mathbf{N}}$: this notation will reflect the orientation of these balls in $\mathbb{R}^{\mathbf{N}}$. We set

$$\begin{aligned} B_{\kappa 1}^* &= \{x = (x_1, \dots, x_{\mathbf{N}-1}, x_{\mathbf{N}})\} : |x| \leq 2^{-\kappa-2}, x_{\mathbf{N}-1} = x_{\mathbf{N}} = 0; \\ B_{\kappa 2}^* &= \{x = (x_1, \dots, x_{\mathbf{N}-1}, x_{\mathbf{N}})\} : |x| \leq 2^{-\kappa-2}, x_{\mathbf{N}-2} = x_{\mathbf{N}-1} = 0; \\ B_{\kappa 3}^* &= \{x = (x_1, \dots, x_{\mathbf{N}-1}, x_{\mathbf{N}})\} : |x| \leq 2^{-\kappa-2}, x_{\mathbf{N}-2} = x_{\mathbf{N}} = 0, \\ B_{\kappa \iota} &= B_{\kappa \iota}^* + \left(\frac{1}{2}(2^{-\kappa-1} + 2^{-\kappa}), 0, \dots, 0 \right), \kappa = 0, 1, \dots, \iota = 1, 2, 3. \end{aligned}$$

The compact set $\mathbf{K} \subset \mathbb{R}^{\mathbf{N}}$ is defined as

$$\mathbf{K} = \left(\bigcup_{\iota=1}^3 \bigcup_{\kappa=0}^{\infty} B_{\kappa \iota} \right) \cup \{0_{\mathbf{N}}\},$$

where $\{0_{\mathbf{N}}\}$ denotes the origin in $\mathbb{R}^{\mathbf{N}}$. It is easy to check that this set is Ahlfors-David-regular of dimension $\mathbf{N}$ -2.

We consider the function $f(x) = |x|^2$, $x \in \mathbf{K}$. The same function, considered in $\mathbb{R}^{\mathbf{N}}$, serves as a local smooth approximation $\Phi_{\delta,x}(y)$ for itself in any neighborhood of any point in $\mathbf{K}$, for any level of smoothness, since $\Phi_{\delta,x}(y) - f(y) \equiv 0$. The only

shortcoming, compared with the conditions of Theorem 1.2, is that the approximating function is not a solution of the Laplace equation.

Theorem 6.1 *It is impossible to approximate $f(x)$ in the sense of Theorem 1.2 with $\mathbf{r} = 2$, and $\omega(\delta) = \delta^\sigma$, $0 < \sigma < 1$, by harmonic functions, this means, by solutions of the equation $\mathcal{L}v_\delta \equiv -\Delta v_\delta = 0$,*

In other words, for such a wild set $\mathbf{K}$, one cannot approximate on $\mathbf{K}$ the non-harmonic function $f(x)$ by harmonic functions, even locally.

Proof Suppose that the approximation in question is possible, thus, for any $\delta \in (0, 1)$, there exists a function v_δ such that,

$$|v_\delta(x) - f(x)| \leq c\delta^{2+\sigma}, \quad x \in \mathbf{K}, \quad (6.1)$$

$$|v_\delta(x) - v_{\frac{\delta}{2}}(x)| \leq c\delta^{2+\sigma}, \quad x \in \mathbf{K}_{\frac{\delta}{2}}, \quad (6.2)$$

with some c not depending on δ , and

$$\Delta v_\delta(x) = 0, \quad x \in \mathbf{K}_\delta. \quad (6.3)$$

We establish the following property.

Lemma 6.2 *Under the assumptions (6.1)–(6.3), the function v_δ must satisfy the estimate:*

$$|\nabla^3 v_{2\delta_\kappa}(x)| \leq c c \delta_\kappa^{\sigma-1}, \quad (6.4)$$

$$x \in U_\kappa := (\mathbf{t}_\kappa + B_{\delta_\kappa}(\mathbb{0}), \mathbf{t}_\kappa = (\frac{1}{2}(2^{-\kappa-1} + 2^{-\kappa}), 0, \dots, 0), \delta_\kappa = 2^{-\kappa-2}.$$

Proof To prove (6.4), we denote by $\phi_k(x)$, $x \in U_\kappa$, the function $\phi_k(x) = v_{2^k\delta_m}(x) - v_{2^{k+1}\delta_m}(x)$, $k = 1, \dots, N$, where N is chosen so that $1 < 2^N\delta \leq 2$. Using this function, we can represent $v_{2\delta_\kappa}$ and its order 3 gradient as

$$v_{2\delta_\kappa} = v_{2^{N+1}\delta} + \sum_{k=1}^N \phi_k, \quad \text{and} \quad \nabla^3 v_{2\delta_\kappa} = \nabla^3 v_{2^{N+1}\delta} + \sum_{k=1}^N \nabla^3 \phi_k. \quad (6.5)$$

