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Portfolio-Level Reliability Analysis of a Multi-Asset System: A Frailty Proportional Intensity Modeling Framework

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ABSTRACT

Understanding and modeling the equipment failure behavior is crucial for reducing downtime, optimizing maintenance strategies, and improving operational safety. This challenge becomes particularly critical in industrial contexts involving multi-asset portfolios, where heterogeneous conditions influence the reliability of individual assets and, consequently, the system as a whole. This study proposes a structured, data-driven framework for reliability analysis of multi-asset portfolios in large-scale operations. Building on this framework, a frailty-based proportional intensity model is applied to a real-world case study of a mining transportation system consisting of 81 trucks operating in different sites with distinct operational and environmental conditions. Model parameters are estimated using penalized partial likelihood, and model adequacy is evaluated using likelihood ratio tests (LRT), Akaike Information Criterion (AIC), and Schoenfeld residual diagnostics. The LRT results indicate that the frailty term is significant ($\theta = 0.50$; $p < 0.001$), confirming the presence of unobserved heterogeneity and supporting the use of the frailty-based model. The analysis identifies key risk factors, including operator skill, working shift, haulage distance, slope direction, and road condition, as major drivers of failure in the case study. The proposed approach provides an improved model fit and a more realistic representation of failure processes in heterogeneous multi-asset systems. These findings also provide practical guidance for managers by enabling targeted interventions, such as operator training, road condition improvement, and haulage route optimization, to reduce failure intensity and enhance system reliability. This study contributes to the reliability literature by extending frailty-based modeling to portfolio-level reliability assessment under heterogeneous conditions. This integration promotes a system-wide perspective on reliability analysis and decision-making beyond traditional single-asset approaches.

1 | Introduction

Ensuring system reliability and optimizing maintenance strategies are essential for minimizing downtime, maintaining productivity, and improving operational safety. Unlike non-repairable systems, repairable systems experience recurrent failures and multiple repair events over their operational lifecycle. This recurrent nature of failures introduces unique challenges for modeling and decision-making due to the wide range of factors that influ-

ence system reliability over time [1], such as the system's repair history and the quality of maintenance actions [2]. Therefore, failure behavior cannot be fully captured by single-event survival models; instead, statistical modeling must account for the repetition of failure events and the dependencies introduced by repair actions. Reliability modeling of repairable systems is further complicated in multi-asset portfolios, such as populations of aircraft, transportation systems, and wind farms, where individual assets operate under heterogeneous conditions and organizations

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must manage them simultaneously [3, 4]. Multi-asset systems are typically categorized as fleets (homogeneous assets) or portfolios (heterogeneous assets) [3]. Within portfolios, differences in environmental exposure, operational demands, maintenance policies, and intrinsic system characteristics often result in significant variability in failure behavior among assets [5, 6]. Given the variable impact of individual asset reliability on overall system performance, traditional asset-level approaches are often insufficient. For instance, in a real industrial portfolio, repairable systems typically experience different types of repairs and maintenance throughout their life cycles. Depending on operational conditions and resource availability, these systems may receive a combination of minimal, imperfect, and perfect repair actions and are therefore considered heterogeneously repaired systems [7]. Moreover, when repairable systems are embedded in portfolios, various forms of dependency, including resource, performance, technical, economic, and stochastic, can arise. These interdependencies cause collective behaviors that deviate from unit-level predictions. Dependencies and interactions within multi-asset systems strongly influence maintenance decision-making [8]. In response to this complexity, asset management has evolved toward a holistic, system-wide perspective that incorporates this variability in asset portfolios [3]. Population-based modeling approaches enable the integration of knowledge across assets and allow patterns to be observed at the portfolio level, supporting more robust and consistent maintenance decisions [9, 10]. While earlier research has primarily focused on individual assets, recent studies have increasingly addressed the complexity of multi-asset systems in maintenance planning [11]. Industrial practice is also shifting from single-asset approaches toward multi-asset applications, where indicators at both the asset and portfolio levels are utilized to improve availability, adaptability, and efficiency while reducing costs and resource consumption [8, 12]. Consequently, a portfolio-level perspective has gained increasing attention in both industry and academia [13]. This shift highlights the need for heterogeneity-aware portfolio-level analyses to support effective reliability assessment and decision-making [14, 15]. To handle heterogeneity within a portfolio, clustering-based approaches have been applied to group assets exhibiting similar failure patterns. These methods approximate all reliability distributions in each cluster by a single representative [16, 17]. Additionally, within survival analysis, frailty models, as an extension of Cox models, were proposed to capture heterogeneity and correlation by introducing random effects. This approach allows the model to account for differences between assets or clusters, recognizing that individuals within the same cluster may share common risk factors/covariates that are not directly observed [18]. Frailty models have become increasingly popular in engineering due to their ability to incorporate both observed and unobserved sources of heterogeneity.

Despite recent methodological advances, important gaps remain in the reliability literature on portfolio-level frailty modeling. First, most prior empirical studies have focused on single assets [19] or small-scale fleets [20, 21], which limits the system-wide perspective required for managing large and heterogeneous industrial portfolios. Second, several valuable contributions remain mainly methodological [6, 22, 23]. Third, existing applications often address either observable risk factors, such as operational, technical, and environmental conditions [24, 25], or unobserved heterogeneity [1, 20] in isolation. This lim-

its their ability to capture the combined effects of observed and unobserved drivers of recurrent failures in multi-asset repairable-system portfolios. A few wind energy applications involve multiple turbines and, therefore, may be interpreted as portfolio-like settings. However, the portfolio-level perspective is not explicitly developed in these studies [26, 27]. Comparable empirical demonstrations in mining transportation systems are even more limited. The closest prior work analyzed trucks across three mining sites but considered trucks of a similar type and clustered the data at the site level, as acknowledged in the study's limitations [28]. The key distinction from this study is the clustering structure. In the earlier research, which used a dataset of a single truck model, shared frailty was assigned at the mine level. As a result, trucks operating within the same mine shared a common frailty effect, limiting the extent to which residual differences between individual trucks could be represented. As a result, empirical demonstrations of such approaches in large-scale, real-world industrial portfolios remain limited. The present study addresses these conceptual and empirical gaps by extending previous frameworks and contributes by operationalizing the portfolio perspective from asset management [3] to portfolio-level reliability analysis. Rather than focusing on single assets, homogeneous fleets, or site-level clusters, this study considers a heterogeneous portfolio of repairable systems comprising different dump truck models operating in multiple mining sites and under distinct environmental and operational conditions. This approach allows trucks within the same mine to exhibit different failure behavior and explicitly accounts for dependence among repeated failures of the same truck after adjusting for observed covariates. The broader empirical coverage of the present study also supports this more detailed analytical structure. To the best of the authors' knowledge, this study analyzes the largest field-collected dataset reported to date for portfolio reliability modeling in mining transportation. The dataset comprises 81 dump trucks of three different models operating in three mining sites. This provides a unique empirical basis for evaluating frailty-based reliability models at the portfolio level and for examining how observed risk factors and unobserved heterogeneity jointly influence recurrent failures.

This study contributes to the field by adapting and extending reliability modeling logics from the literature [28–31] to portfolio-level reliability analysis. It provides structured, data-driven frameworks that support analysts in testing for heterogeneity and in selecting, applying, and evaluating suitable reliability models at the portfolio level. Therefore, this study contributes to the reliability engineering literature in three ways:

- It introduces a heterogeneity-aware, data-driven framework for modeling reliability in multi-asset portfolios, integrating both observed and unobserved factors.
- It empirically demonstrates the framework's applicability through a real-world case study of a mining transportation system in which an organization manages a large group of heterogeneous assets operating in multiple sites under diverse environmental, operational, and logistical conditions.
- From a practical point of view, the findings provide actionable insights by identifying key modifiable risk factors that significantly influence equipment failure occurrence.

The remainder of this paper is structured as follows. Section 2 reviews related literature, and Section 3 introduces the proposed analytical framework. Section 4 describes the case study context, including the portfolio of dump trucks, operational environment, and available data. Section 5 presents the results, and Section 6 discusses the implications of the findings for portfolio reliability. Finally, Section 7 concludes by summarizing the contributions, outlining limitations, and suggesting directions for future research.

2 | Related Literature on Frailty Models in Reliability Engineering

Statistical approaches, such as the Cox proportional hazard model [32], have been extensively applied in reliability engineering to investigate the effect of operational and environmental risk factors on system failure [25, 33–35]. However, these methods often assume independence between failure events or homogeneity across systems, which can lead to biased estimates in the presence of clustering or heterogeneity. Frailty models [18] have long been used in survival analysis and biostatistics. In the reliability domain, several studies have adopted frailty models to highlight the importance of incorporating heterogeneity. For instance, Finkelstein [22] discussed heterogeneity in the context of system repairs. Cha and Finkelstein [6] studied preventive maintenance (PM) for non-repairable components within a heterogeneous population and investigated PM's effect on failure rates. They showed that neglecting existing heterogeneity leads to overestimation of the failure rate in components undergoing PM. Slimacek and Lindqvist [23] applied heterogeneity without the presence of observed covariates to a simulated dataset using a non-homogeneous Poisson process (NHPP), and in another study [27], extended that model to include observed covariates. In line with their work, they analyzed the reliability of wind turbines under both observed and unobserved heterogeneity [36]. The study by Liu et al. [37] proposed an enhanced age-based model by introducing a frailty component to capture variability among systems undergoing imperfect repairs. Their findings revealed that ignoring heterogeneity led to an underestimation of the aging rate and an overestimation of repair effectiveness. Their case study on Norwegian railway signaling systems further validated the model, showing that incorporating heterogeneity resulted in more realistic survival functions and cost-effective maintenance schedules. Allahkarami et al. [38] investigated the combined impact of observed and unobserved risk factors on the reliability of a mining hauling system, focusing on different types of dump trucks. While accounting for measurable variables, the study showed a significant role of unobserved heterogeneity, such as differences in equipment types, in shaping failure patterns. The findings indicated that ignoring hidden risk factors can lead to biased reliability estimates and less effective maintenance planning. Allahkatami et al. [28] proposed a modeling framework in their work. They suggested a frailty modeling approach to failure data from three fleets of dump trucks working under diverse environmental and operational conditions. The research emphasized the effect of site-specific conditions and variability on failure behavior. Using a Weibull model with a frailty term, Zaki et al. [19] addressed latent variables that influence the reliability of excavators in their case study. Brown et al. [39] compared inverse Gaussian and gamma frailty models for hetero-

geneous repairable systems to analyze component-level failure occurrences. Brito et al. [4] presented a framework for analyzing dependent failure data in multiple repairable systems within the context of the perfect repair (PR) assumption. Specifically, they utilized frailty models to estimate the reliability function while accounting for unobserved heterogeneity and correlations among failure times. The findings emphasized the critical role of accurately characterizing the dependence structure among failures by selecting a distribution for the frailty term, thereby improving the estimation of model parameters. The suggested method is mainly helpful when parts are replaced after they fail. Authors in [20, 40] extended the modeling of multiple repairable systems by proposing a class of frailty models that simultaneously account for imperfect repairs (IR) and unobserved heterogeneity in failure processes. Their work generalizes existing models by incorporating both the arithmetic reduction of age and intensity classes of IR. In another study, Gonzatto et al. [41] developed a frailty-based statistical model for analyzing failure data in hierarchically structured, repairable systems subject to competing risks. The model was applied to a dataset of agricultural tractors, considering unobserved heterogeneity and imperfect repairs by the ARA framework. Results showed variation in repair effectiveness among groups and the significance of frailty in certain system clusters.

