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Letter

Toward fault-tolerant quantum memory with universal continuous-variable gates

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Continuous-variable (CV) systems have shown remarkable potential for quantum computation, particularly excelling in scalability and holding great promise for error correction through bosonic encoding. Within this framework, the foundational notion of computational universality was introduced by Lloyd and Braunstein [Phys. Rev. Lett. **82**, 1784 (1999)], and has proven especially successful since it allows for the identification of finite sets of universal CV gates independent of the encoding scheme. However, realizing the full power of this computational model requires achieving fault tolerance, which in turn demands discretization via some form of encoding—and, to date, no proof exists that universal CV gates can give rise to encoded fault tolerance. We present compelling evidence in this direction by utilizing the Gottesman-Kitaev-Preskill (GKP) encoding. Specifically, we numerically optimize the generation of GKP states from vacua using circuits comprised solely of universal CV gates, and demonstrate that the resulting states are of sufficient quality to exhibit error probabilities below the threshold required for fault-tolerant memory via concatenated GKP-stabilizer codes.

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Introduction. In 1999, Lloyd and Braunstein (LB) introduced an approach to quantum computing based on bosonic infinite-dimensional systems [1], as opposed to the more conventional two-dimensional qubit systems [2]. This approach exploits observables with continuous spectra as information carriers [3,4], giving rise to the concept of quantum computation over continuous variables (CVs) [5–7]. This computational model has since ignited an active research field, yielding groundbreaking achievements at the forefront of quantum computing. These include the generation of entangled states comprising billions of individually addressable temporal modes [8,9], the implementation of quantum codes attaining error correction beyond the breakeven point [10–13], and leading-edge experiments pushing toward quantum computational advantage [14–16]. The abundance in nature of physical systems described by quantum CVs is also noteworthy, as evidenced by the diverse experimental platforms that have enabled these advances, including optics [17–22], trapped ions [23,24], and superconducting systems [25–27].

Central to this model of computation is the following notion of universality, which we refer to as LB universality: A physical platform is deemed universal if it can host a finite set of parametrized gates that allow for arbitrarily accurate approximation, through appropriate gate sequences, of any dynamics whose Hamiltonian is polynomial in the field variables [1]. Several studies have sought to identify platforms capable of hosting an LB universal gate set [5–7,28–34], culminating recently with the first experimental demonstration using superconducting circuits [35].

A notion of computational universality is of limited value without a robust strategy for fault tolerance [36], which is indispensable for large-scale quantum computing [37]. Error tolerance in CV systems is generally believed to require some form of discretization (or encoding), mirroring classical computing, where information is digitized despite the continuous nature of underlying physical classical variables [38]. Recently, there has been increasing interest in comparing continuous and discrete quantum computational models, and in the relationship between them provided by different encoding schemes [39–42]. In this context, the concepts of *encoded* universality and fault tolerance become central, leading to the introduction of bosonic codes [43–48]. Theoretical work has primarily focused on designing specific instances of bosonic codes [49–55]. While these efforts have been successful in their own right, they have largely bypassed the notion of universality proposed by Lloyd and Braunstein, whose key feature is precisely the identification of finite universal CV gate sets that are independent of any encoding scheme. Ideally, a computational model should guarantee robust computation regardless of the specific encoding chosen—yet, to date, no proof exists that an LB universal platform can support encoded universality and fault tolerance. The

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central theoretical obstacle is the lack of rigorous control over approximation errors in Trotter-type constructions involving unbounded operators, as used in Ref. [1] to establish universality. Consequently, universality alone does not guarantee that a target code state can be prepared with the fidelity required for fault tolerance, nor that this can be achieved efficiently [42]. Establishing this is the main technical contribution of our work. Specifically, we demonstrate that GKP code states of sufficient quality for fault-tolerant quantum memory can be prepared using a relatively small number of LB gates. While we do not claim immediate practical applicability, this result closes a fundamental gap and provides a key step in elevating the LB model to a fully fledged framework for fault-tolerant quantum computation.