Due to the definition of N , we have $|\nabla^3 v_{2^{N+1}\delta}| \leq c$. For the functions ϕ_k , we have the estimate, by the assumption (6.2):

$$|\phi_k(x)| \leq c c (2^k \delta_\kappa)^{2+\sigma}, \quad x \in \mathbf{K}_{2^k \delta_\kappa}.$$

Derivatives of the function ϕ_k which is harmonic in the ball $U_{k,\kappa} = \mathbf{t}_\kappa + B_{2^k \delta_\kappa}(\mathbb{0}) \subset \mathbf{K}_{2^k \delta_\kappa}$, can be estimated using the Poisson formula:

$$|\nabla^3 \phi_k(x)| \leq c c \frac{(2^k \delta_\kappa)^{2+\sigma}}{(2^k \delta_\kappa)^3} = c c 2^{k(\sigma-1)} \delta_\kappa^{\sigma-1}. \quad (6.6)$$

Now, it follows from (6.5), (6.6) that

$$|\nabla^3 v_{2\delta_{\mathbf{k}}}(x)| \leq c\mathbb{C}\delta_{\mathbf{k}}^{\sigma-1} \sum_{k=1}^M 2^{k(\sigma-1)} + c \leq c'\delta_{\mathbf{k}}^{\sigma-1},$$

and this proves Lemma 6.2. $\square$

Having this estimate, we proceed with our example. We introduce the function

$$P_2(x, \mathbf{t}_{\mathbf{k}}) = v_{2\delta_{\mathbf{k}}}(\mathbf{t}_{\mathbf{k}}) + \sum_{|\alpha|=1} \frac{1}{\alpha!} \partial_{\alpha} v_{2\delta_{\mathbf{k}}}(\mathbf{t}_{\mathbf{k}})(x - \mathbf{t}_{\mathbf{k}})^{\alpha} + \sum_{|\alpha|=2} \frac{1}{\alpha!} \partial_{\alpha} v_{2\delta_{\mathbf{k}}}(\mathbf{t}_{\mathbf{k}})(x - \mathbf{t}_{\mathbf{k}})^{\alpha}; \quad (6.7)$$

this is the second degree Taylor polynomial for $v_{2\delta_{\mathbf{k}}}(x)$ at the point $x = \mathbf{t}_{\mathbf{k}}$. We use the integral form of the remainder term in the Taylor formula, to express the difference of values at the points x and $\mathbf{t}_{\mathbf{k}}$: which, according to (6.7), gives

$$|\nabla^2 v_{2\delta_{\mathbf{k}}}(x) - \nabla^2 P_2(x, \mathbf{t}_{\mathbf{k}})| \leq c\mathbb{C}\delta_{\mathbf{k}} \sup_{y \in \mathbf{t}_{\mathbf{k}} + B_{\delta_{\mathbf{k}}}(\mathbb{0}_{\mathbf{N}})} |\nabla^3 v_{2\delta_{\mathbf{k}}}(y)| \leq c\mathbb{C}\delta_{\mathbf{k}} \cdot \delta_{\mathbf{k}}^{\sigma-1} = c\mathbb{C}\delta_{\mathbf{k}}^{\sigma}.$$

In particular, this implies

$$|\Delta v_{2\delta_{\mathbf{k}}}(x) - \Delta P_2(x, \mathbf{t}_{\mathbf{k}})| \leq c\mathbb{C}\delta_{\mathbf{k}}^{\sigma}, \quad x \in \mathbf{t}_{\mathbf{k}} + B_{\delta_{\mathbf{k}}}(\mathbb{0}_{\mathbf{N}}). \quad (6.8)$$

Combining (6.3) and (6.8), we obtain

$$|\Delta P_2(x, \mathbf{t}_{\mathbf{k}})| \leq c\mathbb{C}\delta_{\mathbf{k}}^{\sigma}, \quad x \in \mathbf{t}_{\mathbf{k}} + B_{\delta_{\mathbf{k}}}(\mathbb{0}_{\mathbf{N}}). \quad (6.9)$$

Next, we represent $P_2(x, \mathbf{t}_{\mathbf{k}})$ in a different form, with some terms, second order homogeneous, separated:

$$P_2(x, \mathbf{t}_{\mathbf{k}}) = \sum_{j=1}^N b_j(\mathbf{t}_{\mathbf{k}}, \delta_{\mathbf{k}})(x_j - \mathbf{t}_{\mathbf{k},j})^2 + \tilde{P}_2(x, \mathbf{t}_{\mathbf{k}}), \quad (6.10)$$

$$\mathbf{t}_{\mathbf{k}} = \left(\frac{1}{2}(2^{-\kappa} + 2^{-\kappa-1}), 0, \dots, 0\right),$$

where the polynomial $\tilde{P}_2(x, \mathbf{t}_{\mathbf{k}})$ contains terms of degree 0 and 1 in $x - \mathbf{t}_{\mathbf{k}}$ as well as mixed terms of the form $c(x_j - \mathbf{t}_{\mathbf{k},j})(x_j - \mathbf{t}_{\mathbf{k},j'})$, $j \neq j'$. Since $\Delta \tilde{P}_2 = 0$, (6.9), (6.10) imply

$$\left| \sum_{j=1}^N b_j(\mathbf{t}_{\mathbf{k}}, \delta_{\mathbf{k}}) \right| \leq c\mathbb{C}\delta_{\mathbf{k}}^{\sigma}. \quad (6.11)$$