Frailty modeling has also extended into adjacent reliability problems, including predicting the maintainability of trucks [42], estimating remaining useful life (RUL) for an excavator [43] and estimating spare parts [44] under heterogeneous conditions.

The reviewed literature demonstrates that frailty models provide a flexible approach for representing heterogeneity in reliability analysis. However, empirical demonstrations at the portfolio level remain limited, particularly in mining transportation systems. This leaves a gap in understanding how frailty-based models can be applied to heterogeneous portfolios composed of different asset models operating across multiple sites and conditions. This gap motivates the present study, which adapts this approach to the portfolio level and empirically examines a heterogeneous collection of mining transportation assets across multiple sites.

3 | Methodological Framework

This study provides a practical framework for modeling the reliability and failure intensity of multi-asset systems and guides the selection of an appropriate modeling approach. However, the proposed framework emphasizes choosing a modeling path aligned with the structure of the collected data and system definition, focusing on three main paths:

1. Time-based models (e.g., power-law process),
2. Time + covariates-based (e.g., Cox models)
3. Time + covariates + heterogeneity-based (e.g., frailty or random-effect extensions)

In practice, this framework operates as a staged modeling sequence. Each of the three modeling paths branches into parametric, non-parametric, and semi-parametric estimation

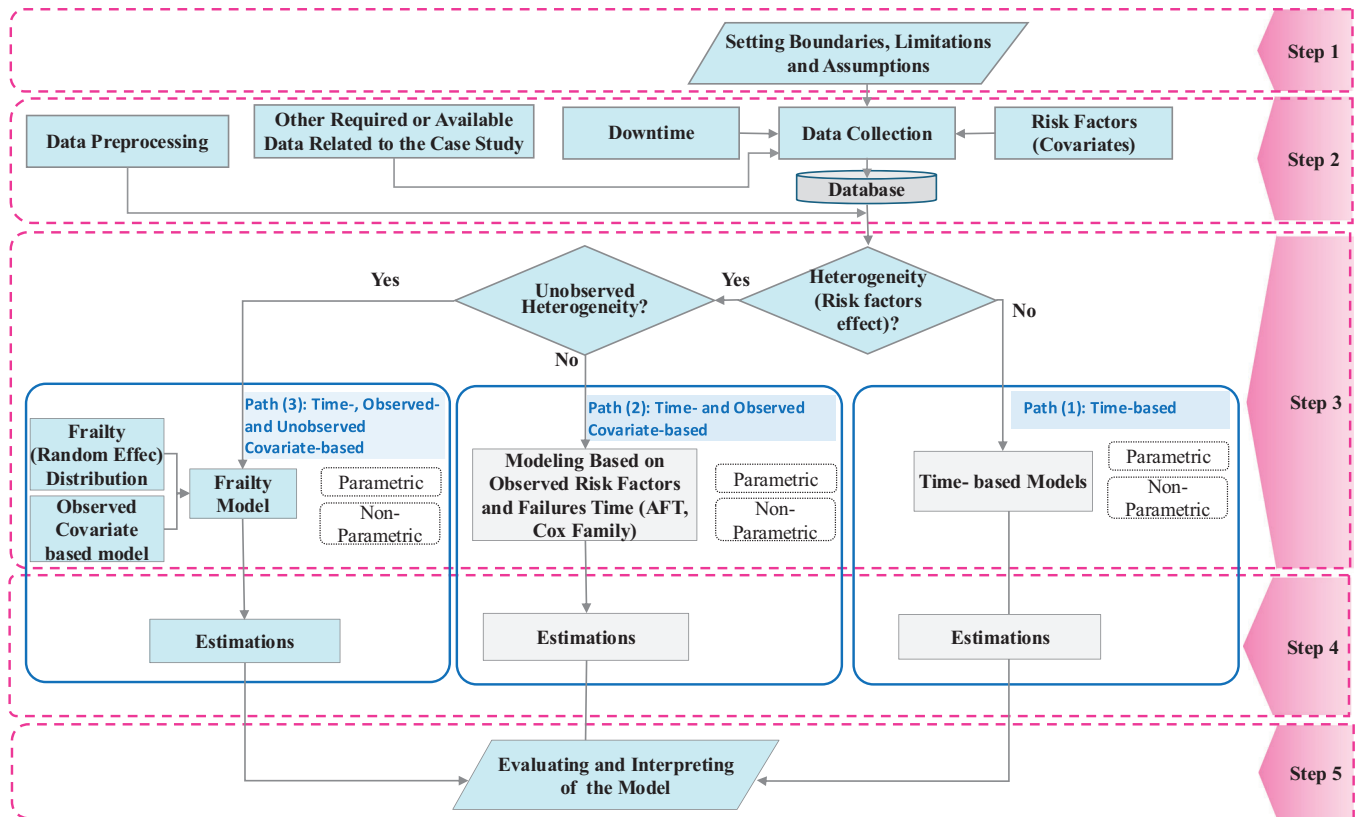


FIGURE 1 | Proposed framework for reliability and failure analysis of repairable multi-asset systems.

approaches, giving different possible combinations. Figure 1 summarizes the process, which consists of five main steps:

3.1 | Step 1: Problem Definition and Scope Delimitation

This step establishes the boundaries of the study by identifying system components, understanding operational contexts, and defining the fundamental assumptions for reliability assessment. The goal is to characterize a multi-asset portfolio and clarify the decision-making objectives for maintenance and asset management.

3.2 | Step 2: Data Collection and Preprocessing

Data is the core for reliability analysis. Step two focuses on collecting, preprocessing and structuring the dataset for the next steps to facilitate accurate analysis and effective decision-making. The collected data can be organized into two main categories:

- Failure time data which includes recurrent failure events for each unit/asset and serves as the basis for estimating reliability and failure intensity functions.
- Risk factors data, which capture operational, environmental, maintenance effects and intrinsic system characteristics that potentially influence failure mechanisms.

3.3 | Step 3: Data Characterization and Modeling Path Selection

The characteristic of the dataset plays an important role in selecting the appropriate analytical path. Following data collection and preprocessing, a preliminary assessment is conducted to evaluate the extent to which failure times among assets are influenced by risk factors and the recurrent nature of failures. If failure times are not significantly impacted by risk factors such as operational and/or repair factors, the data are classified as homogeneous. In such cases, time-based reliability models can be applied (Path 1, Figure 1). When variability in failure pattern is detected across units, indicating the influence of risk factors, the dataset is treated as heterogeneous. This heterogeneity can be partially attributed to measurable covariates such as ambient conditions. In these examples, two modeling paths arise. If observed covariates sufficiently account for the variability in failure times, a covariate-based model such as proportional intensity or a Cox-type model is appropriate, as it accommodates observed risk factors (Path 2). If unexplained variation remains, especially due to unknown, unmeasured, or clustered influences (e.g., varying maintenance regimes or operational management), a frailty model is proposed (Path 3).

The frailty model offers an approach for modeling failure times in repairable systems by treating each asset as a distinct cluster and its associated failure events as members within that cluster. Through this structure, the frailty term captures unobserved factors that may influence failure patterns. The results provide insights into possible correlations among recurrent failures,

thereby accounting for unobservable heterogeneity and dependence [4].

3.4 | Step 4: Estimation

After selecting the modeling path, estimation proceeds according to that approach. Reliability models can be parametric, nonparametric, or hybrids thereof and therefore require different estimation approaches. Nonparametric and semiparametric approaches can estimate reliability without strict distributional assumptions which gives more flexibility.

- Non-parametric approach:

The primary nonparametric tool is the Mean Cumulative Function (MCF), which describes the expected cumulative number of failures per unit up to time t . Nonparametric estimation of the MCF is widely used for repairable systems because it requires no parametric assumptions while providing a clear graphical estimate of the population-average failure accumulation over time [45, 46]. The nonparametric population-based estimation is obtained by aggregating the cumulative history functions across all assets [47, 48].

- Parametric approach:

Parametric forms of counting process models have been widely applied to repairable systems [45]. Key modes include the Renewal Process (RP), which is commonly associated with perfect repair, and the Non-Homogeneous Poisson process (NHPP), which models the failure intensity under a minimal repair assumption [4]. In practice, Weibull-based renewal models and the power-law process (PLP) within the NHPP family are among the most frequently used intensities because they are flexible and can represent a range of failure-time behaviors. Parametric approaches assume specific distributions or intensity functions for event times, which supports likelihood-based estimation (e.g., maximum likelihood) and prediction of reliability metrics [4, 49]. Parametric intensity models can be further extended to account for observed risk factors and heterogeneity through proportional intensity models, in which a parametric baseline is combined with a covariate component [50].