LB universal model. The LB paradigm entails computing over individually addressable bosonic modes. Denoting with $\hat{a}$ and $\hat{a}^\dagger$ the annihilation and creation operators of a mode, the position and momentum operators are given by $\hat{q} = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$ and $\hat{p} = (\hat{a} - \hat{a}^\dagger)/\sqrt{2}i$, respectively. Following Ref. [1], we select a set of CV gates made up of displacements in position $X(c)$, squeezing $S(r)$, rotations $R(\theta)$, and Kerr gates $K_{\rm LB}(\kappa)$, defined as

$$X(c) = \exp[-ic\hat{p}], \tag{1}$$

$$S(r) = \exp\left[i\frac{r}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})\right], \tag{2}$$

$$R(\theta) = \exp\left[i\frac{\theta}{2}(\hat{q}^2 + \hat{p}^2)\right], \tag{3}$$

$$K_{\rm LB}(\kappa) = \exp[i\kappa(\hat{q}^2 + \hat{p}^2)^2], \tag{4}$$

with $\{c, r, \theta, \kappa\} \in \mathbb{R}$. A Fourier transform can be implemented as $R(\pi/2)$ since this swaps the position and momentum axes. The Kerr gate is the only non-Gaussian gate in the set and is the most difficult to realize experimentally. However, this can be achieved using a SNAIL-terminated nonlinear resonator [56].

It is known that this set of gates is LB universal for single-mode computation since unitary operators generated by polynomials of arbitrary order in $\hat{q}$ and $\hat{p}$ can be arbitrarily well approximated by sequences of the above gates [1]. The addition of any nontrivial two-mode gate acting between two generic modes j and k, such as $\mathrm{CZ} = \exp[i\hat{q}_j\hat{q}_k]$, allows for full multimode LB universality. For the LB model to enable computation, it is essential to specify not only a gate set but also a fiducial initial state and a set of CV measurements. Since, in the CV gate set described, all but the Kerr gate are Gaussian [57,58], we use Gaussian inputs and measurements to avoid introducing further resources [59,60]. For simplicity, we will consider vacuum states as input and position-eigenstate projectors as measurements (homodyne measurements). We also include classical feedforward in our definition of the LB model—i.e., the ability to use classical communication to feed measurement results to subsequent gates. Since the LB approach is encoding agnostic (as it directly concerns the universal control of the full CV state), to demonstrate that the above-defined model enables, in turn, *encoded* universality and fault tolerance, it is sufficient to prove this for a specific encoding. For convenience, we adopt the one proposed by Gottesman, Kitaev, and Preskill (GKP) [51]. We will now present a strategy for achieving GKP-encoded universality using the LB universal model defined above.

LB universality implies encoded universal computation. The approximate basis square-GKP states $|\tilde{\mu}\rangle$, for $\mu \in \{0, 1\}$, are defined as [61]

$$|\tilde{\mu}\rangle = N \sum_{n=-\infty}^{+\infty} \exp\left[-\frac{1}{2}\Delta^2([2n+\mu]\sqrt{\pi}\,)^2\right] \times X([2n+\mu]\sqrt{\pi}\,)|\Psi(0)\rangle, \tag{5}$$

where N is a normalization constant and Δ^{-1} is the width of a Gaussian envelope. Here, $|\Psi(x)\rangle$ is the approximate position eigenstate with eigenvalue x,

$$|\Psi(x)\rangle = \int_{-\infty}^{+\infty} dq\, \frac{1}{(\pi\Delta^2)^{1/4}} \exp\left[-\frac{1}{2}\frac{(q-x)^2}{\Delta^2}\right]|q\rangle_q, \tag{6}$$

which is a squeezed state of width Δ, with squeezing in decibels given by $s_{\rm dB} = -10\log_{10}\Delta^2$. In the limit of infinite squeezing, one obtains the so-called ideal GKP states, hereafter indicated as $|\bar{\mu}\rangle$.

The first step is to show that the CV gates in Eqs. (1)–(4) can generate approximate GKP states from vacuum. This is not obvious, as there exists no analytical proof confirming this capability or specifying the quality of the approximation that can be achieved. In general, it is unknown whether an LB universal gate set can generate a dense family of states in the full Hilbert space of the system [62]. A recent related work showed that polynomially generated Hamiltonians can generate any physical state from vacuum, with a rigorous approximation bound [42]. However, the constructive proof uses a universal set that is very different from Ref. [1] and whose size depends on the target accuracy. Extending those results to a *finite* set of universal Hamiltonians, the size of which does not depend on the approximation error, remains an open question.