We recall that the coefficients $b_j(\mathbf{t}_{\mathbf{k}}, \delta_{\mathbf{k}})$ in (6.11) are constant times the second derivatives of $v_{2\delta_{\mathbf{k}}}(\mathbf{t}_{\mathbf{k}})$, $b_j(\mathbf{t}_{\mathbf{k}}, \delta_{\mathbf{k}}) = \frac{1}{2}(\partial_{j,j}^2 v_{2\delta_{\mathbf{k}}})(\mathbf{t}_{\mathbf{k}})$. The set $\mathbf{K} \cap (\mathbf{t}_{\mathbf{k}} + \mathbf{B}_{\delta_{\mathbf{k}}}(\mathbb{0}_{\mathbf{N}}))$ contains

the closed intervals

$$\mathbf{I}_{1,\kappa} = [2^{-\kappa-1}, 2^{-\kappa}] \times \{(0)_{\mathbb{N}-1}\}, \text{ and}$$

$$\mathbf{I}_{j,\kappa} = \{\frac{1}{2}(2^{-\kappa} + 2^{-\kappa-1}), 0, \dots, \overbrace{[-\delta_\kappa, \delta_\kappa]}^j, 0, \dots, 0.\}, \text{ for } j > 1,$$

where the overset $\cdot^j$ denotes the component at the j -th co-ordinate.

Now, it follows from (6.1) that

$$|v_{2\delta_\kappa}(x) - |x|^2| \leq c\mathbb{C}\delta_\kappa^{2+\sigma}, \quad x \in \bigcup_{j=1}^{\mathbb{N}} \mathbf{I}_{j,\kappa}.$$

On the other hand, since P_2 is the quadratic Taylor polynomial for $v_{2\delta_\kappa}(x)$, we should have

$$\begin{aligned} |v_{2\delta_\kappa}(x) - P_2(x, \mathbf{t}_\kappa)| &\leq c\delta_\kappa^3 \sup_{y \in \mathbf{t}_\kappa + B_{\delta_\kappa}(\mathbb{0}_{\mathbb{N}})} |\nabla^3(y)| \\ &\leq c\mathbb{C}\delta_\kappa^3 \delta_\kappa^{\sigma-1} = c\mathbb{C}\delta_\kappa^{\sigma+2}, \quad x \in \mathbf{t}_\kappa + B_{\delta_\kappa}(\mathbb{0}_{\mathbb{N}}). \end{aligned}$$

The latter two inequalities imply

$$|P_2(x, \mathbf{t}_\kappa) - |x|^2| \leq c\mathbb{C}\delta_\kappa^{2+\sigma}. \quad (6.12)$$

Since both polynomials in (6.12) have degree 2 and (6.12) must hold for any δ_κ , we have a contradiction: $P_2(x, \mathbf{t}_\kappa)$ is harmonic, while $|x|^2$ is not. $\square$

Remark It may be interesting to notice that a counterexample as above is impossible for a lesser smoothness, namely, for $\mathbf{r} = 1$. In fact, the sense of our construction for $\mathbf{r} = 2$ is finding a crazy compact set $\mathbf{K}$ such that a certain function admits local approximation, in the sense of the paper, by *smooth functions*, but does not admit an approximation in the same sense at least near one point, by *harmonic functions*. Let now $\mathbf{r} = 1$, and suppose that the local approximation by *smooth* functions $\Phi_{x,\delta}(y)$ satisfying (1.1) is possible. Then, if we replace $\Phi_{x,\delta}(y)$ by its degree 1 Taylor polynomial $\tilde{\Phi}_{x,\delta}(y)$ at the point x , the inequality (1.1) will still hold (with some other constant) for $\tilde{\Phi}_{x,\delta}(y)$. However, the latter function is linear, and therefore, harmonic (and this is THE difference between two values of $\mathbf{r}$), and it provides the local approximation of f . So, it turns out that for $\mathbf{r} = 1$, unlike $\mathbf{r} = 2$, the existence of the required local approximation by smooth functions implies the existence of a local approximation by harmonic functions, thus making a counterexample impossible.