- Semi-parametric approach:

Semi-parametric approaches, most notably the Cox-family models for recurrent failures, specify a parametric form for risk factors' effects while leaving the baseline intensity unspecified. Estimation in these models is typically based on partial likelihood, enabling risk factor coefficients to be estimated without assuming a particular baseline intensity. The baseline cumulative intensity is then obtained through a nonparametric estimator (e.g., Breslow or Nelson-Aalen). In semi-parametric frailty models, the penalized partial likelihood (PPL) modifies the partial likelihood by adding a penalty term for the frailty distribution [51, 52].

For example, in Path 2 (time + covariates) with a parametric model such as power law model, we estimate both the intensity

parameters and the regression coefficients for the observed covariates.

3.5 | Step 5: Model Evaluation and Interpretation

This step evaluates fit, predictive performance, and assumptions for candidate models. Graphical diagnostics, residual analyses, and tests of key assumptions such as proportional intensity are applied. Models can be assessed using Likelihood Ratio Test (LRT) and the Akaike Information Criterion (AIC). The objective is to select the model that reproduces empirical patterns and achieves the best balance of accuracy and simplicity.

Teal elements mark the path adopted in this study (i.e., path 3). The details of the modeling and estimation will be presented in Section 4.3.

4 | Case Study

4.1 | System Description and Data Collection

Simultaneous analysis of units enhances the detection of failure patterns that individual analyses might overlook [5]. In broader asset management structures, the portfolio concept extends to managing a diverse set of assets in a way that optimizes overall performance. The term emphasizes a strategic, holistic view of asset management rather than focusing solely on individual assets. The portfolio concept supports multi-asset decision-making, allowing managers to allocate resources, prioritize investments, and plan maintenance based on the overall composition and performance of the entire asset collection [3]. Therefore, considering the advantages of simultaneous system analysis, this approach is adopted in the present study. Conceptual representation of the portfolio in this study is shown in Figure 2.

In alignment with the objective of this study, modeling the reliability of a heterogeneous multi-asset system, a portfolio of dump trucks from a mining holding company, operating in three active mining sites, was collected over a two-year period. This company supplies the required fleet for various mining operations. The selection criteria for these mines were as follows:

- Different climatic conditions
- Different mineral types
- Accessibility for data collection and the possibility of conducting expert interviews

According to the domain experts consulted during the study, the truck type deployed at a mine is generally constrained by client (i.e., mine management) requirements, mine design, and compatibility with the existing loading and haulage system. Consequently, truck type is often not a freely selectable variable for this mining holding company (i.e., portfolio owner) in the most common operating scenario. When the truck type is fixed, a more practical question for the service provider is which specific trucks of that type, and in what combination, should be

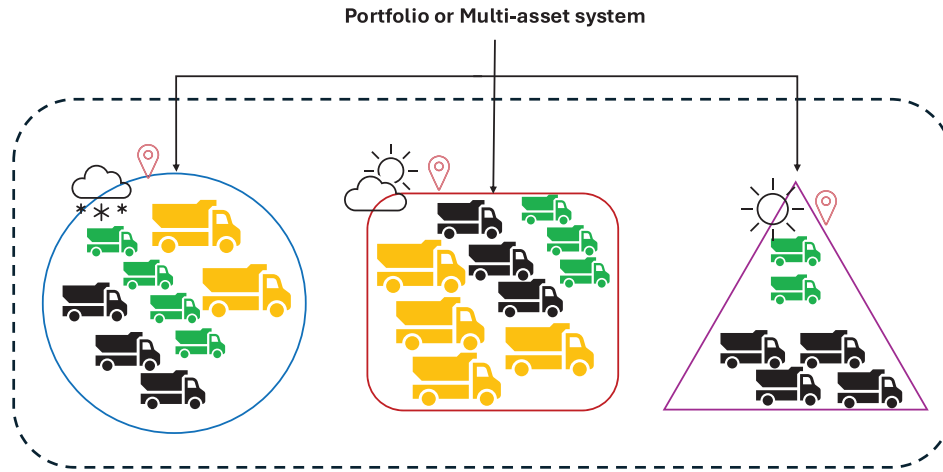


FIGURE 2 | Conceptual representation of the portfolio in this study.

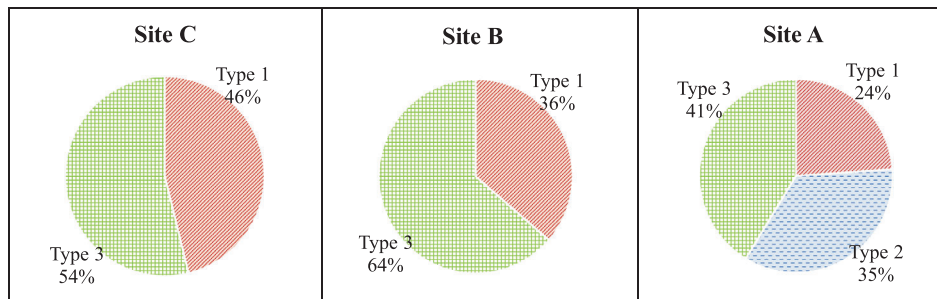


FIGURE 3 | The distribution of dump trucks operating in the studied mines, categorized by truck type.

assigned to fulfill operational needs or determine the necessary maintenance level.

During this timeframe, 81 dump trucks from three different models/types were involved in haulage operations and formed the truck portfolio analyzed in this study. The composition of the portfolio is summarized in Figure 3. The distribution of truck types was uneven across the three mining sites, and truck type 2 was found at only one site.

During the two years observation period, a total of 20,045 recurrent failure events were recorded across the portfolio. Each truck experienced multiple failures during follow-up. If a truck was still in operation at the end of the study window or was lost to observation earlier, its last recorded operating time was labeled as right censored in the status column. This resulted in 81 right-censored observations.

In this study, operator skill was evaluated using a structured assessment form (Figure 4). This form included variables such as age, education level, work experience, and other relevant criteria. The assessment framework was inspired by the company's internal operator testing procedures. Based on the collected data and expert judgment, operators were qualitatively evaluated and ranked based on a four-point scale, with rank "4" representing the highest level of skill (Excellent/Expert).

The operator assignments were recorded for each operational shift, with each operator identified by a unique personnel num-

ber. The operator skill level was extracted from the assessment form and linked to the corresponding failure interval using the shift code, date, truck ID, and personnel number. The personnel identifiers were then anonymized.

Moreover, to evaluate the influence of environmental conditions on system reliability, regional meteorological data such as temperature, wind velocity and precipitation were collected from the nearest weather stations to the selected mining sites. These data were then matched to the corresponding operational records by calendar date. As an example of collected environmental data, the temporal temperature profile is presented in Figure 5, with the first month corresponding to March.

Operational and environmental covariates, such as shift, operator, road condition, haulage, and other recorded conditions, were defined at the recurrent event level. Their values were linked to the corresponding failure records and allowed to vary across successive failure intervals for the same truck. The recurrent event histories were structured in a start-stop counting process format and then incorporated into the proportional intensity model.

Furthermore, operation and maintenance data were collected from the selected mine sites and the main office. Maintenance actions following a recorded failure were carried out based on the type and severity of the failure and could include component repair, replacement of damaged parts, welding or structural repairs, tire-related interventions, or a combination of these.

Row no.	Mine site	Personnel Code	Operator Name	Relevant Work Experience	Education	Attentiveness	Safety Compliance During Work	Start-of-Shift Equipment Inspection	Familiarity with the equipment	Operating Proficiency	Teamwork Spirit
1											
2						1					
3						2					
4						3					
5						4					
6											
7											
8											
9											
10											
11											

FIGURE 4 | Operator skill assessment form.

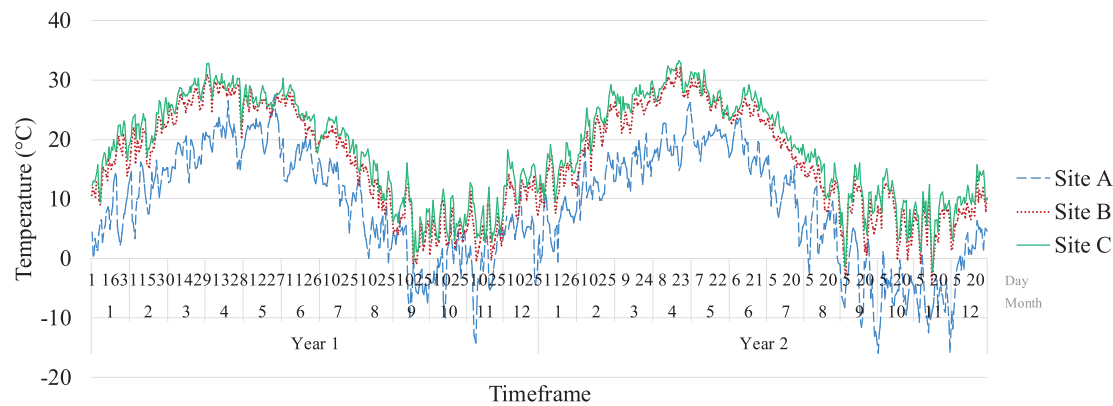


FIGURE 5 | Two-year temperature profile at the studied sites.

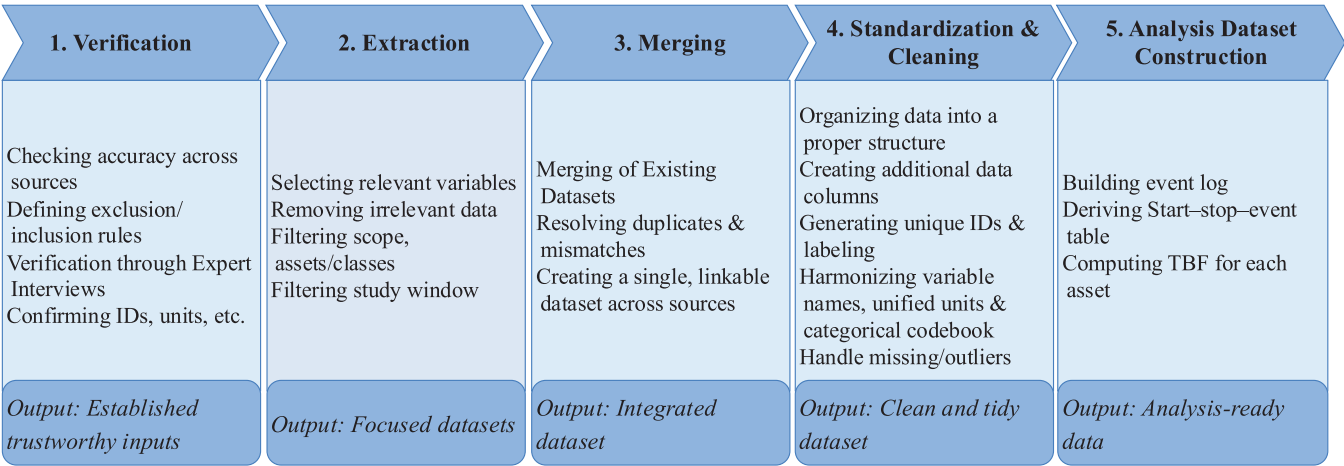


FIGURE 6 | Data preprocessing workflow.