While various proposals exist for generating the standard GKP states, none use an LB universal model [63]. Some methods are probabilistic and conditioned on measurements [51,64–75], while others involve auxiliary qubits [76–79], or higher-dimensional discrete-variable systems [80]. Also, generation strategies based on ground-state preparation of systems modeled by a GKP Hamiltonian or on dissipation engineering have been extensively studied (see Ref. [48] and references therein). Although we are interested in GKP state generation not from a practical perspective but instead as a theoretical means to assess the capability of the LB framework, it is worth noting that recent experiments have succeeded in generating GKP states by a combination of measurements and auxiliary qubits [23–25,27,81]. All of these methods lie outside the LB approach, which is our focus. Protocols involving auxiliary qubits and discrete-variable measurements are clearly beyond the scope of sole CV systems, whereas proposals that include ground-state preparation or dissipation have no known decomposition in terms of an LB universal gate set or are not unitary. Implementations involving conditional CV measurements are nondeterministic, and therefore problematic when considering encoded fault tolerance and the related overhead.

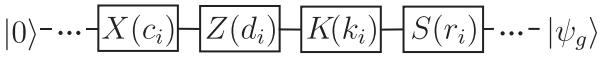


FIG. 1. One block of the circuit of universal CV gates used for generating GKP states. Here, $X(c_i)$ is a displacement in position, $Z(d_i)$ is a displacement in momentum, $K(k_i)$ is the Kerr gate, and $S(r_i)$ is squeezing. The input is the vacuum state $|0\rangle$ and the output is the generated state $|\psi_g\rangle$.

Therefore, we employ a numerical approach to generate GKP states in a deterministic, unitary, and fully CV manner. We use a parametrized quantum circuit of repeating blocks shown in Fig. 1, inspired by Ref. [82], composed of gates from the universal set in Eqs. (1)–(4) or straightforwardly obtainable from them [83]. The gate parameters are vectors $\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{k}$, and $\boldsymbol{r}$, of length equal to the total number of blocks, where the ith element of each gives the parameters for the ith. We take the vacuum as input, and the square GKP $|\tilde{0}\rangle$ in Eq. (5) as the target $|\Psi_t\rangle$. We use the Strawberry Fields Python library [84,85] to simulate this circuit using a truncated Fock representation. We optimize the gate parameters to maximize the fidelity, $F = |\langle\Psi_g|\Psi_t\rangle|^2$, between the generated and the target state. We emphasize that the circuits presented here are not intended as experimental proposals for GKP state preparation; our interest is mainly fundamental—specifically, to investigate whether an LB universal gate set is capable of generating GKP states of sufficient quality for encoded universality and error correction.

An example of the generated states compared to the target can be seen in Fig. 2, which shows the position probability distributions for both, for $\Delta = 0.12$. See the Supplemental Material [86] for the probability distributions in momentum space and the Wigner function. For this case, the fidelity between the target and generated states is 0.987. More generally, we generate states with inverse width Δ between 0.4 and 0.08, obtaining infidelities $1 - F$ below 0.03, as shown in Fig. 3 (dashed line, right axis). Details of our simulations can be found in the Supplemental Material [86] and the code is available in Ref. [87].

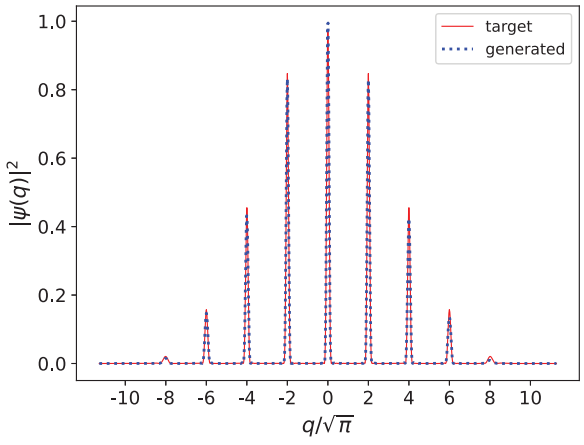


FIG. 2. Probability distributions for a target $|\tilde{0}\rangle$ state, with $\Delta = 0.12$, and the state generated numerically using a Fock basis cutoff of 120.