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Appendix A Estimates for integrals in Sect. 2.2. Proofs

Proof of Lemma 2.2 For $y \in B_{2\delta}(x)$, we have $d(y) \leq c\delta$, therefore,

$$\int_{B_{2\delta}(x)} \frac{d(y)^{r-2}\omega(d(y))}{|x-y|^{N-2}} dy \leq C\delta^r \int_{B_{2\delta}(x)} \frac{d(y)^{-2}\omega(d(y))}{|x-y|^{N-2}} dy.$$

For the last integral, we can use the estimate in [27], see there Lemma 6.2, which gives us the required inequality. $\square$

Proof of Corollary 2.3 If $y \in B_{2\delta}(x)$, then $\delta^{2-N} \leq C|x-y|^{2-N}$, therefore, (2.4) implies

$$\begin{aligned} \int_{B_{2\delta}(x)} d(y)^{r-2}\omega(d(y)) dy &= \delta^{N-2} \int_{B_{2\delta}(x)} \frac{d(y)^{r-2}\omega(d(y))}{\delta^{N-2}} dy \\ &\leq C\delta^{N-2} \int_{B_{2\delta}(x)} \frac{d(y)^{r-2}\omega(d(y))}{|x-y|^{N-2}} dy \leq c\delta^{N-2+r}\omega(\delta). \end{aligned}$$

$\square$

Proof of Lemma 2.4 We set $\delta_1 = c_0\delta$. The integral in (2.6) can be represented as $I_k = \sum_{\kappa=1}^{\infty} I_{(\kappa)}$, where $I_{(\kappa)}$ is the integral over the spherical annulus $\mathcal{T}_{\kappa} = B_{2^{-\kappa+1}\delta_1}(x) \setminus B_{2^{-\kappa}\delta_1}(x)$. For the latter integral, we have

$$\begin{aligned} I_{\kappa} &\leq c \frac{1}{(2^{-\kappa}\delta_1)^{N-2+k}} \int_{\mathcal{T}_{\kappa}} d(y)^{k-2}\omega(d(y)) dy \\ &\leq c \frac{1}{(2^{-\kappa}\delta_1)^{N-2+k}} \int_{B_{2^{-\kappa+1}\delta_1}(x)} d(y)^{k-2}\omega(d(y)) dy \\ &\leq c \frac{1}{(2^{-\kappa}\delta_1)^{N-2+k}} (2^{-\kappa}\delta_1)^k \int_{B_{2^{-\kappa+1}\delta_1}(x)} d(y)^{-2}\omega(d(y)) dy \\ &\leq c \frac{1}{(2^{-\kappa}\delta_1)^{N-2+k}} \int_{B_{2^{-\kappa+1}\delta_1}(x)} d(y)^{k-2}\omega(d(y)) dy \\ &\leq c \frac{1}{(2^{-\kappa}\delta_1)^{N-2+k}} (2^{-\kappa}\delta_1)^k \int_{B_{2^{-\kappa+1}\delta_1}(x)} \frac{(2^{-\kappa}\delta_1)^{N-2}}{|x-y|^{N-2}} d(y)^{-2}\omega(d(y)) dy \\ &\leq c \frac{1}{(2^{-\kappa}\delta_1)^{N-2+k}} (2^{-\kappa}\delta_1)^k (2^{-\kappa}\delta_1)^{N-2} \int_{B_{2^{-\kappa+1}\delta_1}(x)} \frac{d(y)^{-2}\omega(d(y))}{|x-y|^{N-2}} dy \end{aligned} \tag{A.1}$$

$$\leq c\omega(2^{-\kappa}\delta_1)$$

(the last inequality was established in Lemma 6.2 in [27]). We sum over κ the expressions in (A.1), which gives

$$\begin{aligned} I_k &\leq c \sum_{\kappa=1}^{\infty} \omega(2^{-\kappa}\delta_1) \leq C \int_1^{\infty} \omega(2^{-\tau}\delta_1) d\tau \\ &= c \int_0^{\frac{1}{2}} \omega(t\delta_1) \frac{dt}{t} = c \int_0^{\delta_1/2} \omega(t)t^{-1} dt \leq c\omega(\delta_1/2) \leq c\omega(\delta). \end{aligned}$$

□

Proof of Corollary 2.5 In fact, by Lemma 2.4, we have

$$J_k \leq c\delta^{\mathbf{r}-k} \int_{B_{c_0\delta}(x)} \frac{d(y)^{k-2}\omega(d(y))}{|y-x|^{\mathbf{N}-2+k}} dy \leq c\delta^{\mathbf{r}-k}\omega(\delta).$$

□

Proof of Lemma 2.6 Again, we represent the integral in (2.9) as the sum of integrals I_k over, now expanding, spherical annuli $\mathcal{T}_k = B_{2^k\delta(x)} \setminus B_{2^{k-1}\delta(x)}$. For a single integral I_k , we have, similarly to how we derived (A.1),

$$\begin{aligned} I_k &\leq c \frac{1}{(2^k\delta_1)^{\mathbf{N}-1+k}} \int_{\mathcal{T}_k} d(y)^{k-2}\omega(y) dy \\ &\leq c \frac{1}{(2^k\delta_1)^{\mathbf{N}-1+k}} \int_{B_{2^k\delta_1}(x)} d(y)^{k-2}\omega(d(y)) dy \\ &\leq c \frac{1}{(2^k\delta_1)^{\mathbf{N}-1+k}} (2^k\delta_1)^{k+\mathbf{N}-2} \omega(2^k\delta_1). \end{aligned} \tag{A.2}$$