All of these datasets were initially preprocessed to extract relevant information for reliability analysis. This process involved ensuring consistency across multiple data sources and verifying the accuracy of recorded values. The preprocessing workflow is shown in Figure 6.

4.2 | Selected Risk Factors for the Modeling Process

Based on the dataset and purpose of the study, a set of risk factors was selected for further analysis. These factors were categorized into three groups according to their nature: binary, categorical,

and continuous. Table 1 illustrates the hierarchical structure of the risk factors considered in the modeling process.

As the primary objective of this study is to quantify the effects of measurable operational and environmental conditions on recurrent failure of the dump trucks, while accounting for unobserved heterogeneity among individual trucks, the analysis focuses primarily on covariates with direct operational and environmental interpretation rather than on the identity (ID) of a particular mine. Therefore, mine ID is not a primary factor of interest in this case study, as a site label represents a broad combination of operational and environmental factors. This approach provides effects with direct operational interpretation beyond the specific

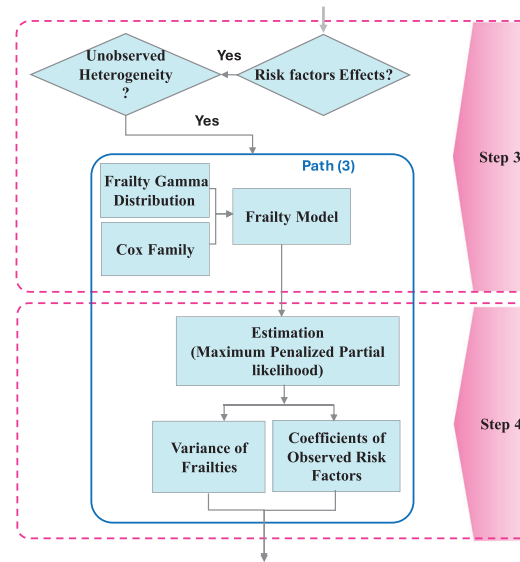
TABLE 1 | Structured breakdown of risk factors used in the analysis.

Type	Risk factor	Levels of risk factor	Code in the database	Label in the model	Description
Categorical	Shift	Morning	1	X_{11}	Based on the assessment form
		Afternoon	2	X_{12}	
		Night	3	X_{13}	
	Operator skill	Beginner	1	X_{21}	
		Competent	2	X_{22}	
		Proficient	3	X_{23}	
		Expert	4	X_{24}	
	Route slope direction	Flat	0	X_{31}	
		Uphill	1	X_{32}	
		Downhill	-1	X_{33}	
Binary	Road condition	Poor/bad	0	X_{41}	Indicates road surface quality, based on the expert daily inspections
		Normal	1	X_{42}	
	Precipitation	No	0	X_{51}	Presence of precipitation, extracted from the weather database
		Yes	1	X_{52}	
Continuous	Haulage distance	—	Numeric	X_6	From loading point to dumping point (in Kilometers)
	Temperature	—	Numeric	X_7	Average daily temperature at the site (in Centigrade)
	Wind velocity	—	Numeric	X_8	Average daily wind velocity at the site (in Meters per hour)

site labels in the present dataset, while acknowledging that unmeasured site-related heterogeneity may remain.

4.3 | Selected Model

Given the structure of the case study, featuring multiple assets working in different operational contexts, the dataset exhibits non-negligible heterogeneity. Consequently, the analytical focus of this research is directed toward the frailty-based modeling framework (Path 3), which is particularly well-suited for analyzing multi-asset repairable systems operating under varying conditions. In practice, path selection is operationalized through a sequence of statistical checks. Path 3 is selected when both covariate effects and a frailty term are statistically supported, indicating that residual unobserved heterogeneity remains after accounting for observed risk factors. Analysts should apply the LRT, AIC, and graphical diagnostics when making the final selection. In this study, a semi-parametric frailty modeling approach is adopted to analyze recurrent failure data. This approach combines the flexibility of the Cox proportional intensity model with a frailty component to account for unobserved heterogeneity among assets, treating each asset as a cluster of recurrent failures. In this formulation, the baseline intensity function is modeled non-parametrically, without assuming a specific distribution, while the covariate effects are modeled parametrically and the frailty term is assigned a specified probability distribution. Parameters are estimated using the Maximum Penalized Partial

**FIGURE 7** | Modeling pathway for the case study.

Likelihood (PPL) method, enabling simultaneous estimation of risk factors' effects and frailty variance. Based on the selected approach, the overall modeling pathway for steps 3 and 4 is summarized in Figure 7. The detailed formulation of the model and estimation procedures are presented in the following section.

4.3.1 | Mathematical Modeling of the Failure Intensity Function

As previously mentioned, this study aims to model the failure intensity and reliability of a multi-asset portfolio operating in different locations. It should be emphasized that each asset may experience recurrent failure events over time and remains assigned to a single site throughout the observation period, with no transfers between locations. In the proposed framework, all recurrent failures associated with a given asset are treated as a cluster. Consequently, its observed failure event (T) or censoring time (C) corresponds to a member within that cluster. Let T_i^* and C_i^* denote the random variables representing the failure and censoring times for the i^{th} event, respectively, and let δ_i be the failure indicator, taking the value 1 if a failure is observed and 0 otherwise (in case of censorship). Considering q clusters, where q is equivalent to the number of assets, with n_j observations in the j^{th} cluster ($n = \sum_{j=1}^q n_j$), the intensity function for the i^{th} observation in cluster j is defined as [52]:

$$\lambda_{i(j)}(t) = \lambda_0(t) e^{X_i \beta + Z_i \omega} \quad (1)$$

where X_i and Z_i are the i^{th} rows of the risk factor matrices $X_{(n \times p)}$ and $Z_{(n \times q)}$, respectively.

X and β refer to the effects of the p observed risk factors, and ω is a vector containing the q random effects associated with unobserved heterogeneity in the model. Z is an indicator variable matrix such that if observation i belongs to cluster j , then $Z_{ij} = 1$; otherwise, this value is zero.

If $\varpi_j = \exp(\omega_j)$ refers to the frailty associated with cluster/asset j , then the above equation can be rewritten as follows:

$$\lambda_{i(j)}(t) = \varpi_j \lambda_0(t) e^{X_i \beta} \quad (2)$$

Equation (2) can be interpreted as a frailty model with two sources of variation, represented by the random variable ϖ with probability density function g , which accounts for unobserved heterogeneity, and covariate-driven variability, modeled by the β , which reflects the influence of observed risk factors on the failure times of individual assets.

4.3.2 | Distributional Assumptions for the Frailty Term

From both analytical and computational viewpoints, Gamma distribution has proven to be an effective choice for modeling frailty in survival and reliability applications [53]. Consequently, it is widely used in the literature and is adopted in this study as well.

Let $\varpi > 0$ be a random variable representing frailty, assumed to follow a Gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\nu > 0$. The probability density function (PDF) of ϖ is given by:

$$g(\varpi) = \frac{\nu^\alpha \varpi^{\alpha-1} e^{-\nu \varpi}}{\Gamma(\alpha)}, \varpi > 0 \quad (3)$$

where:

$$\Gamma(\alpha) = \int_0^\infty \varpi^{\alpha-1} e^{-\nu \varpi} d\varpi \quad (4)$$

The expectation and variance of ϖ are:

$$E(\varpi) = \frac{\alpha}{\nu} \quad (5)$$

$$\text{Var}(\varpi) = \frac{\alpha}{\nu^2} \quad (6)$$

To simplify interpretation and ensure identifiability, the frailty distribution is often parameterized such that $E(\varpi) = 1$ and $\text{Var}(\varpi) = \theta$, where θ quantifies the degree of unobserved heterogeneity. This constraint implies $\alpha = \nu = 1/\theta$, leading to a one-parameter Gamma frailty model with PDF:

$$g(\varpi; \theta) = \frac{\varpi^{\frac{1}{\theta}-1} \exp\left(-\frac{\varpi}{\theta}\right)}{\Gamma\left(\frac{1}{\theta}\right) \theta^{\frac{1}{\theta}}}, \theta > 0 \quad (7)$$

A larger value of θ indicates greater unobserved heterogeneity and stronger within-cluster correlation. In contrast, $\theta = 0$ reduces the model to a standard Cox model, indicating no excess correlation or unobserved risk. Furthermore, the magnitude of the frailty term ω_j can be interpreted as follows [53]:

- $\omega_j > 1$: Indicates that asset j is more failure-prone (weaker) than average.
- $\omega_j < 1$: Indicates that asset j is more robust (less prone to failure) than average.

4.3.3 | Parameter Estimation of Frailty Model

4.3.3.1 | Penalized Partial Likelihood Estimation. The penalized partial likelihood (PPL) function consists of two components:

1. The partial likelihood component, based on observed failure times.
2. A penalty term that incorporates the frailty distribution.