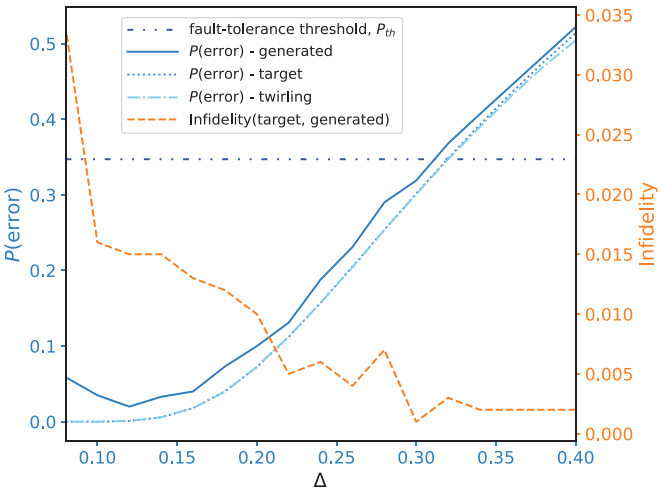


FIG. 3. Error probability, calculated using Eq. (A1), for target GKP $|\tilde{0}\rangle$ and generated states with different values of Δ, the inverse width. We show the error probability for the twirling states used in Ref. [90], as well as the fault-tolerance threshold. The infidelity between the target and generated GKP states is also plotted.

Now that we have seen that the GKP qubits $|\tilde{0}\rangle$ can be prepared approximately within an LB universal model, encoded universality can be proven by showing how to implement universal circuits on the embedded qubits. A conventional choice [2] for the latter circuits comprises the ability to perform (1) preparation and measurement in the qubit computational basis; (2) arbitrary encoded Clifford circuits as sequences of Hadamard $\bar{H}$, phase $\bar{S}$, and control-$\bar{Z}$ gates; and (3) encoded non-Clifford gate such as the $\bar{T}$ gate. Therefore, encoded universality follows from three considerations. First, encoded basis measurements correspond to homodyne measurements with binned outcomes, which are already within our minimally extended LB universal model. Second, encoded Clifford gates can be obtained immediately from the chosen LB universal set, since the control-$\bar{Z}$ gate coincides with CZ, the Hadamard $\bar{H}$ gate corresponds to $R(\pi)$, and the phase gate $\bar{S}(s)$ corresponds to $R(\theta)S(r)R(\theta - \pi/2)$, where [88]

$$s = 2\sinh r, \quad \cos\theta = \frac{1}{\sqrt{1+e^{2r}}}, \quad \sin\theta = \frac{e^r}{\sqrt{1+e^{2r}}}. \tag{7}$$

Third, the $\bar{T}$ gate can be approximately implemented using encoded Clifford gates along with a supply of encoded $|\tilde{H}\rangle$ states [89], where

$$|\tilde{H}\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\frac{\pi}{8}}|\tilde{0}\rangle + e^{i\frac{\pi}{8}}|\tilde{1}\rangle\right). \tag{8}$$

We can generate these $|\tilde{H}\rangle$ states using the same method as for the $|\tilde{0}\rangle$ states, obtaining similar fidelities, as shown in the Supplemental Material [86]. Alternatively, notice that one could implement a $\bar{T}$ gate simply by equipping the reference LB universal set in Eqs. (1)–(4) with an additional cubic-phase gate [51].

Fault tolerance. We now address whether the LB notion of computational universality is compatible with fault tolerance. Specifically, we argue that a fault-tolerant quantum memory can indeed be realized using GKP-encoded qubits prepared

via the LB universal gate set defined in Eqs. (1)–(4). Our primary objective is to show that the imperfections inevitably introduced by the LB framework during state preparation do not, in themselves, preclude fault tolerance. To isolate the effect of encoding imperfections, we assume that all operations from the LB universal set used in the error correction procedure are implemented ideally, with no additional sources of error. Our argument builds on Ref. [90], which showed that infinite-energy GKP states can serve as qubits in a surface code to realize a fault-tolerant quantum memory. Therefore, to establish that LB universal sets enable fault-tolerant quantum error correction, it suffices to demonstrate that such a set can generate a supply of GKP states that, despite inherent imperfections and finite energy, are of sufficient quality to meet the conditions for fault tolerance.

We compute the probability of uncorrectable errors, $P(\text{error})$, upon measuring the quadratures of the generated GKP states [see Eq. (A1)]. Errors are deemed uncorrectable if they violate the Glancy-Knill condition [91], which provides an upper bound on the size of shift errors that can be corrected by repeated rounds of Steane-type GKP error correction. For fault tolerance, we require $P(\text{error}) < P_{\text{th}}$, where P_{th} is the threshold value for a given discrete-variable error correcting code operating at the logical level, for a given noise model. To our knowledge, this is the first error analysis for approximate GKP states that can be prepared with a physically realizable procedure prescribed within the LB framework. Using the twirling approximation (see details in the Appendix), Noh *et al.* [90] identified a 9.9 dB threshold for fault-tolerant quantum memory in terms of the required squeezing, which we convert to a threshold P_{th} in terms of error probability.