Next, (A.2) implies

$$I_{(k)} \leq c \frac{1}{2^k} \omega(2^k\delta_1).$$

Finally, similarly to the preceding calculations, we have

$$\begin{aligned} \sum_{\kappa=1}^{\infty} I_{(\kappa)} &\leq c \sum_{\kappa=1}^{\infty} \frac{1}{2^{\kappa}\delta_1} \omega(2^{\kappa}\delta_1) \leq \int_1^{\infty} (2^{\tau}\delta_1)^{-1} \omega(2^{\tau}\delta_1) d\tau \\ &= c \int_{2\delta_1}^{\infty} t^{-1} \omega(t) \frac{dt}{t} \leq c \frac{\omega(2\delta_1)}{2\delta_1} \leq c \frac{\omega(\delta_1)}{\delta_1}. \end{aligned}$$

□

Proof of Lemma 2.7 Estimates follow directly from Lemmas 2.4 and 2.2. □

Proof of Lemma 2.8 It goes over the same lines as the proof of Lemma 2.6. The integral in (2.10) can be represented as

$$\mathcal{J}(x) = \sum_{j \geq 1} \mathcal{J}_j(x) := \sum_{j=1}^{\infty} \int_{\mathcal{A}_j} d(y)^{\mathbf{r}-2} \omega(d(y)) |x - y|^{1-\mathbf{N}} dy,$$

where $\mathcal{A}_j$ is the spherical annulus,

$$\mathcal{A}_j = B_{2^j \delta}(x) \setminus B_{2^{j-1} \delta}(x)$$

For $j \geq 1$, $y \in \mathcal{A}_j(x)$, by (4.24), we have $|x - y|^{1-\mathbf{N}} \leq \delta(2^j \delta)^{1-\mathbf{N}}$, therefore,

$$\mathcal{J}_j(x) \leq c(2^j \delta)^{1-\mathbf{N}} \int_{\mathcal{A}_j(x)} d(y)^{\mathbf{r}-2} \omega(d(y)) dy.$$

Since the volume of $\mathcal{A}_j(x)$ is no greater than $C(2^j \delta)^{\mathbf{N}-1}$, we have

$$\begin{aligned} \mathcal{J}_j(x) &\leq c \delta (2^j \delta)^{1-\mathbf{N}} (2^j \delta)^{\mathbf{r}+\mathbf{N}-2} \omega(2^j \delta) \\ &= (2^j \delta)^{\mathbf{r}-1} \omega(2^j \delta). \end{aligned} \quad (\text{A.3})$$

The sum in (A.3) is, in fact, finite. it contains only terms with $2^j \delta \leq 1$, therefore $\mathcal{J}_j(x) \leq C(2^j \delta)^{-1}$. After we sum the expression in (A.3) over $j \geq 1$, we obtain

$$\begin{aligned} \mathcal{J}(x) &\leq \delta^{-1} \sum_j 2^{-j} \omega(2^j \delta) \\ &\leq c \delta^{-1} \int_0^\infty \frac{\omega(2^t \delta)}{2^t} dt \\ &= \delta^{-1} c' \int_1^\infty \frac{\omega(\tau \delta)}{\tau^2} d\tau \leq c'' \delta^{-1} \omega(\delta). \end{aligned}$$

□

Appendix B The Green function and its derivatives

B.1 General

In this appendix, we establish some estimates for the Green function $G(x, y; \varsigma)$ of a second order elliptic operator and for its derivatives in the unit ball $B \subset \mathbb{R}^{\mathbf{N}}$ with smooth boundary. Derivatives in the variables x, y are well studied; the estimates for first and second derivatives, needed for our applications, are contained in the classical paper [14]. We, however, need estimates for derivatives of a higher order; moreover, in Sect. 3 we consider operators whose coefficients depend on an extra parameter ς in

a small ball $\mathcal{B} \subset \mathbb{R}^N$, and we need estimates for derivatives of the Green function in all three variables.

The proofs will be presented further on in this Appendix. Since we consider only the unit ball, we omit the superscript $\circ$ in the notation of the Green function.

B.1.1 Schauder estimates

Note that the estimates of derivatives not involving ς are already contained in [18], see Theorem 2.9. So, it is only the derivatives G_ς , $G_{x\varsigma}$, $G_{\varsigma\varsigma}$, $G_{xx\varsigma}$, $G_{xy\varsigma}$, $G_{x\varsigma\varsigma}$ that we need to consider, with points x, y well separated. Namely, the point x should be in a small neighborhood of the centerpoint $\mathbb{O}$ of the ball, while y should be near some point y° on the boundary of the ball. Thus, the singularity of the Green function at $x = y$ is cut-away.