This estimation approach follows an iterative scheme. First, the partial likelihood is maximized, and then the frailty estimates are updated using the penalized likelihood [52]. In this model, each event belongs to a single cluster. Given a fixed frailty variance parameter θ , the penalized log-partial likelihood function (Equation (8)) is derived. Estimates for the regression coefficients β and the frailty effects ω are obtained by differentiating the function with respect to both parameters:

$$PPL(\beta, \omega, \theta) = PL(\beta, \omega) - g(\omega; \theta) \quad (8)$$

where $PL(\beta, \omega)$ is the log partial likelihood of the Cox model:

$$PL(\beta, \omega) = \sum_{i=1}^n \int_0^\infty [Y_i(t)(X_i \beta + Z_i \omega) - \log\{Y_k(t) \exp(X_k \beta + Z_k \omega)\}] dN_i(t) \quad (9)$$

$N_i(t)$ is a function indicating the number of failures in the interval $[0, t]$, 1 in case of a failure and 0 otherwise. $Y_i(t)$ is the at-risk indicator, $Y_i(t) = 1$ if the subject is at risk for the i th event at time t , and 0 otherwise. To estimate ω_j , the following equation should be solved:

$$\frac{\partial PPL}{\partial \omega_j} = \frac{\partial PL}{\partial \omega_j} - \frac{\partial g(\omega; \theta)}{\partial \omega_j} = 0 \quad (10)$$

with:

$$\bar{z}_j(t) = \bar{z}_j(\beta, \omega, t) = \frac{\sum Z_{ij} Y_i(t) \exp[X_i \beta + Z_i \omega]}{\sum Y_i(t) \exp[X_i \beta + Z_i \omega]} \quad (11)$$

Thus:

$$\frac{\partial PL}{\partial \omega_j} = \sum_{i=1}^n \int_0^{\infty} (Z_{ij} - \bar{z}_j(t)) dN_i(t) \quad (12)$$

Considering the Mean Cumulative Function (MCF), the estimator $(\hat{\Lambda}_0(t; \beta, \omega))$ for failures based on parameters β and ω is defined:

$$d\hat{\Lambda}_0(t; \beta, \omega) = \frac{\sum dN_i(t)}{\sum Y_i(t) \exp(X_i \beta + Z_i \omega)} \quad (13)$$

where $\hat{\lambda}_i = \hat{\lambda}_i(\beta, \omega) = \int_0^{\infty} Y_i(s) d\hat{\Lambda}_0(t; \beta, \omega)$. Hence, the simplified PPL form becomes:

$$\frac{\partial PPL}{\partial \omega_j} \sum_{i=1}^n [Z_{ij} \delta_i - Z_{ij} \hat{\lambda}_i e^{X_i \beta + Z_i \omega}] - \frac{\partial g(\omega; \theta)}{\partial \omega_j} = 0 \quad (14)$$

Given the structure of the indicator matrix Z :

$$\frac{\partial PPL}{\partial \omega_j} = [d_j - \hat{A}_j e^{\omega_j}] - \frac{\partial g(\omega; \theta)}{\partial \omega_j} = 0 \quad (15)$$

where n_j is the number of events in cluster j , and $\hat{A}_j = \sum \hat{\lambda}_i \exp(X_i \beta)$ is calculated based on all members of that cluster.

The parameter β in the likelihood function is estimated similarly to the Cox model, where $Z\omega$ is considered as fixed values, since β does not appear in the penalty function.

4.3.3.2 | Estimation of the Gamma Frailty Distribution Parameter. Assuming the frailty terms follow a Gamma distribution with a mean of 1 and variance $\theta = 1/\nu$ ensuring identifiability, the logarithm of the frailty density function:

$$\log [f(\varpi; \nu)] = (\nu - 1) \log(\varpi) - \nu \varpi + \nu \log(\nu) - \log \Gamma(\nu) \quad (16)$$

The log density of the frailty distribution ω is expressed as $\frac{[\omega - \exp(\omega)]}{\theta}$ plus a function of θ .

$$PPL(\beta, \omega, \theta) = PL(\beta, \omega) + \left(\frac{1}{\theta} \right) \sum_{j=1}^q [\omega_j - \exp(\omega_j)] \quad (17)$$

The estimation process for the shared Gamma frailty model is carried out through a nested iterative algorithm comprising an inner and an outer loop. In the inner loop, for each fixed provisional value of θ , the penalized likelihood function is maximized using the Newton-Raphson method to estimate the regression

coefficients β and the frailty terms ω . The outer loop then updates θ by maximizing the corresponding profile likelihood. This two-stage procedure is repeated until the convergence criteria are satisfied.

4.3.4 | Model Evaluation

The Likelihood Ratio Test (LRT) is utilized to evaluate model fit and examine the presence of significant unobserved heterogeneity within the data structure [52].

$$LRT = 2 (\log L_{Mf} - \log L_{Mc}) \quad (18)$$

L_{Mf} and L_{Mc} denote the likelihood of models with and without frailty.

Akaike Information Criterion (AIC) is also used for model comparison and variable selection [54]. It consists of two components, a goodness-of-fit term and a penalty term that accounts for the number of parameters estimated in the model [55]:

$$AIC = -2 \log(L) + 2K \quad (19)$$

In this equation, L denotes the likelihood function, and K represents the number of parameters in the model.

5 | Results

The analysis started with univariable analysis, followed by multivariable analysis. In the final section, the results of the frailty model were compared with those of the model without frailty. Before incorporating observed risk factors, the initial model development focused on the frailty component to examine any potential heterogeneity within the system. Subsequently, the coefficients of the risk factors affecting the reliability and failure intensity of the dump truck portfolio were estimated using the frailty model. For this purpose, first, observed risk factors were entered individually into a frailty model. If a factor was statistically significant at a 90% confidence level, it was included in the multivariable model. Once significant factors were identified, a stepwise forward-backward method was applied to refine the selection of factors. The parameters of the frailty model, incorporating the selected risk factors, were then estimated and interpreted.

5.1 | Frailty Model, Without Considering Observed Risk Factors

Figure 8 presents the mean cumulative number of failures over time for each dump truck. As shown in the figure, the units exhibit different failure patterns, with substantial variation in failure counts. This shows the presence of heterogeneity in the studied system, which likely arises from both observed and unobserved factors. Considering that the trucks operated under varying conditions, the observed differences in failure behavior may have been influenced by factors such as spare-part quality, maintenance practices, staff competency, etc.

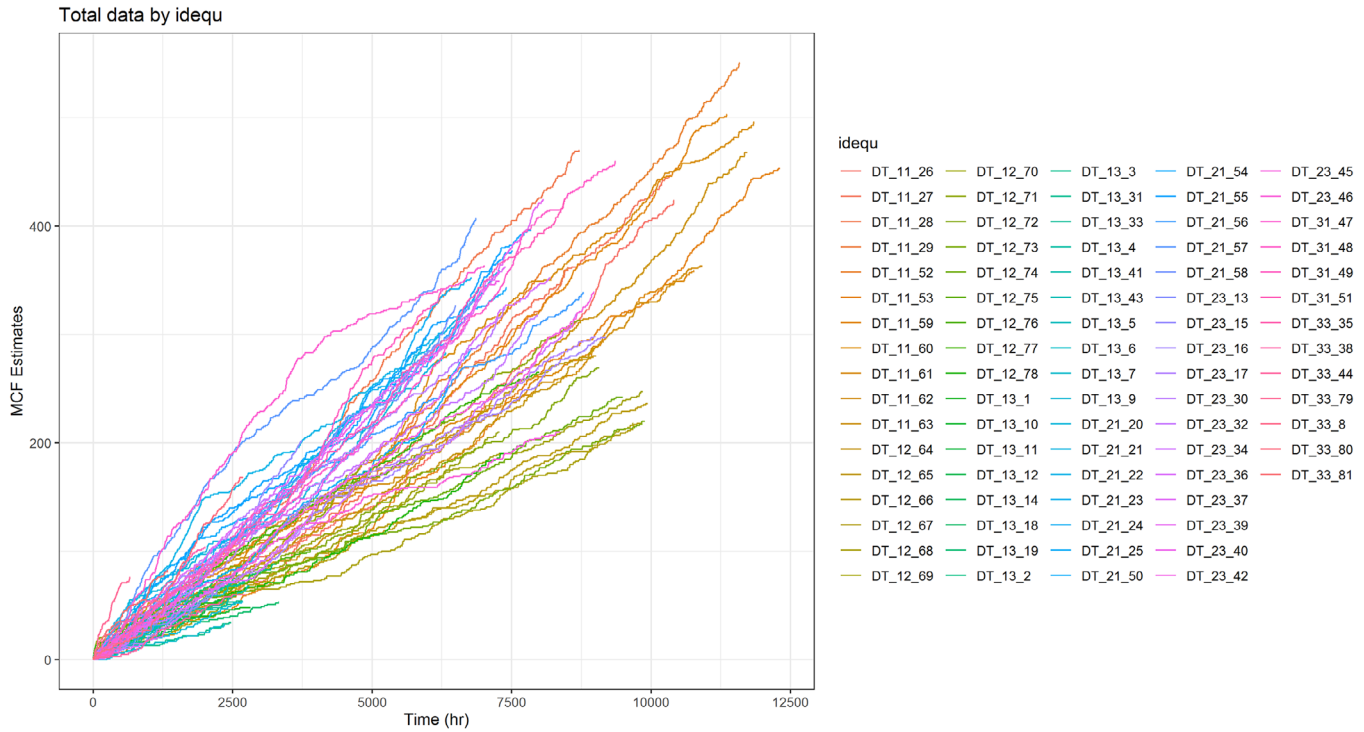


FIGURE 8 | Cumulative average of the number of failures of each dump truck.

TABLE 2 | Hazard model with frailty and without observed risk factors (null model).

θ	Chi-square	p -value	I-likelihood
0.7	1598	<0001	-78368

To examine the presence of heterogeneity within the portfolio, a semi-parametric intensity model including a frailty random effect without entering observed risk factors was considered. The detail of this model is shown in Table 2. The result showed $\theta = 0.7$, which was statistically significant with a p -value < 0.05 and indicated that unobserved risk factors influenced the failure time of the equipment.

5.2 | Univariable Analysis of the Frailty Models

To gain a clearer understanding of each risk factor's effect, each risk factor was modeled individually using a univariable frailty model, with each model including only a single observed covariate. This procedure was repeated for all risk factors, allowing separate estimation of their effects before multivariable modeling.

Categorical risk factors are incorporated into the model using dummy variables. For a risk factor with n levels, $n-1$ dummy variables are defined. For example, for the three-level work shift variable, two dummy variables can effectively represent all three levels of the original variable, as demonstrated in Table 3, using values of zero and one. This procedure was applied to all categorical and binary variables in the dataset. The findings from

TABLE 3 | Coding of the categorical work shift factor using dummy variables.