Following the reasoning of Ref. [90], we assume that, when using imperfect generated GKP states as the qubits in a surface code, the dominant sources of noise are the finite squeezing of the GKP peaks and the imperfect fidelity between the target and generated GKP states. $P(\text{error})$ accounts for both of these factors (see Appendix for justification). To obtain P_{th} for a quantum memory using the GKP surface code with this noise model, we compute $P(\text{error})$ for the twirling-approximated GKP states with squeezing 9.9 dB (namely, $\Delta = 0.32$). Since twirling-approximated GKP states are periodic, with periodicity $2\sqrt{\pi}$, the ratio $P(\text{error}):P(\text{no error})$ over the full infinitely extended wavefunction is equal to that ratio over a single cell $q \in (-\sqrt{\pi}, \sqrt{\pi}\,]$. This allows us to calculate $P(\text{error})$ for different values of Δ. The results are plotted in Fig. 3 (dot-dashed line). In particular, we see that the value for $\Delta = 0.32$ is $P_{\text{th}} = 0.347$, which is also shown in Fig. 3 (horizontal line). Notice that the error probability for the target states in Eq. (5) is also plotted (dotted line), and it is almost indistinguishable from the one corresponding to the twirling-approximated states. This confirms the validity of the twirling approximation in the range of Δ values shown [92], corroborating the results found in Ref. [93]. Therefore, we conclude that a fault-tolerant memory can be achieved if it is possible to generate GKP states with $P(\text{error})$ significantly smaller than $P_{\text{th}} = 0.347$.

The identified threshold can now be used to assess the quality of the states generated within the LB model, as well as any state in general. The error probability for the generated states is plotted in Fig. 3 as well (solid line). Below $\Delta = 0.3$ (squeezing above 10.5 dB), the error probability for the generated states falls below P_{th} reaching a minimum of 0.0174 at $\Delta = 0.12$ [94]. Therefore, we can conclude that the states generated using the LB framework can be of sufficient quality for a fault-tolerant quantum memory. Note that here we only consider fault tolerance in the error correction of the prepared GKP states. We do not analyze how gates could lead to error propagation. Hence, our result does not show fault-tolerant quantum computation, but instead is a demonstration of a fault-tolerant quantum memory.

Assessing the usefulness of our generated states for quantum computation beyond memory operation requires accounting for the fact that even perfect gates act imperfectly on states with finite squeezing and nonunit fidelity [95,96]. For example, applying a perfect X gate to our generated GKP $|\tilde{0}\rangle$ with $\Delta = 0.12$ from Fig. 2 yields $|\tilde{1}\rangle$ with $P(\text{error}) = 0.0313$, compared to 0.0197 for the original $|\tilde{0}\rangle$ state. Although both of these values are well below the fault-tolerance threshold, further analysis is required to establish full fault-tolerant quantum computation.

Conclusions. We have investigated a computational framework regarded as universal in the CV setting, according to the seminal notion introduced by Lloyd and Braunstein [1], and recently experimentally realized [35]. Within this framework, we have demonstrated the ability to generate encoded GKP qubits suitable for performing universal encoded computation and, crucially, that these qubits can support a fault-tolerant quantum memory.

These results show that the long-standing Lloyd-Braunstein framework is not only computationally universal in the original sense of Ref. [1] but also capable of supporting fault-tolerant quantum memories, a key requirement for scalable quantum computation. This marks a significant theoretical advancement in establishing the power of the Lloyd-Braunstein model of CV quantum computing. To fully demonstrate its compatibility with fault-tolerant quantum computation, future work should analyze how imperfections in the GKP qubits propagate through logical gate implementations within this framework.

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Data availability. The data that support the findings of this article are openly available [87].

Appendix: Error probability and fault-tolerance threshold. Consider the protocol for correcting a momentum error in an approximate GKP state, $|\tilde{\mu}\rangle$, as shown in Fig. 4. Note that, in the case where the circuit is made up of perfect gates and the