The result we are going to use systematically is a consequence of Schauder estimates. We formulate the particular cases of interest, for a ball $\mathcal{B}_r$ of radius $r < 1$, derivatives of order 2 ($k = 2$).

Theorem B.1 (Interior estimate) *Let $\mathcal{L}$ be a second order elliptic operator in B_r , with coefficients in the Hölder class $C^{k+\gamma}$, $0 < \gamma < 1$ and let $f \in C^{k+\gamma}(B_{\theta r})$, $0 < \theta < 1$. Then for the solution $u(x)$ of the equation $\mathcal{L}u = f$ in B_r the estimate holds*

$$|u|_{2+k+\gamma, B_{\theta r}} \leq C(|f|_{k+\gamma, B_r} + \|u\|_{C(B_r)}), \quad (\text{B.1})$$

where the notations as $|u|_{\gamma, B_r}$ denote the norm of u in the Hölder class C^{γ, B_r} etc., and the constant in (B.1) depends on the ellipticity constant of the operator $\mathcal{L}$, and Hölder norms of the coefficients and the parameter $\theta < 1$.

Thus, this interior Schauder estimate states that the solution of a second order elliptic equation has, in the Hölder scale, higher, by order 2, smoothness than the right-hand side, with norm controlled by the corresponding norm of f in a larger ball. For the proof, see [13], Theorem 6.2.

We also need the boundary Schauder estimate. We formulate it for the special case when the domain is the intersection of two balls. Let B be the unit ball and B_ϵ be a ball of radius ϵ , centered at a point $y^\circ \in \Gamma \equiv \partial B$. Denote by U_ϵ the intersection $U_\epsilon = B \cap B_\epsilon$ and by $U_{\epsilon/2}$ the intersection of B with the smaller ball with center y° .

Theorem B.2 (Boundary estimate) *Let $u(x)$ be a solution of the equation $\mathcal{L}u = f$ in $\mathcal{D} = U_\epsilon$ with $f \in C^\gamma$, such that $u(y) = 0$ on $\Gamma \cap B_r$. Then*

$$|u|_{k+2+\gamma, U_{\epsilon/2}} \leq C(|f|_{k+\gamma, \mathcal{D}} + \|u\|_{C(B \cap U_\epsilon)}).$$

This result follows immediately from the main local step in the proof of the standard boundary Schauder estimate for a domain having a piece of hyperplane as a part of boundary, by means of a change of variables straightening the boundary, see, e.g., [13], Corollary 6.7.

B.2 Proof of Theorem 2.11

Proof The proof consists of two steps. On the first step, we establish estimates for the first and second order derivatives of the Green function in ς variable; here we use Krasovskii's estimates. On the second step, by using Schauder estimates we perform a kind of bootstrap and derive estimates for derivatives involving variables x and y as well. Below, ∂_ς denotes the derivative with respect to any of variables ς .

We start by finding estimates for derivatives of the Green function $G(x, y; \varsigma)$ in ς variable. We have $\mathcal{L}(\varsigma)\mathbf{G}(\varsigma) = \mathbf{I}$, the latter is the identity operator. We differentiate this equality in ς variable; for the derivative of order one, we have

$$\partial_\varsigma \mathbf{G}(\varsigma) = -\mathbf{G}(\varsigma)\mathcal{L}_\varsigma \mathbf{G}(\varsigma); \quad \mathcal{L}_\varsigma = \partial_\varsigma \mathcal{L}, \quad (\text{B.2})$$

the latter is a second order operator with coefficients $\mathbf{a}_\varsigma = \partial_\varsigma \mathbf{a}(\varsigma)$ in C^{m-1} . For the Green function, (B.2) gives

$$\partial_\varsigma G(x, y; \varsigma) = - \int_{\mathcal{B}} G(x, z; \varsigma) \mathcal{L}_\varsigma(z, \partial_z; \varsigma) G(z, y; \varsigma) dz. \quad (\text{B.3})$$

We integrate in (B.3) by parts; the boundary term vanishing due to the Dirichlet boundary conditions for G , so

$$\partial_\varsigma G(x, y; \varsigma) = \int_{\mathcal{B}} \langle \mathbf{a}_\varsigma(z) \nabla_z G(x, z; \varsigma), \nabla_z G(z, y; \varsigma) \rangle dz. \quad (\text{B.4})$$

For estimating the integral in (B.4), we use (2.11), which gives

$$|\partial_\varsigma G(x, y; \varsigma)| \leq C \int_{\Omega} |x - z|^{1-N} |z - y|^{1-N} dz \leq C |x - y|^{2-N} \leq C,$$

since $|x - y| > r_0 > 0$.