Shift	Data registration code	Virtual variable X_{12}	Virtual variable X_{13}
Morning	1	0	0
Afternoon	2	1	0
Night	3	0	1

this step showed that slope direction significantly impacts the reliability of dump trucks.

Table 4 displays the results of the individual analysis of observed risk factors in univariable frailty models (models 1–8). The findings indicated that shift, operator skill, slope direction, road condition and haulage distance were among influential risk factors affecting the failure and reliability of the dump truck portfolio.

5.3 | Multivariable Frailty Model

The previous step helped predict and check the potential outcomes of the multivariable analysis. In multivariable analysis, the selection of risk factors was performed using a stepwise procedure, based on a forward/backward algorithm, with AIC applied to balance model complexity and goodness of fit. Consequently, five risk factors, i.e., haulage distance, work shift, operator skill, road condition, and slope direction, were significant at the 95% confidence level and were identified as the final influential covariates affecting the reliability and failure occurrence of dump

TABLE 4 | Individual analysis of risk factors in frailty models (models 1 to 8).

Model		Risk factor	Coef.	Exp. Coef.	Lower	Upper	SE (coef.)	p-value
(1)	X ₁₁	Shift (Morning)	0	1	Reference level			
	X ₁₂	Shift (Afternoon)	0.10	1.11	1.08	1.15	0.02	<0.001
	X ₁₃	Shift (Nigth)	0.11	1.12	1.07	1.18	0.03	<0.001
(2)	X ₂₂	Operator skill ^a (Competent)	0	1	Reference level			
	X ₂₃	Operator skill (Proficient)	−0.18	0.84	0.81	0.87	0.02	<0.001
	X ₂₄	Operator skill (Expert)	−0.28	0.76	0.73	0.78	0.02	<0.001
(3)	X ₃₁	Slope (Flat)	0	1	Reference level			
	X ₃₂	Slope (UP)	0.15	1.16	1.10	1.23	0.029	<0.001
	X ₃₃	Slope (Down)	0.17	1.18	1.12	1.25	0.029	<0.001
(4)	X ₄₁	Road condition (Poor)	0	1	Reference level			
	X ₄₂	Road condition (Normal)	−0.14	0.87	0.84	0.91	0.02	<0.001
(5)	X ₅₁	Precipitation (No)	0	1	Reference level			
	X ₅₂	Precipitation (Yes)	0.02	1.00	0.97	1.07	0.02	0.45
(6)	X ₆	Haulage distance	0.09	1.09	1.07	1.12	0.01	<0.001
(7)	X ₇	Temperature	−0.001	0.999	0.997	1.001	0.001	3.10E-01
(8)	X ₈	Wind velocity	0.00	1.00	0.99	1.02	0.01	0.42

^aSince the first level, representing beginner operators, was not observed in the collected data, the second level was selected as the reference category.

TABLE 5 | Results of the multivariable frailty model.

Risk factors		Coef.	Exp.(coef.)	Lower	Upper	SE (coef.)	p-value
X ₁₂	Shift (Afternoon)	0.10	1.11	1.08	1.14	0.02	<0.001
X ₁₃	Shift (Night)	0.10	1.11	1.05	1.16	0.03	<0.001
X ₂₃	Operator skill (Proficient)	−0.18	0.84	0.81	0.87	0.02	<0.001
X ₂₄	Operator skill (Expert)	−0.28	0.76	0.73	0.79	0.02	<0.001
X ₃₂	Slope direction (UP)	0.14	1.15	1.08	1.22	0.03	<0.001
X ₃₃	Slope direction (Down)	0.19	1.21	1.14	1.28	0.03	<0.001
X ₄₂	Road condition [Normal]	−0.14	0.87	0.84	0.91	0.02	<0.001
X ₆	Haulage distance	0.09	1.09	1.07	1.12	0.01	<0.001

trucks. Table 5 presents the results of the final frailty model. Stepwise selection based on AIC was additionally validated against the univariable significance pattern reported above to ensure consistency.

A direct comparison of the null and final models quantifies the relative contributions of observed and unobserved sources of heterogeneity. The frailty variance dropped from $\theta = 0.7$ in the null model (no covariates) to $\theta = 0.5$ in the final model, indicating that incorporating the five observed risk factors (i.e., work shift, operator skill, slope direction, road condition, and haulage distance) explained approximately 30% of the total heterogeneity (see Table 6). The residual frailty ($\theta = 0.5$, $p < 0.001$) captures the remaining unobserved variation among trucks, attributable to factors not directly measurable in routine operational data. As a result, the model minimizes estimation bias and enhances the accuracy of the predictions.

TABLE 6 | The variance of heterogeneity in the final model.

Parameters	Value
Variance of the random effect	0.5
I-likelihood	−78148.6
P-Value	<0.001

Interpreting the coefficients of risk factors in the frailty model is more nuanced than in the standard Cox model. Because the frailty model is conditional, its coefficients are interpreted relative to a fixed level of frailty. In practical terms, this means that comparisons are made either for the same dump truck over time, assuming its frailty remains constant, or between different dump trucks that share the same frailty level. Under this interpretation,

the results indicated that, conditional on the same frailty, work shift timing influenced the rate of occurrence of failures. Both afternoon and night shifts were associated with approximately 11% higher failure intensity compared to the morning shift. This increase could be linked to factors such as operator fatigue, differences in supervision level, and environmental conditions related to different times of day. Moreover, increasing operator skill from the competent level to upper levels, i.e., proficient and expert levels, reduced failure intensity by 16% and 24%, respectively. This reduction could be explained by more efficient handling, avoidance of operational errors, better adherence to protocols and responses to unexpected conditions. As a modifiable factor, operator skill can be improved through targeted training. Slope direction also had a notable effect where both uphill and downhill increased intensity by about 15% and 21%, respectively, compared to flat terrain. This effect may be explained by the additional mechanical load on the drivetrain during ascents, which can accelerate wear of powertrain components, and by the increased braking demands during descents, which may lead to thermal stress and brake system degradation. Mitigation strategies include enhanced slope navigation training, operational speed adjustments, and modifications to ramp design where possible. Road condition, compared with poor roads as the reference category, showed that improved surfaces reduced failure intensity by 13%, likely due to enhanced vehicle stability and lower vibration loads. Maintaining optimal road conditions through regular grading and timely maintenance can substantially reduce failure risk. Hauling distance, modeled as a continuous variable, was significant at the 95% confidence level, with each additional kilometer increasing failure intensity by about 10%. This effect was likely due to the additional cumulative mechanical stress experienced in longer haul cycles. Longer distances may also have increased operator fatigue, further contributing to the elevated failure rate. Potential interventions include route optimization and relocation of loading or dumping points if it is feasible. Collectively, these findings underlined the influence of both observable and unobservable risk factors on failure intensity and emphasized that targeted interventions addressing modifiable factors can substantially improve the reliability and safety of dump trucks in mining operations.

Conditional on the frailty component, ϖ , the frailty model assumes the proportional intensities assumption. Under this framework, the conditional failure intensity for a given dump truck is defined as follows:

$$\begin{aligned} \lambda_{i(j)}(t, X, \varpi) \\ = \lambda_0(t) \varpi_f e^{0.1X_{12} + 0.1X_{13} - 0.18X_{23} - 0.28X_{24} + 0.14X_{32} + 0.19X_{33} - 0.14X_{42} + 0.09X_6} \end{aligned} \quad (20)$$

And considering the relationship between reliability and failure intensity, the conditional reliability model is obtained as follows:

$$\begin{aligned} R(t, X, \varpi) &= R(t; X)^\varpi \\ &= \left(R_0(t)^{\exp(\beta X)} \right)^\varpi \\ &= \exp(-\varpi \Lambda(t; X)) \end{aligned} \quad (21)$$

where R_0 is the baseline reliability function, and $\Lambda(t; X)$ represents the cumulative intensity function. Consequently, the

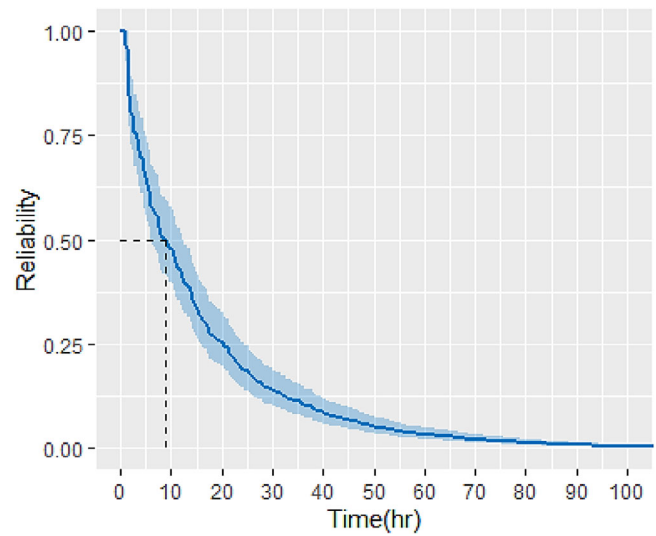


FIGURE 9 | Conditional reliability curve in the frailty mixed effects model.