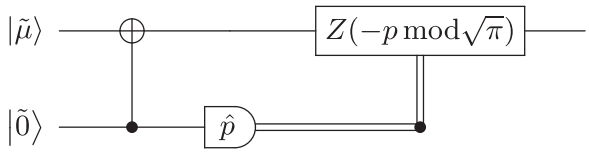


FIG. 4. Quantum circuit for correcting momentum errors in an approximate GKP state, $|\tilde{\mu}\rangle$, using an auxiliary GKP qubit, $|\tilde{0}\rangle$ [51,91]. An inverse-SUM gate, $\exp[i\hat{q}_1\hat{p}_2]$, acts on the inputs. Then, the second mode is measured in the momentum basis and the measurement outcome, p, is fed to a displacement gate, $Z(-p \bmod \sqrt{\pi})$. Notice that all the elements of the circuit are comprised in our LB universal model.

auxiliary GKP qubit is the ideal $|\bar{\mu}\rangle$, this gadget projects the input state $|\tilde{\mu}\rangle$ onto an eigenstate of the modular momentum operator [101]. To correct errors in position, we use this circuit again after a Fourier transform, where this time the state is projected onto an eigenstate of the modular position operator in the ideal case. In the case where the auxiliary GKP states are noisy, the Glancy-Knill condition [91] states that repeated use of this quantum error correction procedure succeeds when the magnitude of each displacement error measured is less than $\sqrt{\pi}/6$. In this context, failure means that either a logical error is introduced or the quality of the data register is decreased by the process of syndrome extraction with imperfect auxiliary states and subsequent correction. The probability that, upon measurement, a general state $|\psi\rangle$ gives outcomes larger than $\sqrt{\pi}/6$ is given by

$$P(\text{error}) = 1 - \int_{-\sqrt{\pi}/6}^{\sqrt{\pi}/6} du \int_{-\sqrt{\pi}/6}^{\sqrt{\pi}/6} dv |\langle\psi|u, v\rangle|^2, \quad \text{(A1)}$$

where $|u, v\rangle$ is the state shifted by u in position and v in momentum from the ideal $|\bar{0}\rangle$, i.e.,

$$|u, v\rangle = \pi^{-1/4} e^{-iu\hat{p}} e^{-iv\hat{q}} |\bar{0}\rangle. \quad \text{(A2)}$$

In Ref. [90], Noh *et al.* showed that when GKP error correction is concatenated with the surface code [102] to correct errors at the qubit level, the dominant noise source is the finite squeezing of the auxiliary GKP states used for physical error correction. They made use of the twirling approximation [103], in which approximate GKP states are modeled as ideal GKP states with the Dirac delta functions replaced by Gaussians with constant nonzero variance. Unlike the approximate GKP states given in Eq. (5), this approximation does not include a surrounding Gaussian envelope and hence the states are unphysical (infinite energy) as their wavefunctions extend to infinity in position and momentum, but this greatly simplifies the error analysis. In particular, this approximation allows for the estimation of the logical error probability of the *overall* concatenated error correction procedure using Monte Carlo methods. The authors found a threshold squeezing of 9.9 dB for fault-tolerant quantum error correction with twirling-approximated GKP states. We note that this threshold is in relatively close agreement with the 10.5 dB threshold found in Ref. [19] for a measurement-based fault-tolerant quantum memory. This threshold of 10.5 dB was considered in Ref. [104], where it was converted to a threshold expressed in terms of a measure of the non-Gaussianity of GKP states that they introduce, called "GKP squeezing."

Since, following Ref. [90], the noise introduced to the error correction procedure by our generated GKP states comes from their finite squeezing and nonunit fidelity, we assume that the error probability $P(\text{error})$ calculated in Eq. (A1) is sufficient to characterize the effective qubit channel induced on the input to the next layer of encoding, which might, for example, be the rotated surface code considered in Ref. [90] or the measurement-based code in Ref. [19]. This is justified by the observation that decoding of concatenated GKP codes is typically performed in two steps: first GKP-level stabilizers are measured, essentially converting CV noise to qubit-level noise, and then qubit-level stabilizers are measured and a higher-level correction is computed. The second step succeeds if the strength of the effective qubit-level noise is low enough after the first step, which is the essence of the Glancy-Knill criterion. Strictly speaking, $P(\text{error})$ alone does not characterize the most general qubit channel, but since the qubit-level noise is local by construction and symmetry between bit flips and phase flips is approximately ensured by the large fidelity with the target state (which is symmetric), we can assume that the true effective qubit channel will not be significantly more detrimental to the operation of the higher-level error correction than a stochastic bit-phase flip channel of equal intensity. In our analysis, we have considered only basic GKP decoding, which is sufficient for our purposes. However, it should be noted that more sophisticated decoders have been introduced, which make use of the analog information in the GKP syndrome measurement [105]. These could be used to reduce the overhead in the GKP state preparation.

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