With some more complications, we handle the second derivatives of G . We differentiate (B.3) again, which gives

$$\begin{aligned} \partial_{\varsigma\varsigma}^2 G(x, y; w) &= - \int_{\mathcal{B}} G(x, z; \varsigma) \mathcal{L}_{\varsigma\varsigma}(z, \partial_z; \varsigma) G(z, y; \varsigma) dz \\ &+ 2 \int_{\mathcal{B}} \int_{\mathcal{B}} G(x, z_1; \varsigma) \mathcal{L}_\varsigma(z_1, \partial_{z_1}; \varsigma) G(z_1, z_2; \varsigma) \mathcal{L}_\varsigma(z_2, \partial_{z_2}; \varsigma) G(z_2, y; \varsigma) dz_1 dz_2. \end{aligned} \quad (\text{B.5})$$

Having $|x - y| > \frac{3}{4}$, we consider three domains in $\mathcal{B} \times \mathcal{B}$:

$$\begin{aligned} \mathcal{D}_1 &= \{(z_1, z_2) : |z_1 - z_2| > \frac{1}{8}\}, \\ \mathcal{D}_2 &= \{(z_1, z_2) : |z_1 - z_2| < \frac{1}{4}, |z_1 - x| > \frac{1}{8}\}, \\ \mathcal{D}_3 &= \{(z_1, z_2) : |z_1 - z_2| < \frac{1}{4}, |z_2 - y| > \frac{1}{8}\}. \end{aligned}$$

These three domains, obviously, cover $\mathcal{B} \times \mathcal{B}$. We introduce the decomposition of unity, subordinated to this cover; $\psi_s(z_1, z_2)$, $s = 1, 2, 3$, $\psi_s \in C^\infty(\overline{\mathcal{B}} \times \overline{\mathcal{B}})$,

$\psi_s = 0$ outside $\overline{\mathcal{D}_s}$, $\sum_s \psi_s(z_1, z_2) = 1$. Order 1 and 2 derivatives of ψ_s are bounded. Correspondingly, the integral in (B.5) splits into three terms, with factors $\psi_s(z_1, z_2)$, $s = 1, 2, 3$. We consider each of them.

For $(z_1, z_2) \in \mathcal{D}_1$, we integrate once by parts both in z_1 and z_2 variables. We omit terms where the derivatives fall on the cut-off function or on the coefficients of $\mathcal{L}$, since these terms have a weaker singularity, and consider the most singular terms, which typically have the form

$$\mathcal{M}_1(x, y) = \int_{\mathcal{D}_1} \psi_1(z_1, z_2) \partial_{z_1} G(x, z_1; \varsigma) \partial_{z_1} \partial_{z_2} G(z_1, z_2; \varsigma) \partial_{z_2} G(z_2, y; \varsigma) dz_1 dz_2, \quad (\text{B.6})$$

with bounded coefficients. Due to the estimate (2.11), the middle term in (B.6) is bounded since $|z_1 - z_2| > \frac{1}{8}$ in $\mathcal{D}_1$, while two other terms have singularity $O|x - z_1|^{1-N}$, $O|x - z_1|^{1-N}$, with coefficients controlled by order 3 derivatives of the coefficients of operator $\mathcal{L}$. This implies that the term in (B.5), corresponding to ψ_1 , is bounded.

Next, we consider the integral over $\mathcal{D}_2$. We integrate by parts twice in z_1 variable and once in z_2 variable. Again, omitting less singular terms, where the derivatives fall on the cut-off function or coefficients, we arrive at the integrals of the form

$$\mathcal{M}_2(x, y) = \int_{\mathcal{D}_2} \psi_2(z_1, z_2) \partial_{z_1}^2 G(x, z_1; \varsigma) \partial_{z_2} G(z_1, z_2; \varsigma) \partial_{z_2} G(z_2, y; \varsigma) dz_1 dz_2. \quad (\text{B.7})$$

The first factor in the integrand in (B.7) is bounded, since here $|x - z_1| > \frac{1}{8}$, while the other terms, containing only order one derivatives, have singularities of order $1 - N$, thus, $\mathcal{M}_2(x, y)$ is bounded.

Finally, we consider the integral over $\mathcal{D}_3$. Here the last factor in (B.5) is bounded. We integrate by parts once in the second factor, and, ignoring again the terms where the derivative falls on the cut-off function or coefficients, we have, as the most singular term

$$\mathcal{M}_3(x, y) = \int_{\mathcal{D}_3} \psi_3(z_1, z_2) \partial_{z_1} G(x, z_1; \varsigma) \partial_{z_1} G(z_1, z_2; \varsigma) \partial_{z_2}^2 G(z_2, y; \varsigma) dz_1 dz_2. \quad (\text{B.8})$$

Here, the last factor is bounded and the first and the second one have singularity of order no greater than $1 - N$; all these terms have coefficients depending on derivatives of coefficients of order not higher than 3. Therefore, all three expressions (B.6), (B.7), (B.8) are bounded.