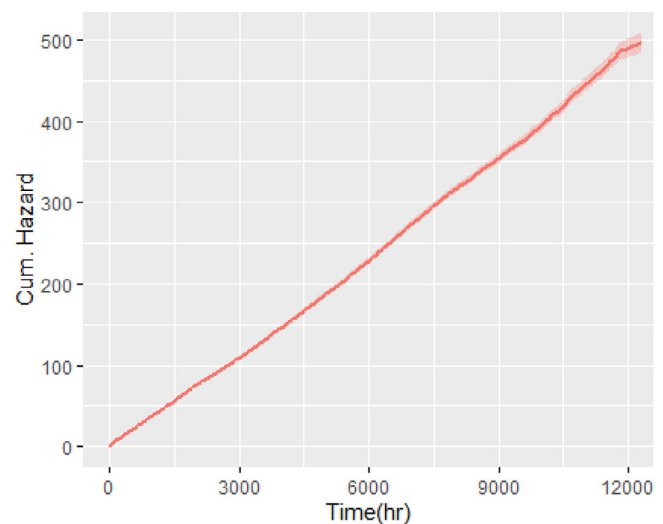


FIGURE 10 | Conditional hazard curve in the frailty mixed effects model.

conditional reliability function can be expressed as:

$$\begin{aligned} R(t, X, \varpi) \\ = \left(R_0(t)^{\exp(0.1X_{12} + 0.1X_{13} - 0.18X_{23} - 0.28X_{24} + 0.14X_{32} + 0.19X_{33} - 0.14X_{42} + 0.09X_6)} \right)^\varpi \end{aligned} \quad (22)$$

In this formulation, the frailty term is assumed to follow a Gamma distribution, specified as:

$$g(\varpi; \theta) = \frac{\varpi^{0.9} \exp(-\varpi/0.5)}{\Gamma(1.9) 0.5^{1.9}}, \theta > 0 \quad (23)$$

Conditional reliability and cumulative failure intensity curves are presented in Figures 9, 10, respectively. The conditional reliability curve for the dump truck, assuming a frailty value of one and controlling for other risk factors, shows an approximate 50 % reduction after 10 h of operation.

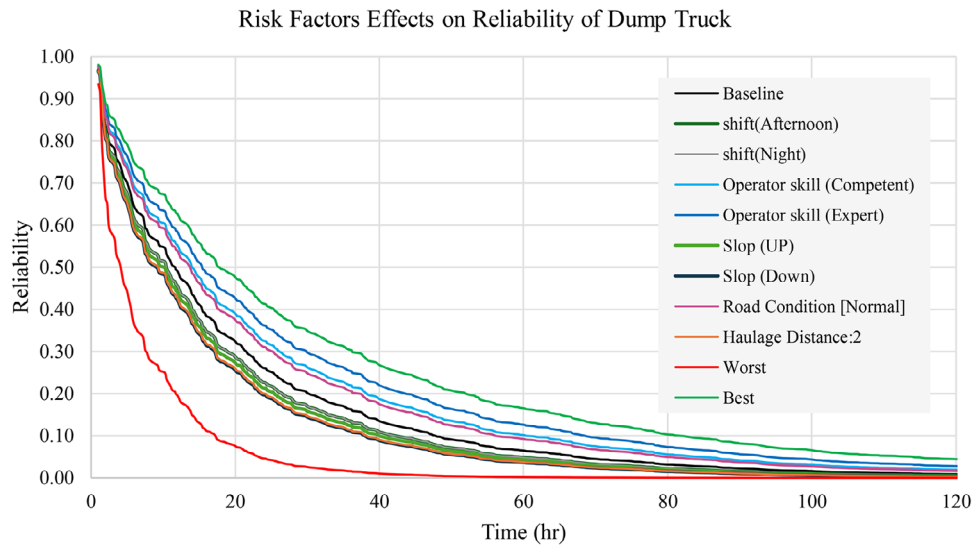


FIGURE 11 | Covariate-specific effects on dump truck reliability, conditional on frailty ($\varpi = 1$).

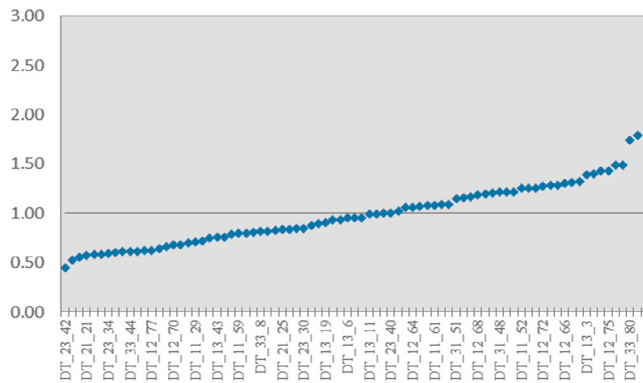


FIGURE 12 | Predicted values of unobserved heterogeneity (frailty) for each dump truck, estimated using a Gamma distribution.

Based on the conditional reliability model, with the frailty term fixed at $\varpi = 1$, the effects of individual risk factors on truck reliability were evaluated relative to the baseline reliability function. In Figure 11, each curve represents the reliability profile obtained by varying a single risk factor while holding all other risk factors at their reference levels. The black curve corresponds to the baseline reliability of the dump truck, whereas the other curves illustrate the impact of specific factors such as work shift, operator skill, slope direction, road condition, and haulage distance on reliability performance.

In the presence of unobserved risk factors, the frailty term in the model captures unexplained heterogeneity among dump trucks. Variation in the frailty values reflects differences in reliability that cannot be explained by the observed covariates alone. Figure 12 presents the predicted frailty values for each dump truck, estimated using a Gamma distribution to model variability due to these unmeasured influences.

Figure 13 illustrates the effect of frailty values on the reliability of dump trucks. Dump trucks with higher frailty values show reduced reliability and are more susceptible to failure compared to the baseline case ($\varpi = 1$). These differences highlight

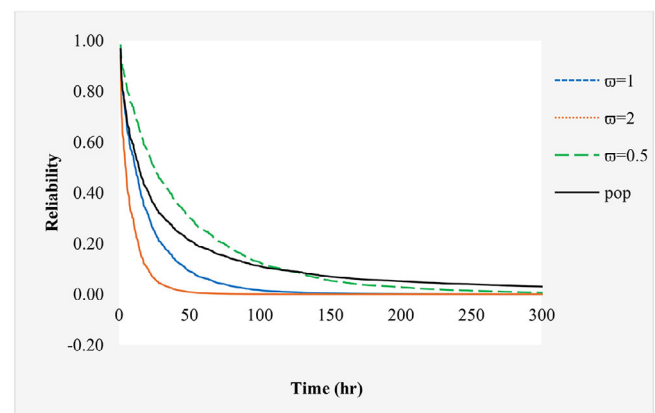


FIGURE 13 | Conditional reliability of dump trucks at different frailty levels ($\omega = 0.5, 1, 2$) compared with the unconditional (population-level) reliability curve.

the substantial role of unobserved heterogeneity in shaping population-level (portfolio) reliability outcomes.

The conditional model described reliability at the unit level (i.e., dump truck). However, system-wide decision-making requires population-level (portfolio) insights that consider heterogeneity across units. To achieve this, the reliability model is extended to the population level. The population-level reliability model is defined as the weighted average of the conditional reliability functions, where the weights are derived from the frailty distribution. To derive the unconditional reliability function, the frailty component must be integrated out. Assuming a one-parameter Gamma distribution for the frailty term, $g(\varpi; \theta)$, with a variance parameter of $\theta = 0.5$, the following formulation is obtained:

$$R(t; X)_{\theta} = (1 + \theta \Lambda(t; X))^{-1/\theta} \quad (24)$$

In this context, the subscript θ is used to denote the unconditional reliability model:

$$R(t; X)_{\theta}$$

$$= \left(1 + 0.5 \left[\Lambda_0(t) e^{0.1X_{12} + 0.1X_{13} - 0.18X_{23} - 0.28X_{24} + 0.14X_{32} + 0.19X_{33} - 0.14X_{42} + 0.09X_6} \right] \right)^{-1.9} \quad (25)$$

where $\Lambda_0(t)$ denotes the cumulative baseline hazard function.

In this case study, the average frailty of the remaining units decreases over time, reflecting the tendency for more reliable trucks to persist.

5.4 | Model Evaluation

5.4.1 | Heterogeneity Test

To test the presence of unobserved heterogeneity, the LRT was applied to compare the model with frailty (Mf) to the model without frailty (Mc) [52]. The null hypothesis assumes no frailty variance ($H_0: \theta = 0$), while the alternative hypothesis allows for unobserved heterogeneity ($H_A: \theta > 0$). The LRT statistic is defined as:

$$LRT = 2 (\log L_{Mf} - \log L_{Mc}) \quad (26)$$

which follows a chi-squared distribution with one degree of freedom. Based on the fitted models, the LRT statistic was 850 with a p-value approaching zero, providing strong evidence for heterogeneity. This confirms that the frailty model is more appropriate than the Cox model (e.g., without frailty) for the dump truck portfolio.

Additionally, the Schoenfeld residuals test is utilized to verify the proportional hazards assumption for each risk factor within the model.

5.4.2 | Proportional Hazards Assumption Test

Following the collection of data and identification of observed risk factors, it is essential to assess the potential time dependency of these covariates. To this end, the Schoenfeld residuals method is employed, which provides a statistical test of the proportionality assumption [56]. If the proportionality assumption holds for a specific risk factor conditional on frailty, Schoenfeld residuals associated with that risk factor should be independent of time. Rejection of the null hypothesis indicates a violation of the proportionality assumption. The test was conducted at the 5% significance level. The results (Table 7) show that the p-values for all risk factors exceed 0.05, indicating no violation of the proportionality assumption.

Schoenfeld residual plots (Figure 14) further support this finding, as residuals appear randomly scattered over time without a systematic trend or pattern.

6 | Discussion

This study shows that a heterogeneity-aware reliability analysis of multi-asset portfolios is essential for realistically characterizing

TABLE 7 | Results of the Schoenfeld residual test for the proportional intensity assumption.

Risk factors	Chi-square	P value
Haulage distance	0.73	0.39
Operator's skill	1.11	0.57
Shift	0.58	0.75
Road condition	2.79	0.10
Slope direction (Dip)	4.27	0.12

the failure behavior of repairable assets operating under diverse conditions. In systems composed of multiple assets, such as mining transportation fleets, heterogeneity naturally arises as units, whether similar or different in design, are exposed to various operational and environmental risk factors. As demonstrated in the empirical analysis, each dump truck working at distinct mining sites exhibited different failure intensities, confirming that heterogeneity is an attribute of industrial portfolios. Consequently, analytical approaches that ignore heterogeneity may lead to suboptimal operational and maintenance decisions.

The heterogeneity test (LRT) supported this observation, showing that the proposed frailty-based proportional intensity model is particularly effective in capturing unobserved differences among assets. These findings align with earlier work in reliability research that shows unobserved heterogeneity must be explicitly accounted for to avoid biased conclusions and ineffective maintenance planning [6, 20, 27, 35]. However, the present work extends these insights by demonstrating their relevance in a multi-asset, portfolio-level context, an analytical dimension that prior research has rarely examined in depth through real portfolio data.

This work makes important contributions to reliability modeling literature in several ways. It provides a framework that guides analysts in identifying when heterogeneity is present and in selecting an appropriate modeling strategy. This structured guidance supports more consistent and justifiable modeling decisions in complex multi-asset settings. While earlier studies have predominantly focused on isolated assets, single fleets, or limited forms of variability [4, 20, 28, 38], the current study demonstrates the suitability of frailty-based approaches for multi-asset portfolios. It further extends their application by illustrating how frailty can represent covariate effects and latent variation across large and diversified asset portfolios using a real-world mining transport system as a case study. In addition, the study generates actionable, covariate-level insights that directly support maintenance decision-making. By identifying key operational and environmental risk factors, the proposed framework enables targeted interventions at the portfolio level. In this way, reliability analysis is positioned as a decision-support tool, enhancing its practical value for managers of heterogeneous multi-asset systems. The identified risk factors can be classified according to their degree of controllability. Some factors, such as operator skill, are directly modifiable through interventions including training and operational standardization. Others, such as weather conditions, are largely unmodifiable and must be treated as external operational uncertainties. Certain factors are only partially

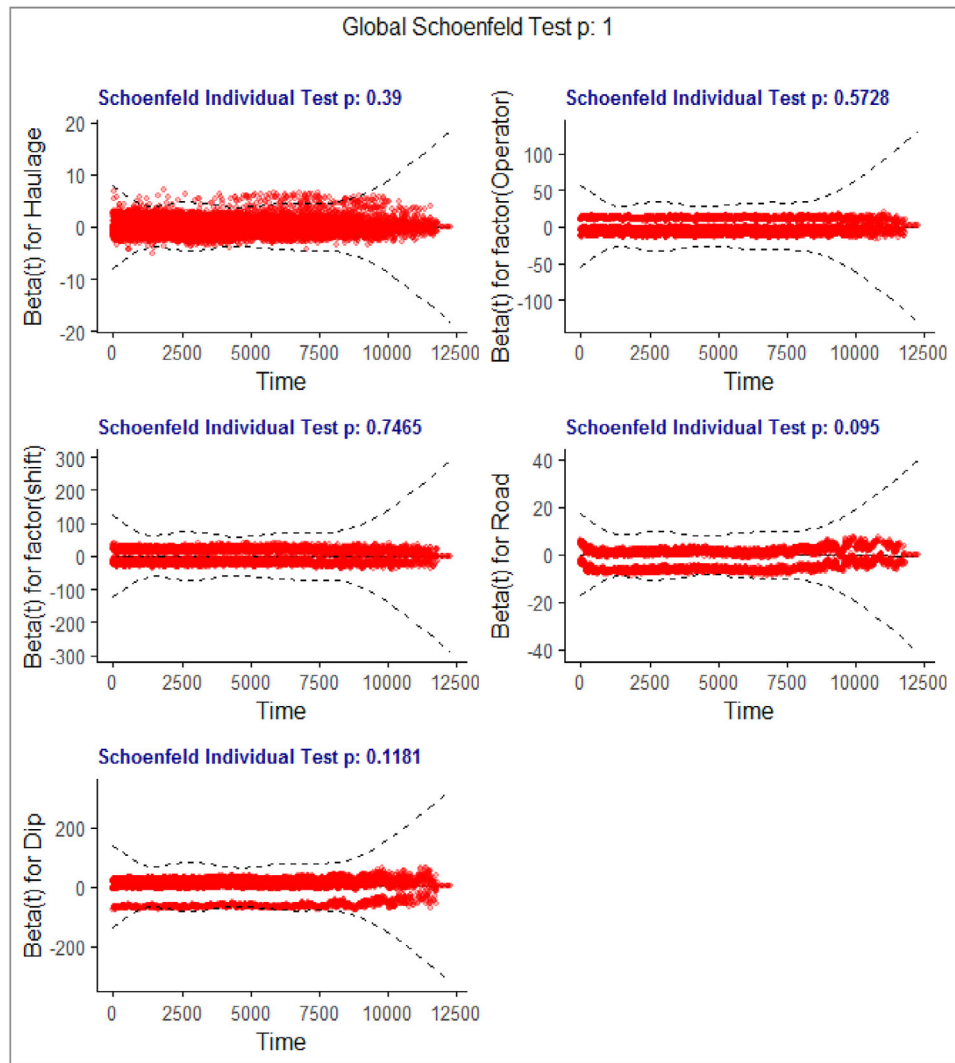


FIGURE 14 | Schoenfeld residual plots for risk factors.

modifiable due to operational or environmental constraints. For example, haulage distance and slope may be constrained by mine geometry and production levels, although alternative routing or dumping strategies may still be feasible under specific conditions. This classification supports maintenance decision-making by helping managers prioritize interventions based on their level of operational control and feasibility.

This system-wide approach also has other significant managerial implications and is particularly relevant for organizations managing large groups of assets, geographically distributed operations, or service-based portfolios, where assets encounter variable duty cycles and different maintenance regimes. First, the integration of covariate effects with asset-specific frailty scores enables more targeted maintenance prioritization by identifying units with elevated intrinsic or context-driven risk. This supports more efficient allocation of maintenance resources, contributing to improved system reliability and operational stability. Second, by quantifying how failure risk evolves as operating conditions change, the model facilitates dynamic decision-making related to maintenance scheduling, staffing levels, and budgeting. Third, the portfolio-based view enhances strategic asset planning by helping

organizations determine the optimal fleet size and composition needed to meet availability and service-level targets. Beyond these operational benefits, the model promotes organizational learning: the ability to evaluate assets collectively and compare behavior across sites allows managers to benchmark practices, identify drivers of variability, and transfer lessons across assets and sites. This is particularly valuable in industries that rely on asset redeployment or seasonal shifts in workload. The practical relevance of these insights is illustrated in Figure 15, which presents reliability estimates under extreme operational and environmental scenarios. The worst-case scenario combines the levels of risk factors that most adversely affect reliability, whereas the best-case scenario combines those that most favorably influence reliability, both relative to the baseline.

The final model focused on measurable operational and environmental covariates and included a truck-specific frailty term to account for residual unobserved heterogeneity and dependence among recurrent failures from the same truck. Nevertheless, some differences associated with mine site or truck type may contribute to the between-truck heterogeneity represented by the frailty term.

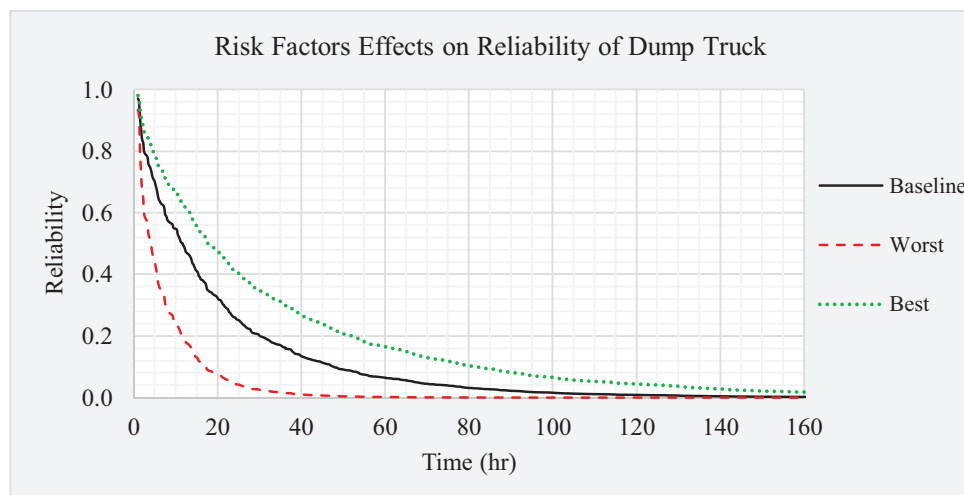


FIGURE 15 | Comparative reliability analysis of dump trucks under best- and worst-case scenarios, conditional on frailty ($\varpi = 1$).

This study builds on prior research by applying the portfolio perspective within reliability analysis, and the Cox frailty model provides the statistical foundation for implementing this portfolio perspective to address a key industry challenge faced by a mining company. It enables trucks within the same mine to show different failure patterns and explicitly considers the dependence among repeated failures of the same truck, after accounting for observed covariates. Beyond the mining context, the proposed framework contributes a generalizable approach for reliability analysis of repairable, multi-unit systems. By linking asset-level failure processes with portfolio-level performance, it supports more effective heterogeneity-aware maintenance strategies and offers a scalable methodology relevant to asset-intensive sectors such as transportation and aviation.

Several limitations of this study warrant acknowledgment and motivate directions for future research. First, the analysis does not incorporate inter-asset interactions such as load sharing or operational dependencies. This omission may limit the framework's applicability in highly interdependent systems. Second, the current model does not include interaction effects between covariates; future work should investigate whether interaction effects further improve model fit and interpretability. Addressing these limitations would enhance the applicability of the framework and provide a more realistic representation of complex multi-asset systems.

7 | Conclusion

This study developed a framework for reliability modeling of multi-asset portfolios, addressing both observed and unobserved sources of heterogeneity. Using a semi-parametric frailty-based proportional intensity model, the methodology was applied to real data from a large-scale portfolio of mining dump trucks operating under diverse conditions and locations. The results showed that incorporating frailty improves model fit compared to a model without frailty. Specifically, operational and environmental risk factors, including operator skill, working shift, haulage distance, slope direction, and road condition, were found to significantly affect failure intensity. Conditional on frailty, afternoon and night

shifts were associated with an approximately 11% increase in failure intensity relative to morning shifts, while improvements in operator skill reduced it by up to 24%. Similarly, adverse road conditions and haulage distance were shown to substantially increase the risk of failure. These insights enable targeted interventions, such as enhanced operator training, improved road maintenance, and optimized haulage management, to reduce operational downtime. The proposed framework can be adapted to other repairable systems where recurrent failures and heterogeneous conditions play a critical role in asset performance. The study contributes to both methodology and practice by extending reliability modeling beyond single-unit analyses toward a system-wide, portfolio-oriented perspective, thereby supporting more effective asset health management.

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