Next we pass to the study of derivatives of the Green function in x and y variables.

With y fixed, and x near the center of B , we consider the balls $B_r = B_r(0)$ with radius r less than $3/4$ and $B_{\theta r} = B_{\theta r}(0)$. For $x \in B_r$, we consider the functions $u_0(x) = G(x, y; \varsigma)$, $u_1(x) = \partial_{\varsigma} G(x, y; \varsigma)$, $u_2(x) = \partial_{\varsigma}^2 G(x, y; \varsigma)$.

As it is already proved, the functions u_0, u_1, u_2 are bounded in B_r . We also know that $u_0(x) \in C^3(\mathcal{D})$. After differentiating in ς the equality $\mathcal{L}(\varsigma)G(x, y; \varsigma) = 0$, $x \in B_r$, we see that $u_1(x)$ is a solution of the equation $\mathcal{L}u_1 = f_1$, with $f_1(x) = -(\partial_{\varsigma} \mathcal{L})u_0$. Thus, $f_1(x) \in C^{1,\gamma}(B_{r/2})$, moreover, $f_1(x) = 0$ on ∂B . We apply the Schauder

estimate to the equation $\mathcal{L}u_1 = f_1$, which gives an estimate for the x -derivatives of u_1 ,

$$|u_1|_{3+\gamma, B_{\theta r}} \leq C \left(\|u_1\|_{L_\infty(B_r)} + |f_1|_{1+\gamma, B_r} \right). \quad (\text{B.9})$$

The inequality (B.9) shows that the derivatives in x , up to the third order, are bounded in $B_{\theta r}$, in fact, they are order 2 Hölder better than f_1 .

In a similar way, we consider the function $u_2(x)$. It is a solution of the equation

$$\mathcal{L}(\varsigma)u_2(x) = f_2(x) \equiv -[(\partial_\varsigma^2 \mathcal{L}(\varsigma))u_0 - 2[\partial_\varsigma \mathcal{L}(\varsigma)]u_1]. \quad (\text{B.10})$$

We have just proved that $u_1 \in C^{2+k+\gamma}(B_{\theta r})$; so, by our assumptions on $\mathcal{L}$, we have $f_2 \in C^{k+\gamma}(B_{\theta 2r})$. Again, applying Schauder's estimates, we obtain an inequality, similar to (B.9), for second derivatives of u_2 .

In order to handle derivatives involving the differentiation in y , we just repeat the reasoning above for the function $v_0(y) = G(x, y; \varsigma)$ for a fixed $x \in B_r$ near the centerpoint $\mathbb{O}$ as a function of $y \in U_{(\epsilon)} = B_\epsilon(y^\circ) \cap B$. This function is a solution of the equation $\mathcal{L}(y, \partial_y; \varsigma)v_0(y) = 0$, and after differentiation in ς , we arrive at the equation $\mathcal{L}(y, \partial_y; \varsigma)\partial_\varsigma v_0 = -[\partial_\varsigma \mathcal{L}]v_0 \equiv g_1$. Now we can apply the boundary Schauder estimate, since $\partial_\varsigma v_0 = 0$ on the boundary, to obtain the estimate for $v_1 = \partial_\varsigma v_0$,

$$|v_1|_{2+\gamma, U_{(\epsilon/2)}} \leq C(\|v_1\|_{L_\infty(U_\epsilon)} + |g_1|_{\gamma, U_\epsilon}). \quad (\text{B.11})$$

The function g_1 on the right-hand side in (B.11) is bounded, uniformly in $x \in B_r(\mathbb{O})$, moreover, has bounded derivatives in x , by Theorem 2.9, since x, y are separated. This gives the required estimate for v_1 .

Finally, we consider the mixed derivative, $G_{\varsigma xy}(x, y; \varsigma) = \partial_x \partial_y \partial_\varsigma G(x, y; \varsigma)$ of the Green function, for $x \in B_r$ and $y \in U_\epsilon$.

We apply again Schauder estimates, this time in the following setting. The function $v_2(x) = \partial_y G(x, y, \varsigma)$ satisfies $\mathcal{L}(x, \partial_x; \varsigma)v_2 = 0$. We differentiate the latter equality in ς and for $v_3 = \partial_\varsigma v_2$ obtain the equation

$$\mathcal{L}(x, \partial_x; \varsigma)_x v_3 = g_2 = -[\partial_\varsigma \mathcal{L}(x, \partial_x; \varsigma)] \quad (\text{B.12})$$

The right-hand side in (B.12) belongs to $C^{k-2+\gamma}$ uniformly in y variable in U_ϵ , therefore, v_3 belongs to $C^{k+\gamma}$ in x variable which implies the required estimate

$$G_{\varsigma xy}(x, y; \varsigma) \leq C.$$

In a similar way, we can establish this kind of estimates for higher order derivatives in x, y , provided the coefficients are sufficiently smooth. $\square$

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